Mason Blanford
CS-225: Discrete Structures in CS
Assignment 3, Part 1
Exercise Set 4.2
Page 1

Problem 28

Determine if the statement is true or false, with a proof or counterexample. Use the terms' definitions and assumptions on page 161 — not any prior-established properties.

Statement:
For all integers n and m, if $n - m$ is even, then $n^3 - m^3$ is even.

Proof:

Supposition:

Suppose n and m are any two particularly but arbitrarily chosen integers where $n - m$ is even.

Goal: We must show $n^3 - m^3$ is even.

Deductions:

Since $n - m$ is an even integer, there exists an integer p such that $n - m = 2p$ by the definition of even integers.

Then:
(1) $n^3 - m^3 = (n - m)(n^2 + m^2 + nm)$ by factoring
(2) $= 2p(n^2 + m^2 + nm)$ by substitution

Now, $n^2 + nm + m^2$ is an integer because it's a sum of products of integers. Let $r = (n^2 + nm + m^2)$, where r is some integer. Thus, $n^3 - m^3 = 2p(r)$, where r is some integer.

Conclusion:

By the definition of even integers, $n^3 - m^3$ is an even integer, as was to be shown.

Problem 36

Determine if the statement is true or false, with a proof or counterexample. Use the terms' definitions and assumptions on page 161 — not any prior-established properties.

Statement:
The difference of the squares of any two consecutive integers is odd.

Proof:

Supposition:

Suppose n and m are any two particularly but arbitrarily chosen integers, where m is the consecutive integer of n.

Goal: We must show $n^2 - m^2$ is odd.

Deductions:

Since n and m are consecutive integers, either n is even and m is odd, or n is odd and m is even.

There exists an integer p such that $n = 2p$, and since m is a consecutive integer and odd, $m = n + 1 = 2p + 1$.

Then:
(1) $n^2 - m^2 = (n - m)(n + m)$ by factoring
(2) $= (n - (n + 1))(n + (n + 1))$ by substitution
(3) $= (2p - (2p + 1))(2p + (2p + 1))$ by substitution
(4) $= (2p - 2p - 1)(2p + 2p + 1)$ by distribution
(5) $= -1(4p + 1)$ by combining like terms
(6) $= -4p - 1$ by distribution
(7) $= -4p - 1$ by factoring -4

Now, $-4p$ is an integer because sums, differences, and products of integers are integers. Let $r = -4p$, where r is some even integer by the definition of even integers.

Thus, $n^2 - m^2 = r - 1$, where r is some integer.

Conclusion:

By the definition of odd integers, $n^2 - m^2$ is odd, as was to be shown.

Canvas Problem

Prove the statement using only the definition of rational numbers, closure properties of integers, and the zero product property.

Statement: $9s^4 + \frac{3}{7}s - 2$

Proof:

Supposition: Suppose s is any particularly but arbitrarily chosen rational number.

Goal: We must show $9s^4 + \frac{3}{7}s - 2$ is rational.

Deductions:

By the definition of rational numbers, $r = \frac{p}{q}$, where p and q are some integers and $q \neq 0$.

Then:
(1) $9s^4 + \frac{3}{7}s - 2 = 9(\frac{p}{q})^4 + \frac{3}{7}(\frac{p}{q}) - 2$ by substitution
(2) $= \frac{9p^4}{q^4} + \frac{3p}{7q} - 2$ by combining like terms
(3) $= \frac{(7q^4)(9p^4)}{7q^4} + \frac{(7q^4)(3p)}{7q} - \frac{(7q^4)(2)}{7q^4}$ by multiplying $7q^4$ as a common denominator
(4) $= \frac{7(9p^4)+q^3(3p)-7q^4(2)}{7q^4}$ by reducing common variables
(5) $= \frac{63p^4-14q^4+3p(q^3)}{7q^4}$ by distribution

Now, $63p^4 - 14q^4 + 3p(q^3)$ and $7q^4$ are integers, as sums, differences, squares and products of integers are integers, and $7q^4 \neq 0$ by the zero product property.

Conclusion:

By the definition of rational numbers, $\frac{63p^4-14q^4+3p(q^3)}{7q^4}$ is rational, as was to be shown.

Problem 39

Find the mistakes in the "proof" that the sum of any two rational numbers is a rational number.

