3.2 Typical two dimensional electric field calculations by finite elements method

Here we have find the potential at the free nodes in the potential system using the finite elements method.

The solution region is divided into 25 three-node triangular elements with the total number of nodes being 21, shown in figure 3.3.

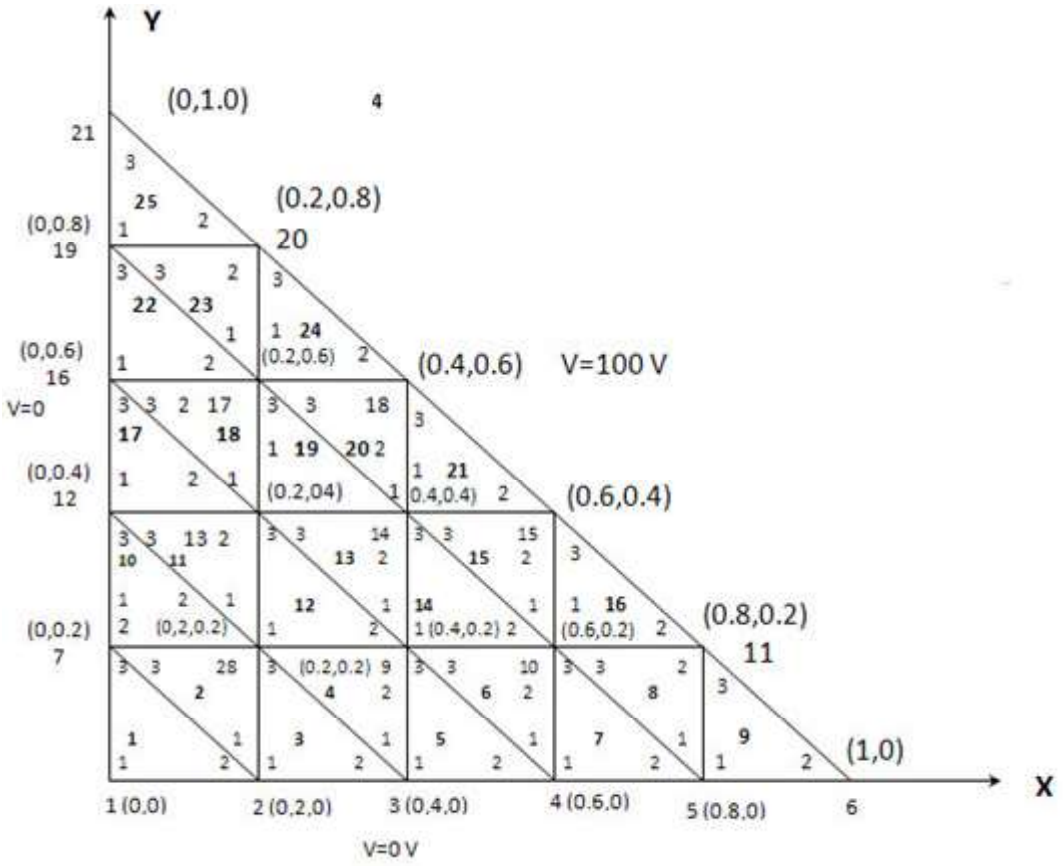
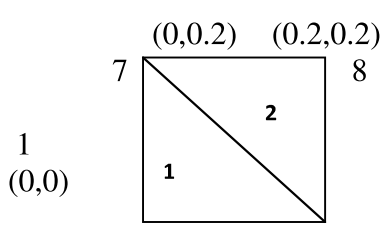
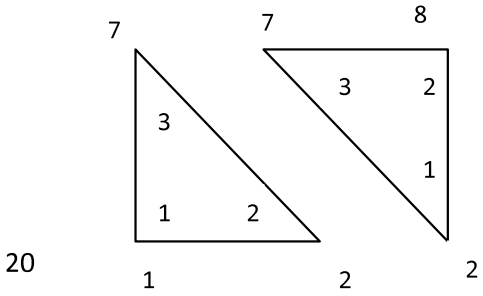


Figure 3.3, solution region divided into 25 triangular elements.

