Newton Raphson

Lec_8

Newton - Raphson Method

- **Newton Raphson is the most widely used method for solving simultaneous non-linear algebraic equations.**

- **It is a successive approximation procedure based on an initial estimate of the unknown and the use of the Taylor's expansion.**

- **Consider the solution of the one-dimensional equation given by:** $f(x) = c$

- **If $x^{(0)}$ is an initial estimate of the solution and $\Delta x^{(0)}$ is a small deviation from the correct solution, we must have:**

$$f(x^{(0)} + \Delta x^{(0)}) = c$$

- **Taylor series expansion yields:**

$$f(x)^{(0)} + \left(\frac{df}{dx}\right)^{(0)} \Delta x^{(0)} + \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\right)^{(0)} \left(\Delta x^{(0)}\right)^2 + \ldots = c$$

Newton - Raphson Method

- **Assume that $\Delta x(0)$ is a small, then the 2nd and higher order terms can be neglected.**

$$\Delta c^{(0)} = c - f(x^{(0)}) = \left(\frac{df}{dx}\right)^{(0)} \Delta x^{(0)}$$

- **The second approximation will be:**

$$x^{(1)} = x^{(0)} + \Delta x^{(0)} = x^{(0)} + \frac{\Delta c^{(0)}}{\left(\frac{df}{dx}\right)^{(0)}}$$

- **Successive use of the above procedure yields the Newton – Raphson algorithm.**

$$\Delta c^{(k)} = c - f(x^{(k)}) = \left(\frac{df}{dx}\right)^{(k)} \Delta x^{(k)}$$

$$x^{(k+1)} = x^{(k)} + \Delta x^{(k)}$$

Example 1

Use the Newton Raphson technique to find a root of the following equation, assume initial estimate = 6: $f(x) = x^3 - 6x^2 + 9x - 4 = 0$

Solution: $\dfrac{df(x)}{dx} = 3x^2 - 12x + 9$

$$\Delta c^{(0)} = c - f(x^{(0)}) = 0 - \left[(6)^3 - 6(6)^2 + 9(6) - 4 \right] = -50$$

$$\left(\dfrac{df}{dx} \right)^{(0)} = 3(6)^2 - 12(6) + 9 = 45$$

$$\Delta x^{(0)} = \dfrac{\Delta c^{(0)}}{\left(\dfrac{df}{dx} \right)^{(0)}} = \dfrac{-50}{45} = -1.11$$

$$x^{(1)} = x^{(0)} + \Delta x^{(0)} = 6 - 1.111 = 4.8889$$

Example 1, continue

The subsequent iterations result in:

$$x^{(2)} = x^{(1)} + \Delta x^{(1)} = 4.8889 - \frac{13.4431}{22.037} = 4.2789$$

$$x^{(3)} = x^{(2)} + \Delta x^{(2)} = 4.2789 - \frac{2.9981}{12.5797} = 4.0405$$

$$x^{(4)} = x^{(3)} + \Delta x^{(3)} = 4.0405 - \frac{0.3748}{9.4914} = 4.0011$$

Example 2

Use the Newton-Raphson technique to find the solution of the following equation, Assume initial solution = 1

$$f(x) = x^2 - 2 = 0$$

Solution:

$$\frac{df(x)}{dx} = 2x$$

$$\Delta c^{(0)} = c - f(x^{(0)}) = 0 - \left[(1)^2 - 2\right] = +1$$

$$\left(\frac{df}{dx}\right)^{(0)} = 2(1) = 2$$

$$\Delta x^{(0)} = \frac{\Delta c^{(0)}}{\left(\frac{df}{dx}\right)^{(0)}} = \frac{1}{2} = 0.5$$

$$x^{(1)} = x^{(0)} + \Delta x^{(0)} = 1 + 0.5 = 1.5$$

Example 2

$$\Delta c^{(1)} = c - f(x^{(1)}) = 0 - \left[(1.5)^2 - 2\right] = -.25$$

$$\left(\frac{df}{dx}\right)^{(1)} = 2(1.5) = 3$$

$$\Delta x^{(1)} = \frac{\Delta c^{(1)}}{\left(\frac{df}{dx}\right)^{(1)}} = \frac{-0.25}{3} = -0.0833$$

$$x^{(2)} = x^{(1)} + \Delta x^{(1)} = 1.5 - 0.0833 = 1.416$$

An then go to the next iteration and so on.

Newton - Raphson Method

- **For n-dimensional equations:**

$$f_1(x_1, x_2, \ldots x_n) = c_1$$
$$f_2(x_1, x_2, \ldots x_n) = c_2$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$f_n(x_1, x_2, \ldots x_n) = c_n$$

Writing the Taylor's series expansion of the left hand side and neglecting the higher order terms:

$$(f_1)^{(0)} + \left(\frac{\partial f_1}{\partial x_1}\right)^{(0)} \Delta x_1^{(0)} + \left(\frac{\partial f_1}{\partial x_2}\right)^{(0)} \Delta x_2^{(0)} + \ldots \left(\frac{\partial f_1}{\partial x_n}\right)^{(0)} \Delta x_n^{(0)}) = c_1$$

$$(f_2)^{(0)} + \left(\frac{\partial f_2}{\partial x_1}\right)^{(0)} \Delta x_1^{(0)} + \left(\frac{\partial f_2}{\partial x_2}\right)^{(0)} \Delta x_2^{(0)} + \ldots \left(\frac{\partial f_2}{\partial x_n}\right)^{(0)} \Delta x_n^{(0)}) = c_2$$

$$\cdot$$

$$(f_n)^{(0)} + \left(\frac{\partial f_n}{\partial x_1}\right)^{(0)} \Delta x_1^{(0)} + \left(\frac{\partial f_n}{\partial x_2}\right)^{(0)} \Delta x_2^{(0)} + \ldots \left(\frac{\partial f_n}{\partial x_n}\right)^{(0)} \Delta x_n^{(0)}) = c_n$$

Newton - Raphson Method

- **Or in matrix form:**

$$\begin{bmatrix} c_1 - (f_1)^{(0)} \\ c_2 - (f_2)^{(0)} \\ . \\ . \\ . \\ c_n - (f_n)^{(0)} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial f_1}{\partial x_1}\right)^{(0)} & \left(\dfrac{\partial f_1}{\partial x_2}\right)^{(0)} & \cdots\cdots & \left(\dfrac{\partial f_1}{\partial x_n}\right)^{(0)} \\ \left(\dfrac{\partial f_2}{\partial x_1}\right)^{(0)} & \left(\dfrac{\partial f_2}{\partial x_2}\right)^{(0)} & \cdots\cdots & \left(\dfrac{\partial f_2}{\partial x_n}\right)^{(0)} \\ . \\ \left(\dfrac{\partial f_n}{\partial x_1}\right)^{(0)} & \left(\dfrac{\partial f_n}{\partial x_2}\right)^{(0)} & \cdots\cdots & \left(\dfrac{\partial f_n}{\partial x_n}\right)^{(0)} \end{bmatrix} \underbrace{}_{\text{Jacobian Matrix}} \begin{bmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \\ . \\ \Delta x_n^{(0)} \end{bmatrix}$$

