THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Computer Exercise Week 7

STAT3023: Statistical Inference — Semester 2, 2021

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Consider the test of $H_0\colon \theta = \theta_0$ against $H_1\colon \theta \neq \theta_0$ based on $X \sim f(\cdot;\gamma_0,\theta)$ where

$$f(x;\gamma,\theta) = \begin{cases} \dfrac{x^{\gamma-1}e^{-x/\theta}}{\Gamma(\gamma)\theta^{\gamma}} & \text{for } x > 0 \\ 0 & \text{otherwise;} \end{cases} \tag{1}$$

that is X has a gamma distribution with known *shape* parameter γ_0 but unknown *scale* parameter θ.

1. Consider first the exponential case where $\gamma_0 = 1$ and suppose the hypothesised value of the scale parameter (also the mean in this case) is $\theta_0 = 1$.

(a) The "equal-tailed" test at level α rejects for $X \le a$ or $X \ge b$ where

$$P_1(X \le a) = P_1(X \ge b) = \frac{\alpha}{2}\,. \tag{2}$$

Taking $\alpha = 0.05$, determine values `a` and `b` satisfying (2) above (**hint:** use `qexp()`).

(b) We shall plot the power function of the equal-tailed test. Define a vector of θ-values: `th=(250:1500)/1000` and obtain a corresponding vector of values of the power (the probability of rejecting) for each such θ-value; that is

$$P_\theta(X \le a) + P_\theta(X \ge b)\,.$$

Finally plot the power against `th` and add a horizontal dashed line at $\gamma = 0.05$. Add an informative heading, etc.. Remember to use `type='l'`.

(c) This is a 1-parameter exponential family with sufficient statistic X and so (since it is continuous) the *uniformly most powerful unbiased* (UMPU) test is of the form

$$\delta(X) = \begin{cases} 1 & \text{for } X \ge d \\ 0 & \text{for } c < X < d \\ 1 & \text{for } X \le c \end{cases} \tag{3}$$

where c and d are chosen so that

$$E_{\theta_0}\left[\delta(X)\right] = \alpha \tag{4}$$

$$E_{\theta_0}\left[X\delta(X)\right] = \alpha E_{\theta_0}(X) = \alpha \tag{5}$$

since $E_{\theta_0}(X) = 1$. We show in a tutorial exercise these are equivalent to

$$\begin{aligned} 1 - e^{-c} + e^{-d} &= \alpha \\ ce^{-c} &= de^{-d}\,. \end{aligned}$$

From the first equation we get

$$\begin{aligned} e^{-d} &= \alpha - 1 + e^{-c} \\ d &= -\log\left[\alpha - 1 + e^{-c}\right]\,. \end{aligned}$$

so once c is determined we can compute d. To determine c we need to solve the equation

$$ce^{-c} - de^{-d} = ce^{-c} + \left\{\log\left[\alpha - 1 + e^{-c}\right]\right\}\left[\alpha - 1 + e^{-c}\right] = 0$$

We can use the R function `uniroot()` to determine c *numerically.*

(i) Write an R function of the form

```
fn=function(c,alpha) {
   ...
}
```

which computes the middle member of the equation above (i.e. the function whose root we wish to find).

(ii) Noting that c can be no bigger than the lower 0.05-quantile of the exponential(1) distribution, execute the command

```
eps=1e-5
uniroot(fn,lower=0,upper=qexp(0.05)-eps,alpha=0.05)
```

(the use of `eps` here is to stay away from the upper bound, since there the function is trying to evaluate `log(0)`).

(iii) Write an R function

```
expon.umpu=function(alpha) {
    ...
}
```

which takes as an argument the level `alpha` and returns a list with elements `$c` and `$d`, corresponding to the desired values c and d defining the UMPU test (3) above for $\theta_0 = 1$ and $\alpha = 0.05$.

(d) Recreate your plot from part (b) above and add to it a red curve of the power of the UMPU test. Add an informative heading and legend. **Comment** on what feature of the plot indicates that the UMPU test is unbiased. The power function of the UMPU test never goes below the 0.05 level, thus it is unbiased.

2. Consider now the case where $Y_1, \dots, Y_n$ are iid exponential with mean θ and we again wish to test $H_0 \colon \theta = \theta_0$ against a two sided $H_1 \colon \theta \neq \theta_0$. The likelihood is

$$\prod_{i=1}^{n} \left[\frac{1}{\theta} e^{-Y_i/\theta} \right] = \exp\left\{ -\frac{1}{\theta} \sum_{i=1}^{n} Y_i - n \log \theta \right\}$$

and so clearly $X = \sum_{i=1}^{n} Y_i$ is a sufficient statistic; indeed this is a 1-parameter exponential family. The UMPU test is thus of the form (3) where c and d are chosen to satisfy the two conditions (4) and (5).

However since X itself has a gamma distribution with shape parameter n and scale parameter θ, the UMPU above is the same as for a single observation from the density (1) above with $\gamma_0 = n$. We show in the tutorial that in the present case the two conditions (4) and (5) are equivalent to

$$\int_0^c f(x; n, 1)\, dx + \int_d^\infty f(x; n, 1)\, dx = \alpha = \int_0^c f(x; n+1, 1)\, dx + \int_d^\infty f(x; n+1, 1)\, dx\,.$$

Below we shall write a function to determine the UMPU test, for the case $n = 5$ and $\alpha = 0.05$.

(a) By adapting your solution to part (c) of the previous question, write a function

```
gamma.root=function(c,n,alpha) {
    ...
}
```

(playing the same role as the function `fn()` above; it will use `pgamma()` and `qgamma()`) the root of which gives the desired value of c to solve the above equations (**hint:** in the body of this function you will need to first find `d` in terms of `c` and `alpha` using one of the two constraints). Then use `uniroot()` to actually find the root. Wrap this all in a function

```
gamma.umpu=function(alpha,n) {
    ...
}
```

which takes as input values of `alpha` and `n` and outputs a list with elements `$c` and `$d`, the lower and upper critical values of the desired UMPU test.

(b) Use your `gamma.umpu()` function to determine the appropriate c and d for the UMPU test for this problem with $n = 5$ and $\alpha = 0.05$. Plot the power as a function of θ and graphically verify that the test is unbiased and of level 0.05.