Proof of the Conservation of Surface Brightness in Gravitational Lensing

1 Conservation of Specific intensity

1.1 The Liouville Theorem

Consider a swarm of particles that travel from the star to us. Particle positions and momenta are completely described by the 6-D distribution function, the density of state in the phase-space:

$$f = f(\boldsymbol{x}, \boldsymbol{p}, t)$$

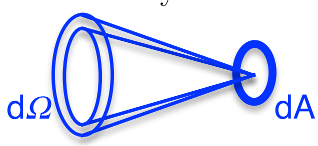
The total number of these particles then is the integration in the whole phase space.

$$N(t) = \int f d^3p d^3x \tag{1}$$

If the Hamiltonian of this system is conserved, the Liouville Theorem tells us that:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + [f, H] = 0 \tag{2}$$

According to this theorem, the "local" density of the representative points, as viewed by an observer moving with a representative point, stays constant in time. Thus, the swarm of the representative points moves in the phase space in essentially the same manner as an incompressible fluid moves in the physical space!



1.2 Particle Specific-Intensity

Define the number-specific intensity $J(U)$: 单位立体角，单位能量，单位时间，单位面积

$$dN = J \; \mathrm{d}\Omega \; \mathrm{d}U \; \mathrm{d}t \; \mathrm{d}A \tag{3}$$

where the energy U is the total relativistic energy, $U = mc^2 = E_k + m_0c^2$.

dN in the phase-space:

$$dN = f \; \underbrace{p^2 \; \mathrm{d}p \; \mathrm{d}\Omega_\mathrm{p}}_{d^3\,p} \underbrace{v \; \mathrm{d}t \; \mathrm{d}A}_{d^3\,q} \tag{4}$$

The axes in the spatial and momentum spaces are coaligned (p_x-axis coaligned with x-axis, etc.). Thus, the solid angles are equivalent, $\Omega = \Omega_p$.

$$J\mathrm{d}U = f \; vp^2 \; \mathrm{d}p \tag{5}$$

Note that the momentum p and total energy U are related by $U^2 = p^2c^2 + (m_0c^2)^2$. The differentiation of U and p then follows:

$$UdU = pc^2dp \tag{6}$$

The velocity and momentum are related because $p = \gamma \overset{m0}{\cancel{m}} v$, where $\gamma = U/\overset{m0}{\cancel{m}} c^2$. Thus,

$$v = \frac{c^2}{U} p \tag{7}$$

Finally, the relation between particle-specific intensity and phase-space density:

$$J = p^2 f \tag{8}$$

1.3 Energy Specific-Intensity

The energy-specific intensity $I(\nu)$ (W m^{-2} s^{-1} Hz^{-1}) describes the energy flux of **photons** per unit frequency interval(Hz). Its relation with total flux F is:

$$F = \int_\nu I_\nu d\nu \Delta\Omega \tag{9}$$

The energy flow (flux) may be expressed in terms of both I and J:

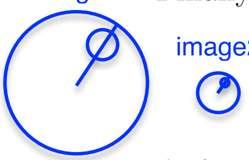
$$I(\nu) d\nu = U J(U) dU \quad \text{能量*粒子束} \tag{10}$$

For photons, $U = h\nu$ which gives that,

$$I(\nu) = J(U) h^2 \nu \tag{11}$$

image1

image2



Finally, eliminate$J(U)$with the relation$J = p^2 f$ and then eliminate p with $p = h\nu/c$ to obtain

$$I(\nu) = \frac{h^4 \nu^3}{c^2} f \tag{12}$$

两个像间的某一对应的点，面亮度相同。这种对应关系就是“物像”对应关系，你可以在像1上找一个微元（对应一个真实的物理客体），把剩下的部分全部去掉，则在像2上也剩下一个微元，这两个微元的面亮度相同。这两个微元是同一批光子“成像”的，或者说我们在两个位置（这批光子传播过程中的两个时刻）接受这一批光子，得到的面亮度都相同，这正是上面证明的东西。对引力透镜，不存在透镜时看源上的某一微元发出的光（不一定在有透镜时被观测到）和存在透镜时看像上对应微元发出的光不是同一批光子，但是可以假设同一微元发出的不同光子面亮度相同。

1.4 Conservation of Surface Brightness

不考虑红移，频率不变，f不变，则I不变

In gravitational lensing there is no chromatic dispersion. In local universe the redshift is negligible. From the Liouville Theorem, we know that f is conserved. Then the specific intensity I_ν is a conserved quantity.

功率/望远镜面积

As we can see from the definition of I_ν and its relation with flux F, the specific intensity I_ν has the same meaning as angular surface brightness. Thus, the surface brightness is a conserved quantity in gravitational lensing.

Other Example: galaxy surface brightness.

先看一个简单的例子：太阳的面亮度与我们与太阳的距离无关(Flux/D^2)/(S/D^2)，用面亮度守恒解释：在太阳光子运动到我们的过程中，角面亮度不变，这意味着我们在光子运动过程的任意时刻去观测（即移动观测者的位置）面亮度不变。

1.5 Relativity Connection(special relativity)

证明f是换系不变

Here we discuss how phase-space density f transforms from this moving inertial frame to the laboratory (stationary) frame. The number of particles within phase volume element dN is Lorentz invariant. And we have $dN = f d^3x d^3p$. Assume particles travel in the x direction. Lorentz transformation tells us

$$\cancel{dx = \gamma^{-1} dx'} \tag{13}$$

$$\cancel{dp_x = \gamma(dp'_x + v dE'/c^2) = \gamma dp'_x} \tag{14}$$

Then $d^3x d^3p = d^3x' d^3p'$ is Lorentz invariant. And thus the phase-space density f is also invariant!

dN=fd^3xd^3p，其中体元dx∧dy∧dz∧dpx∧dpy∧dpz是洛伦兹不变量，于是f=I/ν^3是洛伦兹不变量。

计算放大率的几何方法，这和假设面亮度守恒的结果一样，即验证面亮度守恒。

2 Geometrical Method

As an extreme case, suppose a pitch of the source's surface has an area of S(S→ 0), and its solid angle related to observer is $\Delta\Omega$. The area of the receiver in observer plane is A. So only the light within the solid angle can reach the observer.

