Exercise 1. (Component Skill 8.1)

The hyperbolic cosine function, denoted as $\cosh(x)$, is defined by the following equation:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

To calculate the fourth-degree Taylor polynomial of $\cosh(x)$ at $x = 0$, we first need to calculate the first four derivatives of $\cosh(x)$ at $x = 0$.

The derivatives of $\cosh(x)$ are as follows:

- First derivative: $\cosh'(x) = \sinh(x)$
- Second derivative: $\cosh''(x) = \cosh(x)$
- Third derivative: $\cosh'''(x) = \sinh(x)$
- Fourth derivative: $\cosh''''(x) = \cosh(x)$

Evaluating these at $x = 0$ gives:

- $\cosh'(0) = \sinh(0) = 0$
- $\cosh''(0) = \cosh(0) = 1$
- $\cosh'''(0) = \sinh(0) = 0$
- $\cosh''''(0) = \cosh(0) = 1$

Therefore, the fourth-degree Taylor polynomial of $\cosh(x)$ at $x = 0$ is:

$$P_4(x) = \cosh(0) + \cosh'(0)x + \frac{\cosh''(0)x^2}{2!} + \frac{\cosh'''(0)x^3}{3!} + \frac{\cosh''''(0)x^4}{4!}$$

Substituting the values we found gives:

$$P_4(x) = 1 + 0x + \frac{1x^2}{2!} + 0x^3 + \frac{1x^4}{4!}$$

Simplifying this gives:

$$P_4(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$$

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Exercise 2. (Component Skill 8.2)

Consider the integral $I = \int_0^{\pi/2} \sin(x)dx$.

a. The exact value of the integral I is 1. This is obtained by evaluating the antiderivative of $\sin(x)$, which is $-\cos(x)$, at the limits of integration.

b. To approximate I with the left-hand rule using $N = 4$ subintervals, we divide the interval $[0, \pi/2]$ into 4 equal subintervals. The width of each subinterval is $\Delta x = (\pi/2 - 0)/4 = \pi/8$. The left-hand rule approximation is then given by

$$I \approx \Delta x \left[\sin(0) + \sin(\pi/8) + \sin(\pi/4) + \sin(3\pi/8)\right]$$

Evaluating this gives $I \approx 0.79077$ (rounded to the nearest fourth decimal place).

c. The left-hand rule underestimates the exact value of I. This is because the function $\sin(x)$ is increasing on the interval $[0, \pi/2]$, so the left endpoint of each subinterval gives a lower estimate for the area under the curve.

d. The maximum possible error of the left-hand rule is given by the formula $(M/2)(\Delta x)^2 N$, where M is the maximum value of the absolute value of the derivative on the interval, Δx is the width of each subinterval, and N is the number of subintervals. In this case, $M = 1$, $\Delta x = \pi/8$, and $N = 4$, so the maximum possible error is $(1/2)(\pi/8)^2 \cdot 4 = 0.30843$.

e. To approximate I with the right-hand rule using $N = 4$ subintervals, we use the right endpoint of each subinterval. The right-hand rule approximation is then given by

$$I \approx \Delta x \left[\sin(\pi/8) + \sin(\pi/4) + \sin(3\pi/8) + \sin(\pi/2)\right]$$

Evaluating this gives $I \approx 1.18347$ (rounded to the nearest fourth decimal place).

f. The right-hand rule overestimates the exact value of I. This is because the function $\sin(x)$ is increasing on the interval $[0, \pi/2]$, so the right endpoint of each subinterval gives a higher estimate for the area under the curve.

g. The maximum possible error of the right-hand rule is the same as for the left-hand rule, which is 0.30843

h. To approximate I with the midpoint rule using $N = 4$ subintervals, we use the midpoint of each subinterval. The midpoint rule approximation is then given by

$$I \approx \Delta x \left[\sin(\pi/16) + \sin(3\pi/16) + \sin(5\pi/16) + \sin(7\pi/16)\right]$$

Evaluating this gives $I \approx 1.9986$ (roundedto the nearest fourth decimal place).

i. The maximum possible error of the midpoint rule is given by the formula $(K/24)(\Delta x)^3 N$, where K is the maximum value of the absolute value of the second derivative on the interval. In this case, the second derivative of $\sin(x)$ is $-\sin(x)$, which has a maximum absolute value of 1 on the interval $[0, \pi/2]$. So $K = 1$, $\Delta x = \pi/8$, and $N = 4$, and the maximum possible error is $(1/24)(\pi/8)^3 \cdot 4 = 0.01009$.

Exercise 3. (Component Skills 8.3-8.4)

Consider an ellipse of the form $x^2/a^2 + y^2/b^2 = 1$. The circumference of this ellipse is given by the integral $C = \int_0^{2\pi} \sqrt{a^2 \sin^2(\theta) + b^2 \cos^2(\theta) d\theta}$. This integral can be solved exactly if the ellipse is a circle. Then, a and b are equal to the radius of the circle r, and the circumference $C = 2\pi r$, as you learned in your grade school days. If a and b are not equal to each other, then we can't solve the integral above exactly.

a. To estimate C by the trapezoidal rule with $N = 4$ subintervals, we divide the interval $[0, 2\pi]$ into 4 equal subintervals. The width of each subinterval is $\Delta\theta = (2\pi - 0)/4 = \pi/2$. The trapezoidal rule approximation is then given by

$$C \approx \Delta\theta \left[\frac{1}{2}\sqrt{a^2 \sin^2(0) + b^2 \cos^2(0)} + \sqrt{a^2 \sin^2(\pi/2) + b^2 \cos^2(\pi/2)} + \sqrt{a^2 \sin^2(\pi) + b^2 \mathrm{c} \right.$$
$$\left. + \sqrt{a^2 \sin^2(3\pi/2) + b^2 \cos^2(3\pi/2)} + \frac{1}{2}\sqrt{a^2 \sin^2(2\pi) + b^2 \cos^2(2\pi)} \right]$$

Evaluating this gives an approximate formula for C as a function of a and b:

$$C \approx 0.7854(4a + 4b)$$

b. To estimate C by Simpson's 1/3 rule with $N = 4$ subintervals, we use the midpoints of each subinterval. The Simpson's 1/3 rule approximation is then given by

$$C \approx \frac{\Delta\theta}{3} \left[\sqrt{a^2 \sin^2(0) + b^2 \cos^2(0)} + 4\sqrt{a^2 \sin^2(\pi/4) + b^2 \cos^2(\pi/4)} + 2\sqrt{a^2 \sin^2(\pi/2) \cdot \right.$$
$$\left. + 4\sqrt{a^2 \sin^2(3\pi/4) + b^2 \cos^2(3\pi/4)} + \sqrt{a^2 \sin^2(\pi) + b^2 \cos^2(\pi)} \right]$$

Evaluating this gives an approximate formula for C as a function of a and b:

$$ C \approx 0.5236 (2a + 2b + 8 \sqrt{0.5a^2 + 0.5b^2})$$