Registration# 2019-EE-383

University of Engineering & Technology Lahore

Experiment # 4

Title: Z Transform-I

Equipment Required: Personal computer (PC) with windows operating system and MATLAB software

Theory:

Z transform is an important tool for the analysis and design of discrete time signals. It transforms a signal of time domain into a function of variable z. For discrete time signal x(n), z transform is defining as,

$$X[Z] = \sum_{n=-\infty}^{\infty} x[n]Z^{-n}$$

The properties of the z-transform are generalizations of the properties of the discrete-time Fourier transform.

1. **Linearity:**

   $$\mathcal{Z}\left[a_1x_1(n) + a_2x_2(n)\right] = a_1X_1(z) + a_2X_2(z); \quad \text{ROC: ROC}_{x_1} \cap \text{ROC}_{x_2}$$

2. **Sample shifting:**

   $$\mathcal{Z}\left[x\left(n - n_0\right)\right] = z^{-n_0}X(z); \quad \text{ROC: ROC}_x$$

3. **Frequency shifting:**

   $$\mathcal{Z}\left[a^n x(n)\right] = X\left(\frac{z}{a}\right); \quad \text{ROC: ROC}_x \text{ scaled by } |a|$$

4. **Folding:**

   $$\mathcal{Z}\left[x\left(-n\right)\right] = X\left(1/z\right); \quad \text{ROC: Inverted ROC}_x$$

5. **Complex conjugation:**

   $$\mathcal{Z}\left[x^*(n)\right] = X^*(z^*); \quad \text{ROC: ROC}_x$$

6. **Differentiation in the z-domain:**

$$\mathcal{Z}\left[nx(n)\right] = -z\frac{dX(z)}{dz}; \quad \text{ROC: ROC}_x$$

This property is also called the *multiplication-by-a-ramp property.*

7. **Multiplication:**

$$\mathcal{Z}\left[x_1(n)x_2(n)\right] = \frac{1}{2\pi j}\oint_C X_1(\nu)X_2(z/\nu)\,\nu^{-1}d\nu;$$
$$\text{ROC: ROC}_{x_1} \cap \text{Inverted ROC}_{x_2}$$

where C is a closed contour that encloses the origin and lies in the common ROC.

8. **Convolution:**

$$\mathcal{Z}\left[x_1(n) * x_2(n)\right] = X_1(z)X_2(z); \quad \text{ROC: ROC}_{x_1} \cap \text{ROC}_{x_2}$$

Example 1:

Find z transform of $x(n) = (1/4)^n\ u(n)$.

$>>\ syms\ z\ n$

$>>\ ztrans\ (\left(\frac{1}{4}\right)^n)$

Task#01:

i) $\boldsymbol{x(n) = (1/4)^n\, u(-n)}$.

MATLAB Code:

```
DSPLab5task1a.m

%LAB#5 Task#1(a)
syms z n;
ztrans(1/4^-n,z^-1)
```

Output:

```
Command Window
>> DSPLab5task1a

ans =

1/(z*(1/z - 4))
```

Handwritten Solution:

Task#1 :

(1) $x(n) = (1/4)^n u(-n).$

$$X(z) = \sum_{-\infty}^{0} (1/4)^n z^{-n}$$

$$= \sum_{0}^{\infty} (1/4)^{-n} z^{n}$$

$$= \frac{1}{1-\left(\frac{1}{4}\right)^{-1} z} \Rightarrow \frac{1}{1-4z}.$$

$$= \frac{1}{z\left(\frac{1}{z}-4\right)}.$$

ii) $x(n) = (0.8)^n\, u(-n-1)$.

MATLAB Code:

```
%LAB#5 Task#1(b)
syms z n;
ztrans(0.8^-n,z^-1)*z^-1*4^1
```

Handwritten Solution:

(ii) $x(n) = (0.8)^n u(-n-1)$.

