Chapter 2

Solution of Algebraic and Transcendental Equations

2.1 INTRODUCTION

One of the basic problems in science and engineering is the computation of roots of an equation in the form, $f(x) = 0$. The equation $f(x) = 0$ is called an *algebraic equation*, if it is purely a polynomial in x; it is called a *transcendental equation* if $f(x)$ contains trigonometric, exponential or logarithmic functions. For example,

$$x^3 + 5x^2 - 6x + 3 = 0$$

is an algebraic equation, whereas

$$M = E - e \sin E \qquad \text{and} \qquad ax^2 + \log (x - 3) + e^x \sin x = 0$$

are transcendental equations.

To find the solution of an equation $f(x) = 0$, we find those values of x for which $f(x) = 0$ is satisfied. Such values of x are called the *roots* of $f(x) = 0$. Thus a is a root of an equation $f(x) = 0$, if and only if, $f(a) = 0$.

Before, we develop various numerical methods, we shall list below some of the basic properties of an algebraic equation:

(i) Every algebraic equation of nth degree, where n is a positive integer, has n and only n roots.

(ii) Complex roots occur in pairs. That is, if $(a + ib)$ is a root of $f(x) = 0$, then $(a - ib)$ is also a root of this equation.

(iii) If $x = a$ is a root of $f(x) = 0$, a polynomial of degree n, then $(x - a)$ is a factor of $f(x)$. On dividing $f(x)$ by $(x - a)$ we obtain a polynomial of degree $(n - 1)$.

(iv) Descartes rule of signs: The number of positive roots of an algebraic equation $f(x) = 0$ with real coefficients cannot exceed the number of changes in sign of the coefficients in the polynomial $f(x) = 0$. Similarly, the number of negative roots of $f(x) = 0$ cannot exceed the number of changes in the sign of the coefficients of $f(-x) = 0$. For example, consider an equation

$$x^3 - 3x^2 + 4x - 5 = 0$$

As there are three changes in sign, also, the degree of the equation is three, and hence the given equation will have all the three positive roots.

(v) Intermediate value property: If $f(x)$ is a real valued continuous function in the closed interval $a \le x \le b$. If $f(a)$ and $f(b)$ have opposite signs, then the graph of the function $y = f(x)$ crosses the x-axis at least once; that is $f(x) = 0$ has at least one root ξ such that $a < \xi < b$.

Broadly speaking, all the known numerical methods for solving either a transcendental equation or an algebraic equation can be classified into two groups: *direct methods* and *iterative methods*. Direct methods require no knowledge of the initial approximation of a root of the equation $f(x) = 0$, while iterative methods do require first approximation to initiate iteration. How to get the first approximation? We can find the approximate value of the root of $f(x) = 0$ either by a *graphical method* or by an *analytical method* as explained below:

Graphical method

Often, the equation $f(x) = 0$ can be rewritten as $f_1(x) = f_2(x)$ and the first approximation to a root of $f(x) = 0$ can be taken as the abscissa of the point of intersection of the graphs of $y = f_1(x)$ and $y = f_2(x)$. For example, consider,

$$f(x) = x - \sin x - 1 = 0$$

It can be written as $x - 1 = \sin x$. Now, we shall draw the graphs of

$$y = x - 1 \qquad \text{and} \qquad y = \sin x$$

as shown in Fig. 2.1. The approximate value of the root is found to be 1.9.

