Gravitational lensing

12_11

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lens equation in general cases

- $G^{\alpha\beta} := R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = \frac{8\pi G}{c^4}T^{\alpha\beta}$
    - given $T^{\alpha\beta}$, we can yield metric $g^{\alpha\beta}$

弱场近似：gαβ=ηαβ+hαβ, 1≫h1

-1,1,1,1

“无迹化”，一种习惯

- $g_{\alpha\beta} = \left(1 - \frac{1}{2}\tilde{h}\right)\eta_{\alpha\beta} + \tilde{h}_{\alpha\beta}$
  $\tilde{h} := \eta^{\alpha\beta}\tilde{h}_{\alpha\beta} \quad , \quad |\tilde{h}_{\alpha\beta}| \ll 1$
    - weak-field approximate matrixs for $g_{\alpha\beta}$

> $$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h^{\alpha\beta} = \frac{16\pi G}{c^4}T^{\alpha\beta} \tag{1}$$
>
> 类似延迟解 $$\$h^{\alpha\beta}(t,\mathbf{x}) = \frac{-4G}{c^4}\int\frac{T^{\alpha\beta}\left(t - \frac{|\mathbf{y}|}{c}, \mathbf{x}+\mathbf{y}\right)}{|\mathbf{y}|}d^3y\$ \tag{2}$$

- $T^{00} \approx \varrho c^2, T^{0i} \approx c\varrho v^i, T^{ij} \approx \varrho v^i v^j + p\delta^{ij}$
    - approximation for slowly moving , perfect fluid sources

- $|\vec{v}| \ll c$, $|p| \ll \varrho c^2$

$$
\begin{aligned}
ds^2 =& g_{\alpha\beta} dx^\alpha dx^\beta \\
&\approx \left(1+\frac{2U}{c^2}\right)c^2dt^2 - 8cdt\frac{\mathbf{V}\cdot d\mathbf{x}}{c^3} - \left(1-\frac{2U}{c^2}\right)d\mathbf{x}^2
\end{aligned}
$$

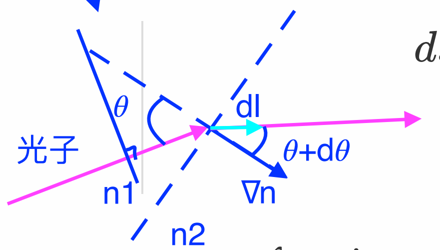
$$U(t,\mathbf{x}) \approx -G\int \frac{\varrho(t,\mathbf{x}+\mathbf{y})}{|\mathbf{y}|}d^3y$$ 势函数

$$\mathbf{V}(t,\mathbf{x}) \approx -G\int \frac{(\varrho\mathbf{v})(t,\mathbf{x}+\mathbf{y})}{|\mathbf{y}|}d^3y$$

(cdt)^2=(1-2U/c^2)^2 dx^2
光程：cdt=(1-2U/c^2)dl

U<<c^2

V = 0



$$
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&\approx \left(1+\frac{2U}{c^2}\right)c^2dt^2 - \left(1-\frac{2U}{c^2}\right)d\mathbf{x}^2\$
\end{aligned}
\quad (3)
$$

- refraction : $n = 1 - \frac{2U}{c^2}$

定义等效光程

n2=n1+∇n · dl
n1sinθ=n2sin(θ+dθ)
(n1-n2)/n2=(-∇n · dl)/n2=(cosθ/sinθ)dθ
近似：n2≈1, ∇n · dl≈l∇nl ldll cosθ
→l∇nl ldll sinθ=dθ

l∇Ul sinθ垂直于光线方向

$$\hat{\boldsymbol{\alpha}} = \frac{2}{c^2}\int \nabla_\perp U dl \quad (4)$$

lens equation : $\boldsymbol{\eta} = \frac{D_s}{D_l}\boldsymbol{\xi} - D_{ls}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ (in sourse plane)

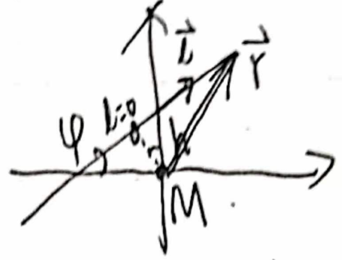
质量分布

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2}\int_{\mathbf{R}^2}\frac{(\boldsymbol{\xi}-\boldsymbol{\xi}')\Sigma(\boldsymbol{\xi}')}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2}d^2\boldsymbol{\xi}' \quad (5)$$

normalized lens equation:

$\hat{\alpha} = \frac{2}{c^2}\int \nabla_\perp U \, dl$.

