

Homework 4. Supervised learning

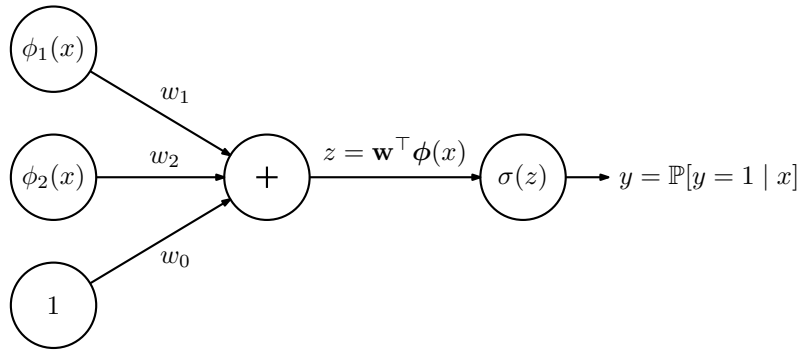


Figure 1: Logistic regression as a neural network with a single neuron.

In Lab 4 you will use supervised learning to solve a real-world classification problem. To prepare for the lab, in this homework you will perform some preliminary computations that will be useful in the lab.

Logistic regression is an approach to classification that estimates the probability of each of two actions, $\mathcal{A} = \{-1, 1\}$, given a set of examples, $\mathcal{D} = \{(x_1, a_1), \dots, (x_N, a_N)\}$, with $a_n \in \mathcal{A}, n = 1, \dots, N$ and where each state x_n is described by a number of features ϕ_1, \dots, ϕ_K . In logistic regression, we assume that the probability of the action $a = 1$ takes the form

$$\pi(1 | x) \stackrel{\text{def}}{=} \mathbb{P}[a = 1 | \mathbf{x} = x] = \sigma(z),$$

where σ is the logistic function

$$\sigma(z) = \frac{1}{1 + \exp(-z)},$$

and z is a weighted sum of the input features, i.e.,

$$z = w_0 + \sum_{k=1}^K w_k \phi_k(x) = \mathbf{w}^\top \phi(x),$$

with

$$\mathbf{w} = \begin{bmatrix} w_0 & w_1 & \dots & w_K \end{bmatrix}^\top, \quad \phi(x) = \begin{bmatrix} 1 & \phi_1(x) & \dots & \phi_K(x) \end{bmatrix}^\top.$$

The logistic regression classifier can also be seen as a very simple neural network with a single neuron, as depicted in Fig. 1.

In practice, training logistic regression consists of finding the parameters \mathbf{w} that minimize the empirical risk

$$\hat{L}_N(\pi) = -\frac{1}{N} \sum_{n=1}^N \log \pi(a_n | x_n)$$

which can be done, for example, using gradient descent.

Exercise 1.

(a) Show that

$$\hat{L}_N(\pi) = \frac{1}{N} \sum_{n=1}^N \log (\exp(-a_n \mathbf{w}^\top \phi(x_n)) + 1).$$

(b) Using the expression from (a), show that

$$\mathbf{g} = \nabla_{\mathbf{w}} \hat{L}_N(\pi) = \frac{1}{N} \sum_{n=1}^N a_n \phi(x_n) (\pi(a_n | x_n) - 1).$$

(c) Show that

$$\mathbf{H} = \nabla_{\mathbf{w}}^2 \hat{L}_N(\pi) = \frac{1}{N} \sum_{n=1}^N \phi(x_n) \phi^\top(x_n) \pi(a_n | x_n) (1 - \pi(a_n | x_n)).$$