SOLUTION:

Determination of different elements coefficient matrix:





Node	(x,y)	$P_1 = (y_2-y_3)$, $P_2= (y_3-y_1)$
1	(0,0)	$P_3= (y_1-y_2)$
2	(0.2,0)	$Q1= (x_3-x_2)$,$Q_2 =(x_1-x_3)$
7	(0,0.2)	$Q_3 = (x_2-x_1)$
8	(0.2,0.2)	

For element 1

1-2-7→1-2-3

P_1 = 0-0.2 = -0.2 Q_1 = 0-0.2 = -0.2

P_2 = 0.2-0 = 0.2 Q_2 = 0-0 = 0

P_3 = 0-0 = 0 Q_3 = 0.2 -0 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A} (P_iP_j + Q_iQ_j)$

$C^{(1)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$

For element 2

2-8-7→1-2-3

P_1 = 0.2-0.2 = 0 Q_1 = 0-0.2 = -0.2

P_2 = 0.2-0 = 0.2 Q_2 = 0.2-0 = 0.2

P_3 = 0-0 =- 0.2 Q_3 = 0.2 -0.2 = 0

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A} (P_iP_j + Q_iQ_j)$

$$C^{(2)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

For element 3

2-3-8→1-2-3

$P_1 = 0\text{-}0.2 = -0.2$ $Q_1 = 0\text{-}0.2 = -0.2$

$P_2 = 0.2\text{-}0 = 0.2$ $Q_2 = 0\text{-}0 = 0$

$P_3 = 0\text{-}0 = 0$ $Q_3 = 0.2\text{ -}0 = 0.2$

$A = ½ \{0.2 \text{ x } 0.2 - 0\text{x } 0\} = \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A} (P_iP_j + Q_iQ_j)$

$$C^{(3)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

For element 4

3-9-8→1-2-3

$P_1 = 0\text{-}0.2 = 0$ $Q_1 = 0.2\text{-}0.4 = -0.2$

$P_2 = 0.2\text{-}0 = 0.2$ $Q_2 = 0.4\text{-}0\ .2 = 0.2$

$P_3 = 0\text{-}0 = -0.2$ $Q_3 = 0.4\text{ -}0.4 = 0$

$A = ½ \{0.2 \text{ x } 0.2 - 0\text{x } 0\} = \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A} (P_iP_j + Q_iQ_j)$

$$C^{(4)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

For element 5

3-4-9→1-2-3

P_1 = 0-0.2 = -0.2 Q_1 = 0.4-0.6 = -0.2

P_2 = 0.2-0 = 0.2 Q_2 = 0.4-0.4 = 0

P_3 = 0-0 = 0 Q_3 = 0.6 -0.4 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(5)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$

For element 6

4-10-9→1-2-3

P_1 = 0.2-0.2 = 0 Q_1 = 0.4-0.6 = -0.2

P_2 = 0.2-0 = 0.2 Q_2 = 0.6-0.4 = 0.2

P_3 = 0-0.2 = -0.2 Q_3 = 0.6 -0.6 = 0

A = ½ {0.2 x 0 + 0.2x 0.2} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(6)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$

For element 7

4-5-10→1-2-3

P_1 = 0-0.2 = -0.2 Q_1 = 0.6-0.8 = -0.2

P_2 = 0.2-0 = 0.2 Q_2 = 0.6-0.6 = 0

P_3 = 0-0 = 0 Q_3 = 0.8 -0.6 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(7)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

For element 8

5-11-10→1-2-3

$P_1 = 0\text{-}0 = 0$ $Q_1 = 0.6\text{-}0.8 = -0.2$

$P_2 = 0.2\text{-}0 = 0.2$ $Q_2 = 0.8\text{-}0.6 = 0.2$

$P_3 = 0\text{-}0.2 = 0.2$ $Q_3 = 0.8\ \text{-}0.8 = 0$

$A = ½ \{0.2 \text{ x } 0.2 + 0\text{x } 0\} = \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(8)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

For element 9

5-6-11→1-2-3

$P_1 = 0\text{-}0.2 = -0.2$ $Q_1 = 0.8\text{-}1 = -0.2$

$P_2 = 0.2\text{-}0 = 0.2$ $Q_2 = 0.8\text{-}0.8 = 0$

$P_3 = 0\text{-}0 = 0$ $Q_3 = 1\ \text{-}0.8 = 0.2$

$A = ½ \{0.2 \text{ x } 0.2 - 0\text{x } 0\} = \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(9)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

For element 10

7-8-12→1-2-3

P_1 = 0.2-0.4 = -0.2 Q_1 = 0-0.2 = -0.2

P_2 = 0.4-0.2 = 0.2 Q_2 = 0-0 = 0

P_3 = 0.2-0.2 = 0 Q_3 = 0.2 -0 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(10)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & 0.5 & 0 \end{bmatrix}$

