# EE242 HW6

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## 1.

a. Fourier transform of the signal $x(t) = \delta(t) + 2\delta(t+3) + 2\delta(t-3)$:

The Fourier transform of a delta function $\delta(t)$ is 1. For a shifted delta function $\delta(t-a)$, the Fourier transform is $e^{-i\omega a}$. Applying these rules gives:

$X(\omega) = 1 + 2e^{3i\omega} + 2e^{-3i\omega}$

b. Fourier transform of the signal $x(t) = 2\sin(2t) - \cos^2(\pi t)$:

The Fourier transform of $\sin(at)$ is $\frac{i}{2}(\delta(\omega-a) - \delta(\omega+a))$, and the Fourier transform of $\cos(at)$ is $\frac{1}{2}(\delta(\omega-a) + \delta(\omega+a))$. The Fourier transform of $\cos^2(at)$ can be obtained using the power-reduction identity $\cos^2(at) = \frac{1+\cos(2at)}{2}$. Applying these rules gives:

$X(\omega) = i\delta(\omega-2) - i\delta(\omega+2) - \frac{1}{2}\delta(\omega) - \frac{1}{4}\delta(\omega-2\pi) - \frac{1}{4}\delta(\omega+2\pi)$

c. Fourier transform of the signal $x(t) = 2e^{-3t}u(t) - 4e^{5t}u(-t)$:

The Fourier transform of $e^{-at}u(t)$ is $\frac{1}{a+i\omega}$, and the Fourier transform of $e^{at}u(-t)$ is $\frac{1}{a-i\omega}$. Applying these rules gives:

$X(\omega) = \frac{2}{3+i\omega} - \frac{4}{5-i\omega}$

## 2.

a) $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$

This property is known as the modulation property of the Fourier transform. It states that multiplying a signal in the time domain by a complex exponential results in a shift in the frequency domain.

Proof:

The Fourier transform of $x(t)e^{j\omega_0 t}$ is given by

$$\int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt$$

which is $X(\omega-\omega_0)$ by the definition of the Fourier transform.

b) $(-jt)^n x(t) \leftrightarrow \frac{d^n X(\omega)}{d\omega^n}$

This property is known as the differentiation property in the frequency domain. It states that multiplying a signal in the time domain by $(-jt)^n$ results in the nth derivative in the frequency domain.

Proof:

The Fourier transform of $(-jt)^n x(t)$ is given by

$$\int_{-\infty}^{\infty} (-jt)^n x(t) e^{-j\omega t} dt$$

Differentiating under the integral sign n times gives

$$\frac{d^n}{d\omega^n} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \frac{d^n X(\omega)}{d\omega^n}$$

by the definition of the Fourier transform.

## 3.

$te^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^2}$

This property can be derived using the differentiation property in the frequency domain from the previous question.

Proof:

The Fourier transform of $e^{-at}u(t)$ is $\frac{1}{a+j\omega}$.

According to the differentiation property in the frequency domain, multiplying a signal in the time domain by $(-jt)$ results in the derivative in the frequency domain.

Therefore, the Fourier transform of $-jte^{-at}u(t)$ is $\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$.

Simplifying the right hand side gives

$$\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right) = -\frac{1}{(a+j\omega)^2}$$

However, we want the Fourier transform of $te^{-at}u(t)$, not $-jte^{-at}u(t)$.

Multiplying both sides by $-j$ gives the desired result:

$$te^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^2}$$

## 4.

Suppose the Fourier transform of a signal $x(t)$ is given by $X(\omega) = e^{-2|\omega|}$. Find the energy of the signal between the frequency band $-3 < \omega < 3$. Use Parseval's relation.

Parseval's relation states that the energy of a signal in the time domain is equal to the energy of its Fourier transform in the frequency domain. Mathematically, this is expressed as

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Given that $X(\omega) = e^{-2|\omega|}$, the energy of the signal between the frequency band $-3 < \omega < 3$ is given by

$$E = \frac{1}{2\pi} \int_{-3}^{3} |e^{-2|\omega|}|^2 d\omega$$

Solving this integral gives

$$E = \frac{-1 + e^{24}}{8\pi e^{24}}$$

So, the energy of the signal in the frequency band $-3 < \omega < 3$ is $\frac{-1+e^{24}}{8\pi e^{24}}$.

## 5.

Suppose the Fourier transform of a signal $x(t)$ is given by $X(\omega) = \frac{j\omega - 1}{3 - \omega^2} + j4\omega$. Using the properties of the Fourier transform, find the Fourier transform expressions of the following signals.

a) $x(-2t+1)$:

Using the time-shifting property of the Fourier transform, if $X(\omega)$ is the Fourier transform of $x(t)$, then the Fourier transform of $x(-2t+1)$ is $\frac{1}{2} X\left(\frac{\omega}{2}\right) e^{j\omega}$.

b) $x(t) * x(t-1)$:

Using the convolution property of the Fourier transform, if $X(\omega)$ is the Fourier transform of $x(t)$, then the Fourier transform of $x(t) * x(t-1)$ is $X(\omega) \cdot X(\omega)$, where $\cdot$ denotes multiplication in the frequency domain.

## 6.

Compute the convolution of the following pairs of signals $x(t)$ and $h(t)$ in the time domain by using the convolution property of the Fourier transform.

$x(t) = e^{-3t}u(t)$ $h(t) = e^{5t}u(-t)$

Using the convolution property of the Fourier transform, the convolution of $x(t)$ and $h(t)$ is given by the inverse Fourier transform of $X(\omega) \cdot H(\omega)$, where $X(\omega)$ and $H(\omega)$ are the Fourier transforms of $x(t)$ and $h(t)$, respectively.

From Problem 1(c), we obtained the Fourier transforms of $x(t)$ and $h(t)$ as:

$$X(\omega) = \frac{i\sqrt{\frac{2}{\pi}}(i+3\omega)}{(3i+\omega)(-5i+\omega)} \; H(\omega) = \frac{i\sqrt{\frac{2}{\pi}}(i-5\omega)}{(5i+\omega)(3i+\omega)}$$

Multiplying $X(\omega)$ and $H(\omega)$ gives:

$$X(\omega) \cdot H(\omega) = \frac{-10(3\omega-i)}{(3i+\omega)(5i+\omega)^2(-5i+\omega)}$$

Taking the inverse Fourier transform of $X(\omega) \cdot H(\omega)$ will give us the convolution of $x(t)$ and $h(t)$ in the time domain.