1. Fourier coefficients of a periodic signal

Let's compute Fourier coefficients of a periodic signal using Euler's formula. Using the approach, find the Fourier coefficients of the following signal. Also, write the signal in terms of complex exponentials (i.e., using Fourier series).

$$x(t) = -2 + 4\cos(3t)\cos(t - \pi/2) + 3e^{j4t}$$

First, rewrite the signal using the trigonometric identity:

$$x(t) = -2 + 2\cos(4t - \pi/2) + 2\cos(2t + \pi/2) + 3e^{j4t}$$

Now, rewrite the signal in terms of complex exponentials using Euler's formula:

$$x(t) = -2 + 2e^{j(2t+\pi/2)} + 2e^{-j(2t+\pi/2)} + 2e^{j(4t-\pi/2)} + 2e^{-j(4t-\pi/2)} + 3e^{j4t}$$

Find the Fourier coefficients by comparing the terms of the signal with the general Fourier series formula:

$$C_{-4} = e^{-j\pi/2}$$

$$C_{-2} = e^{j\pi/2}$$

$$C_0 = -2$$

$$C_2 = e^{-j\pi/2}$$

$$C_4 = 3 + e^{j\pi/2}$$

For all other n, $C_n = 0$.

Now we have the Fourier coefficients and the signal representation in terms of complex exponentials:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

With the Fourier coefficients C_{-4}, C_{-2}, C_0, C_2, and C_4 as computed above, and all other C_n being zero.

2. Synthesize the original signal from the Fourier coefficients

We have the ability to synthesize the original signal from the Fourier coefficients. Consider a real periodic signal $x(t)$ whose fundamental period is given as $T = \pi$. The non-zero Fourier coefficients of $x(t)$ for $k \in \mathbb{N}$ are given as follows (rest of the Fourier coefficients are zero for $k \in \mathbb{N}$).

$$c_0 = 1, \quad c_1 = \frac{3}{2}, \quad c_3 = 2j$$

Write the signal $x(t)$ in the time domain and in simplified form (without any complex exponentials).

To synthesize the original signal from the Fourier coefficients, we use the inverse Fourier series formula:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Where $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$.

Given the non-zero Fourier coefficients, we have:

$$x(t) = c_0 + c_1 e^{j(1)(2t)} + c_{-1} e^{j(-1)(2t)} + c_3 e^{j(3)(2t)} + c_{-3} e^{j(-3)(2t)}$$

Since $x(t)$ is a real signal, we know that $c_{-k} = c_k^*$, where $*$ denotes the complex conjugate. Thus, $c_{-1} = c_1^* = \frac{3}{2}$ and $c_{-3} = c_3^* = -2j$. Now we can write $x(t)$ as:

$$x(t) = 1 + \frac{3}{2} e^{j2t} + \frac{3}{2} e^{-j2t} + 2je^{j6t} - 2je^{-j6t}$$

Now, we convert the complex exponentials to trigonometric functions using Euler's formula:

$$x(t) = 1 + \frac{3}{2}(\cos(2t) + j\sin(2t)) + \frac{3}{2}(\cos(-2t) + j\sin(-2t)) + 2j(\cos(6t) + j\sin(6t)) \\ - 2j(\cos(-6t) + j\sin(-6t))$$

Simplify the expression by combining the terms and using the properties of cosine and sine functions ($\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$):

$$x(t) = 1 + \frac{3}{2}(\cos(2t) + \cos(-2t)) + 2j(\sin(6t) - \sin(-6t))$$

Further simplifying the expression using the properties of sine and cosine functions:

$$x(t) = 1 + 3\cos(2t) + 4\sin(6t)$$

Now we have the signal $x(t)$ in the time domain and in simplified form, without any complex exponentials.