Jacobian Matrix

- **In short form:** $\Delta C^{(k)} = J^{(k)} \Delta X^{(k)}$

So: $\Delta X^{(k)} = \left[J^{(k)}\right]^{-1} \Delta C^{(k)}$

And the Newton-Raphson algorithm becomes $\qquad X^{(k+1)} = X^{(k)} + \Delta X^{(k)}$

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Example 3

Use the Newton Raphson method to solve the following system of equations, assume initial estimate $= 1$:

$$x_1{}^2 - x_2{}^2 + x_3{}^2 = 11$$

$$x_1 x_2 + x_2{}^2 - 3x_3 = 3$$

Solution:

$$x_1 - x_1 x_3 + x_2 x_3 = 6$$

$$J = \begin{bmatrix} \left(\dfrac{\partial f_1}{\partial x_1}\right) & \left(\dfrac{\partial f_1}{\partial x_2}\right) & \left(\dfrac{\partial f_1}{\partial x_3}\right) \\ \left(\dfrac{\partial f_2}{\partial x_1}\right) & \left(\dfrac{\partial f_2}{\partial x_2}\right) & \left(\dfrac{\partial f_2}{\partial x_3}\right) \\ \left(\dfrac{\partial f_3}{\partial x_1}\right) & \left(\dfrac{\partial f_3}{\partial x_2}\right) & \left(\dfrac{\partial f_3}{\partial x_3}\right) \end{bmatrix} = \begin{bmatrix} 2x_1 & -2x_2 & 2x_3 \\ x_2 & x_1 + 2x_2 & -3 \\ 1 - x_3 & x_3 & -x_1 + x_2 \end{bmatrix}$$

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Example 3 ,continue

$$J^{(1)} = \begin{bmatrix} 2(1) & -2(1) & 2(1) \\ (1) & (1)+2(1) & -3 \\ 1-(1) & (1) & -(1)+(1) \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 3 & -3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Delta C^{(1)} = \begin{bmatrix} c_1 - (f_1)^{(0)} \\ c_2 - (f_2)^{(0)} \\ c_3 - (f_3)^{(0)} \end{bmatrix} = \begin{bmatrix} 11-1 \\ 3-(-1) \\ 6-(1) \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 5 \end{bmatrix}$$

$$\Delta X^{(1)} = [J^1]^{-1} \Delta C^{(1)} = \begin{bmatrix} 4.75 \\ 5.00 \\ 5.25 \end{bmatrix} \quad \text{And} \quad X^{(1)} = X^{(0)} + \Delta X^{(1)} = \begin{bmatrix} 5.75 \\ 6.00 \\ 6.25 \end{bmatrix}$$

And the process continues till the error is within certain limits.

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Example 4

Use the Newton Raphson method to solve the following system of equations, assume initial estimate = 1:

$$f_1(x) = 2x_1^2 + x_2^2 - 8 = 0$$
$$f_2(x) = x_1^2 - x_2^2 + x_1 x_2 - 4 = 0$$

Solution:

$$J = \begin{bmatrix} \left(\dfrac{\partial f_1}{\partial x_1}\right) & \left(\dfrac{\partial f_1}{\partial x_2}\right) \\ \left(\dfrac{\partial f_2}{\partial x_1}\right) & \left(\dfrac{\partial f_2}{\partial x_2}\right) \end{bmatrix} = \begin{bmatrix} 4x_1 & +2x_2 \\ 2x_1 + x_2 & x_1 - 2x_2 \end{bmatrix}$$

$$J^{(1)} = \begin{bmatrix} 4(1) & -2(1) \\ 2(1)+1 & 1-2(1) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

Example 4-continue

$$J^{(1)-1} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}^{-1} = \frac{1}{-4-6} \begin{bmatrix} -1 & -2 \\ -3 & +4 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & -0.4 \end{bmatrix}$$

$$\Delta C^{(1)} = \begin{bmatrix} c_1 - (f_1)^{(0)} \\ c_2 - (f_2)^{(0)} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\Delta X^{(1)} = [J^1]^{-1} \Delta C^{(1)} = \begin{bmatrix} 1.1 \\ 0.3 \end{bmatrix} \quad \text{And} \quad X^{(1)} = X^{(0)} + \Delta X^{(1)} = \begin{bmatrix} 2.1 \\ 1.3 \end{bmatrix}$$

And the process continues till the error is within certain limits.

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Newton - Raphson Method for power system

• **As previously derived:**

$$P_i = |V_i| \sum_{p=1}^{n} |Y_{ip}| |V_p| \cos(\delta_i - \delta_p - \gamma_{ip}) \qquad (1)$$

$$Q_i = |V_i| \sum_{p=1}^{n} |Y_{ip}| |V_p| \sin(\delta_i - \delta_p - \gamma_{ip}) \qquad (2)$$

These two equations constitute a set of nonlinear algebraic equations in term of independent variables:

a) Voltage magnitude in per unit.

b) Phase angle in radian.

We have two equations for each load bus and one equation for each voltage controlled bus.

Newton - Raphson Method for power system

• Expanding the previous equations using Taylor's series and neglecting all higher order terms results in the following:

$$
\begin{bmatrix} \Delta P_2^{(k)} \\ \cdot \\ \cdot \\ \Delta P_n^{(k)} \\ \hline \Delta Q_2^{(k)} \\ \cdot \\ \cdot \\ \Delta Q_n^{(k)} \end{bmatrix}
=
\left[
\begin{array}{ccc|ccc}
\left(\dfrac{\partial P_2}{\partial \delta_2}\right)^{(k)} & \cdots & \left(\dfrac{\partial P_2}{\partial \delta_n}\right)^{(k)} & \left(\dfrac{\partial P_2}{\partial |V_2|}\right)^{(k)} & \cdots & \left(\dfrac{\partial P_2}{\partial |V_n|}\right)^{(k)} \\
\cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot \\
\left(\dfrac{\partial P_n}{\partial \delta_2}\right)^{(k)} & \cdots & \left(\dfrac{\partial P_n}{\partial \delta_n}\right)^{(k)} & \left(\dfrac{\partial P_n}{\partial |V_2|}\right)^{(k)} & \cdots & \left(\dfrac{\partial P_n}{\partial |V_n|}\right)^{(k)} \\
\hline
\left(\dfrac{\partial Q_2}{\partial \delta_2}\right)^{(k)} & \cdots & \left(\dfrac{\partial Q_2}{\partial \delta_n}\right)^{(k)} & \left(\dfrac{\partial Q_2}{\partial |V_2|}\right)^{(k)} & \cdots & \left(\dfrac{\partial Q_2}{\partial |V_n|}\right)^{(k)} \\
\cdot & & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & & \cdot \\
\left(\dfrac{\partial Q_n}{\partial \delta_2}\right)^{(k)} & \cdots & \left(\dfrac{\partial Q_n}{\partial \delta_n}\right)^{(k)} & \left(\dfrac{\partial Q_n}{\partial |V_2|}\right)^{(k)} & \cdots & \left(\dfrac{\partial Q_n}{\partial |V_n|}\right)^{(k)}
\end{array}
\right]
\begin{bmatrix} \Delta \delta_2^{(k)} \\ \cdot \\ \cdot \\ \Delta \delta_n^{(k)} \\ \Delta |V_2^{(k)}| \\ \cdot \\ \cdot \\ \Delta |V_n^{(k)}| \end{bmatrix}
\qquad (3)
$$

