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Inoculation strategies for bounded degree graphs ☆

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A B S T R A C T

We study the inoculation game, a game-theoretic abstraction of epidemic containment played on an undirected graph G: each player is associated with a node in G and can either acquire protection from a contagious process or risk infection. After decisions are made, an infection starts at a random node v and propagates through all unprotected nodes reachable from v. It is known that the price of anarchy (PoA) in n-node graphs can be as large as $\Theta(n)$. Our main result is a tight upper bound of $O(\sqrt{n\Delta})$ on the PoA, where Δ is the *maximum degree* of the graph. Indeed, we provide constructions of graphs with maximum degree Δ for which the PoA is $\Omega(\sqrt{n\Delta})$. We also study additional factors that can reduce the PoA, such as higher thresholds for contagion and varying the costs of becoming infected vs. acquiring protection.

1. Introduction

Networks can be conducive to the spread of undesirable phenomena such as infectious diseases, computer viruses, and false information. A great deal of research has been aimed at studying computational challenges that arise when trying to contain a contagious process [26,3,4].

One factor that can contribute to the spread of contagion is the discrepancy between *locally* optimal behavior of rational agents and *globally* optimal behavior that minimizes the total cost to the agents in the network. For example, individuals in a computer network may prefer not to install anti-virus software because it is too expensive, whereas a network administrator may prefer to install copies at key points, limiting the distance a virus could spread and the global damage to the network. The former strategy is a *locally optimal* solution if each individual minimizes their own cost, whereas the latter strategy is a *socially optimal* solution if it minimizes the total cost (i.e., the sum of costs) to all individuals in the network.

Locally optimal solutions and how they compare to the social optimum can be quantified by the classical game theoretic notions of *Nash equilibria* [27] and *price of anarchy* (PoA) [31,29]. Informally, given a multiplayer game, a strategy is a Nash equilibrium if no player can improve her individual cost by unilaterally switching to another strategy. Then, the PoA is the ratio of the total costs of the *worst* Nash equilibrium and the social optimum. The larger the PoA, the larger the potential cost players in the game may experience due to selfish, uncoordinated behavior. Hence, it is of interest to investigate methods of reducing the PoA in games.

We study the PoA of a game-theoretic abstraction of epidemic containment introduced by [1]. The *inoculation game* is an n-player game where each player is a node in an undirected graph $G = (V, E)$. A node can buy security against infection at a cost of $C > 0$, or they can choose to accept the risk of infection. If a node is infected, it must pay a cost $L > 0$. After all nodes have made their decisions, a random node in V is infected. The infection then propagates through the graph; any unsecured node that is adjacent to an infected node is infected as well.

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Understanding the links between properties of the inoculation game, the underlying graph, and the PoA may shed light on methods for designing networks that are less susceptible to contagion. It may also shed light on the effectiveness of interventions (e.g., changing the cost of acquiring inoculations) aimed at controlling contagion. For instance, it was shown by [1] that the PoA of the inoculation game in an n-vertex graph is at most $O(n)$, and this is the best possible upper bound: When $C = L(1 - 1/n)$, the n-star (a node connected to $n-1$ other nodes) has PoA equal to $n/2$. Hence, on the star graph, the potential cost to society is much higher when individuals make selfish decisions.

Motivated by the relationship between "superspreaders" and contagion, we analyze how the *maximum degree* influences the PoA: The star lower bound crucially depends on the star having the largest possible maximum degree, $n-1$, whereas we obtain an asymptotically tight bound in terms of the number of nodes n and the maximum degree Δ.

One can try to reduce the PoA by changing the values of C and L (e.g., by making inoculations less costly). Previous results regarding the PoA typically assume $C \geq L - \epsilon$ for small ϵ [1,7], and the star lower bound also depends on the costs being nearly identical. However, there is good reason to study the regime where $C < L$: In practical settings, the cost of infection may be significantly larger than the cost of inoculation (e.g., getting a flu vaccine as opposed to catching a flu and being sick for 5 days). Further, whether or not the PoA can change significantly when the cost of inoculation is reduced should be of interest to the inoculation provider. Indeed, prior work has studied this regime; [25] shows that the PoA of the grid graph is $\Theta(n^{1/3})$ for all $C < L$. We analyze the case where $C = \alpha L$ for a constant $\alpha \in (0, 1)$ that does not depend on n. As it turns out, lower bounds for the PoA when C is noticeably smaller than L (e.g., $C = L/10$) are more involved and require different constructions from those in prior works [1,7].

Social contagions often follow a model of *complex contagion* [11], where there is a higher *threshold* for infection: a node becomes infected only if multiple neighbors are infected. We initiate the study of the inoculation game in the case where the threshold for contagion is 2 (as opposed to 1) and provide a simple analysis of the PoA in this contagion model.

1.1. Preliminaries

Unless stated otherwise, we consider graphs with n nodes and identify the set of nodes (vertices) by integers $[n] := \{1, \cdots, n\}$. An *automorphism* of a graph G is a permutation σ of its vertex set $[n]$ such that i is adjacent to j if and only if $\sigma(i)$ is adjacent to $\sigma(j)$. A graph $G = (V, E)$ is *vertex transitive* if for every two nodes $i, j \in V$ there is automorphism f of G such that $f(i) = j$.

1.1.1. Inoculation game

Following [1], we describe the infection model of the inoculation game and a useful characterization of Nash equilibria.

Definition 1. The inoculation game is a one-round, n-player game, played on a connected, undirected graph $G = (V, E)$, where each node in V is a player in the game. Every node has two possible actions: Inoculate against an infection, or do nothing and risk being infected. The cost of inoculation is a real number $C > 0$ and the cost of infection is $L > 0$.

Remark 1. We always assume that C and L are positive constants independent of n, unless otherwise stated. In particular, our lower bounds generally require that the *ratio* $C/L = \Theta(1)$.

The *strategy* of each node i is the probability of inoculating, denoted by $a_i \in [0, 1]$, and the *strategy profile* for the nodes in G is represented by the vector $\vec{\boldsymbol{a}} \in [0, 1]^n$. If $a_i \in \{0, 1\}$, we call the strategy *pure*, and otherwise *mixed*.

