Topology
Lecture notes

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February 26, 2024

Contents

1 Lecture 1

Definition:
For a metric space (X,d), a set $U \subseteq X$ is **open** if for every $a \in U$, there is an open r-ball $a \in B(x;r) \subseteq U$, where
$$B(x;r) := \{y \in X : d(x,y) < r\}.$$

Theorem 1.1: Open sets in a metric space

For a metric space X.

1. $\varnothing$ and X are open sets.
2. Every union of open sets is open.
3. Every finite intersection of open sets is open.

Definitions:

1. For a metric space X, a set $F \subseteq X$ is **closed** if $X \setminus F$ is open.
2. The **closed** r**-ball** is
$$\overline{B}(a;r) := \{y \in X : d(a,y) \leq r\}.$$
3. The **sphere** is
$$S(a;r) := \{x \in X : d(a,x) = r\}$$

Note:

1. The closure of an open ball is not always the closed ball (for certain metrics/topologies).
2. $S(a;r)$ is a closed set.

Theorem 1.2: Exercise

For a set X with metrics d_1, d_2, assume there exist constants $c, c' > 0$ such that
$$cd_1(x,y) \leq d_2(x,y) \leq c'd_1(x,y)$$
for all $x, y \in X$. Then, the open sets for (X,d_1) and (X,d_2) are the same.

Definition:
A sequence (x_n) in X converges to $x \in X$ if
$$\lim_{n\to\infty} d(x_n, x) = 0.$$

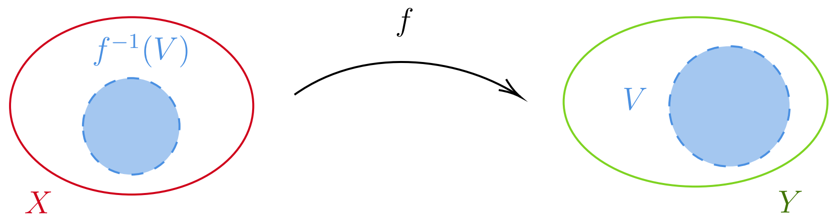
Definition:
$$d_p(x,y) = \left(\sum_i |x_i - y_i|^p\right)^{1/p}$$
$$d_\infty(x,y) = \max_{i \leq i \leq n} |x_i - y_i|.$$

2 Lecture 2

Theorem 2.1: Characterization of continuity

Let (X, d_X), (Y, d_Y) be metric spaces, and $f : X \to Y$ be a map. The following are equivalent.

1. f is continuous.
2. For any $V \subseteq Y$ open, the set $f^{-1}(V)$ is open.

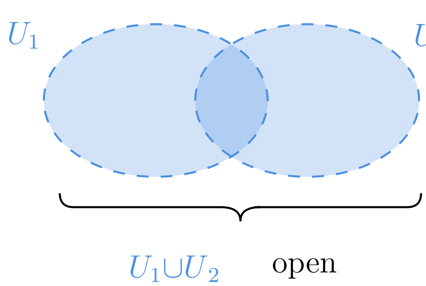


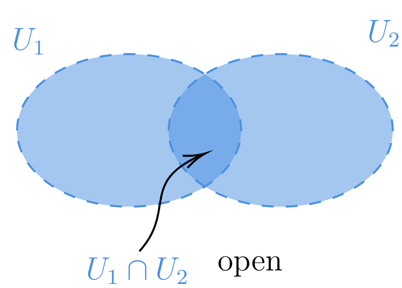
Definition: a topology

Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X such that

1. $\varnothing, X \in \mathcal{T}$.
2. An *arbitrary* union of elements of $\mathcal{T}$ is in $\mathcal{T}$, i.e. if $U_\alpha \in \mathcal{T}$ for each $\alpha \in I$, then
$$\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$$
3. A *finite* intersection of elements of $\mathcal{T}$ is in $\mathcal{T}$, i.e. if $U_1, U_2, \dots, U_n \in \mathcal{T}$, then
$$\bigcap_{k=1}^{n} U_k \in \mathcal{T}$$
.

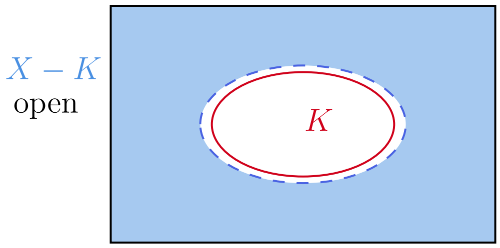
These subsets are called the **open sets**, and $(X, \mathcal{T})$ is a **topological space**.



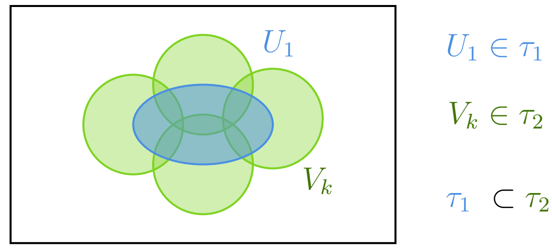


Definitions: closed sets and comparing topologies

1. A set is **closed** if its complement is open, i.e. for a topological space X, a set $F \subset X$ is closed if $X - F$ is open.
2. Consider a set X with topologies τ_1, τ_2. If $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$ then they are comparable. If $\tau_1 \subset \tau_2$, then τ_1 is **coarser** than τ_2, or equivalently, τ_2 is **finer** than τ_1.



(1)



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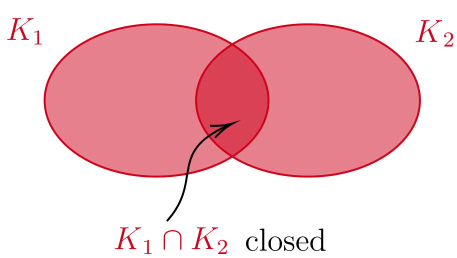
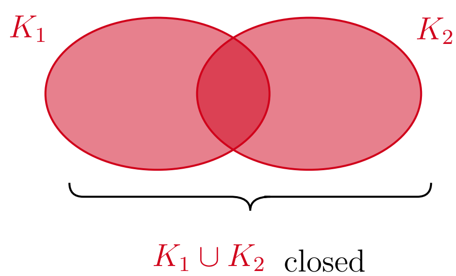
Theorem 2.2: Closed sets

For a topological space X.

1. $X, \varnothing$ are closed sets.
2. Arbitrary intersection of closed sets are closed.
3. Finite union of closed sets are closed.



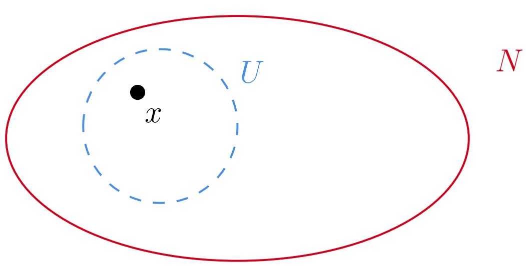


Definitions: neighborhoods

For a topological space X and a point $x \in X$, a set $N \subset X$ is a **neighborhood** of x if there exists an open set U such that

$$x \in U \subset N.$$

Note: a neighborhood is not necessarily open.



