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5/1/2023

HW5

Exercise 1. (Component Skill 5.1)

Solve the following least-squares problem:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

To solve this least-squares problem, we need to minimize the following expression:

$$||AX - B||^2$$

To minimize it, we need to derive the normal equation:

$$A^T AX = A^T B$$

Compute $A^T A$ and $A^T B$:

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now, we can solve the linear system:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The solution of the linear system is:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Exercise 2. (Component Skill 5.2)

The following data represents the temperature T (in °C) in Seattle t hours after sunrise:

t	T
0	0
1	0
2	2

We want to find the equation of the line that fits these data points and minimizes the least-squares error. Let's represent the line as $T = a + bt$.

First, create the matrix A and vector B:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Now, we can derive the normal equation:

$$A^T A X = A^T B$$

Compute $A^T A$ and $A^T B$:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Now, we can solve the linear system:

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The solution of the linear system is:

$$X = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Thus, the equation of the line that fits these data points and minimizes the least-squares error is:

$$T = -2 + 2t$$

Exercise 3. (Component Skill 5.3)

Suppose we sample $y = sin^2(x)$ at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We obtain the following data:

x	y
0	0
π/4	1/2
π/2	1

(a) Fit the data points above to the curve $y = a + b\cos(2x)$. Determine a and b that minimize the least-squares error.

First, create the matrix A and vector B:

$$A = \begin{bmatrix} 1 & \cos(0) \\ 1 & \cos(\pi) \\ 1 & \cos(2\pi) \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$$

Now, we can derive the normal equation:

$$A^T AX = A^T B$$

Compute $A^T A$ and $A^T B$:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Now, we can solve the linear system:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

The solution of the linear system is:

$$X = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Thus, the curve that fits these data points and minimizes the least-squares error is:

$$y = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

(b) The values of a and b you found in (a) should not surprise you. Give a one-sentence explanation for why you should not be surprised.

The values of a and b are not surprising because, for the given data points, the curve $y = a + b\cos(2x)$ is similar to the original function $y = \sin^2(x)$, which is a periodic function with a mean value of $\frac{1}{2}$, and the $\cos(2x)$ term can capture the shape of the original function.