Three Ways to Prove the Conservation of Surface Brightness in Gravitational Lensing

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Abstract

Here I present three different ways to calculate the magnification rate of surface brightness in gravitational lensing. The First one is based on Liouville's Theorem, where I'll begin with specific intensity and demonstrate how Liouville's Theorem leads to the conservation of surface brightness. Some interesting deductions are also presented. The second is a purely geometrical method. I start with some most basic principles and use geometrical relations between the object and image to calculate the magnification rate. Its methodology, especially the differential of the light path, can be applied to more complicated systems and is easy to use code to implement. The third method rises from a beautiful symmetry in the gravitational lensing equation, where I proved the conservation without any mathematical calculation. These methods provide very different angles to understand the conservation of surface brightness in gravitational lensing. And it's fun to find so many ways leading to this basic law.

1 Introduction

The surface brightness magnification of the source is a characteristic feature of gravitational lensing events, especially in micro-gravitational lensing. The magnification of brightness provides crucial information about this system and contains almost all the information accessible in microlensing. To decode its underlying physical meaning, the foremost thing is to understand why it's magnified and how it's magnified. General relativity and geometry tell us gravitational lensing makes the source larger in area. Then every paper says, due to the conservation of surface brightness in gravitational lensing, the magnification rate equals the Jacobi determinant from the source plane to the image plane.

While authors are always willing to spend time on the proof of lens equation, proof for the conservation of surface brightness receives unfairly cold attention, even though it's equally crucial and intriguing for curious people. Moreover, the conservation of surface brightness seems a little counter-intuitive (at least I once felt very uncomfortable to accept it firstly). But almost in every book, every review, and every paper, authors uniformly use the same arcane sentence to explain it: "Because of Liouville's theorem, gravitational lensing conserves surface brightness" without any further discussion. More authors even do not bother to write such a sentence and merely present the conclusion. The puzzling conservation always perplexed me until I set out to find out how Liouville's Theorem works here (with a surprise that this theorem, which is fundamental in Theoretical Mechanics and Equilibrium Statistical Physics, is also elemental in astrophysics).

In the journey of investigating Liouville's Theorem, I serendipitously came up with two other ways to prove the conservation of surface brightness. These two methods are more elementary, whose principles are more lucid than Liouville's Theorem. It's fun to find that so many ways could lead to this fundamental law. These different proofs from distinct angles could also conduce to understand this phenomenon better.

2 Analytical Method

2.1 The Liouville Theorem

Consider a swarm of particles that travel from the star to us. Particle positions and momenta are completely described by the 6-D distribution function, the density of state in the phase-space:

$$f = f(\boldsymbol{x}, \boldsymbol{p}, t)$$

The total number of these particles then is the integration in the whole phase space.

$$N(t) = \int f d^3p d^3x \tag{1}$$

If the Hamiltonian of this system is conserved, the Liouville Theorem tells us that:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + [H, f] = 0$$

According to this theorem, the "local" density of the representative points, as viewed by an observer moving with a representative point, stays constant in time. Thus, the swarm of the representative points moves in the phase space in essentially the same manner as an incompressible fluid moves in the physical space!

2.2 Particle Specific-Intensity

Knowledge of the density of state f is equivalent to knowledge of the specific intensity. Two important specific intensities are the *particle-specific intensity* J and the *energy-specific intensity* I. Note that the expressions below are relativistically correct as well.

Define the number-specific intensity $J(U)$ to be the number of particles crossing **unit surface per unit time** into **unit solid angle** within **unit interval of particle energy** U as expressed by:

$$dN = J \ \mathrm{d}\Omega \ \mathrm{d}U \ \mathrm{d}t \ \mathrm{d}A$$

Define the energy U to be the total relativistic energy, the sum of the rest and kinetic energies of a given particle, $U = E + m_0c^2$.