Or in short

$$
\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta |V| \end{bmatrix}
$$

$$P_i = |V_i| \sum_{P=1}^{n} |Y_{iP}| |V_P| \cos(\delta_i - \delta_P - \gamma_{iP})$$

$$Q_i = |V_i| \sum_{P=1}^{n} |Y_{iP}| |V_P| \cos(\delta_i - \delta_P - \gamma_{iP})$$

$P_2 =$

$Q_2 =$

$P_3 =$

$Q_3 =$

$P_4 =$

Q_4

$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\}$

$V_2 \angle \delta_2$

$V_3 \angle \delta_3$

$V_4 \angle \delta_4$

$$P_2 \qquad \delta_2$$
$$P_3 \qquad \delta_3$$
$$P_4 \qquad \delta_4$$
$$\overline{\qquad\qquad} \quad \overline{\qquad\qquad}$$
$$Q_2 \qquad |V_2|$$
$$Q_3 \qquad |V_3|$$
$$Q_4 \qquad |V_4|$$

$$
\begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \\ \Delta Q_2 \\ \Delta Q_3 \\ \Delta Q_4 \end{bmatrix}
=
\begin{bmatrix} \end{bmatrix}
\begin{bmatrix} \Delta \delta_2 \\ \Delta \delta_3 \\ \Delta \delta_4 \\ \Delta |V_2| \\ \Delta |V_3| \\ \Delta |V_4| \end{bmatrix}
$$

Jacobian.

$$
\begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \\ \Delta \delta_2 \\ \Delta \delta_3 \\ \Delta \delta_4 \end{bmatrix}
=
\left[
\begin{array}{ccc|ccc}
\dfrac{\partial P_2}{\partial \delta_2} & \dfrac{\partial P^2}{\partial \delta_3} & \dfrac{\partial P^2}{\partial \delta_4} & \dfrac{\partial P_2}{\partial |V_2|} & \dfrac{\partial P_2}{\partial |V_3|} & \dfrac{\partial P_2}{\partial |V_4|} \\[2mm]
\dfrac{\partial P_3}{\partial \delta_2} & \dfrac{\partial P_3}{\partial \delta_3} & \dfrac{\partial P_3}{\partial \delta_4} & \dfrac{\partial P_3}{\partial |V_2|} & \dfrac{\partial P_3}{\partial |V_3|} & \dfrac{\partial P_3}{\partial |V_3|} \\[2mm]
\dfrac{\partial P_4}{\partial \delta_2} & \dfrac{\partial P_4}{\partial \delta_3} & \dfrac{\partial P_4}{\partial \delta_4} & \dfrac{\partial P_4}{\partial |V_2|} & \dfrac{\partial P_4}{\partial |V_3|} & \dfrac{\partial P_4}{\partial |V_3|} \\[2mm]
\hline
\dfrac{\partial \delta_2}{\partial \delta_2} & \dfrac{\partial Q_2}{\partial \delta_3} & \dfrac{\partial \delta_2}{\partial \delta_4} & \dfrac{\partial \delta_2}{\partial \delta_{42}} & \dfrac{\partial \delta_2}{\partial |V_3|} & \dfrac{\partial Q_2}{\partial |V_4|} \\[2mm]
\dfrac{\partial Q_3}{\partial \delta_2} & \dfrac{\partial Q_3}{\partial \delta_3} & \dfrac{\partial Q_3}{\partial \delta_4} & \dfrac{\partial Q_3}{\partial |V_2|} & \dfrac{\partial Q_3}{\partial |V_3|} & \dfrac{\partial Q_3}{\partial |V_4|} \\[2mm]
\dfrac{\partial Q_4}{\partial \delta_2} & \dfrac{\partial Q_4}{\partial \delta_3} & \dfrac{\partial Q_4}{\partial \delta_4} & \dfrac{\partial Q_4}{\partial |V_2|} & \dfrac{\partial Q_4}{\partial |V_3|} & \dfrac{\partial Q_4}{\partial |V_4|}
\end{array}
\right]
\begin{bmatrix} \Delta \delta_2 \\ \Delta \delta_3 \\ \Delta \delta_4 \\ \delta |V_2| \\ \delta |V_3| \\ \delta |V_4| \end{bmatrix}
$$

Newton - Raphson Method for power system

- The terms ΔP_i and ΔQ_i are the difference between the scheduled and calculated values. Therefore:

$$\Delta P_i^{(k)} = P_i^{sch} - P_i^{(k)} \qquad (4)$$

$$\Delta Q_i^{(k)} = Q_i^{sch} - Q_i^{(k)} \qquad (5)$$

- The new estimates for bus voltages are:

$$\delta_i^{(k+1)} = \delta_i^{(k)} + \Delta \delta_i^{(k)} \qquad (6)$$

$$\left|V_i^{(k+1)}\right| = \left|V_i^{(k)}\right| + \Delta\left|V_i^{(k)}\right| \qquad (7)$$

Note:
For voltage controlled buses, the voltage magnitudes are known. Therefore, if m buses of the system are voltage controlled, m equations involving ΔQ and ΔV and the corresponding columns of the Jacobian matrix are eliminated.

