Canvas Problem 1

Use iteration to guess an explicit formula for the sequence

$$e_k = 4e_{k-1} + 3 \textit{ for all integers } k \geq 2, \textit{ where } e_1 = 2$$

Use the summation property formulas to simply answers.

$e_1 = 2$

$e_2 = 4e_{2-1} + 3$
$= 4e_1 + 3$
$= 4(2) + 3$
$= 4^1 \cdot 2 + 4^0 \cdot 3$

$e_3 = 4e_{3-1} + 3$
$= 4e_2 + 3$
$= 4(4^1 \cdot 2 + 4^0 \cdot 3) + 3$
$= 4^2(2) + 4^1(3) + 4^0(3)$

$e_4 = 4e_{4-1} + 3$
$= 4e_3 + 3$
$= 4(4^2(2) + 4^1(3) + 4^0(3)) + 3$
$= 4^3(2) + 4^2(3) + 4^1(3) + 4^0(3)$

Guess:

$$e_n = 4^{n-1}(2) + 4^{n-2}(3) + 4^{n-3}(3) + ... + 4^2(3) + 4^1(3) + 4^0(3)$$

$$e_n = 4^{n-1}(2) + 3(4^{n-2} + 4^{n-3} + ... + 4^2 + 4^1 + 4^0)$$

$$= 4^{n-1}(2) + 3 \cdot \sum_{i=0}^{n-2} 4i$$

$$= 2(4^{n-1}) + 3\left(\frac{4^{n-2+1} - 1}{4 - 1}\right) \text{ by the formula } \sum_{k=0}^{n} r^k = \left(\frac{r^{n+1} - 1}{r - 1}\right)$$

$$= 2(4^{n-1}) + 3\left(\frac{4^{n-1} - 1}{3}\right) \text{ by the sums and differences of integers}$$

$$= 2(4^{n-1}) + \frac{3(4^{n-1} - 1)}{3} \text{ by the products of numerators}$$

$$= \frac{3(2)(4^{n-1})}{3} + \frac{3(4^{n-1} - 1)}{3} \text{ by the products of denominators}$$

$$= \frac{3(2)(4^{n-1}) + 3(4^{n-1} - 1)}{3} \text{ by the sums of fractions with equal denominators}$$

$$= \frac{6(4^{n-1}) + 3(4^{n-1} - 1)}{3} \text{ by the products of integers}$$

$$= \frac{6(4^{n-1}) + 3(4^{n-1}) - 3}{3} \text{ by distribution}$$

$$= \frac{9(4^{n-1}) - 3}{3} \text{ by the sums of integers}$$

$$= \frac{3(3(4^{n-1}) - 1)}{3} \text{ by factoring 3}$$

$$= 3(4^{n-1}) - 1 \text{ by canceling the denominator } \textbf{(Answer)}$$

Canvas Problem 2

A sequence $e_1, e_2, e_3, e_4 ...$ *is defined as*

$$e_1 = 2$$

$$e_k = 4e_{k-1} + 3$$

for all integers $k \geq 2$.

Show $e_k = 3(4^{n-1}) - 1$ *for all integers* $n \geq 1$.

Proof:
Let $e_1, e_2, e_3, e_4 ...$ be a sequence defined as:

$$e_1 = 2 \text{ and } e_k = 4e_{k-1} + 3 \text{ for all integers } k \geq 2$$

Let the property $P(n)$ be the equation:

$$e_k = 3(4^{n-1}) - 1$$

for each integer $n \geq 1$.

We must prove $P(n)$ is true for each integer $n \geq 1$.

Base: We must show $P(1)$ is true.
$P(1) = 3(4^{1-1}) - 1$ by substitution
$P(1) = 3(4^0) - 1$ by differences of integers
$P(1) = 3(1) - 1$ by exponentiation
$P(1) = 3 - 1$ by products of integers
$P(1) = 2$ by differences of integers

So, $P(1)$ is true.

Inductive Hypothesis: Let k be any integer with $k \geq 1$. Suppose $P(k)$ is true.

$P(k) \equiv e_k = 3(4^{k-1}) - 1$

Inductive Step: If $P(k)$ is true, then $P(k+1)$ is true for all integers $k \geq 1$.

We must show:

$$P(k) \equiv e_{k+1} = 3(4^{(k+1)-1}) - 1 \text{ by inductive hypothesis substitution}$$

$$P(k) \equiv e_{k+1} = 3(4^{k}) - 1 \text{ by differences of integers}$$

The left side of $P(k+1)$ is:

$$e_{k+1} = 4e_{(k+1)-1} + 3 \text{ by the sequence's recursive definition}$$

LHS: $= 4e_k + 3$ by differences of integers
LHS: $= 4[3(4^{k-1}) - 1] + 3$ by substitution
LHS: $= 4(3)(4^{k-1}) - 4(1) + 3$ by distributing 4
LHS: $= 4(3)(4^{k-1}) - 4 + 3$ by products of integers
LHS: $= 4(3)(4^{k-1}) - 1$ by differences of integers
LHS: $= 3(4)(4^{k-1}) - 1$ by the associative property
LHS: $= 3(4^1)(4^{k-1}) - 1$ by the power rule of exponents, as $4^1 = 4$
LHS: $= 3(4^{k-1+1}) - 1$ by the product rule of exponents
LHS: $= 3(4^k) - 1$ by the differences of integers

which is the right side of $P(k+1)$, making the property true for $n = k+1$.

Conclusion: The basis and inductive steps have been proven, so $P(n)$ is true for all integers $n \geq 1$.