Meanwhile, we can express dN in the phase-space:

$$dN = f \ p^2 \ \mathrm{d}p \ \mathrm{d}\Omega_{\mathrm{p}} v \ \mathrm{d}t \ \mathrm{d}A$$

The axes in the spatial and momentum spaces are coaligned (p_x-axis coaligned with x-axis, etc.). Thus, the solid angles are equivalent, $\Omega = \Omega_p$.

$$J\mathrm{d}U = f \ vp^2 \ \mathrm{d}p$$

Note that the momentum p and total energy U are related by $U^2 = p^2c^2 + (m_0c^2)^2$. The differentiation of U and p then follows:

$$UdU = pc^2dp$$

The velocity and momentum are related because $p = \gamma m v$, where $\gamma = U/mc^2$. Thus,

$$v = \frac{c^2}{U} p$$

Finally we obtain the relation between particle-specific intensity and phase-space density:

$$J = p^2 f$$

2.3 Energy Specific-Intensity

The energy-specific intensity $I(\nu)$ (W m^{-2} s^{-1} Hz^{-1}) describes the energy flux of **photons** per unit frequency interval(Hz), which is widely used in astronomy. Its relation with total flux F is:

$$F = \int_\nu I_\nu d\nu \Delta\Omega$$

The energy flow (flux) may be expressed in terms of both I and J. Then we have the relation as follows:

$$I(\nu) d\nu = U J(U) dU$$

For photons, $U = h\nu$ which gives that,

$$I(\nu) = J(U) h^2 \nu$$

Finally, eliminate$J(U)$with the relation$J = p^2 f$ and then eliminate p with $p = h\nu/c$ to obtain

$$I(\nu) = \frac{h^4 \nu^3}{c^2} f$$

2.4 Conservation of Surface Brightness

In gravitational lensing there is no chromatic dispersion. When the cosmological redshift is negligible, the frequency of photons ν is invariant. From the Liouville Theorem, we know that f is conserved. Then the specific intensity I_ν is a conserved quantity.

As we can see from the definition of I_ν and its relation with flux F, the specific intensity I_ν has the same meaning as angular surface brightness. Thus, the surface brightness is a conserved quantity along the light path of gravitational lensing.

When there's no lensing, our observed surface brightness at a distance equals to that we observe close to the source. If we place a lens object along the light path, our observed surface brightness behind the lens also equals to that we observe close to the source. Therefore, the existence of the lens do not change our observed surface brightness, which means the surface brightness of source and image is exactly the same.

It has many interesting deductions. For example, the measured specific intensity of the sun's surface would not change if the sun were to be moved to twice its current distance. This is usually explained as the cancellation of the increased size (as distance squared) of the patch within the observer's beam and the decreased radiation (inverse square of distance) received from each element of the patch. Nevertheless, it is a consequence of Liouville's theorem!

2.5 Relativity Connection

The observer of Liouville's theorem follows the particles even as they are guided along curved paths by magnetic fields. This observer would thus be in a noninertial (i.e., accelerating) frame of reference to which the Lorentz transformations of special relativity do not apply. However, one can adopt an inertial (constant velocity) frame of reference that, at some moment, is instantaneously at rest with respect to the particles. In this case, the Lorentz transformations would apply.

Here we discuss how phase-space density f transforms from this moving inertial frame to the laboratory (stationary) frame. The number of particles within phase volume element dN is Lorentz invariant. And we have $dN = f d^3x d^3p$. Assume particles travel in the x direction. Lorentz transformation in special relativity tells us

$$dx = \gamma^{-1} dx'$$

$$dp_x = \gamma(dp'_x + vdE'/c^2) = \gamma dp'_x$$

Then $d^3xd^3p = d^3x'd^3p'$ is Lorentz invariant. And thus the phase-space density f is also invariant!

In general relativity, d^3xd^3p is the product of displacement dual vector(dx_i) and momentum vector($p^i = m(\frac{\partial}{\partial\tau})^i$). Therefore, d^3xd^3p is a covariant scalar. So the phase-space density f is Lorentz invariant.

Since the Lorentz transformation for frequency ν is simple, we can easily calculate the surface brightness and flux in the relativistic case!

2.6 Cosmological dimming

Surface Brightness $B_\nu = I_\nu d\Omega$. Combining the angular size and flux-density relations thus gives the cosmological version of surface-brightness conservation.:

$$B_\nu \propto \frac{I_\nu}{1+z} \propto \frac{\nu^3 f}{1+z} \propto \frac{f}{(1+z)^4}$$

The phase-space density f is conserved along the light path. Therefore, the surface brightness is dimmed by a factor of $(1+z)^{-4}$. This is the famous cosmological dimming!

