Homework 7.

Amath 301
Beginning Scientific Computing

Due: 5/16/23 at 11:59pm to Gradescope

Directions:

For the **written exercises** of the homework, please do the following:

- Write your name, student number, and section at the top of your written exercises. A half point will be deducted if these are not included.
- Complete all written exercises to the best of your ability and as neatly as possible.
- For those who are writing their answers by hand, please scan your work and save it as a pdf file.
- You are encouraged to type your written homework using LaTeX. Doing so will earn you a half bonus point. Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.

For the **coding exercises** of the homework, please do the following:

- Create a MATLAB script titled `hw7.m` and divide your script into sections according to the various coding exercises.
- Make sure your final answers in each coding exercise are assigned to the correct variable names.
- Publish your script as a pdf file. You can do this by typing `publish('hw7.m','pdf')` in the Command Window.
- Attach your published code to the end of your written exercises. Submit your entire homework assignment (written + coding exercises) to Gradescope.

Pro-Tips:

- You can check many of your answers to the written exercises of this homework by having MATLAB, Wolfram Alpha, or Chat GPT do the calculations for you.
- Teamwork makes the dream work, but please make sure you write up your solutions.
- Don't wait until the last minute. ☺

Written Exercises

(13 points total)

Exercise 1. (Component Skill 7.1)

Consider $f(x) = 3x^3 - 2x^2 + 3x - 2$ over the interval $0 \le x \le 4$.

a. What is the exact root of f in this interval?

   Hint: Factor by grouping.

b. In what interval is the root guaranteed to be after three iterations of the bisection method?

c. Based on your answer to (b), what is the best guess for the root of f after three iterations of the bisection method?

d. What is the maximum possible error of your answer to (c)?

e. How many iterations of the bisection method are needed to guarantee that the maximum possible error of the best guess of the root of f is less than 10^{-6}?

Exercise 2. (Component Skills 7.2-7.3)

Consider again $f(x) = 3x^3 - 2x^2 + 3x - 2$.

a. Perform one iteration of Newton's method with initial guess $x_0 = 1$.

b. A grad student is concerned that there is an initial guess x_0 such that $f'(x_k) = 0$ for some $k \ge 0$. In such a situation, Newton's method will fail. Explain why the grad student has nothing to worry about.

   Hint: Consider the values of x such that $f'(x) = 0$.

c. Perform one iteration of the secant method with initial guesses $x_0 = 0$ and $x_1 = 1$.

Exercise 3. (Component Skill 7.4)

Consider yet again $f(x) = 3x^3 - 2x^2 + 3x - 2$.

a. A grad student wants to solve $f(x) = 0$ by fixed point iteration. They naively choose the following fixed point form:

$$x = 3x^3 - 2x^2 + 4x - 2,$$

which inspires the iteration scheme

$$x_{k+1} = 3x_k^3 - 2x_k^2 + 4x_k - 2.$$

Does this fixed point iteration converge for any initial guess x_0 (assuming x_0 is not exactly the root of f)? Perform a small calculation to justify your answer.

b. A postdoc tries to fix the grad student's mistake by proposing the following fixed point form:

$$x = -x^3 + \frac{2}{3}x^2 + \frac{2}{3},$$

which inspires the iteration scheme

$$x_{k+1} = -x_k^3 + \frac{2}{3}x_k^2 + \frac{2}{3}.$$

Does this fixed point iteration converge for all initial guesses sufficiently close to the root of f? Perform a small calculation to justify your answer.

c. A professor tries to improve convergence of the postdoc's scheme by proposing the following fixed point form:

$$x = \left(\frac{2}{3}x^2 - x + \frac{2}{3}\right)^{\frac{1}{3}},$$

which inspires the iteration scheme

$$x_{k+1} = \left(\frac{2}{3}x_k^2 - x_k + \frac{2}{3}\right)^{\frac{1}{3}}.$$

Does this fixed point iteration converge faster than the postdoc's for all initial guesses sufficiently close to the root of f? Perform a small calculation to justify your answer.

Hint: Use WolframAlpha to calculate and/or evaluate any derivatives you need to take.

