Time Series Analysis : Problem Set - Week 9 (Tutorial and Computer Problems)

Attempt these questions by yourself
Monday 25 (Week 9) April is a public holiday

1. Suppose that $\{X_t\}$ is a stationarprocess with autocovariance function γ_k. The sdf for $\{X_t\}$ is given by
$$f_X(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k e^{-i\omega k},\ -\pi < \omega < \pi.$$
   (i) Show that $f_X(\omega)$ can be written as $\frac{1}{2\pi}[\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k \cos(k\omega)],\ -\pi < \omega < \pi$.
   (ii) Suppose that $\{X_t\}$ follows an MA(2) process given by $X_t = Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2}$, where $\{Z_t\} \sim WN(0, \sigma^2)$. Write down its autocovariance function. Hence, find the sdf $f_X(\omega)$.
   (iii) What is the corresponding normalized spectrum, $f_X^*(\omega)$?

2. Write down the MA(2) in Q1(ii) in the form of $X_t = \psi(B)Z_t$. By considering the transfer function of this MA(2), find $|\psi(e^{-i\omega}|^2$. Hence verify the theorem that $f_X(\omega) = |\psi(e^{-i\omega}|^2 f_Z(\omega)$, where $f_Z(\omega)$ is the sdf of $\{Z_t\}$ given by $f_Z(\omega) = \frac{\sigma^2}{2\pi},\ -\pi < \omega < \pi$.

3. Consider the stationary AR(2) given by $X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t,$ where $\{Z_t\} \sim WN(0, \sigma^2)$.
   (i) Write down this AR(2) in the form of $X_t = \psi(B)Z_t$.
   (ii) Using (i), find $f_X(\omega)$ of $\{X_t\}$ using the theorem $f_X(\omega) = |\psi(e^{-i\omega}|^2 f_Z(\omega)$.

4. Write down the autocovariance function of the ARMA(1,1) process given by
$$X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1},$$
where $|\alpha| < 1$ and $\{Z_t\} \sim WN(0, \sigma^2)$. Show that its sdf can be expressed as
$$f_X(\omega) = \frac{\sigma^2}{2\pi}[\gamma_0 + 2\gamma_1 \cos\omega + \frac{2\gamma_1}{\alpha}\sum_{k=2}^{\infty}\alpha^k \cos(\omega k)].$$

5. Suppose that $\{X_t\}$ is a stationary time series with the spectral density function (sdf) $h_X(\omega),\ -\pi < \omega < \pi$. This time series is smoothed by a filter of the form
$$Y_t = aX_{t-1} + bX_t + cX_{t+1},$$
where a, b and c are constant weights such that $a + b + c = 1$.
   (i) Write down the sdf $h_Y(\omega)$ of $\{Y_t\}$ in terms of the sdf $h_X(\omega)$ and the corresponding transfer function. Show that the sdf of $\{Y_t\}$ is given by
$$f_Y(\omega) = [a^2 + b^2 + c^2 + 2b(a+c)\cos\omega + 2ac\cos(2\omega)]f_X(\omega).$$
   (ii) When $a = b = c = \frac{1}{3}$ and using (i), deduce that
$$f_Y(\omega) = \frac{1}{9}(1 + 2\ \cos\omega)^2 f_X(\omega),\ -\pi < \omega < \pi.$$
   (iii) Sketch the sdf $f_Y(\omega),\ -\pi < \omega < \pi$ when
   (a) $X_t \sim WN(0, 2^2)$,
   (b) $X_t = Z_t + 0.6Z_{t-1},\ \{Z_t\} \sim WN(0, 2^2)$,
   (c) $X_t = 0.6X_{t-1} + Z_t,\ \{Z_t\} \sim WN(0, 2^2)$.

6. Let ω_j given by $\omega_j = \frac{2\pi j}{n},\ j = -\frac{n}{2}, \cdots, -1, 0, 1, \cdots, \frac{n}{2}$, where n is an even integer. Prove that $\sum_{t=1}^{n}\cos(t\omega_j) = \sum_{t=1}^{n}\sin(t\omega_j) = 0$ and $\sum_{t=1}^{n}\cos^2(t\omega_j) = \sum_{t=1}^{n}\sin^2(t\omega_j) = \frac{n}{2}, \quad \forall j$.

PTO FOR THE COMPUTER EXERCISE

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Computer Exercise - Working with R

- Download and install the package astsa in RStudio (or on a new/recent version of R OR use install.packages('astsa')
- library(astsa) [to recall the library]
- arma.spec(ar=a,ma=b) [to get the theoretical spectrum of ARMA(1,1)] [replace **a** by a=c(a1,a2,...,ap) and **b** by b=c(b1,b2,...,bq) for an ARMA(p,q)]
- spectrum(d) [to get the Periodogram for the data in d]

Practice question

(i) Simulate 1000 readings from $N(0,1)$, and store the last 300 values in Z.

(ii) Display the tsplot, acf and pacf.

(iii) Draw the sample periodogram of Z.

(iv) Generate 1000 values in X given by $X_t = 1+2t+Z_t$, where Z_t from $N(0,1)$. Store the last 300 values in $X1$.

(v) Obtain the tsplot, acf, pacf and the periodogram of the data in $X1$.

Computer question for W9 - Submit Q6 to Q7 by 23.59 on 29 April (optional)

1. Draw the theoretical spectrum for an AR(1) with $\alpha = 0.6$.
2. Simulate 500 values of an AR(1) with $\alpha = 0.6$. After discarding the first 300 values, store the remainder in d. Draw the sample periodogram for the data in d. Comment on the spectrum in (i) and the periodogram in (ii).
3. Repeat the work in Q1 and Q2 when $\alpha = -0.6,\ 0.99$ and -0.99.
4. Simulate 500 values of an ARMA(1,1) with $\alpha = 0.7$. and $\beta = 0.4$ After discarding the first 200 values, store the remainder in $d1$. Draw the sample periodogram for the data in $d1$ and comment.
5. Repeat the work in Q4 for $\alpha = -0.7,\ 0.99$ and -0.99 by keeping the same β as in Q4.
6. Draw the theoretical spectrum for an ARMA(1,1) with $\alpha = 0.6$ and $\beta = 0.4$.
7. Refer to Beer Data given below:

   (i) Draw a sample periodogram for this data set (see below).

   (ii) Difference the data at lag 12. Draw a periodogram for this differenced data.

   (iii) Fit AR(1) and MA(1) models to the data in (ii). Draw the corresponding theoretical spectrums for AR(1) and MA(1) using your estimated parameters. Compare these spectrums with the periodogram in (ii) and comment.

Beer data

18.705 20.232 20.467 22.123 25.036 26.839 29.640 30.935 28.278 24.235
22.370 21.224 21.061 19.598 21.463 23.287 24.065 27.447 30.413 32.307
32.974 29.973 23.986 26.953 24.250 23.518 20.816 23.743 25.152 28.804
31.158 31.540 32.849 33.748 31.910 27.609 25.170 24.040 25.368 21.260
24.109 26.320 27.701 34.502 33.297 31.252 35.173 36.207 31.511 28.560
26.828 26.660