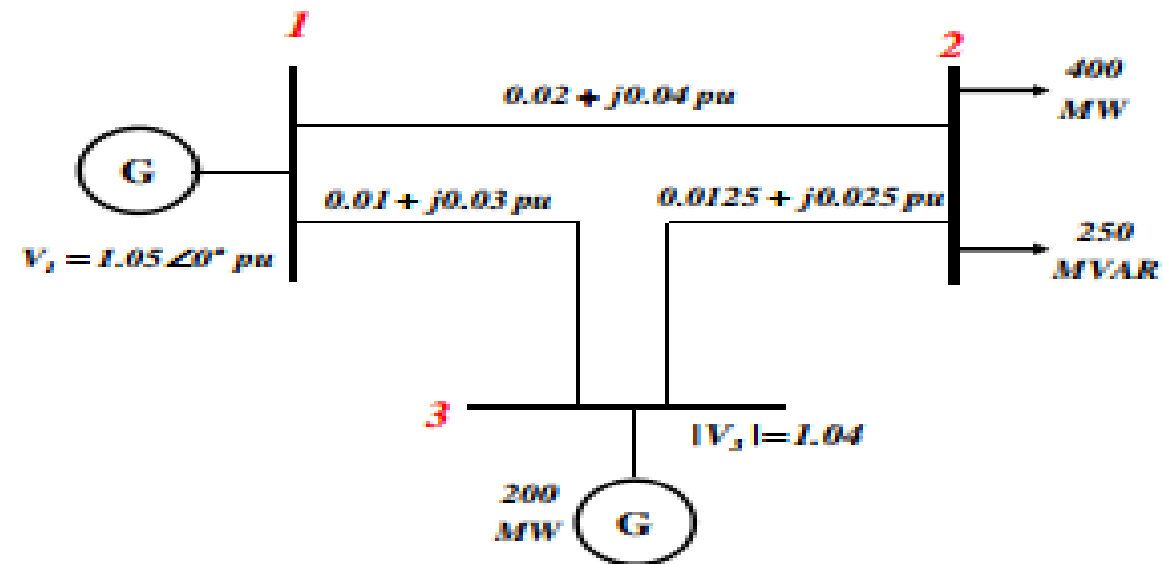## *Newton - Raphson Method for power system*

The procedure for power flow solution by the Newton-Raphson method is as follows:

1. For load buses, voltage magnitudes and angles are set. For voltage regulated buses, only the voltage angles are set.
2. For load buses, ΔP and ΔQ and for voltage controlled ΔP are calculated from the equations 4 and 5.
3. The elements of the Jacobian matrix are calculated.
4. The linear equations are solved by optimally ordered triangular factorization and Gaussian elimination.
5. The new voltage magnitudes and phases are computed using equations 6 and 7.
6. The process is continued until:

$$\left| \Delta P_i^{(k)} \right| \le \varepsilon$$

$$\left| \Delta Q_i^{(k)} \right| \le \varepsilon$$

Example 5:

The line impedances are as indicated in per unit on 100MVA base. Using Newton Raphson method find the power flow solution of the system.

1

G

$V_1 = 1.05 \angle 0°$ pu

0.02 + j0.04 pu

0.01 + j0.03 pu

0.0125 + j0.025 pu

2

400 MW

250 MVAR

3

$|V_3| = 1.04$

200 MW G

The expressions for the real and reactive power are as follows:

$$P_2 = |V_2| \sum_{p=1}^{3} |Y_{2p}| |V_p| \cos(\delta_2 - \delta_p - \gamma_{2p})$$

$$P_2 = |V_2||V_1||Y_{21}|\cos(\delta_2 - \delta_1 - \gamma_{21}) + |V_2|^2|Y_{22}|\cos(-\gamma_{22}) + |V_2||V_3||Y_{23}|\cos(\delta_2 - \delta_3 - \gamma_{23})$$

$$P_3 = |V_3| \sum_{p=1}^{3} |Y_{3p}| |V_p| \cos(\delta_3 - \delta_p - \gamma_{3p})$$

$$P_3 = |V_3||V_1||Y_{31}|\cos(\delta_3 - \delta_1 - \gamma_{31}) + |V_3||V_2||Y_{32}|\cos(\delta_3 - \delta_2 - \gamma_{32}) + |V_3|^2|Y_{31}|\cos(-\gamma_{33})$$

$$Q_2 = |V_2| \sum_{p=1}^{3} |Y_{2p}| |V_p| \sin(\delta_2 - \delta_p - \gamma_{2p})$$

$$Q_2 = |V_2||V_1||Y_{21}|\sin(\delta_2 - \delta_1 - \gamma_{21}) + |V_2|^2|Y_{22}|\sin(-\gamma_{22}) + |V_2||V_3||Y_{21}|\sin(\delta_2 - \delta_3 - \gamma_{23})$$

$$Q_2 = |V_2||V_1||Y_{21}|\sin(\delta_2 - \delta_1 - \gamma_{21}) - |V_2|^2|Y_{22}|\sin(\gamma_{22}) + |V_2||V_3||Y_{21}|\sin(\delta_2 - \delta_3 - \gamma_{23})$$

The Jacobian Matrix

$$J = \begin{bmatrix} \dfrac{\partial P_2}{\partial \delta_2} & \dfrac{\partial P_2}{\partial \delta_3} & \dfrac{\partial P_2}{\partial |V_2|} \\[2ex] \dfrac{\partial P_3}{\partial \delta_2} & \dfrac{\partial P_3}{\partial \delta_3} & \dfrac{\partial P_3}{\partial |V_2|} \\[2ex] \dfrac{\partial Q_2}{\partial \delta_2} & \dfrac{\partial Q_2}{\partial \delta_3} & \dfrac{\partial Q_2}{\partial |V_2|} \end{bmatrix}$$

Elements of the Jacobian matrix can be calculated as follows:

$$\frac{\partial P_2}{\partial \delta_2} = |V_2||V_1||Y_{21}|\sin(\gamma_{21} - \delta_2 + \delta_1) + |V_2||V_3||Y_{23}|\sin(\gamma_{23} - \delta_2 + \delta_3)$$

$$\frac{\partial P_2}{\partial \delta_3} = -|V_2||V_3||Y_{23}|\sin(\gamma_{23} - \delta_2 + \delta_3)$$

$$\frac{\partial P_2}{\partial |V_2|} = |V_1||Y_{21}|\cos(\gamma_{21} - \delta_2 + \delta_1) + 2|V_2||Y_{22}|\cos(\gamma_{22}) + |V_3||Y_{23}|\cos(\gamma_{23} - \delta_2 + \delta_3)$$

The other Jacobian elements will be calculated to finalize the Jacobian matrix

$$P_2 = |V_2||V_1||Y_{21}| \cos(\delta_2 - \delta_1 - \gamma_{21})$$

$$+ |V_2|^2 |Y_{22}| \cos(-\gamma_{22})$$

$$+ |V_2||V_3| \overset{|Y_{23}|}{} \cos(\delta_2 - \delta_3 - \gamma_{23})$$

— — — — — — — — —

*

$$\frac{\partial P_2}{\partial \delta_2} = -|V_2||V_1||Y_{21}| \sin(\delta_2 - \delta_1 - \gamma_{21})$$