For element 11

8-13-12→1-2-3

P_1 = 0.4-0.4 = 0 Q_1 = 0-0.2 = -0.2

P_2 = 0.4-0.2 = 0.2 Q_2 = 0.2-0 = 0.2

P_3 = 0.2-0.4 =- 0.2 Q_3 = 0

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(11)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$

For element 12

8-9-13→1-2-3

P_1 = 0.2-0.4 = -0.2 Q_1 = 0.2-0.4 = -0.2

P_2 = 0.4-0.2 = 0.2 Q_2 = 0.2-0.2 = 0

$P_3 = 0.2\text{-}0.2 = 0$ $Q_3 = 0.4\ \text{-}0.2 = 0.2$

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(12)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$

For element 13

9-14-13→1-2-3

$P_1 = 0.4\text{-}0.4 = 0$ $Q_1 = 0.2\text{-}0.4 = -0.2$

$P_2 = 0.4\text{-}0.2 = 0.2$ $Q_2 = 0.4\text{-}0.2 = 0.2$

$P_3 = 0.2\text{-}0.4 = 0$ $Q_3 = 0$

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(13)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & 0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$

For element 14

9-10-14→1-2-3

$P_1 = 0.2\text{-}0.4 = -0.2$ $Q_1 = 0.4\text{-}0.6 = -0.2$

$P_2 = 0.4\text{-}0.2 = 0.2$ $Q_2 = 0.4\text{-}0.4 = 0$

$P_3 = 0\text{-}0 = 0$ $Q_3 = 0.6\ \text{-}0.4 = 0.2$

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(14)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

For element 15

10-15-14→1-2-3

$P_1 = 0$ $Q_1 = 0\text{-}0.2 = -0.2$

$P_2 = 0.4\text{-}0.2 = 0.2$ $Q_2 = 0.6\text{-}0.4 = 0$

$P_3 = 0.2\text{-}0\ .4 = -0.2$ $Q_3 = 0.6\ \text{-}0\ .6 = 0$

$A = ½\ \{0.2 \text{ x } 0.2 - 0\text{x } 0\} = \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(15)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

For element 16

10-11-15→1-2-3

$P_1 = 0.2\text{-}0.4 = -0.2$ $Q_1 = 0.6\text{-}0.8 = -0.2$

$P_2 = 0.4\text{-}0\ .2 = 0.2$ $Q_2 = 0.6\text{-}0.6 = 0$

$P_3 = 0\text{-}0 = 0$ $Q_3 = 0.8\ \text{-}0.6 = 0.2$

$A = ½\ \{0.2 \text{ x } 0.2 - 0\text{x } 0\} = \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(16)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

For element 17

12-13-16→1-2-3

P_1 = 0.4-0.6 = -0.2 Q_1 = 0-0.2 = -0.2

P_2 = 0.6-0.4 = 0.2 Q_2 = 0-0 = 0

P_3 = 0-0 = 0 Q_3 = 0.2 -0 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(17)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$

For element 18

13-17-16→1-2-3

P_1 = 0.6-0.6 = 0 Q_1 = 0-0.2 = -0.2

P_2 = 0.6-0.4 = 0.2 Q_2 = 0.2-0 = 0.2

P_3 = 0.4-0.6 =- 0.2 Q_3 = 0.2 -0.2 = 0

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(18)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & 0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$

For element 19

13-14-17→1-2-3

P_1 = 0.4-0.6 = -0.2 Q_1 = 0.2-0.4 = -0.2

P_2 = 0.6-0.4= 0.2 Q_2 = 0.2-0.2 = 0

P_3 = 0-0 = 0 Q_3 = 0.4 -0.2 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(19)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$

For element 20

14-18-17→1-2-3

P_1 = 0.6-0.6 = 0 Q_1 = 0.2-0.4 = -0.2

P_2 = 0.4-0.6 = -0.2 Q_2 = 0.4-0.2= 0.2

P_3 = 0.4-0.6 = -0.2 Q_3 = 0.4 -0.4 = 0

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(20)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$

For element 21

14-15-18→1-2-3

P_1 = 0.4-0.6 = -0.2 Q_1 = 0.4-0.6 = -0.2

P_2 = 0.6-0.4= 0.2 Q_2 = 0.4-0.4 = 0

P_3 = 0.4-0.4 = 0 Q_3 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(21)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

