1 Geometrical Method

1.1 observed Brightness

Before our proof, let's review the definition of observed brightness. As a limiting case, suppose a pitch of the source's surface has an area of S(S$\rightarrow$ 0), and its solid angle related to observer is $\Delta\Omega$. The area of the receiver in observer plane is A. So only the light within the solid angle can reach the observer. And it is shown in Figure 1.

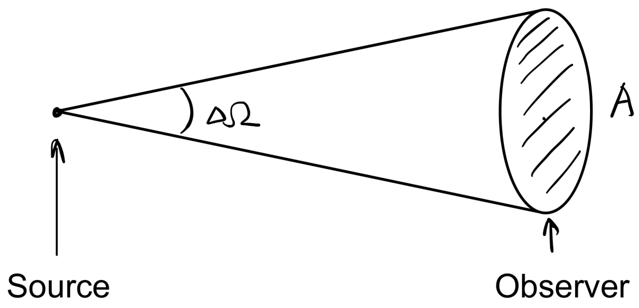

Figure 1: Observed brightness

Assume the total luminosity of this pitch of source is I, and its radiation is isotropic. So the received flux is $F = \frac{\Delta\Omega}{4\pi} I$. And the brightness is:

$$B = \lim_{A \to 0} \frac{F}{A} = \lim_{A \to 0} \frac{\Delta\Omega I}{4\pi A} \tag{1}$$

While $'I'$ is an intrinsic character of the source and independent of the track of the light, the observed brightness only relies on $\Delta\Omega$. If there is a lense lies between the source and observer, the light doesn't travel along a line, and the solid angle of the light at the source that can reach the observer is changed to $\Delta\Omega'$, as shown in Figure2. According to equation (1), the magnification rate is:

$$\mu = \frac{B'}{B} = \frac{\Delta\Omega'}{\Delta\Omega} \tag{2}$$

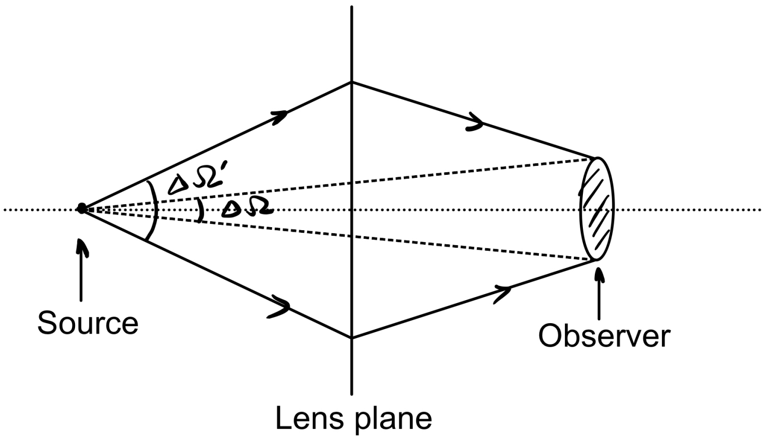

Figure 2: light track with lens

(in this case the $\Delta\Omega$ is enlarged and hence luminosity is magnified)

Then the core part of this proof is the calculation of $\Delta\Omega'$.

1.2 Situation in 2D

First, let's begin our calculation with 2-D situation. The magnification rate of brightness then is $\mu = \frac{\Delta u'}{\Delta u}$, where Δu is the plane angle, while in 3-D case it's solid angle $\Delta\Omega$. Despite that it's a little different with our 3-D situation, it's much more explicit and clear, and may help as to understand the magnification better. So just take it as our beginning.

