EE-220 Signals and Systems

Session 2019

Week 2

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Last Week

- Signals Definition
  - Classification of signals
- System Definition
- Transformation of Signals
- Even and Odd Signals
- Exponential Signals
  - Real valued

Periodic Complex Exponential and Sinusoidal Signals

- A second important class of complex exponentials is obtained by constraining *a* to be purely imaginary.

$$x(t) = e^{j\omega_0 t}.$$

- An important property of this signal is that it is periodic

$$e^{j\omega_0 t} = e^{j\omega_0(t+T)}. \qquad e^{j\omega_0(t+T)} = e^{j\omega_0 t}e^{j\omega_0 T},$$

$$e^{j\omega_0 T} = 1. \qquad T_0 = \frac{2\pi}{|\omega_0|}.$$

- Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period T_0 and fundamental frequency will be f_0.

Periodic Complex Exponential and Sinusoidal Signals

- A signal closely related to the periodic complex exponential is the **sinusoidal signal**

$$A\cos(\omega_0 t + \phi) = A\,\mathcal{R}e\{e^{j(\omega_0 t+\phi)}\}, \qquad A\sin(\omega_0 t + \phi) = A\,\mathcal{I}m\{e^{j(\omega_0 t+\phi)}\}.$$

$$x(t) = A\cos(\omega_0 t + \phi),$$

- Like the complex exponential signal, the sinusoidal signal is periodic with fundamental period T_0
- If we decrease the magnitude of ω_0, we slow down the rate of oscillation and therefore increase the period. Exactly the opposite effects occur if we increase the magnitude of ω_0. What about ω_0=0?

Energy of Periodic Complex Exponential and Sinusoidal Signals

- The complex periodic exponential signal and the sinusoidal signal provide signals with infinite total energy but finite average power

$$E_{\text{period}} = \int_0^{T_0} |e^{j\omega_0 t}|^2\, dt = \int_0^{T_0} 1 \cdot dt = T_0,$$

- Since there are an infinite number of periods as *t* ranges from -∞ to +∞, the total energy integrated over all time is infinite.

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |e^{j\omega_0 t}|^2\, dt = 1. \qquad P_{\text{period}} = \frac{1}{T_0}E_{\text{period}} = 1.$$

Periodic Complex Exponential and Sinusoidal Signals

- A necessary condition for a complex exponential $e^{j\omega_0 t}$ to be periodic with period T_0 is

$$e^{j\omega T_0} = 1,$$

- which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

$$\phi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

- For k = 0, *signal* is a constant, while for any other value of *k, it* is periodic with fundamental frequency $k\omega_0$.

Periodic Complex Exponential and Sinusoidal Signals

$$\phi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots .$$

- Signal is still periodic with fundamental period

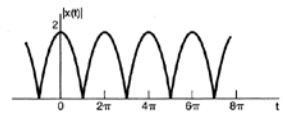
$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}.$$

- The k_{th} harmonic is still periodic with period T_0 as well, as it goes through exactly k of its fundamental periods during any time interval of length T_0

Example 1.5

- Plot the magnitude of the signal

$$x(t) = e^{j2t} + e^{j3t},$$



General Complex Exponential Signals

- Specifically, consider a complex exponential Ce^{at}, where C is expressed in polar form and a in rectangular form. That is,

$$C = |C|e^{j\theta}$$

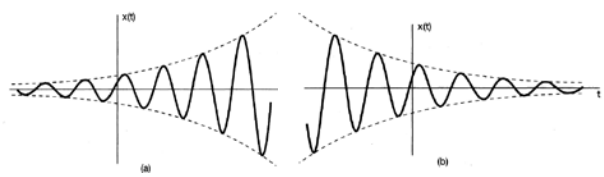
$$a = r + j\omega_0.$$

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}.$$

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta).$$

General Complex Exponential Signals

- r>0   r<0

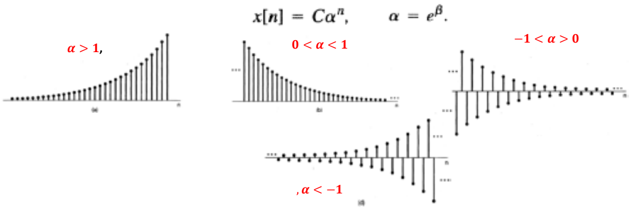


$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta)$$

Discrete-Time Complex Exponential and Sinusoidal Signals

- As in continuous time, an important signal in discrete time is the *complex exponential signal* or *sequence,* defined by

