THE UNIVERSITY OF SYDNEY

Mathematical Statistics: STAT3925/STAT4025 - Semester 1 - 2022

Time Series Analysis : Problem Set - Week 8 (Tutorial and Computer Problems)

Attempt these questions before your class and discuss any issues with your tutor
Go to your assigned tutorial class/Lab and record your attendance

1.

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

(a) (i)

$$\begin{aligned}
\hat{X}_{t+\ell} &= E\left(X_{t+\ell}|X_t, X_{t-1}, \dots\right) \\
&= E\left[\sum_{j=0}^{\infty} \psi_j Z_{t+\ell-j}|X_t, X_{t-1}, \dots\right] \\
&= \sum_{j=\ell}^{\infty} \psi_j Z_{t+\ell-j} \\
&= \sum_{j=0}^{\infty} \psi_{\ell+j} Z_{t-j}
\end{aligned}$$

(ii)

$$\begin{aligned}
\epsilon_{t+\ell} &= X_{t+\ell} - \hat{X}_{t+\ell} \\
&= \sum_{j=0}^{\ell-1} \psi_j Z_{t+\ell-j} \\
\therefore Var(\epsilon_{t+\ell}) &= \sigma^2 \sum_{j=0}^{\ell-1} \psi_j^2
\end{aligned}$$

(b) (i) Use the results: $\gamma_Z(0) = \sigma^2$ and $\gamma_Z(k) = 0,\ k \neq 0,\ -\pi < \omega < \pi$. Hence

$$\begin{aligned}
f_Z(\omega) &= \frac{1}{2\pi} \sum_{k=0}^{\infty} \gamma_Z(k) e^{-i\omega k} \\
&= \frac{1}{2\pi}
\end{aligned}$$

(ii) $f_X(\omega) = |\psi_j e^{-i\omega j}|^2 f_Z(\omega),\ -\pi < \omega < \pi.$

2. (i) $\alpha_1 = 1,\ \alpha_2 = -0.5$ satisfy $|\alpha_2| < 1$, $\alpha_2 - \alpha_1 < 1$ and $\alpha_1 + \alpha_2 < 1$. Therefore, $\{X_t\}$ is stationary.

(ii) Let $E\left(X_t\right) = \mu$ for all t. Therefore,

$$\mu = \star + \mu - 0.5\mu \Rightarrow \star = 0.5\mu = 18$$

(iii)

$$\begin{aligned}
\hat{x}_{t+1} &= 18 + x_t - 0.5x_{t-1} \\
\therefore \hat{x}_{85} &= 18 + x_{84} - 0.5x_{83} = 28.5
\end{aligned}$$

Therefore, a 90% FI for x_{t+1} is $\hat{x}_{t+1} \pm 1.645\hat{\sigma}$ or, in the case of $t = 85$,

$$\hat{x}_{85} \pm 1.645 \times 2 = (25.21, 31.79)$$

(iv)

$$\hat{x}_{t+2} = 18 + \hat{x}_{t+1} - 0.5x_t$$
$$\therefore \hat{x}_{86} = 18 + \hat{x}_{85} - 0.5x_{84} = 32.25$$

We also have that the error is

$$\begin{aligned} \epsilon_{t+2} &= X_{t+2} - \hat{X}_{t+2} \\ &= X_{t+1} - \hat{X}_{t+1} + Z_{t+2} \\ &= Z_{t+1} + Z_{t+2} \end{aligned}$$

Therefore, $Var(\epsilon_{t+2}) = 2\sigma^2 = 8$. A 90% FI for x_{t+2} is $\hat{x}_{t+2} \pm 1.645\sqrt{8} = 32.25 \pm 4.653$.

3. (ia)

$$\begin{aligned} \hat{X}_{t+\ell} &= E\left(X_{t+\ell}|Z_t, Z_{t-1}, \dots\right) \\ &= E\left(50 + 0.5X_{t+\ell-1} + Z_{t+\ell} - 0.2Z_{t+\ell-1} - 0.1Z_{t+\ell-3}|Z_t, Z_{t-1}, \dots\right) \end{aligned}$$

We have the cases of

$$\hat{X}_{t+\ell} = \left\{ \begin{array}{ll} 50 + 0.5X_t + 0.2Z_t - 0.1Z_{t-2} & ;\ell = 1 \\ 50 + 0.5\hat{X}_{t+1} - 0.1Z_{t-1} & ;\ell = 2 \\ 50 + 0.5\hat{X}_{t+2} - 0.1Z_t & ;\ell = 3 \end{array} \right\}$$

(ib) It is clear that for $\ell \geq 4$,

$$\hat{X}_{t+\ell} = 50 + 0.5\hat{X}_{t+\ell-1}$$
$$\Rightarrow \hat{X}_{t+\ell} - 100 = 0.5\left[\hat{X}_{t+\ell-1} - 100\right] \text{ since } E\left(X_t\right) = 100$$

Thus, as $\ell \to \infty$, $E\left(\hat{X}_{t+\ell-1} - 100 \to 0\right)$. In the long run, we always obtain the mean as our forecast. This is not very satisfactory. The model is good for short-term forecasting.

