Exercise 1

Given the atmospheric pressure function: $p(t) = 1 - \frac{12t}{57+16(t-2)^2}$

a) To determine if $p(t)$ is unimodal, we need to find the critical points by taking the first derivative $p'(t)$ and set it to 0.

Using Wolfram Alpha, we find that: $p'(t) = \frac{864-864t+192t^2}{(57+16(t-2)^2)^2}$

Setting $p'(t) = 0$ and solving for t, we find that there is only one critical point $t = 1$ over the interval $0 \leq t \leq 3$.

b) To determine if the critical point is a local minimum, we use the second derivative test.

Using Wolfram Alpha, we find that: $p''(t) = \frac{96(47-48t+16t^2)}{(57+16(t-2)^2)^3}$

Evaluating $p''(1)$, we get: $p''(1) = \frac{96(47-48+16)}{(57+16(1-2)^2)^3} = \frac{96 \cdot 15}{57^3} > 0$

Since $p''(1) > 0$, the critical point $t = 1$ is a local minimum.

c) To find the minimum pressure, we evaluate $p(t)$ at the local minimum and the interval boundaries:

$p(0) = 1 - \frac{12 \cdot 0}{57+16(0-2)^2} = \frac{1}{57}$

$p(1) = 1 - \frac{12 \cdot 1}{57+16(1-2)^2} = \frac{-11}{57}$

$p(3) = 1 - \frac{12 \cdot 3}{57+16(3-2)^2} = \frac{-35}{185}$

The minimum pressure is $p(1) = \frac{-11}{57}$.

d) To find the maximum pressure, we compare the pressure values at the interval boundaries:

$p(0) = \frac{1}{57}$

$p(3) = \frac{-35}{185}$

The maximum pressure is $p(0) = \frac{1}{57}$.

e) The tornado is right on top of the weather station when the pressure is minimized. The minimum pressure occurs at $t = 1$ minute.

Exercise 2

Given the function: $f(x) = |x - 1|$ over the interval $-2 \leq x \leq 6$

a) Three-point equal interval search:

1st iteration: Divide the interval into 3 equal parts: $x_1 = -2$, $x_2 = 0$, $x_3 = 2$, $x_4 = 4$, $x_5 = 6$ Evaluate $f(x)$ at x_2, x_3, and x_4: $f(-2) = |-2-1| = 3$ $f(0) = |0-1| = 1$ $f(2) = |2-1| = 1$ Since $f(0) = f(2) < f(-2)$, we choose the interval $[-2, 2]$.

2nd iteration: Divide the interval into 3 equal parts: $x_1 = -2$, $x_2 = -\frac{2}{3}$, $x_3 = \frac{2}{3}$, $x_4 = 2$ Evaluate $f(x)$ at x_2, x_3, and x_4: $f(-\frac{2}{3}) = |-\frac{2}{3}-1| = \frac{5}{3}$ $f(\frac{2}{3}) = |\frac{2}{3}-1| = \frac{1}{3}$ $f(2) = |2-1| = 1$ Since $f(\frac{2}{3}) < f(-\frac{2}{3})$ and $f(\frac{2}{3}) < f(2)$, we choose the interval $[-\frac{2}{3}, 2]$.

3rd iteration: Divide the interval into 3 equal parts: $x_1 = -\frac{2}{3}$, $x_2 = \frac{2}{9}$, $x_3 = \frac{22}{9}$, $x_4 = 2$ Evaluate $f(x)$ at x_2, x_3, and x_4: $f(\frac{2}{9}) = |\frac{2}{9}-1| = \frac{7}{9}$ $f(\frac{22}{9}) = |\frac{22}{9}-1| = \frac{13}{9}$ $f(2) = |2-1| = 1$ Since $f(\frac{2}{9}) < f(\frac{22}{9})$ and $f(\frac{2}{9}) < f(2)$, we choose the interval $[-\frac{2}{3}, \frac{22}{9}]$.

After three iterations, the interval $[-\frac{2}{3}, \frac{22}{9}]$ is guaranteed to contain the value of x that corresponds to the minimum of f.

b) Successive parabolic interpolation

Successive parabolic interpolation:

Initial samples: $x_1 = 0$, $x_2 = 2$, $x_3 = 4$ Evaluate $f(x)$ at x_1, x_2, and x_3: $f(0) = |0-1| = 1$ $f(2) = |2-1| = 1$ $f(4) = |4-1| = 3$

Let's perform one iteration of successive parabolic interpolation:

$$x_{min} = x_2 - \frac{1}{2}\frac{(x_2-x_1)^2(f(x_2)-f(x_3)) - (x_2-x_3)^2(f(x_2)-f(x_1))}{(x_2-x_1)(f(x_2)-f(x_3)) - (x_2-x_3)(f(x_2)-f(x_1))}$$

Substitute the values and calculate x_{min}:

$$x_{min} = 2 - \frac{1}{2}\frac{(2)^2(1-3) - (2)^2(1-1)}{(2)(1-3) - (2)(1-1)} = \frac{5}{3}$$

Based on this one iteration, the minimum value of $f(x)$ occurs at $x = \frac{5}{3}$.

Exercise 3

a) Gradient descent with $a = 1$, $x_0 = 1$, and $\gamma = 1/4$:

$f(x) = ax^2 = x^2$

$f'(x) = 2ax = 2x$

Iteration 1: $x_1 = x_0 - \gamma f'(x_0) = 1 - \frac{1}{4}(2) = \frac{1}{2}$

Iteration 2: $x_2 = x_1 - \gamma f'(x_1) = \frac{1}{2} - \frac{1}{4}(2 \cdot \frac{1}{2}) = \frac{1}{4}$

Iteration 3: $x_3 = x_2 - \gamma f'(x_2) = \frac{1}{4} - \frac{1}{4}(2 \cdot \frac{1}{4}) = \frac{1}{8}$

Gradient descent appears to be converging.

b) Gradient descent with $a = 1$, $x_0 = 1$, and $\gamma = 1$:

Iteration 1: $x_1 = x_0 - \gamma f'(x_0) = 1 - 1(2) = -1$

Iteration 2: $x_2 = x_1 - \gamma f'(x_1) = -1 - 1(-2) = 1$

Iteration 3: $x_3 = x_2 - \gamma f'(x_2) = 1 - 1(2) = -1$

Gradient descent does not appear to be converging.

c) The second derivative of $f(x)$ is:

$f''(x) = 2a = 2$

The learning rate γ must be strictly less than $\frac{1}{L} = \frac{1}{2}$ for gradient descent to converge. In part (a), $\gamma = \frac{1}{4}$, which is less than $\frac{1}{2}$, so gradient descent converges. In part (b), $\gamma = 1$, which is not less than $\frac{1}{2}$, so gradient descent does not converge.

d) Given $f(x) = x^2$, $x_0 = 1$, $x^* = 0$, and $\gamma = \frac{1}{4}$, find k such that $|f(x_k) - f(x^*)| < 10^{-15}$:

$$\frac{(x_0 - x^*)^2}{2\gamma k} < 10^{-15}$$

$$\frac{(1-0)^2}{2 \cdot \frac{1}{4} k} < 10^{-15}$$

$$\frac{1}{2k} < 10^{-15}$$

Multiply both sides by $2k$:

$$1 < 2k \cdot 10^{-15}$$

Divide both sides by $2 \cdot 10^{-15}$:

$$k > \frac{1}{2 \cdot 10^{-15}}$$

$$k > 5 \cdot 10^{14}$$

k must be greater than $5 \cdot 10^{14}$ for the error $|f(x_k) - f(x^*)|$ to be less than 10^{-15}.