Mason Wheeler

5/1/2023

HW5

Exercise 1. (Component Skill 5.1)

Solve the following least-squares problem:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

To solve this least-squares problem, we need to minimize the following expression:

$$||AX - B||^2$$

To minimize it, we need to derive the normal equation:

$$A^T AX = A^T B$$

Compute $A^T A$ and $A^T B$:

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now, we can solve the linear system:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The solution of the linear system is:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Exercise 2. (Component Skill 5.2)

The following data represents the temperature T (in °C) in Seattle t hours after sunrise:

t	T
0	0
1	0
2	2

We want to find the equation of the line that fits these data points and minimizes the least-squares error. Let's represent the line as $T = a + bt$.

First, create the matrix A and vector B:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Now, we can derive the normal equation:

$$A^T AX = A^T B$$

Compute $A^T A$ and $A^T B$:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Now, we can solve the linear system:

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The solution of the linear system is:

$$X = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Thus, the equation of the line that fits these data points and minimizes the least-squares error is:

$$T = -2 + 2t$$

Exercise 3. (Component Skill 5.3)

Suppose we sample $y = sin^2(x)$ at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We obtain the following data:

x	y
0	0
π/4	1/2
π/2	1

(a) Fit the data points above to the curve $y = a + b\cos(2x)$. Determine a and b that minimize the least-squares error.

First, create the matrix A and vector B:

$$A = \begin{bmatrix} 1 & \cos(0) \\ 1 & \cos(\pi) \\ 1 & \cos(2\pi) \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$$

Now, we can derive the normal equation:

$$A^T AX = A^T B$$

Compute $A^T A$ and $A^T B$:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Now, we can solve the linear system:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

The solution of the linear system is:

$$X = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Thus, the curve that fits these data points and minimizes the least-squares error is:

$$y = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

(b) The values of a and b you found in (a) should not surprise you. Give a one-sentence explanation for why you should not be surprised.

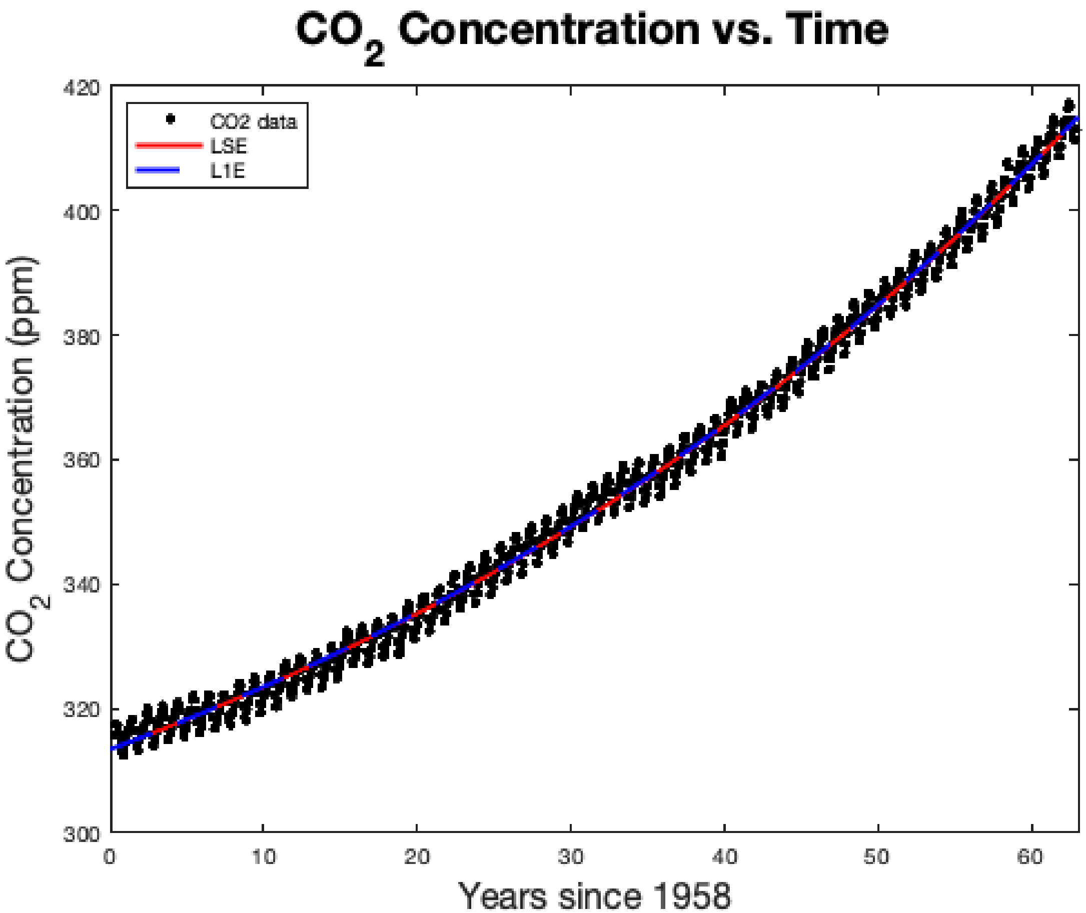
The values of a and b are not surprising because, for the given data points, the curve $y = a + b\cos(2x)$ is similar to the original function $y = \sin^2(x)$, which is a periodic function with a mean value of $\frac{1}{2}$, and the $\cos(2x)$ term can capture the shape of the original function.

```
% Mason Wheeler
% HW5 Code
% Exercise 1 (Component Skill 5.3) - Part a

% Load the salmon_data.mat file
load('salmon_data.mat');

% first-degree polynomial that best fits the salmon data in the least-squares
 sense
A1 = polyfit(year, salmon, 1);

% Display the coefficients of the first-degree polynomial
disp('Coefficients of the first-degree polynomial:');
disp(A1);

% Exercise 1 (Component Skill 5.3) - Part b

% Use the MATLAB built-in function 'polyfit' to find the coefficients of the
% third-degree polynomial that best fits the salmon data in the least-squares
 sense
% Assign the row vector of coefficients to the variable A2
A2 = polyfit(year, salmon, 3);

% Display the coefficients of the third-degree polynomial
disp('Coefficients of the third-degree polynomial:');
disp(A2);

% Exercise 1 (Component Skill 5.3) - Part c

% Use the MATLAB built-in function 'polyfit' to find the coefficients of the
% fifth-degree polynomial that best fits the salmon data in the least-squares
 sense
% Assign the row vector of coefficients to the variable A3
A3 = polyfit(year, salmon, 5);

% Display the coefficients of the fifth-degree polynomial
disp('Coefficients of the fifth-degree polynomial:');
disp(A3);

% Exercise 1 (Component Skill 5.3) - Part d

% True salmon count in 2021
true_salmon_2021 = 489523;

% Calculate the predicted salmon count in 2021 using the polynomials p1, p3,
 and p5
p1_2021 = polyval(A1, 2021);
p3_2021 = polyval(A2, 2021);
p5_2021 = polyval(A3, 2021);

% Calculate the relative errors (epsilon1, epsilon2, epsilon3)
epsilon1 = abs(p1_2021 - true_salmon_2021) / true_salmon_2021;
```

