# Problem 8

*Let*

$$T = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

*Will two integers sum* 10 *from five integers chosen?*

**Answer:**
No. By pigeonhole theory, the problem requires five "pigeons" — the chosen integers [...]

$$a_1, a_2, a_3, a_4, a_5$$

[...] but there are five "pigeonholes" — the paired disjoint subsets of set $T$.

$$\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \text{ and } \{5\}$$

The function $P$ — defined for each integer $a_i$ in the domain — is one-to-one, so there are five possible integers that don't sum 10.

Consider [...]

$$1, 2, 3, 4, 5$$

The largest possible sum from any two of these integers is 9, so the pigeonhole principle doesn't apply.

# Problem 11

*If $n + 1$ integers are chosen from the set [...]*

$$\{1, 2, 3, ..., 2n\}$$

*[...] where $n$ is a positive integer, must at least one chosen integer be even?*

**Answer:**
Yes, the pigeonhole principle applies.

The "pigeons" are the $n + 1$ chosen integers.

The "pigeonholes" are the disjoint subsets of set $T$, where [...]

$$T = \{1, 2, 3, ..., 2n\}$$

$$\{1, 2n\}, \{2, 2n - 1\}, \{3, 2n - 2\}, ..., \{n - 2, n - 1\}, \{n - 1, n\}, \{n, n + 1\}$$

There are only $n$ "pigeonholes" or disjoint subsets, where each subset has one even integer.

Since $n + 1 > n$, there are always more chosen integers than disjoint subsets.

## Problem 30

*A penny collection has*
*twelve 1967 pennies,*
*seven 1968 pennies, and*
*eleven 1971 pennies.*
*If one picks some pennies without looking at the dates, how many must one pick to get at least five pennies from the same year?*

**Answer:**
Thirteen pennies must number at least five from the same year.

The generalized pigeonhole principle applies, where $n$ "pigeons" (chosen pennies) fly to $m$ "pigeonholes" or three disjoint subsets of all pennies by year.

$$\{12 \cdot 1967\}, \{7 \cdot 1968\}, \text{ and } \{11 \cdot 1971\}$$

Make $k$ the four chosen pennies from each subset and $n$ some integer, with $n = km + 1$.

$$km = (4)(3) = 12$$

$$n = km + 1 = (12) + 1 = 13$$

$$13 > 12$$

The next penny chosen must number five pennies from one disjoint subset.

# Problem 31

*Fifteen executives share 5 assistants.*
*Each executive has exactly one assistant.*
*No assistant has more than four executives.*
*Show how at least three assistants have three or more executives.*

Let $k$ be the number of assistants with three or more executives. Show $k \geq 3$.

No assistant has more than four executives, so the assistants have at most $4k$ executives.

Each remaining $5 - k$ assistant has at most two executives.

$$2(5-k) = 10 - 2k \text{ by the contrapositive of the generalized pigeonhole principle}$$

The assistants serve at maximum [...]

$$4k + 10 - 2k = 2k + 10$$

[...] executives, but 15 executives have assistants.

$$2k + 10 \geq 15$$

$$= 2k \geq 5$$

$$k = 2.5$$

But $k$ is an integer, so $k = 3$.

# Problem 34

*Let $S$ be a set of 10 integers* 1 *through* 50.
*Show the set has at least two different — but not necessarily disjoint — subsets of four integers that sum to the same number.*

**Answer:**
Make $k$ the number of sets with different — not necessarily disjoint — four-integer subsets that sum to the same number. Show $k \geq 1$.

There are 210 subsets of four integers of a set of 10 integers (given). Make $n$ the "pigeons," which are these 210 subsets.

Make $m$ the possible number of sums of four-integer subsets, which are the "pigeonholes."

Find the smallest and largest possible sums.

$$1 + 2 + 3 + 4 =$$

$$10\text{, the smallest possible sum from integers 1 through 50}$$

$$47 + 48 + 49 + 50 =$$

$$194 \text{ the largest possible sum from integers 1 through 50}$$

The distinct sums possible are [...]

$$m = 194 - 10$$

$$= 184 \text{ different sums possible}$$

The generalized pigeonhole principle shows the function is not one-to-one.

$$n > km$$

$$210 > (1)(184)$$

$$\frac{210}{184} > 1$$

$$1\frac{26}{184} > 1\text{, as } \frac{210}{184} \text{ has a remainder of 26}$$

$$1\frac{13}{92} > 1 \text{ by simplifying fractions}$$

Since $\frac{n}{m} > k$, the generalized pigeonhole principle applies.