3 Geometrical Method

3.1 observed Brightness

Before our proof, let's review the definition of observed brightness. As a limiting case, suppose a pitch of the source's surface has an area of S(S→ 0), and its solid angle related to observer is $\Delta\Omega$. The area of the receiver in observer plane is A. So only the light within the solid angle can reach the observer. And it is shown in Figure 1.

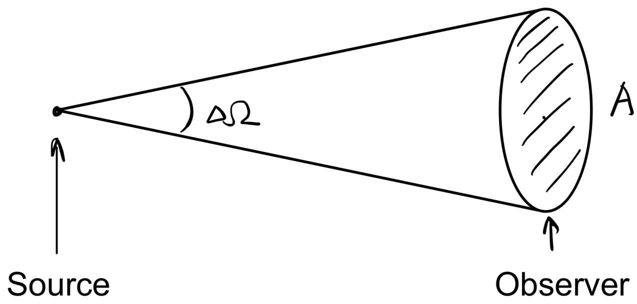

Figure 1: Observed brightness

Assume the total luminosity of this pitch of source is I, and its radiation is isotropic. So the received flux is $F = \frac{\Delta\Omega}{4\pi} I$. And the brightness is:

$$B = \lim_{A \to 0} \frac{F}{A} = \lim_{A \to 0} \frac{\Delta\Omega I}{4\pi A} \tag{2}$$

While $'I'$ is an intrinsic character of the source and independent of the track of the light, the observed brightness only relies on $\Delta\Omega$. If there is a lense lies between the source and observer, the light doesn't travel along a line, and the solid angle of the light at the source that can reach the observer is changed to $\Delta\Omega'$, as shown in Figure2. According to equation (2), the magnification rate is:

$$\mu = \frac{B'}{B} = \frac{\Delta\Omega'}{\Delta\Omega} \tag{3}$$

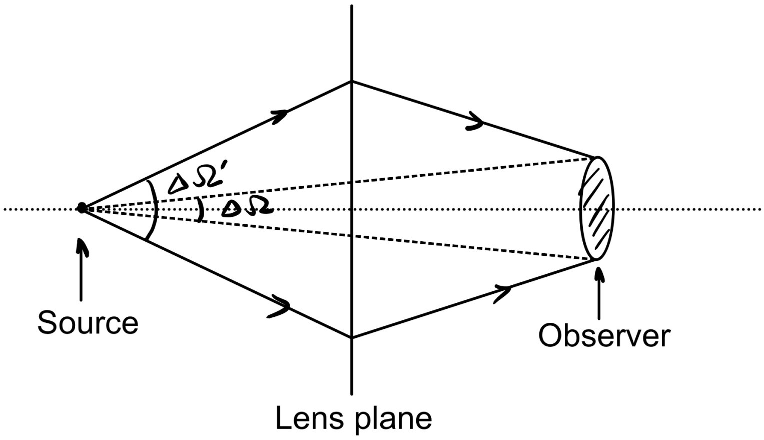

Figure 2: light track with lens

(in this case the $\Delta\Omega$ is enlarged and hence luminosity is magnified)

Then the core part of this proof is the calculation of $\Delta\Omega'$.

3.2 Situation in 2D

First, let's begin our calculation with 2-D situation. The magnification rate of brightness then is $\mu = \frac{\Delta u'}{\Delta u}$, where Δu is the plane angle, while in 3-D case it's solid angle $\Delta\Omega$. Despite that it's a little different with our 3-D situation, it's much more explicit and clear, and may help as to understand the magnification better. So just take it as our beginning.