$$- |V_2||V_3| \overset{|Y_{23}|}{} \sin(\delta_2 - \delta_3 - \gamma_{23})$$

$$= |V_2||V_1||Y_{21}| \sin(\gamma_{21} - \delta_1 - \delta_2)$$

$$+ |V_2||V_3| \overset{|Y_{23}|}{} \sin(\gamma_{23} + \delta_3 - \delta_2)$$
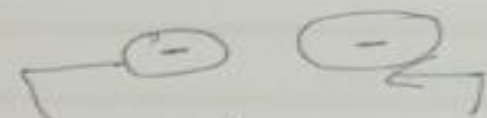
$$P_2 = |V_2||V_1||Y_{21}| \cos (\theta_{21} - \delta_2 + \delta_1)$$
$$+ |V_2^2||Y_{22}| \cos (\theta_{22}$$
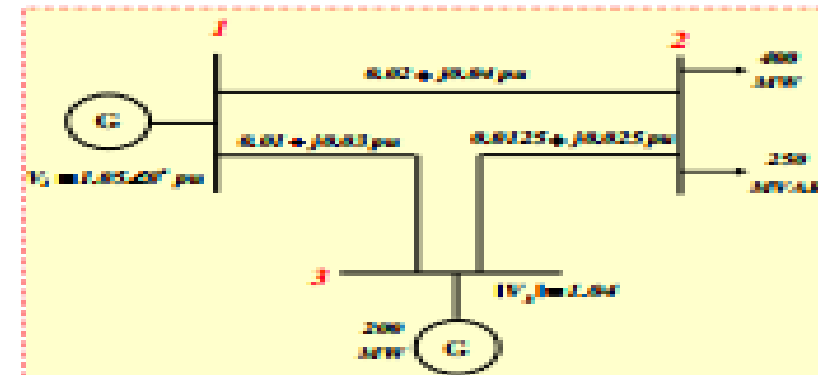$$+ |V_2||V_3||Y_{23}| \cos (\theta_{23} - \delta_2 + \delta_3)$$

$$P_2 = |V_2||V_1||Y_{21}| \sin (\theta_{21} - \delta_2 + \delta_1)$$
$$+ |V_2||V_3||Y_{23}| \sin (\theta_{23} - \delta_2 + \delta_3)$$

Classification of buses:

Bus 1: Slack Bus

$$V_1 = 1.05 \angle 0° \ pu$$

Bus 2: Load Bus (*PQ bus*)

P_2 and Q_2 are known V_2 and δ_2 are unknown

$$S_{2,sch} = \frac{(P_{2,g} - P_{2,d}) + j(Q_{2,g} - Q_{2,d})}{Base \ MVA} \ pu$$

$$S_{2,sch} = \frac{(0 - 400) + j(0 - 250)}{100} \ pu$$

$$S_{2,sch} = -4 - j2.5 \ pu$$

Bus 3: Voltage Controlled Bus (*PV bus*)

$|V_3|$ and $P_{g,3}$ are known $Q_{3,sch}$ and δ_3 are unknown

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$$P_{3,sch} = 2.0 \ pu$$

Using N-R method, select the initial values for the unknowns as:

$$V_1 = 1.05\angle 0^\circ \; pu \qquad V_2^o = 1\angle 0 \qquad |V_3| = 1.04$$

$$\delta_2^o = 0^\circ \qquad \delta_3^o = 0^\circ$$

$$\Delta P_2^{(0)} = P_2^{sch} - P_2^{(0)} = -4.0 - (-1.14) = -2.86$$

$$\Delta P_3^{(0)} = P_3^{sch} - P_3^{(0)} = 2.0 - (0.5616) = 1.4384$$

$$\Delta Q_2^{(0)} = Q_2^{sch} - Q_2^{(0)} = -2.5 - (-2.28) = -0.220$$

We are ready now to use the N-R technique:

$$
\begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \Delta Q_3 \end{bmatrix} =
\begin{bmatrix}
\dfrac{\partial P_2}{\partial \delta_2} & \dfrac{\partial P_2}{\partial \delta_3} & \dfrac{\partial P_2}{\partial |V_2|} \\[2mm]
\dfrac{\partial P_3}{\partial \delta_2} & \dfrac{\partial P_3}{\partial \delta_3} & \dfrac{\partial P_3}{\partial |V_2|} \\[2mm]
\dfrac{\partial Q_2}{\partial \delta_2} & \dfrac{\partial Q_2}{\partial \delta_3} & \dfrac{\partial Q_2}{\partial |V_2|}
\end{bmatrix}
\begin{bmatrix} \Delta \delta_2 \\ \Delta \delta_3 \\ \Delta |V_2| \end{bmatrix}
$$

$$
\begin{bmatrix} -2.86 \\ 1.4384 \\ -0.22 \end{bmatrix} =
\begin{bmatrix}
54.28 & -33.28 & 24.86 \\
-33.28 & 66.04 & -16.64 \\
-27.14 & 16.64 & 49.72
\end{bmatrix}
\begin{bmatrix} \Delta \delta_2^{(0)} \\ \Delta \delta_3^{(0)} \\ \Delta |V_2|^{(0)} \end{bmatrix}
$$

$\Delta \delta_2^{(0)} = -0.045263$ $\Longrightarrow$ $\delta_2^{(1)} = 0 + (-0.045263) = -0.045263$

$\Delta \delta_3^{(0)} = -0.007718$ $\Longrightarrow$ $\delta_3^{(1)} = 0 + (-0.007718) = -0.007718$

$\Delta |V_2|^{(0)} = -0.026548$ $\Longrightarrow$ $|V_2|^{(1)} = 1 + (-0.026548) = 0.97345$

This is the end of the fist iteration, the process will be repeated for the second iteration and so on.

Example 6:

For the two bus power system shown below, use the Newton-Raphson power flow to determine the voltage magnitude and angle at bus two. Assume that bus one is the slack and SBase = 100 MVA.

Line Z = 0.1j

One   1.000 pu   Two   1.000 pu

0 MW
0 MVR

200 MW
100 MVR

1033
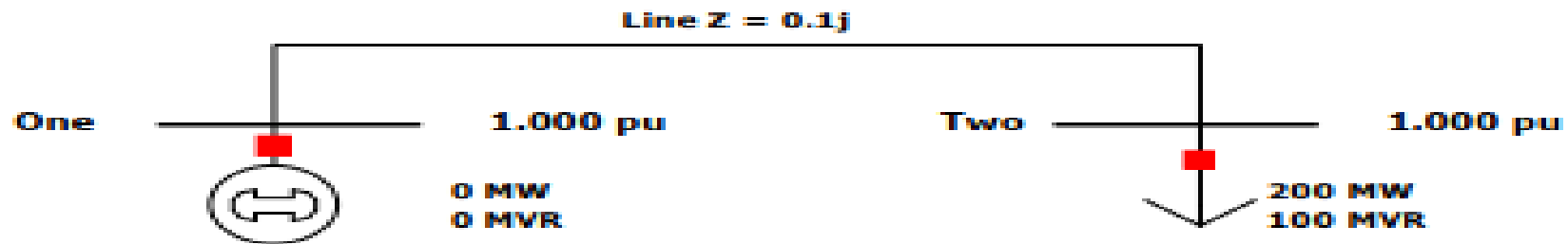
Example 6-solution:

