when m is even use the integration by parts: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$E(X^m) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{m-1}(xe^{-x^2/2})dx$$

$$u = x^{m-1} \quad ; \quad \frac{dv}{dx} = xe^{-x^2/2} \Rightarrow v = -e^{-\frac{1}{2}x^2}$$

$$\therefore E(X^m) = \frac{1}{\sqrt{2\pi}}\left(\left[-x^{m-1}e^{-\frac{1}{2}x^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\cdot(m-1)x^{m-2}dx\right)$$

$$= 0 + E(X^{m-2})$$

$$= (m-1)(m-3)E(X^{m-4})$$

$$= (m-1)(m-3)(m-5)\cdots\times 3\times 1$$

$$= \frac{(m-1)(m-3)(m-5)\cdots 3\times2\times1\times m(m-2)(m-4)\times\cdots\times4\times2}{m(m-2)(m-4)\cdots\cdots 4\times 2}$$

$$= \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)\left(\frac{m}{2}-1\right)\left(\frac{m}{2}-2\right)\cdots 2\times1}$$

$$= \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!}$$

5 (i) $E(X_t) = E(\sigma_t\xi_t) = E[E(\sigma_t\xi_t|F_{t-1})] = E[\sigma_t E(\xi_t|F_{t-1})] = E[\sigma_t\times 0] = 0$

$E(X_t|F_{t-1}) = E(\sigma_t\xi_t|F_{t-1}) = \sigma_t E(\xi_t|F_{t-1}) = \sigma_t\times 0 = 0$

(ii) $Var(X_t) = E(X_t^2) = E[E(\sigma_t^2\xi_t^2|F_{t-1})] = E[\sigma_t^2 E(\xi_t^2|F_{t-1})] = E(\sigma_t^2)$

$Var(X_t|F_{t-1}) = Var(\sigma_t\xi_t|F_{t-1}) = \sigma_t^2 Var(\xi_t|F_{t-1}) = \sigma^2$

(iii) $E(\sigma_t^2) = \alpha_0 + \alpha_1 E(X_{t-1}^2) \Rightarrow E(X_t^2) = \alpha_0 + \alpha_1 E(X_t^2)$ since $\{X_t\}$ is stationary

$$\therefore E(X_t^2) = \frac{\alpha_0}{1-\alpha_1}$$

Since $E(X_t^2) > 0$, $\alpha_0 > 0 \Rightarrow 1-\alpha_1 > 0$ or $0 < \alpha_1 < 1$

(iv) $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 \Rightarrow X_t^2 - \eta_t = \alpha_0 + \alpha_1 X_{t-1}^2$

$$\therefore X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \eta_t$$

Since $\{\eta_t\}$ is a martingale difference, we have $X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \eta_t$ is an AR(1) process.

② $f_X(\omega) = \frac{\sigma^2}{\pi} \left(2\sin\frac{\omega}{2}\right)^{-2d}$

$\ln f_X(\omega) = \ln\left(\frac{\sigma^2}{\pi}\right) - d \ln\left(2\sin\frac{\omega}{2}\right)^2$

$\ln[I_X(\omega)] = -d\ln\left(4\sin^2\frac{\omega}{2}\right) + \ln I_X(\omega) - \ln f_X(\omega) + \ln\left(\frac{\sigma^2}{\pi}\right)$

$= -d\ln\left(4\sin^2\frac{\omega}{2}\right) + \ln\left[\frac{I_n(\omega)}{f_X(\omega)}\right] + \ln\left(\frac{\sigma^2}{\pi}\right)$

Let $y_j = \ln[I_X(\omega_j)]$; $x_j = \ln\left[4\sin^2\frac{\omega_j}{2}\right]$; $b = -d$; $\alpha = \ln\left(\frac{\sigma^2}{\pi}\right)$

Taking $\omega_j = \frac{\pi j}{n}$; $j = 1, 2, \cdot\cdot$; $n = n^\alpha$; $0 < \alpha < 1$
we have the regression model

$$y_j = \alpha + bx_j + e_j \quad ; \quad e_j = \ln\left[\frac{I_n(\omega_j)}{f_X(\omega_j)}\right]$$

$$\hat{b} = \frac{\sum_{j=1}^{m} (x_j - \bar{x}) y_j}{\sum_{j=1}^{m} (x_j - \bar{x})^2}; \quad m < n^\alpha; \; 0 < \alpha < 1$$

③ $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$

$\hat{X}_{t+l} = E(X_{t+l} \mid F_{t-1}) = \sum_{j=l}^{\infty} \psi_j Z_{t+l-j}$

$\varepsilon_{t+l} = X_{t+l} - \hat{X}_{t+l} = \sum_{j=0}^{l-1} \psi_j Z_{t+l-j}$

④ $E(X^m) = \int_{-\infty}^{\infty} x^m \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^m e^{-\frac{1}{2}x^2} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x)\,dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\,dx$

$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{0} f(-y)(-dy) + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\,dx$

$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(-y)\,dy + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\,dx$

when m is odd $f(-y) = f(y) \Rightarrow E(X^m) = 0$

(iv) $\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{(-1)^k \sigma^2 \Gamma(1-2d)}{\Gamma(-d+k+1)\Gamma(-d-k+1)} \times \frac{\Gamma(-d+1)\Gamma(-d+1)}{\sigma^2 \Gamma(1-2d)} = \frac{(-1)^k \sigma^2 \Gamma^2(1-d)}{\Gamma(-d+k+1)\Gamma(-d-k+1)}$

