THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Computer Exercise Week 9

STAT3023: Statistical Inference — Semester 2, 2021

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Interval estimation of a Poisson mean

Suppose we have a single Poisson observation X with unknown mean $\theta \in \Theta = (0, \infty)$ and we are interested in forming a fixed-width interval estimate of θ (rather than a "confidence interval"; the main difference is that we are fixing the width, not the coverage probability of the procedure).

Consider the following motivating example: A study of several thousand Australian fire fighters found 8 cases of testicular cancer in the years 1990-1996 compared with the national average which would predict 3 cases for this population over this period. Fire fighters were not found to suffer from any other types of cancer more often than the population at large*.

Two possibilities are a simple interval of the form $X \pm C$, and a Bayes procedure using a decision space $\mathcal{D} = \Theta$ and loss function $L(d|\theta) = 1\{|d - \theta| > C\}$. We shall compare the coverage probabilities/risks of these two procedures (the risk $E_\theta[L(d(X)|\theta)]$ of any "decision" $d(X)$ is the *non*-coverage probability.

The Bayes procedure formally involves firstly

- deciding on a weight function/prior;
- identifying the posterior density $p(\theta|X)$;
- finding the level set of this density of the appropriate width, i.e.
  - if $p(\theta|X)$ is *unimodal*, in that it increases to some mode m, then decreases, find d such that
  $$p(d - C|X) = p(d + C|X) .$$
  - if $p(\theta|X)$ simply decreases over $(0, \infty)$, quote the interval as $(0, 2C)$, so that $d = C$ (the midpoint of the interval).

We shall, for simplicity, take $C = 1$ and use as the Bayes weight function $w(\theta) \equiv 1$, the "flat prior".

The likelihood is

$$f_\theta(X) = \frac{\theta^X e^{-\theta}}{X!}$$

thus the product

$$w(\theta) f_\theta(X) = \frac{\theta^X e^{-\theta}}{X!} = \frac{\theta^{(X+1)-1} e^{-\theta}}{\Gamma(X+1)} = p(\theta|X)$$

since when viewed as a function of θ this is precisely the gamma$(X + 1, 1)$ density. For $X \geq 1$, this is unimodal, in that it increases to the mode at X and then decreases. If $X = 0$, it simply decreases. For $X \geq 1$ we need to find d such that

$$\begin{aligned}
(d-1)^X e^{-(d-1)} &= (d+1)^X e^{-(d+1)} \\
(d-1)^X e^{1} &= (d+1)^X e^{-1} \\
(d-1)^X e^{2} &= (d+1)^X \\
(d-1) e^{2/X} &= (d+1) \\
d\left(e^{2/X} - 1\right) &= e^{2/X} + 1 \\
d &= \frac{e^{2/X} + 1}{e^{2/X} - 1} .
\end{aligned}$$

*Taken from `http://www.statsci.org/data/oz/firefigh.html`

The corresponding interval estimate is

$$(d-1, d+1) = \left(\frac{2}{e^{2/X}-1}, \frac{2e^{2/X}}{e^{2/X}-1} \right) ,$$

which looks rather different to the simple interval $X \pm 1$!

1. Obtain the two interval estimates based on the observation of $X = 8$. Call them `simple` and `bayes`.

2. We shall now write functions that compute these two intervals, so we can approximate their risk functions by simulation. Note that the procedure for constructing the interval is different (in *both* cases) if the Poisson observation is zero (which is certainly possible!); in both cases for an observation of zero the interval is $(0, 2)$. Write two functions `simple()` and `bayes()`. They should both be of the same *conditional* form:

```
simple=function(X){
    if (X==0) {
        out=...
    } else {
        ...
        out=...
    }
    out
}
```

Once you have written them, test them out by executing both `simple(8)` and `bayes(8)`.

3. Define `th=(1:1000)/250`, `L=length(th)` and `B=1000`. Also define

```
noncoverage.simple= noncoverage.bayes=0
```

Perform a double loop: at the `i`-th iteration of the outer loop

- define matrices `s.mat=matrix(0,B,2)` and `b.mat=matrix(0,B,2)`;
- perform the inner loop: at the `j`-th iteration of the inner loop
  - generate a single Poisson pseudo-random observation `X`;
  - obtain `simple` and `bayes` intervals, saving them in the `j`-th row of `s.mat` and `b.mat` respectively;
- save in the `i`-th element of `noncoverage.simple` the number of times the simple interval did **not** cover `th[i]`; similarly for `bayes`;

Convert the counts in the noncoverage vectors to proportions. Plot (as lines) these proportions against `th` (red for `simple`, blue for `bayes`). Add an informative heading and legend.

4. For which values of `th` did

- the `simple` interval do better;
- the `bayes` interval do better;
- the two intervals have similar performance?