3. Output signal for a given LTI system

Recall that in Lecture 19 we were introduced to the transfer function $H(s)$ which is the Laplace transform of impulse response $h(t)$. We also learned that LTI system preserves the frequency (see slides 17-30). Consider the signal $x(t)$ in Problem 2. Suppose the signal

$x(t)$ is applied as an input to an LTI system whose transfer function $H(s)$ for $s = j\omega$ is given by:

$$H(j\omega) = \begin{cases} 2 & \text{for } |\omega| \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Write the output signal $y(t) = x(t) * h(t)$ in the time domain and in simplified form. [Hint: $y(t) = H(j\omega)e^{j\omega t}$ when $x(t) = e^{j\omega t}$]

From Problem 2, we know that:

$$x(t) = 1 + 3\cos(2t) + 4\sin(6t)$$

The output $y(t)$ can be found by applying the transfer function to each component of $x(t)$:

$$y(t) = H(j\omega)e^{j\omega t}\Big|_{\omega=0} + H(j\omega)e^{j\omega t}\Big|_{\omega=2} + H(j\omega)e^{j\omega t}\Big|_{\omega=-2} + H(j\omega)e^{j\omega t}\Big|_{\omega=6} + H(j\omega)e^{j\omega t}\Big|_{\omega}$$

Since $H(j\omega) = 2$ for $|\omega| \leq 3$ and $H(j\omega) = 0$ otherwise, we have:

$$y(t) = 2e^{j(0)t} + 2e^{j(2)t} + 2e^{-j(2)t} + 0e^{j(6)t} + 0e^{-j(6)t}$$

Now, we can simplify the expression:

$$y(t) = 2 + 2\cos(2t)$$

Thus, the output signal $y(t) = x(t) * h(t)$ in the time domain and in simplified form is:

$$y(t) = 2 + 2\cos(2t)$$

4. Fourier series coefficients of a half-wave signal

Recall that in Lecture 21 we obtained the Fourier coefficients using the formula of c_k. Consider the half-wave signal shown in Figure 1. Find the expression for the Fourier series coefficients of this signal.

The signal is given as:

$$x(t) = \begin{cases} \sin(\frac{\pi}{5}t) & \text{for } 0 \leq t \leq 25 \\ 0 & \text{otherwise} \end{cases}$$

To find the Fourier series coefficients for the given signal, we need to compute the complex exponential Fourier series coefficients c_k. For a periodic signal with period T, the coefficients are given by:

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

Where $\omega_0 = \frac{2\pi}{T}$. In this case, $T = 25$ and $\omega_0 = \frac{\pi}{5}$.

For $k = 0$, we have:

$$c_0 = \frac{1}{25}\int_0^{25} \sin(\frac{\pi}{5}t)dt$$

Integrating and substituting the limits, we get:

$$c_0 = \frac{1}{25}\left[\frac{-5}{\pi}\cos(\frac{\pi}{5}t)\right]_0^{25} = \frac{1}{25}\left(\frac{-5}{\pi}\cos(5\pi) + \frac{5}{\pi}\cos(0)\right) = \frac{1}{5}$$

For $k \neq 0$, we have:

$$c_k = \frac{1}{25}\int_0^{25} \sin(\frac{\pi}{5}t)e^{-jk\frac{\pi}{5}t}dt$$

Integrating by parts, we get:

$$c_k = \frac{1}{25}\left[\frac{-j}{k\pi}e^{-jk\frac{\pi}{5}t}(\frac{\pi}{5}\cos(\frac{\pi}{5}t) + jk\sin(\frac{\pi}{5}t))\right]_0^{25}$$

Substituting the limits, we obtain:

$$c_k = \frac{1}{25}\left[\frac{-j}{k\pi}e^{-5jk\pi}(\frac{\pi}{5} + jk\sin(0)) - \frac{-j}{k\pi}(\frac{\pi}{5} + jk\sin(0))\right] = \frac{1}{25}\left[\frac{-j}{k\pi}(\frac{\pi}{5})\right] = \frac{-j}{5k\pi}$$

Thus, the Fourier series coefficients of the half-wave signal are:

$$c_0 = \frac{1}{5}$$

$$c_k = \frac{-j}{5k\pi} \quad \text{for } k \neq 0$$

5. Fourier series coefficients of a full-wave rectified signal

Recall that in Lecture 22 we learned several properties of Fourier series. Using these properties and your results in Problem 4, find the expression for the Fourier series coefficients of the full-wave rectified signal shown below.