Theorem 2.3: Properties of neighborhoods

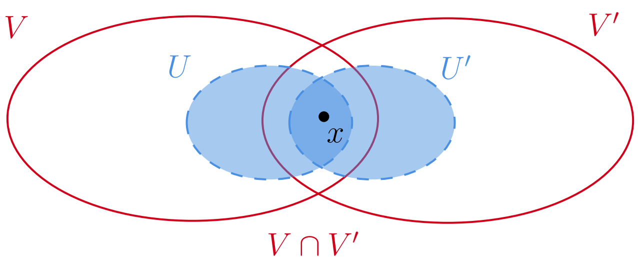
Let X be a topological space, with $x \in X$. Then,

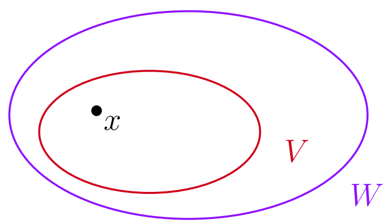
1. If V, V' are neighborhoods of x, then $V \cap V'$ too is a neighborhood of x.
2. If V is a neighborhood of x, and $V \subset W$, then W is a neighborhood of x.

Proof sketch: For (1), if $x \in U \subset V$, and $x \in U' \subset V'$, where U, U' are open, then

$$x \in U \cap U' \subset V \cap V',$$

because U and U' are open.





Theorem 2.4: Open sets and neighborhoods

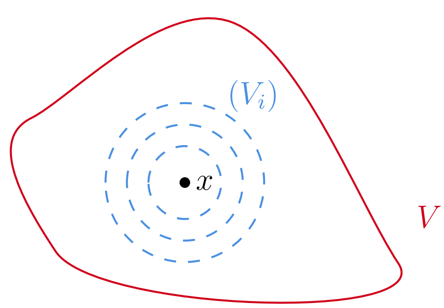
For a topological space X, with a subset $A \subset X$. The following are equivalent.

1. A is open.
2. A is a neighborhood of each point in A.

Definition: Fundamental system of neighborhoods

For a topological space X, a **fundamental system of neighborhoods** of $x \in X$ is a family (V_i) of neighborhoods of x such that every neighborhood of x contains a V_i, i.e. for every neighborhood of x, there exists an $i \in I$ such that

$$x \in V_i \subset V.$$



Example:
Let $X = \mathbb{R}$, $x \in \mathbb{R}$, and define

$$B_n := B\left(x; \frac{1}{n}\right)$$

with $n \in \mathbb{Z}_{\geq 0}$. Then, $B_1, B_2, \ldots$ is a fundamental system of neighborhoods of x. This is because if V is a neighborhood of X, then there exists an open set U such that $x \in U \subset V$. Then, since U is an open set in $\mathbb{R}$, we use the metric topology and

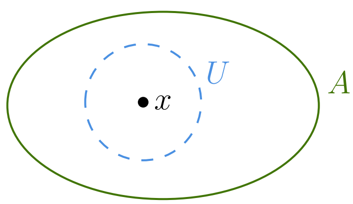
$$\exists r > 0 : x \in B(x; r) \subset U \subset V.$$

If $n > 1/r$, then $x \in B(x; 1/n) \subset B(x; r) \subset V$. Thus any given neighborhood of $x \in \mathbb{R}$ must include one of these B_n.

Definition: Interior points

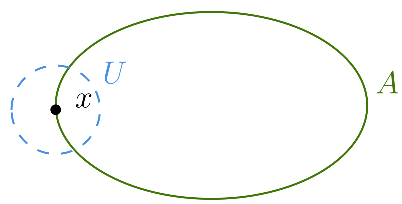
A point $x \in X$ is an **interior point** of A if A itself is a neighborhood of x, i.e. there exists an open set $U \subset A$ such that $x \in U$. The set of interior points of A is called the **interior** of A, written

$$\operatorname{int}(A) \text{ or } \mathring{A}.$$



$U \subset A$

$x \in \operatorname{int} A$



$U \not\subset A$

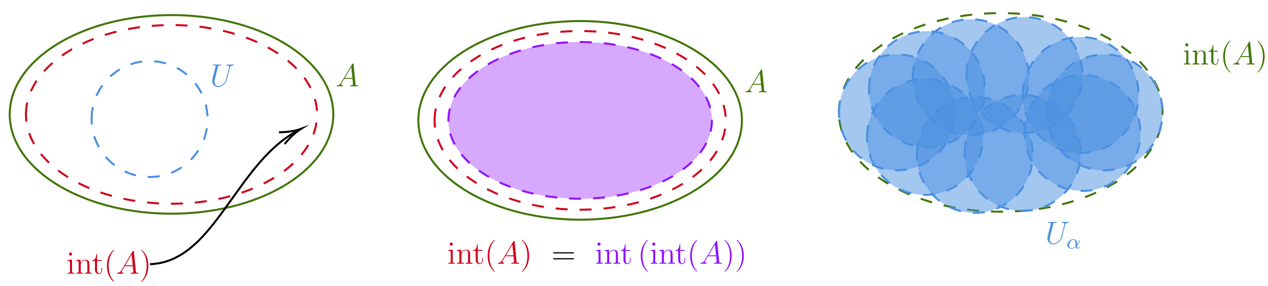
$x \notin \operatorname{int} A$

3 Lecture 3

Theorem 3.1: Properties of interior

For a topological space X, and a subset $A \subset X$.

1. $\text{int}(A)$ is the largest open set contained in A, i.e. if $\text{int}(A) \subset U \subset A$ with U open, then
$$\text{int}(A) = U.$$
2. $$\text{int}(\text{int}(A)) = \text{int}(A).$$
3. $$\text{int}(A) = \bigcup \{U \subset A : \ U \text{ open}\}.$$



Proof sketch:
(2) and (3) follow from (1). To prove (1), show that $\text{int}(A)$ is open, and $\text{int}(A)$ contains every open subset of A.

Observation: As a result of (1), if $U \subset A$, and U is open, then
$$U \subset \text{int}(A).$$

Theorem 3.2: Open sets and the interior

For a topological space X and a subset $A \subset X$. The following are equivalent.

1. A is open.
2. $A = \text{int}(A)$.

Proposition: properties of interior

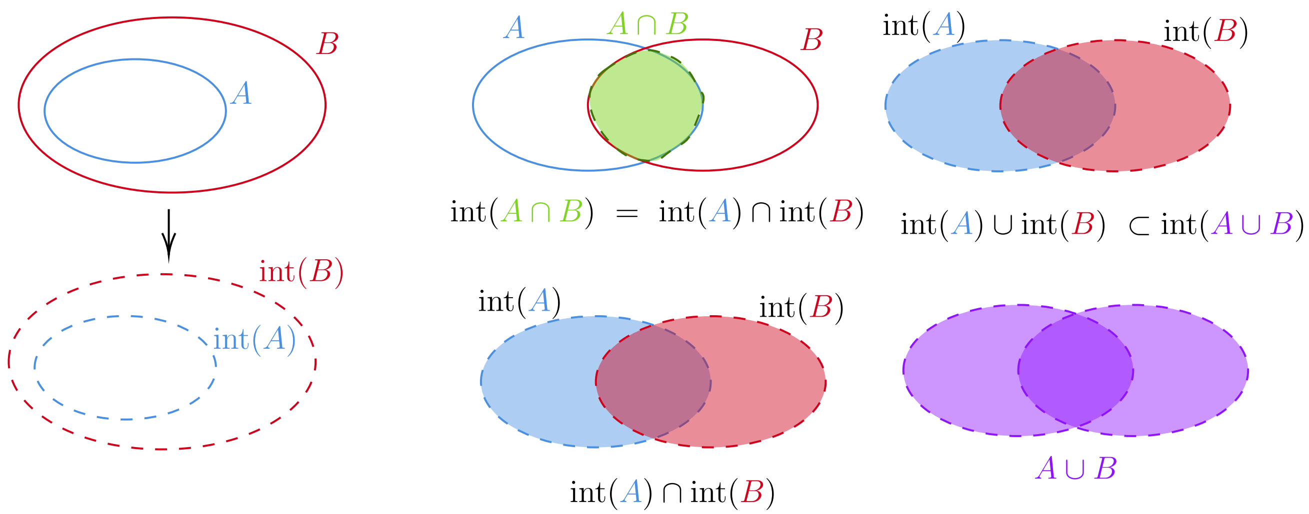
For a topological space X, and subsets $A, B \subset X$.

