THE UNIVERSITY OF SYDNEY

Mathematical Statistics: STAT3925/STAT4025 - Semester 1 - 2022

Time Series Analysis : Solution Set - Week 9 (Tutorial and Computer Problems)

1. (i) Since $\{X_t\}$ is stationary, $\gamma_{-k} = \gamma_k$. Therefore, we have that

$$\begin{aligned} f_X(\omega) &= \frac{1}{2\pi}\left[\gamma_0 + \sum_{k=-\infty}^{-1}\gamma_k e^{-i\omega k} + \sum_{k=1}^{\infty}\gamma_k e^{i\omega k}\right] \\ &= \frac{1}{2\pi}\left[\gamma_0 + \sum_{k=1}^{\infty}\left(\gamma_k e^{i\omega k} + \gamma_k e^{-i\omega k}\right)\right] \\ &= \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k \cos(\omega_k)\right] \end{aligned}$$

The last line follows from the fact that $e^{i\omega k} + e^{-i\omega k} = 2\cos(\omega_k)$.

(ii) We have that

$$\gamma_k = \begin{cases} \left(1+\beta_1^2+\beta_2^2\right)\sigma^2 & ; k = 0 \\ \beta_1(1+\beta_2)\sigma^2 & ; k = 1 \\ \beta_2\sigma^2 & ; k = 2 \\ 0 & ; k \geq 3 \end{cases}$$

Therefore,

$$\begin{aligned} f_X(\omega) &= \frac{1}{2\pi}\left[\gamma_0 + 2\left(\gamma_1\cos(\omega) + \gamma_2\cos(2\omega)\right)\right] \\ &= \frac{\sigma^2}{2\pi}\left[1+\beta_1^2+\beta_2^2 + 2\left(\beta_1(1+\beta_2)\cos(\omega) + \beta_2\cos(2\omega)\right)\right], \pi < \omega < \pi \end{aligned}$$

(iii)

$$f_X^*(\omega) = \frac{h_X(\omega)}{\gamma_0} = \frac{1}{2\pi}\left[1 + \frac{2\left(\beta_1(1+\beta_2)\cos(\omega) + \beta_2\cos(2\omega)\right)}{1+\beta_1^2+\beta_2^2}\right]$$

2. We can see that $\beta(B) = 1 + \beta_1 B + \beta_2 B^2 = \psi(B)$. Therefore,

$$\psi\left(e^{-i\omega}\right) = 1 + \beta_1 e^{-i\omega} + \beta_2 e^{-2i\omega} = 1 + \left(\beta_1\cos(\omega) + \beta_2\cos(2\omega) + i\left(\beta_1\sin(\omega) + \beta_2\sin(2\omega)\right)\right)$$

In addition,

$$\begin{aligned} \left|\psi\left(e^{i\omega}\right)\right|^2 &= \left(1+\beta_1\cos(\omega)+\beta_2\cos(2\omega)\right)^2 + \left(\beta_1\sin(\omega)+\beta_2\sin(2\omega)\right)^2 \\ &= 1 + \beta_1^2\cos^2(\omega) + \beta_2^2\cos^2(2\omega) + 2\beta_1\cos(\omega) + 2\beta_2\cos(2\omega) + \\ &\quad 2\beta_1\beta_2\cos(\omega)\cos(2\omega) + \beta_1^2\sin^2(\omega) + \beta_2^2\sin^2(2\omega) + 2\beta_1\beta_2\sin(\omega)\sin(2\omega) \\ &= 1+\beta_1^2+\beta_2^2 + 2\beta_1\cos(\omega) + 2\beta_2\cos(2\omega) + 2\beta_1\beta_2\left[\cos(2\omega)\cos(\omega) + \sin(2\omega)\sin(\omega)\right] \\ &= 1+\beta_1^2+\beta_2^2 + 2\beta_1\cos(\omega) + 2\beta_2\cos(2\omega) + 2\beta_1\beta_2\cos(\omega), \text{ using } \cos(A-B) = \cos A\cos B + \sin A\sin B \\ &= 1+\beta_1^2+\beta_2^2 + 2\left[\beta_1(1+\beta_2)\cos(\omega) + \beta_2\cos(2\omega)\right]. \end{aligned}$$

Therefore, $f_X(\omega) = \left|\psi\left(e^{-i\omega}\right)\right|^2 f_Z(\omega)$.

