THE UNIVERSITY OF SYDNEY

Mathematical Statistics: STAT3925/STAT4025 - Semester 1 - 2022

Time Series Analysis : Problem Set - Week 10 (Tutorial and Computer Problems)

Attempt these questions before your class and discuss any issues with your tutor
Go to your assigned tutorial class/Lab and record your attendance

1. The periodogram for a stationary zero mean time series, based on n observations $X_1, X_2, \dots, X_n$ is defined as $I_{n,X}(\omega) = \dfrac{2}{n}\left[\left(\sum_{t=1}^{n} X_t \cos\omega\, t\right)^2 + \left(\sum_{t=1}^{n} X_t \sin\omega\ t\right)^2\right]$, where $\omega \in (-\pi,\ \pi)$.

   (i) Prove that $I_{n,X}(\omega) = 2\left\{\hat{R}_X(0) + 2\sum_{k=1}^{n-1}\hat{R}_X(k)\cos\omega\, k\right\}$, where $\hat{R}_X(k)$ is the standard estimate of the autocovariance function at lag k.

   (ii) Show that $I^*_{n,X}(\omega)$ is asymptotically unbiased for the non-normalized spectral density function $f_X(\omega)$, of $\{X_t\}$, where $I^*_{n,X}(\omega) = \dfrac{1}{4\pi} I_{n,X}(\omega)$.

2. Let $W_n(\theta) = \frac{1}{2\pi}\sum_{s=-(n-1)}^{n-1}\lambda_s e^{-i\theta s}$ be the spectral window corresponding to a lag window $\{\lambda_s\}$. Write down the spectral window corresponding to the lag window

$$\lambda(s) = \begin{cases} 1 - 2a + 2a\cos(\frac{\pi s}{m}) & ;\ |s| \le m \\ 0 & ;\ \text{otherwise} \end{cases}$$

   and show that it is given by $W(\theta) = aD_m(\theta - \frac{\pi}{m}) + (1-2a)D_m(\theta) + aD_m(\theta + \frac{\pi}{m})$, where

$$D_m(\alpha) = \frac{1}{2\pi}\sum_{|k|\le m} e^{-ik\alpha} = \frac{1}{2\pi}\,\frac{\sin(m+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}$$

   is the Dirichlet kernel.

3. Consider the following model for a time series $\{X_t\}$ is given by $(1-\alpha B)(1-B)^{\delta}X_t = (1+\theta B)Z_t$, where $-1 < \delta < 1$ is a real valued parameter and $Z_t \sim WN(0, \sigma^2)$.

   (i) Using the result that the acf at lag k, $\rho_k \sim k^{2\delta-1}$ as $k \to \infty$, *investigate* the behavior of ρ_k as $k \to \infty$ in each case below:

   (a) $\delta < 0$, (b) $0 < \delta < 0.5$, (c) $\delta \ge 0.5$.

   (ii) Find the sdf $f_X(\omega)$. *Investigate* the behavior of $f_X(\omega)$ as $\omega \to 0$, in each case (a), (b) and (c) in (i).

   (iii) *Deduce that* the sdf $f_X(\omega)$ exists near $\omega = 0$ if $\delta < 0$.

   (iv) *State* the values of δ such that the sdf $f_X(\omega)$ is unbounded as $\omega \to 0$.

   (v) *Deduce that* the process $\{X_t\}$ has long memory if $0 < \delta < 0.5$.

PTO for the computer exercise

Computer Exercise - W10

Submit Q4 by 23.59 on Monday 2 May

1. Write down the theoretical spectrum of $\{X_t\}$ given by $X_t = 0.4X_{t-1}+0.4X_{t-2}+Z_t+0.5Z_{t-1}+0.6Z_{t-2}$, where $\{Z_t\} \sim WN(0, 2.5^2)$.

   (i) Draw the spectrum in $(-\pi, \pi)$ using the formula.

   (ii) Compare the spectrum in (i) using the command arma.spec in **R**.

2. Simulate 500 values from the above model using:

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arima.sim(n = 500, list(ar = c(0.4, 0.4), ma = c(0.5, 0.6)), sd = sqrt(2.5^2))
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   After discarding the first 250 values, draw the sample periodogram of this data. Comment on this periodogram comparing with the spectrum in Q1.

3. Suppose that the spectrum of a stationary process is given by $f_X(\omega) = [2\sin(\omega/2)]^{-2d}\frac{\sigma^2}{2\pi},\ 0 < \omega < \pi$. Plot this sdf for $\omega > 0,\ d = 0.4$ and $\sigma^2 = 3$. Investigate its behaviour as $\omega \to 0$ by taking $\omega = 0.1, 0.01, 0.001, 0.0001, 0.00001$. What do you notice?