1. $$\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A).$$

2. $$A \subset B \implies \operatorname{int}(A) \subset \operatorname{int}(B).$$

3. $$\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B).$$

4. $$\operatorname{int}(A) \cup \operatorname{int}(B) \subset \operatorname{int}(A \cup B).$$



Definition: exterior

For a topological space X, with a subset $A \subset X$. A point $x \in X$ is **exterior to** A if it is interior to $X - A$. The **exterior** of A is the set of exterior points. Then,

$$\operatorname{ext}(A) := \operatorname{int}(X - A),$$

i.e. "the exterior of A is the interior of the complement". Likewise,

$$\operatorname{ext}(X - A) = \operatorname{int}(A).$$

Definition: boundary

The sets $\text{ext}(A)$, $\text{int}(A)$ are disjoint. The **boundary** of A, ∂A, is the complement of this union:

$$\partial A = X - (\text{int}(A) \sqcup \text{ext}(A)).$$

This means we have a partition of X as follows.

$$X = \text{int}(A) \sqcup \partial A \sqcup \text{ext}(A).$$

Definition: closure

For a topological space X and a subset $A \subset X$, the **closure of** A (denoted $\overline{A}$ or $c\ell(A)$) is the set of all points adherant to A, where $x \in X$ is **adherant** to A if A meets every neighborhood of x.
Remark: We have

$$A \subset \overline{A}$$

because for each $x \in A$, if U is a neighborhood of x,

$$\{x\} \subset U \cap A \neq \varnothing.$$

Theorem 3.3: Relation between closure, exterior, boundary

For a topological space X, and a subset $A \subset X$,

$$\begin{aligned}\overline{A} &= X - \text{ext}(A)\\ &= \text{int}(A) \sqcup \partial A\\ &= A \cup \partial A\end{aligned}$$

Theorem 3.4: Closure and closed sets

For a topological space X, and a subset $A \subset X$,

1. $\overline{A}$ is the *smallest closed set* containing A.
2. $\overline{(\overline{A})} = \overline{A}$.
3. $\overline{A} = \bigcap \{A \subset F : F \text{ closed}\}$.

Note that we always have $A \subset \overline{A}$.

Remark:

$$\overline{A} = \partial A \cup \text{int}(A)$$

and

$$\partial A = \overline{A} \cap \overline{X - A}.$$

Observation:

$$\partial A = \varnothing \iff A \text{ is both open and closed.}$$

and

$$A \text{ is closed} \iff \partial A \subset A.$$

Theorem 3.5: Summary of closure properties

For subsets $A, B \subset X$ of a topological space X,

1. $$\overline{\left(\overline{A}\right)} = \overline{A}.$$

2. $$A \subset B \implies \overline{A} \subset \overline{B}.$$

3. $$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

4. $$\overline{A \cap B} = \overline{A} \cap \overline{B}.$$

4 Lecture 4

Theorem 4.1: Archimedean property of $\mathbb{R}$

Let $\varepsilon > 0$, then for any $M \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that

$$n\varepsilon > M.$$

Proposition:
For any $x \in \mathbb{R}$ and for all $\varepsilon > 0$,

$$(x - \varepsilon, x + \varepsilon) \cap \mathbb{Q} \neq \varnothing.$$

Theorem 4.2: supremum in $\mathbb{R}$

Let $A \subseteq \mathbb{R}$, with A bounded above. Then,

$$\sup(A) = \max\left(\overline{A}\right).$$

Definition: convergence in a topological space

A sequence (x_n) of points in a topological space is said to **converge** to $x \in X$ if for each open neighborhood U of x, there exists an $N \in \mathbb{N}$, such that

$$n \geq N \implies x_n \in U.$$

Remark:
In a metric topological space, if (x_n) converges with respect to the metric space, it also converges in the topological definition, because by definition of open sets in a metric space, the open balls are contained in the open set.

Theorem 4.3: Convergent sequences in a topological space

If $S \subset X$ for a topological space X, and (x_n) converges to $x \in X$, then

$$x \in \overline{S}.$$

4.1 Bases

Definition: Base

For a set X, a **base** for a topology on X is a collection $\mathcal{B} \subset \mathcal{P}(X)$ such that

1. $$\bigcup_{B \in \mathcal{B}} B = X$$
   i.e.
   $$\forall x \in X : \exists B \in \mathcal{B} : x \in B.$$
2. If $B_1, B_2 \in \mathcal{B}$, with $x \in B_1 \cap B_2$, then there exists $B \in \mathcal{B}$ such that
   $$x \in B \subset B_1 \cap B_2.$$

Examples:

1. For a metric space (X, d),
   $$\mathcal{B} = \{B(x; \varepsilon) : x \in X, \varepsilon > 0\}$$
   is a base.
2. For a set X,
   $$\{\varnothing, X\} \text{ is a topology.}$$
   Examples of a base for this topology could be
   $$\mathcal{B} = \mathcal{P}(X)$$
   because it is the set of all subsets, or
   $$\mathcal{B} = \{\{x\} : x \in X\}$$
   because it clearly covers all of X and
   $$\{x\} \cap \{y\} = \varnothing \text{ or } \{x\}.$$
   In either case, condition (2) is met.

Definition: topology generated by a base

If $\mathcal{B}$ is a base on X, then the **topology generated** by $\mathcal{B}$, denoted by $\mathcal{T}(\mathcal{B})$, is defined as

$$U \in \mathcal{T}(\mathcal{B}) \iff \forall x \in U : \exists B \in \mathcal{B} : x \in B \subset U.$$

We see that

$$\mathcal{B} \subset \mathcal{T}(\mathcal{B}).$$

Proposition: $\mathcal{T}(\mathcal{B})$ is a topology.

Proposition

For a set X, and a base $\mathcal{B}$, the topology generated by $\mathcal{B}$ can be written as the set of all unions of elements of $\mathcal{B}$, i.e.

$$\mathcal{T}(\mathcal{B}) = \mathcal{T}_U := \left\{ \bigcup_{\alpha \in I} B_\alpha : B_\alpha \in \mathcal{B} \right\}$$

Corollary

$\mathcal{T}(\mathcal{B})$ is the smallest topology containing $\mathcal{B}$. (Because any topology containing $\mathcal{B}$ must contain $\mathcal{T}_U = \mathcal{T}(\mathcal{B})$.

Remark:
A base for a topological space is not unique, and the decomposition is not unique. For example, you could generate $\mathbb{R}^2$ using either open disks or open rectangles. They both give the same metric topology.

