EE-220 Signals and Systems

Session 2019

Week 7

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Last Week

- Fourier Series
  - Examples
  - Properties

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Example 3.9

- Find x(t) such that

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$.

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1} e^{-j\pi t/2}.$$

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Example 3.9

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1} e^{-j\pi t/2}.$$

$$a_1 = a*_{-1}$$

$$x(t) = a_0 + a_1 e^{j\pi t/2} + (a_1 e^{j\pi t/2})^* = a_0 + 2\Re e\{a_1 e^{j\pi t/2}\}.$$

Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd

$$b_0 = 0 \text{ and } b_{-1} = -b_1$$

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = 1/2. \quad |b_1|^2 + |b_{-1}|^2 = 1/2.$$

Substituting $b_1 = -b_{-1}$ in this equation, we obtain $|b_1| = 1/2$. $j/2$ or $-j/2$ $a_1 = \frac{1}{2}, -\frac{1}{2}$

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FOURIER SERIES REPRESENTATION OF DISCRETE-TIME PERIODIC SIGNALS

- Periodic signal if

$$x[n] = x[n+N].$$

- The fundamental period is the smallest positive integer *N*, and ωo = 2π/*N* is the fundamental frequency

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \; k = 0, \pm 1, \pm 2, \ldots.$$

$$\phi_k[n] = \phi_{k+rN}[n].$$

FOURIER SERIES REPRESENTATION OF DISCRETE-TIME PERIODIC SIGNALS

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}.$$

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

$$x[0] = \sum_{k=\langle N\rangle} a_k,$$
$$x[1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N},$$
$$\vdots$$
$$x[N-1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}.$$

- For example, *k* could take on the values *k* = 0, 1, ... , *N* - 1, or *k* = 3, 4, . . . , *N* + 2.
- ***Discrete-time Fourier series*** and the coefficients a_k as **the *Fourier series coefficients.***

FOURIER SERIES REPRESENTATION OF DISCRETE-TIME PERIODIC SIGNALS

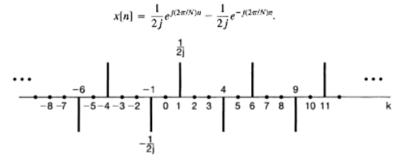
$$\boxed{\begin{aligned} x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}, \\ a_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}. \end{aligned}}$$

$$a_k = a_{k+N}.$$

Example 3.10

$$x[n] = \sin \omega_0 n,$$

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}.$$

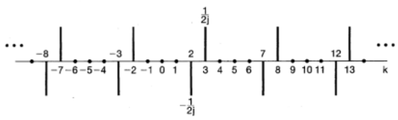


Example 3.10

ωo = 2π/N $\quad \omega_0 = \frac{2\pi M}{N}$.

$$x[n] = \sin 3(2\pi/5)n.$$

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n},$$



Example 3.11

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

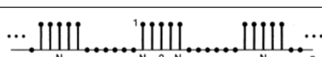
$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}]$$
$$+ \frac{1}{2}[e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}].$$

$$x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j(2\pi/N)n}$$
$$+ \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

$$a_0 = 1,$$
$$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j,$$
$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}j,$$
$$a_{-2} = -\frac{1}{2}j.$$

Example 3.12

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n},$$
$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}.$$



$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}.$$

Letting $m = n + N_1$, we observe that eq. (3.102) becomes

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)}$$
$$= \frac{1}{N}e^{jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}. \qquad y[n] = \sum_{k=0}^{n}\alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$a_k = \frac{1}{N}e^{jk\omega_0 N_1}\sum_{m=0}^{2N_1} e^{-jk\omega_0 m}$$

$$= \frac{1}{N}e^{jk\omega_0 N_1}\frac{1 - e^{-jk\omega_0(2N_1+1)}}{1 - e^{-jk\omega_0}}$$

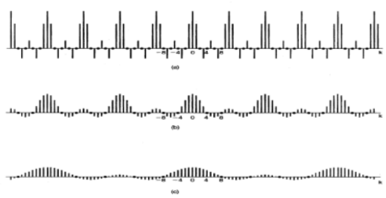
$$= \frac{1}{N}\frac{1\cdot e^{jk\omega_0 N_1} - e^{-jk\omega_0(2N_1+1-N_1)}}{1 - e^{-jk\omega_0}} \quad N_1+1$$

$$= \frac{1}{N}\frac{\cancel{e^{-jk\omega_0/2}}\left[e^{jk\omega_0/2}\cdot e^{jk\omega_0 N_1} - e^{-jk\omega_0/2}\cdot e^{-jk\omega_0 N_1}\right]}{\cancel{e^{-jk\omega_0/2}}\left[e^{jk\omega_0/2} - e^{-jk\omega_0/2}\right]} \quad N_1+1/2 \quad N+1/2$$

$$a_k = \frac{1}{N}\frac{\sin k\omega_0(N_1+1/2)}{\sin k\omega_0/2} \qquad \omega_0 = \frac{2\pi}{N}$$

Example 3.12

$$\frac{1}{N}\frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, \quad k \neq 0, \pm N, \pm 2N, \ldots$$

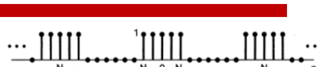


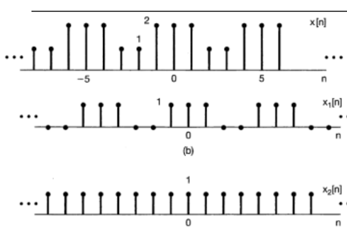
Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l}x[n]\\y[n]\end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l}a_k\\b_k\end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k$ $\left(\begin{array}{l}\text{viewed as periodic}\\ \text{with period } mN\end{array}\right)$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ $\left(\begin{array}{l}\text{finite valued and periodic only}\\ \text{if } a_0 = 0\end{array}\right)$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Example 3.13





$$a_k = b_k + c_k.$$

$$\frac{1}{N}\frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, \quad k \neq 0, \pm N, \pm 2N, \ldots$$

$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \dfrac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1.$$

$$a_k = \begin{cases} b_k = \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \dfrac{8}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

Example 3.14

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n},$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}.$$

Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7}(-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

THE RESPONSE OF LTI SYSTEMS TO COMPLEX EXPONENTIALS

- The importance of complex exponentials in the study of LTI systems stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude;

$$\text{continuous time: } e^{st} \longrightarrow H(s)e^{st},$$
$$\text{discrete time: } z^{n} \longrightarrow H(z)z^{n},$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau. \qquad H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}.$$

Frequency Response

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau. \qquad H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}.$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt. \qquad H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}.$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}.$$

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}.$$

DIY

- Example 3.16 and 2.17