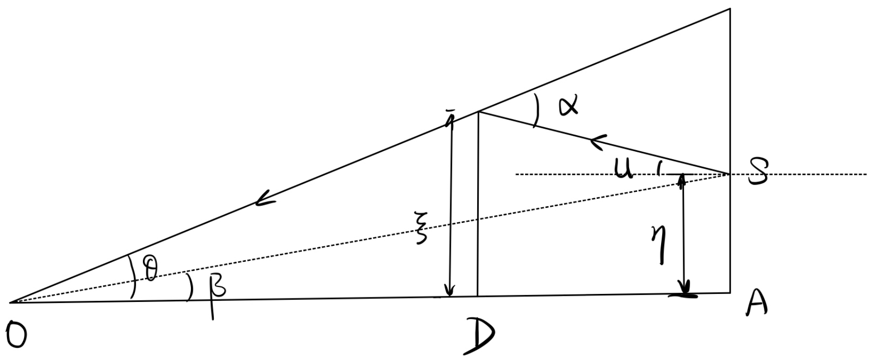

Figure 3: Gravitational Lensing Geometry

As shown in Figure 3, S is the source, O is the observer and D is the lensing object which makes the light curved. $D_d, D_s$and D_d are the distance to the lens (deflector), distance to the source and distance between the lens (deflector) and the source. η is the source position (distance perpendicular to the line connecting the observer and the lens), ξ is the image position,and α is the deflection angle. β, θ, u is shown in graph. And we can immediately get the some relations from geometry:

$$u = \alpha - \theta \tag{4}$$

$$\xi = D_d\theta, \quad \eta = D_s\beta \tag{5}$$

The result got from general relativity is:

$$\alpha = \frac{4GM}{c^2\xi} = \frac{4GM}{c^2 D_d}\frac{1}{\theta} \tag{6}$$

where M is the mass of lensing object.

From(4)(5)(6), we have:

$$u(\theta) = \frac{4GM}{c^2 D_d}\frac{1}{\theta} - \theta \tag{7}$$

And there is a characteristic angle θ_E, which is the angular Einstein radius:

$$\theta_E = \sqrt{\frac{4GMD_{ds}}{c^2 D_d D_s}} \tag{8}$$

And the lens equation for a point lens in angles is:

$$\beta + \frac{\theta_E^2}{\theta} = \theta \tag{9}$$

Since the gravitational force is a central force, the curve of light must lie on one plane, and the lense object is also in such plane. Thus point O, S, D and the light curve are all in one plane no matter what situation it is.

Now let us suppose the observer is not a single point but has some stretch in this plane. As shown in Figure 4, the observer is a line segment OO' with a length of $D_d\delta$ ($\delta \to 0$). Then there is a little of changes in geometry. And if we compare it with our standard situation is Figure 3, it's equivalently to have some variation in η and β. And only the light within the field angel of ray 1 and ray 2(shown in graph) can reach observer. So the Δu mentioned above is the field angle of this two ray.

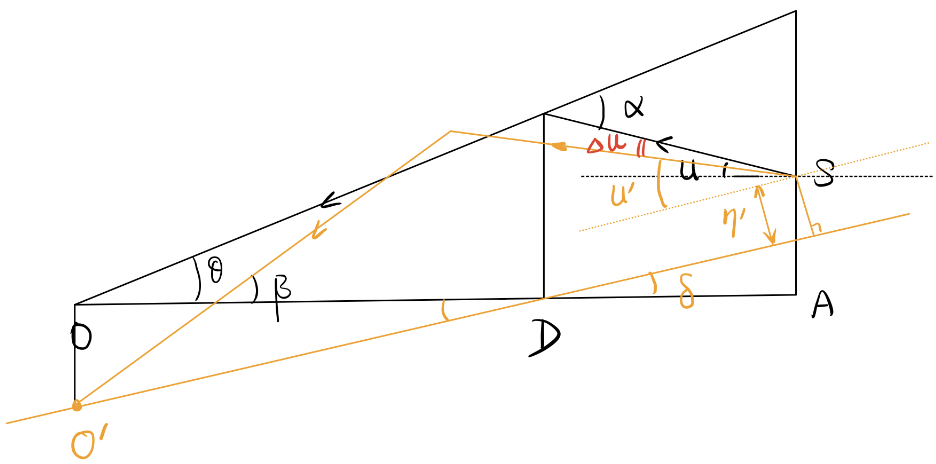

Figure 4: Observer stretch in 2D plane

Then we have:

$$d\eta = \eta' - \eta = -D_{ds}\delta \tag{10}$$

Differentiate eq.(7) and eq.(9) and notice that $\theta_E^2 = \frac{4GMD_{ds}}{c^2 D_d D_s}$:

