STAT3023 Statiscal Inference

Lab Week 5

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We shall compare three different estimators of a binomial success probability. If $Y \sim B(2,\theta)$ then we have: $P(Y=0) = (1-\theta^2)$, $P(Y=1) = 2\theta(1-\theta)$, $P(Y=2) = \theta^2$. Moreover, if we have an iid sample $Y_1, Y_2, ..., Y_n$ then if we define:

- $N_0 = \sum_{i=1}^n 1_{\{Y_i=0\}}$ as the number of 0's $\implies N_0 \sim B(n,(1-\theta)^2)$
- $N_1 = \sum_{i=1}^n 1_{\{Y_i=1\}}$ as the number of 1's $\implies N_1 \sim B(n,2\theta(1-\theta))$
- $N_2 = \sum_{i=1}^n 1_{\{Y_i=2\}}$ as the number of 2's $\implies N_2 \sim B(n,\theta^2)$

The usual estimator of θ based on an iid sample $Y_1, Y_2, ..., Y_n$ is a function of $\overline{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$

1. Determine an unbiased estimator of θ which is a *linear* function of $\overline{Y}$. Call it $\hat{\theta}_1$

Solution

To find an unbiased estimator of θ we first note that:

$$\begin{aligned}\mathbb{E}[\overline{Y}] &= \frac{1}{n}\sum_{i=1}^n \mathbb{E}[Y_1] \\ &= \mathbb{E}[Y_1] \\ &= 2\theta\end{aligned}$$

Hence we should define an unbiased estimator $\hat{\theta}_1$ by:

$$\hat{\theta}_1 = \frac{1}{2}\overline{Y}$$

2. Determine an unbiased estimator of θ which is a *nonlinear* function of N_0 (hint: use method of moments, i.e. set equal to expectation and solve for θ). Call it $\hat{\theta}_0$

Solution

To find an unbiased estimator of θ we recall the method of moments. We have that $1_{\{Y_i=0\}} \sim$ *bernoulli*$((1-\theta)^2)$. Hence we have that $\mathbb{E}\left(1_{\{Y_i=0\}}\right) = (1-\theta)^2$. With the random sample $1_{\{Y_1=0\}}, 1_{\{Y_2=0\}}, ..., 1_{\{Y_n=0\}}$. We have that (by the method of moments) $(1-\theta)^2 = \frac{1}{n}\sum_{i=1}^n 1_{\{Y_i=0\}} \implies (1-\theta)^2 = \frac{1}{n}N_0 \implies \hat{\theta}_0 = 1 - \sqrt{\frac{N_0}{n}}$

3. Determine an unbiased estimator of θ which is a *nonlinear* function of N_2 (hint: use method of moments, i.e. set equal to expectation and solve for θ). Call it $\hat{\theta}_2$

Solution

To find an unbiased estimator of θ we recall the method of moments. We have that $1_{\{Y_i=2\}} \sim$ *bernoulli*(θ^2). Hence we have that $\mathbb{E}\left(1_{\{Y_i=2\}}\right) = \theta^2$. With the random sample $1_{\{Y_1=2\}}, 1_{\{Y_2=2\}}$, ..., $1_{\{Y_n=2\}}$. We have that (by the method of moments) $\theta^2 = \frac{1}{n}\sum_{i=1}^n 1_{\{Y_i=2\}} \implies \theta^2 = \frac{1}{n}N_0 \implies \hat{\theta}_2 = \sqrt{\frac{N_0}{n}}$

4. We shall simulate a sample if $n = 100$ iid such Y_is and compute the values of these three estimators and then compare their mean squared errors, for a fine grid of θ values.

Solution

We want to now compare the variance of $\hat{\theta}_0, \hat{\theta}_1$ and $\hat{\theta}_2$ with the CRLB of $\frac{\theta[1-\theta]}{2n}$