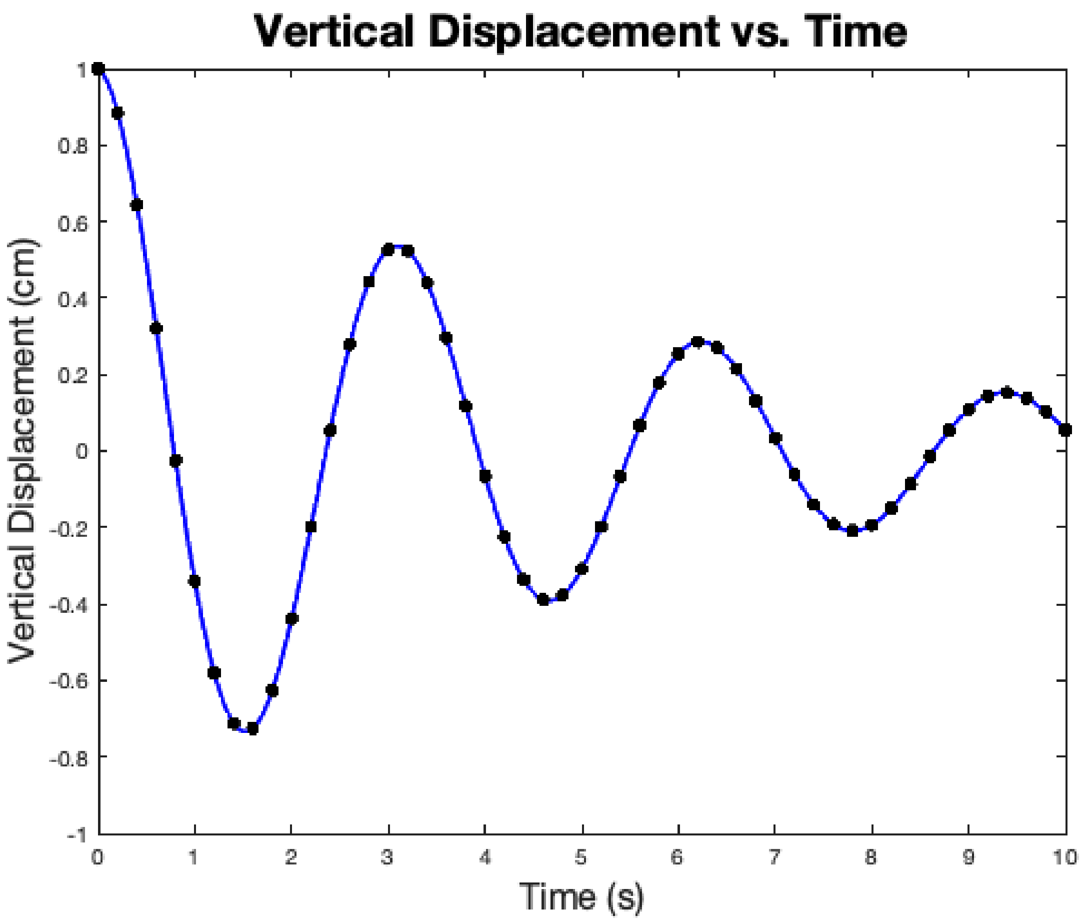
```
epsilon2 = abs(p3_2021 - true_salmon_2021) / true_salmon_2021;
epsilon3 = abs(p5_2021 - true_salmon_2021) / true_salmon_2021;

% Assign the variable A4 to the 3x1 row vector [epsilon1, epsilon2, epsilon3]
A4 = [epsilon1; epsilon2; epsilon3];

% Display the relative errors
disp('Relative errors:');
disp(A4);

% Exercise 2 (Component Skill 5.3)

% Load CO2_data.mat file
load('CO2_data.mat');

% Part a
% Find the values of a, r, and b that minimize the least-squares error
initial_guess = [30; 0.03; 300];
A5 = fminsearch(@(params) lse(params, year, CO2), initial_guess);

% Part b
% Find the values of a, r, and b that minimize the L1 error
A6 = fminsearch(@(params) l1e(params, year, CO2), initial_guess);

% Plot the results
plot_results(year, CO2, A5, A6);

% Exercise 3 (Component Skill 5.3)

% Load spring_data.mat file
load('spring_data.mat');

% Part a
% Estimate the value of the toy displacement at t = pi/2 seconds using cubic
 spline interpolation
A7 = interp1(time, deltaz, pi/2, 'spline');

% Part b
% Plot the data and cubic spline interpolation
plot_spline(time, deltaz);



% Create separate functions for least-squares error and L1 error
function error = lse(params, year, CO2)
    error = sum((CO2 - (params(1) * exp(params(2) * year) + params(3))).^2);
end

function error = l1e(params, year, CO2)
    error = sum(abs(CO2 - (params(1) * exp(params(2) * year) + params(3))));
end


function plot_results(year, CO2, A5, A6)
```

```
    % Plot CO2 data in black
    figure;
    plot(year, CO2, '.', 'MarkerSize', 10, 'Color', 'k', 'LineStyle', 'none');
    hold on;

    % Plot least-squares error exponential curve in red
    x = linspace(0, 63, 1000);
    y_lse = A5(1) * exp(A5(2) * x) + A5(3);
    plot(x, y_lse, 'r', 'LineWidth', 2);

    % Plot L1 error exponential curve in blue
    y_l1e = A6(1) * exp(A6(2) * x) + A6(3);
    plot(x, y_l1e, 'b--', 'LineWidth', 2);