波恩近似：假设积分路径为直线 ； single Lens ： $U = -\frac{GM}{r}$



直线方程：$\begin{cases} x = l\cos\theta \\ y = l\sin\theta + h \end{cases}$

$r^2 = l^2 + h^2 + 2lh\sin\varphi = l^2 + b^2$

$\nabla U = \frac{GM\vec{r}}{r^3}$ ，垂直于光线：$\nabla_\perp U = \frac{GMb}{r^3}$ b：碰撞参数

$$\hat{\alpha} = \frac{2}{c^2}\int_{-\infty}^{+\infty} \frac{GMb}{r^3}\, dl = \frac{4GMb}{c^2}\int_0^{+\infty} \frac{1}{r^3}\, dl = \frac{4GMb}{c^2}\int_0^{+\infty} \frac{1}{(b^2+l^2)^{\frac{3}{2}}}\, dl = \frac{4GM}{c^2 b} .$$

$$
\begin{gathered}
\text{源的位置}\quad \mathbf{y}=\mathbf{x}-\boldsymbol{\alpha}(\mathbf{x})\\
\boxed{\alpha(\mathbf{x})=\frac{1}{\pi}\int_{\mathbf{R}^2} d^2x'\kappa\left(\mathbf{x}'\right)\frac{\mathbf{x}-\mathbf{x}'}{\left|\mathbf{x}-\mathbf{x}'\right|^2}}\\
\text{透镜的位置}\\
\text{无量纲化的质量密度}\,\kappa(\mathbf{x})=\frac{\Sigma\left(\xi_0\mathbf{x}\right)}{\Sigma_{\mathbf{cr}}};\quad \mathbf{x}=\frac{\boldsymbol{\xi}}{\xi_0}\quad;\quad \mathbf{y}=\frac{\boldsymbol{\eta}}{\eta_0}\\
\Sigma_{\mathrm{cr}}=\frac{c^2D_{\mathrm{s}}}{4\pi GD_{\mathrm{l}}D_{\mathrm{ls}}}
\end{gathered}\tag{6}
$$

- For lensing in cosmoslogical scale , we have to introduce RW metric that we skipped here.

Properties of the mapping

“透镜势”

- let $\psi(\mathbf{x})=\frac{1}{\pi}\int_{\mathbf{R}^2}d^2x'\kappa\left(\mathbf{x}'\right)\ln\left|\mathbf{x}-\mathbf{x}'\right|$, then $\alpha=\nabla\psi$

$$
\mathbf{y}=\nabla\left(\frac{1}{2}\mathbf{x}^2-\psi(\mathbf{x})\right)\tag{7}
$$

- arrive time: 光从发出到观测到的时间。观测量：同一个源由多个像，源有一个光变，但是像在光变时有一个时间差，就来自于到达时间的差。
  - $cdt=(1-\frac{2U}{c^2})\mathrm{d}l$, integral along the propagation path
  - $c\Delta t=\frac{D_{\mathrm{l}}D_{\mathrm{s}}}{2D_{\mathrm{ls}}}\left(\frac{\xi}{D_{\mathrm{l}}}-\frac{\eta}{D_{\mathrm{s}}}\right)^2-\hat{\psi}(\xi)+const$

    where $\hat{\psi}(\boldsymbol{\xi})=\frac{4G}{c^2}\int d^2\xi'\Sigma\left(\boldsymbol{\xi}'\right)\ln\left(\left|\boldsymbol{\xi}-\boldsymbol{\xi}'\right|\right)$
  - normalized : (dominated by distance)

$$
c\Delta t=\frac{1}{\pi_{rel}}(1+\underset{\text{红移}}{z_d})(\frac{1}{2}\Delta\theta^2-\psi(\mathbf{x}))+const\quad,\quad where\ \xi_0=D_l\tag{8}
$$