$$du = -(1 + \frac{4GM}{c^2 D_d}\frac{1}{\theta^2})d\theta = -(1 + \frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta^2})d\theta \tag{11}$$

$$d\theta = (1 + \frac{\theta_E^2}{\theta^2})^{-1} d\beta \tag{12}$$

since $d\beta = D_s d\eta$, we can get the relation between du and $d\eta$:

$$du = -(1 + \frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta^2})(1 + \frac{\theta_E^2}{\theta^2})^{-1}\frac{d\eta}{D_s} \tag{13}$$

substitute $d\eta$ and we have:

$$du = \frac{(\frac{D_{ds}}{D_s} + \frac{\theta_E^2}{\theta^2})}{(1 + \frac{\theta_E^2}{\theta^2})}\delta$$

Finally we have:

$$\Delta u' = |u' - u - \delta| = \delta - du = (1 + \frac{\theta_E^2}{\theta^2})^{-1}\frac{D_d}{D_s}\delta \tag{14}$$

if there is no lens, the field angle Δu should be:

$$\Delta u = \frac{D_d \delta}{D_s}$$

So the magnification rate is:

$$\mu = \frac{\Delta u'}{\Delta u} = (1 + \frac{\theta_E^2}{\theta^2})^{-1} \tag{15}$$

3.3 Situation in 3D

In 3D situation, the receiver stretches in a plane. Assume it's a disc with radius of $D_d\delta$ ($\delta \to 0$). For a point named O' at the rim of the disc, the geometry relation also changes as in 2D case for this point observer. But as proved in section 2.2.2, O'(point observer), D(lense), S(source) and the light reaches O' are all in a plane. So the key point is to find this plane and the difference in geometry parameters. The graph of this situation shows below, and the observer is not shown here. The meaning of η, η' and other parameters are in accord with Figure 3 and Figure 4.

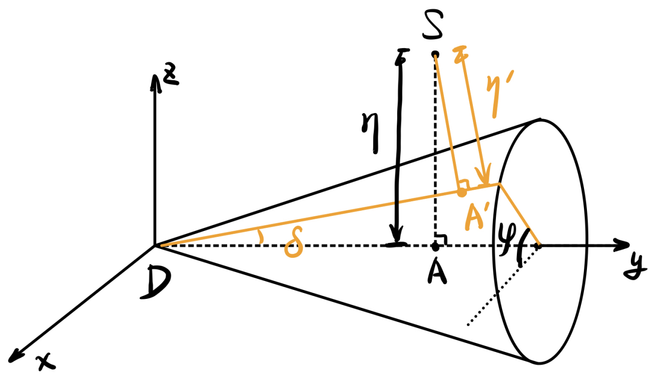

Figure 5: Geometry in 3D plane

We use vectors to calculate the geometrical relations. For simplicity, we scale all length parameters with D_{ds}. Namely, in our calculation we let D_{ds} to be 1, and η is scaled to $\tilde{\eta} = \eta/D_{ds}$. But for convenience, we still use η to stand $\tilde{\eta}$. The we have:

$$\overrightarrow{DA} = (0, 1, 0)$$

$$\overrightarrow{DS} = (0, 1, \eta)$$

We assume the distance between point A' and D is a. Then we have:

$$\overrightarrow{DA'} = (a\cos\varphi\sin\delta, a\cos\delta, a\sin\varphi\sin\delta)$$

$$\overrightarrow{SA'} = \overrightarrow{DA'} - \overrightarrow{DS} = (a\cos\varphi\sin\delta, a\cos\delta - 1, a\sin\varphi\sin\delta - \eta)$$

Notice that $\overrightarrow{SA'}$ and $\overrightarrow{DA'}$ are perpendicular, $\overrightarrow{SA'} \cdot \overrightarrow{DA'} = 0$, then we can get the value of a:

$$a = \frac{\cos\delta + \eta\sin\varphi\sin\delta}{\cos^2\delta + \sin^2\varphi\sin^2\delta} = 1 + o(\delta)$$

