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CS-225: Discrete Structures in CS
Assignment 3, Part 2
Exercise Set 4.7

Problem 22

Write proofs of contraposition and contradiction for the statement.

Statement: For every real number r, if r^2 is irrational, then r is irrational.

Contraposition Proof:

The statement's contrapositive is, "For every real number r, if r is rational, then r^2 is rational.

Supposition: Suppose r is any particularly but arbitrarily chosen integer, where r is rational.

Goal: We must show r^2 is rational.

Deductions:

By the definition of rational numbers, $r = \frac{p}{q}$, where p and q are some integers and $q \neq 0$.

Then:
(1) $r^2 = (\frac{p}{q})^2$ by substitution
(2) $= \frac{p^2}{q^2}$

Now, p^2 and q^2 are integers because the exponentiations of integers are integers, and $q^2 \neq 0$ by the zero product property.

Conclusion:

By the definition of rational numbers, $= \frac{p^2}{q^2}$ is rational, as was to be shown.

Problem 22 (Continued)

Contradiction Proof:

The statement's contradiction is, "There is an integer r such that r^2 is irrational and r is rational."

Supposition: Suppose r is any particularly but arbitrarily chosen integer such that r is rational.

Goal: Show the supposition leads to a contradiction.

Deductions:

Since r is rational, then by definition of rational numbers, $r = \frac{p}{q}$, where p and q are some integers and $q \neq 0$.

Then:
(1) $r^2 = (\frac{p}{q})^2$ by substitution
(2) $= \frac{p^2}{q^2}$ by exponential distribution

Now, p^2 and q^2 are integers, as the products of integers are integers, and $q \neq 0$ by the zero product property. So, by the definition of rational numbers, r^2 is rational, and we have a contradiction that r is rational and r^2 is irrational.

Conclusion:

That means the assumption is not correct, and the original statement is true, as was to be shown.

Problem 27

Prove the statement by contraposition or contradiction.

Statement: For all positive real numbers r and s, $\sqrt{r+s} \neq \sqrt{r} + \sqrt{s}$.

Contradiction Proof:

Supposition: Suppose r and s are any particularly but arbitrarily chosen integers, where r and s are positive real numbers.

Goal: Show the supposition leads to a contradiction.

Deductions:

Since r and s are positive real numbers, $p > 0$ and $q > 0$. Let $r = p$ and $s = q$, where p and q are some positive integers.

Then:
(1) $\sqrt{p+q} = \sqrt{p} + \sqrt{q}$ by substitution
(2) $\sqrt{p+q}^2 = (\sqrt{p} + \sqrt{q})^2$ by squaring both sides
(3) $p + q = p + \sqrt{pq} + \sqrt{pq} + q$ by foiling
(4) $p + q = p + q + 2\sqrt{pq}$ by combining like terms

Now, $p + q$ is an integer, as the sums of integers are integers. Let $p + q = x$, where x is a positive integer, as the sums and products of positive integers are positive integers, and say $x > 0$ by the definition of positive integers.

Let $p + q + 2\sqrt{pq} = y$, where y is some positive real number, as the sums and products of positive integers are positive integers, but the squares of positive integer are positive real numbers. Also, say $y > 0$ by the definition of positive real numbers.

Since $x < y$, we have a contradiction that $x = y$.

Conclusion:

That means the assumption is not correct, and the original statement is true, as was to be shown.

Problem 29

Prove the statement by contraposition or contradiction.

Statement: For all integers m and n, if $m + n$ is even, then m and n are both even or m and n are both odd.

Contraposition Proof:

The statement's contrapositive is, "For all integers m and n, if one is even and the other odd, then $m + n$ is odd."

Supposition:

Suppose m and n are any particularly but arbitrarily chosen integers such that m is even, and n is odd.

Goal: We must show $m + n$ is odd.

Deductions:

Since m is even and n is odd, $m = 2p$ by the definition of even integers, and $n = 2q+1$ by the definition of odd integers.

Then:
(1) $m + n = 2p + (2q + 1)$ by substitution
(2) $= 2(p + q) + 1$ by factoring 2

Now, $2(p + q) + 1$ is an integer, as products and sums of integers are integers. Hence, $2(p + q) + 1$ is odd by the definition of odd integers.

Conclusion:

This proves the contraposition, so the original statement is also true, as was to be shown.

Canvas Problem 1

Prove the statement by contraposition or contradiction, showing both cases.

Statement: Suppose $x, y, z \in Z$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

Contradiction Proof:

The statement's contradiction is, "For all integers x, y, and z, where $x \neq 0$, if $x \nmid yz$, then $x \mid y$ or $x \mid z$.

Supposition:

Suppose x, y, and z are any particularly but arbitrarily chosen integers, where $x \neq 0$ and $x \mid y$ or $x \mid z$.

Goal: We must show that $x \nmid yz$.

Deductions:

By the definition of divisibility, $y = xr$ or $z = xs$, where r and s are some integers.

Then:
(1) $yz \neq xt$, where t is some integer

For $x \mid y$:
(2) $xrz \neq xt$ by substitution
(3) $rz \neq t$ by dividing x from both sides

For $x \mid z$:
(4) $yxs \neq xt$ by substitution
(5) $ys \neq t$ by dividing x from both sides

Now, rz, t, and ys are some integers, as products of integers are integers, and $t = rz$ and $t = ys$ by the definition of divisibility. We have a contradiction that $yz \neq x$(some integer).

Conclusion:

That means the assumption is not correct, and the original statement is true, as was to be shown.

Canvas Problem 2

Prove the statement by contraposition or contradiction.

Statement: $\sqrt{6}$ is irrational

Contradiction Proof:

The statement's negation is, "$\sqrt{6}$ is rational."

<u>Supposition:</u> Suppose $\sqrt{6}$ is rational, then $\sqrt{6} = \frac{p}{q}$, where p, q are integers, and $q \neq 0$.

<u>Goal:</u> To show the supposition leads to a contradiction.

<u>Deductions:</u>

By the definition of rational numbers, p and q are integers with no common factors, since the expression cannot be simplified further.

Then:
(1) $\sqrt{6}^2 = (\frac{p}{q})^2$ by squaring both sides
(2) $6 = \frac{p^2}{q^2}$ by exponential distribution
(3) $6q^2 = p^2$ by transposition
(4) $2(3q^2) = p^2$ by factoring 2

Now, p^2 is an integer, as the products of integers are integers, and $p = 2n$ because, by the definition of even integers, p is equal to 2 times something.

Then:
(5) $2(3q^2) = (2n)^2$ by substitution
(6) $2(3q^2) = 4n^2$ by squaring
(7) $3q^2 = n^2$ by dividing by 2

Now, n^2 is even, but $3q^2$ is odd, as $\frac{p}{q}$ has no common factors. We have a contradiction because the two integers, by the definitions of even and odd integers, cannot be equal to one another or have a common factor of 2.

<u>Conclusion:</u>

That means the assumption is not correct. Therefore, the original statement is true, as was to be shown.