    % Set axis limits
    xlim([0, 63]);
    ylim([300, 420]);

    % Add legend, labels, and title
    legend('CO2 data', 'LSE', 'L1E', 'Location', 'NorthWest');
    xlabel('Years since 1958', 'FontSize', 16);
    ylabel('CO_2 Concentration (ppm)', 'FontSize', 16);
    title('CO_2 Concentration vs. Time', 'FontSize', 20);

    % Hold off
    hold off;
end

function plot_spline(time, deltaz)
    % Create cubic spline interpolation with sample points t = 0:0.01:10
    t_interp = 0:0.01:10;
    z_interp = interp1(time, deltaz, t_interp, 'spline');

    % Plot cubic spline interpolation in blue
    figure;
    plot(t_interp, z_interp, 'b', 'LineWidth', 2);
    hold on;

    % Superimpose the displacement data in black
    plot(time, deltaz, 'k.', 'MarkerSize', 15, 'LineStyle', 'none');

    % Set axis limits
    xlim([0, 10]);
    ylim([-1, 1]);

    % Add labels and title
    xlabel('Time (s)', 'FontSize', 16);
    ylabel('Vertical Displacement (cm)', 'FontSize', 16);
    title('Vertical Displacement vs. Time', 'FontSize', 20);

    % Hold off
    hold off;
end
```

```
Coefficients of the first-degree polynomial:
   1.0e+06 *

    0.0042   -7.8654

Warning: Polynomial is badly conditioned. Add points
with distinct X values, reduce the degree of the
polynomial, or try centering and scaling as
described in HELP POLYFIT.
Coefficients of the third-degree polynomial:
   1.0e+10 *

    0.0000   -0.0000    0.0016   -1.0637

Warning: Polynomial is badly conditioned. Add points
with distinct X values, reduce the degree of the
polynomial, or try centering and scaling as
described in HELP POLYFIT.
Coefficients of the fifth-degree polynomial:
   1.0e+14 *

   -0.0000    0.0000   -0.0000    0.0000   -0.0103    4.0754

Relative errors:
    0.2233
    0.6722
    0.1778
```



CO2 Concentration vs. Time
CO2 data
LSE
L1E
420
400
380
360
340
320
300
CO2 Concentration (ppm)
0
10
20
30
40
50
60
Years since 1958



Published with MATLAB® R2022b