Canvas Problem 3

Find an explicit formula for the sequence

$$t_k = t_{k-1} + 5k^2 + 7k + 2 \text{ for all integers } k \geq 1 \text{, where } t_0 = 0$$

Use the summation property laws to simplify.

$t_0 = 0$

$t_1 = t_{1-1} + 5(1)^2 + 7(1) + 2$
$= t_0 + 5(1)^2 + 7(1) + 2$
$= 0 + 5(1)^2 + 7(1) + 2$
$= 5(1)^2 + 7(1) + 2$

$t_2 = t_{2-1} + 5(2)^2 + 7(2) + 2$
$t_2 = t_1 + 5(2)^2 + 7(2) + 2$
$t_2 = [5(1)^2 + 7(1) + 2] + 5(2)^2 + 7(2) + 2$
$= 5(1^2 + 2^2) + 7(1 + 2) + 4$

$t_3 = t_{3-1} + 5(3)^2 + 7(3) + 2$
$= t_2 + 5(3)^2 + 7(3) + 2$
$= [5(1^2 + 2^2) + 7(1 + 2) + 2 + 2] + 5(3)^2 + 7(3) + 2$
$= 5(1^2 + 2^2 + 3^2) + 7(1 + 2 + 3) + 6$

$t_4 = t_{4-1} + 5(4)^2 + 7(4) + 2$
$= t_3 + 5(4)^2 + 7(4) + 2$
$= [5(1^2 + 2^2 + 3^2) + 7(1 + 2 + 3) + 6] + 5(4)^2 + 7(4) + 2$
$= 5(1^2 + 2^2 + 3^2 + 4^2) + 7(1 + 2 + 3 + 4) + 8$

$$t_n = 5(1^2 + 2^2 + 3^2 + ... + (n-1)^2 + n^2) + 7(1 + 2 + 3 + ... + (n-1) + n) + 2n$$

$1^2 + 2^2 + 3^2 + ... + (n-1)^2 + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
$1 + 2 + 3 + ... + (n-1) + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

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\begin{aligned}
t_n &= 5 \cdot \frac{n(n+1)(2n+1)}{6} + 7 \cdot \frac{n(n+1)}{2} + 2n \text{ by substitution} \\
&= \frac{5n(n+1)(2n+1)}{6} + \frac{7n(n+1)}{2} + 2n \text{ by the products of numerators} \\
&= \frac{5n(n+1)(2n+1)}{6} + \frac{7n(n+1)}{2} + \frac{2 \cdot 2n}{2} \text{ by the product of the denominator} \\
&= \frac{5n(n+1)(2n+1)}{6} + \frac{7n(n+1) + 2 \cdot 2n}{2} \text{ by the sum of the fractions} \\
&= \frac{5n(n+1)(2n+1)}{6} + \frac{7n(n+1) + 4n}{2} \text{ by the products of integers} \\
&= \frac{(2)(5n)(n+1)(2n+1)}{(6)(2)} + \frac{6[(7n)(n+1) + 4n]}{(6)(2)} \text{ by the products of the denominators} \\
&= \frac{(2)(5n)(n+1)(2n+1) + 6[(7n)(n+1) + 4n]}{12} \text{ by the sum of the fractions} \\
&= \frac{(2)(5n)(n+1)(2n+1) + 6(7n^2 + 7n + 4n)}{12} \text{ by distribution} \\
&= \frac{(2)(5n)(n+1)(2n+1) + 6(7n^2 + 11n)}{12} \text{ by the sums of integers} \\
&= \frac{10n(n+1)(2n+1) + 6(7n^2 + 11n)}{12} \text{ by the products of integers} \\
&= \frac{10n(2n^2 + n + 2n + 1) + 6(7n^2 + 11n)}{12} \text{ by foiling} \\
&= \frac{10n(2n^2 + 3n + 1) + 6(7n^2 + 11n)}{12} \text{ by the sums of integers} \\
&= \frac{20n^3 + 30n^2 + 10n + 6(7n^2 + 11n)}{12} \text{ by distribution} \\
&= \frac{20n^3 + 30n^2 + 10n + 42n^2 + 66n}{12} \text{ by distribution} \\
&= \frac{20n^3 + 72n^2 + 76n}{12} \text{ by the sums of integers} \\
&= 4n \cdot \frac{5n^2 + 18n + 19}{12} \text{ by factoring } 4n \textbf{ (Answer)}
\end{aligned}
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Canvas Problem 4

Give a recursive definition for the set of all strings of a's and b's, where all strings contain exactly two a's that are consecutive.

Assume S is the set of all strings of a's and b's, where all strings contain two consecutive a's.

$$S = \{aa, aab, baa, aabb, baab, baab, bbaa, aabbb, baabb, ...\}$$

<u>I. Base:</u> $aa \in S$

<u>II. Recursion:</u> If $x \in S$:
(a) $bx \in S$
(b) $xb \in S$
(c) $bxb \in S$

<u>III. Restriction:</u> Nothing is in S other than the objects defined in I and II above, where x is string aa.

Canvas Problem 5

Give a recursive definition for the set of all strings of a's and b's, where every string has odd length.

Assume S is the set of all strings of a's and b's where all strings are odd lengths:

$$S = \{a, b, aaa, aba, aab, abb, baa, bba, bab, bbb, aaaaa, ...\}$$

I. Base:
(a) $a \in S$
(b) $b \in S$

II. Recursion: If $x \in S$:
(a) aax
(b) xaa
(c) axa
(d) bbx
(e) xbb
(f) bxb
(g) abx
(h) xab
(i) axb
(j) bax
(k) xba
(l) bxa

III.Restriction: Nothing is in S other than the objects defined in I and II above, where s is a string of odd length.