AMATH 301 Mason Wheeler Homework 1

Exercise 1. (Component Skill 1.1)

A chemist must determine the coefficients x_j (for $1 \leq j \leq 5$) that balance the reaction:

$$x_1 Cu + x_2 HNO_3 \rightarrow x_3 Cu(NO_3)_2 + x_4 NO + x_5 H_2O.$$

In order to balance this reaction, there must be an equal number of each element on both sides of the reaction.

a. Use this information to derive a linear system satisfied by the unknown coefficients x_j. Hint: Derive an equation that balances each element.

Balancing each element, we derive the following linear equations:

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - 2x_5 = 0 \\ x_2 - x_4 - 2x_3 = 0 \\ 3x_2 - 6x_3 - x_4 - x_5 = 0 \end{cases}$$

b. What are the dimensions of this system?

The dimensions of this system are 4 (number of equations) by 5 (number of unknown coefficients).

c. Is this system overdetermined, square, or underdetermined? Based on your answer, is it most likely this system has no solution, a unique solution, or infinitely many solutions?

This system is underdetermined since the number of unknown coefficients (5) is greater than the number of equations (4). It is most likely that this system has infinitely many solutions.

Matrix representation of the system:

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 3 & -6 & -1 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, the matrix-vector form of the linear system is:

$$Ax = b$$

To represent the linear system as an augmented matrix, we combine the matrix A and the vector b:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & -6 & -1 & -1 & 0 \end{array}\right]$$

Exercise 2. (Component Skill 1.2)

Express the linear system in Exercise 1 in matrix-vector form and as an augmented matrix.

The linear system in Exercise 1 is:

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_5 = 0 \\ x_2 - 2x_3 - x_4 = 0 \\ 3x_2 - 6x_3 - x_5 = 0 \end{cases}$$

To express this system in matrix-vector form, we represent the coefficients of the unknowns as a matrix A, the unknowns as a column vector x, and the constants on the right-hand side as a column vector b:

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 3 & -6 & 0 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, the matrix-vector form of the linear system is:

$$Ax = b$$

To represent the linear system as an augmented matrix, we combine the matrix A and the vector b:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & -6 & 0 & -1 & 0 \end{array}\right]$$

Exercise 3. (Component Skill 1.3)

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

a. Does $A + B$ exist? Why or why not?

$A + B$ does not exist because the matrices A and B have different dimensions. A is a 2×2 matrix, while B is a 3×2 matrix.

b. Compute $\frac{1}{2}A + \frac{1}{2}D$

We are given matrices A and D as follows:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

To compute $\frac{1}{2}A + \frac{1}{2}D$, we first multiply each matrix by $\frac{1}{2}$ and then add the resulting matrices:

$$\frac{1}{2}A = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad \frac{1}{2}D = \begin{bmatrix} 0.5 & -1 \\ 0 & 0.5 \end{bmatrix}$$

Now, add the two matrices:

$$\frac{1}{2}A + \frac{1}{2}D = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.5 & -1 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. Does AB exist? Why or why not?

AB does not exist because the number of columns in matrix A does not match the number of rows in matrix B. A has 2 columns, while B has 3 rows.

d. Compute AB^T.

$$AB^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 7 \\ 2 & 0 & 1 \end{bmatrix}$$

e. Do B and C commute? (That is, does $BC = CB$?)

To check if B and C commute, we need to compute BC and CB:

$$BC = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 & -1 \\ -3 & -1 & 1 \\ 10 & 5 & -2 \end{bmatrix}$$

$$CB = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$$

Since $BC \neq CB$, matrices B and C do not commute.

f. Compute $(AC)^T - BA$.

This computation is not possible because both AB and AC do not exist, as explained in parts c and e.

g. Are A and D inverses of each other? Why or why not?

To check if A and D are inverses of each other, we need to compute AD and DA:

$$AD = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$DA = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $AD = DA = I$, where I is the identity matrix, matrices A and D are inverses of each other.