Theorem 4.4: Comparing bases

Let $\mathcal{B}, \mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively, i.e. $\mathcal{T} = \mathcal{T}(\mathcal{B})$, $\mathcal{T}' = \mathcal{T}(\mathcal{B}')$. The following are equivalent,

1. $\mathcal{T}'$ is finer than $\mathcal{T}$, i.e.
$$\mathcal{T} \subset \mathcal{T}'$$

2. For each $x \in X$ and for each $B \in \mathcal{B}$ such that $x \in B$, there exists $B' \in \mathcal{B}'$ such that
$$x \in B' \subset B$$

3. $$\mathcal{B} \subset \mathcal{T}' = \mathcal{T}(\mathcal{B}').$$

5 Lecture 5

Definition: fundamental system of open sets

A **fundamental system of open sets** for X is a collection $\mathcal{C}$ of open sets which forms simultaneously a fundamental system of neighborhoods of every point of X.

Exercise: Let X be a topological space, and $\mathcal{C}$ be a collection of open sets. Then, $\mathcal{C}$ is a base for the topology of X *if and only if* $\mathcal{C}$ is a fundamental system of open sets.

Definition: subbase

For a set X, with $A \subset \mathcal{P}(X)$, define $\mathcal{T}(A)$ to be the smallest topology on X containing A, i.e. the intersection of all topologis containing A.

$$\mathcal{T}(A) := \bigcap_{A \subset \mathcal{T}_\alpha} \mathcal{T}_\alpha.$$

Then, A is a **subbase** for $\mathcal{T}(A)$.

Proposition

Let $\mathcal{B}$ be the set of all finite intersections of elements of A, plus $\varnothing$ and X, i.e.

$$\mathcal{B} := \left\{ \bigcap_{k=1}^{n} S_k \ : \ S_1, \ldots, S_n \in A \right\} \cup \{\varnothing, X\}.$$

Then, $\mathcal{B}$ is a base, and

$$\mathcal{T}(A) = \mathcal{T}(\mathcal{B})$$

Proof sketch: (B_1) is satisfied since $X \in \mathcal{B}$. Also, (B_2) is true because given

$$S_1 \cap \cdots \cap S_n \in \mathcal{B}$$

$$T_1 \cap \cdots \cap T_n \in \mathcal{B}$$

It follows that

$$S_1 \cap \cdots \cap S_n \cap T_1 \cap \cdots \cap T_n \in \mathcal{B}.$$

Definition: filter

For a set X, a **filter** on X is a collection $\mathcal{F} \subset \mathcal{P}(X)$ such that

1. $\mathcal{F}$ does not include the empty set, i.e.
$$\varnothing \notin \mathcal{F}$$
2. If $A, B \in \mathcal{F}$ then
$$A \cap B \in \mathcal{F}$$
3. If $A \in \mathcal{F}$ and $A \subset B$, then
$$B \in \mathcal{F}$$

Definition: filter base

A **filter base** is a collection $\mathcal{F} \subset \mathcal{P}(X)$ such that

1. $\mathcal{F}$ does not include the empty set,
$$\varnothing \notin \mathcal{F}$$
2. If $A, B \in \mathcal{F}$ then there exists $C \in \mathcal{F}$ such that
$$C \subset A \cap B$$

Remarks:

1. If $\mathcal{F}$ is a filter, then $\mathcal{F}$ is a filter base.
2. A base for a topological space is a filter base.

Examples:

1. The set of all neighborhoods of a point forms a filter base.
2. A fundamental system of neighborhoods forms a filter base.
3. If $X = \mathbb{R}$, and $x_0 \in \mathbb{R}$, then
$$\big\{(x_0 - \varepsilon, x_0 + \varepsilon) \ : \ \varepsilon > 0\big\}$$
is a filter base.
4. All of the following are filter bases:

$$\{[x_0, x_0 + \varepsilon) \ : \ \varepsilon > 0\} \tag{1}$$
$$\{(x_0, x_0 + \varepsilon) \ : \ \varepsilon > 0\} \tag{2}$$
$$\{(x_0 - \varepsilon, x_0] \ : \ \varepsilon > 0\} \tag{3}$$
$$\{[x_0 - \varepsilon, x_0) \ : \ \varepsilon > 0\} \tag{4}$$
$$\{(x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon) \ : \ \varepsilon > 0\} \tag{5}$$
$$\{[a, \infty) \ : \ a \in \mathbb{R}\} \tag{6}$$
$$\tag{7}$$

Definition: convergence

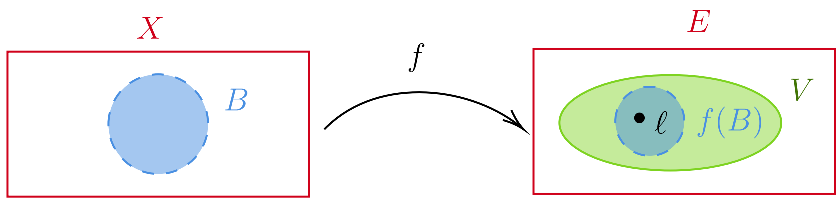
Let X be a set, E be a topological space, and $f : X \to E$ be a map. Further, let $\mathcal{B}$ be a filter base on X, with $\ell \in E$. Then, define

$$\lim_{\mathcal{B}} f = \ell,$$

if for all neighborhoods V of ℓ, there exists $B \in \mathcal{B}$,

$$f(B) \subset V,$$

i.e. "f tends to ℓ along $\mathcal{B}$".



Notation for sequences

If (x_n) is a sequence in a topological space, we write

$$\lim_{n\to\infty} x_n = \ell$$

for convergence, i.e. in a metric space E,

$$\lim_{n\to\infty} x_n = \ell \iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, \ell) \leq \varepsilon.$$

Definition: more general convergence

More generally, for a topological space X and a set $Y \subset X$ with $x_0 \in \overline{Y}$, if

$$\{Y \cap V \ : \ V \text{ is a neighborhood of } x_0\}$$

is a filter, then we write

$$\lim_{\substack{x\to x_0\\ x\in Y}} f(x)$$

for the limit of a function.

Theorem 5.1: Convergent sequences and fundamental systems of neighborhoods

Let X, E be topological spaces, $f : X \to E$ be a map, and $x_0 \in X$, $\ell \in E$. Consider $(W_i)_{i \in I}$ a fundamental system of neighborhoods of x_0, and $(V_j)_{j \in I}$ a fundamental system of neighborhoods of ℓ, then the following are equivalent

1. $$\lim_{x \to x_0} f(x) = \ell.$$

2. for all $j \in I$, there exists $i \in I$ such that
$$f(W_i) \subset V_j.$$

Proof sketch (of (1) $\implies$ (2)**):** if

$$\lim_{x \to x_0} f(x) = \ell$$

then there exists W a neighborhood of x_0 such that

$$f(W) \subset V_j.$$

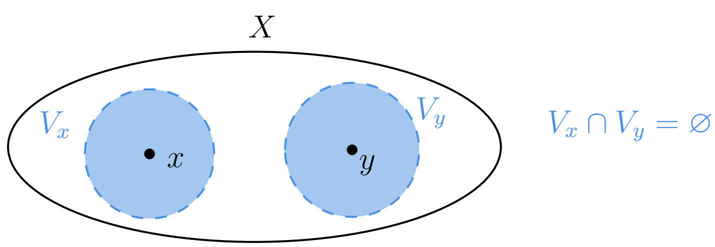
Thus there exists $i \in I$ such that

$$W_i \subset W \implies f(W_i) \subset V_j.$$

Definition: Hausdorff

A topological space is **Hausdorff** (or separated), if any 2 distinct points are contained in disjoint neighborhoods, i.e. for any $x, y \in X$, with $x \neq y$, there exist neighborhoods V_x, V_y of x and y such that

$$V_x \cap V_y = \varnothing.$$



Examples:

1. If X is a metric space then X is Hausdorff, because if $x \neq y$ for some $x, y \in X$, then let $\varepsilon := d(x, y)$ and observe that
$$B\left(x; \frac{\varepsilon}{2}\right) \cap B\left(y; \frac{\varepsilon}{2}\right) = \varnothing.$$

2. If X is a discrete topological space then it is Hausdorff, because for any $x, y \in X$ with $x \neq y$, then
$$x \in \{x\}, y \in \{y\}$$
so
$$\{x\} \cap \{y\} = \varnothing.$$

Theorem 5.2: Singletons are closed sets in a Hausdorff space

Consider a topological space X such that X is Hausdorff. Take a point $x \in X$. Then, $\{x\}$ is closed.

