EE-220 Signals and Systems

Session 2019

Week 10

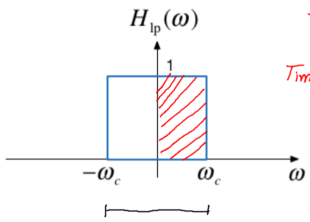
Instructor : **Engr. Ali Raza**
ealiraza1@gmail.com

Last Week

- ❑ Fourier Transform
- ❑ Examples
- ❑ Properties

Low Pass Filter

$f \rightarrow$ bounded
$t \rightarrow$ Unbound



rect $\rightarrow$ low pass filter
Time domain $\rightarrow$ Sinc

Example 4. 20

$x(t) = \frac{\sin w_i t}{\pi t}$, $h(t) = \frac{\sin w_c t}{\pi t}$

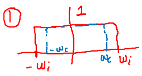
$Y(jw) = X(jw) \cdot H(jw)$

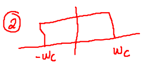
$X(jw) = \begin{cases} 1 & |w| \le w_i \\ 0 & \text{otherwise} \end{cases}$

$H(jw) = \begin{cases} 1 & |w| \le w_c \\ 0 & \text{otherwise} \end{cases}$

w_i , w_c
① $w_i < w_c$, ② $w_c < w_i$
$w_o = w_i$ $w_o = w_c$

$Y(jw) = \begin{cases} 1 & |w| \le w_o \\ 0 & \text{otherwise} \end{cases}$

①


②


Smaller value $= w_o$

Example 4. 20

- Example 4.20 $x(t) = \frac{\sin \omega_i t}{\pi t}$ $h(t) = \frac{\sin \omega_c t}{\pi t}$

$$X(j\omega) = \begin{cases} 1, & |\omega| \le \omega_i \\ 0, & elsewhere \end{cases} \qquad H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & elsewhere \end{cases}$$

$$Y(j\omega) = \begin{cases} 1, & |\omega| \le \omega_0 \\ 0, & elsewhere \end{cases} \qquad \omega_o = smaller\ (\omega_i, \omega_c)$$

$$y(t) = \begin{cases} \frac{\sin \omega_c t}{\pi t}, & if \omega_c \le \omega_i \\ \frac{\sin \omega_i t}{\pi t}, & if \omega_i \le \omega_c \end{cases}$$

THE MULTIPLICATION PROPERTY

$$\boxed{r(t) = s(t)p(t) \quad \overset{\mathfrak{F}}{\longleftrightarrow} \quad R(\omega) = \frac{1}{2\pi}[S(\omega) * P(\omega)]}$$

Example 4.21

$$r(t) = s(t)p(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} R(\omega) = \frac{1}{2\pi}[S(\omega) * P(\omega)]$$

- Example 4.21

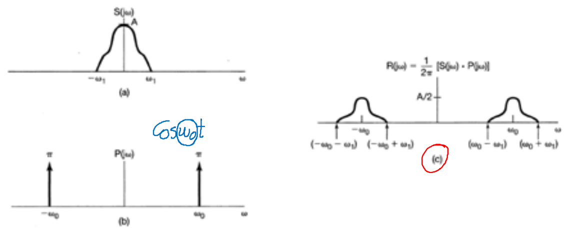
$$p(t) = \cos \omega_0 t$$

$$P(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$R(\omega) = \frac{1}{2\pi}S(\omega) * P(\omega) = \tfrac{1}{2}S(\omega - \omega_0) + \tfrac{1}{2}S(\omega + \omega_0)$$

Example 4.21

Amplitude Modulation



Information shifted to a new frequency without loss of intelligibility.

