Problem 14

Prove the statement by induction.

Statement:
$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$ for every integer $n \geq 0$

Proof:
In the statement, the property $P(n)$ is the equation:

$$P(n) = \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

We must prove $P(n)$ is true for every integer $n \geq 0$.

1. Base Case: We must prove $P(2)$ is true.

LHS: $P(2) = \sum_{i=1}^{1} i \cdot 2^i = (1) \cdot 2^{(1)}$, by substitution
LHS: $P(2) = 1 \cdot 2 = 2$, by order of operations

RHS: $P(2) = \sum_{i=1}^{0} = n \cdot 2^{n+2} + 2 = (0) \cdot 2^{(0)+2} + 2$, by substitution
RHS: $P(2) = 0 \cdot 2^2 + 2 = 0 \cdot 4 + 2 = 0 + 2 = 2$, by order of operations

$$\sum_{i=1}^{n+1} 2 = 2$$

As $2 = 2$, $P(2)$ is true.

2. Inductive Hypothesis: Let k be any integer where $k \geq 0$. Suppose $P(k)$ is true.

$$\sum_{i=1}^{(k)+1} i \cdot 2^i = (k) \cdot 2^{(k)+2} + 2, \text{ by inductive hypothesis substitution}$$

We must show $P(k+1)$ is true.

$$\sum_{i=1}^{(k+1)+1} i \cdot 2^i = (k+1) \cdot 2^{(k+1)+2} + 2, \text{ by substitution}$$

$$\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2, \text{ by combining like terms}$$

Problem 14 (Continued)

3. Inductive Step:
LHS: $\sum_{i=1}^{k+2} i \cdot 2^i = \sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{(k+2)}$, by adding the last term

RHS: $\sum_{i=1}^{k+2} k \cdot 2^{k+2} + 2 = \sum_{i=1}^{k+1} k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{(k+2)}$, by inductive hypothesis substitution

RHS: $= k \cdot 2^{k+2} + 2 + k \cdot 2^{k+2} + 2 \cdot 2^{k+2}$, by distribution
RHS: $= k \cdot 2^{k+2} + 2 + k \cdot 2^{k+2} + 2^{k+3}$, by multiplying $2^1 \cdot 2^{k+2}$
RHS: $= 2(k \cdot 2^{k+2}) + 2 + 2^{k+3}$, by combining like terms
RHS: $= k \cdot 2^{k+3} + 2 + 2^{k+3}$, by product of $2^1 \cdot (k \cdot 2^{k+3})$
RHS: $= k \cdot 2^{k+3} + 2^{k+3} + 2$, by associative property
RHS: $= (k+1) \cdot 2^{k+3} + 2$, by factoring $(k+1)$

With the base case and inductive hypothesis proven,

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

must be true for every integer $n \geq 0$.

Problem 16

Prove the statement by induction.

Statement:
$(1-\frac{1}{2^2})(1-\frac{1}{3^2})\ ...\ (1-\frac{1}{n^2})=\frac{n+1}{2n}$, for every integer $n \geq 2$

Proof:
The property $P(n)$ is the equation:

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})\ ...\ (1-\frac{1}{n^2})=\frac{n+1}{2n}$$

We must prove the $P(n)$ is true for every integer $n \geq 2$.

1. Base Case: We must prove $P(2)$ is true.

LHS: $P(2)=(1-\frac{1}{2^2})(1-\frac{1}{3^2})\ ...\ (1-\frac{1}{n^2})=1-\frac{1}{(2)^2}$ by substitution
LHS: $P(2)=(1-\frac{1}{2^2})(1-\frac{1}{3^2})\ ...\ (1-\frac{1}{4})$ by exponentiation, then subtraction
LHS: $P(2)=(1-\frac{1}{2^2})(1-\frac{1}{3^2})\ ...\ (\frac{3}{4})$ by exponentiation, then subtraction

RHS: $P(2)=\frac{n+1}{2n}=\frac{(2)+1}{2(2)}$, by substitution
RHS: $P(2)=\frac{3}{4}$, by addition and multiplication

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ ...\ \left(\frac{3}{4}\right)=\frac{3}{4}$$

As $\frac{3}{4}=\frac{3}{4}$, $P(2)$ is true.

2. Inductive Hypothesis: Let k be any integer where $k \geq 2$. Suppose $P(k)$ is true.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ ...\ \left(1-\frac{1}{(k)^2}\right)=\frac{(k)+1}{2(k)}, \text{ by inductive hypothesis substitution}$$

We must show $P(k+1)$ is true.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ ...\ \left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)}, \text{ by substitution}$$

$$1-\frac{1}{(k+1)^2}=\frac{k+2}{2(k+1)}, \text{ by combining like terms}$$

Problem 16 (Continued)

3. Inductive Step:

LHS: $(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})$ ... $(1 - \frac{1}{n^2})(1 - \frac{1}{(k+1)^2})$, by adding the last term

RHS: $\frac{k+1}{2k}(1 - \frac{1}{(k+1)^2})$, by inductive hypothesis substitution
RHS: $\frac{k+1}{2k}(\frac{1 \cdot (k+1)^2}{1 \cdot (k+1)^2} - \frac{1}{(k+1)^2})$
RHS: $\frac{k+1}{2k}\left(\frac{(k+1)^2 - 1}{(k+1)^2}\right)$
RHS: $\frac{k+1}{2k}\left(\frac{(k+1) - 1}{k+1}\right)$

Problem 12

Prove the statement by induction.

Statement:
For any integer $n \geq 0$, $7^n - 2^n$ is divisible by 5.