$$x[n] = C\alpha^n, \qquad \alpha = e^{\beta}.$$



Sinusoidal Signals

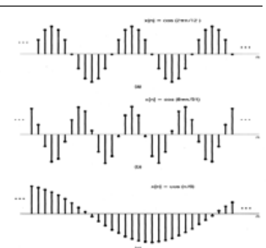
- If beta is pure imaginary

$$x[n] = C\alpha^n, \qquad \alpha = e^{\beta}.$$

$$x[n] = e^{j\omega_0 n}.$$

$$x[n] = A\cos(\omega_0 n + \phi).$$

- Units of ω_0 and phi are radians



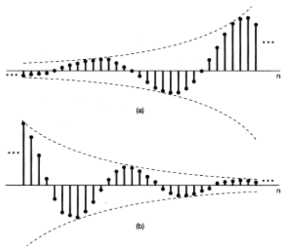
General Complex Exponential Signals

$$x[n] = C\alpha^n,$$

$$C = |C|e^{j\theta}$$

$$\alpha = |\alpha|e^{j\omega_0},$$

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$



Periodicity Properties of Discrete-Time Complex Exponentials

- There are many similarities between continuous-time and discrete-time signals, there are also a number of important differences
- In continuous signal $e^{j\omega_0 t}$ *or* $\cos(\omega_0 t)$, we know that
  - The larger the magnitude of ω_0, the higher is the rate of oscillation in the signal
  - Signal is periodic for any value of ω_0

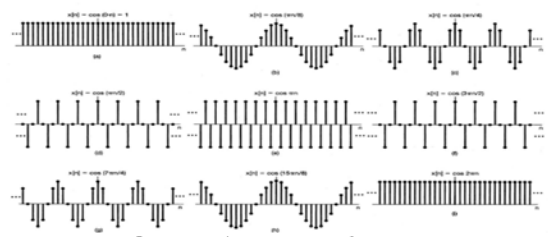
Periodicity Properties of Discrete-Time Complex Exponentials

- Specifically, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}.$$

- we see that the exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0.
- $e^{j\omega_0 t}$ signal in continues time domain produces distinguished signals for different values of ω_0. This is not the case of discrete.
- **MATLAB tutorial**

Periodicity Properties of Discrete-Time Complex Exponentials



$$0 \le \omega_0 < 2\pi \text{ or the interval } -\pi \le \omega_0 < \pi.$$

Periodicity Properties of Discrete-Time Complex Exponentials

- The second property we wish to consider concerns the periodicity of the discrete time complex exponential.
- In order for the signal $e^{j\omega_0 n}$ to be periodic with period *N*, we must have

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}, \qquad e^{j\omega_0 N} = 1.$$

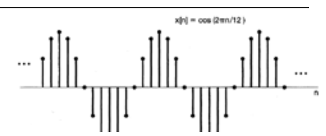
- $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer *m* such that

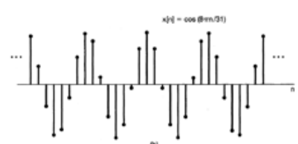
$$\omega_0 N = 2\pi m, \qquad \frac{\omega_0}{2\pi} = \frac{m}{N}.$$

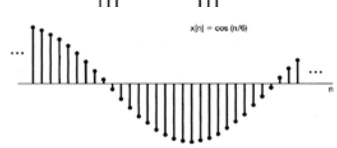
Periodicity Properties of Discrete-Time Complex Exponentials

$$\frac{\omega_0}{2\pi} = \frac{m}{N}.$$

MATLAB tutorial







Periodicity Properties of Discrete-Time Complex Exponentials

- we define the fundamental frequency of a discrete-time periodic signal as we did in continuous time.
- That is, if *x[n]* is periodic with fundamental period N, its fundamental frequency is $2\pi/N$.
- $x[n] = e^{j\omega_0 n}$, the fundamental frequency and period will be

$$\frac{2\pi}{N} = \frac{\omega_0}{m}. \qquad N = m\left(\frac{2\pi}{\omega_0}\right).$$

