The z-Transform

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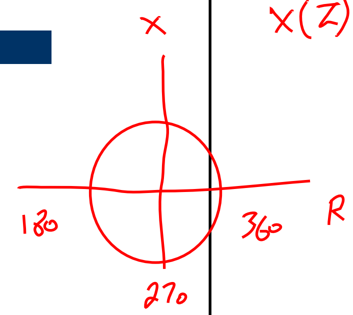
Why z-Transform?

- A generalization of Fourier transform
- Why generalize it?
  - FT does not converge on all sequence
  - Notation good for analysis

e^{jw}

$X(e^{jw})$

$X(Z)$



Definition

- The z-transform of sequence $x(n)$ is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Let $z = e^{j\omega}$.

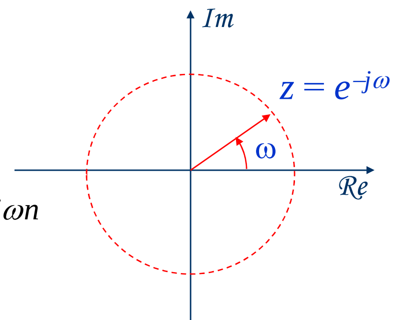
Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

z-Plane

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$



Fourier Transform is to *evaluate z-transform on a unit circle*.

Definition ROC

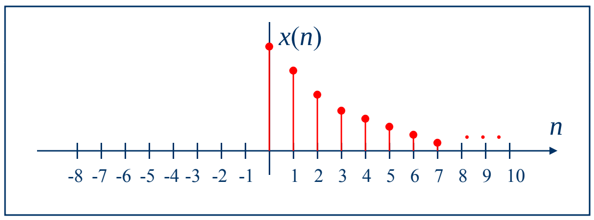
- Give a sequence, the set of values of z for which the z-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence.

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)| |z|^{-n} < \infty$$

ROC is centered on origin and consists of a set of rings.

Example: A right sided Sequence

$$\boxed{x(n) = a^n u(n)}$$



Example: A right sided Sequence

$$\boxed{x(n) = a^n u(n)}$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}| < \infty \Rightarrow |az^{-1}| < 1$$

$$\Rightarrow |z| > |a|$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

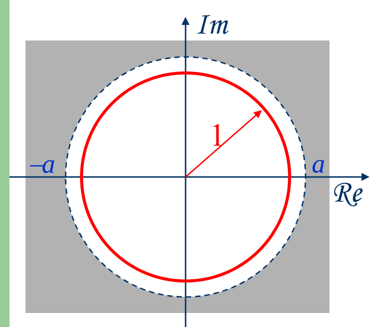
$$|z| > |a|$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

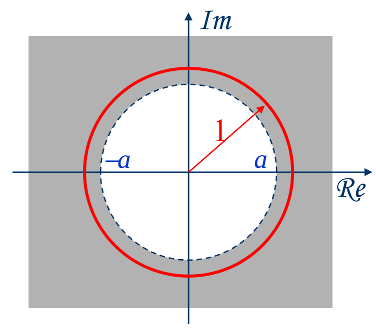
Example: A right sided Sequence ROC for $x(n)=a^n u(n)$

$$X(z) = \frac{z}{z-a}, \qquad |z| > |a|$$

Which one is stable?



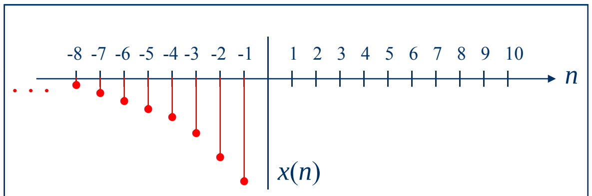
$a>1$



$a<1$

Example: A left sided Sequence

$$x(n) = -a^n u(-n-1)$$



Example: A left sided Sequence

$$x(n) = -a^n u(-n-1)$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^{n}$$

$$= 1 - \sum_{n=0}^{\infty} a^{-n} z^{n}$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |a^{-1}z| < \infty \Rightarrow |a^{-1}z| < 1$$

