EE-220 Signals and Systems

Session 2019

Week 9

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Last Week

- Representation of **Periodic Signals** in linear combinations of complex exponentials
- Describing the effect of LTI systems on signals

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Chapter 4

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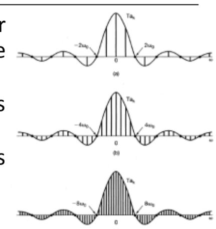
Fourier Transform

- Fourier series (FS) for Aperiodic Signal, where time period is infinity.
- Frequency harmonics spacing becomes closer
- Summation becomes Integral



REPRESENTATION OF APERIODIC SIGNALS: THE CONTINUOUS-TIME FOURIER TRANSFORM

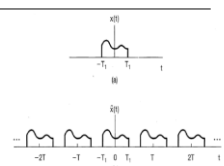
$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\,dt,$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}\,dt.$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt, \qquad a_k = \frac{1}{T}X(jk\omega_0).$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t}, \qquad \tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0.$$



$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega}$$

REPRESENTATION OF APERIODIC SIGNALS: THE CONTINUOUS-TIME FOURIER TRANSFORM

- **Inverse Fourier Transform:**

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega}$$

- **Fourier Transform:**

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt.}$$

Convergence of Fourier Transforms

1. $x(t)$ be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty. \tag{4.13}$$

2. $x(t)$ have a finite number of maxima and minima within any finite interval.
3. $x(t)$ have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

Example 4.1

$$x(t) = e^{-at}u(t) \quad a > 0.$$

Example 4.2

$$x(t) = e^{-a|t|}, \quad a > 0.$$

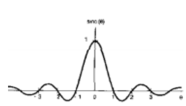
Example 4.3

$$x(t) = \delta(t).$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}dt = 1.$$

Sinc Function

$$\operatorname{sinc} x = \frac{\sin x}{x}. \qquad \operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}.$$

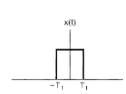


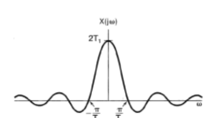
- Other than θ=0, 'sinc' is a 'sine' function with decaying amplitude.
- Zero crossing point: θ = ±1, ±2, ±3, . . . (OR) θ = ±π, ± 2π, ± 3π, . . .
- While plotting Sinc function, always calculate zero crossing points.

Example 4.4

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases},$$

$$X(j\omega) = 2\frac{\sin \omega T_1}{\omega}$$





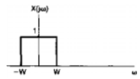
Example 4.5

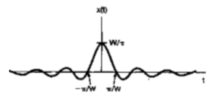
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega}$$

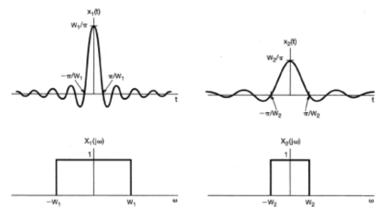
$$x(t) = \frac{1}{2\pi}\int_{-W}^{+W} e^{j\omega t}d\omega$$

$$x(t) = \frac{\sin Wt}{\pi t}$$

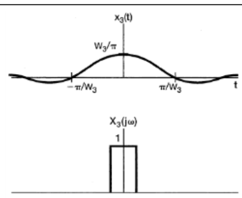




Example 4.5



Sinc Function



THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

- The Fourier Transform a Periodic Signal with spectral coefficients 'a_k' can be interpreted as a train of impulses occurring at the harmonically related frequencies.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}. \qquad X(j\omega) = 2\pi\delta(\omega - \omega_0).$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega$$
$$= e^{j\omega_0 t}.$$

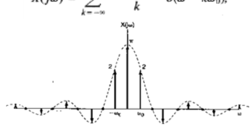
$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0),$$

Example 4.6

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0), \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}.$$

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k},$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0),$$

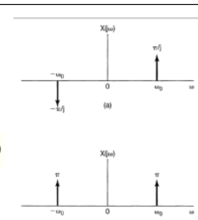


Example 4.7

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0),$$

$$x(t) = \sin\omega_0 t.$$

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)(\frac{1}{2j}) + 2\pi\delta(\omega + \omega_0)(\frac{1}{2j})$$



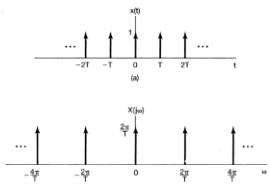
Example 4.8

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \quad X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0),$$