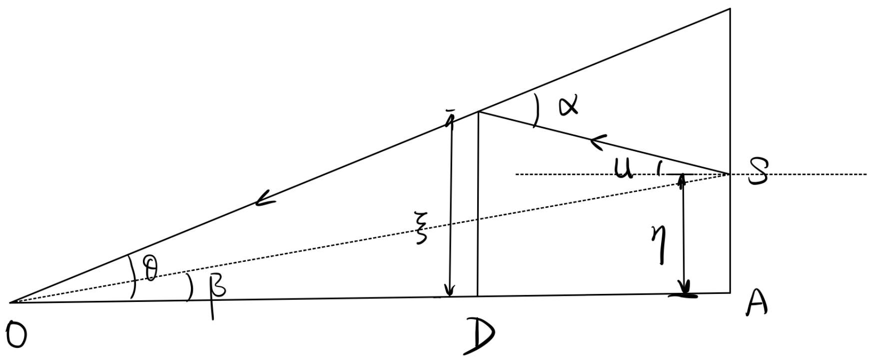

Figure 3: Gravitational Lensing Geometry

As shown in Figure 3, S is the source, O is the observer and D is the lensing object which makes the light curved. $D_d, D_s$and D_d are the distance to the lens (deflector), distance to the source and distance between the lens (deflector) and the source. η is the source position (distance perpendicular to the line connecting the observer and the lens), ξ is the image position,and α is the deflection angle. β, θ, u is shown in graph. And we can immediately get the some relations from geometry:

$$u = \alpha - \theta \tag{3}$$

$$\xi = D_d\theta, \quad \eta = D_s\beta \tag{4}$$

The result got from general relativity is:

$$\alpha = \frac{4GM}{c^2\xi} = \frac{4GM}{c^2 D_d}\frac{1}{\theta} \tag{5}$$

where M is the mass of lensing object.

From(3)(4)(5), we have:

$$u(\theta) = \frac{4GM}{c^2 D_d}\frac{1}{\theta} - \theta \tag{6}$$

And there is a characteristic angle θ_E, which is the angular Einstein radius:

$$\theta_E = \sqrt{\frac{4GMD_{ds}}{c^2 D_d D_s}} \tag{7}$$

And the lens equation for a point lens in angles is:

$$\beta + \frac{\theta_E^2}{\theta} = \theta \tag{8}$$

Since the gravitational force is a central force, the curve of light must lie on one plane, and the lense object is also in such plane. Thus point O, S, D and the light curve are all in one plane no matter what situation it is.

Now let us suppose the observer is not a single point but has some stretch in this plane. As shown in Figure 4, the observer is a line segment OO' with a length of $D_d\delta$ ($\delta \to 0$). Then there is a little of changes in geometry. And if we compare it with our standard situation is Figure 3, it's equivalently to have some variation in η and β. And only the light within the field angel of ray 1 and ray 2(shown in graph) can reach observer. So the Δu mentioned above is the field angle of this two ray.

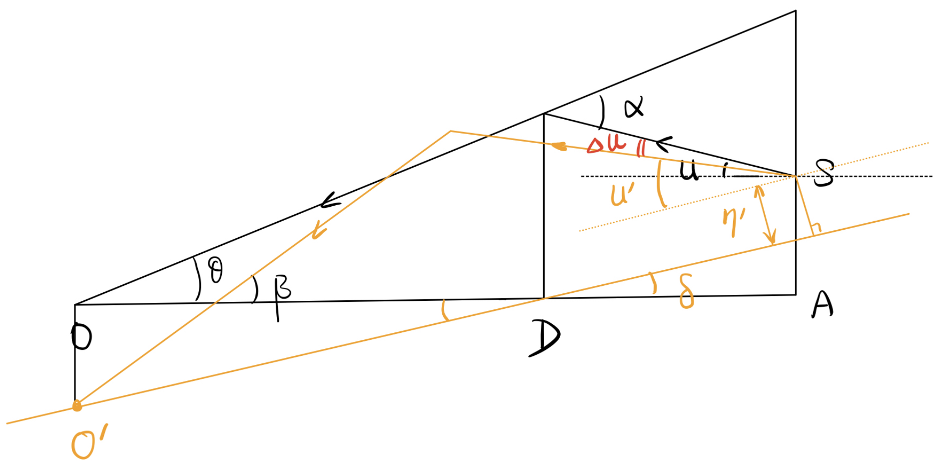

Figure 4: Observer stretch in 2D plane

Then we have:

$$d\eta = \eta' - \eta = -D_{ds}\delta \tag{9}$$

Differentiate eq.(6) and eq.(8) and notice that $\theta_E^2 = \frac{4GMD_{ds}}{c^2 D_d D_s}$:

$$du = -(1 + \frac{4GM}{c^2 D_d}\frac{1}{\theta^2})d\theta = -(1 + \frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta^2})d\theta \tag{10}$$