(ii) We are given

$$x_t = 109;\ x_{t-1} = 98; x_{t-2} = 90$$
$$\hat{x}_t = 108; x_{t-1} = 100; \hat{x}_{t-2} = 94$$
$$z_t = 1; z_{t-1} = -2; z_{t-2} = -4$$

We require the one-step forecast value using the model. Thus,

$$\begin{aligned} \hat{x}_{t+1} &= 50 + 0.5x_t - 0.2z_t - 0.1z_{t-2} \\ &= 50 + 0.5 - 0.2 \times 1 - 0.1 \times -4 \\ &= 104.7 \end{aligned}$$

The forecast error is

$$\epsilon_{t+1} = X_{t+1} - \hat{X}_{t+1} = Z_{t+1}$$

Therefore, $Var\left(\epsilon_{t+1}\right) = 1$ and a 95% FI for X_{t+1} is

$$104.7 \pm 1.96 or \left(102.74, 106.66\right)$$

4. We have that

$$\begin{aligned}
\gamma_0 &= \left(1+0.7^2\right) \times 1.5^2 = 3.3525 \\
\gamma_1 &= (0.7) \times 1,5^2 = 1.575 \\
\gamma_k &= 0,\ \ k \geq 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_X(\omega) &= \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{j=1}^{\infty}\gamma_j \cos(\omega_j)\right] \\
&= \frac{1}{2\pi}\left[\gamma_0 + 2\gamma_1 \cos(\omega)\right] \\
&= \frac{1}{2\pi}\left[3.3525 + 3.15\cos(\omega)\right].
\end{aligned}$$

and

$$f_X^*(\omega) = \frac{f_X(\omega)}{\gamma_0} = \frac{1}{2\pi \times 3.3535}\left[3.3525 + 3.15\cos(\omega)\right]$$

(ii) Use calculus to sketch.

Computer Exercise Computer Exercise: Submit your answers to Q3 by 11.59 on Monday 11 April

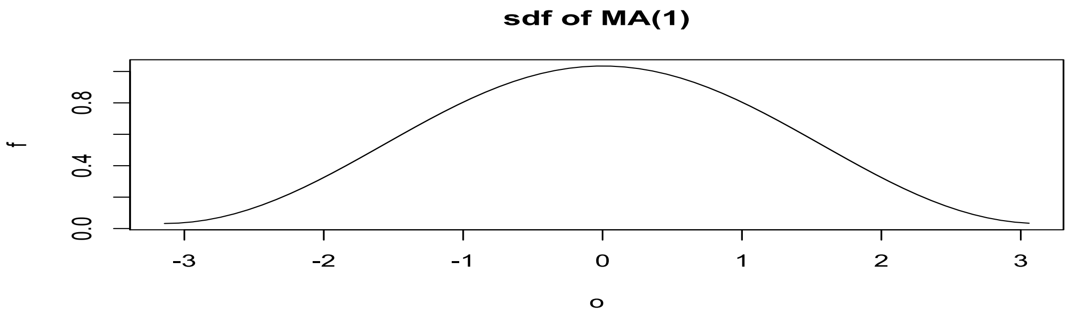
1. (i)
```
x=scan()
```
(ii)
```
d1=diff(x)
ts.plot(d1)
acf(d1)
pacf(d1)
```
The series look non-stationary with a trend.