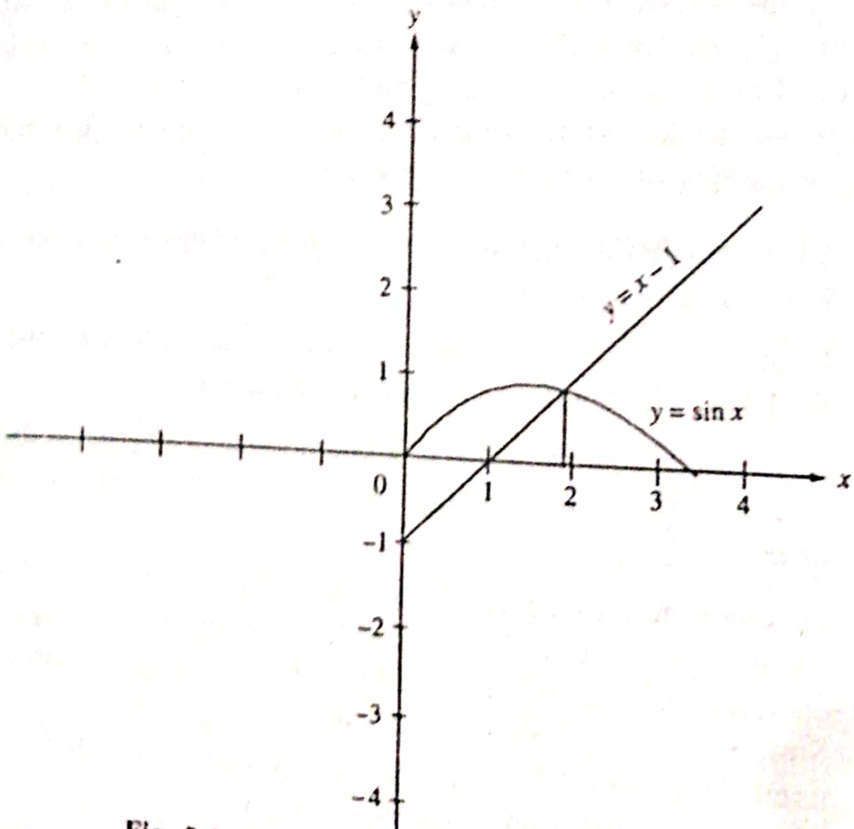

Fig. 2.1 Illustration by graphical method.

Analytical method

This method is based on 'intermediate value property'. We shall illustrate it through an example. Let,

$$f(x) = 3x - \sqrt{1 + \sin x} = 0$$

We can easily verify

$$f(0) = -1$$

$$f(1) = 3 - \sqrt{1 + \sin\left(1 \times \frac{180}{\pi}\right)} = 3 - \sqrt{1 + 0.84147} = 1.64299$$

We observe that $f(0)$ and $f(1)$ are of opposite signs. Therefore, using intermediate value property we infer that there is at least one root between $x = 0$ and $x = 1$. This method is often used to find the first approximation to a root of either transcendental equation or algebraic equation. Hence, in analytical method, we must always start with an initial interval (a, b), so that $f(a)$ and $f(b)$ have opposite signs.

2.2 BISECTION METHOD

This method is due to Bolzano. Suppose, we wish to locate the root of an equation $f(x) = 0$ in an interval, say (x_0, x_1). Let $f(x_0)$ and $f(x_1)$ are of opposite signs, such that $f(x_0) f(x_1) < 0$.

Then the graph of the function crosses the x-axis between x_0 and x_1, which guarantees the existence of at least one root in the interval (x_0, x_1). The desired root is approximately defined by the mid-point

$$x_2 = \frac{x_0 + x_1}{2}$$

If $f(x_2) = 0$, then x_2 is the desired root of $f(x) = 0$. However, if $f(x_2) \neq 0$, then the root may be between x_0 and x_2 or x_2 and x_1. Now, we define the next approximation by

$$x_3 = \frac{x_0 + x_2}{2}$$

provided $f(x_0) f(x_2) < 0$, then the root may be found between x_0 and x_2 or by

$$x_3 = \frac{x_1 + x_2}{2}$$

provided $f(x_1) f(x_2) < 0$, then the root lies between x_1 and x_2 etc.

Thus, at each step, we either find the desired root to the required accuracy or narrow the range to half the previous interval as depicted in Fig. 2.2. This process of halving the intervals is continued to determine a smaller and smaller interval within which the desired root lies. Continuation of this process eventually gives us the desired root. This method is illustrated in the following example.

