THE UNIVERSITY OF SYDNEY

Mathematical Statistics: STAT3925/STAT4025 - Semester 1 - 2022

Time Series Analysis : Problem Set - Week 11 (Tutorial and Computer Problems)

Attempt these questions before your class and discuss any issues with your tutor
Go to your assigned tutorial class/Lab and record your attendance

1. Suppose that $(I-B)^d X_t = Z_t,\ -0.5 < d < 0.5,\ \{Z_t\} \sim WN(0,\sigma^2)$ is a FDWN.
   (i) Find the sdf of this ARFIMA(0,d,0) for $0 < \omega < \pi$ when $0 < d < 0.5$ and show that it is unbounded as $\omega \to 0$.
   (ii) Find $\psi_k,\ k \geq 0$ in terms of the gamma function such that $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$.
   (iii) Let γ_k be the acf at lag k. Show that $\gamma_k = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{k+j} = \frac{(-1)^k \Gamma(1-2d)}{\Gamma(-d+k+1)\Gamma(-d-k+1)}$. *Hint:* You may use the fact that $\sum_{j=0}^{\infty} \frac{\Gamma(j+a)\Gamma(j+b)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(j+c)\Gamma(j+1)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ for all $c-a-b > 0$.
   (iv) Using (iii), deduce that the acf at lag k, $\rho_k = \frac{\Gamma(k+d)\Gamma(1-d)}{\Gamma(k-d+1)\Gamma(d)}$.
   (v) Find the pacf at lags 1 and 2 of of this FDWN.

2. Briefly explain a regression method in the frequency domain to estimate the fractional degree of differencing d in ARFIMA(0,d,0) of Q1.

3. Find the ℓ-step-ahead forecast function generated by $(1-B)^d X_t = Z_t$, where $0 < d < 0.5$ and $\{Z_t\} \sim WN(0,\sigma^2)$. What is the corresponding forecast error?

4. Suppose that $X \sim N(0,1)$. Show that

$$E(X^m) = \begin{cases} 0, & \text{when } m \text{ odd} \\ 2^{-m/2} \frac{m!}{(m/2)!}, & \text{when } m \text{ is even} \end{cases} \tag{1}$$

5. A stationary process $\{X_t\}$ satisfies

$$X_t = \sigma_t \epsilon_t,\ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2,$$

   where $\{\epsilon_t\}$ is a sequence of iid random variables with mean zero and variance 1, $\alpha_0,\ \alpha_1 > 0$.
   (i) *Find* the unconditional mean, $E(X_t)$ and the conditional mean, $E(X_t|F_{t-1})$, where F_{t-1} stands for the information set available at $t-1, t-2, \ldots$.
   (ii) *Find* the unconditional variance, $Var(X_t)$ and the conditional variance, $Var(X_t|F_{t-1})$.
   (iii) *Show* that $E(X_t^2) = \frac{\alpha_0}{1-\alpha_1}$. Hence *give* an upper bound for α_1.
   (iv) Let $\eta_t = X_t^2 - \sigma_t^2$. Asuuming $\{\eta_t\}$ is a martingale difference sequence, *show* that X_t^2 follows an AR(1) process.
   (v) Find the acf of $\{X_t^2\}$.

6. Find the ℓ-step-ahead forecast function of X_t and X_t^2 for the process given in Q5.

PTO for W11 Computer Exercise

Computer Exercise

Working with R

- Download and install the package fracdiff. The type

  library(fracdiff)

  To simulate 500 vlues from ARFIMA(p,d,q) use

  - b= fracdiff.sim(500, ar = c($\alpha_1, \ldots, \alpha_p$), ma = c($\beta_1, \ldots, \beta_q$), d = d)
  - c=b$series (to extract the time series component in b)

- Download and Install the package TSA. Then type

  library(TSA)

  To simulate 300 values from $X_t = \sigma_t \epsilon_t$; $\sigma_t^2 = 0.5 + 0.3X_{t-1}^2$ and store in b use:

  - d=garch.sim(alpha=c(0.5, 0.3), n = 300)

To get the ts.plot acf, pacf of d use

- ts.plot(d)
- acf(d)
- pacf(d)

Computer Problems - Week 11

Submit Q2 and Q6 by 23.59 on Monday 9 May

1. Simulate 1300 values from ARFIMA(1,0.35,1) with $\alpha = 0.7$ and $\beta = 0.4$. After discarding the first 300 values store the remaining values in s. Obtain the tsplot, acf, pacf and the periodogram of the data in s and comment.

2. Now suppose that Peter wants to fit an ARFIMA(1,d,1) model for the data in s and estimate the corresponding parameters. Estimate all parameters of this model and their corresponding se's. Report 95% CI for these parameters.

3. Using set.seed(100) to simulate 1000 values from $X_t = \sigma_t \epsilon_t$; $\sigma_t^2 = 0.7 + 0.4X_{t-1}^2$ After discarding the first 200 values, store the remainder in $d1$. Draw the ts plot, sample acf, sample pacf for the data in $d1$ and comment.

4. Draw the sample periodogram of your data in Q3 and comment.

5. Store the squared values of $d1$ from Q3 in $d2$. Draw the ts plot, sample acf, sample pacf and periodogram for the data in $d2$ and comment.

6. Consider the data in $d2$. Fit AR(1), AR(2), MA(1), MA(2) and ARMA(1,1) models for this data set. Select the best possible fit for the data in $d2$ based on the aic criterion.