Proof: If $y \in X - \{x\}$, then since X is Hausdorff, there exist neighborhoods $V_x, V_y \subset X$ such that $x \in V_x$, $y \in V_y$, where

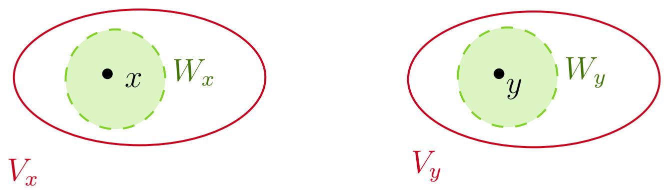
$$V_x \cap V_y = \varnothing,$$

and

$$x \in W_x \subset V_x$$

$$y \in W_y \subset V_y$$

for some open sets W_x, W_y.



Then, $x \notin W_y$ since $W_y \cap W_x = \varnothing$. Therefore, for all $y \in X - \{x\}$, there exists an open W_y such that

$$W_y \subset X - \{x\},$$

i.e. $X - \{x\}$ is open.

$\square$

Exercise: Let $f : X \to E$ be a map, where X is a set with a filterbase $\mathcal{B}$, and E is a Hausdorff topological space. Show that if f has a limit along $\mathcal{B}$, then the limit is unique.

6 Lecture 6

6.1 Limits and continuity

Recall: for a map $f : X \to E$, where E is a topological space, let $\mathcal{B}$ be a filter base on X. If $\ell \in E$ then

$$\lim_{\mathcal{B}} f = \ell$$

if and only if for any neighborhood V of ℓ in E, there exists $B \in \mathcal{B}$ such that

$$f(B) \subset V$$

Definition: limit towards a point

For topological spaces X, E, a map $f : X \to E$ and $\ell \in E$, $x_0 \in X$.

$$\lim_{x \to x_0} f(x) = \ell$$

if and only if for all neighborhoods V of ℓ in E, there exists a neighborhood W of x_0 such that

$$f(W) \subset V.$$

Theorem 6.1: Limits are unique in a Hausdorff space

Let $f : X \to E$ be a map, where E is a topological space and X is a set. For a filter base $\mathcal{B}$ on X, if E is Hausdorff then the limit, if it exists, is unique.

Proof: Let ℓ, ℓ' be two limits along $\mathcal{B}$, then there exist V, V' disjoint neighborhoods of ℓ and ℓ' such that

$$V \cap V' = \varnothing$$

and

$$\ell \in V, \quad \ell' \in V'.$$

In particular then, there exist $B, B' \in \mathcal{B}$ such that

$$f(\mathcal{B}) \subset V$$

and since $\mathcal{B}$ is a filter base, there exists $B'' \in \mathcal{B}$ such that

$$B'' \subset B \cap B'$$

Then,

$$f(B'') \subset f(B \cap B') \subset f(B) \cap f(B') \subset V \cap V' = \varnothing.$$

Thus,

$$\ell = \ell'.$$

Definition: limit towards a point

Let X be a set, $\mathcal{B}$ be a filter base on X, and E be a topological space. Finally, let

$$f : X \to E$$

be a map. Then, $\ell \in E$ is an **adherance value of f along $\mathcal{B}$** if for each neighborhood V of ℓ,

$$\forall B \in \mathcal{B} : f(B) \cap V \neq \varnothing.$$

CHECK ($= \varnothing$ or $\neq \varnothing$)
Note:

- $f(B)$ is a neighborhood of $f(x_0)$.
- $\ell \in f(B)$.
- This set is related to the closure.

Example:
Let $X = \mathbb{N}$. Then, ℓ is an adherance value for (x_n) iff for each neighborhood V of ℓ,

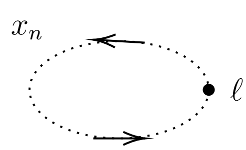
$$\forall N \in \mathbb{N} : \exists n \geq N : x_n \in V$$

i.e.

$$\forall \varepsilon > 0 : \forall N \in \mathbb{N} : \exists n \geq N : d(x_n, \ell) \leq \varepsilon.$$

Here, if $B = \{n, n+1, \dots\}$, then $f(B) = \{x_n, x_{n+1}, \dots\}$.
Notes:

- Pay attention to the quantifiers.
- Adherance is weaker than convergence.
- It tells you that the sequence keeps coming back to the point.



Example: The sequence

$$\begin{cases} x_{2n} = \frac{1}{2n} \\ x_{2n+1} = 1 - \frac{1}{2n+1} \end{cases}$$

does not have a limit, but 0 and 1 are adherance values.

Theorem 6.2: Adherance values and limits

Let
$$f : X \to E$$
be a map, for a set X and Hausdorff space E, and let $\ell \in E$. If
$$\lim_{\mathcal{B}} f = \ell$$
then ℓ is the unique adherance value of f along $\mathcal{B}$.

Proof: proven.

Theorem 6.3

Let $f : X \to E$ be a map. Then, the set of adherance values of f along $\mathcal{B}$ is
$$\bigcap_{B \in \mathcal{B}} \overline{f(B)}$$

6.2 Continuous maps

Definition: continuous map

Let X and Y be topological spaces, and
$$f : X \to Y$$
be a map. Then, f is **continuous at** $x_0 \in X$ if and only if
$$\lim_{x \to x_0} f(x) = f(x_0)$$
i.e. for each neighborhood W of $f(x_0)$, there exists a neighborhood V of x_0 such that
$$f(V) \subset W.$$

Example: If X, Y are metric spaces, then
$$f : X \to Y$$
is continuous at x_0 if
$$\forall \varepsilon > 0 : \exists \delta > 0 : d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon.$$

Proposition: composition of continuous maps is continuous

If f is continuous at x_0, and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof: If W is a neighborhood of $(g \circ f)(x_0) = g(f(x_0))$, then there exists a neighborhood V of $f(x_0)$ such that
$$g(V) \subset W,$$

and also there exists a neighborhood V' of x_0 such that

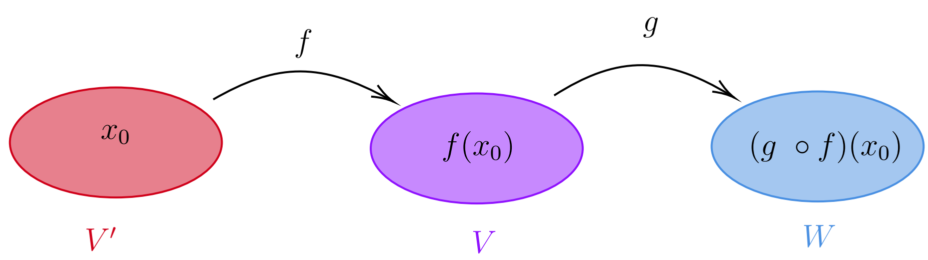
$$f(V') \subset V.$$

Thus,

$$g(f(V')) \subset g(V),$$

i.e.

$$(g \circ f)(V') \subset W.$$



Example: If

$$\lim_{n\to\infty} x_n = a$$

then

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(a)$$

if and only if f is continuous.

