Mason Blanford
CS-225: Discrete Structures in CS
Assignment 4, Part 1
Exercise Set 6.1
Page 1

Problem 7

Prove or disprove the statements.

Let:
$A = \{x \in \mathbb{Z} \mid x = 6a + 4 \text{ for some integer } a\}$,
$B = \{y \in \mathbb{Z} \mid y = 18b - 2 \text{ for some integer } b\}$, and
$C = \{z \in \mathbb{Z} \mid z = 18c + 16 \text{ for some integer } z\}$

Statements:
(a) $A \subseteq B$
(b) $B \subseteq A$
(c) $B = C$

Answer (a):
Let sets A and B be given. To disprove $A \subseteq B$, suppose r is a particular element of A. We must show r is not an element of B.

Assume:
(1) $r \in A$, and $r = 10$
(2) $r = 6(1) + 4 = 10$, as $1 \in \mathbb{Z}$, so $10 \in A$.

Then:
(3) $18b - 2 = 10$, where b must be an integer
(4) $18b = 12$
(5) $b = \frac{12}{18} = \frac{2}{3}$, but $b = \frac{2}{3}$ is not an integer $\in \mathbb{Z}$

So, $10 \in A$, but $10 \notin B$, so $A \notin B$.

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Answer (b):
Suppose r is a particular but arbitrarily chosen element of B. We must show r is an element of A.

(1) By definition of B, there is an integer b such that $r = 18b - 2$.
(2) With $r \in A$, $r = 6a + 4$, where a is an integer.

Then:
(3) $6a + 4 = 18b - 2$
(4) $a = 3b - 1$

With the value of a from (4):
(5) $6a + 4 = 6(3b - 1) + 4$
(6) $= 18b - 6 + 4$
(7) $= 18b - 2 = r$

So, $B \subseteq A$.

Answer (c):
To prove $B = C$, we must prove $B \subseteq C$ and $C \subseteq B$.

Proof of $B \subseteq C$
Suppose r is a particular but arbitrarily chosen element of B. We must show r is an element of C.

(1) By definition of B, there is an integer b such that $r = 18b - 2$.
(2) With $r \in C$, $r = 18c + 16$, where c is an integer.

Then:
(3) $18c + 16 = 18b - 2$
(4) $c = b - 1$

With the value of c from (4):
(5) $18c + 16 = 18(b - 1) + 16$
(6) $= 18b - 18 + 16$
(7) $= 18b - 2 = r$

So, $B \subseteq C$.

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Proof of $C \subseteq B$

Suppose r is a particular but arbitrarily chosen element of C. We must show r is an element of B.

(1) By definition of C, there is an integer c such that $r = 18c + 16$.
(2) With $r \in B$, $r = 18b - 2$, where b is an integer.

Then:
(3) $18b - 2 = 18c + 16$
(4) $b = c + 1$

With the value of b from (4):
(5) $18b - 2 = 18(c + 1) - 2$
(6) $= 18c + 18 - 2$
(7) $= 18c + 16 = r$

So, $C \subseteq B$ and $B \subseteq C$, meaning $B = C$.

Mason Blanford
CS-225: Discrete Structures in CS
Assignment 4, Part 1
Exercise Set 6.1
Page 4

Problem 26(a, b, c)

Let $S_i = \{x \in \mathbb{R} \mid 1 < x < 1 + \frac{1}{i}\} = (1, 1 + \frac{1}{i})$ for each positive integer i.

(a) $\bigcup_{i=1}^{4} S_i$
(b) $\bigcap_{i=1}^{4} S_i$
(c) Are $S_1, S_2, S_3...$ mutually disjoint? Explain.

Answers:
$S_1 = (1, 1 + \frac{1}{1}) = (1, 1 + 1) = (1, 2)$
$S_2 = (1, 1 + \frac{1}{2}) = (1, 1\frac{1}{2})$
$S_3 = (1, 1 + \frac{1}{3}) = (1, 1\frac{1}{3})$
$S_4 = (1, 1 + \frac{1}{4}) = (1, 1\frac{1}{4})$

(a) $\bigcup_{i=1}^{4} S_i = \{x \in \mathbb{R} \mid x$ is in at least one of the intervals $(1, 1 + \frac{1}{i})$, where i is a positive integer$\}$.

$= (1, 2) \cup (1, 1\frac{1}{2}) \cup (1, 1\frac{1}{3}) \cup (1, 1\frac{1}{4})$

$= (1, 2]$, as all elements in every interval $(1, 1 + \frac{1}{i})$ are in $(1, 2]$

(b) $\bigcap_{i=1}^{4} S_i = \{x \in \mathbb{R} \mid x$ is in all of the intervals $(1, 1 + \frac{1}{i})$, where i is a positive integer$\}$.

