Proof of the Conservation of Surface Brightness in Gravitational Lensing

1 Conservation of Specific intensity

1.1 The Liouville Theorem

Consider a swarm of particles that travel from the star to us. Particle positions and momenta are completely described by the 6-D distribution function, the density of state in the phase-space:

$$f = f(\boldsymbol{x}, \boldsymbol{p}, t)$$

The total number of these particles then is the integration in the whole phase space.

$$N(t) = \int f d^3p d^3x \tag{1}$$

If the Hamiltonian of this system is conserved, the Liouville Theorem tells us that:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + [H, f] = 0 \tag{2}$$

According to this theorem, the "local" density of the representative points, as viewed by an observer moving with a representative point, stays constant in time. Thus, the swarm of the representative points moves in the phase space in essentially the same manner as an incompressible fluid moves in the physical space!

1.2 Particle Specific-Intensity

Define the number-specific intensity $J(U)$:

$$dN = J \ \mathrm{d}\Omega \ \mathrm{d}U \ \mathrm{d}t \ \mathrm{d}A \tag{3}$$

where the energy U is the total relativistic energy, $U = mc^2 = E_k + m_0c^2$.

dN in the phase-space:

$$dN = f \ p^2 \ \mathrm{d}p \ \mathrm{d}\Omega_{\mathrm{p}} v \ \mathrm{d}t \ \mathrm{d}A \tag{4}$$

The axes in the spatial and momentum spaces are coaligned (p_x-axis coaligned with x-axis, etc.). Thus, the solid angles are equivalent, $\Omega = \Omega_p$.

$$J\mathrm{d}U = f \ vp^2 \ \mathrm{d}p \tag{5}$$

Note that the momentum p and total energy U are related by $U^2 = p^2c^2 + (m_0c^2)^2$. The differentiation of U and p then follows:

$$UdU = pc^2dp \tag{6}$$

The velocity and momentum are related because $p = \gamma m v$, where $\gamma = U/mc^2$. Thus,

$$v = \frac{c^2}{U} p \tag{7}$$

Finally, the relation between particle-specific intensity and phase-space density:

$$J = p^2 f \tag{8}$$

1.3 Energy Specific-Intensity

The energy-specific intensity $I(\nu)$ (W m^{-2} s^{-1} Hz^{-1}) describes the energy flux of **photons** per unit frequency interval(Hz). Its relation with total flux F is:

$$F = \int_\nu I_\nu d\nu \Delta\Omega \tag{9}$$

The energy flow (flux) may be expressed in terms of both I and J:

$$I(\nu) d\nu = U J(U) dU \tag{10}$$

For photons, $U = h\nu$ which gives that,

$$I(\nu) = J(U) h^2 \nu \tag{11}$$

Finally, eliminate$J(U)$with the relation$J = p^2 f$ and then eliminate p with $p = h\nu/c$ to obtain

$$I(\nu) = \frac{h^4 \nu^3}{c^2} f \tag{12}$$

1.4 Conservation of Surface Brightness

In gravitational lensing there is no chromatic dispersion. In local universe the redshift is negligible. From the Liouville Theorem, we know that f is conserved. Then the specific intensity I_ν is a conserved quantity.

As we can see from the definition of I_ν and its relation with flux F, the specific intensity I_ν has the same meaning as angular surface brightness. Thus, the surface brightness is a conserved quantity in gravitational lensing.

Other Example: galaxy surface brightness.

1.5 Relativity Connection(special relativity)

Here we discuss how phase-space density f transforms from this moving inertial frame to the laboratory (stationary) frame. The number of particles within phase volume element dN is Lorentz invariant. And we have $dN = f d^3x d^3p$. Assume particles travel in the x direction. Lorentz transformation tells us

$$dx = \gamma^{-1} dx' \tag{13}$$

$$dp_x = \gamma(dp'_x + v dE'/c^2) = \gamma dp'_x \tag{14}$$

Then $d^3x d^3p = d^3x' d^3p'$ is Lorentz invariant. And thus the phase-space density f is also invariant!

2 Geometrical Method

As an extreme case, suppose a pitch of the source's surface has an area of S(S$\rightarrow$ 0), and its solid angle related to observer is $\Delta\Omega$. The area of the receiver in observer plane is A. So only the light within the solid angle can reach the observer.