- magnification:
  - usually use complex form:

$$
\begin{aligned}
x_c &= x_1 + i x_2 \\
y_{\mathrm{c}} &= x_{\mathrm{c}} - I_{\mathrm{c}}^{*}\left(x_{\mathrm{c}}\right) \\
I_{\mathrm{c}}\left(x_{\mathrm{c}}\right) &= \frac{1}{\pi} \int_{\mathbb{C}} \kappa\left(x_{\mathrm{c}}'\right) \frac{1}{x_{\mathrm{c}} - x_{\mathrm{c}}'} d^2 x'
\end{aligned} \tag{9}
$$

- Jacobian matrix for this mapping (distortion)

$$
A(\mathbf{x}) = \frac{\partial \mathbf{y}}{\partial \mathbf{x}},
$$

$$
A_{ij} = \frac{\partial y_i}{\partial x_j} = \delta_{ij} - \alpha_{ij} \quad \text{定义}\alpha\text{ij}
$$

$$
notice : tr(\alpha_{ij}) = \nabla \cdot \alpha = \nabla^2 \psi = 2\kappa(\mathbf{x}) \text{ 先分离出}\alpha\text{的迹}
$$

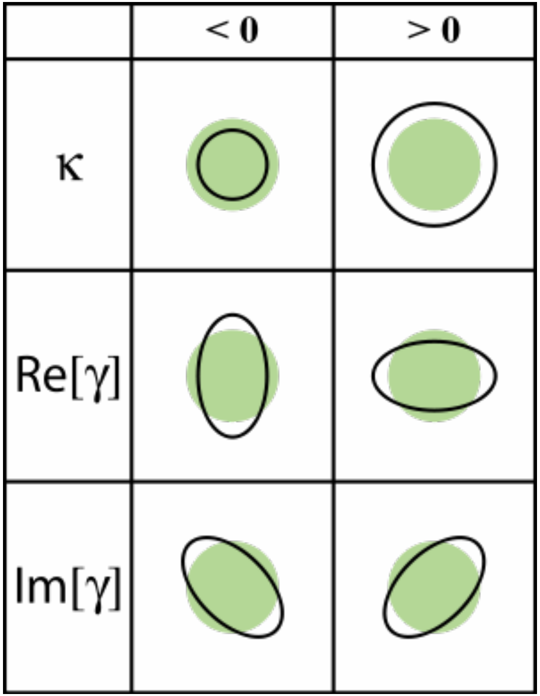
$$
\tag{10}
$$

$$
A = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}
$$

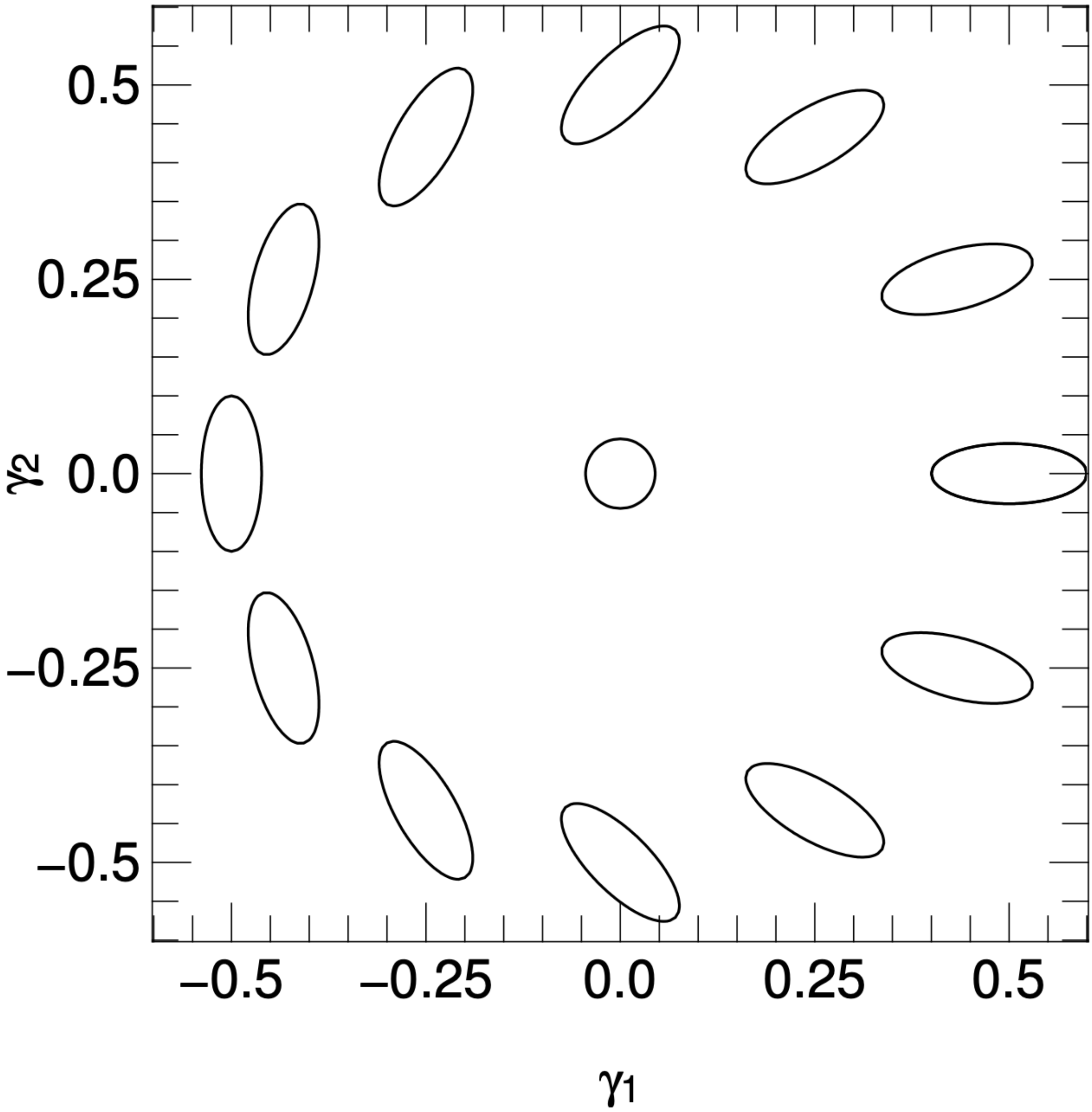
$$
\gamma_1 = \frac{1}{2}(\psi_{11} - \psi_{22}) \quad ; \quad \gamma_2 = \psi_{12} = \psi_{21}
$$

$$
\det A = (1 - \kappa)^2 - \overset{\gamma 1\text{^}2+\gamma 2\text{^}2}{\gamma^2}
$$

- magnification factor $\mu(\mathbf{x}) = \frac{1}{\det A(\mathbf{x})}$