Proposition: composition over a filter base

If

$$\lim_{\mathcal{B}} f = \ell$$

and g is continuous at ℓ, then

$$\lim_{\mathcal{B}} g \circ f = g\left(\lim_{\mathcal{B}} f\right) = g(\ell)$$

Proof: Exercise!

Definition: continuous over a whole space

Consider a map

$$f : X \to Y$$

between topological spaces X, Y. Then, f is **continuous** if and only if f is continuous at every $x \in X$. Further, we can define

$$\mathcal{C}(X,Y) := \left\{ f : X \to Y \ : \ f \text{ is continuous} \right\}$$

Theorem 6.4: Properties of continuous functions

Let $f : X \to Y$ be a map of topological spaces. The following are equivalent.

1. f is continuous.
2. $f^{-1}(V)$ is open for all V open in Y.
3. $f^{-1}(K)$ is closed for all K closed in Y.
4. for all $A \subset K$, $f\left(\overline{A}\right) \subset \overline{f(A)}$.

Proof: Given. Shown

- (1) $\implies$ (4),
- (4) $\implies$ (3),
- (3) $\implies$ (2),
- (2) $\implies$ (1).

FILL

7 Lecture 7

7.1 Continuous functions

Proposition

A map $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X for all U open in Y.

Proof: Given last lecture.

Definition: open & closed maps

- If $f(U)$ is open for all open $U \subset X$, then f is called an **open map**.
- If $f(K)$ is closed for all closed $K \subset X$, then f is called a **closed map**.

Theorem 7.1

Let $f : X \to Y$ be a bijection. The following are equivalent.

1. f, f^{-1} are continuous.
2. U is open if and only if $f(U)$ is open.
3. K is closed if and only if $f(K)$ is closed.

Definition: homeomorphisms

A map $f : X \to Y$ is a **homeomorphism** if f, f^{-1} are continuous and f is bijective. If such an f exists, then X and Y are **homeomorphic**.

Example: $(0, 1)$ is homeomorphic to (a, b) for any real $a < b$.

7.2 Constructing topological spaces

7.2.1 Subspace topology

Definition

Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$. Define

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}.$$

Proposition

$\mathcal{T}_Y$ is a topology on Y, called the **subspace topology**.

Proof: First we observe that $\varnothing = \varnothing \cap Y$, so

$$\varnothing \in \mathcal{T}_Y$$

as we wanted. Next, observe that $Y = Y \cap Y$, so

$$Y \in \mathcal{T}_Y$$

which is also what we wanted to check because $\mathcal{T}_Y$ is a topology on Y, so Y must be an open set. Next, we check that intersections of open sets are open. Let $V_1, \dots, V_n \in \mathcal{T}_Y$, then there exist $U_1, \dots, U_n \in \mathcal{T}$ such that

$$V_i = U_i \cap Y.$$

Thus,

$$\begin{aligned} \bigcap_{i=1}^{n} V_i &= \bigcap_{i=1}^{n} (U_i \cap Y) \\ &= Y \cap \left(\bigcap_{i=1}^{n} U_i \right) \in \mathcal{T}_Y. \end{aligned}$$

This is what we wanted to show.

Lemma: base of a subspace topology

If $\mathcal{B}$ is a base for the topology on X, with $Y \subset X$, then

$$\mathcal{B}_Y := \{B \cap Y : B \in \mathcal{B}\}$$

is a base for some subspace topology.

Proof: If U is open in X, and $y \in U \cap Y$, then there exists $B \in \mathcal{B}$ such that

$$y \in B \subset U$$

$$\implies y \in B \cap Y \subset U \cap Y$$

where $B \cap Y \in \mathcal{B}_Y$, and $U \cap Y$ is open in Y.

Theorem 7.2

Let X be a metric space with $Y \subset X$. Then, $U \subset Y$ is open in Y if and only if $U = V \cap Y$ for some open subset $V \subset X$.

Note that the metrics over X and Y are

$$d : X \times X \to \mathbb{R}$$

$$d' : Y \times Y \to \mathbb{R}$$

So if U is an open ball in Y, it is also an open ball in X. Next, let's try to construct a topological space. For any ball $B(x; r) \subset X$,

$$B(x; r) \cap Y \subset Y$$

is open in Y. Thus, for any open set $U \subset Y$ (open in Y), take $y \in U$, then there exists $r_y > 0$ such that

$$y \in B(y; r_y) \subset U.$$

Thus,

$$U = \bigcup_{y \in U} B(y; r_y)$$

Note: In general, we have the following. If $Y \subset X$ and Y is open, with a subset $A \subset Y$. If A is open in X, then A is open in Y.

Theorem 7.3

Let $Y \subset X$. Then, $A \subset Y$ is closed in Y if and only if $A = K \cap Y$ for some K closed in X.

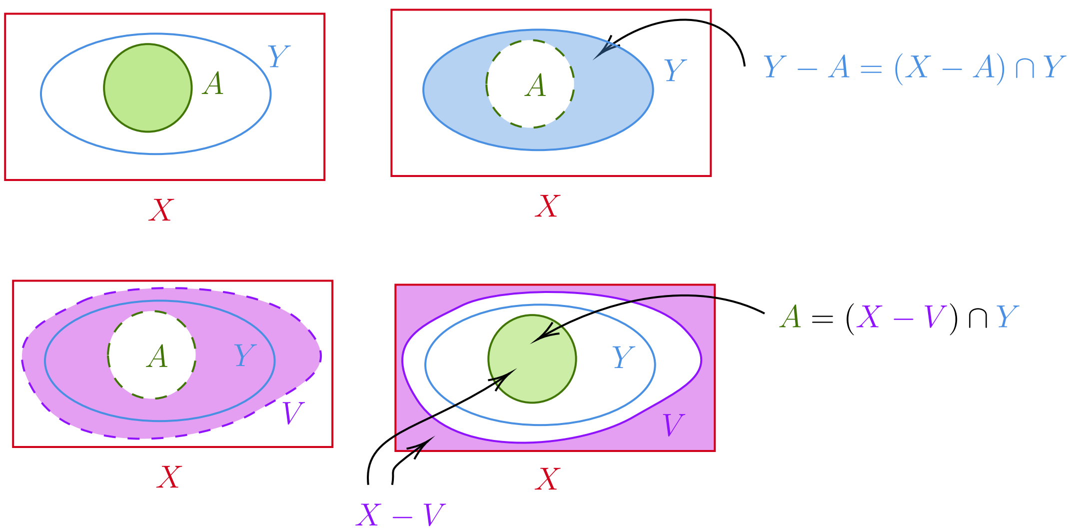
Proof: $(\rightarrow)$

If A is closed in Y, then $Y - A$ is open in Y. Thus, there exists V open in X such that

$$Y - A = V \cap Y. \tag{8}$$

Note further that

$$Y - A = (X - A) \cap Y.$$



So (8) becomes

$$(X - A) \cap Y = V \cap Y.$$

Thus,

$$A = (X - V) \cap Y,$$

by the picture and $X - V$ is closed in X because $V \subset X$ is open.