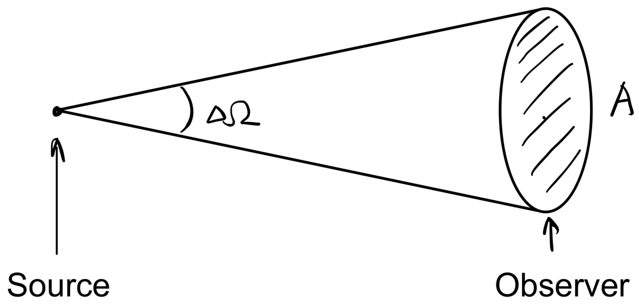

Figure 1: Observed brightness

Assume the total luminosity of this pitch of source is I, and its radiation is isotropic. The received flux is $F = \frac{\Delta\Omega}{4\pi} I$. And the brightness is:

$$B = \lim_{A\to 0} \frac{F}{A} = \lim_{A\to 0} \frac{\Delta\Omega I}{4\pi A} \tag{15}$$

Since I is the intrinsic property of the source independent of the path of the light, the observed brightness only depends on the $\Delta\Omega$. If there is a lense lies between the source and observer, the light doesn't travel along a line. The solid angle of the light at the source that can reach the observer changes to $\Delta\Omega'$, as shown in Figure2. According to equation (15), the magnification rate is:

$$\mu = \frac{B'}{B} = \frac{\Delta\Omega'}{\Delta\Omega} \tag{16}$$

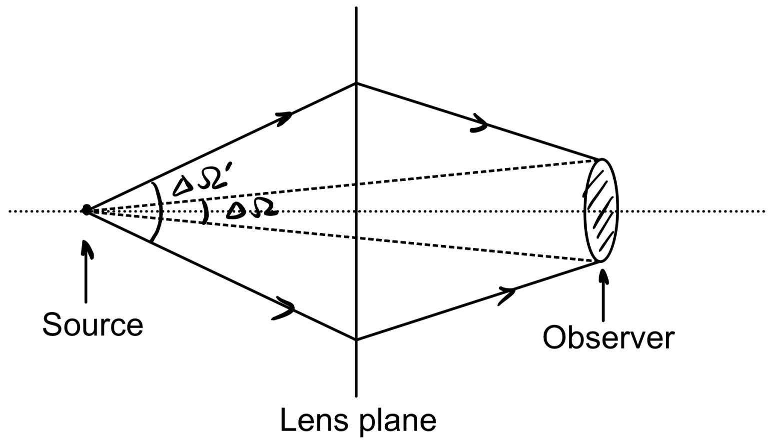

Figure 2: light track with lens

(in this case the $\Delta\Omega$ is enlarged and hence luminosity is magnified)

Then the core part of this proof is the calculation of $\Delta\Omega'$. It can be calculated by the differential of the lens equation.

3 Symmetry in Lens Equation

The lensing equation:

$$r^2 - r_0 r - r_E^2 = 0$$

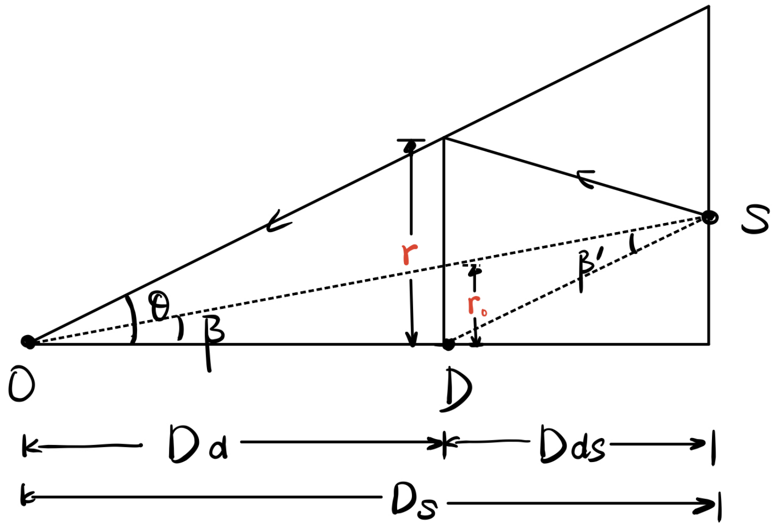

Figure 3: Projection to lens plane