  - two terms in formula:
    - $(1-\kappa)^2$ depends only on the surface mass density \kappa within the beam -----> convergence / Ricci focusing
    - γ^2 depends on the mass distribution outside of the beam ------> shear



< 0
> 0
κ
Re[γ]
Im[γ]



- image distortion

两个本征方向的“伸长”就是本征值

  - for uniform disk : the eigenvalue of the matrix is

以前是个椭圆，成像变成圆

$$1 - \kappa \mp \gamma \tag{11}$$

so the disk become a ellipse , $\epsilon = \dfrac{\gamma}{1-\kappa}$

  - for more complicated cases

we define ellipticity of a galaxy :

$$
\begin{aligned}
\chi &= \frac{1-q^2}{1+q^2} e^{2i\phi} = \frac{a^2-b^2}{a^2+b^2} e^{2i\phi} \\
\epsilon &= \frac{1-q}{1+q} e^{2i\phi} = \frac{a-b}{a+b} e^{2i\phi}
\end{aligned} \tag{12}
$$

statistically , we use image data to get ellipicity:

$$
Q_{ij} = \frac{\int \mathrm{d}^2\theta I(\boldsymbol{\theta}) q_I[I(\boldsymbol{\theta})] \left(\theta_i - \bar{\theta}_i\right)\left(\theta_j - \bar{\theta}_j\right)}{\int \mathrm{d}^2\theta I(\boldsymbol{\theta}) q_I[I(\boldsymbol{\theta})]}, \quad i,j \in \{1,2\} \tag{13}
$$