The signal is given as:

$$y(t) = |\sin(\frac{\pi}{5}t)|$$

with an upper bound of 25.

A full-wave rectified signal can be expressed as the sum of its even and odd harmonics. The half-wave signal found in Problem 4 can be considered as the odd harmonic component of the full-wave rectified signal. We need to find the even harmonic component to obtain the Fourier series coefficients of the full-wave rectified signal.

The even harmonic component of the full-wave rectified signal can be obtained by shifting the half-wave signal to the right by $\frac{T}{2} = \frac{25}{2}$ and then taking the absolute value:

$$y_{even}(t) = \left| \sin\left(\frac{\pi}{5}(t - \frac{25}{2})\right) \right|$$

Using the linearity and time shift properties of Fourier series, we can find the Fourier coefficients for the even harmonic component $y_{even}(t)$ based on the coefficients of the half-wave signal $x(t)$ found in Problem 4:

$$c'_k = \frac{-j}{5k\pi} e^{-j\frac{25}{2}k\frac{\pi}{5}}$$

Now, the Fourier series coefficients of the full-wave rectified signal $y(t)$ can be found by summing the coefficients of the odd harmonic component $x(t)$ and the even harmonic component $y_{even}(t)$:

$$c''_k = c_k + c'_k$$

For $k = 0$, we have:

$$c''_0 = c_0 = \frac{1}{5}$$

For $k \neq 0$, we have:

$$c''_k = \frac{-j}{5k\pi} + \frac{-j}{5k\pi} e^{-j\frac{25}{2}k\frac{\pi}{5}}$$

Thus, the Fourier series coefficients of the full-wave rectified signal are:

$$c''_0 = \frac{1}{5}$$

$$c''_k = \frac{-j}{5k\pi} + \frac{-j}{5k\pi} e^{-j\frac{25}{2}k\frac{\pi}{5}} \quad \text{for } k \neq 0$$

6. Fourier series coefficients for a discrete-time signal

Recall in Lecture 23 and 24, we have learned the discrete-time version of Fourier series. Now find the Fourier series coefficients for the discrete-time signal $x[n]$ shown in Figure 3. Consider $a = \frac{1}{3}$ and $N_0 = 6$ for your computation.

The signal is given as:

$$x[n] = a^n, \quad 0 \leq n \leq N_0 - 1$$

with $a = \frac{1}{3}$ and $N_0 = 6$.

To find the Fourier series coefficients for the discrete-time signal, we need to compute the discrete Fourier series coefficients $X[k]$. For a periodic signal with period N_0, the coefficients are given by:

$$X[k] = \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi}{N_0}kn}$$

Substitute the given signal and parameters:

$$X[k] = \sum_{n=0}^{5} \left(\frac{1}{3}\right)^n e^{-j\frac{2\pi}{6}kn}$$

Now, let's compute the coefficients for each k.

For $k = 0$:

$$X[0] = \sum_{n=0}^{5} \left(\frac{1}{3}\right)^n = \frac{1 - (\frac{1}{3})^6}{1 - \frac{1}{3}} = \frac{243 - 1}{242} = 1$$

For $k \neq 0$:

$$X[k] = \sum_{n=0}^{5} \left(\frac{1}{3}\right)^n e^{-j\frac{2\pi}{6}kn}$$

As the summation is a geometric series, we can use the geometric series formula:

$$X[k] = \frac{1 - (\frac{1}{3})^6 e^{-j\frac{2\pi}{6}k6}}{1 - \frac{1}{3}e^{-j\frac{2\pi}{6}k}}$$

Simplifying the expression, we get:

$$X[k] = \frac{1 - (\frac{1}{3})^6}{1 - \frac{1}{3}e^{-j\frac{2\pi}{6}k}} \quad \text{for } k \neq 0$$

Thus, the Fourier series coefficients for the discrete-time signal are:

$$X[0] = 1$$

$$X[k] = \frac{1 - (\frac{1}{3})^6}{1 - \frac{1}{3}e^{-j\frac{2\pi}{6}k}} \quad \text{for } k \neq 0$$

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