THE UNIVERSITY OF SYDNEY

Mathematical Statistics: STAT3925/STAT4025 - Semester 1 - 2022

Time Series Analysis : Problem Set - Week 12 (Tutorial and Computer Problems)

Reminder:

- There will be a Computer Quiz on Monday 23 May (Week 13) at 16.00 in your class time. Submit your answers through turnitin by due time.
- There will be a Non-computer Quiz on Friday 27 May (Week 13) at 11.00 in your lecture time. Submit your answers through turnitin by due time.

Attempt these questions before your class and discuss any issues with your tutor
Go to your assigned tutorial class/Lab and record your attendance

1. Suppose that $X_t,\ t = 1, 2, \cdots, n$ follows an ARCH(1) process $X_t = \sigma_t \epsilon_t,\ \sigma_t^2 = \alpha_0 + \alpha X_{t-1}^2$, where $E(\epsilon_t|F_{t-1}) = 0,\ Var(\epsilon_t|F_{t-1}) = 1$ (F_{t-1} is to history of the process).
   (a) Let $\eta_t = X_t^2 - \sigma_t^2$ be a martingale difference sequence. Given that X_t^2 follows a second order stationary AR(1) process, derive the corresponding Yule-Walker type for the autocovariance function at lag k, γ_k, where $\gamma_k = \text{Cov}(X_t^2, X_{t-k}^2)$.
   (b) Find the Yule-Walker estimator for α.
   (c) Find an estimate of α_0.
   (d) When ϵ_t follows a Gaussian distribution, find $Var(\eta_t)$.
2. Find the ℓ-step-ahead forecast function for σ_t^2 and X_t^2 for a second-order stationary ARCH(p) process satisfying $X_t = \sigma_t \epsilon_t,\ \sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2$, where ϵ_t are as in Q1.
3. Suppose that the return R_t of a particular stock follows $R_t = \theta R_{t-1} + X_t$, where $|\theta| < 1$ and X_t follows the GARCH(1,1) model given by $X_t = \sigma_t \epsilon_t,\ \sigma_t^2 = \alpha_0 + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2$ ($\{\epsilon_t\} \sim NID(0, 1),\ \alpha_0 > 0,\ \alpha > 0,\ \beta > 0$) and X_t is uncorrelated with R_t for all t.

   Assuming $\{X_t^2\}$ is stationary, find $E(R_t)$ and $Var(R_t)$.
4. Find the ℓ-step-ahead forecast function for σ_t^2 and X_t^2 for a second-order stationary GARCH(p,q) process satisfying $X_t = \sigma_t \epsilon_t,\ \sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2 + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2$, where ϵ_t are as in Q1.
5. Refer Q3. Following Q4, find the ℓ-step-ahead forecast function for R_t and R_t^2.
6. A stationary process $\{X_t\}$ is said to be a $GARCH(2, 1)$ process if it satisfies

$$X_t = \sigma_t \epsilon_t,\ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2, \qquad (*)$$

   where $\alpha_0 > 0,\ \alpha_1, \alpha_2 \geq 0,\ \beta_1 > 0$ and $\{\epsilon_t\}$ is a sequence of iid random variables with mean zero and variance 1.
   (i) Given that $\eta_t = X_t^2 - \sigma_t^2$ is a martingale difference series, *find* the values of $r,\ s$ such that X_t^2 follows an $ARMA(r, s)$ process.
   (ii) Let $\delta_i,\ i = 1, 2, \ldots, r$ be the corresponding AR coefficients δ_i in (i). If X_t is weakly stationary, *find* $E(X_t^2)$ and *show that* the corresponding AR coefficients δ_i satisfy $\sum_{i=1}^{r} \delta_i < 1$.
7. Using the ARMA representation for X_t^2 in Q6(i)
   (i) Find Yule-Walker type equations for $\gamma_k = \text{Cov}(X_t^2, X_{t-k}^2)$ for all $k \geq 0$.
   (ii) Find the ℓ-step-ahead forecast function for R_t^2 and σ_t^2.

PTO for computer problems

Week12 - Working with R

- library(TSA)
- library(tseries)
- library(fGarch)

GARCH(p,q) process: $X_t = \sigma_t \epsilon_t;\ \sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2 + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2$.

In R: **garch.sim(alpha=c(alpha0,alpha1,......,alphap), beta=c(beta1, beta2,.....,betaq))**

Example: To simulate 500 values from the GARCH(1,1) $X_t = \sigma_t \epsilon_t;\ \sigma_t^2 = 0.5 + 0.3X_{t-1}^2 + 0.4\sigma_{t-1}^2$ and store the values in g, use:

- g=garch.sim(alpha=c(0.5, 0.3), beta=0.4, n = 500)
- Store the last 300 values in $g1$ using $g1 = g[201 : 500]$
- To get the ts.plot acf, pacf, sample periodogram of $g1$ use: ts.plot(g1); acf(1); pacf(g1); spectrum(1)
- To fit a GARCH(p,q) model for the data in g1 using **fGarch** use:

  fit=garchFit(˜ garch(p,q)) [see Notes Set 4: PP229-232 for worked examples]
- **summary(fit)**

  gives Diagnostic Tests including **Jarque Bera Test, AIC** and **Box-Ljung Test.** When the fitted model is satisfactory, p-value in each test must be greater than 0.05.