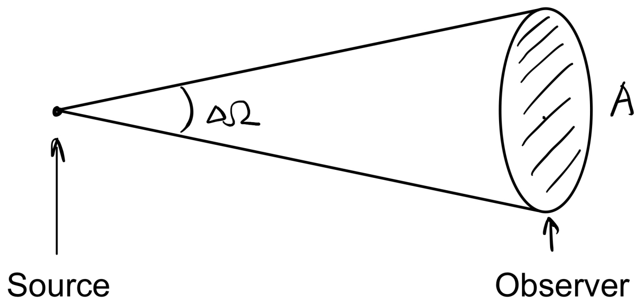


Figure 1: Observed brightness

Assume the total luminosity of this pitch of source is L and its radiation is isotropic. The received flux is $F = \frac{\Delta\Omega}{4\pi} I$. And the brightness is:

$$B = \lim_{A \to 0} \frac{F}{A} = \lim_{A \to 0} \frac{\Delta\Omega L}{4\pi A} \quad \text{描述源面亮度是不依赖观测面积的} \tag{15}$$

Since L is the intrinsic property of the source independent of the path of the light, the observed brightness only depends on the $\Delta\Omega$. If there is a lense lies between the source and observer, the light doesn't travel along a line. The solid angle of the light at the source that can reach the observer changes to $\Delta\Omega'$, as shown in Figure2. According to equation (15), the magnification rate is:

$$\mu = \frac{B'}{B} = \frac{\Delta\Omega'}{\Delta\Omega} \tag{16}$$

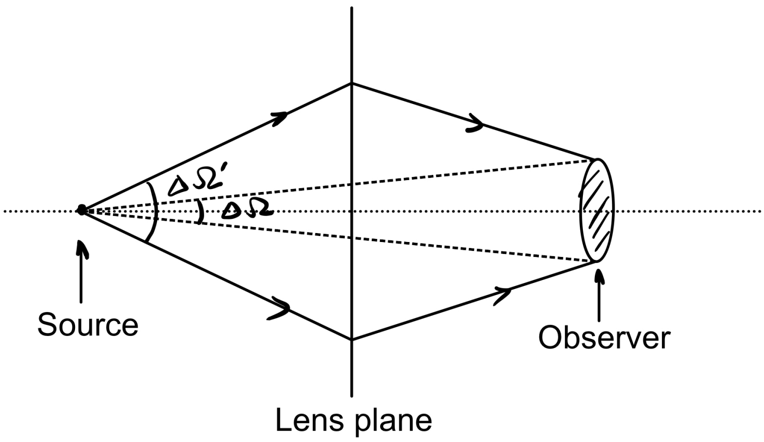


Figure 2: light track with lens

(in this case the $\Delta\Omega$ is enlarged and hence luminosity is magnified)

Then the core part of this proof is the calculation of $\Delta\Omega'$. It can be calculated by the differential of the lens equation.

3 Symmetry in Lens Equation

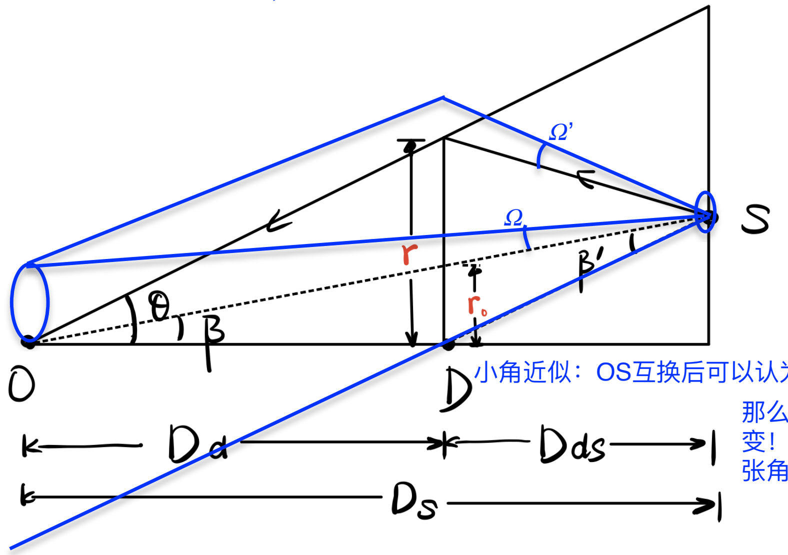
证面亮度守恒

投影在mass sheet上，这些量都不改变。

The lensing equation:

$$r^2 - r_0 r - r_E^2 = 0$$

放大率：到达我们的源的面积微元发出的光线夹角在有无透镜存在时的比，它等于观测者上某一点发出的光线到达源的面积微元的夹角之比，这个正好是源在观测者看来有无透镜的面积之比。



小角近似：OS互换后可以认为mass sheet还是这样竖着的。

那么，变换后r0不变，rE不变，则r不变！而几何上看，观察到的面积之比/张角之比是r的函数。

Figure 3: Projection to lens plane