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}.$$

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$



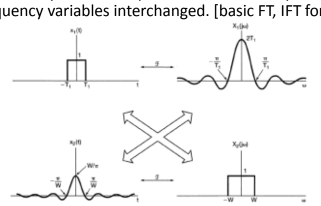
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$$

Duality

- Duality: For any transform pair, there is a dual pair with the time and frequency variables interchanged. [basic FT, IFT formulae]

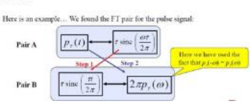


Property Table of FT

Suppose we have a FT table that a FT Pair A… we can get the dual Pair B using the general Duality Property:

1. Take the FT side of (known) Pair A and replace ω by t and move it to the time-domain side of the table of the (unknown) Pair B.
2. Take the time-domain side of the (known) Pair A and replace t by $-\omega$, multiply by 2π, and then move it to the FT side of the table of the (unknown) Pair B.

Here is an example… We found the FT pair for the pulse signal:



Example 4. 1 3

$$g(t) = \frac{2}{1 + t^2}.$$

Then, from Example 4.2,

$$x(t) = e^{-|t|} \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1 + \omega^2}.$$

$$\mathfrak{F}\left\{\frac{2}{1 + t^2}\right\} = 2\pi e^{-|\omega|}.$$

Properties of FT

$$\boxed{-jtx(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.}$$

$$\boxed{e^{j\omega_0 t}x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j(\omega - \omega_0))}$$

$$\boxed{-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{\mathfrak{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)d\eta.}$$

Parseval's Relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.$$

1 2

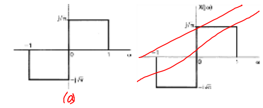
$x(t)$ $X(j\omega)$

Example 4.14

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.$$

$\longrightarrow E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$D = \left.\frac{d}{dt}x(t)\right|_{t=0}$



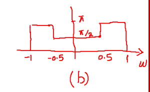
$y(t) = D = \frac{d}{dt}x(t) \xrightarrow{F} \underline{j\omega\, X(j\omega) = Y(j\omega)}$

$\underline{y(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)\cdot d\omega$

$\searrow j\omega\cdot X(j\omega)$

$-1 \longrightarrow 0$ $-j\sqrt{\pi}$

$0 \longrightarrow 1$ $j\sqrt{\pi}$



THE CONVOLUTION PROPERTY

$$y(t) = h(t) * x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).$$

Example 4.15

$H(j\omega) = \int_{-\infty}^{\infty} \delta(t-t_0)\cdot e^{-j\omega t}\cdot dt$

$\underline{e^{-j\omega t_0}}$

$h(t) = \delta(t - t_0).$

$H(j\omega) = e^{-j\omega t_0}$

$Y(j\omega) = X(j\omega)\cdot H(j\omega)$

$= X(j\omega)\cdot e^{-j\omega t_0}$

$y(t) = x(t-t_0)$

Example 4. 16

$h(t) \longrightarrow$ Impulse Response

$H(j\omega) \longrightarrow$ frequency Response

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property of Section 4.3.4,

$$Y(j\omega) = j\omega \underline{X(j\omega)}. \tag{4.61}$$

Consequently, from eq. (4.56), it follows that the frequency response of a differentiator is

$$\boxed{H(j\omega) = j\omega.} \tag{4.62}$$

Example 4. 1 7

Consider an integrator—that is, an LTI system specified by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

The impulse response for this system is the unit step $u(t)$, and therefore, from Example 4.11 and eq. (4.33), the frequency response of the system is

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Then using eq. (4.56), we have

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(j\omega)\delta(\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega), \end{aligned}$$

Example 4. 19

$$h(t) = e^{-at}u(t), \quad a > 0, \longrightarrow H(j\omega) = \frac{1}{a+j\omega}$$

$$x(t) = e^{-bt}u(t), \quad b > 0. \longrightarrow X(j\omega) = \frac{1}{b+j\omega}$$

$$Y(j\omega) = \frac{1}{a+j\omega} \cdot \frac{1}{b+j\omega} = \frac{1}{(a+j\omega)(b+j\omega)} \quad — \text{①}$$

$$Y(j\omega) = \frac{A}{a+j\omega} + \frac{B}{b+j\omega} \quad — \text{②} \Longrightarrow \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right] \overset{F^{-1}}{\Longrightarrow} \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]u(t)$$

$$\frac{1}{(a+j\omega)(b+j\omega)} = \frac{A}{a+j\omega} + \frac{B}{b+j\omega} \Rightarrow A = \frac{1}{b-a}, \; B = \frac{-1}{b-a}$$