Coding Exercises

(7 points total)

Exercise 1. (Component Skills 7.1-7.3, 7.5)

A former TA of this class studied caterpillar life cycles. She encountered the following equation in her research:

$$6 - 3x(1 + e^{3(1-x)}) = 0. \tag{1}$$

Here, x represents the population of caterpillars in thousands. By guess and check, one can see that $x = 1$ is a solution of (1). There are two other nontrivial solutions x_1 and x_2. We take $x_2 > x_1$ without loss of generality.

a. Use the MATLAB built-in function `fzero` with initial guess 0.1 to determine the value of x_1. Assign `A1` to the value of x_1.

   Answer:

```
A1 =

    0.141440363359890
```

b. Use the MATLAB built-in function `fzero` with initial guess 1.9 to determine the value of x_2. Assign `A2` to the value of x_2.

   Answer:

```
A2 =

    1.858559636640110
```

c. Copy the MATLAB function `bisectionMethod` from the weekly lecture notes. Paste this function at the very bottom of your `hw7.m` script. Use this function to perform the bisection method on the function $f(x)$ given by the LHS of (1). Take the initial interval to be $0 \le x \le 0.5$. Set the stop criterion to 10^{-15}. Assign `A3` to the approximation of the root x_1 according to the bisection method. Assign `A4` to the number of iterations necessary to satisfy the stop criterion.

   Answer:

```
A3 =

   0.141440363359890


A4 =

    49
```

d. Repeat (c) but take the initial interval to be $1.5 \le x \le 2$. Set the stop criterion to 10^{-15}. Assign `A5` to the approximation of the root x_2 according to the bisection method. Assign `A6` to the number of iterations necessary to satisfy the stop criterion.

Answer:

```
A5 =

   1.858559636640110


A6 =

    49
```

e. Copy the MATLAB function `newtonMethod` from the weekly lecture notes. Modify the function so that the Cauchy error is used as the stopping criterion. Have your function return the approximation of the root according to Newton's method as well as the number of iterations necessary to satisfy the stop criterion.

Once your function has been modified appropriately, paste this function at the very bottom of your `hw7.m` script. Use this function to perform the Newton's method on the function $f(x)$ given by the LHS of (1). Take the initial guess to be $x_0 = 0.1$. Set the stop criterion to 10^{-15}. Assign `A7` to the approximation of the root x_1 according to Newton's method. Assign `A8` to the number of iterations necessary to satisfy the stop criterion.

Note: You will need to compute $f'(x)$ by hand and write this as a function handle to run Newton's method in MATLAB.

Answer:

```
A7 =

   0.141440363359890


A8 =

     6
```

f. Repeat (e) but take the initial guess to be $x_0 = 1.9$. Set the stop criterion to 10^{-15}. Assign `A9` to the approximation of the root x_2 according to Newton's method. Assign `A10` to the number of iterations necessary to satisfy the stop criterion.

Answer:

```
A9 =

   1.858559636640111


A10 =

     5
```

g. Create your own MATLAB function `secantMethod`. The function should have the following inputs:

- a function handle for the function whose root is to be found,
- an initial guess for the root,
- a second guess for the root, and
- a threshold for the Cauchy error.

The function should output the following:

- the approximate value of the root according to the secant method and
- the number of iterations necessary to satisfy the threshold for the Cauchy error.

Paste your function at the very bottom of your `hw7.m` script. Use this function to perform the secant method on the function $f(x)$ given by the LHS of (1). Take the initial guess to be $x_0 = 0.1$, the second guess to be $x_1 = 0.11$, and the stop criterion to be 10^{-15}. Assign `A11` to the approximation of the root x_1 according to the secant method. Assign `A12` to the number of iterations necessary to satisfy the stop criterion.

Answer:

```
A11 =

   0.141440363359890


A12 =

     7
```

h. Repeat (g) but take the initial guess to be $x_0 = 1.9$ and second guess to be $x_1 = 1.8$. Set the stop criterion to 10^{-15}. Assign `A13` to the approximation of the root x_2 according to the secant method. Assign `A14` to the number of iterations necessary to satisfy the stop criterion.

Answer:

```
A13 =

   1.858559636640110


A14 =

     6
```