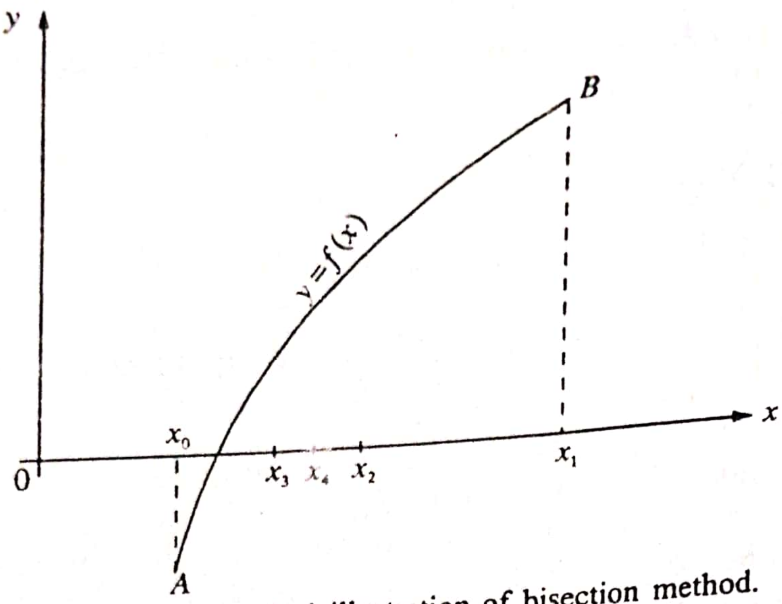

Fig. 2.2 Geometrical illustration of bisection method.

Example 2.1 Solve $x^3 - 9x + 1 = 0$ for the root between $x = 2$ and $x = 4$ by the bisection method.

Solution Given $f(x) = x^3 - 9x + 1$. We can verify $f(2) = -9$, $f(4) = 29$. Therefore, $f(2)\,f(4) < 0$ and hence the root lies between 2 and 4. Let $x_0 = 2$, $x_1 = 4$. Now, we define

$$x_2 = \frac{x_0 + x_1}{2} = \frac{2 + 4}{2} = 3$$

as a first approximation to a root of $f(x) = 0$ and note that $f(3) = 1$, so that $f(2)\,f(3) < 0$. Thus, the root lies between 2 and 3. We further define,

$$x_3 = \frac{x_0 + x_2}{2} = \frac{2 + 3}{2} = 2.5$$

and note that $f(x_3) = f(2.5) < 0$, so that $f(2.5)\,f(3) < 0$. Therefore, we define the mid-point,

$$x_4 = \frac{x_3 + x_2}{2} = \frac{2.5 + 3}{2} = 2.75, \text{ etc.}$$

Similarly, we find that

$$x_5 = 2.875 \quad \text{and} \quad x_6 = 2.9375$$

and the process can be continued until the root is obtained to the desired accuracy. These results are presented in the table.

n	x_n	$f(x_n)$
2	3	1.0
3	2.5	−5.875
4	2.75	−2.9531
5	2.875	−1.1113
6	2.9375	−0.0901

2.3 REGULA–FALSI METHOD

This method is also known as the *method of false position.* In this method, we choose two points x_n and x_{n-1} such that $f(x_n)$ and $f(x_{n-1})$ are of opposite signs. Intermediate value property suggests that the graph of $y = f(x)$ crosses the x-axis between these two points, and therefore, a root say $x = \xi$ lies between these two points. Thus, to find a real root of $f(x) = 0$ using Regula-Falsi method, we replace the part of the curve between the points $A[x_n, f(x_n)]$ and $B[x_{n-1}, f(x_{n-1})]$ by a chord in that interval and we take the point of intersection of this chord with the x-axis as a first approximation to the root (see Fig. 2.3).