Observe that a mixed strategy defines a probability distribution $\mathcal{D}$ over pure strategies. The cost of a mixed strategy profile to node i is equal to the expected cost over $\mathcal{D}$,

$$\begin{aligned}\text{cost}_i(\vec{\boldsymbol{a}}) &= C \cdot \Pr(i \text{ inoculates}) + L \cdot \Pr(i \text{ is infected}) \\ &= C \cdot a_i + L \cdot (1 - a_i) p_i(\vec{\boldsymbol{a}}),\end{aligned}$$

where $p_i(\vec{\boldsymbol{a}})$ denotes the probability that i becomes infected given strategy profile $\vec{\boldsymbol{a}}$ *conditioned on i not inoculating*. The total (social) cost of $\vec{\boldsymbol{a}}$ is equal to the sum of the individual costs,

$$\text{cost}(\vec{\boldsymbol{a}}) = \sum_{i=1}^{n} \text{cost}_i(\vec{\boldsymbol{a}}).$$

Definition 2. Given a strategy profile $\vec{a}$, let $I_{\vec{a}}$ denote the set of secure nodes (nodes which have inoculated). The *attack graph*, which we denote by $G_{\vec{a}}$, is the sub-graph of G induced by removing all nodes in $I_{\vec{a}}$ (and their edges) (see Fig. 1):

$$G_{\vec{a}} = G[V \setminus I_{\vec{a}}].$$

After every node has decided whether or not to inoculate, a node $i \in V$ is chosen uniformly at random (over all nodes) as the starting point of the infection. If i is not inoculated, then i and every insecure node reachable from i in $G_{\vec{a}}$ are infected. When strategies are pure, [1] derive the following formula for the social cost:

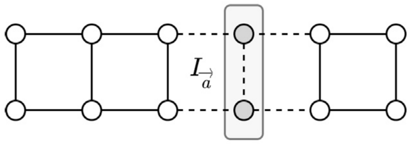

Fig. 1. The two nodes in the shaded region are inoculated. The remaining nodes form the attack graph consisting of two connected components of sizes 4 and 6 respectively.

Theorem 1 *([1]). Let $\vec{a}$ be a pure strategy profile for a graph G. Then,*

$$\mathrm{cost}(\vec{a}) = C|I_{\vec{a}}| + \frac{L}{n}\sum_{i=1}^{\ell} k_i^2,$$

where $k_1, \cdots, k_\ell$ denote the sizes of the connected components in $G_{\vec{a}}$.

1.1.2. Nash equilibria

Definition 3. A strategy profile $\vec{a}$ is a *Nash equilibrium* (NE) if no nodes can decrease their individual cost by changing their own strategy.

Formally, let $\vec{a}^* := (a_i^*, \vec{a}_{-i}^*)$ be a strategy profile where $\vec{a}_{-i}^*$ denotes the strategy profile of all players except node i. Then, $\vec{a}^*$ is a Nash equilibrium if, for all i,

$$\mathrm{cost}_i((a_i^*, \vec{a}_{-i}^*)) \leq \mathrm{cost}_i((a_i, \vec{a}_{-i}^*)) \text{ for all } a_i \neq a_i^*.$$

The cost of a strategy $\vec{a}$ is always the sum of individual costs, regardless of whether $\vec{a}$ is an NE. In the inoculation game, Nash equilibria can be characterized by the expected sizes of the *connected components* in the attack graph.

Theorem 2 *([1]). Let $S(i)$ denote the expected size of the component containing node i in the attack graph* ***conditioned on*** *i* ***not inoculating****, and let $t = Cn/L$. A strategy $\vec{a}$ is a Nash equilibrium if and only if every node satisfies the following:*

1. *if $a_i = 0$, then $S(i) \leq t$*
2. *if $a_i = 1$, then $S(i) \geq t$*
3. *if $0 < a_i < 1$, then $S(i) = t$*

Note that $p_i(\vec{a})$ (i.e., the probability i is infected given strategy $\vec{a}$) is equal to $S(i)/n$. Hence, Theorem 2 follows from the fact that t is the threshold on the expected component size where the cost of inoculating equals the cost of not inoculating, $C = L \cdot S(i)/n$. This observation upper-bounds the cost of any NE in the inoculation game:

Corollary 1 *([1]). The cost of a Nash equilibrium is at most* $\min\{C, L\}n$.

Proof. If $C > L$, then the only Nash equilibrium is the strategy $\vec{a} = 0^n$ where no node inoculates, which has cost Ln. Otherwise, if $C \leq L$, then the individual cost to any node is at most C (any node will switch its strategy to inoculate, if preferable.) □

1.1.3. Price of anarchy

Definition 4. The *Price of Anarchy* (PoA) of the inoculation game played on a graph G is equal to the ratio between (i) the cost of the *worst* Nash equilibrium and (ii) the cost of the socially optimal strategy,[1]

$$\mathrm{PoA}(G) = \frac{\max_{\vec{a}:\text{Nash eq.}} \mathrm{cost}(\vec{a})}{\min_{\vec{a}} \mathrm{cost}(\vec{a})}.$$

To upper bound the price of anarchy, we must lower bound the cost of the socially optimal strategy and upper bound the cost of the worst Nash equilibrium. By Corollary 1, we have the simple upper bound,

$$\mathrm{PoA}(G) \leq \frac{\min\{C, L\}n}{\min_{\vec{a}} \mathrm{cost}(\vec{a})} \tag{1}$$

[1] Observe that the cost is always strictly positive as $C, L > 0$.

Remark 2. As we show later, for many graph families, the upper bound of Corollary 1 is tight; for all $C, L > 0$, there exists a Nash equilibrium with cost $\min\{C, L\}n$. This makes (1) into an equality, yielding tight bounds on the PoA.

1.2. Related work

The seminal paper of [1] introduced the inoculation game and showed constructively that a pure Nash equilibrium always exists, and some instances have many. In the same paper, it is shown that the price of anarchy for an arbitrary graph is at most $O(n)$, and that there exists a graph with price of anarchy $n/2$. Subsequent work has studied PoA on graph families such as grid graphs [24] and expanders [21].

Several works have extended the basic model of [1] to analyze the effect of additional *behaviors* on the PoA. For instance, [24] extend the model to include *malicious* players whose goal is to maximize the cost to society. They prove that the social cost deteriorates as the number of malicious players increases, and the effect is magnified when the selfish players are *unaware* of the malicious players. Somewhat conversely, [7] extend the model to include *altruistic* players who consider a combination of their individual cost and the social cost (weighted by a parameter β). They prove that the social cost does indeed decrease as β increases. Finally, [23] consider a notion of *friendship* in which players care about the welfare of their immediate neighbors. Interestingly, the cost of the worst Nash equilibrium when the friendship factor $F > 0$ is always less than when $F = 0$, but does not decrease monotonically as F increase (and can actually increase with F in some instances).