For element 22

16-17-19→1-2-3

$P_1 = 0.6-0.8 = -0.2$ $Q_1 = 0-0.2 = -0.2$

$P_2 = 0.8-0.6 = 0.2$ $Q_2 = 0-0 = 0$

$P_3 = 0.6-0.6 = 0$ $Q_3 = 0.2 -0 = 0.2$

$A = ½ \{0.2 \text{ x } 0.2 – 0\text{x } 0\} = \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(22)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

For element 23

17-20-19→1-2-3

$P_1 = 0$ $Q_1 = 0-0.2 = -0.2$

$P_2 = 0.8-0.6= 0.2$ $Q_2 = 0-0 = 0$

$P_3 = 0.6-0.8 =- 0.2$ $Q_3 = 0.2 -0 = 0.2$

$A = ½ \{0.2 \text{ x } 0.2 – 0\text{x } 0\} = \frac{0.2^2}{2} = \frac{0.04}{2} = 0.02$

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$$C^{(23)} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

For element 24

17-18-20→1-2-3

P_1 = 0.6-0.8 = -0.2 Q_1 = 0.2-0.4 = -0.2

P_2 = 0.8-0.6 = 0.2 Q_2 = 0.2-0.2 = 0

P_3 = 0 Q_3 = 0.4 -0.2 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(24)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$

For element 25

19-20-21→1-2-3

P_1 = 0.8-1 = -0.2 Q_1 = 0-0.2=- 0.2

P_2 = 1-0.8 = 0.2 Q_2 = 0-0 = 0

P_3 = 0-0 = 0 Q_3 = 0.2 -0 = 0.2

A = ½ {0.2 x 0.2 – 0x 0} = $\frac{0.2^2}{2} = \frac{0.04}{2}$ = 0.02

A= 0.02

$Cij^{(e)} = \frac{1}{4A}(P_iP_j + Q_iQ_j)$

$C^{(25)} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$

$$[Cff] = \begin{bmatrix} C_{88} & C_{89} & C_{810} & C_{813} & C_{814} & C_{817} \\ C_{98} & C_{99} & C_{910} & C_{913} & C_{914} & C_{917} \\ C_{108} & C_{109} & C_{1010} & C_{1013} & C_{1014} & C_{1017} \\ C_{138} & C_{139} & C_{1310} & C_{1313} & C_{1314} & C_{1317} \\ C_{148} & C_{149} & C_{1410} & C_{1413} & C_{1414} & C_{1417} \\ C_{178} & C_{179} & C_{1710} & C_{1713} & C_{1714} & C_{1717} \end{bmatrix}$$

$$[\mathrm{Cff}] = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & & -1 \\ 0 & -1 & 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & -1 & 0 & 4 \end{bmatrix}$$

$$[\mathrm{B}] = \begin{bmatrix} 0 \\ 0 \\ -200 \\ 0 \\ -200 \\ -200 \end{bmatrix}$$

$$[\mathrm{V}] = [A]^{-1}[\mathrm{B}]$$

$$[V]^T = \begin{bmatrix} 18.1818 \\ 36.3636 \\ 59.0909 \\ 36.3636 \\ 68.1818 \\ 59.0909 \end{bmatrix} \quad \text{(ANSWER)}$$

MATLAB PROGRAM:

APPENDIX A: A.2 MATLAB program for 2D problem using finite element method.

Result:

node	X	Y	potential
1.0000	0	0	0
2.0000	0.2000	0	0
3.0000	0.4000	0	0
4.0000	0.6000	0	0
5.0000	0.8000	0	0
6.0000	1.0000	0	50.0000
7.0000	0	0.2000	0
8.0000	0.2000	0.2000	18.1818
9.0000	0.4000	0.2000	36.3636
10.0000	0.6000	0.2000	59.0909
11.0000	0.8000	0.2000	100.0000
12.0000	0	0.4000	0
13.0000	0.2000	0.4000	36.3636
14.0000	0.4000	0.4000	68.1818
15.0000	0.6000	0.4000	100.0000
16.0000	0	0.6000	0
17.0000	0.2000	0.6000	59.0909
18.0000	0.4000	0.6000	100.0000
19.0000	0	0.8000	0
20.0000	0.2000	0.8000	100.0000
21.0000	0	1.0000	50.0000