4. Consider the time series of 500 observations given below:

   11.60 11.38 10.67 9.18 10.02 10.12 10.36 11.43 10.51 9.32 9.73 9.95 7.66 7.10 7.90 6.71 8.32 9.25 10.03 9.24 9.12 10.59 10.26 11.63 12.03 11.20 12.15 12.98 13.85 14.13 14.47 13.81 14.72 14.19 14.30 13.63 14.08 14.69 14.67 15.66 16.50 16.54 16.10 13.46 11.85 13.05 12.80 13.08 12.57 13.50 13.08 12.44 12.11 11.70 12.37 12.65 11.55 10.88 12.11 12.17 13.32 12.84 12.26 11.27 11.34 10.99 11.07 11.53 12.22 10.36 12.69 11.47 9.78 9.51 10.33 11.55 13.34 12.06 12.09 12.42 10.78 9.67 9.41 10.21 10.33 10.73 9.11 9.82 11.86 11.34 12.52 12.58 12.10 13.08 12.09 14.03 14.95 15.18 16.29 15.85 14.09 13.62 13.25 13.64 13.06 12.04 12.26 12.00 10.37 8.87 8.38 9.54 10.83 9.52 8.78 8.06 7.92 8.65 9.59 10.48 11.70 11.32 10.65 12.29 13.55 14.43 14.39 14.44 13.45 13.12 13.46 13.88 14.32 12.69 12.86 12.01 11.36 10.62 10.87 11.16 11.56 12.41 12.21 12.95 12.29 11.32 11.30 11.93 12.22 12.80 13.51 12.77 13.51 14.67 13.65 13.07 15.91 16.20 17.43 16.67 17.20 16.95 17.98 16.92 16.12 13.14 13.90 13.43 13.26 14.95 14.81 14.24 14.76 14.21 14.40 14.77 15.26 15.81 16.92 16.45 16.82 16.97 16.78 16.78 16.21 15.83 15.89 15.29 14.07 13.28 12.22 11.97 11.93 12.86 12.69 12.58 12.76 13.86 13.37 11.07 10.18 9.26 8.73 8.78 9.18 9.53 11.11 11.30 10.22 10.68 12.23 11.25 10.83 11.69 9.95 8.79 8.65 9.36 9.05 8.68 9.63 8.40 8.92 10.47 10.07 9.80 9.27 10.14 9.68 10.75 9.55 10.42 9.72 8.52 8.92 6.69 6.62 8.44 8.50 10.01 10.54 11.43 12.95 12.26 13.10 12.46 12.94 12.55 11.96 11.63 10.52 9.50 9.33 8.46 7.96 6.97 7.37 7.94 9.13 9.13 8.40 9.35 10.60 10.94 11.75 10.76 9.74 9.07 9.00 10.03 10.66 10.04 10.18 11.09 11.89 13.56 13.20 13.34 14.10 13.57 12.94 10.02 8.47 8.77 8.35 7.82 8.92 8.49 7.93 7.52 8.35 8.79 10.47 8.39 8.68 9.01 9.51 8.56 9.59 9.97 7.66 8.02 7.60 7.97 7.22 8.36 8.90 8.91 7.27 6.20 7.13 7.22 6.98 7.35 7.66 9.85 9.78 11.21 10.14 10.33 9.23 8.87 7.94 8.42 8.70 8.29 9.06 8.81 9.02 9.21 8.86 9.13 10.22 10.44 11.38 10.91 8.89 7.83 7.60 7.18 7.64 9.26 9.09 9.85 9.71 7.76 6.55 6.64 8.59 9.97 8.65 9.06 8.52 7.03 4.72 5.19 6.08 7.25 9.01 9.46 9.37 9.94 10.73 11.25 11.50 10.93 10.36 10.93 11.17 11.67 10.88 11.29 12.12 11.08 10.63 9.36 8.37 8.45 8.64 8.85 9.03 7.14 5.58 5.84 6.04 5.58 4.76 4.26 5.61 4.89 4.59 4.56 5.62 4.62 4.38 4.58 5.33 7.07 7.70 9.04 9.76 9.11 7.62 6.47 4.71 5.13 4.51 5.12 7.30 6.52 7.70 8.28 9.97 9.13 9.03 7.99 7.95 7.56 8.57 8.11 7.36 5.67 4.42 3.98 3.13 5.99 6.10 4.89 5.00 5.72 6.47 7.90 6.12 5.21 5.32 3.84 3.60 4.48 5.64 7.06 6.82 5.75 4.01 2.49 2.29 3.99 4.67 6.14 7.79 10.06 9.41 6.70 6.62 5.02 4.40 5.65 5.46 5.60 7.65 6.94 6.53 7.31 8.80 10.42 10.87 11.22 12.39 11.72 10.96 11.65 10.50 10.34 9.71 9.02 8.24 7.98 7.34 8.27 8.89 8.73 7.23 6.96 5.22 3.74 2.13 2.30 3.27 2.80 4.38 5.58 6.87 6.54 6.02 5.71 6.72 6.71 5.55 3.55 3.71 2.60

   (i) Plot the acf and the periodogram of this data and comment.

   (ii) Show that the large values of the periodogram occur near the origin. Give the first five values of the spectrum near the origin. What are the corresponding frequencies?