THE UNIVERSITY OF SYDNEY

Mathematical Statistics: STAT3925/STAT4025 - Semester 1 - 2022

Time Series Analysis : Solution Set - Week 10 (Tutorial and Computer Problems)

1. (i)

$$\begin{aligned}
I_{n,X}(\omega) &= \frac{2}{n}\left[\left(\sum_{t=1}^{n} X_t \cos(\omega t)\right)^2 + \left(\sum_{t=1}^{n} X_t \sin(\omega t)\right)^2\right] \\
&= \frac{2}{n}\sum_{t=1}^{n} \left|X_t\left(\cos(\omega t) + i\sin(\omega t)\right)\right|^2 \\
&= \frac{2}{n}\sum_{t=1}^{n} \left|X_t \exp^{i\omega t}\right|^2 \\
&= \frac{2}{n}\left(\sum_{t=1}^{n} X_t \exp^{i\omega t}\right)\left(\sum_{u=1}^{n} X_u \exp^{-i\omega u}\right); \text{ since } |a|^2 = a\bar{a},\ \bar{a} \text{ is the complex conjugate} \\
&= \frac{2}{n}\sum_{s=-(n-1)}^{n-1}\left(\sum_{t=1}^{n-|s|} X_t X_{t-|s|}\right)\cos(|s|\,\omega) \\
&= \frac{2}{n}\times n\sum_{s=-(n-1)}^{n-1}\left(\frac{1}{n}\sum_{t=1}^{n-|s|} X_t X_{t-|s|}\right)\cos(s\omega); \text{ since } \cos(-\theta) = \cos(\theta) \\
&= 2\sum_{s=-(n-1)}^{n-1} C_X(|s|) \\
&= 2\{C_X(0) + 2\sum_{s=1}^{n-1} C_X(s)\cos(s\omega)\}
\end{aligned}$$

(ii)

$$\begin{aligned}
I^*_{n,X}(\omega) &= \frac{1}{4\pi} I_{n,X}(\omega) \\
&= \frac{1}{2\pi}\{C_X(0) + 2\sum_{s=1}^{n-1} C_X(s)\cos(s\omega)\}
\end{aligned}$$

Recall that $C_X(s) \to \gamma_X(s)$ as $n \to \infty$ for all $s \geq 0$. Therefore,

$$\begin{aligned}
I^*_{n,X} &\to \frac{1}{2\pi}\{\gamma_X(0) + 2\sum_{s=1}^{n-1} \gamma_X(s)\cos(s\omega)\} \\
&= f_X(\omega)
\end{aligned}$$

$\therefore I^*_{n,X}$ is asymptotically unbiased for $f_X(\omega)$.

2.

$$\begin{aligned}
W(\theta) &= \frac{1}{2\pi}\sum_{s=1}^{n} \lambda(s)\exp^{-i\theta s}, \quad \pi \leq \theta \leq \pi \\
&= \frac{1}{2\pi}\sum_{|s|\leq m}\left(1 - 2a + 2a\cos\left(\frac{\pi s}{m}\right)\right)\exp^{-i\theta s} \\
&= \frac{1}{2\pi}\sum_{|s|\leq m}\left(1 - 2a + 2a\{\frac{\exp^{\frac{i\pi s}{m}} + \exp^{-\frac{i\pi s}{m}}}{2}\}\right)\exp^{-\theta s} \\
&= \frac{1}{2\pi}\sum_{|s|\leq m}\left((1-2a)\exp^{-\theta s} + a\exp^{-i\left(\theta - \frac{\pi}{n}\right)s} + a\exp^{-i\left(\theta + \frac{\pi}{m}\right)s}\right)
\end{aligned}$$

Since $D_m(\alpha) = \frac{1}{2\pi}\sum_{|k|\le m} \exp^{-i\alpha k}$, we have

$$\begin{aligned} W(\theta) &= (1-2a)\,D_m(\theta) + aD_m\left(\theta - \frac{\pi}{m}\right) + aD_m\left(\theta + \frac{\pi}{m}\right) \\ &= aD_m\left(\theta - \frac{\pi}{m}\right) + (1-2a)\,D_m(\theta) + aD_m\left(\theta + \frac{\pi}{m}\right) \end{aligned}$$

