① a) $X_t^2 = \alpha_0 + \alpha X_{t-1}^2 + \eta_t$

$X_t^2 - \mu = \alpha(X_{t-1}^2 - \mu) + \eta_t$, where $\mu = E(X_t^2)$

$Y_t - \mu = \alpha(Y_{t-1} - \mu) + \eta_t$

$\gamma_k = E[(Y_t - \mu)(Y_{t-k} - \mu)]$

Multiplying by $Y_{t-k} - \mu$ and taking expectations, we have

$E[(Y_t-\mu)(Y_{t-k}-\mu)] = \alpha E[(Y_{t-1}-\mu)(Y_{t-k}-\mu)] + E[\eta_t(Y_{t-k}-\mu)]$

$k \geq 1$, $\gamma_k = \alpha\gamma_{k-1}$

b) $k=1 \Rightarrow \gamma_1 = \alpha\gamma_0 \Rightarrow \hat{\alpha} = \frac{\hat{\gamma}_1}{\hat{\gamma}_0}$

c) $E(X_t^2) = \alpha_0 + \alpha E(X_{t-1}^2)$

$\therefore \mu = \alpha_0 + \alpha\mu \Rightarrow \hat{\alpha}_0 = \hat{\mu}(1-\hat{\alpha})$

d) $\eta_t = X_t^2 - \sigma_t^2 = (1-\varepsilon_t^2)\sigma_t^2$

$\therefore Var(\eta_t) = E(\eta_t^2) = E[(1-\varepsilon_t^2)^2\sigma_t^4]$

$= E[E\{(1-\varepsilon_t^2)^2\sigma_t^4 \mid F_{t-1}\}]$

$= E[\sigma_t^4 E(1-2\varepsilon_t^2+\varepsilon_t^4 \mid F_{t-1})]$

$= E[\sigma_t^4(1-2+K)]$, $K = E(\varepsilon_t^4)$

$= (K-1)E(\sigma_t^4)$

$= 2E(\sigma_t^4)$ for a Gaussian $\{\varepsilon_t\}$ with $K=3$.

② We $\hat{\sigma}_{t+l}^2 = E(\sigma_{t+l}^2 \mid F_t)$ & $\hat{X}_{t+l}^2 = E(X_{t+l}^2 \mid F_t)$

$= E[\alpha_0 + \alpha_1 X_{t-1+l}^2 + \cdots + \alpha_p X_{t-p+l}^2 \mid F_t]$

$l=1$, $\hat{\sigma}_{t+1}^2 = \alpha_0 + \alpha_1 X_t^2 + \alpha_2 X_{t-1}^2 + \cdots + \alpha_p X_{t-p+1}^2$

$l=2$, $\hat{\sigma}_{t+2}^2 = \alpha_0 + \alpha_1 \hat{X}_{t+1}^2 + \alpha_2 X_t^2 + \cdots + \alpha_p X_{t-p+2}^2$

$\vdots$

$\hat{\sigma}_{t+p}^2 = \alpha_0 + \alpha_1 \hat{X}_{t-1+p}^2 + \cdots + \alpha_{p-1}\hat{X}_{t-p+1}^2 + \alpha_p X_t^2$

$l \geq p+1$, $\hat{\sigma}_{t+l}^2 = \alpha_0 + \alpha_1 \hat{X}_{t-1+l}^2 + \cdots + \alpha_p \hat{X}_{t-p+l}^2$

③ $E(x_t)=0 \Rightarrow E(R_t)=\theta E(R_{t-1}) \Rightarrow E(R_t)=0$

$$Var(R_t)=\theta^2 Var(R_{t-1})+Var(x_t)$$

since $Var(R_t)=Var(R_{t-1})$ & $Var(x_t)=E(x_t^2)=\frac{\alpha_0}{1-(\alpha+\beta)}=\mu^*$, say

$$Var(R_t)=\frac{\mu^*}{1-\theta^2}$$

④ let $r=\max(p,q)$ & write

$$\sigma_t^2=\alpha_0+\sum_{j=1}^{r}\alpha_j x_{t-j}^2+\sum_{k=1}^{r}\beta_k\sigma_{t-k}^2$$

use: $\hat{\sigma}_{t+l}^2=\alpha_0+\sum_{j=1}^{r}\alpha_j E(x_{t+l-j}^2|F_t)+\sum_{k=1}^{r}\beta_k E(\hat{\sigma}_{t+l-k}^2|F_t)$

$l=1,2,\cdots,r$

$$\hat{\sigma}_{t+l}^2=\alpha_0+\sum_{j=1}^{r}\alpha_j x_{t+l-j}^2+\sum_{k=1}^{r}\beta_j\sigma_{t+l-k}^2$$

$l\geq r+1$, $\hat{\sigma}_{t+l}^2=\alpha_0+\sum_{j=1}^{r}\alpha_j\hat{x}_{t+l-j}^2+\sum_{k=1}^{r}\beta_j\hat{\sigma}_{t+l-k}^2$.