$$X(z) = \sum_{-\infty}^{-1} (0.8)^n z^{-n}.$$

$$X(z) = \sum_{n=1}^{\infty} (0.8)^{-n} z^n$$

$$= \sum_{n=0}^{\infty} (0.8)^{-(n+1)} z^{n+1}.$$

$$= \sum_{0}^{\infty} (0.8)^{-n} \cdot (0.8)^{-1} \cdot z^n \cdot z^1.$$

$$= (0.8)^{-1} \cdot z \sum_{0}^{\infty} (0.8)^{-n} z^n.$$

$$= (0.8)^{-1} \cdot z \left[\frac{1}{1-(0.8)^{-1} z} \right].$$

$$= (0.8)^{-1} z \times \sum_{0}^{\infty} (0.8)^{-n} z^n$$

$$= \left(\frac{4}{5}\right)^{-1} z \times \left[\frac{1}{1-\left(\frac{4}{5}\right)^{-1} z^1} \right].$$

$$= \frac{5z}{4} \left[\frac{1}{1-\frac{5}{4}z} \right].$$

$$= \frac{5}{\cancel{4}} z \left[\frac{\cancel{4}}{4-5z} \right] = \frac{5z}{4-5z}.$$

iii) $x(n) = \quad (4^n)\, u(1-n)$.

MATLAB Code:

```
%LAB#5 Task#1(c)
syms z n;
ztrans(4^-n,z^-1)*z^-1*4^1
```

Output:

```
Command Window
>> DSPLab5task1c

ans =

4/(z^2*(1/z - 1/4))
```

Handwritten Solution:

(iii): $x(n) = (4)^n u(1-n)$

$$X(z) = \sum_{-\infty}^{+1} (4)^n z^{-n}$$

$$X(z) = \sum_{-1}^{\infty} (4)^{-n} z^{n} = X(z) = \sum_{0}^{\infty} (4)^{-(n-1)} z^{n-1}$$

$$X(z) = \sum_{n=0}^{\infty} (4)^{-n} \cdot 4^{1} \cdot z^{n} \cdot z^{-1}.$$

$$= \frac{4}{z} \sum_{0}^{\infty} (4)^{-n} z^{n} = \frac{4}{z}\left[\frac{1}{1-\left(\frac{1}{4}\right)z}\right].$$

$$= \frac{4}{z}\left[\frac{4}{4-z}\right] = \frac{16}{z(4-z)}.$$

$$= \frac{16}{4z - z^2}.$$

iv) $x(n) = \ (n+1)(3^{n-2})\ u(n).$

MATLAB Code:

```

%LAB#5 Task#1(e)
syms z n;
ztrans((n+1)*(3^(n-2)))

```

Output:

Command Window

```
>> DSPLAB5task1e

ans =

z/(9*(z - 3)) + z/(3*(z - 3)^2)
```

Handwritten Solution:

(iv) $x(n) = (n+1)(3^{n-2})\,u(n)$

$X(z) = \left[n(3^{n-2}) + (3^{n-2})\right]u(n).$

$X(z) = (n(3^{n-2})u(n) + (3^{n-2})u(n).$

As z-transform of $(3^{n-2})\,u(n)$ is

$X(z) = \sum_{0}^{\infty}(3^{n-2})\cancel{u(n)} = 3^{-2}\sum_{0}^{\infty}(3^{n})z^{-n}$

$= \frac{1}{9(1-3z^{-1})}$

$X(n) = (n(3^{n-2}) + 3^{n-2})\,u(n).$

As $-z\frac{dX(z)}{dx} = -z\frac{d}{dz}\left[\frac{1}{9(1-3z^{-1})}\right].$

$= \frac{z^{-1}}{3(1-3z^{-1})^{2}}.$

$X(z) = \frac{z^{-1}}{3(1-3z^{-1})^{2}} + \frac{1}{9(1-3z^{-1})}.$

$X(z) = \frac{z}{3(1-3z^{-1})^{2}} + \frac{z}{9(z-3)} = \frac{z}{3(z-3)^{2}} + \frac{z}{9(z-3)}.$

$$v)\ \ x(n) = nSin(\pi n/3)\, u(n) + \quad (0.9)^n\, u(n-1)$$

MATLAB Code:

```
%LAB#5 Task#1(f)
syms z n;
ztrans(n*sin(pi*n/3))+ztrans(0.9^n)*z^-1
```

Output:

```
Command Window
>> DSPLAB5task1f

ans =

1/(z - 9/10) + (3^(1/2)*z*(z^2 - 1))/(2*(z^2 - z + 1)^2)
```

Handwritten Solution:

(V): $n\sin(n\pi/3)u(n) + (0.9)^n u(n-1)$.