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ |V_2| \end{bmatrix} \qquad \mathbf{Y}_{bus} = \begin{bmatrix} -j10 & j10 \\ j10 & -j10 \end{bmatrix}$$

$$P_i = |V_i| \sum_{p=1}^{n} |Y_{ip}| |V_p| \cos(\delta_i - \delta_p - \gamma_{ip})$$

$$Q_i = |V_i| \sum_{p=1}^{n} |Y_{ip}| |V_p| \sin(\delta_i - \delta_p - \gamma_{ip})$$

$$P_2 = |V_2| \sum_{p=1}^{2} |Y_{2p}| |V_p| \cos(\delta_2 - \delta_p - \gamma_{2p})$$

$$P_2 = |V_2| |Y_{21}| |V_1| \cos(\delta_2 - \delta_1 - \gamma_{21}) + |V_2| |Y_{22}| |V_2| \cos(\delta_2 - \delta_2 - \gamma_{22})$$

$$P_2 = |V_2| |10| \sin(\delta_2) = 0$$

$$Q_2 = |V_2| \sum_{p=1}^{2} |Y_{2p}| |V_p| \sin(\delta_2 - \delta_p - \gamma_{2p})$$

$$Q_2 = |V_2| |Y_{21}| |V_1| \sin(\delta_2 - \delta_1 - \gamma_{21}) + |V_2|^2 |Y_{22}| \sin(\delta_2 - \delta_2 - \gamma_{22})$$

$$Q_2 = |V_2|(-10) \cos(\delta_2) + |V_2|^2 10 = 0$$

Example 6-solution:

The Jacobian Matrix

$$J = \begin{bmatrix} \dfrac{\partial P_2}{\partial \delta_2} & \dfrac{\partial P_2}{\partial |V_2|} \\ \dfrac{\partial Q_2}{\partial \delta_2} & \dfrac{\partial Q_2}{\partial |V_2|} \end{bmatrix}$$

$$J = \begin{bmatrix} 10|V_2|\cos\delta_2 & 10\sin\delta_2 \\ 10|V_2|\sin\delta_2 & -10\cos\delta_2 + 20|V_2| \end{bmatrix}$$
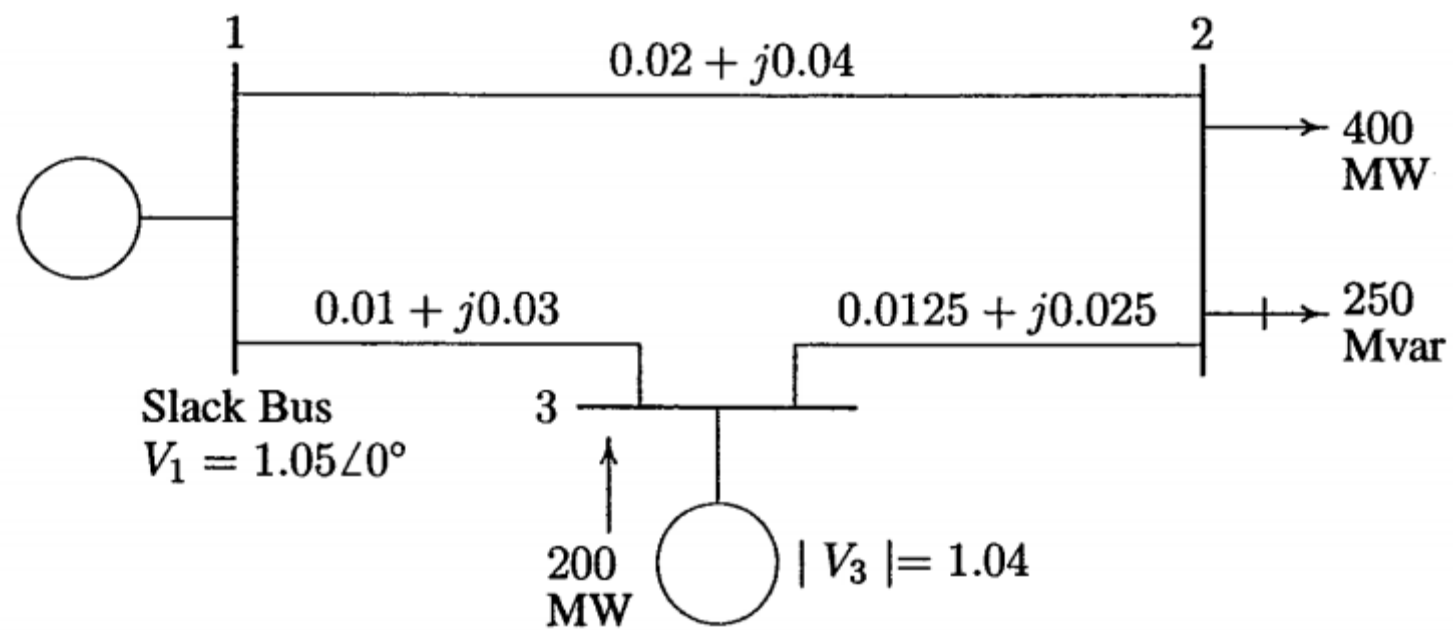
$$X^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad J^{(0)} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \qquad \begin{bmatrix} \Delta P_2^{(0)} \\ \Delta Q_2^{(0)} \end{bmatrix} = \begin{bmatrix} (0-2)-0 \\ (0-1)-0 \end{bmatrix}$$

$$X^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.9 \end{bmatrix}$$

This is the end of the fist iteration, the process will be repeated for the second iteration and so on.

1

$0.02 + j0.04$

2

→ 400 MW

$0.01 + j0.03$

$0.0125 + j0.025$

→ 250 Mvar

Slack Bus
$V_1 = 1.05\angle 0°$

3

200 MW

$|V_3| = 1.04$

$$Y_{bus} = \begin{bmatrix} 20 - j50 & -10 + j20 & -10 + j30 \\ -10 + j20 & 26 - j52 & -16 + j32 \\ -10 + j30 & -16 + j32 & 26 - j62 \end{bmatrix}$$