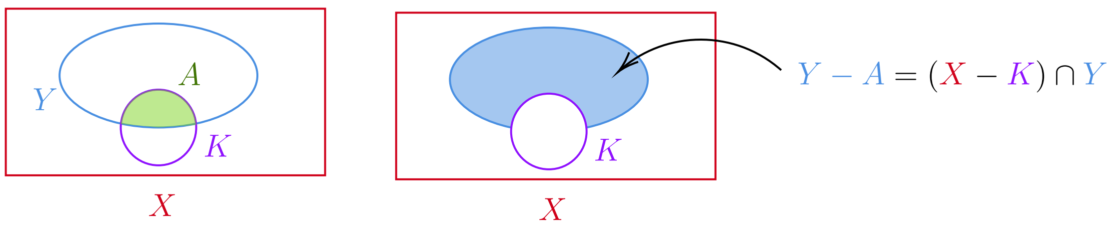
$(\leftarrow)$ If there exists K closed in X such that

$$A = K \cap Y$$

then we want to show that $Y - A$ is open. Observe that

$$\begin{aligned} Y - A &= Y - (K \cap Y) \\ &= Y - K \\ &= (X - K) \cap Y \end{aligned}$$

which is open in Y. Thus, $A \subset Y$ is closed in Y.



7.2.2 Product topology

Definition

Consider $X \times Y$. An **elementary open set** in $X \times Y$ is a set of the form $U \times V$ such that U is open in X and V is open in Y.

Definition

The **product topology** is the topology on $X \times Y$ generated by elementary open sets.

Proposition

The elementary open sets form a base.

Note:

$$\begin{aligned}(U_1 \times V_1) \cap (U_2 \times V_2) &= (U_1 \cap U_2) \times (V_1 \cap V_2) \\ &= \{(x, y) \ : \ x \in U_1, U_2, \ y \in V_1, V_2\}\end{aligned}$$

From this, the proposition follows.

Definition

Define

$$\begin{aligned}\pi_x : X \times Y &\to X \\ (x, y) &\mapsto x\end{aligned}$$

and

$$\begin{aligned}\pi_y : X \times Y &\to Y \\ (x, y) &\mapsto y.\end{aligned}$$

These are the **projection maps**.

Proposition

The projection maps π_x, π_y are continuous.

Proof: Take any U open in X. Then,

$$\pi_x^{-1}(U) = U \times Y$$

this is an elementary open set, and so is open. Likewise for π_y.

Proposition

Let $\mathcal{B}$ be the set of elementary open sets. Then, $\mathcal{T}(\mathcal{B})$ is the smallest topology for which π_x, π_y are continuous.

Proof: Let $\mathcal{T}$ on $X \times Y$ be another topology such that π_x, π_y are continuous. Then, for all U open in X,

$$\pi_x^{-1}(U) \in \mathcal{T}.$$

Take any such open $U \subset X$ and $V \subset Y$, then

$$\begin{aligned} U \times V &= (U \cap X) \times (V \cap Y) \\ &= (U \times Y) \cap (X \times V) \\ &= \pi_x^{-1}(U) \cap \pi_y^{-1}(V). \end{aligned}$$

Then, since π_x, π_y are continuous, it follows that $\pi_x^{-1}(U)$ and $\pi_y^{-1}(V)$ are both open, and so

$$U \times V \in \mathcal{T}.$$

Finally, since $\mathcal{B} \subset \mathcal{T}$, it also follows that

$$\mathcal{T}(\mathcal{B}) \subset \mathcal{T}$$

i.e. $\mathcal{T}(\mathcal{B})$ is smaller than $\mathcal{T}$.

Exercise: Prove that π_x, π_y are open maps.

8 Lecture 8

8.1 More on subspace and product topologies

Example: Consider the metric space $\mathbb{R}^2$. Then,

$$d_1, d_2, d_\infty$$

are metrics. They are equivalent, i.e. generate the same metric topology. But also, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ could inherit the product topology from those on $\mathbb{R}$.

Question: How does this product topology from $\mathbb{R}$ compare with the metric topology in $\mathbb{R}^2$?

We have to compare the bases. Call $\mathcal{T}_\pi$ and $\mathcal{T}_d$ the product and metric topologies, respectively. The base for $\mathcal{T}_d$ is the set of open balls, and the base for $\mathcal{T}_\pi$ is the set of open rectangles. In general, for a product topology, the base is the set of elementary open sets,

$$\{A \times B \ : \ A, B \text{ are open in respective factors}\}.$$

So

$$\mathcal{T}_\pi = \mathcal{T}(\mathcal{B})$$

where

$$\begin{aligned} \mathcal{B} &= \{U \times V \ : \ U, V \text{ are open in } \mathbb{R}\} \\ &= \{(a,b) \times (c,d) \ : \ a, b, c, d \in \mathbb{R}\} \end{aligned}$$

Note that the set of open squares in $\mathbb{R}^2$ is a subset of the set of open rectangles in $\mathbb{R}^2$, i.e.

$$\mathcal{T}_{d_\infty} \subset \mathcal{T}_\pi$$

because rectangles can be used to generate squares.

8.2 Product of metric spaces

Let $X_1, \ldots, X_n$ be metric spaces, and let

$$X = \prod_{i=1}^{n} X_i.$$

Define on $X \times X$, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,

$$d(x,y) = \left(\sum_{i=1}^{n} d_i(x_i, y_i)^2 \right)^{1/2}.$$

Note: If $X = \mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$, this is d_2.
Exercise: Prove that the following metrics are equivalent.

$$d(x,y) = \max_{1 \le i \le n} d(x_i, y_i)$$

$$d(x,y) = \sum_{i=1}^{n} d(x_i, y_i).$$

Question: With

$$X = \prod_{i=1}^{n} X_i,$$

how does the product metric space $\mathcal{T}_d$ compare to the product topological space $\mathcal{T}_\pi$?

Proposition

$$\mathcal{T}_d = \mathcal{T}_\pi.$$

Proof: Let $\mathcal{B}_d$ and $\mathcal{B}_\pi$ be the bases for $\mathcal{T}_d$ and $\mathcal{T}_\pi$ respectively, where $\mathcal{B}_d$ is the set of open balls, and $\mathcal{B}_\pi$ is the set of products of base elements. We want to show that

$$\mathcal{T}(\mathcal{B}_d) = \mathcal{T}(\mathcal{B}_\pi)$$

Note that

$$\forall B \in \mathcal{B}_d : \forall x \in B : \exists B' \in \mathcal{B}_\pi : x \in B' \subset B.$$

This is equivalent to

$$\mathcal{T}(\mathcal{B}_d) \subset \mathcal{T}(\mathcal{B}_\pi).$$

Why? Let $B := B_d(x; \varepsilon)$. If

$$y \in B\left(x_1; \frac{\varepsilon}{n}\right) \times \cdots \times B\left(x_n; \frac{\varepsilon}{n}\right),$$

then

$$\begin{aligned} d(x,y)^2 &= \sum_{i=1}^{n} d(x_i, y_i)^2 < \sum_{i=1}^{n} \left(\frac{\varepsilon}{n}\right)^2 = \frac{\varepsilon^2}{n} \\ \iff d(x,y) &< \frac{\varepsilon}{\sqrt{n}} \end{aligned}$$

So

$$y \in B_d\left(x; \frac{\varepsilon}{\sqrt{n}}\right) \subset B_d(x; \varepsilon)$$

i.e.

$$\mathcal{B}_d \subset \mathcal{B}_\pi.$$

The reverse inclusion proceeds in the same way. Observe that

$$B(x_1; \varepsilon_1) \times \cdots \times B(x_n; \varepsilon_n) \in \mathcal{B}_\pi.$$

Consider $B(x; \varepsilon)$, where $\varepsilon = \min_{1 \le i \le n} \varepsilon_i$. If $y \in B(x; \varepsilon)$, then

$$d(x_i, y_i) < d(x, y) < \varepsilon$$

So

$$\sum_{i=1}^{n} d(x_i, y_i)^2 < \sum_{i=1}^{n} d(x,y)^2 < n\varepsilon^2$$

From this, we conclude

$$\mathcal{B}_\pi \subset \mathcal{B}_d.$$

Theorem 8.1: Product topology = subspace topology

Consider $X \times Y$, with subsets $A \subset X$, $B \subset Y$. We could have $A \times B$ with the product topology, or

$$A \times B \subset X \times Y$$

as the subspace topology. Both topologies are the same.