$$d\theta = (1 + \frac{\theta_E^2}{\theta^2})^{-1} d\beta \tag{11}$$

since $d\beta = D_s d\eta$, we can get the relation between du and $d\eta$:

$$du = -(1 + \frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta^2})(1 + \frac{\theta_E^2}{\theta^2})^{-1}\frac{d\eta}{D_s} \tag{12}$$

substitute $d\eta$ and we have:

$$du = \frac{(\frac{D_{ds}}{D_s} + \frac{\theta_E^2}{\theta^2})}{(1 + \frac{\theta_E^2}{\theta^2})}\delta$$

Finally we have:

$$\Delta u' = |u' - u - \delta| = \delta - du = (1 + \frac{\theta_E^2}{\theta^2})^{-1}\frac{D_d}{D_s}\delta \tag{13}$$

if there is no lens, the field angle Δu should be:

$$\Delta u = \frac{D_d \delta}{D_s}$$

So the magnification rate is:

$$\mu = \frac{\Delta u'}{\Delta u} = (1 + \frac{\theta_E^2}{\theta^2})^{-1} \tag{14}$$

1.3 Situation in 3D

In 3D situation, the receiver stretches in a plane. Assume it's a disc with radius of $D_d\delta$ ($\delta \to 0$). For a point named O' at the rim of the disc, the geometry relation also changes as in 2D case for this point observer. But as proved in section 2.2.2, O'(point observer), D(lense), S(source) and the light reaches O' are all in a plane. So the key point is to find this plane and the difference in geometry parameters. The graph of this situation shows below, and the observer is not shown here. The meaning of η, η' and other parameters are in accord with Figure 3 and Figure 4.

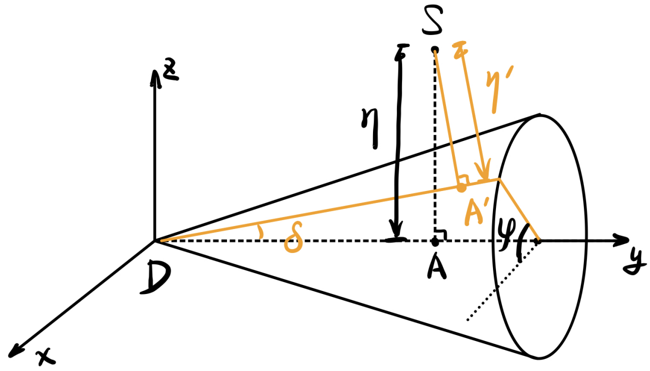

Figure 5: Geometry in 3D plane

We use vectors to calculate the geometrical relations. For simplicity, we scale all length parameters with D_{ds}. Namely, in our calculation we let D_{ds} to be 1, and η is scaled to $\tilde{\eta} = \eta/D_{ds}$. But for convenience, we still use η to stand $\tilde{\eta}$. The we have:

$$\overrightarrow{DA} = (0, 1, 0)$$

$$\overrightarrow{DS} = (0, 1, \eta)$$

We assume the distance between point A' and D is a. Then we have:

$$\overrightarrow{DA'} = (a\cos\varphi\sin\delta, a\cos\delta, a\sin\varphi\sin\delta)$$

$$\overrightarrow{SA'} = \overrightarrow{DA'} - \overrightarrow{DS} = (a\cos\varphi\sin\delta, a\cos\delta - 1, a\sin\varphi\sin\delta - \eta)$$

Notice that $\overrightarrow{SA'}$ and $\overrightarrow{DA'}$ are perpendicular, $\overrightarrow{SA'} \cdot \overrightarrow{DA'} = 0$, then we can get the value of a:

$$a = \frac{\cos\delta + \eta\sin\varphi\sin\delta}{\cos^2\delta + \sin^2\varphi\sin^2\delta} = 1 + o(\delta)$$