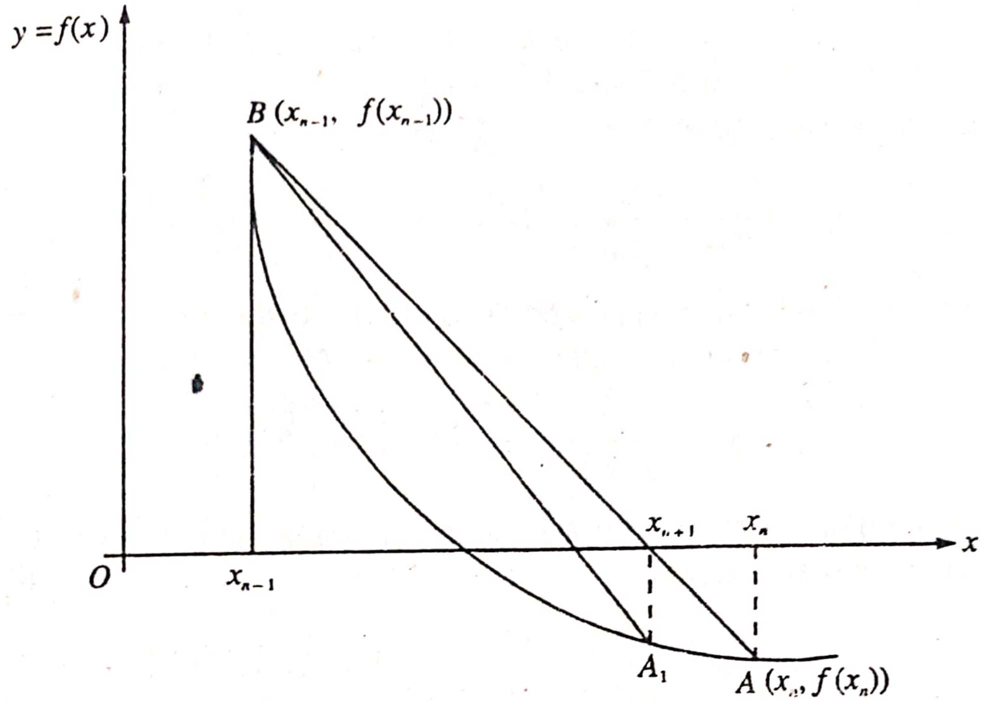

Fig. 2.3 Geometrical illustration of Regula-Falsi method.

Now, the equation of the chord joining the points A and B is

$$\frac{y - f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n} \tag{2.1}$$

Setting $y = 0$ in Eq. (2.1), we get

$$x = x_n - \frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)} f(x_n)$$

Hence, the first approximation to the root of $f(x) = 0$ is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \tag{2.2}$$

From Fig. 2.3, we observe that $f(x_{n-1})$ and $f(x_{n+1})$ are of opposite sign. Thus, it is possible to apply the above procedure, to determine the line through B and A_1 and so on. Hence, the successive approximations to the root of $f(x) = 0$ is given by Eq. (2.2). This method can best be understood through the following examples.

Example 2.2 Use the Regula-Falsi method to compute a real root of the equation $x^3 - 9x + 1 = 0$,

(i) if the root lies between 2 and 4
(ii) if the root lies between 2 and 3.

Comment on the results.

Solution Let $f(x) = x^3 - 9x + 1$.

(i) $f(2) = -9$ and $f(4) = 29$. Since $f(2)$ and $f(4)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 4. Taking $x_1 = 2$, $x_2 = 4$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 4 - \frac{2 \times 29}{38} = 2.47368$$

and $f(x_3) = -6.12644$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3. The second approximation to the root is given as

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.73989$$

and $f(x_4) = -3.090707$. Now, since $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.86125$$

and $f(x_5) = -1.32686$. This procedure can be continued till we get the desired result. The first three iterations are shown as in the table.

n	x_{n+1}	$f(x_{n+1})$
2	2.47368	–6.12644
3	2.73989	–3.090707
4	2.86125	–1.32686

(ii) $f(2) = -9$ and $f(3) = 1$. Since $f(2)$ and $f(3)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 3. Taking $x_1 = 2$, $x_2 = 3$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{1}{10} = 2.9$$

and $f(x_3) = -0.711$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3. The second approximation to the root is given as