Converting the bus admittance matrix to polar form with angles in radian yields

$$Y_{bus} = \begin{bmatrix} 53.85165\angle{-1.9029} & 22.36068\angle{2.0344} & 31.62278\angle{1.8925} \\ 22.36068\angle{2.0344} & 58.13777\angle{-1.1071} & 35.77709\angle{2.0344} \\ 31.62278\angle{1.8925} & 35.77709\angle{2.0344} & 67.23095\angle{-1.1737} \end{bmatrix}$$

$$P_2 = |V_2||V_1||Y_{21}| \cos(\theta_{21} - \delta_2 + \delta_1) + |V_2^2||Y_{22}| \cos \theta_{22} + |V_2||V_3||Y_{23}| \cos(\theta_{23} - \delta_2 + \delta_3)$$

$$P_3 = |V_3||V_1||Y_{31}| \cos(\theta_{31} - \delta_3 + \delta_1) + |V_3||V_2||Y_{32}| \cos(\theta_{32} - \delta_3 + \delta_2) + |V_3^2||Y_{33}| \cos \theta_{33}$$

$$Q_2 = -|V_2||V_1||Y_{21}| \sin(\theta_{21} - \delta_2 + \delta_1) - |V_2^2||Y_{22}| \sin \theta_{22} - |V_2||V_3||Y_{23}| \sin(\theta_{23} - \delta_2 + \delta_3)$$

$$\frac{\partial P_2}{\partial \delta_2} = |V_2||V_1||Y_{21}| \sin(\theta_{21} - \delta_2 + \delta_1) + |V_2||V_3||Y_{23}| \sin(\theta_{23} - \delta_2 + \delta_3)$$

$$\frac{\partial P_2}{\partial \delta_3} = -|V_2||V_3||Y_{23}| \sin(\theta_{23} - \delta_2 + \delta_3)$$

$$\frac{\partial P_2}{\partial |V_2|} = |V_1||Y_{21}| \cos(\theta_{21} - \delta_2 + \delta_1) + 2|V_2||Y_{22}| \cos\theta_{22} + |V_3||Y_{23}| \cos(\theta_{23} - \delta_2 + \delta_3)$$

$$\frac{\partial P_3}{\partial \delta_2} = -|V_3||V_2||Y_{32}| \sin(\theta_{32} - \delta_3 + \delta_2)$$

$$\frac{\partial P_3}{\partial \delta_3} = |V_3||V_1||Y_{31}| \sin(\theta_{31} - \delta_3 + \delta_1) + |V_3||V_2||Y_{32}| \sin(\theta_{32} - \delta_3 + \delta_2)$$

$$\frac{\partial P_3}{\partial |V_2|} = |V_3||Y_{32}| \cos(\theta_{32} - \delta_3 + \delta_2)$$

$$\frac{\partial Q_2}{\partial \delta_2} = |V_2||V_1||Y_{21}| \cos(\theta_{21} - \delta_2 + \delta_1) + |V_2||V_3||Y_{23}|$$
$$\cos(\theta_{23} - \delta_2 + \delta_3)$$

$$\frac{\partial Q_2}{\partial \delta_3} = -|V_2||V_3||Y_{23}| \cos(\theta_{23} - \delta_2 + \delta_3)$$

$$\frac{\partial Q_2}{\partial |V_2|} = -|V_1||Y_{21}| \sin(\theta_{21} - \delta_2 + \delta_1) - 2|V_2||Y_{22}| \sin\theta_{22} -$$
$$|V_3||Y_{23}| \sin(\theta_{23} - \delta_2 + \delta_3)$$

$$S_2^{sch} = -\frac{(400 + j250)}{100} = -4.0 - j2.5 \quad \text{pu}$$

$$P_3^{sch} = \frac{200}{100} = 2.0 \quad \text{pu}$$

The slack bus voltage is $V_1 = 1.05\angle 0$ pu, and the bus 3 voltage magnitude is $|V_3| = 1.04$ pu. Starting with an initial estimate of $|V_2^{(0)}| = 1.0$, $\delta_2^{(0)} = 0.0$, and $\delta_3^{(0)} = 0.0$, the power residuals are computed from (6.63) and (6.64)

$$\Delta P_2^{(0)} = P_2^{sch} - P_2^{(0)} = -4.0 - (-1.14) = -2.8600$$

$$\Delta P_3^{(0)} = P_3^{sch} - P_3^{(0)} = 2.0 - (0.5616) = 1.4384$$

$$\Delta Q_2^{(0)} = Q_2^{sch} - Q_2^{(0)} = -2.5 - (-2.28) = -0.2200$$

$$
\begin{bmatrix} -2.8600 \\ 1.4384 \\ -0.2200 \end{bmatrix} = \begin{bmatrix} 54.28000 & -33.28000 & 24.86000 \\ -33.28000 & 66.04000 & -16.64000 \\ -27.14000 & 16.64000 & 49.72000 \end{bmatrix} \begin{bmatrix} \Delta\delta_2^{(0)} \\ \Delta\delta_3^{(0)} \\ \Delta|V_2^{(0)}| \end{bmatrix}
$$

Obtaining the solution of the above matrix equation, the new bus voltages in the first iteration are

$$\Delta\delta_2^{(0)} = -0.045263 \qquad \delta_2^{(1)} = 0 + (-0.045263) = -0.045263$$

$$\Delta\delta_3^{(0)} = -0.007718 \qquad \delta_3^{(1)} = 0 + (-0.007718) = -0.007718$$

$$\Delta|V_2^{(0)}| = -0.026548 \qquad |V_2^{(1)}| = 1 + (-0.026548) = 0.97345$$

Fig below shows the single line diagram of a simple 3 bus power system with generation at buses 1 and 2. The voltage at bus 1 is $V = 1.0\angle 0^0$ per unit. Voltage magnitude at bus 2 is fixed at 1.05 pu with a real power generation of 400 MW. A load consisting of 500 MW and 400 MVAR is taken from bus 3. Line admittances are marked in per unit on a 100 MVA base. For the purpose of hand calculations, line resistances and line charging susceptances are neglected.
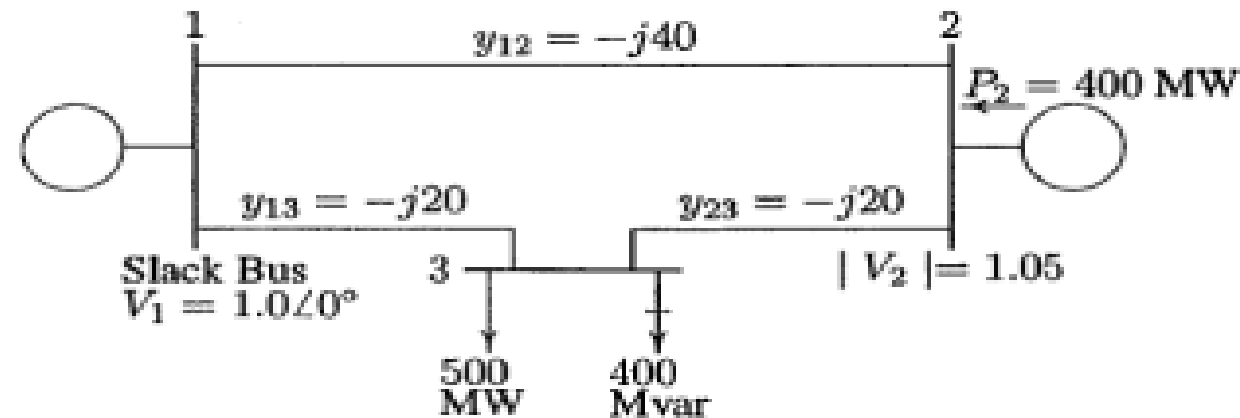
a. Show that the expression for the real power at bus 2 and real and reactive power at bus 3 are

$$P_2 = 40|V_2||V_1|\cos(90^\circ - \delta_2 + \delta_1) + 20|V_2||V_3|\cos(90^\circ - \delta_2 + \delta_3)$$
$$P_3 = 20|V_3||V_1|\cos(90^\circ - \delta_3 + \delta_1) + 20|V_3||V_2|\cos(90^\circ - \delta_3 + \delta_2)$$
$$Q_3 = -20|V_3||V_1|\sin(90^\circ - \delta_3 + \delta_1) - 20|V_3||V_2|\sin(90^\circ - \delta_3 + \delta_2) + 40|V_3|^2$$

b. Using the Newton-Raphson method, start with the initial estimates of $V_2^{(0)} = 1.0 + j0$ and $V_3^{(0)} = 1.0 + j0$, and keeping $|V_2| = 1.05$ per unit, determine the phasor values of V_2 and V_3. (perform only two iterations)

c. Find the slack bus real and reactive power after second iteration

d. Determine the line flows and line losses after second iteration.