$$\Rightarrow |z| < |a|$$

$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{z}{z-a}$$

$$|z| < |a|$$

$$1 - \frac{1}{1-a^{-1}z}$$

$$\frac{\cancel{1} - a^{-1}z - \cancel{1}}{1-a^{-1}z}$$

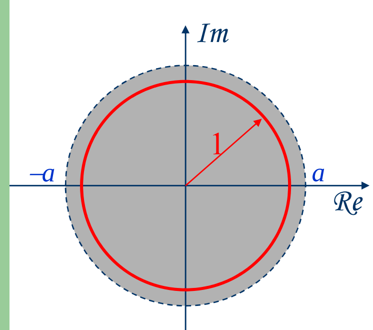
$$= \frac{-a^{-1}z}{1-a^{-1}z}$$

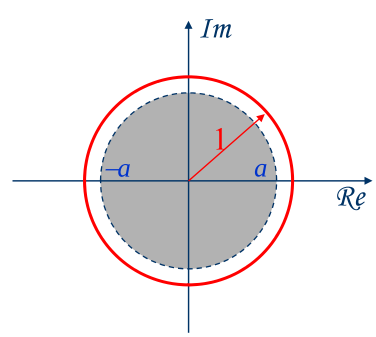
$$= \frac{-z}{a-z} = \frac{z}{z-a}$$

Example: A left sided Sequence ROC for $x(n)=-a^n u(-n-1)$

$$X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$

Which one is stable?





Represent *z*-transform as a Rational Function

$X(z) = \frac{z}{z-a}$

$$X(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials in z.

Zeros: The values of z's such that $X(z) = 0$

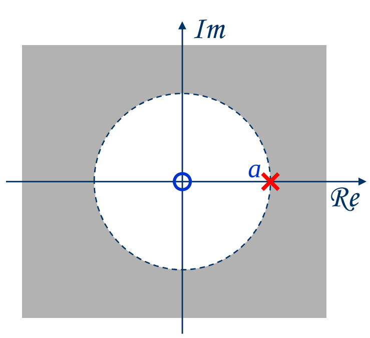
Poles: The values of z's such that $X(z) = \infty$

Example: A right sided Sequence

$$x(n) = a^n u(n) \Rightarrow X(z) = \frac{z}{z-a}, \qquad |z| > |a|$$

Zeros o

$z = 0$



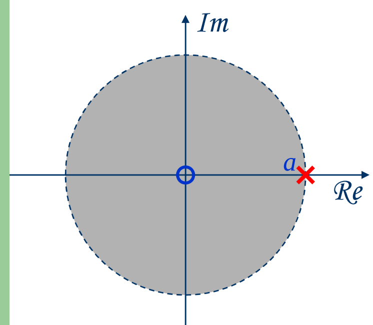
ROC is bounded by the pole and is the exterior of a circle.

Poles x

$z = a$

Example: A left sided Sequence

$$x(n) = -a^n u(-n-1) \Rightarrow X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$



ROC is bounded by the pole and is the interior of a circle.

Example: Sum of Two Right Sided Sequences

$a^n u(n)$

$X(Z) = \frac{Z}{Z-a}$

$|Z| > |a|$

$$x(n) = (\tfrac{1}{2})^n u(n) + (-\tfrac{1}{3})^n u(n)$$

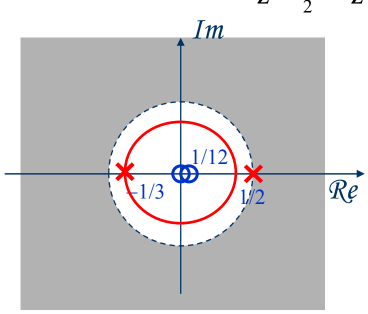
$$\Rightarrow X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

$\frac{Z}{Z-1/2} + \frac{Z}{Z+1/3}$

$\frac{Z^2 + 1/3\, Z + Z^2 - 1/2\, Z}{(Z-1/2)(Z+1/3)}$

$= \frac{2Z^2 - 1/6\, Z}{''}$

$= \frac{2Z(Z-\frac{1}{12})}{(Z-1/2)(Z+1/3)}$



ROC is bounded by poles and is the exterior of a circle.

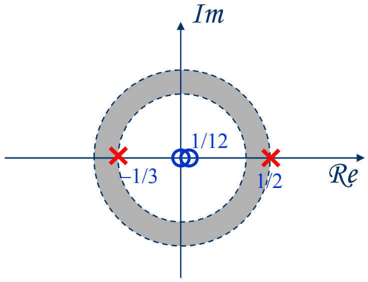
ROC does not include any pole.