$\Gamma(1-d) = (-d)(-d-1)\ldots(-d-k+1)\Gamma(-d-k+1)$

$\therefore \frac{\Gamma(1-d)}{\Gamma(-d-k+1)} = (-1)^k \frac{(d+k-1)\ldots(d)\Gamma(d)}{\Gamma(d)} = (-1)^k \frac{\Gamma(d+k)}{\Gamma(d)}$

$\therefore \rho_k = \frac{(-1)^{2k}\Gamma(1-d)\Gamma(d+k)}{\Gamma(k-d+1)\Gamma(d)} = \frac{\Gamma(1-d)\Gamma(d+k)}{\Gamma(k-d+1)\Gamma(d)}$

(v) $\rho_1 = \frac{\Gamma(1-d)\Gamma(1+d)}{\Gamma(2-d)\Gamma(d)} = \frac{\Gamma(1-d)\, d\, \Gamma(d)}{(1-d)\Gamma(1-d)\Gamma(d)} = \frac{d}{1-d}$

$\therefore \pi_1 = \rho_1$

$\rho_2 = \frac{\Gamma(1-d)\Gamma(2+d)}{\Gamma(3-d)\Gamma(d)} = \frac{\Gamma(1-d)(d+1)\, d\, \Gamma(d)}{(2-d)(1-d)\Gamma(1-d)\Gamma(d)} = \frac{d(d+1)}{(2-d)(1-d)}$

$\pi_2 = \frac{\rho_2 - \rho_1}{1-\rho_1^2}$

$= \frac{\frac{d(d+1)}{(2-d)(1-d)} - \left(\frac{d}{1-d}\right)^2}{1-\left(\frac{d}{1-d}\right)^2}$

$= \frac{\frac{d(d+1)(1-d) - d^2(2-d)}{(2-d)(1-d)^2}}{\frac{(1-d)^2 - d^2}{(1-d)^2}}$

$= \frac{d - \cancel{d^3} - 2d^2 + \cancel{d^3}}{(1-2d)(2-d)}$

$= \frac{d(1-2d)}{(1-2d)(2-d)}$

$= \frac{d}{2-d}$

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① (i) Notes: $f_X(\omega) = \frac{\sigma^2}{\pi}\left(2\sin\frac{\omega}{2}\right)^{-2d}$; $0<\omega<\pi$

as $0<\omega<\pi$, $\sin\frac{\omega}{2}>0$ & $\sin\frac{\omega}{2}\to 0$ as $\omega\to 0$.

$\therefore \left(\sin\frac{\omega}{2}\right)^{-2d} = \frac{1}{\left(\sin\frac{\omega}{2}\right)^{2d}}$ is undefined when $d>0$, as $\omega\to 0$

(ii) $(1-B)^{-d} = \sum_{j=0}^{\infty}\psi_j B^j$,

where $\psi_j = (-1)^j\binom{-d}{j} = (-1)^j\frac{(-d)(-d-1)\cdots(-d-j+1)}{j!}$

$$= \frac{(-1)^{2j}(d)(d+1)\cdots(d+j-1)}{j!}$$

$$= \frac{(d+j-1)\cdots(d+1)(d)\,\Gamma(d)}{j!\,\Gamma(d)}$$

$$= \frac{\Gamma(d+j)}{j!\,\Gamma(d)}$$

(iii) $\gamma_k = E(X_tX_{t-k})$ since $E(X_t)=0$

$$= E\left[\left(\sum_{j=0}^{\infty}\psi_j Z_{t-j}\right)\left(\sum_{l=0}^{\infty}\psi_l Z_{t-k-l}\right)\right]$$

$$= \sigma^2(\psi_k\psi_0+\psi_{k+1}\psi_1+\cdots) = \sigma^2\sum_{j=0}^{\infty}\psi_j\psi_{j+k}$$

$$= \sigma^2\sum_{j=0}^{\infty}\frac{\Gamma(j+d)\Gamma(j+k+d)}{j!\,\Gamma(d)(j+k)!\,\Gamma(d)}$$

$$= \sigma^2\cdot\frac{\Gamma(b)}{\Gamma(a)\Gamma(c)}\sigma^2\sum_{j=0}^{\infty}\frac{\Gamma(j+a)\Gamma(j+b)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(j+c)\Gamma(j+1)}$$; $j! = \Gamma(j+1)$; $(j+k)! = \Gamma(j+k+1)$

where $a=d$, $b=k+d$, $c=k+1$)

$$= \sigma^2\frac{\Gamma(b)}{\Gamma(a)\cancel{\Gamma(c)}}\cdot\frac{\cancel{\Gamma(k+1)}\Gamma(1-2d)}{\Gamma(k-d+1)\Gamma(1-d)}$$

$$= \sigma^2\frac{\Gamma(k+d)\Gamma(1-2d)}{\Gamma(d)\Gamma(k-d+1)\Gamma(1-d)}$$

Now $\Gamma(k+d) = (k+d-1)(k+d-2)\cdots d\,\Gamma(d)$

$\Gamma(-d+1) = (-d)(-d-1)\cdots\cdots(-d-k+1)\Gamma(-d-k+1)$

$= (-1)^k(d)(d+1)\cdots(d+k-1)\Gamma(-d-k+1)$

$= (-1)^k\frac{\Gamma(k+d)}{\Gamma(d)}\Gamma(-d-k+1)$

$$\therefore \gamma_k = \frac{\sigma^2\cancel{\Gamma(k+d)}\Gamma(1-2d)\cancel{\Gamma(d)}}{\cancel{\Gamma(d)}\Gamma(k-d+1)(-1)^k\cancel{\Gamma(k+d)}\Gamma(-d-k+1)} = \frac{(-1)^k\sigma^2\Gamma(1-2d)}{\Gamma(k-d+1)\Gamma(-d-k+1)}$$