Computer Exercise - Week 12 - Submit Q1 and Q5 by 23.59 on 16 May

1. Simulate 500 values of GARCH(1,1) with $\alpha_0 = 0.7,\ \alpha_1 = 0.4$ and $\beta_1 = 0.5$. After discarding the first 200 values, store the remainder in $d1$. Draw the
   (i) ts plot, sample acf, sample pacf and the sample periodogram for the data in $d1$ and comment.
   (ii) ts plot, sample acf, sample pacf and the sample periodogram for the squared data in $d1$ and comment.
2. Repeat the work in Q1(i) and (ii) when $\alpha_0 = 0.5$ and $\alpha_1 = 0.2, \beta_1 = 0.7$;
   $\alpha_1 = 0.5, \beta_1 = 0.4;\ \alpha_1 = 0.6, \beta_1 = 0.37$.
3. Let x_t be the simulated data in each case of Q2. Draw the spectrum in each case for x_t and x_t^2 and comment.
4. Refer the simulated data $d1$ in Q1.
   (i) Fit ARCH(1), ARCH(2), GARCH(1,1) and GARCH(1,2) to the data d1. What is the best possible model from this pool of models based on Jarque Bera Test and Box-Ljung Test? Give your reason.
   (ii) Fit AR(1), AR(2), ARMA(1,1) and ARMA(2,2) for the data in $d1^2$. Select the best possible model from this pool of models based on aic values. Give your reason.

5.
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10.01 8.73 7.87 8.59 12.47 10.75  9.32  9.69 11.02 10.10  9.56  9.64 10.07
9.66 11.04 9.28  9.23 8.89 8.42   7.91 13.13  5.50 11.10  7.07 12.52  8.28
6.88  4.77 12.21 12.72 12.87 7.05 10.14 10.84  8.71 10.28 9.18  8.76 10.42
7.45  6.94  5.94 10.82  8.54  7.25 10.96 11.42  7.89  9.03 11.00 12.33 11.27
12.59 12.34 10.39 12.77 10.52  9.11 12.12  9.22 10.17 9.78 10.45  9.70 10.24
10.23 10.39 10.51 10.11 10.88  8.95  9.26 10.49 10.00 11.79  9.10 12.86 12.01
12.04 10.79 10.35  9.81 10.36  9.18  8.86 11.92  9.41 11.01  8.66 11.68 12.13
11.32  6.80  8.88 10.77 10.24 10.50 10.69 11.18  9.46 10.47 10.47 11.02  9.80
10.91  8.77  9.30 10.21 9.29 10.83  8.83  8.73 12.59 11.89 11.25 10.26  9.96
12.13  9.63 12.37  8.66  8.78  8.87 10.80 10.33 10.65 11.02 10.61  9.59 10.20
8.84  9.82 11.00 10.59 10.02 10.35  8.47 11.21  9.74 11.09  9.11 10.71 10.72
9.77 10.25  9.98  9.72 11.10 10.13 10.28  9.52 10.60 10.55  9.15  9.42  9.44
8.90 11.88  8.76  7.26 10.05 12.03 10.31  9.12  8.98  6.56  7.66  9.56 10.37
8.76  8.42  8.79 10.27 10.51  9.79  9.70  9.95  9.55 11.00 10.39 9.11 10.97
10.35 10.78  9.28 10.21 9.91  9.18  9.33  9.11  8.79  9.20 10.52 11.04 11.15
12.14 10.53  8.90 9.97 10.01
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   (i) Save the mean corrected data in d and comment on this data based on the tsplot, the acf and pacf.

   (ii) Fit ARCH(1), GARCH(1,1), GARCH(2,1), GARCH(1,2) and GARCH(2,2) models to the data in d. Select the best possible model from this pool of models based on Jarque Bera and Box-Ljung Tests.

   (iii) Fit AR(1), ARMA(1,1), ARMA(2,1) and ARMA(2,2) for the data in d^2. Select the best possible model from this pool of models based on the aic values.

6. (**See P229 of Notes 11**) Simulate 1000 values of GARCH(1,1) with $\alpha_0 = 0.07$, $\alpha_1 = 0.3$ and $\beta_1 = 0.4$. After discarding the first 200 values, store the remainder in $d6$. Now use fGarch to fit a GARCH(1,1) model to this data in d6 and report all estimated parameters $\hat{u} = (\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_1)$. Now repeat this process another 2 times and report the estimates $\hat{u}^i = (\hat{\alpha}_0^i, \hat{\alpha}_1^i, \hat{\beta}_1^i)$ for each simulation $i = 1, 2, 3$. What do you notice? Find the average of these three estimates for each parameter such that $\bar{\hat{u}} = \frac{1}{3}\sum_{i=1}^{3} \hat{u}^i = \frac{1}{3}\sum_{i=1}^{3}(\hat{\alpha}_0^i, \hat{\alpha}_1^i, \hat{\beta}_1^i)$. Comment on this estimated average parameter vector $\bar{\hat{u}}$ and $\hat{u}$ (from the first simulation). What is the standard error (se) of $\bar{\hat{u}}$? Recall that $se(\bar{\hat{u}}) = \sqrt{Var(\bar{\hat{u}})} = \sqrt{\frac{1}{9}\sum_{i=1}^{3} Var(\hat{u}^i)}$.