Example: A Two Sided Sequence

$|z|>|a|$ $|z|<|a|$

$$x(n) = \boxed{(-\tfrac{1}{3})^n u(n)} - \boxed{(\tfrac{1}{2})^n u(-n-1)}$$

$$X(z) = \frac{z}{z+\frac{1}{3}} + \frac{z}{z-\frac{1}{2}} = \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})}$$



ROC is bounded by poles and is a ring.

ROC does not include any pole.

$$\delta[n] \xleftrightarrow{z} \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = 1, \qquad \delta[n-1] \xleftrightarrow{z} \sum_{n=-\infty}^{+\infty} \delta[n-1] z^{-n} = z^{-1}. \qquad \delta[n+1] \xleftrightarrow{z} \sum_{n=-\infty}^{+\infty} \delta[n+1] z^{-n} = z.$$

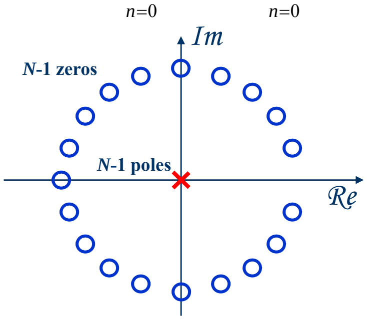
Example: A Finite Sequence

$$x(n) = a^n, \qquad 0 \le n \le N-1$$

$$y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

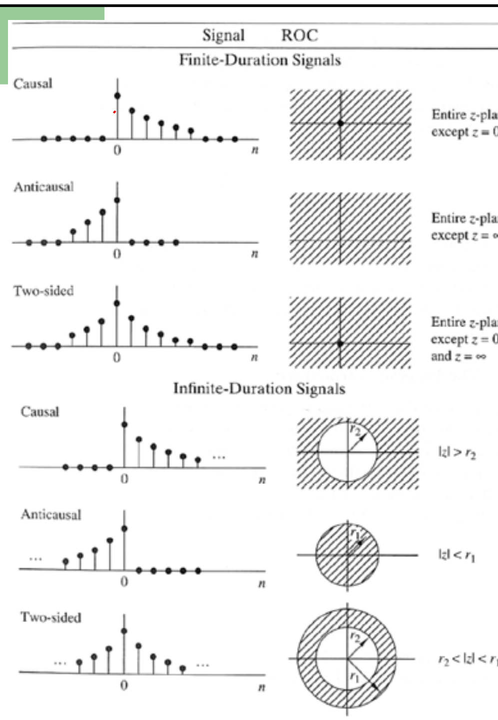
$$\frac{z^n - a^N z^{-N} z^N}{z^N - a z^{N-1}}$$



ROC: $0 < z < \infty$

ROC does not include any pole.

Always Stable



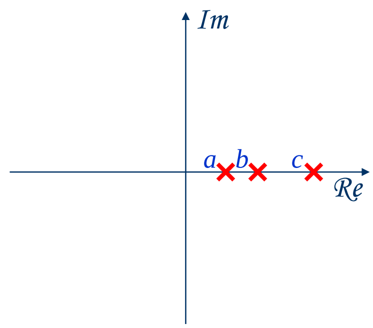
Properties of ROC

- A ring or disk in the z-plane centered at the origin.
- The Fourier Transform of $x(n)$ is converge absolutely iff the ROC includes the unit circle.
- The ROC cannot include any poles
- Finite Duration Sequences: The ROC is the entire z-plane except possibly z=0 or z=∞.
- Right sided sequences: The ROC extends outward from the outermost finite pole in $X(z)$ to z=∞.
- Left sided sequences: The ROC extends inward from the innermost nonzero pole in $X(z)$ to z=0.