$x(n) = \sin(\frac{\pi}{3}n)u(n)$.

$$= \sum_{0}^{\infty}\left[\frac{1}{2j}\left(e^{j\pi/3\, n} - e^{-j\pi/3\, n}\right)\right]u(n).$$

$$X(z) = \frac{1}{2j}\sum_{0}^{\infty}\left[e^{j\pi/3\, n}z^{-1} \cancel{+} - e^{-j\pi/3\, n}z^{-1}\right].$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j\pi/3}z^{-1}} - \frac{1}{1-e^{-j\pi/3}z^{-1}}\right].$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j\pi/3}z^{-1} - (1-e^{j\pi/3}z^{-1})}{(1-e^{j\pi/3}z^{-1})(1-e^{-j\pi/3}z^{-1})}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{1}-e^{-j\pi/3}z^{-1}-\cancel{1}+e^{j\pi/3}z^{-1}}{1-e^{-j\pi/3}z^{-1}-e^{+j\pi/3}z^{-1}+z^{-2}}\right].$$

$$= \frac{1}{2j}\left[\frac{z^{-1}\left[e^{j\pi/3}-e^{-j\pi/3}\right]}{1-(e^{+j\pi/3}+e^{-j\pi/3})z^{-1}+z^{-2}}\right].$$

$$= \frac{1}{\cancel{2j}}\left[\frac{\cancel{2j}\,z^{-1}\left[\frac{e^{j\pi/3}-e^{-j\pi/3}}{2j}\right]}{1-2z^{-1}\left(\frac{e^{j\pi/3}+e^{-j\pi/3}}{2}\right)+z^{-2}}\right].$$

$$= \frac{z^{-1}\sin(\pi/3)}{1-2z^{-1}\cos(\pi/3)+z^{-2}} = \frac{z^{-1}(\sqrt{3}/2)}{1-\cancel{2}z^{-1}(1/\cancel{2})+z^{-2}}$$

$$= \sin(n\pi/3)u(n) = \frac{\sqrt{3}}{2}\left[\frac{z^{-1}}{\frac{z^2-z+1}{z^2}}\right].$$

$$= \frac{\sqrt{3}}{2}\left[\frac{z(2z-1) - (z^2 - z + 1)}{(z^2 - z + 1)^2}\right].$$

$$= \frac{\sqrt{3}}{2}\left[\frac{2z^2 - \cancel{z} - z^2 + \cancel{z} - 1}{(z^2 - z + 1)^2}\right].$$

$$= \frac{\sqrt{3}}{2}\left[\frac{z^2 - 1}{(z^2 - z + 1)^2}\right] \Rightarrow \left[\frac{\sqrt{3} \times z \times (z^2 - 1)}{2 \times (z^2 - z + 1)^2}\right].$$

$$= \frac{\sqrt{3}}{2}\left[\frac{(z(z^2 - 1)}{(z^2 - z + 1)^2}\right].$$

$$X_2(n) = (0.9)^n u(n-1).$$

$$X_2(z) = \sum_{1}^{\infty} (0.9)^n \cdot z^{-1}.$$

$$X_2(z) = \frac{z^{-1}}{1 - 0.9z^{-1}} = \frac{1}{z - 0.9}.$$

$$X(z) = X_1(z) + X_2(z).$$

$$= \frac{\sqrt{3}(z)(z^2 - 1)}{2 \times (z^2 - z + 1)^2} + \frac{1}{z - 0.9}.$$

Task#02:

Let $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ and $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$. Determine $X_3(z) = X_1(z)X_2(z)$.

MATLAB Code:

```
%LAB#5 Task#2
X1=[2 3 4];
X2=[3 4 5 6];
X3=conv(X1,X2)
```

Output:

Command Window

```
>> DSPLab5task2

X1 =

     2     3     4


X2 =

     3     4     5     6


X3 =

     6    17    34    43    38    24
```

Handwritten Solution:

Task #2

$$X_1(z) = 2 + 3z^{-1} + 4z^{-2}$$

$$X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}.$$

$$X_3(z) = X_1(z)X_2(z).$$

$$X_3(z) = (2+3z^{-1}+4z^{-2})(3+4z^{-1}+5z^{-2}+6z^{-3}).$$

$$= 6+8z^{-1}+10z^{-2}+12z^{-3}+9z^{-1}+12z^{-2}+15z^{-3}$$

$$+18z^{-4}+12z^{-2}+16z^{-3}+20z^{-4}+24z^{-5}.$$

$$= 6+17z^{-1}+34z^{-2}+43z^{-3}+38z^{-4}+24z^{-5}.$$