The question of how to lower the PoA has been studied before (e.g., [30]). For a survey regarding methods to reduce the PoA, see [32]. The general question here is the following: how can we modify some aspect of a game to lower the PoA? To this end, variations on the *infection process* (rather than the players) have also been studied; [21] examine the price of anarchy in terms of the *distance*, d, that the infection can spread from the starting point. They prove that when $d = 1$, the price of anarchy is at most $\Delta + 1$, where Δ is the maximum degree of the graph. In this work, we comment on a *complex contagion* extension of the model, where nodes only become infected if *multiple* neighbors are infected. We show that this modification does not unilaterally decrease the PoA. There is a vast literature on complex contagion for variety of graph families [10,35,14,13,17]. We are not aware of previous work that studies the PoA in our setting where every node has threshold 2 for infection.

There are many classical models of epidemic spread. One of the most popular of these is the *SIS model* [16], which simulates infections like the flu, where no immunity is acquired after having been infected (as opposed to the *SIR model* [19], in which individuals recover with permanent immunity). In this model, it was shown by [12] that the strategy which inoculates the highest-degree nodes in power-law random graphs has a much higher chance of eradicating viruses when compared to traditional strategies. It is also known that the price of anarchy here increases as the expected proportion of high-degree nodes decreases [34]. Furthermore, it is established by [36,15,28] that epidemics die out quickly if the *spectral radius* (which is known to be related to the maximum degree [9]) is below a certain threshold. This initiated the development of graph algorithms dedicated to minimizing the spectral radius by inoculating nodes [33].

[7] consider the PoA in the inoculation game in graphs with maximum degree Δ. They state in their paper: "Indeed, we can show that even in the basic model of Aspnes et al. without altruism, the Price of Anarchy is bounded by $\sqrt{n\Delta}$ if all degrees are bounded by Δ (whereas the general bound is $\Theta(n)$)." A similar statement is made in the PhD thesis [8]. Both [7] and [8] do not include proofs of these statements. We are not aware of a published proof of either a lower bound or an upper bound for the PoA in graph in terms of the maximum degree Δ and the number of nodes.

2. PoA in terms of maximum degree

[1] proved that the price of anarchy can be as large as $\Omega(n)$. Their lower bound is based on the star graph $K_{1,n-1}$ where the optimal strategy (inoculating the root) has cost $O(1)$. Note that inoculating the central node of the star is maximally "efficient," in that it splits the attack graph into $n - 1$ components. This notion of efficiency (i.e., number of components created per inoculation) is the crux of the following result.

Remark 3. As a mixed strategy is simply a distribution over pure strategies, the optimal cost can always be realized by a pure strategy. This enables the use of Theorem 1 to bound the optimal cost as we do in Theorem 3.

Theorem 3. *Let G be a graph with maximum degree Δ. Then,* $\mathrm{PoA}(G) = O(\sqrt{n\Delta})$ *for all $C, L > 0$.*

Proof. Suppose the optimal strategy $\vec{\boldsymbol{a}}^*$ inoculates γ nodes. Note that if $\gamma = 0$, then $\mathrm{PoA}(G) = 1$: In this case, $C \geq L$ (otherwise, a better strategy could inoculate all nodes) so $\vec{\boldsymbol{a}}^*$ is also the only NE by Corollary 1. Hence, we assume $\gamma > 0$.

Let $k_1 \leq \cdots \leq k_\ell$ denote the sizes of the connected components in $G_{\vec{\boldsymbol{a}}^*}$. Then

$$\mathrm{cost}(\vec{\boldsymbol{a}}^*) = C\gamma + \frac{L}{n}\sum_{i=1}^{\ell} k_i^2 \geq \min\{C, L\} \cdot \min_{\gamma}\left(\gamma + \frac{1}{n}\frac{(n-\gamma)^2}{\ell}\right) \tag{2}$$

where the second expression follows from the Cauchy-Schwarz inequality: $\left(\sum_{i=1}^{\ell} k_i\right)^2 \leq \ell \cdot \sum_{i=1}^{\ell} k_i^2$. Observe that every inoculation adds at most $\Delta - 1$ components to the attack graph (i.e., $\ell < \gamma\Delta$), so (2) becomes

$$\mathrm{cost}(\vec{\boldsymbol{a}}^*) > \min\{C, L\} \cdot \min_\gamma \left(\gamma + \frac{\gamma}{\Delta n} + \frac{n}{\gamma\Delta} - \frac{2}{\Delta}\right)$$

The function $f(\gamma) = \gamma\left(1 + \frac{1}{\Delta n}\right) + \frac{n}{\gamma\Delta} - \frac{2}{\Delta}$ is convex, minimized by $\gamma = n/\sqrt{1+\Delta n}$. Plugging this value in, we get

$$\mathrm{cost}(\vec{\boldsymbol{a}}^*) > \min\{C, L\} \cdot \frac{2\sqrt{n\Delta+1}-1}{\Delta} = \min\{C, L\} \cdot \Omega(\sqrt{n/\Delta}).$$

Corollary 1 completes the proof, since $n/\Omega(\sqrt{n/\Delta}) = O(\sqrt{n\Delta})$. Note that the $\min\{C, L\}$ term is canceled out; the price of anarchy bound is independent of C, L. □

We also show that Theorem 3 is the strongest possible upper bound; for arbitrary values of n and Δ, we can construct a graph G with price of anarchy $\Omega(\sqrt{n\Delta})$. First, we prove a similar, but more restrictive result to illustrate the general flavor of PoA lower bounds. (Here we return to the regime where C, L are constants independent of n – See Remark 1.)

