Computer Exercise Week 10

STAT3023: Statistical Inference   Semester 2, 2021

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Interval estimation of a Binomial success probability

Suppose we have a single $B(m, \theta)$ observation X for m known but the success probability parameter $\theta \in \Theta = (0, 1)$ unknown. Consider the decision problem where the decision space is $\mathcal{D} = \Theta$ and the loss function $L(d|\theta) = 1\{|d - \theta| > C\}$. Then a "decision" $d(X)$ may be viewed as an estimator of θ and the risk is precisely the probability that the resultant *interval estimator* $d(X) \pm C$ does *not* cover the true θ, i.e. the *noncoverage probability* of the interval. In other words, we are interested in forming a fixed-width interval estimate of θ (rather than a "confidence interval"; the main difference is that we are fixing the width, not the coverage probability of the procedure).

We shall consider two different procedures:

- a simple procedure centred at the observed proportion $\hat{\theta}_{\mathrm{ML}} = \frac{X}{m}$;

- a Bayes procedure using the "flat" prior $w(\theta) \equiv 1$; note that in this case this is a "proper prior", in fact the beta$(1, 1)$ density, an example of a "conjugate prior" for this problem.

For simplicity, we shall take $m = 10$ and suppose we wish to form an interval of width 0.2, so $C = 0.1$.

Since there are only 11 different possible observations, we can exhaustively list all possible intervals using both procedures. It is then possible to compute "exactly" the risk functions of the two procedures. Note also that when X takes the (extreme possible) values 0 and 10, we have a modified rule for constructing the interval, to ensure it remains within (0,1).

1. **The simple interval**

   For observed values $x = 1, 2, \ldots, 9$, the simple interval is $\left[\frac{x-1}{10}, \frac{x+1}{10}\right]$. But for $x = 0$ the interval is the same as for $x = 1$; for $x = 10$ the interval is the same as for $x = 9$.

   (a) Define `x=0:10` and also define two vectors `simple.lower` and `simple.upper` giving the lower and upper endpoints of the simple interval.

   (b) Define now `th=(1:1000)/1001`. The following code computes the risk of the simple interval for each value in `th`:

   ```
   th=(1:1000)/1001
   L=length(th)
   risk=0
   for (i in 1:L){
       probs=dbinom(x,10,th[i])
       lower.miss=th[i]<simple.lower
       upper.miss=th[i]>simple.upper
       risk[i]=sum(probs[lower.miss])+sum(probs[upper.miss])
   }
   ```

   Plot `risk` against `th` (as a red line plot).

2. **The Bayes interval**

   The product of the likelihood and the weight function/prior is

   $$\binom{m}{X}\theta^X(1-\theta)^{m-X} = \underbrace{\frac{1}{m+1}}_{\text{const.}} \underbrace{\frac{\theta^{(X+1)-1}(1-\theta)^{(m-X+1)-1}}{B(X+1, m-X+1)}}_{p(\theta|X)}$$

   where the $B(X+1, m-X+1) = \frac{X!(m-X)!}{(m+1)!}$ in the denominator here denotes the beta function; the posterior is thus the beta$(X+1, m-X+1)$ density.

The interval is determined by finding the *level set* of the posterior density of width 0.2. For $X = 0$, the posterior is just the decreasing function

$$(m + 1)(1 - \theta)^m$$

and so the level set is simply $[0, 0.2]$. Similarly for $X = 10$, the interval is $[0.8, 1]$.

For all other values $X = 1, 2, \ldots, 9$, the posterior density increases and then decreases and we need to use the computer (specifically, the R function `uniroot()`) to find the level set; that is, we need to find the value d such that

$$p(d - C|X) = p(d + C|X);$$

then the desired interval is $d \pm C$.

(a) First define an objective function for which to find the root

```
beta.obj=function(d,X,m=10,C=0.1){
    dbeta(d-C,X+1,m-X+1)-dbeta(d+C,X+1,m-X+1)
}
```

(b) Try solving this manually for $X = 1$:

```
uniroot(beta.obj,lower=0.1,upper=0.9,X=1)
```

(c) Verify that we have indeed found the right value:

- extract the root;
- form the interval by defining `lower.endpt` and `upper.endpt`;
- plot the posterior (beta(2,10)) density (try using `curve()`);
- add vertical lines (using `lty=2`) at `lower.endpt` and `upper.endpt`;
- add a horizontal line (using `lty=3`) showing the level set (i.e. at the height $p(\theta|X)$ where $\theta = d \pm C$ and $X = 1$).

(d) Now iterate through the values $X = 1, \ldots, 9$ to find the lower and upper endpoints of the Bayes intervals, saving these in vectors `bayes.lower` and `bayes.upper`. Complete the vectors with appropriate endpoints for $X = 0$ and $X = 10$.

(e) Adapt the code from question 1 part (b) above to compute the risk of the Bayes interval. Recreate your plot from 1(b) and add to it the Bayes interval risk in blue. Add an informative heading and legend.

3. For which values of `th`

- does the `simple` interval do better;
- does the `bayes` interval do better;
- do the two intervals have similar performance?