EE-220 Signals and Systems

Session 2019

Week 6

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Last Week

- Linear Constant-Coefficient Differential Equations
- Block Diagram Representation
- Basic Functions
  - Sine, Exponential

Fourier Series of a Periodic Continuous-Time Signal

- This pair of equations, then, defines the Fourier series of a **periodic continuous-time signal**:

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \\ a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt. \end{aligned}}$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt,$$

Example 3.3

$$x(t) = \sin\omega_0 t,$$

$$\sin\omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}.$$

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$
$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

Example 3.4

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right).$$

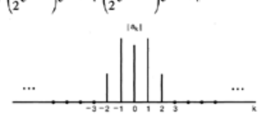
$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Example 3.4

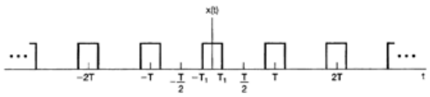
$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$
$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$
$$a_k = 0, \quad |k| > 2.$$



Example 3.5

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}. \quad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}, \quad a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right].$$

Example 3.5

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right].$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0,$$

- *For T= 4T1, x(t)* is a square wave that is unity for half the period and zero for half the period

$$\omega_0 T_1 = \pi/2$$

Example 3.5

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \quad k \neq 0,$$

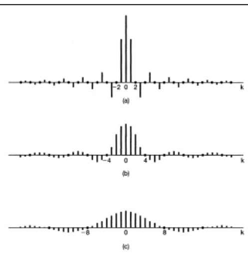
$$a_0 = \frac{1}{2},$$

$$a_1 = a_{-1} = \frac{1}{\pi},$$

$$a_3 = a_{-3} = -\frac{1}{3\pi},$$

$$a_5 = a_{-5} = \frac{1}{5\pi},$$

$$\vdots$$

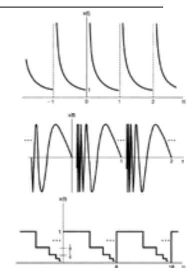


CONVERGENCE OF THE FOURIER SERIES

1. Over any period, *x(t)* must be *absolutely integrable;* that is,

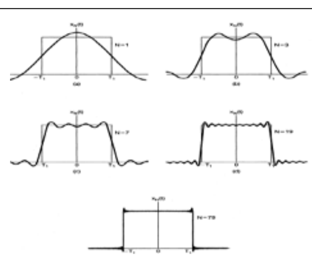
$$\int_T |x(t)|\,dt < \infty.$$

2. In any finite interval of time, *x(t)* is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.
3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.



Gibbs Phenomenon

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t},$$



Gibbs Phenomenon

- Thus, as *N* increases, the ripples in the partial sums become compressed toward the discontinuity, but for *any* finite value of *N,* the peak amplitude of the ripples remains constant. This behavior has come to be known as the *Gibbs phenomenon.*
- The implication is that the truncated Fourier series approximation $X_N(t)$ of a discontinuous signal *x(t)* will in general exhibit high-frequency ripples and overshoot *x(t)* near the discontinuities.
- A large enough value of *N* should be chosen so as to guarantee that the total energy in these ripples is insignificant.

PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

- **Linearity:**

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k.$$
$$z(t) = Ax(t) + By(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k.$$

- **Time Shifting:**

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k.$$
$$x(t - t_0) \stackrel{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

- **Time Reversal:**

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{-k}.$$

- **Time Scaling:**
  - *x(αt),* where *α* is a positive real number, is periodic with period *T/α* and fundamental frequency *αw₀*

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

- **Multiplication:**

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k.$$
$$x(t)y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

- Multiplication in time-domain=Convolution in Frequency Domain
- Convolution in time-domain= Multiplication in Frequency Domain

PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

- **Conjugation and Conjugate Symmetry:**

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$x^*(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}.$$

- For real x(t) $\quad x(t) = x^*(t) \qquad a_{-k} = a^*_k.$
- For real and even x(t) $\quad a_k = a_{-k} \qquad a_k = a^*_k$

Parseval's Relation for Continuous-Time Periodic Signals

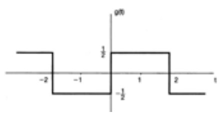
- Total average power in a periodic signal equals the sum of the average powers in all of its harmonic components

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2,$$

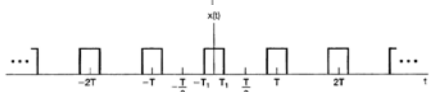
$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2,$$

Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Example 3.6

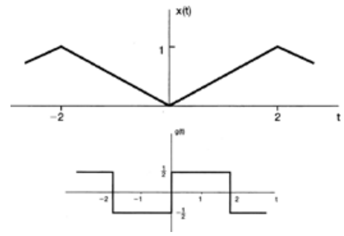


$$g(t) = x(t-1) - 1/2.$$



$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0,$$

Example 3.7



Example 3.8

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT);$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi t/T} dt = \frac{1}{T}.$$

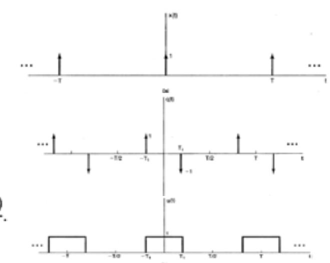
In other words, all the Fourier series coefficients of the impulse train are identical.

Example 3.8

$$q(t) = x(t+T_1) - x(t-T_1).$$

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k,$$

$$b_k = \frac{1}{T}[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}] = \frac{2j\sin(k\omega_0 T_1)}{T}.$$



Example 3.8

$$q(t) = \frac{dg(t)}{dt}$$

$$b_k = (jk\omega_0)c_k$$

$$c_k = (1/jk\omega_0)b_k$$

$$c_k = \left(\frac{1}{jk\omega_0}\right)\frac{1}{T}[2j\sin(k\omega_0 . T_1)]$$

$$c_k = \frac{1}{k\pi}[\sin(k\omega_0 . T_1)], \quad k \neq 0$$

$$c_0 = \frac{2T_1}{T}, \quad k = 0$$