Theorem 4. *Let G be a complete k-ary tree with odd height. For sufficiently large k,*

$$\mathrm{PoA}(G) = \Omega\left(\sqrt{n\Delta}\right).$$

Proof. Denote the height of G by h and note that $n = \sum_{i=0}^h k^i = \frac{k^{h+1}-1}{k-1}$. To upper bound the cost of the social optimum, consider the strategy which secures layer ℓ of the tree (k^ℓ inoculations). The attack graph then consists of several disjoint sub-trees: the first $\ell - 1$ layers of the original tree (on $\frac{k^\ell-1}{k-1}$ nodes), and $k^{\ell+1}$ disjoint sub-trees with height $h - \ell - 1$ (each having $\frac{k^{h-\ell}-1}{k-1}$ nodes). This strategy has cost equal to

$$Ck^\ell + \frac{L}{n}\left[\left(\frac{k^\ell - 1}{k-1}\right)^2 + k^{\ell+1}\left(\frac{k^{h-\ell}-1}{k-1}\right)^2\right]$$

Substituting $\ell = \frac{h-1}{2}$, we obtain an upper bound on the optimal cost:

$$\begin{aligned}
\mathrm{cost}(\overrightarrow{a^*}) &= Ck^{(h-1)/2} + L \cdot \frac{k-1}{k^{h+1}-1} \cdot \left(\frac{k^{h-1} - 2k^{(h-1)/2} + 1 + k^{3(h+1)/2} - 2k^{h+1} + k^{(h+1)/2}}{(k-1)^2}\right) \\
&= Ck^{(h-1)/2} + L \cdot O\left(\frac{k^{3(h+1)/2}}{k^{h+2}}\right) \\
&= (C+L) \cdot O(\sqrt{n/k})
\end{aligned}$$

If $C \geq L$, then the only Nash Equilibrium has cost Ln as in Corollary 1. Otherwise, assume $C < L$ and let $t = Cn/L$. To lower bound the cost of the worst Nash equilibrium, consider the strategy which secures $n - \lfloor t \rfloor$ leaves, leaving one insecure component of size $\lfloor t \rfloor$. There are enough leaves to do so given our assumption that $C/L = \Theta(1)$ (See Remark 1), so we may write $C = \alpha L$ for some constant $\alpha < 1$. In particular,

$$n - \lfloor t \rfloor \leq k^h \iff \frac{k^h - 1}{k-1} \leq \alpha \frac{k^{h+1}-1}{k-1} \iff k^h(\alpha k - 1) \geq 1 - \alpha$$

Indeed, the latter expression clearly holds for k large enough, say, $k \geq 2/\alpha = 2L/C$.

This strategy is a Nash equilibrium by Theorem 2 – every insecure node i has $S(i) = \lfloor t \rfloor \leq t$, and every secure node (leaf) j has $S(j) = \lfloor t \rfloor + 1 > t$ – with cost

$$\begin{aligned}
&C(n - \lfloor t \rfloor) + L\lfloor t \rfloor^2/n \\
&\geq Cn - C(t - \{t\}) + \frac{L}{n}(t - \{t\})^2 \\
&= Cn - Ct + C\{t\} + Ct - 2C\{t\} + \frac{L}{n}\{t\}^2 \\
&\geq C(n - \{t\}) + \frac{L}{n}\{t\}^2 = C \cdot \Omega(n)
\end{aligned}$$

where $\{t\} = t - \lfloor t \rfloor \in [0, 1)$ is the fractional part of t. Again by virtue of Remark 1,

$$\mathrm{PoA}(G) = \frac{\min\{C, L\} \cdot \Omega(n)}{(C+L) \cdot O(\sqrt{n/k})} = \Omega(\sqrt{n/k}). \quad \square$$

Theorem 4 only holds for certain values of $\Delta \ll n$, as Δ is constrained by the requirement that there is complete $(\Delta - 1)$-ary tree with n nodes. Next, we give a more involved construction illustrating the tightness of the upper bound on the PoA of graphs with maximal degree Δ that is valid for wider range of the parameters n, Δ.

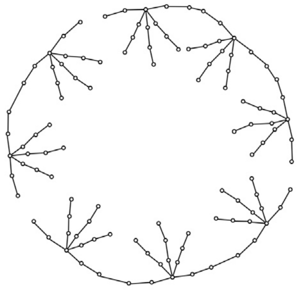

Fig. 2. Rough illustration of the attack graph induced by the Nash equilibrium from Theorem 5. Here, $c = 3$: The midpoint of each path is inoculated, aside from the outer cycle where every third midpoint is inoculated.

Idea. Given n and Δ, we construct an instance of the inoculation game where the optimal strategy has cost at most $O(\sqrt{n/\Delta})$ and the worst-case Nash equilibria has cost at least $\Omega(n)$.

The upper bound follows when it is possible to secure $\gamma \sim \sqrt{n/\Delta}$ nodes and create $\gamma\Delta \sim \sqrt{n\Delta}$ equally-sized components, as in Theorem 3. The lower bound clearly follows for $C \geq L$ as the only NE has cost Ln, but the case when $C < L$ is slightly more involved:

Recall that components in the attack graph must have size at most $t = Cn/L$, but suppose every component still has size $\Omega(t)$. Then, this strategy has cost

$$\Omega\left(C\gamma + \frac{L}{n} \cdot \frac{n-\gamma}{t} t^2\right) = \Omega(C\gamma + C(n-\gamma)) = C \cdot \Omega(n).$$

Since $t = Cn/L = \Omega(n)$ by assumption (Remark 1), we construct a graph consisting of many components of size $s = o(n)$ which are all connected along a path. Nash equilibria may (roughly) split the path into sub-paths of length t/s such that every insecure component has size $\Omega(t)$. (See Fig. 2.)

Theorem 5. *For all $n > \Delta^3/4$, there exists a graph G with* $\mathrm{PoA}(G) = \Omega(\sqrt{n\Delta})$.

Proof. Let $m = 2\left\lfloor \sqrt{n/\Delta} \right\rfloor$ and construct a graph G as follows: Arrange nodes $v_1, \cdots, v_m$ in a cycle and connect each node to its $\Delta/2$ nearest neighbors on either side (and its opposite node in the cycle if Δ is odd). Observe that G is Δ-regular. Note that this construction is only feasible if $m > \Delta$, so we must have $n > \Delta^3/4$. The inoculation game will take place on a new graph G' constructed by adding $n - m$ new nodes to G, replacing all $m\Delta/2$ edges with paths on $\ell = \lfloor (n-m)/(m\Delta/2) \rfloor$ or $\ell + 1$ nodes (not including the endpoints), so that the total number of nodes is n. That is, $n - m = \ell \cdot (m\Delta/2) + r$ for some $r < m\Delta/2$, so r paths have $\ell + 1$ nodes.