⑤ use $\hat{R}_{t+l}=E(R_{t+l}|F_t)=$ [illegible]$(R_{t+l-1}|F_t)+E(x_{t+l}|F_t)$

$l=1$, $\hat{R}_{t+1}=\theta R_t$ since $E(x_{t+1}|F_t)=0$

$l\geq 2$, $\hat{R}_{t+l}=\theta\hat{R}_{t+l-1}$

$$\hat{R}_{t+l}^2=E[(\theta R_{t+l-1}+x_t)^2|F_t]$$
$$=E[\theta^2R_{t+l-1}^2+2\theta R_{t+l-1}x_t+x_{t+l}^2|F_t]$$

$l=1$, $\hat{R}_{t+1}^2=\theta^2R_t^2+\hat{x}_{t+1}^2$

$l\geq 2$, $\hat{R}_{t+l}^2=\theta^2\hat{R}_{t+l-1}^2+\hat{x}_{t+l}^2$,

where $\hat{x}_{t+l}^2$ are obtained from

$$x_t^2=\alpha_0+(\alpha+\beta)x_{t-1}^2+\eta_t-\beta\eta_{t-1} \quad ; \quad \eta_t=x_t^2-\sigma_t^2$$

$$\hat{x}_{t+1}^2=\alpha_0+(\alpha+\beta)x_t^2-\beta\eta_t$$

$l\geq 2$ $\hat{x}_{t+l}^2=\alpha_0+(\alpha+\beta)\hat{x}_{t+l-1}^2$

⑥ (i) $x_t^2 - \eta_t = \alpha_0 + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \beta_1 (x_{t-1}^2 - \eta_{t-1})$

$$x_t^2 = \alpha_0 + (\alpha_1+\beta_1) x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \eta_t - \beta_1 \eta_{t-1} \qquad -(*)$$

this is an ARMA(2,1) model for $\{x_t^2\}$.

Since $E(x_t^2)$ is a constant, we have

$$E(x_t^2) = \alpha_0 + (\alpha_1+\beta_1) E(x_t^2) + \alpha_2 E(x_t^2)$$

$$\therefore E(x_t^2) = \frac{\alpha_0}{1-(\alpha_1+\beta_1)-\alpha_2}$$

(ii) Since $E(x_t^2) > 0$ & $\alpha_0 > 0$, we have $1-(\alpha_1+\beta_1)-\alpha_2 > 0$

or $0 < \alpha_1 + \beta_1 + \beta_2 < 1$ or $0 < 1 + \delta_1 + \delta_2 < 1$, $\delta_1 = \alpha_1 + \beta_1$; $\delta_2 = \alpha_2$

⑦ (i) Let $\mu = E(x^2)$

(*) in Q6: $\mu = \alpha_0 + (\alpha_1+\beta_1)\mu + \alpha_2 \mu \qquad -②$

①-②: $x_t^2 - \mu = (\alpha_1+\beta_1)(x_{t-1}^2 - \mu) + \alpha_2 (x_{t-2}^2 - \mu) + \eta_t - \beta_1 \eta_{t-1}$

Multiply by $x_{t-k}^2 - \mu$ and take expected values.

$$E[(x_t^2-\mu)(x_{t-k}^2-\mu)] = (\alpha_1+\beta_1) E[(x_{t-1}^2-\mu)(x_{t-k}^2-\mu)] + \alpha_2 E[(x_{t-2}^2-\mu)(x_{t-k}^2-\mu)] + E[\eta_t (x_{t-k}^2-\mu)] - \beta_1 E[\eta_{t-1}(x_{t-k}^2-\mu)].$$

For $k \geq 2$, $\gamma_k = (\alpha_1+\beta_1)\gamma_{k-1} + \alpha_2 \gamma_{k-2}$,

where $\gamma_k = \operatorname{Cov}(x_t^2, x_{t-k}^2)$.

(ii) See Q4.