3. (i) (a) When $\delta < 0$,

$$\rho_k \sim \frac{1}{k^p}, \quad p > 0$$

Therefore,

$$\rho_k \to 0 \text{ as } k \to \infty$$

(b) When $0 < \delta < 0.5$,

$$0 < \delta < 1 \Rightarrow \rho_k \sim \frac{1}{k^q};\ q > 0$$

Therefore,

$$\rho_k \to 0 \text{ as } k \to \infty$$

(c) When $\delta > 0.5$, $2\delta > 1$ which then implies $\rho_k \sim k^r$ for $r > 0$. Therefore,

$$\rho_k \to \infty \text{ as } k \to \infty$$

(ii) Since $X_t = \frac{1+\theta B}{(1-\alpha B)(1-B)^\delta} Z_t$, the spectrum is

$$f_X(\omega) = \frac{\left|1+\theta\exp^{-i\omega}\right|^2}{\left|1-\alpha\exp^{-i\omega}\right|^2\left|1-\exp^{-i\omega}\right|^{2\delta}} h_Z(\omega)$$

$$\begin{aligned} \left|1+\theta\exp^{-i\omega}\right|^2 &= (1+\theta\cos(\omega))^2 + (-\theta\sin(\omega))^2 = 1+\theta^2+2\theta\cos(\theta\omega) \\ \left|1-\omega\exp^{-i\omega}\right|^2 &= (1-\alpha\cos(\omega))^2 + (\alpha\sin(\omega))^2 = 1+\alpha^2-2\alpha\cos(\omega) \\ \left|1-\exp^{-i\omega}\right|^2 &= (1-\cos(\omega))^2 + (\sin(\omega))^2 = 2(1-\cos(\omega)) = 4\sin^2\left(\frac{\omega}{2}\right). \end{aligned}$$

Using $f_Z(\omega) = \frac{\sigma^2}{2\pi}$ and combining the above yields

$$f_X(\omega) = \frac{1+\theta^2+2\theta\cos(\omega)}{1+\alpha^2-2\alpha\cos(\omega)}\left[2\sin\left(\frac{\omega}{2}\right)\right]^{-2\delta}$$

The sdf $f_X(\omega) = \frac{1+\theta^2+2\theta\cos(\omega)}{1+\alpha^2-2\alpha\cos(\omega)}\left[2\sin\left(\frac{\omega}{2}\right)\right]^{-2\delta}$.
Therefore, when $\delta < 0$, as $\omega \to 0$, $f_X(\omega) \to 0$ since $\sin\left(\frac{\omega}{2}\right) \to 0$. When $\delta > 0$,

$$f_X(\omega) = \frac{1+\theta^2+2\theta\cos(\omega)}{1+\alpha^2-2\alpha\cos(\omega)}\left[2\sin\left(\frac{\omega}{2}\right)\right]^t, \qquad t = 2\delta < 0$$

Therefore, since $\left(\sin\left(\frac{\omega}{2}\right)\right)^t \to \infty$ as $\omega \to 0$ $\sin\left(\frac{\omega}{2}\right) \to 0$, we have that

$$f_X(\omega) \to \infty \text{ as } \omega \to 0$$

(iii) Therefore $f_X(\omega)$ exists near $\omega = 0$ when $\delta < 0$.

(iv) $f_X(\omega)$ is unbounded as $\omega \to 0$ when $0 < \delta < 0.5$.

(v) When $0 < \delta < 0.5$, the acf $\rho_k \to 0$ & $f_X(\omega) \to \infty$ as $\omega \to 0$. Therefore, $\{X_t\}$ has long memory.

PTO for the computer exercise

Computer Exercise - W10

Q3.

$\omega = c(0.00001, 0.0001, 0.001, 0.01, 0.1)$

$f =$ ....

plot(o,f,type=”l”)

```
Q4.f=spectrum(d)
     freq=f$freq
     0.002 0.004 0.006 0.008 0.010

     sdf=f$spec
     1.084509e+02 4.049064e+01 1.522470e+02 1.524457e+01 1.286409e+02
```