EE-220 Signals and Systems

Session 2019

Week 11

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Chapter: 4

- CT FT
- Properties of FT
- Important Transforms of Functions
- Examples
- Differential Equation in Frequency Domain

THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

- The DT-FT for a Periodic Signal with spectral coefficients ‘ak’ can be interpreted as a train of impulses occurring at the harmonically related frequencies. (There are only ‘N’ unique harmonics!)

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$\boxed{X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}[2\pi\delta(\omega-k\omega_0)]a_k}$$

$$\boxed{X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}[2\pi\delta(\omega-k(\frac{2\pi}{N}))]a_k}$$

Example 5.5

$$a_k e^{jk(2\pi/N)n} \qquad \boxed{X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}[2\pi\delta(\omega-k\omega_0)]a_k}$$

$$x[n]=\cos\omega_0 n=\frac{1}{2}e^{j\omega_0 n}+\frac{1}{2}e^{-j\omega_0 n}, \qquad \text{with} \quad \omega_0=\frac{2\pi}{5}.$$

$$\boxed{X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}[2\pi\delta(\omega-k(\frac{2\pi}{N}))]a_k}$$

$$\boxed{X(e^{j\omega})=\pi\delta(\omega-\frac{2\pi}{5})+\pi\delta(\omega+\frac{2\pi}{5}), \quad -\pi\le\omega<\pi}$$

Example 5.6

$$\boxed{X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}[2\pi\delta(\omega-k\omega_0)]a_k}$$

$$x[n]=\sum_{k=-\infty}^{+\infty}\delta[n-kN],$$

$$a_k=\frac{1}{N}\sum_{n=\langle N\rangle}x[n]e^{-jk(2\pi/N)n}.$$

$$a_k=\frac{1}{N}.$$

$$X(e^{j\omega})=\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right).$$

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix}X(e^{j\omega})\\ Y(e^{j\omega})\end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n=\text{multiple of } k\\ 0, & \text{if } n\neq\text{multiple of } k\end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases}X(e^{j\omega})=X^*(e^{-j\omega})\\ \Re e\{X(e^{j\omega})\}=\Re e\{X(e^{-j\omega})\}\\ \Im m\{X(e^{j\omega})\}=-\Im m\{X(e^{-j\omega})\}\\ \lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert\\ \sphericalangle X(e^{j\omega})=-\sphericalangle X(e^{-j\omega})\end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n]=\mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n]=\mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re e\{X(e^{j\omega})\}$ $j\Im m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty}|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2d\omega$$

Example 5.8

- Example: 5.8 $x[n] = u[n]$

$$g[n] = \delta[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} G(e^{j\omega}) = 1.$$

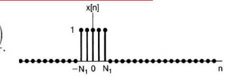
$$x[n] = \sum_{m=-\infty}^{n} g[m]$$

$$X(e^{j\omega}) = \frac{1}{(1-e^{-j\omega})} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$X(e^{j\omega}) = \frac{1}{(1-e^{-j\omega})} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Example 5.9/5.3

$$X(e^{j\omega}) = \frac{\sin \omega \left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}.$$



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

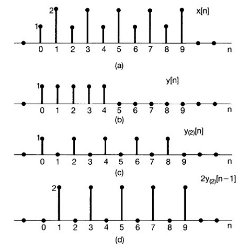
$$Y(e^{j\omega}) = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

Shifting

$$y_{(2)}[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j4\omega} \frac{\sin(5\omega)}{\sin(\omega)},$$

$$2y_{(2)}[n-1] \stackrel{\mathfrak{F}}{\longleftrightarrow} 2e^{-j5\omega} \frac{\sin(5\omega)}{\sin(\omega)}.$$

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega}) \left(\frac{\sin(5\omega)}{\sin(\omega)}\right).$$



Example 5.11

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$h[n] = \delta[n - n_0]$$

$$H(e^{j\omega}) = e^{-j\omega n_0}$$

$$y[n] = x[n] * h[n]$$

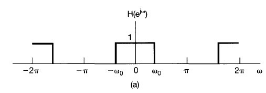
$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

$$y[n] = x[n - n_0]$$

Example 5.12

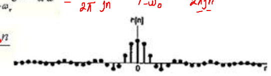
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$



$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$h[n] = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega n} d\omega = \frac{1}{2\pi} \frac{1}{jn} e^{j\omega n} \Big|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi j n} e^{j\omega_0 n} - e^{-j\omega_0 n}$$

$$h[n] = \frac{\sin \omega_0 n}{\pi n}$$



Example 5.13/5.1

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1 \qquad x[n] = \beta^n u[n], \quad |\beta| < 1$$

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \qquad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}) \qquad Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

$$Y(e^{j\omega}) = \frac{A}{(1 - \alpha e^{-j\omega})} + \frac{B}{(1 - \beta e^{-j\omega})}$$