Given "Proof:"

Suppose r and s are rational numbers. If $r + s$ is rational, then by definition of rational $r + s = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Also, since r and s are rational, $r = \frac{i}{j}$ and $s = \frac{m}{n}$ for some integers i, j, m, and n with $j \neq 0$ and $n \neq 0$. It follows that:

$$r + s = \frac{i}{j} + \frac{m}{n} = \frac{a}{b},$$

which is a quotient of integers with a nonzero denominator. Hence, it is a rational number. This is what was to be shown.

Answer: This proof assumes what is to be proved (indicating confusion between what is known and what is still to be shown) and must show how $a = i + m$ and $b = jn$ (because sums and products of integers are integers) and $jn \neq 0$ (by the zero product property).

The proof also uses the conditional *if*, which in this context is more egregious, as the relatively minor mistake is paired with stating the proof's goal rather than its supposition. It calls into question the sum of two rational numbers being rational, which is the proof's sole objective.

Additionally, the proof uses an unclear antecedent in its last sentence, "This is what was to be shown," substituting "This" for a specific noun. While this error wouldn't necessarily negate a correct proof, the grammatical ambiguity weakens the overall argument.

The correct proof is on the next page.

Problem 39 (Continued)

Correct Proof:

Supposition: Suppose r and s are any particularly but arbitrarily chosen rational numbers.

Goal: We must show that $r + s$ is also rational.

Deductions:

By the definition of rational numbers, $r = \frac{a}{b}$ and $s = \frac{c}{d}$, where a, b, c, and d are some integers, with $b \neq 0$ and $c \neq 0$.

Then:

(1) $r + s = \frac{a}{b} + \frac{c}{d}$ by substitution
(2) $= \frac{a+c}{bd}$ by adding fractions

Now, $a + c$ and bd are integers, as sums and products of integers are integers, and $ab \neq 0$ by the zero product property.

Conclusion:

By the definition of rational numbers, $\frac{a+c}{bd}$ is rational, as was to be shown.

Mason Blanford
CS-225: Discrete Structures in CS
Assignment 3, Part 1
Exercise Set 4.4
Page 6

Problem 26

Determine if the statement is true or false, with a proof or counterexample.

Statement: For all integers a, b, and c, if $ab \mid c$, then $a \mid c$ and $b \mid c$.

Proof:

Supposition:

Suppose a, b, and c are particular but arbitrarily chosen integers such that $ab \mid c$.

Goal: We must show that $a \mid c$ and $b \mid c$.

Deductions:

By the definition of divisibility, $c = abr$, $c = as$, and $c = bt$, where r, s, and t are some integers. Then:

For $a \mid c$: (1) $c = abr$ and $c = as$ by substitution
(2) $abr = as$ by the definition of equality
(3) $br = s$ by dividing a from both sides

For $b \mid c$:
(1) $c = bar$ and $c = bt$ by substitution
(2) $bar = bt$ by the definition of equality
(3) $ar = t$ by dividing b from both sides

Now, br, s, ar, and t are all integers, as sums of integers are integers.

Conclusion:

By the definition of divisibility, $a \mid c$ and $b \mid c$, as was to be shown.

Canvas Problem

Prove the statement.

Statement: For all integers a, b, and c, if $a \mid b$ and $a \mid (b^2 - c)$, then $a \mid c$.

Proof:

Supposition:

Suppose a, b, and c are any particularly but arbitrarily chosen integers such that $a \mid b$ and $a \mid b^2 - c$.

Goal: We must show that $a \mid c$.

Deductions:

By the definition of divisibility, $b = ar$ and $b^2 - c = as$ where r and s are some integers.

Then:
(1) $as = b^2 - c$ by substitution
(2) $as + c = b^2$ by transposition
(3) $c = b^2 - as$ by transposition
(4) $c = (ar)^2 - as$ by substitution
(5) $c = (a^2)(r^2) - a(s)$ by exponential distribution
(6) $c = a(ar^2 - s)$ by factoring a

Now, $ar^2 - s$ is an integer, as products and differences of integers are integers, and $c = a(\text{some integer})$ by the definition of divisibility.

Conclusion:

By the definition of divisibility, $a \mid c$, as was to be shown.