In further calculation, we find that $o(\delta)$ has no contributions. So we can safely view a as 1. And we calculate the length of $\overrightarrow{SA'}$:

$$|\overrightarrow{SA'}| = \eta' = \eta - \sin\varphi\sin\delta + o(\delta^2)$$

We keep the same definition of u as above(u is also very small), and we can get the vector of the ray of light at the source which reaches point O' (the point at the rim of source disc):

$$\begin{aligned} \overrightarrow{n} &= \frac{\overrightarrow{A'S}}{|\overrightarrow{A'S}|}\sin u + \frac{\overrightarrow{A'D}}{|\overrightarrow{A'D}|}\cos u && (16) \\ &= (-(\frac{u}{\eta} + 1)\delta\cos\varphi, 1, u - \sin\varphi\delta) && (17) \end{aligned}$$

For the light that reaches the center of source disc, the vector is:

$$\overrightarrow{n} = (0, 1, u_0) \qquad (18)$$

the difference vector:

$$\overrightarrow{\Delta n} = \overrightarrow{n} - \overrightarrow{n_0} = (-(\frac{u}{\eta} + 1)\delta\cos\varphi, 0, u - u_0 - \sin\varphi\delta) \qquad (19)$$

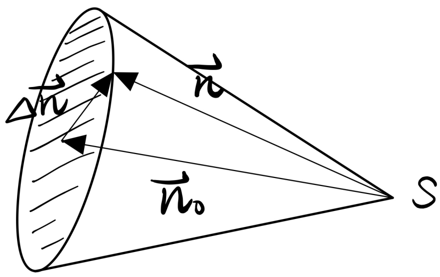

Figure 6: Light cone at the source

From eq.(4), eq.(13), we have:

$$u - u_0 = du = -(1 + \frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta^2})(1 + \frac{\theta_E^2}{\theta^2})^{-1}\frac{d\eta}{D_s} \tag{20}$$

$$= (1 + \frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta^2})(1 + \frac{\theta_E^2}{\theta^2})^{-1}\frac{D_{ds}}{D_s}\sin\varphi \cdot \delta \tag{21}$$

$$u/\eta = (\alpha - \theta)/\eta = (\frac{D_s}{D_{ds}}\frac{\theta_E^2}{\theta} - \theta)/\eta \tag{22}$$

$$= (\frac{\theta_E^2}{\theta} - \frac{D_{ds}}{D_s}\theta)/\beta \tag{23}$$

$$= -(\frac{\theta_E^2}{\theta} - \frac{D_{ds}}{D_s}\theta)(\frac{\theta_E^2}{\theta} - \theta)^{-1} \tag{24}$$

and we get:

$$\overrightarrow{\Delta n} = (\frac{D_d/D_s}{\theta_E^2/\theta^2 - 1}\delta\cos\varphi, 0, \frac{-D_d/D_s}{\theta_E^2/\theta^2 + 1}\delta\cos\varphi) \tag{25}$$

So the solid open angel is:

$$\Delta\Omega' = \frac{1}{(\theta_E^2/\theta^2 - 1)(\theta_E^2/\theta^2 + 1)}\left(\frac{D_d}{D_s}\delta\right)^2 \tag{26}$$

If there isn't a lens, the solid open angel is:

$$\Delta\Omega = \left(\frac{D_d}{D_s}\delta\right)^2 \tag{27}$$

We finally get the magnification rate:

$$\mu = \frac{\Delta\Omega'}{\Delta\Omega} = \frac{1}{(\theta_E^2/\theta^2 - 1)(\theta_E^2/\theta^2 + 1)} \tag{28}$$

The most frequently used expression of magnification rate is given by:

$$\mu = \frac{dr \times r\Delta\phi}{dr_{\mathrm{s}} \times r_{\mathrm{s}}\Delta\phi} = \frac{r}{r_{\mathrm{s}}}\frac{dr}{dr_{\mathrm{s}}} = \frac{\theta}{\beta}\frac{d\theta}{d\beta} \tag{29}$$

and we have:

$$\beta = \theta - \frac{\theta_E^2}{\theta} = (1 - \frac{\theta_E^2}{\theta^2})\theta$$

$$d\beta = (1 + \frac{\theta_E^2}{\theta^2})d\theta$$

if we substitute β and $d\beta$ in eq.(29) ,we find that eq.(29) is equivalent to eq.(28)

As shown above, we presented a pure geometrical way to derive the magnification rate of a point source in gravitational lensing. And it could also serve as a proof of the conservation of surface brightness in such system.