Example 4.22

Amplitude deModulation

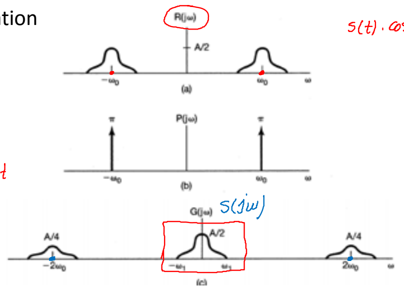
- Amplitude Demodulation

$g(t) = s(t) \cdot \cos\omega_0 t \cdot \cos\omega_0 t$

$= s(t)\left[\frac{1 + \cos 2\omega_0 t}{2}\right]$

$= \frac{1}{2}s(t) + \frac{s(t) \cdot \cos 2\omega_0 t}{2}$

$s(t) \cdot \cos\omega_0 t$



Example 4.23

$$r(t) = s(t)p(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} R(\omega) = \frac{1}{2\pi}[S(\omega) * P(\omega)]$$

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}.$$

$+\infty$

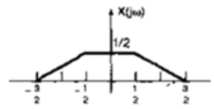
$-\infty$

Sinc rect

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right).$$

Sinc rect

$$X(j\omega) = \frac{1}{2}\mathfrak{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathfrak{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$



Signal	Fourier transform	(if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \ \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \ \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

LINEAR CONSTANT -COEFFICIENT DIFFERENTIAL EQUATIONS

$$Y(j\omega) = H(j\omega)X(j\omega),$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)},$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}.$$

$$\sum_{k=0}^{N} a_k \mathfrak{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{\frac{d^k x(t)}{dt^k}\right\},$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega),$$

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right].$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}.$$

Example 4.24

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \qquad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}.$$

$$\frac{dy(t)}{dt} + a y(t) = x(t),$$

$$H(j\omega) = \frac{1}{j\omega + a}.$$

$$h(t) = e^{-at}u(t).$$

Example 4.25

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \qquad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}.$$

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}.$$

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}.$$

$$H(j\omega) = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}. \qquad h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

Example 4.26

$$x(t) = e^{-t}u(t).$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}\right]\left[\frac{1}{j\omega + 1}\right]$$

$$= \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}.$$

$$Y(j\omega) = \frac{A_{11}}{j\omega + 1} + \frac{A_{12}}{(j\omega + 1)^2} + \frac{A_{21}}{j\omega + 3},$$

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3}. \qquad y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$

Chapter 4

- Extension of Fourier Analysis for Aperiodic Signals.
- Aperiodic signal can be thought of as a periodic signal with Infinite time period.
- As time tends to infinity, frequency harmonics becomes continuously closer.
- For Aperiodic Signals, the complex exponentials occur at the continuum of frequencies with amplitude" X(jω)/2π"
- Summation turns into Integral.

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Chapter 5

UNIVERSITY OF ENGINEERING AND TECHNOLOGY, LAHORE.

Instructor : **Engr. Ali Raza**
ealiraza1@gmail.com

DT FT

- Synthesis/Inverse Fourier Transform Equation

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega}$$

- Analysis/Fourier Transform Equation/ Spectrum of 'x[n]'

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}}$$

- For derivation/proof, consult Text Book.

DTFT Convergence

- x[n] be absolutely summable; OR the sequence has finite energy.

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

- No convergence issues with the Synthesis equation
  - Integration is on a finite interval only.

Example 5.1

$$x[n] = a^n u[n], \quad |a| < 1$$

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}}$$

$$\boxed{X(e^{j\omega}) = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n}$$

$$\boxed{X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}}$$

Example 5.2

Example 5.2 $\quad x[n] = a^{|n|}, \quad |a| < 1$

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}.$$

Making the substitution of variables $m = -n$ in the second summation, we obtain

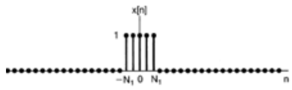
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m.$$

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}.$$

Example 5.3

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases},$$



$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}.$$

Using calculations similar to those employed Example 3.12, we can write

$$a_k = \frac{1}{N} \frac{\sin k\omega_0 (N_1 + \frac{1}{2})}{\sin k\omega_0/2}$$

$$X(e^{j\omega}) = \frac{\sin\omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}.$$

Example 5.4

$$x[n] = \delta[n].$$

$$X(e^{j\omega}) = 1.$$

THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

- The DT-FT for a Periodic Signal with spectral coefficients ‘ak’ can be interpreted as a train of impulses occurring at the harmonically related frequencies. (There are only ‘N’ unique harmonics!)

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} [2\pi\delta(\omega - k\omega_0)] a_k}$$

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} [2\pi\delta(\omega - k(\frac{2\pi}{N}))] a_k}$$