由于光度分布中的背景星光/天光不太好减去，需要在星系光强较暗的地方截断

where $q_I(I)$ is a suitably chosen weight function; θ_i would be the center of light within a limiting isophote of the image.

$$
\chi \equiv \frac{Q_{11} - Q_{22} + 2\mathrm{i}Q_{12}}{Q_{11} + Q_{22}} \quad \text{and } \epsilon \equiv \frac{Q_{11} - Q_{22} + 2\mathrm{i}Q_{12}}{Q_{11} + Q_{22} + 2\left(Q_{11}Q_{22} - Q_{12}^2\right)^{1/2}} \tag{14}
$$

what we care is the ellipicity of the source and the image

$$
\begin{aligned}
\chi^{(i)} &\equiv \frac{Q_{11}^{(i)} - Q_{22}^{(i)} + 2\mathrm{i}Q_{12}^{(i)}}{Q_{11}^{(i)} + Q_{22}^{(i)}} \quad \text{成像后的} \\
\chi^{(s)} &\equiv \frac{Q_{11}^{(s)} - Q_{22}^{(s)} + 2\mathrm{i}Q_{12}^{(s)}}{Q_{11}^{(s)} + Q_{22}^{(s)}} \quad \text{成像前的}
\end{aligned} \tag{15}
$$

源平面的角

$$
Q_{ij}^{(\mathrm{s})} = \frac{\int \mathrm{d}^2\beta I^{(\mathrm{s})}(\boldsymbol{\beta}) q_I\left[I^{(\mathrm{s})}(\boldsymbol{\beta})\right]\left(\beta_i - \bar{\beta}_i\right)\left(\beta_j - \bar{\beta}_j\right)}{\int \mathrm{d}^2\beta I^{(\mathrm{s})}(\beta) q_I\left[I^{\mathrm{s})}(\boldsymbol{\beta})\right]}, \quad i,j \in \{1,2\} \tag{16}
$$

分别变成列向量和行向量

$$
\mathrm{d}^2\beta = \det A \;\; \mathrm{d}^2\theta, \boldsymbol{\beta} - \overline{\boldsymbol{\beta}} = A(\boldsymbol{\theta} - \overline{\boldsymbol{\theta}}) \tag{17}
$$

$$
Q^{(\mathrm{s})} = AQA^T = AQA
$$

$$
A = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} = (1-\kappa)\begin{pmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{pmatrix} \tag{18}
$$

$$
g = g_1 + ig_2 \quad , \quad g = \gamma/(1-\kappa)
$$

$$\chi^{(\mathrm{s})} = \frac{\chi - 2g + g^2\chi^*}{1 + |g|^2 - 2\mathcal{R}e\,(g\chi^*)}; \quad \epsilon^{(\mathrm{s})} = \begin{cases} \frac{\epsilon - g}{1 - g^*\epsilon} & \text{if } |g| \leq 1 \\ \frac{1 - g\epsilon^*}{\epsilon^* - g^*} & \text{if } |g| > 1 \end{cases} \tag{19}$$

reversed form :

$$\begin{aligned} \chi &= \frac{\chi^{(s)} + 2g + g^2\chi^{(s)*}}{1 + |g|^2 + 2\,\mathrm{Re}\left(g\chi^{(s)*}\right)} \\ \epsilon &= \frac{\epsilon^{(s)} + g}{1 + g^*\epsilon^{(s)}} \end{aligned} \tag{20}$$

