Exercise 1. (Component Skill 3.1)

a. The computational complexity of calculating the mean of n real numbers by adding up the x_j first and then dividing by n is **O(n)**. This is because we need to perform n additions and then one division operation.

b. The computational complexity of calculating the mean of n real numbers by dividing each x_j by n first and then adding up the x_j/n is also **O(n)**. This is because we need to perform n division operations and n-1 addition operations. Both methods have the same computational complexity, so neither is less expensive to calculate the mean.

c. The computational complexity to calculate the standard deviation of n real numbers, assuming we first calculate x by method (a), then calculate $\sum_{j=1}^{n}(x_j - x)^2$, then divide by n, and then take the square root, is **O(n)** for calculating the mean, **O(n)** for calculating the sum of squares, **O(1)** for dividing by n, and **O(1)** for taking the square root. Thus, the overall computational complexity is **O(n)**.

d. The asymptotic computational complexity of the answer to (c) is **O(n)**.

e. Based on the answers to (a) and (d), computing the standard deviation is not more computationally complex than computing the mean. Both have a computational complexity of **O(n)**.

Exercise 2. (Component Skills 3.2-3.3)

a. Gaussian elimination and back substitution for the linear system:

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Gaussian elimination:

Step 1: Eliminate the first element of the second and third rows.

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Step 2: Eliminate the second element of the third row.

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Back substitution:

Step 3: Solve for x_3.

$x_3 = -1$

Step 4: Solve for x_2.

$x_2 - x_3 = -1 \Rightarrow x_2 = 0$

Step 5: Solve for x_1.

$2x_1 + 2x_2 + x_3 = 1 \Rightarrow x_1 = 1$

Solution: $(x_1, x_2, x_3) = (1, 0, -1)$

b. The given linear system:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Gaussian elimination:

Step 1: Eliminate the first element of the second and third rows.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

Since the third row is all zeros except for the constant term on the right-hand side, this system has **no solution**.

c. The given linear system:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Gaussian elimination:

Step 1: Eliminate the first element of the second and third rows.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

Step 2: Solve for x_2.

$x_2 = \frac{-2}{-3} = \frac{2}{3}$

Step 3: Substitute x_2 back into the first equation and solve for x_1 and x_3.

$$x_1 + 2\left(\frac{2}{3}\right) + x_3 = 1$$

$$x_1 + \frac{4}{3} + x_3 = 1$$

Since there is no unique solution for x_1 and x_3, this system has **infinitely many solutions**.

Exercise 3. (Component Skill 3.4)

a. The elementary matrix that adds two times the first row of A to the third row of A is:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b. The elementary matrix that subtracts three times the fourth row of A from the second row of A is:

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c. The elementary matrix that interchanges the second and third rows of A is:

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

d. The inverse of the matrix obtained in (a) is:

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

e. The inverse of the matrix obtained in (c) is:

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

f. Let E represent the elementary matrix that subtracts 2 times the second row of A from the first row of A:

$$E = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let P represent the elementary matrix that interchanges the third and fourth rows of A:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The inverse of the product of these matrices is:

$$(PE)^{-1} = E^{-1}P^{-1}$$

We find the inverses of E and P:

$$E^{-1} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Now, we can calculate the inverse of the product of E and P:

$$(PE)^{-1} = E^{-1}P^{-1} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Exercise 4. (Component Skill 3.5)

Calculate the LU decomposition of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Major elimination steps:

1. Subtract 4 times the first row from the second row, and subtract 7 times the first row from the third row:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

1. Subtract 2 times the second row from the third row:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

LU decomposition:

The lower triangular matrix L is:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$$

The upper triangular matrix U is:

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

So, the LU decomposition of matrix A is: A = LU