The upper bound on the optimal cost comes from the pure strategy which secures $v_1, \cdots, v_m$, separating G' into $m\Delta/2$ paths of length at most $\ell + 1 \sim \sqrt{n/\Delta}$. (These are the paths with which we replaced the edges of G.) This has cost at most

$$Cm + \frac{L}{n} \cdot \frac{m\Delta}{2}(\ell+1)^2 = (C + L) \cdot O(\sqrt{n/\Delta})$$

The lower bound on the cost of the worst Nash equilibrium follows immediately if $C \geq L$, as worst Nash equilibrium has cost Ln. Otherwise, there is some threshold $t = Cn/L < n$ on the expected insecure component size, per Theorem 2.

Designate a unique midpoint for each path in G' connecting nodes of G (breaking ties arbitrarily for even-length paths). Then, G' is a collection of m disjoint star graphs — a node connected to Δ paths of length at least $\lfloor (\ell-1)/2 \rfloor$ (and at most $\lfloor (\ell+1)/2 \rfloor$) — with $m\Delta/2$ midpoints connecting their leaves. In particular, stars have size at least $s = 1 + \Delta \lfloor (\ell-1)/2 \rfloor \sim \sqrt{n\Delta}$. Let c be a positive integer such that

$$c \cdot (s+1) - 1 \leq t \leq (c+1) \cdot (s+1) - 1. \tag{3}$$

Recall that G contains a cycle through $v_1, \cdots, v_m$, say H. Let H' be the result of removing every c'th edge of H, creating $\lfloor m/c \rfloor$ disjoint paths of length c (and possibly one of length $c' < c$ if c does not divide m). Then, consider the following strategy:

1. Inoculate the midpoint of every path connecting *distinct* components of H'. This splits G' into $\lfloor m/c \rfloor$ insecure components, each consisting of c stars connected by (at minimum) $c - 1$ midpoints (and possibly one with c' stars, $c' - 1$ midpoints). That is, $\lfloor m/c \rfloor$ components of size at least $c \cdot (s+1) - 1$ (and possibly one of size $c' \cdot (s+1) - 1$).
2. Iteratively inoculate midpoints in insecure components of size greater than t until all components have size at most t. (This is possible without disconnecting the component; the minimal set of $(c-1)$ midpoints results in a connected component of size at most t, as in (3).)

Clearly, every component has size at most t, and un-inoculating any node either increases the size of an insecure component to $t + 1$, or merges two components, creating a new insecure component of size at least $(c+1) \cdot (s+1) - 1 \geq t$. Therefore, this strategy is a Nash equilibrium and has cost at least

$$C \cdot \Omega\left(n - \left\lfloor \frac{m}{c} \right\rfloor t\right) + \frac{L}{n} \cdot \Omega\left(\left\lfloor \frac{m}{c} \right\rfloor \cdot t^2\right) = C \cdot \Omega(n)$$

where the number of inoculations is taken by subtracting the size of the attack graph from n, and we have invoked Remark 1 to hide a factor of $\frac{C}{L}$. Hence,

$$\mathrm{PoA}(G) = \frac{\min\{C, L\} \cdot \Omega(n)}{(C+L) \cdot O(\sqrt{n/\Delta})} = \Omega(\sqrt{n\Delta}) \quad \square$$

3. The relationship between PoA, C and L

By Theorem 2, when $C \geq L$, the pure strategy in which no node inoculates is a Nash equilibrium with cost Ln. Therefore, in this case, the price of anarchy is equal to Ln divided by the cost of the optimal strategy. When $C < L$, it is not necessarily the case that there exists a Nash equilibrium with maximal cost (Cn, by Corollary 1), which makes it more technically challenging to lower bound the price of anarchy.

We first prove that there are graph families for which the worst-case Nash equilibrium has cost Cn for all $C \in (0, L]$. We say that a Nash equilibrium is *fractional* if no node has a pure strategy; every node i chooses her action with probability differing from 0, 1.

Lemma 1. *The cost of every fractional Nash equilibrium is equal to Cn.*

Proof. Suppose strategy $\vec{a}$ is a Nash equilibrium with $a_i \in (0,1)$ for all i. As a consequence of Theorem 2, the expected component size $S(i) = Cn/L$ for all i. Thus, probability of infection for any node i is equal to $p_i(\vec{a}) = C/L$. By definition,

$$\mathrm{cost}(\vec{a}) = \sum_{i=1}^{n} \left(C \cdot a_i + L \cdot (1-a_i)\frac{C}{L}\right) = Cn. \quad \square$$

This implies that lower bounds on the asymptotic PoA for these graphs remain the same so long as $C/L = \Theta(1)$.

Theorem 6. *For a graph G, let $f(n) := \mathrm{PoA}(G)$ when $C = L$ and suppose, for all $C \in (L/n, L)$, that G has a fractional equilibrium. Then, for all $C, L \geq 0$,*

$$\mathrm{PoA}(G) \geq \frac{\min\{C, L\}}{\max\{C, L\}} f(n)$$

Proof. Let $\mathrm{cost}^{[C,L]}(\vec{a})$ denote the cost of $\vec{a}$ when the cost of inoculation is C and the cost of infection is L. We make two observations:

1. $\mathrm{cost}^{[\lambda C, \lambda L]}(\vec{a}) = \lambda \cdot \mathrm{cost}^{[C,L]}(\vec{a})$
2. If $C' \leq C$ and $L' \leq L$, then $\min_{\vec{a}} \mathrm{cost}^{[C',L']}(\vec{a}) \leq \min_{\vec{a}} \mathrm{cost}^{[C,L]}(\vec{a})$.