More on Rational *z*-Transform

Consider the rational *z*-transform
with the pole pattern:

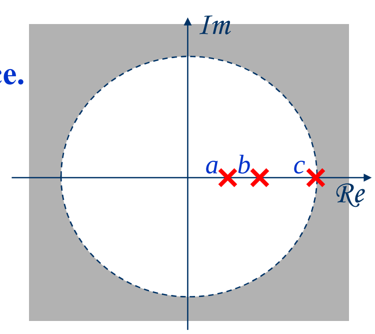
Find the possible ROC's



More on Rational *z*-Transform

Consider the rational *z*-transform
with the pole pattern:

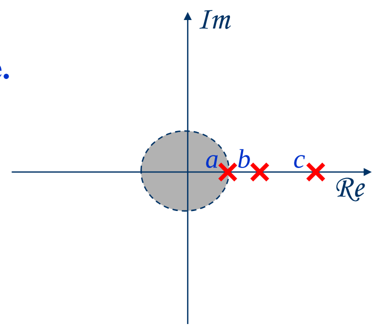
Case 1: A right sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

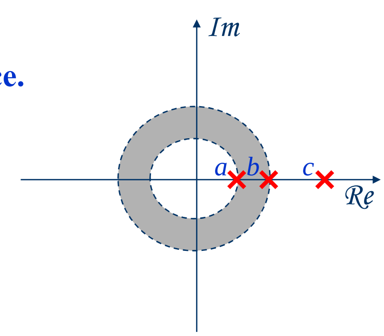
Case 2: A left sided Sequence.



More on Rational *z*-Transform

Consider the rational *z*-transform with the pole pattern:

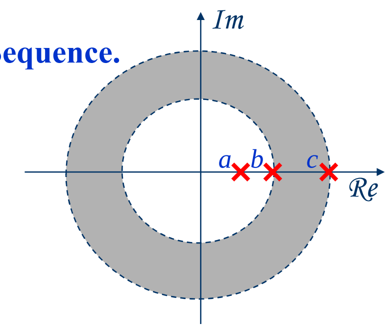
Case 3: A two sided Sequence.



More on Rational z-Transform

Consider the rational z-transform with the pole pattern:

Case 4: Another two sided Sequence.



(1) $X(z)=\frac{1}{1+\frac{5z^{-1}}{4}+\frac{3z^{-2}}{8}}$ (2) $X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$. Find possible ROC...

Z-Transform Pairs

Sequence	z-Transform	ROC
$\delta(n)$	1	All z
$\delta(n-m)$	z^{-m}	All z except 0 (if m>0) $\longrightarrow \frac{1}{z^m}$ or ∞ (if m<0) $\longrightarrow z^m$
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert<1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>\lvert a\rvert$
$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert<\lvert a\rvert$

Z-Transform Pairs

Sequence	z-Transform	ROC
$[\cos\omega_0 n]u(n)$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\|>1$
$[\sin\omega_0 n]u(n)$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\|>1$
$[r^n\cos\omega_0 n]u(n)$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\|>r$
$[r^n\sin\omega_0 n]u(n)$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\|>r$
$\begin{cases} a^n & 0\le n\le N-1 \\ 0 & \text{otherwise}\end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\|>0$

The z-Transform

z-Transform Theorems and Properties

Linearity

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[y(n)] = Y(z), \qquad z \in R_y$$

$$\mathcal{Z}[ax(n) + by(n)] = aX(z) + bY(z), \qquad z \in \underbrace{R_x \cap R_y}$$

Overlay of the above two ROC's

Shift

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\mathcal{Z}[x(n+n_0)] = z^{n_0} X(z) \qquad z \in R_x$$

Multiplication by an Exponential Sequence

$$\mathcal{Z}[x(n)] = X(z), \qquad R_{x-} < |z| < R_{x+}$$

$$\Downarrow$$

$$\mathcal{Z}[a^n x(n)] = X(a^{-1}z) \qquad z \in |a| \cdot R_x$$

Differentiation of *X*(*z*)

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\Downarrow$$

$$\mathcal{Z}[nx(n)] = -z\frac{dX(z)}{dz} \qquad z \in R_x$$

Conjugation

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\Downarrow$$

$$\mathcal{Z}[x^*(n)] = X^*(z^*) \qquad z \in R_x$$

Reversal

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\Downarrow$$

$$\mathcal{Z}[x(-n)] = X(z^{-1}) \qquad z \in 1/R_x$$

Real and Imaginary Parts

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$

$$\Downarrow$$

$$\mathcal{R}e[x(n)] = \tfrac{1}{2}[X(z) + X^*(z^*)] \qquad z \in R_x$$
$$Im[x(n)] = \tfrac{1}{2j}[X(z) - X^*(z^*)] \qquad z \in R_x$$