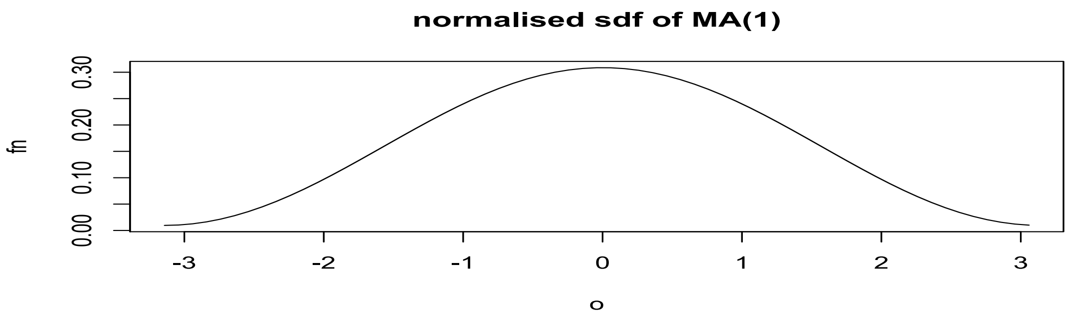
(iii)
```
d2=diff(d1)
ts.plot(d2)
acf(d2)
pacf(d2)
```
The series in d2 looks stationary.

(iv) $d = 2$ is the best value for d.

(v)
```
ts.plot(d2)
acf(d2)
pacf(d2)
```

2. (i)
```
o=c(-32:32)/10   # to create a vector for omega OR o= seq(-pi,pi,0.1)
s2=1.5^2              # variance of the noise
f=(3.3525+3.15*cos(o))/(2 pi) # take from Q4
plot(o,f, type="l", main="sdf of MA(1)")
```





3. (i)

```
fit1=arima(d2, order=c(1,0,0))
fit1

Call:
arima(x = d2, order = c(1, 0, 0))

Coefficients:
ar1  intercept
0.7318     0.0045
s.e.  0.0483     0.0045

sigma^2 estimated as 0.0002977:  log likelihood = 522.5,  aic = -1038.99

fit2=arima(d2, order=c(2,0,0))
fit2

Call:
arima(x = d2, order = c(2, 0, 0))

Coefficients:
ar1     ar2  intercept
0.564  0.2293     0.0043
s.e.  0.069  0.0692     0.0056

sigma^2 estimated as 0.0002819:  log likelihood = 527.84,  aic = -1047.67

fit3=arima(d2, order=c(3,0,0))
fit3

Call:
arima(x = d2, order = c(3, 0, 0))

Coefficients:
ar1     ar2      ar3  intercept
0.6275  0.3843  -0.2754     0.0045
s.e.  0.0680  0.0766   0.0681     0.0043

sigma^2 estimated as 0.0002601:  log likelihood = 535.67,  aic = -1061.35

fit4=arima(d2, order=c(0,0,3))
fit4
```

```
Call:
arima(x = d2, order = c(0, 0, 3))

Coefficients:
ma1     ma2     ma3  intercept
0.5341  0.6813  0.3719     0.0045
s.e.  0.0608  0.0564  0.0580     0.0030

sigma^2 estimated as 0.0002709:  log likelihood = 531.58,  aic = -1053.15

fit5=arima(d2, order=c(1,0,2))
fit5

Call:
arima(x = d2, order = c(1, 0, 2))

Coefficients:
ar1      ma1     ma2  intercept
0.6810  -0.1112  0.4202     0.0044
s.e.  0.0716   0.0794  0.0757     0.0046

sigma^2 estimated as 0.0002539:  log likelihood = 537.99,  aic = -1065.98
```

(ii) The model with the smallest AIC value is ARIMA(1,2) for the data in d2. The constant can be dropped out since its se is 0.0046 and the t-ratio is 0.957 (small). The fitted model is

$$Y_t = 0.6810Y_{t-1} + Z_t - 0.1112Z_{t-1} + 0.4202Z_{t-2},\ Var(Z_t) = 0.0002593$$

(iii)
```
tsdiag(fit1)
tsdiag(fit2)
tsdiag(fit3)
tsdiag(fit4)
tsdiag(fit5)
```

Largest p-values for the GOF statistic is for fit5. This confirms the ARMA(1,2) as the best model.

(iv)
```
forecast=predict(fit5,n.ahead=5,se.fit=T)
forecast


$pred
Time Series:
Start = 199
End = 203
Frequency = 1
[1] 0.012828029 0.012496967 0.009917557 0.008160866 0.006964485

$se
Time Series:
Start = 199
End = 203
Frequency = 1
[1] 0.01593570 0.01834138 0.02241245 0.02406806 0.02479846
```

(v)
```
ts.plot(cbind(forecast$pred, forecast$se), lty=c(1,2))
legend("topleft", c("Forecast Values", "---SEs"))
```

(vi)
```
ts.plot(cbind(forecast$pred,forecast$pred+1.96*forecast$se, forecast$pred,forecast$pred-1.96*forecast$se),
lty=c(1,2,3))
legend("bottomleft", c("Forecast Values", "---UL", "---LL"))
```