The first observation implies that $f(n) = n / \min_{\vec{a}} \mathrm{cost}^{[1,1]}(\vec{a})$ regardless of the magnitude of $C = L$. Then, if $C \geq L$,

$$\mathrm{PoA}(G) = \frac{Ln}{C \cdot \min_{\vec{a}} \mathrm{cost}^{[1,L/C]}(\vec{a})} \geq \frac{L}{C} f(n)$$

If $L/n < C < L$, then there exists a fractional equilibrium with cost Cn by assumption. Furthermore, if $C \leq L/n$, then $t = Cn/L \leq 1$, so (by Theorem 2) the pure strategy in which every node inoculates is a Nash equilibrium with (maximal) cost Cn. In either case,

$$\mathrm{PoA}(G) = \frac{Cn}{L \cdot \min_{\vec{a}} \mathrm{cost}^{[C/L,1]}(\vec{a})} \geq \frac{C}{L} f(n) \quad \square$$

It is non-trivial to show that a fractional equilibrium exists (note that Nash's theorem does not guarantee existence because the space of fractional strategies is not compact). However, it is possible to show that some graphs will always exhibit one. Our first example is the star, which is the prime example for linear PoA assuming $C \geq L$. We show that a similar lower bound on the PoA holds for all $C < L$.

Theorem 7. *Let $G = K_{1,n-1}$. For all $C \in (L/n, L)$, there exists a fractional Nash equilibrium.*

Proof. Let $\vec{a}$ be the strategy in which every leaf inoculates with probability p and the root inoculates with probability q. Then,

- $S(\mathrm{root}) = 1 + (1-p)(n-1)$,
- $S(\mathrm{leaf}) = q + (1-q)[2 + (1-p)(n-2)]$.

It is easy to verify that $S(\text{root}) = S(\text{leaf}) = t$ when

$$p = \frac{n-t}{n-1},\ q = \frac{n-t}{nt-2t+1}$$

Indeed, p and q are positive probabilities for all $t \in (1, n)$. Thus, for all $C \in (L/n, L)$, there exist $p, q \in (0, 1)$ such that $S(\text{root}) = S(\text{leaf}) = \frac{Cn}{L}$ (i.e., $\vec{\boldsymbol{a}}$ is a fractional Nash equilibrium). □

Remark 4. We could find an *asymptotically* maximal pure equilibrium by inoculating $n - \lfloor Cn/L \rfloor$ leaves of the star, but fractional equilibria provide the *exact* maximal cost.

In fact, we can show that fractional equilibria always exist for graphs with a sufficient amount of symmetry. For example, consider the most symmetric graph: a complete graph on n nodes. Here, if every node inoculates with the same probability p, then the expected component size of every node is equal to $1 + p(n-1)$. Hence, if $p = (\frac{C}{L}n - 1)/(n-1)$, then this strategy is a fractional Nash equilibrium by Theorem 2.

Theorem 8. *Suppose G is vertex-transitive. Then, for all $C \in (L/n, L)$, there exists a fractional Nash equilibrium.*

Proof. We prove that there exists a $p \in (0, 1)$ such that the strategy $\vec{\boldsymbol{a}}_p = p^n$ is a fractional Nash equilibrium. By the definition of vertex-transitivity, for any two nodes $i \neq j$, there exists an automorphism $f : G \to G$ such that $f(i) = j$.

Consider an arbitrary set of inoculated nodes, A, and their image, $f(A)$. Let $S(i|A)$ denote the size of the connected component containing i in the attack graph $G \setminus A$ (assuming i is not inoculated, as usual). By vertex-transitivity, $S(i|A) = S(f(i)|f(A)) = S(j|f(A))$, and

$$\begin{aligned} &\Pr(I_{\vec{\mathbf{a}}} = A | a_i = 0) = \Pr(I_{\vec{\mathbf{a}}} = f(A) | a_j = 0) \\ &= p^{|A|}(1-p)^{n-|A|-1}. \end{aligned}$$

Then, by linearity of expectation,

$$\begin{aligned} S(i) &= \sum_A S(i|A) \Pr(I_{\vec{\mathbf{a}}} = A | a_i = 0) \\ &= \sum_A S(j|f(A)) \Pr(I_{\vec{\mathbf{a}}} = f(A) | a_j = 0) = S(j). \end{aligned}$$

Note that $S(i)$ is a polynomial in p satisfying $\lim_{p\to 0} S(i) = n$ and $\lim_{p\to 1} S(i) = 1$. Therefore, for all $C \in (L/n, L)$, there exists a $p \in (0, 1)$ such that $S(i) = Cn/L$ for all i. □

It follows that for transitive graphs such as cycles and torus grid graphs there always exists a Nash equilibrium of cost Cn, which is the largest possible.

3.1. Non-existence of fractional equilibria

A natural question is whether or not we can prove similar results for arbitrary graphs via the existence of fractional NE. We prove that this is not the case even for graphs obtained from transitive graphs by deleting a single edge; the path graph.

Theorem 9. *There is a choice of $C < L$ such that the path graph on 5 nodes does not have a fractional Nash equilibrium.*

Proof. We apply the same method that we used in Theorem 7 to compute fractional equilibria by solving the system $\{S(i) = t\}_{i \le n}$ in terms of $a_1, \cdots, a_n$. (Recall that fractional equilibria *must* satisfy this system by Theorem 2). In particular, the system corresponding to a path of length 5 is the following:

$$\begin{aligned} &1 + x_2 + x_2x_3 + x_2x_3x_4 + x_2x_3x_4x_5 \\ &= x_1 + 1 + x_3 + x_3x_4 + x_3x_4x_5 \\ &= x_1x_2 + x_2 + 1 + x_4 + x_4x_5 \\ &= x_1x_2x_3 + x_2x_3 + x_3 + 1 + x_5 \\ &= x_1x_2x_3x_4 + x_2x_3x_4 + x_3x_4 + x_4 + 1 \\ &= t. \end{aligned}$$

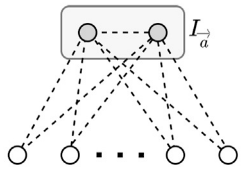

Fig. 3. Inoculating the two central nodes of the graph constructed in Theorem 11 splits the rest of the graph into $n-2$ singleton components. This has a much lower social cost than the Nash equilibrium in which no node inoculates.

When $t = 3/2$ ($C/L = 3/10$), standard computational methods[2] show that this system has no solution. □

4. Larger thresholds of infection

Complex contagion [11] refers to a higher threshold of infection; multiple infected neighbors are required for contagion. We consider the case where the threshold of infection for every node is 2. After inoculation decisions, two random nodes are infected initially, and an insecure node becomes infected if 2 of its neighbors are infected. Intuitively, a higher threshold of contagion makes it harder for a contagious phenomena to spread and one might expect a unilaterally lower PoA as a result. We prove that, although the PoA does decrease for certain graphs, it can still be as large as $\Omega(n)$.