Proof:
The property $P(n)$ is the sentence "$7^n - 2^n$ is divisible by 5." We must prove $P(n)$ is true for any integer $n \geq 0$.

1. Base Case: First, we must show $P(0)$ is true.

$P(n) = 7^n - 2^n = 7^0 - 2^0$, by substitution
$P(n) = 1 - 1 = 0$, by difference

As 0 is divisible by 5, where $0 = 5 \cdot 0$, then $P(0)$ is true.

2. Inductive Hypothesis: Let k be any integer with $k \geq 0$. Suppose $P(k)$ is true, and $7^k - 2^k$ is divisible by 5.

$7^k - 2^k = 5p$, where p is some integer
$7^k = 5p + 2^k$, by transposition

We must show $P(k+1)$ is true, where $7^{k+1} - 2^{k+1}$ is divisible by 5.

$7^{k+1} - 2^{k+1}$
$= 7^k \cdot 7^1 - 2^{k+1}$, by separating exponents
$= 7^k \cdot 7 - 2^{k+1}$, as $7^1 = 7$

Now, $7k = 5p + 2^k$.

$7^k(7) - 2^{k+1}$
$= (5p + 2^k)(7) - 2^{k+1}$, by substitution
$= 35p + 2^k \cdot 7 - 2^{k+1}$, by distribution
$= 35p + 2^k \cdot 7 - 2^k \cdot 2$, by separating exponents, as $2^1 = 2$
$= 35p + (7 - 2) \cdot 2^k$, by factoring 7 and 2
$= 35p + 5 \cdot 2^k$, by difference
$= 5(7p + 2^k)$, by factoring 5

As $7p + 2^k$ is an integer under closure properties, $7^n - 2^n$ is divisible by 5 by the definition of divisibility.

Problem 15

Prove the statement by induction.

Statement:
$n(n^2 + 5)$ is divisible by 6, for each integer $n \geq 0$

Proof:
The property $P(n)$ is the sentence "$n(n^2 + 5)$ is divisible by 6." We must prove $P(n)$ is true for each integer $n \geq 0$.

1. Base Case: First, we must show $P(0)$ is true.

$P(n) = n(n^2 + 5) = (0)((0)^2 + 5)$, by substitution
$P(n) = 0(0 + 5) = 0(5) = 0$, by order of operations

As 0 is divisible by 6, where $0 = 6 \cdot 0$, then $P(0)$ is true.

2. Inductive Hypothesis:
Let k be any integer with $k \geq 0$. Suppose $P(k)$ is true, and $n(n^2 + 5)$ is divisible by 6.

$n(n^2 + 5) = 6p$, where p is some integer

We must show $P(k + 1)$ is true, where $(n + 1)((n + 1)^2 + 5)$ is divisible by 6.

$(k + 1)((k + 1)^2 + 5)$, by substitution
$= (k + 1)((k^2 + k + k + 1) + 5)$, by foiling
$= (k + 1)(k^2 + 2k + 6)$, by combining like terms
$= k^3 + 2k^2 + 6k + k^2 + 2k + 6$, by distribution
$= k^3 + 2k^2 + k^2 + 6k + 2k + 6$, by associative property
$= k^3 + 3k^2 + 8k + 6$, by combining like terms
$= k(k^2 + 5) + 3(k^2 + k + 2)$, by factoring $(k + 3)$

Now, $k(k^2 + 5) = 6p$.

$= 6p + 3(k^2 + k + 2)$, by substitution
$= 6p + 3k^2 + 3k + 6$, by distribution
$= 3(2p + k^2 + k + 2)$, by factoring 3

As $(2p + k^2 + k + 2)$ is an integer under closure property, let $r = (2p + k^2 + k + 2)$, where r is some integer and $n(n^2 + 5) = 3r$. As 3 is a factor of 6, $n(n^2 + 5)$ is divisible by 6, for each integer $n \geq 0$.

Problem 23b

Solve the statement by induction.

Statement:
$n! > n^2$, for each integer $n \geq 4$

Proof:
The property $P(n)$ is the inequality $n! > n^2$. We must prove $P(n)$ is true for each integer $n \geq 4$.

1. Base Case: First, we must prove $P(4)$ is true.

$P(4) = n! > n^2 = (4)! > (4)^2$, by substitution
$P(4) = 24 > 16$, by factorization and exponentiation

As $24 > 16$, $P(4)$ is true.

2. Inductive Hypothesis:
Let k be any integer where $k \geq 4$. Suppose $P(k)$ is true, and $k! > k^2$.

We must prove $P(k+1)$ is true.

$$(k+1)! > (k+1)^2\text{, by substitution}$$

$$(k+1)k! > (k+1)^2\text{, by the definition of factorials}$$

Problem 23b (Continued)

3. Inductive Step:

$(k+1)k! > k^2(k+1)$, by inductive hypothesis substitution

$k^2(k+1)$
$= k^3 + k^2$, by distribution

Now, we must prove $k^3 + k^2 > (k+1)^2$.

$k^3 + k^2 > k^2 + 2k + 1$, by foiling
$k^3 > 2k + 1$, by derivation of k^2
$k^2 - 2k > 1$, by transposition

We know $k \geq 4$.
$(4)^2 - 2(4) > 1$, by substitution
$16 - 8 > 1$, by product, and $8 > 1$, by difference, which is true

Now, we've proven $k^3 > 2k + 1$. So:

$$k^2(k+1) > (k+1)^2$$

$$(k+1)k! > k^2(k+1)$$

With the basis case and inductive hypothesis proven, $P(k+1)$ must be true for every integer $n \geq 4$.