3. (i) Let $X_t = \psi(B)Z_t$, where $\psi(B) = \frac{1}{1-\alpha_1 B - \alpha_2 B^2}$. Then $X_t = \frac{1}{1-\alpha_1 B-\alpha_2 B^2}Z_t$. Now

$$f_X(\omega) = \left|\frac{1}{1-\alpha_1\exp^{-i\omega} - \alpha_2\exp^{-2i\omega}}\right|^2 f_Z(\omega).$$

(ii) Replacing $\beta_1 = -\alpha_1$ & $\beta_2 = -\alpha_2$ in Q2, we have

$$f_X(\omega) = \frac{1}{1+\alpha_1^2+\alpha_2^2 - 2\left[\alpha_1(1-\alpha_2)+\alpha_2\cos(2\omega)\right]} \times \frac{\sigma^2}{2\pi}$$

4. From Tutorial 6 Q2, we have

$$\gamma_0 = \frac{1+\beta^2+2\alpha\beta}{1+\beta^2};\ \gamma_1 = \alpha\gamma_0 + \beta;\ \gamma_k = \alpha\gamma_{k-1},\ k \geq 2$$

Therefore, $\gamma_k = \alpha^{k-1}\gamma_1$. We then have for the spectrum

$$\begin{aligned}
f_X(\omega) &= \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k\cos(\omega k)\right] \\
&= \frac{1}{2\pi}\left[\gamma_0 + 2\gamma_1\cos(\omega) + 2\sum_{k=2}^{\infty}\gamma_k\cos(\omega k)\right] \\
&= \frac{1}{2\pi}\left[\gamma_0 + 2\gamma_1\cos(\omega) + 2\sum_{k=2}^{\infty}\alpha^{k-1}\gamma_1\cos(\omega k)\right] \\
&= \frac{1}{2\pi}\left[\gamma_0 + 2\gamma_1\cos(\omega) + 2\frac{\gamma_1}{\alpha}\sum_{k=2}^{\infty}\alpha^{k}\cos(\omega k)\right].
\end{aligned}$$

5. (i)

$$\begin{aligned}
Y_t &= \left(aB + b + cB^{-1}\right)X_t \\
\therefore f_Y(\omega) &= \left|a\exp^{-i\omega} + b + c\exp^{i\omega}\right|^2 f_X(\omega) \\
&= \left|b + (a+c)\cos(\omega) + i(a+c)\sin(\omega)\right|^2 f_X(\omega) \\
&= \{\left[b + (a+c)\cos(\omega) + [(a-c)\sin(\omega)]^2\right]\}f_X(\omega) \\
&= \left[b^2 + 2b(a+c)\cos(\omega) + (a+c)^2\cos^2(\omega) + (a-c)^2\sin^2(\omega)\right]f_X(\omega) \\
&= \left[a^2+b^2+c^2+2b(a+c)\cos(\omega)+2ac\cos(2\omega)\right]f_X(\omega)
\end{aligned}$$

(ii)

$$\begin{aligned}
f_Y(\omega) &= \left[\frac{3}{9} + \frac{4}{9}\cos(\omega) + \frac{2}{9}\cos(2\omega)\right]f_X(\omega) \\
&= \frac{1}{9}\left[3 + 4\cos(\omega) + 2\left(2\cos^2(\omega)-1\right)\right]f_X(\omega) \\
&= \frac{1}{9}\left[1 + 4\cos(\omega) + 4\cos^2(\omega)\right]f_X(\omega) \\
&= \frac{1}{9}\left(1+2\cos(\omega)\right)^2 f_X(\omega)
\end{aligned}$$

(iii) (a) Since $f_X(\omega) = \frac{2^2}{2\pi} = \frac{2}{\pi}$, we have $f_Y(\omega) = \frac{2}{9\pi}\left(1+2\cos(\omega)\right)^2$.

(b)

$$\begin{aligned}
f_X(\omega) &= \frac{2^2}{2\pi}\left(1 + 1.2\cos(\omega) + 0.36\right);\ \text{for } f_X(\omega) = \frac{\sigma^2}{2\pi}\left(1+2\beta\cos(\omega)+\beta^2\right) \\
&= \frac{2}{\pi}\left(1.36 + 1.2\cos(\omega)\right)
\end{aligned}$$

Therefore, $f_Y(\omega) = \frac{2}{9\pi}\left(1+2\cos(\omega)\right)^2\left(1.36 + 2\cos(\omega)\right)$.

(c)

$$\begin{aligned}
f_X(\omega) &= \frac{\sigma^2}{2\pi}\frac{1}{1-2\alpha\cos(\omega)+\alpha^2}; \text{ for } X_t = \alpha X_{t-1} + Z_t \\
&= \frac{2}{\pi}\frac{1}{1-1.2\cos(\omega)+0.36} \\
&= \frac{2}{\pi(1.36-1.2\cos(\omega))}
\end{aligned}$$

Therefore, $f_Y(\omega) = \frac{2}{9\pi}\frac{(1+2\cos(\omega))^2}{1.36-1.2\cos(\omega)}$.