We first show that the price of anarchy can dramatically decrease when all thresholds are 2. Recall that [1] proved that the star graph $K_{1,n-1}$ has price of anarchy $\Omega(n)$ (for threshold 1). In contrast, we have the following:

Theorem 10. *Suppose that the threshold of every node is* 2. *Then,* $\mathrm{PoA}(K_{1,n-1}) = O(1)$.

Proof. Any leaf node has only one neighbor and can only be infected if chosen at the start. Thus,

$$p_{\mathrm{leaf}}(\vec{\boldsymbol{a}}) = \frac{n-1}{\binom{n}{2}} = \frac{2}{n},$$

and $\mathrm{cost}_{\mathrm{leaf}}(\vec{\boldsymbol{a}}) = Ca_i + \dfrac{2L(1-a_i)}{n}$. Then, for large enough n, no leaf node will inoculate in a Nash equilibrium. Therefore, the worst case Nash equilibrium has cost at most

$$\max\{L, C\} + L \cdot (n-1)\frac{2}{n} = \max\{L, C\} + 2L\left(1 - \frac{1}{n}\right)$$

Now consider the optimal strategy. If at least one node inoculates with probability $a_i \geq 1/2$, then $\mathrm{cost}(\vec{\boldsymbol{a}}^*) \geq C/2$. Otherwise, the root (if insecure) is infected with probability at least $1/4$; either it is chosen at the start, or two insecure nodes are chosen. As the root inoculates with probability at most $1/2$, the optimal social cost is at least $L/8$. □

However, even when all thresholds equal 2, there are still cases where the price of anarchy is $\Omega(n)$.

Theorem 11. *If $C \geq L$, then there exists a graph G for which* $\mathrm{PoA}(G) = \Omega(n)$. *(See Fig. 3.)*

Proof. Consider the graph $G = K_{2,n-2}$ with an edge between the two nodes on the smaller side. A Nash equilibrium where every node chooses not to inoculate has cost Ln as every two nodes will infect the entire graph. On the other hand, the strategy which inoculate both nodes on the smaller side upper bounds the cost of the social optimum by

$$\mathrm{cost}(\vec{\boldsymbol{a}}^*) \leq 2C + (n-2) \cdot L \cdot \frac{2}{n} \leq 2(C + L)$$

Recall that $C/L = \Theta(1)$ by Remark 1, so $\mathrm{PoA}(G) = \Omega(n)$. □

The example above can be extended to the case where the threshold of contagion is an arbitrary constant (independent of n) larger than 2.

5. PoA for additional graph families

Here we describe price of anarchy bounds for planar graphs, trees, expanders and Erdős-Renyi random graphs. The first two bounds, in contrast to other results in the paper, assume $C \geq L$ so that the only Nash equilibrium has cost Ln. Whether or not these results generalize to the case when $C < L$ is an interesting open question. This simplifies the problem of computing the PoA to computing the cost of the optimal strategy.

[2] The *Groebner basis* of the ideal generated by $\{S(i) - t\}_{i\leq 5}$ is $\{1\}$ (See [6]).

5.1. Planar graphs

Theorem 12. *Let G be a planar graph. If $C \geq L$, then* $\mathrm{PoA}(G) = \Omega(n^{1/3})$.

This is the best lower bound possible for planar graphs, as it was shown in [24] that a (planar) two-dimensional grid has price of anarchy $O(n^{1/3})$. Clearly, planar graphs can have price of anarchy $\Omega(n)$, as the classic star graph example is planar.

Before proving Theorem 12, we state the *planar separator theorem*:

Theorem 13 *([22]). Let G be a planar graph on n nodes. There exists a partition of the nodes $V = A \cup S \cup B$ such that*

- $|A|, |B| \leq n/2$
- $|S| = O(\sqrt{n})$
- *There does not exist an edge between A and B.*

Proof of Theorem 12. By Theorem 13, any planar graph G can be split into 2 components of size at most $n/2$ by removing at most $O(\sqrt{n})$ nodes. Iteratively, these components can both be split into 4 total components of size at most $n/4$ by removing $2 \cdot O(\sqrt{n/2})$ more nodes. Generally, the number of inoculations required to create $\ell := 2^k$ components of size at most $2^{-k}n$ is upper bounded by

$$\sum_{i=1}^{k} 2^i \cdot O(\sqrt{n2^{-i}}) = O(\sqrt{n}) \cdot \sum_{i=1}^{k} 2^{i/2} = O(\sqrt{n\ell}).$$

By optimizing over ℓ, we have that for any planar graph, such a strategy upper bounds the cost of the optimal strategy by

$$\begin{aligned} & C \cdot O(\sqrt{n\ell}) + \frac{L}{n} \cdot \ell \cdot \left(\frac{n}{\ell}\right)^2 \\ &= O\left(C\sqrt{n\ell} + L\frac{n}{\ell}\right) \\ &= (C+L) \cdot O(n^{2/3}) \end{aligned}$$

Since $C \geq L$, the result follows as the only Nash equilibrium has cost Ln. By Remark 1,

$$\mathrm{PoA}(G) \geq \frac{Ln}{(C+L)n^{2/3}} = \Omega(n^{1/3}) \quad \square$$

5.2. Trees

We apply a similar method to trees, obtaining a stronger lower bound for this type of planar graph.

Theorem 14. *Let T be an arbitrary tree. If $C \geq L$, then* $\mathrm{PoA}(T) = \Omega(\sqrt{n})$.

This is the best possible lower bound for general trees; a matching upper bound follows from Theorem 3 when the maximum degree Δ is a constant. The proof essentially follows from a slightly more efficient separator theorem than Theorem 13.

Theorem 15 *([18]). In any tree on n nodes, there exists a node whose removal leaves components of size at most $n/2$.*

Proof of Theorem 14. By Theorem 15, T can be split into multiple components of size at most $n/2$ by removing one node. Each of these components, in turn, can be split into components of size at most $n/4$ by removing one node each. This process can be repeated until all components have at most $\sqrt{n}$ nodes. This process requires $O(\sqrt{n})$ removals. (This is likely known, but we include a proof for completeness).