Proof: Consider $U \times V$ an elementary open set in $X \times Y$. Then,

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Therefore, the topologies are the same.

8.3 More on continuous functions

Theorem 8.2

Let $f : A \to X \times Y$ be a map given by

$$f(a) := (f_1(a), f_2(a))$$

for all $a \in A$. The maps f_1, f_2 are the "coordinates" of f. Then, f is continuous if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Note: $f_1 : A \to X$ is precisely

$$f_1 = \pi_1 \circ f$$

where $\pi_1 : X \times Y \to X$ is the projection map.
Remark: We can also write

$$f = (f_1 \times f_2) \circ \Delta$$

where

$$\begin{aligned} \Delta : A &\to A \times A \\ a &\mapsto (a, a) \end{aligned}$$

and

$$\begin{aligned} f_1 \times f_2 : A \times A &\to X \times Y \\ (x, y) &\mapsto (f_1(x), f_2(y)) \end{aligned}$$

Proof of theorem: ($\rightarrow$) Note that π_1, π_2 are continuous. So if f is continuous, then $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

($\leftarrow$) If $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$ are continuous, then let $U \times V$ be an elementary open set, and observe that

$$\begin{aligned} a \in f^{-1}(U \times V) &\iff f(a) \in U \times V \\ &\iff f_1(a) \in U \text{ and } f_2(a) \in V \\ &\iff a \in f_1^{-1}(U) \text{ and } a \in f_2^{-1}(V) \\ &\iff a \in f_1^{-1}(U) \cap f_2^{-1}(V). \end{aligned}$$

Therefore,

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V).$$

This completes the proof with the following exercise.

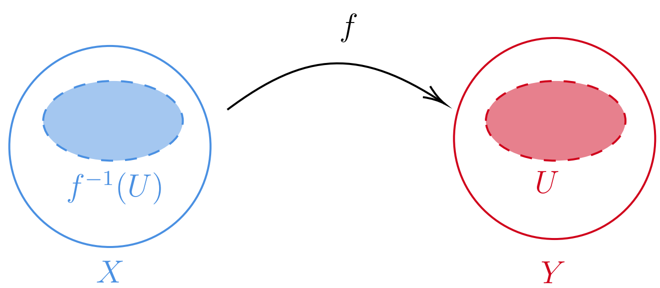
Exercise: If $f^{-1}(U)$ is open in X whenever U is open in Y for U in a sub-base then $f : X \to Y$ is continuous.

8.4 Quotient Topology

Definition

Let $f : X \to Y$ be a map. Assume f is surjective. We call f a **quotient map** if

$$U \subset Y \text{ is open} \iff f^{-1}(U) \text{ is open.}$$



Definition: quotient topology

The **quotient topology** is precisely the unique topology for which a surjective map $X \to Y$ becomes a quotient.

Theorem 8.3

Let X be a topological space, and let R be an equivalence relation on X. Let X/R be the quotient set. Then, $\pi : X \to X/R$ is a quotient map when the topology on X/R is given by

$$\mathcal{T}_{X/R} = \left\{ A \subset X/R \ : \ \pi^{-1}(A) \text{ is open in } X \right\}$$

and π is continuous.

Note:

$$X/R = \{[x]_R \ : \ x \in X\}$$

is the set of equivalence classes under the relation R.

Example: Let $X = \mathbb{R}$ as a topological space. Let $\sim$ be an equivalence relation on $\mathbb{R}$ given by

$$a \sim b \iff a - b \in \mathbb{Z}.$$

Definition: 1D torus

Let $\mathbb{T} := \mathbb{R}/\sim$. This is sometimes denoted $\mathbb{R}/\mathbb{Z}$. This is the **1-dimensional torus**.

Note: For any $a \in \mathbb{R}$,

$$\begin{aligned}[a] &= a + \mathbb{Z} \\ &= \{a + b \ : \ b \in \mathbb{Z}\}\end{aligned}$$

So we want to consider $[0, 1)$ as a representative for $\mathbb{R}/\mathbb{Z}$.

Question: How does the product topology interact with the quotient topology?

Definition: n-D torus

Let $\mathbb{T}^n$ be the n-dimensional torus, i.e.

$$\mathbb{T}^n = (\mathbb{R}/\sim) \times \cdots \times (\mathbb{R}/\sim) = \mathbb{T} \times \cdots \times \mathbb{T}$$

or

$$\mathbb{T}^n = \mathbb{R}^n/\sim,$$

with

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} - \mathbf{y} \in \mathbb{Z}^n.$$

Exercise: compare the topologies (hint: compare boxes in $\mathbb{R}^2$ and generalize.

Example: Let

$$U = \{z \in \mathbb{C} : |z| = 1\}.$$

and define

$$\begin{aligned} g : \mathbb{R} &\to U \\ x &\mapsto e^{2\pi x}\end{aligned}$$

Note that g is surjective. Note also that

$$g(x) = g(y) \iff x - y \in \mathbb{Z}.$$

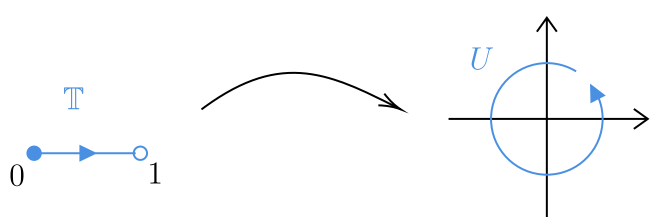
Finally, note that we can partition $\mathbb{R}$ into pre-images under g:

$$R = \bigcup_{c \in g(\mathbb{R})} g^{-1}(c)$$

The following diagram commutes.

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "g"] \arrow[dr, "\pi"'] & & U \arrow[dl, "\overline{g}"] \\ & \mathbb{T} & \end{tikzcd}$$

Intuition:



9 Lecture 9

Example: The torus $\mathbb{T} = \mathbb{T}^1$, i.e.

$$\alpha : X \to Y.$$

FILL

Claim: T is Hausdorff.

Proof: Given. FILL

9.1 Quotient Constructions

Definition topological group

A **topological group** is a group plus a topological space plus compatibility, i.e. the maps

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

and

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are continuous.

Then, if $H \subset G$ is a subgroup,

$$G/H = \{[g]_H \ : \ g \in G\}$$

such that

$$g \sim_H h \iff g^{-1}h \in H.$$

Further, a group action on a set

9.2 Compactness

10 Lecture 10

10.1 More on compactness

10.2 Metric spaces