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.94156$$

and $f(x_4) = -0.0207$. Now, we observe that $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.94275$$

and $f(x_5) = -0.0011896$. This procedure can be continued till we get the desired result. The first three iterations are shown as in the table.

n	x_{n+1}	$f(x_{n+1})$
2	2.9	−0.711
3	2.94156	−0.0207
4	2.94275	−0.0011896

From the above computations, we observe that the value of the root as a third approximation is evidently different in both the cases, while the value of x_5, when the interval considered is (2, 3), is closer to the root. Hence, an important observation in this method is that the interval (x_1, x_2) chosen initially in which the root of the equation lies must be sufficiently small.

Example 2.3 Use Regula-Falsi method to find a real root of the equation

$$\log x - \cos x = 0$$

accurate to four decimal places after three successive approximations.

Solution Given $f(x) = \log x - \cos x$. We observe that

$$f(1) = 0 - 0.5403 = -0.5403$$

and

$$f(2) = 0.69315 + 0.41615 = 1.1093$$

Since $f(1)$ and $f(2)$ are of opposite signs, the root lies between $x_1 = 1$, $x_2 = 2$. The first approximation is obtained from

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2 - \frac{1.1093}{1.6496} = 1.3275$$

and

$$f(x_3) = 0.2833 - 0.2409 = 0.0424$$

Now, since $f(x_1)$ and $f(x_3)$ are of opposite signs, the second approximation is obtained as

$$x_4 = 1.3275 - \frac{(.3275)\,(.0424)}{0.0424 + 0.5403} = 1.3037$$

and

$$f(x_4) = 1.24816 \times 10^{-3}$$

Similarly, we observe that $f(x_1)$ and $f(x_4)$ are of opposite signs, so, the third approximation is given by

$$x_5 = 1.3037 - \frac{(0.3037)\,(0.001248)}{0.001248 + 0.5403} = 1.3030$$

and

$$f(x_5) = 0.62045 \times 10^{-4}$$

Hence, the required real root is 1.3030.

Example 2.4 Using Regula-Falsi method, find the real root of the following equation correct to three decimal places:

$$x \log_{10} x = 1.2$$

Solution Let $f(x) = x \log_{10} x - 1.2$. We observe that $f(2) = -0.5979$, $f(3) = 0.2314$. Since $f(2)$ and $f(3)$ are of opposite signs, the real root lies between $x_1 = 2$, $x_2 = 3$. The first approximation is obtained from

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{0.2314}{0.8293} = 2.72097$$

and $f(x_3) = -0.01713$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root of $f(x) = 0$ lies between x_2 and x_3. Now, the second approximation is given by

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.7402$$

and $f(x_4) = -3.8905 \times 10^{-4}$. Thus, the root of the given equation correct to three decimal places is 2.740.

2.4 METHOD OF ITERATION

The method of iteration can be applied to find a real root of the equation $f(x) = 0$ by rewriting the same in the form,

$$x = \phi(x) \tag{2.3}$$

For example, $f(x) = \cos x - 2x + 3 = 0$. It can be rewritten as

$$x = \frac{1}{2}(\cos x + 3) = \phi(x)$$

Let $x = \xi$ is the desired root of Eq. (2.3). Suppose x_0 is its initial approximation. The first and successive approximations to the root can be obtained as

$$\left.\begin{aligned} x_1 &= \phi(x_0) \\ x_2 &= \phi(x_1) \\ &\vdots \\ x_{n+1} &= \phi(x_n) \end{aligned}\right\} \tag{2.4}$$

Definition 2.1 Let $\{x_i\}$ be the sequence obtained by a given method and let $x = \xi$ denotes the root of the equation $f(x) = 0$. Then, the method is said to be ***convergent***, if and only if

$$\operatorname*{Lt}_{n\to\infty} |x_n - \xi| = 0$$

The convergence of the above sequence to the root is stated as in Theorem 2.1.