Let N_i denote the number of components in the attack graph with size greater than $n \cdot 2^{-i}$ but at most $n \cdot 2^{-i+1}$. Removing a node from such a component (per Theorem 15) will decrease N_i by one and increase some N_j's where $j > i$. Note that $N_i \leq 2^i$ as component sizes sum to at most n. Therefore, the number of removals required to reduce N_i to 0 for all $i \leq \log_2(\sqrt{n})$ is upper bounded by

$$\sum_{i=1}^{\log_2(\sqrt{n})} 2^i = 2\sqrt{n} - 2.$$

Hence, there exists a strategy that inoculates $O(\sqrt{n})$ nodes, leaving attack graph components of size at most $\sqrt{n}$. This leaves $n - O(\sqrt{n}) \leq n$ insecure nodes whose probability of infection is at most $\sqrt{n}/n = 1/\sqrt{n}$. Thus, the cost of this strategy upper bounds the social optimum by

$$C \cdot O(\sqrt{n}) + Ln \cdot \frac{1}{\sqrt{n}} = (C+L) \cdot O(\sqrt{n}).$$

Since $C \geq L$, the result follows as the only Nash equilibrium has cost Ln. By Remark 1,

$$\mathrm{PoA}(T) \geq \frac{Ln}{(C+L) \cdot O(\sqrt{n})} = \Omega(\sqrt{n}) \quad \square$$

5.3. Random graphs

Finally, we show that the price of anarchy for well-connected graphs is $O(1)$. These are graphs with the property that even the optimal strategy has cost $\Omega(n)$. That is, the situation requires that a majority of individuals either inoculate or get infected.

Theorem 16. *Let $G = G(n,p)$ for $p \geq \frac{1+\epsilon}{n}$ with $\epsilon > 0$. Then, $\mathrm{PoA}(G) = O(1)$ with high probability.*

The proof of Theorem 16 amounts to the following two lemmas regarding *expander* graphs.

Definition 5. An n-node undirected graph $G = (V, E)$ is called an α-expander if for every subset S of nodes with $|S| \leq n/2$, there are at least $\alpha|S|$ nodes in $V \setminus S$ neighboring a node in S. (Here, $\alpha > 0$ is a fixed constant independent of n.)

The following lemma may be folklore, but we include the proof here for completeness.

Lemma 2. *Suppose G is an α-expander. Then there exist constants $\delta, \epsilon \in (0,1)$ such that for every subset A of δn nodes, removing A from G leaves a connected component of size at least ϵn.*

Proof. Suppose that there is a subset of nodes A of size $\alpha n/4$ such that removing A results with connected components $S_1 \ldots S_m$ all of size smaller than ϵn where $\epsilon > 0$ is a constant to be determined later. This implies that there is a subset S of nodes satisfying

$$\left(\frac{1}{2} - \epsilon\right) n \leq |S| \leq \frac{n}{2}$$

with at most $\alpha n/4$ neighbors in $V \setminus S$.

Indeed, initialize S as the empty set and keep adding components S_i (that were not added already) until S has size at least $(1/2 - \epsilon)n$. Note that S has cardinality no larger than $n/2$ and has at most $\alpha n/4$ neighbors. For ϵ sufficiently small, this is a contradiction to G being an α-expander. $\square$

Remark 5. Observe that the assertion in Lemma 2 applies also if G contains a sub-graph with $\Omega(n)$ nodes which is an α-expander.

Lemma 3. *For a graph G, suppose there exist constants $\delta, \epsilon > 0$ such that for every subset A of δn nodes, removing A from G leaves a connected component of size at least ϵn. Then, $\mathrm{PoA}(G) = O(1)$.*

Proof. If removing δn nodes leaves a component of size at least ϵn, then, for any pure strategy $\vec{\boldsymbol{a}}$,

- If $|I_{\vec{a}}| \leq \delta n$, then $\mathrm{cost}(\vec{\boldsymbol{a}}) \geq L\epsilon^2 n$.
- If $|I_{\vec{a}}| > \delta n$, then $\mathrm{cost}(\vec{\boldsymbol{a}}) > C\delta n$.

Since mixed strategies are just distributions over pure strategies, we have that the cost of the optimal strategy is lower-bounded by $\min\{C\delta, L\epsilon^2\}n$. Then, since $\max_{\vec{a}:\text{Nash eq.}} \mathrm{cost}(\vec{\boldsymbol{a}}) \leq \min\{C, L\}n$ by Corollary 1, we have

$$\mathrm{PoA}(G) \leq \frac{\min\{C, L\}n}{\min\{C\delta, L\epsilon^2\}n} = O(1). \quad \square$$

Proof of Theorem 16. Note that $G(n,p)$ may not be an α-expander for fixed $\alpha > 0$: If edge probability $p = (1+\epsilon)/n$, then $G(n,p)$ contains an induced path of length $\Omega(\log n)$ with high probability [5]. Nevertheless, it is known that if $p \geq (1+\epsilon)/n$, then $G(n,p)$ contains an α-expander with $\Omega(n)$ nodes with high probability (see [20]). Lemmas 2 and 3 complete the proof. $\square$

6. Conclusion

We have studied the PoA of the inoculation game and have shown that while some properties (bounded maximal degree) lead to sublinear upper bounds in terms of the number of nodes, others (inoculation costs smaller than infection costs, larger threshold for contagion) do not. Several questions remain for future study. While we have shown the existence of Nash equilibria with high costs, we suspect that they are unlikely to be predictive of human behavior in experiments. For example, in a 10 by 10 grid where the cost of inoculation is 1 USD and the cost of infection is 8 USD, we suspect the vast majority of players will choose to inoculate with probability 1 – Factors such as social responsibility may supersede the results of a mathematical analysis, especially when the cost of inoculation is so low. Characterizing how resource constrained players (e.g., players with limited memory) play the inoculation game is an interesting direction for future research.

Our results regarding random graphs might generalize to additional models of random graphs such as preferential attachment [2] and small world networks [37] as these may contain subgraphs of size $\Omega(n)$ that are expanding. We suspect that for such graphs that PoA is upper bounded by a constant with high probability. Proving this is left for future work.

We have introduced the concept of fractional equilibria. It remains an open question for which graphs such an equilibrium exists. More broadly, future research could examine that computational complexity of deciding whether a fractional equilibrium exists for a normal form game.

CRediT authorship contribution statement

Mason DiCicco: Writing – review & editing, Writing – original draft, Conceptualization. **Henry Poskanzer:** Writing – review & editing, Writing – original draft. **Daniel Reichman:** Writing – review & editing, Writing – original draft, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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