4 Non-calculation Method

Here we will use no mathematical calculation to prove the conservation of surface brightness in gravitational lensing.

This proof is mainly based on two very interesting facts in gravitational lensing.

First, the image and the object lie in the same plane.Second, there is a swap-symmetry for the source and observer in the lens equation(in length version, not angular version. You'll see it's important in swapping), which means $r_E = \sqrt{\frac{4GMD_{ds}D_d}{D_s}}$ is unchanged if we swap the position of source and observer.

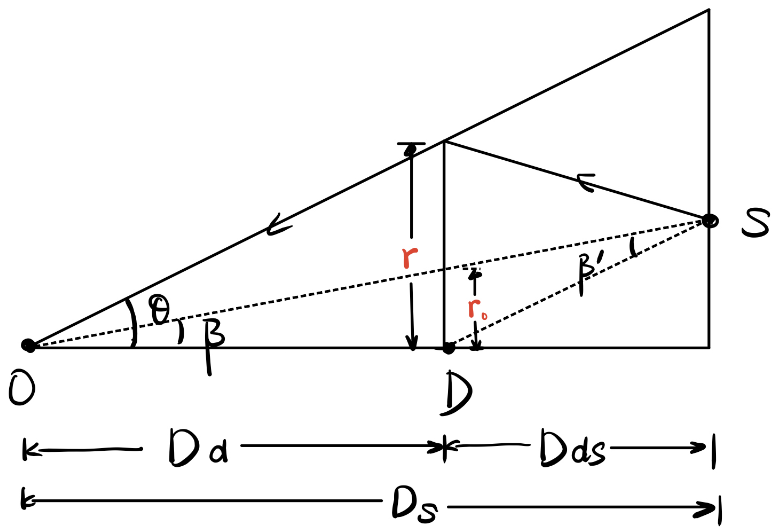

Figure 7: Projection to lens plane

If we project all length parameters to the lens plane. As shown in Figure 7. r is the projected distance to the lens object of image and r_0 is that of source. The lensing equation is:

$$r^2 - r_0 r - r_E^2 = 0$$

When we swap source and observer, while the angular position of source varies ($\beta' = \frac{D_d}{D_{ds}}\beta$),the r_E is unchanged(as illustrated above). The r_0 does not change as well, because it's the distance from the intersection point of the source and observer to the lens. Therefore, according to the lensing equation, the projected image position is unchanged. So the magnification rate of the area of the source is the same as before. Namely, if the observer finds the area of the source is magnified μ_a times, another observer, if there is, at the position of the source will also find the area of the former observer being enlarged μ_a times! According to the natural definition of magnification(as demonstrated in section 3, the magnification rate is proportional to the opening angle of the observer to the source), we get the conclusion:

The magnification of an image equals the ratio of the image area and source area.

References

[1] Bradt, H. (2008). Equations of state. In Astrophysics Processes: The Physics of Astronomical Phenomena (pp. 87-116). Cambridge: Cambridge University Press.

doi:10.1017/CBO9780511802249.004

[2] Misner CW, Thorne KS, Wheeler JA. Gravitation. Macmillan; 1973 Sep 15.

[3] Bartelmann, M. and Schneider, P., 2001. Weak gravitational lensing. Physics Reports, 340(4-5), pp.291-472.

[4] Mao, S., 2008. Introduction to gravitational microlensing. arXiv preprint. arXiv: 0811.0441.

[5] Bartelmann, M., 2010. Gravitational lensing. Classical and Quantum Gravity, 27(23), p.233001.

[6] Schneider, P., Kochanek, C. and Wambsganss, J., 2006. Gravitational lensing: strong, weak and micro: Saas-Fee advanced course 33 (Vol. 33). Springer Science Business Media.

[7] Narayan, R. and Bartelmann, M., 1996. Lectures on gravitational lensing. arXiv preprint astro-ph/9606001.

[8] Treu, T., 2010. Strong lensing by galaxies. Annual Review of Astronomy and Astrophysics, 48, pp.87-125.