$= (1, 2) \cap (1, 1\frac{1}{2}) \cap (1, 1\frac{1}{3}) \cap (1, 1\frac{1}{4})$

$= (1, 1\frac{1}{4})$, as the elements in the interval $(1, 1\frac{1}{4})$ are in every interval $(1, 1 + \frac{1}{i})$

(c) $S_1, S_2, S_3...$ are not mutually disjoint, as $S_{i+1} \subseteq S_i$

Problem 29

Answer the question.

Question:
Let $\mathbb{R}$ be the set of all real numbers. Is $\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$ a partition of $\mathbb{R}$?

Answer:
Yes, $\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$ is a partition of $\mathbb{R}$; a real number can't be both positive and negative, and 0 is neither a positive nor a negative real number.

Problem 30

Answer the question.

Let:
$\mathbb{Z}$ be the set of all integers.
$A_0 = \{n \in \mathbb{Z} \mid n = 4k$, for some integer $k\}$
$A_1 = \{n \in \mathbb{Z} \mid n = 4k + 1$, for some integer $k\}$
$A_2 = \{n \in \mathbb{Z} \mid n = 4k + 2$, for some integer $k\}$, and
$A_3 = \{n \in \mathbb{Z} \mid n = 4k + 3$, for some integer $k\}$

Question:
Is $\{A_0, A_1, A_2, A_3\}$ a partition of $\mathbb{Z}$?

Answer:
Yes, $\{A_0, A_1, A_2, A_3\}$ is a partition of $\mathbb{Z}$ by the quotient-remainder theorem:

Given any integer n and positive integer d, there exists unique integers k and r such that:

$$n = dk + r \text{ and } 0 \leq r < d,$$

where every integer n is represented in one of four forms:

$$n = 4k, n = 4k + 1, n = 4k + 2 \text{ or } n = 4k + 3$$

for some integer k.

(1) No integer is in any two of the sets A_0, A_1, A_2, or A_3.
(2) So, A_0, A_1, A_2, and A_3 are mutually disjoint.
(3) Every integer is in one of the sets A_0, A_1, A_2, or A_3.

So, $\mathbb{Z} = A_0 \cup A_1 \cup A_2 \cup A_3$

Problem 33(b, c)

Find the power sets.

(a) $\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\})$, as all sets are subsets of themselves, so $\{\emptyset\} \in \mathcal{P}(\{\emptyset\})$

$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$, as $\emptyset$ is a subset of all sets, so $\emptyset \in \mathcal{P}(\{\emptyset\})$

(b) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \mathcal{P}(\mathcal{P}(\{\emptyset\})) = \mathcal{P}(\{\emptyset, \{\emptyset\}\})$

$\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\{\{\emptyset\}, \emptyset\}, \{\{\emptyset\}\}, \{\emptyset\}, \emptyset\}$, as:

(1) $\{\{\emptyset\}\} \in \mathcal{P}(\{\emptyset, \{\emptyset\}\})$,
(2) $\{\emptyset\} \in \mathcal{P}(\{\emptyset, \{\emptyset\}\})$, and
(3) $\emptyset \in \mathcal{P}(\{\emptyset, \{\emptyset\}\})$

Mason Blanford
CS-225: Discrete Structures in CS
Assignment 4, Part 1
Exercise Set 6.1
Page 8

Problem 34b

Find the sets.

Let:
$A_1 = \{1\}$,
$A_2 = \{u, v\}$, and
$A_3 = \{m, n\}$

Set:
(b) $(A_1 \cup A_2) \cdot A_3$

$= (\{1\} \cup \{u, v\}) \cdot \{m, n\}$
$= \{u, v, 1\} \cdot \{m, n\}$
$= \{(u, m), (u, n), (v, m), (v, n), (m, 1), (n, 1)\}$

Mason Blanford
CS-225: Discrete Structures in CS
Assignment 4, Part 1
Exercise Set 6.1
Page 9

Problem 35d

Find the set.

Let:
$A = \{a, b\}$
$B = \{1, 2\}$, and
$C = \{2, 3\}$

Set:
(d) $(A \cdot B) \cap (A \cdot C)$

$= (\{a, b\} \cdot \{1, 2\}) \cap (\{a, b\} \cdot \{2, 3\})$
$= (\{(a, 1), (a, 2), (b, 1), (b, 2)\}) \cap (\{(a, 2), (a, 3), (b, 2), (b, 3)\})$
$= \{(a, 2), (b, 2)\}$