weak lensing limit : 弱引力透镜极限下，

$\kappa << 1$, $\gamma << 1$, $g \simeq \gamma$

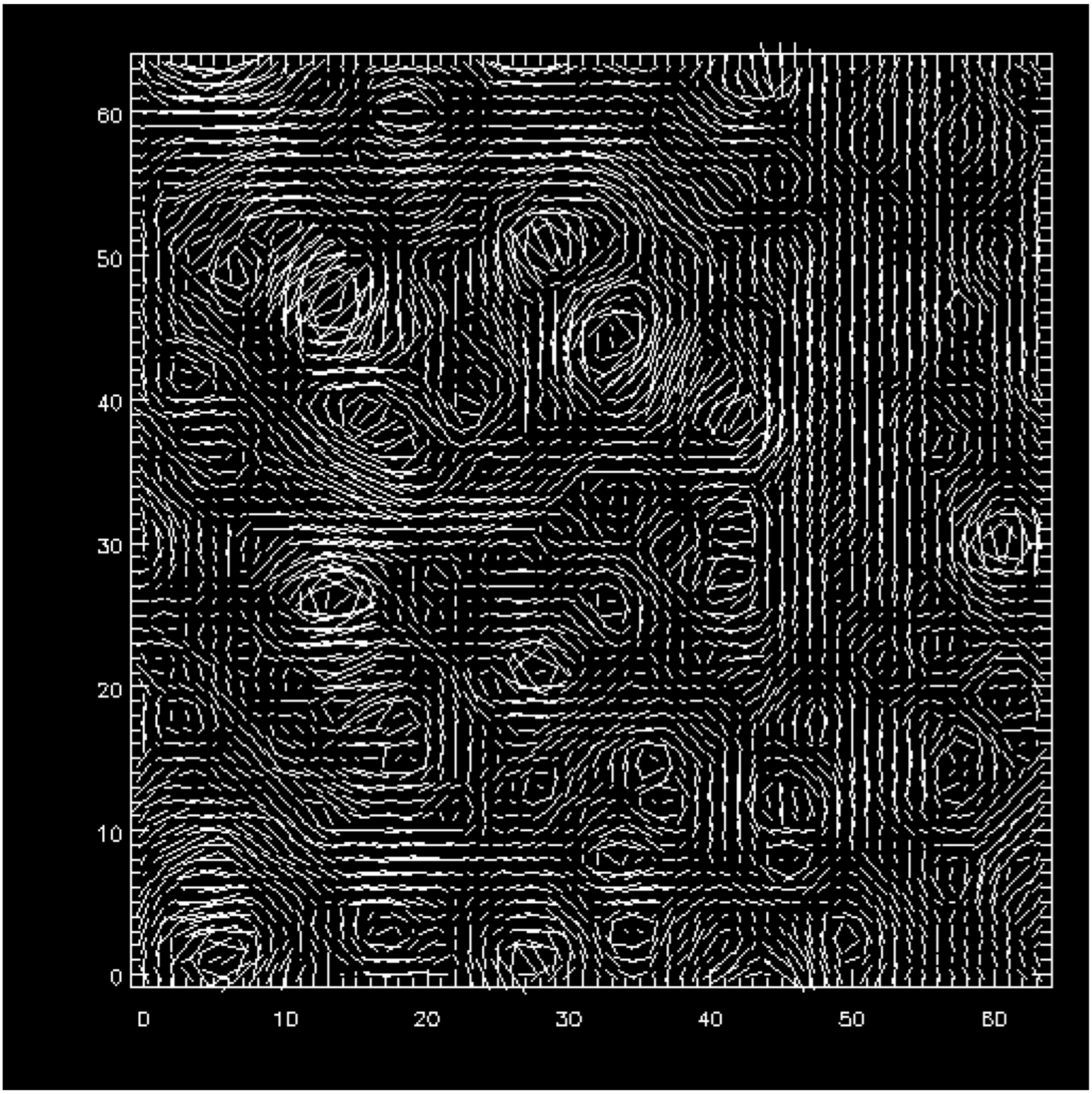
=> $\epsilon = \epsilon^{(s)} + g$

$< \epsilon^{(s)} >= 0$, so $\gamma = g =< \epsilon >$

对大量的星系，椭率的期望为0，得到这一片区域的sheer

=> shear map => mass distributation (weak lensing area ; since we can't resolve κ , the case is degenerated)

比如在一个点透镜附近，不能用weak lensing limit，就不存在这一步推导。



坐标是位置，上面的矢量是sheer