Example 5.13

- Example: 5.13

$$A = \frac{\alpha}{\alpha - \beta}, \quad B = \frac{-\beta}{\alpha - \beta}$$

$$Y(e^{j\omega}) = \frac{A}{(1 - \alpha e^{-j\omega})} + \frac{B}{(1 - \beta e^{-j\omega})}$$

$$y[n] = \frac{\alpha}{\alpha - \beta} \alpha^n u[n] - \frac{\beta}{\alpha - \beta} \beta^n u[n]$$

$\alpha = \beta$

$$Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \alpha e^{-j\omega})}$$

Example 5.13

Example: 5.13

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$\alpha^n u[n] \leftrightarrow \frac{1}{1-\alpha e^{-j\omega}}$$

$$\boxed{nx[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.}$$

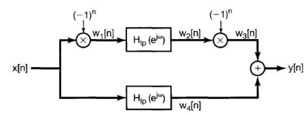
$$n\alpha^n u[n] \leftrightarrow j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

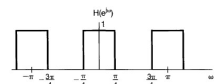
To account for the factor $e^{j\omega}$, we use the time-shifting property to obtain

$$(n+1)\alpha^{n+1}u[n+1] \leftrightarrow je^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

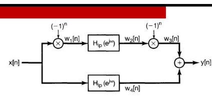
Example 5. 14

- Consider the system shown in Figure with input *x[n]* and output *y[n]*. The LTI systems with frequency response $H_{lp}(e^{jw})$ are ideal low-pass filters with cutoff frequency *pi/4* and unity gain in the passband.





Example 5. 14



$$(-1)^n = e^{j\pi n}$$

$$w_1[n] = e^{j\pi n}x[n].$$

$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)}).$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)}).$$

Since $w_3[n] = e^{j\pi n}w_2[n]$, we can again apply the frequency-shifting property to obtain

$$W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}).$$

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}).$$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega}).$$

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega}).$$

$$H(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]$$

Example 5.15

$$\boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.}$$

$$x[n] = x_1[n]x_2[n],$$

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.$$

- Solve using same convolution concepts. (Refer book)

DUALITY

- In considering the continuous-time Fourier transform, we observed a symmetry or duality between the analysis equation and the synthesis equation.
- No corresponding duality exists between the analysis equation and the synthesis equation for the discrete-time Fourier transform
- Duality exist for DT-FS
- See Table 5.1 and 5.2

SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality
duality
duality

Example 5.16/3.12

- Periodic signal with a period of $N = 9$

$$x[n] = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ \dfrac{5}{9}, & n = \text{multiple of } 9 \end{cases}$$

- In Chapter 3, we found that a rectangular square wave has Fourier coefficients in the above form. Let *g[n]* be a rectangular square wave with period $N = 9$, $N1=2$ such that

$$a_k = \frac{1}{N}\,\frac{\sin k\omega_0 (N_1 + 1/2)}{\sin k\omega_0/2}$$

Example 5.16/3.12

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4. \end{cases}$$

$$x[n] = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ \dfrac{5}{9}, & n = \text{multiple of } 9 \end{cases}$$

$$b_k = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ \dfrac{5}{9}, & k = \text{multiple of } 9 \end{cases}$$

$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$		$a_k = \frac{1}{N}\sum_{k=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$
discrete time periodic in time	duality	discrete frequency periodic in frequency

$$b_k = \frac{1}{9}\sum_{n=-2}^{2}(1)e^{-j2\pi nk/9}. \qquad x[n] = \frac{1}{9}\sum_{k=-2}^{2}(1)e^{-j2\pi nk/9}.$$

$$x[n] = \frac{1}{9}\sum_{k'=-2}^{2} e^{+j2\pi nk'/9}. \qquad a_k = \begin{cases} 1/9, & |k| \leq 2 \\ 0, & 2 < |k| \leq 4, \end{cases}$$

Example 5.17/3.5

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

$$g(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & T_1 < |t| \leq \pi \end{cases}, \qquad a_k = \frac{\sin(kT_1)}{k\pi}$$

- Using Duality Relationship between DT-FT and CT-FS

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}. \qquad X(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \pi/2 \\ 0 & \pi/2 < |\omega| \leq \pi \end{cases}.$$

Linear Constant-Coeff. Difference Equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}),$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

Example 5.18

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

$$y[n] - ay[n-1] = x[n],$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

$$h[n] = a^n u[n].$$

Example 5.19

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]. \qquad (5.84)$$

From eq. (5.80), the frequency response is

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}. \qquad (5.85)$$

As a first step in obtaining the impulse response, we factor the denominator of eq. (5.85):

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}. \qquad (5.86)$$

$H(e^{j\omega})$ can be expanded by the method of partial fractions, as in Example A.3 in the appendix. The result of this expansion is

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}. \qquad (5.87)$$

The inverse transform of each term can be recognized by inspection, with the result that

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]. \qquad (5.88)$$

Example 5.20

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-e\omega})^2}.$$

$$Y(e^{j\omega}) = \frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}} + \frac{B_{12}}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}, \quad Y(e^{j\omega}) = -\frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}.$$

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\} u[n].$$