6. (i) Recall that

$$\begin{aligned}
\sum_{t=1}^{n}\cos(t\theta) &= Re\left(\sum_{t=1}^{n} e^{it\theta}\right), \text{ where Re stands for thr real part of a complex number} \\
&= Re\left(\frac{e^{2i\theta}\left(1-e^{in\theta}\right)}{1-e^{i\theta}}\right) \\
&= Re\left(\frac{e^{i\theta}\exp^{\frac{in\theta}{2}}\left(e^{-\frac{in\theta}{2}}-e^{\frac{in\theta}{2}}\right)}{e^{\frac{i\theta}{2}}\left(e^{-\frac{i\theta}{2}}-e^{\frac{i\theta}{2}}\right)}\right) \\
&= Re\left(e^{\frac{i(n+1)\theta}{2}} \times \frac{2i\sin\left(\frac{n\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)}\right) \\
&= \cos\left(\frac{(n+1)\theta}{2}\right)\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.
\end{aligned}$$

Similarly,

$$\sum_{t=1}^{\infty}\sin(t\theta) = Im\left(\sum_{t=1}^{n} e^{it\theta}\right) = \sin\left(\frac{(n+1)\theta}{2}\right)\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}, \text{ Im is the imaginary part}$$

When $\theta = \omega_j = \frac{2\pi j}{n} \Rightarrow \sin\left(\frac{n\theta}{2}\right) = \sin(\pi j) = 0$ Therefore,

$$\sum_{t=1}^{n}\cos(t\omega_j) = \sum_{t=1}^{n}\sin(t\omega_j) = 0.$$

(ii) We use the rules

$$\begin{aligned}
\cos(2A) = 2\cos^2(A) - 1 &\Rightarrow \cos^2(t\theta) = \frac{1+\cos(2t\theta)}{2} \\
\cos(2A) = 1 - 2\sin^2 A &\Rightarrow \sin^2(t\theta) = \frac{1-\cos(2t\theta)}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{t=1}^{n}\cos^2(t\theta_j) &= \frac{n}{2} + \frac{1}{2}\sum_{t=1}^{n}\cos(2t\theta_j) \\
&= \frac{n}{2} + \frac{1}{2}\left[\cos((n+1)\theta)\frac{\sin(n\theta)}{\sin\left(\frac{\theta}{2}\right)}\right].
\end{aligned}$$

When $\theta = \omega_j = \frac{2\pi j}{n} \Rightarrow \sin(n\theta) = \sin(2\pi j) = 0$, it is clear that

$$\sum_{t=1}^{n}\cos^2(t\omega_j) = \sum_{t=1}^{n}\sin^2(t\omega_j) = \frac{n}{2}.$$

Computer Exercise - Working with R

```
install.packages("astsa")
library(astsa)
```

Computer question for W9 - Submit Q6 to Q7 by 23.59 on 26 April

1.
```
arma.spec(0.6)
```

2.
```
d=arima.sim(list(ar=0.6), n=500)[301:500]
spectrum(d)
```

   Comment: arma.spec() and spectrum (d) are on different scales. However, they begin with large values and tail off towards small values.

3. Inspect each case. When alpha=0.99, both plots show a very large value near the origin. This indicates that the series is almost non-stationary.

4.
```
d=arima.sim(list(ar=0.6), n=500)[301:500]
spectrum(d)
```

5. Repeat the work as before.

6.
```
arma.spec(ar=0.6, 0.4)
```

7. Refer to Beer Data:

   (i)
```
d7=scan()
spectrum(d7)
```

   (ii)
```
d8=diff(d7, lag=12)
spectrum(d8)
fit1=arima(d8, order=c(1,0,0))
```

   (iii)
```
fit1

Call:
arima(x = d8, order = c(1, 0, 0))
Coefficients:
ar1  intercept
0.3515     1.8557
s.e.  0.1487     0.6372
sigma^2 estimated as 6.994:  log likelihood = -95.72,  aic = 197.45

fit2=arima(d7, order=c(0,0,1))
fit2

Call:
arima(x = d7, order = c(0, 0, 1))
Coefficients:
ma1  intercept
0.7544    26.7904
s.e.  0.0775     0.7416
sigma^2 estimated as 9.445:  log likelihood = -132.59,  aic = 271.18
```

Taking approximate estimates to 2dp for ar and ma coefficients, use the following commands draw the theoretical spectrums:

```
arma.spec(ar=0.35)
arma.spec(ma=0.33)
```