In further calculation, we find that $o(\delta)$ has no contributions. So we can safely view a as 1. And we calculate the length of $\overrightarrow{SA'}$:

$$|\overrightarrow{SA'}| = \eta' = \eta - \sin\varphi\sin\delta + o(\delta^2)$$

We keep the same definition of u as above(u is also very small), and we can get the vector of the ray of light at the source which reaches point O' (the point at the rim of source disc):

$$\begin{aligned} \overrightarrow{n} &= \frac{\overrightarrow{A'S}}{|\overrightarrow{A'S}|}\sin u + \frac{\overrightarrow{A'D}}{|\overrightarrow{A'D}|}\cos u & (15) \\ &= (-(\frac{u}{\eta} + 1)\delta\cos\varphi, 1, u - \sin\varphi\delta) & (16) \end{aligned}$$

For the light that reaches the center of source disc, the vector is:

$$\overrightarrow{n} = (0, 1, u_0) \tag{17}$$

the difference vector:

$$\overrightarrow{\Delta n} = \overrightarrow{n} - \overrightarrow{n_0} = (-(\frac{u}{\eta} + 1)\delta\cos\varphi, 0, u - u_0 - \sin\varphi\delta) \tag{18}$$

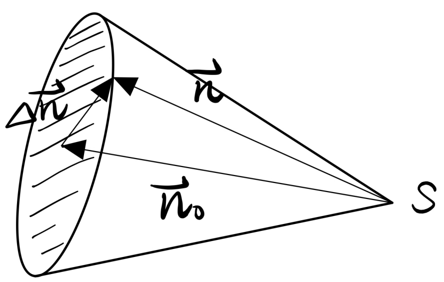

Figure 6: Light cone at the source

From eq.(3), eq.(12), we have:

$$u - u_0 = du = -(1 + \frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta^2})(1 + \frac{\theta_E^2}{\theta^2})^{-1}\frac{d\eta}{D_s} \tag{19}$$

$$= (1 + \frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta^2})(1 + \frac{\theta_E^2}{\theta^2})^{-1}\frac{D_{ds}}{D_s}\sin\varphi \cdot \delta \tag{20}$$

$$u/\eta = (\alpha - \theta)/\eta = (\frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta} - \theta)/\eta \tag{21}$$

$$= (\frac{\theta_E^2}{\theta} - \frac{D_{ds}}{D_s}\theta)/\beta \tag{22}$$

$$= -(\frac{\theta_E^2}{\theta} - \frac{D_{ds}}{D_s}\theta)(\frac{\theta_E^2}{\theta} - \theta)^{-1} \tag{23}$$

and we get:

$$\overrightarrow{\Delta n} = (\frac{D_d/D_s}{\theta_E^2/\theta^2 - 1}\delta\cos\varphi, 0, \frac{-D_d/D_s}{\theta_E^2/\theta^2 + 1}\delta\cos\varphi) \tag{24}$$

So the solid open angel is:

$$\Delta\Omega' = \frac{1}{(\theta_E^2/\theta^2 - 1)(\theta_E^2/\theta^2 + 1)}\left(\frac{D_d}{D_s}\delta\right)^2 \tag{25}$$

If there isn't a lens, the solid open angel is:

$$\Delta\Omega = \left(\frac{D_d}{D_s}\delta\right)^2 \tag{26}$$

We finally get the magnification rate:

$$\mu = \frac{\Delta\Omega'}{\Delta\Omega} = \frac{1}{(\theta_E^2/\theta^2 - 1)(\theta_E^2/\theta^2 + 1)} \tag{27}$$

The most frequently used expression of magnification rate is given by:

$$\mu = \frac{dr \times r\Delta\phi}{dr_{\mathrm{s}} \times r_{\mathrm{s}}\Delta\phi} = \frac{r}{r_{\mathrm{s}}}\frac{dr}{dr_{\mathrm{s}}} = \frac{\theta}{\beta}\frac{d\theta}{d\beta} \tag{28}$$

and we have:

$$\beta = \theta - \frac{\theta_E^2}{\theta} = (1 - \frac{\theta_E^2}{\theta^2})\theta$$

$$d\beta = (1 + \frac{\theta_E^2}{\theta^2})d\theta$$

if we substitute β and $d\beta$ in eq.(28) ,we find that eq.(28) is equivalent to eq.(27)

As shown above, we presented a pure geometrical way to derive the magnification rate of a point source in gravitational lensing. And it could also serve as a proof of the conservation of surface brightness in such system.