$$Y_{Bus} = \begin{bmatrix} 60\angle{-90} & 40\angle{90} & 20\angle{90} \\ 40\angle{90} & 60\angle{-90} & 20\angle{90} \\ 20\angle{90} & 20\angle{90} & 40\angle{-90} \end{bmatrix}$$

i) $P_2 = |Y_{21}||V_2||V_1|\cos(\theta_{21} - \delta_2 + \delta_1) + |Y_{22}||V_2|^2\cos(\theta_{22})$

$\quad + |Y_{23}||V_2||V_3|\cos(\theta_{23} - \delta_2 + \delta_3)$

$\quad = 40|V_2||V_1|\cos(90 - \delta_2 + \delta_1) + 60|V_2|^2\cos(-90)$

$\quad + 20|V_2||V_3|\cos(90 - \delta_2 + \delta_3)$

$P_2 = 40|V_2||V_1|\cos(90 - \delta_2 + \delta_1) + 20 N_2||V_3|\cos(90 - \delta_2 + \delta_3)$

Similarly by putting values in P_3 and Q_3 it can be shown that

$$P_3 = 20|v_3||v_1|\cos(90 - \delta_3 + \delta_1) + 20|v_3||v_2|\cos(90 - \delta_3 + \delta_2)$$

$$Q_3 = -20|v_3||v_1|\sin(90 - \delta_3 + \delta_1) - 20|v_3||v_2|\sin(90 - \delta_3 + \delta_2) + 40|v_3|^2$$

b)

$$\frac{\partial P_2}{\partial \delta_2} = 40|v_2||v_1|\sin(90 - \delta_2 + \delta_1) + 20|v_2||v_3|\sin(90 - \delta_2 + \delta_3)$$

$$\frac{\partial P_2}{\partial \delta_3} = -20|v_2||v_3|\sin(90 - \delta_2 + \delta_3)$$

$$\frac{\partial P_2}{\partial |v_3|} = 20|v_2|\cos(90 - \delta_2 + \delta_3)$$

$$\frac{\partial P_3}{\partial \delta_2} = -20|v_3||v_2|\sin(90 - \delta_3 + \delta_2)$$

$$\frac{\partial P_3}{\partial \delta_3} = 20|v_3||v_1|\sin(90 - \delta_3 + \delta_1) + 20|v_3||v_2|\sin(90 - \delta_3 + \delta_2)$$

$$\frac{\partial P_3}{\partial |v_3|} = 20|v_1|\cos(90 - \delta_3 + \delta_1) + 20|v_2|\cos(90 - \delta_3 + \delta_2)$$

$$\frac{\partial Q_3}{\partial \delta_2} = -20|v_3||v_2|\cos(90 - \delta_3 + \delta_2)$$

$$\frac{\partial Q_3}{\partial \delta_3} = 20|v_3||v_1|\cos(90 - \delta_3 + \delta_1) + 20|v_3||v_2|\cos(90 - \delta_3 + \delta_2)$$

$$\frac{\partial Q_3}{\partial |v_3|} = -20|v_1|\sin(90 - \delta_3 + \delta_1) - 20|v_2|\sin(90 - \delta_3 + \delta_2) + 80|v_3|$$

$$P_{2,sch} = \frac{400}{100} = 4 \text{ pu}$$

$$S_{3,sch} = -\frac{500 + j400}{100} = -5 - j4 \quad pu$$

$$\Delta P_2^{(0)} = P_{2,sch} - P_2^{(0)} = 4 - 0 = 4$$

$$\Delta P_3^{(0)} = P_{3,sch} - P_3^{(0)} = -5 - 0 = -5$$

$$\Delta Q_3^{(0)} = Q_{3,sch} - Q_3^{(0)} = -4 - (-1) = -3$$

$$J = \begin{bmatrix} 63 & -21 & 0 \\ -21 & 41 & 0 \\ 0 & 0 & 39 \end{bmatrix}$$

1st iteration

$$\begin{bmatrix} -2.8600 \\ 1.4384 \\ -0.2200 \end{bmatrix} = \begin{bmatrix} 63 & -21 & 0 \\ -21 & 41 & 0 \\ 0 & 0 & 39 \end{bmatrix} \begin{bmatrix} \Delta \delta_2^{(0)} \\ \Delta \delta_3^{(0)} \\ \Delta |v_3^{(0)}| \end{bmatrix}$$

$$\Delta \delta_2^{(0)} = 0.0275 \qquad \delta_2^{(1)} = 0 + 0.0275 = 0.0275 \text{ rad}$$

$$\Delta \delta_3^{(0)} = -0.1078 \qquad \delta_3^{(1)} = 0 + (-0.1078) = -0.1078 \text{ rad}$$

$$\Delta |v_3^{(0)}| = -0.0769 \qquad |v_3^{(1)}| = 1 + (-0.0769) = 0.9231 \text{ pu}$$

second iteration

$$\begin{bmatrix} 0.2269 \\ -0.3965 \\ -0.5213 \end{bmatrix} = \begin{bmatrix} 61.1913 & -19.2072 & 2.8345 \\ -19.2072 & 37.5615 & -4.9871 \\ 2.6164 & -4.6035 & 33.1545 \end{bmatrix} \begin{bmatrix} \Delta \delta_2^{(1)} \\ \Delta \delta_3^{(1)} \\ \Delta |v_3^{(1)}| \end{bmatrix}$$

$$\Delta \delta_2^{(1)} = 0.0006 \qquad \delta_2^{(2)} = 0.0275 + 0.0006 = 0.0281 \text{ rad}$$

$$\Delta \delta_3^{(1)} = -0.0126 \qquad \delta_3^{(2)} = -0.1078 + (-0.0126) = -0.1204 \text{ rad}$$

$$\Delta |v_3^{(1)}| = -0.0175 \qquad |v_3^{(2)}| = 0.9231 + (-0.0175) = 0.9056 \text{ pu}$$