- **Solve Example 1.6**

Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

Periodicity Properties of Discrete-Time Complex Exponentials

$$\frac{2\pi}{N} = \frac{\omega_0}{m}.$$

- For periodic exponentials with a common period *N*,

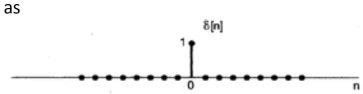
$$\phi_k[n] = e^{jk(2\pi/N)n}, \qquad k = 0, \pm 1, \ldots.$$

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} = e^{jk(2\pi/N)n}e^{j2\pi n} = \phi_k[n].$$

The Discrete-Time Unit Impulse and Unit Step Sequences

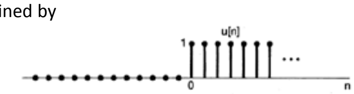
- One of the simplest discrete-time signals is the *unit impulse* (or *unit sample),* which is defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



- A second basic discrete-time signal is the discrete-time *unit step,* denoted by *u[n]* and defined by

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}.$$



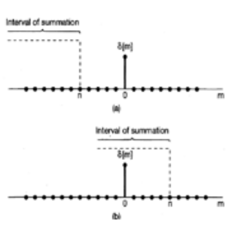
The Discrete-Time Unit Impulse and Unit Step Relationship

- There is a close relationship between the discrete-time unit impulse and unit step. In particular, the discrete-time unit impulse is the *first difference* of the discrete-time step

$$\delta[n] = u[n] - u[n-1].$$

- Conversely, the discrete-time unit step is the *running sum* of the unit sample. That is,

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]. \quad u[n] = \sum_{k=0}^{\infty} \delta[n-k].$$



Sampling Using Impulse

- The unit impulse sequence can be used to sample the value of a signal at $n = 0$. In particular, since *impulse* is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n].$$

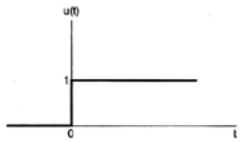
$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0].$$

The Continuous-Time Unit Step and Unit Impulse Functions

- The continuous-time *unit step function* *u(t)* is defined in a manner similar to its discrete time counterpart. Specifically,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases},$$

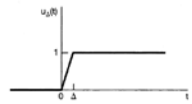
$$u[n] = \sum_{m=-\infty}^{n} \delta[m].$$

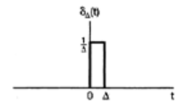


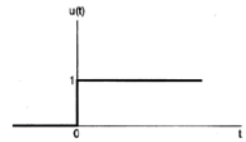
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau. \qquad \int_{-\infty}^{t} k\delta(\tau)\, d\tau = ku(t). \qquad \delta(t) = \frac{du(t)}{dt}.$$

$u(t)$ is the limit of $u_\Delta(t)$ as $\Delta \to 0$.

- Since *u(t)* is discontinuous at *t* = 0 and consequently is formally not differentiable.





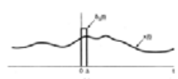


$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}, \qquad \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t), \qquad \delta(t) = \frac{du(t)}{dt}.$$

Unit Step Function

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k].$$

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma.$$



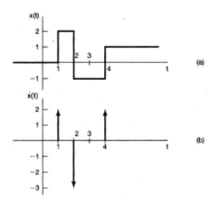
$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t).$$

Since $\delta(t)$ is the limit as $\Delta \to 0$ of $\delta_\Delta(t)$, it follows that

$$x(t)\delta(t) = x(0)\delta(t).$$

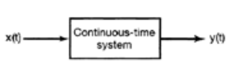
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

Example 1.7

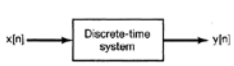


CONTINUOUS-TIME AND DISCRETE-TIME SYSTEMS

- **System** is a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs.

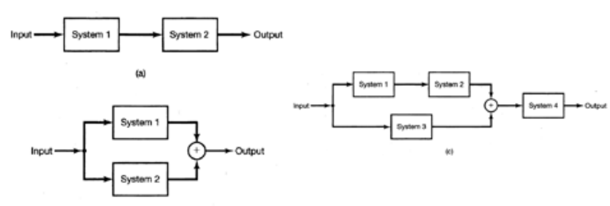


$$x(t) \to y(t).$$



$$x[n] \to y[n].$$

Interconnections of Systems



Interconnections of Systems