Initial Value Theorem

$$x(n) = 0, \qquad \text{for } n < 0$$

$$\Downarrow$$

$$x(0) = \lim_{z\to\infty} X(z)$$

Convolution of Sequences

$$\mathcal{Z}[x(n)] = X(z), \qquad z \in R_x$$
$$\mathcal{Z}[y(n)] = Y(z), \qquad z \in R_y$$

$$\Downarrow$$

$$\mathcal{Z}[x(n) * y(n)] = X(z)Y(z) \qquad z \in R_x \cap R_y$$

Inverse z-Transform

$$\boxed{x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz,}$$

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}.$$

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}.$$

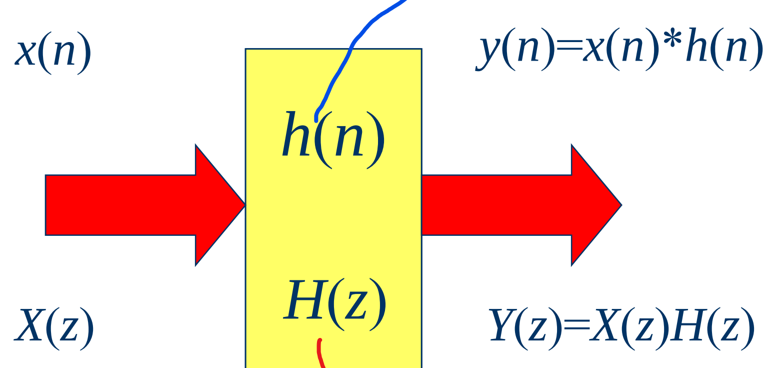
$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n].$$

The z-Transform

System Function

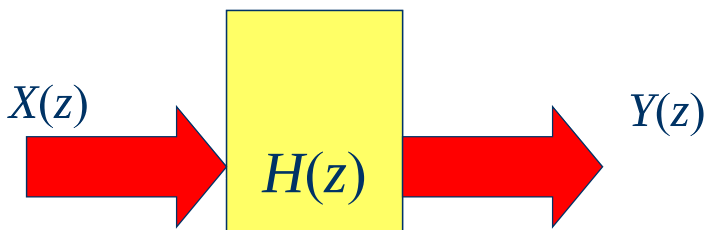
LTI System

Impulse Response



Frequency Response

LTI System



$$H(z) = \frac{Y(z)}{X(z)}$$

N^{th}-Order Difference Equation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{r=0}^{M} b_r x(n-r)$$

$y(n) \longrightarrow Y(z)$

$x(n) \longrightarrow X(z)$

$$\Rightarrow \quad Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{r=0}^{M} b_r z^{-r}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

$$H(z) = \sum_{r=0}^{M} b_r z^{-r} \Big/ \sum_{k=0}^{N} a_k z^{-k}$$

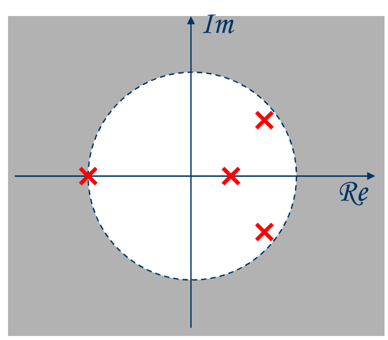
$z = e^{j\omega}$

Representation in Factored Form

$$H(z) = \frac{A\prod_{r=1}^{M}(1 - c_r z^{-1})}{\prod_{k=1}^{N}(1 - d_r z^{-1})}$$

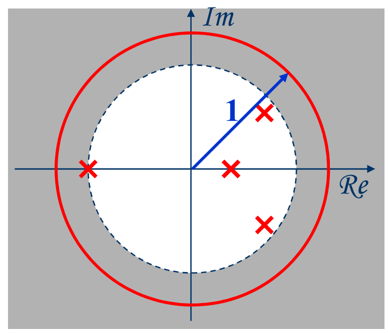
Stable and Causal Systems

Causal Systems : ROC extends outward from the outermost pole.



Stable and Causal Systems

Stable Systems : ROC includes the unit circle.



$x(n) = a^n$

$X(z) = \frac{1}{1-az^{-1}}$

Example

Consider the causal system characterized by

$$y(n) = ay(n-1) + x(n)$$

$$\boxed{H(z) = \frac{1}{1-az^{-1}}}$$

$$h(n) = a^n u(n)$$