# Problem 7b(iii), 7c

*A team has 13 members: seven women and six men. Find the number of seven-member groups:*
*(b)(iii) Containing at most three women; and*
*(c) When two members refuse to work together.*

**Answer (b)(iii):**
Use the addition rule to find the number of groups containing at most three women.

$$[\text{groups with at most three women}]$$

$$= [\text{groups with no women}] + [\text{groups with one woman}]$$
$$+ [\text{groups with two women}] + [\text{groups with three women}]$$

The order of the members doesn't matter, so calculate for r-combinations.

There are seven group members, and the number of women range zero to three.

$$C(7,0) + C(7,1) + C(7,2) + C(7,3)$$
$$= \frac{n!}{r!(n-r)!} + \frac{n!}{r!(n!-r)!} + \frac{n!}{r!(n!-r)!} + \frac{n!}{r!(n!-r)!}$$
$$= \frac{7!}{0!(7-0)!} + \frac{7!}{1!(7-1)!} + \frac{7!}{2!(7-2)!} + \frac{7!}{3!(7-3)!} \text{ by substitution}$$
$$= \frac{7!}{0! \cdot 7!} + \frac{7!}{1! \cdot 6!} + \frac{7!}{2! \cdot 5!} + \frac{7!}{3! \cdot 4!} \text{ by differences of integers}$$
$$= \frac{7!}{7!} + \frac{7!}{6!} + \frac{7!}{2! \cdot 5!} + \frac{7!}{3! \cdot 4!} \textbf{ or } \frac{5,040}{5,040} + \frac{5,040}{720} + \frac{5,040}{240} + \frac{5,040}{144} \text{ by products of integers}$$
$$= 1 + 7 + 21 + 35 \text{ by quotients of integers}$$
$$= 64 \text{ by sums of integers}$$

**Answer:** There are 64 possible teams with at most three women.

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# Problem 7c (Continued)

Call the two uncooperative people members $A$ and $B$.

Use the addition rule to find the number of groups without both members who refuse to work together.

$$[\text{groups without members } A \text{ or } B]$$

$$= [\text{groups without member } A] + [\text{groups without member } B] + [\text{groups without} A \text{ or } B]$$

Groups with members $A$ or $B$ have six other possible members of $13 - 2 = 11$ people: $C(11, 6) \cdot 2$.

Groups without members $A$ or $B$ have seven possible members of 11 people: $C(11, 7)$.

$$(C(11, 6) \cdot 2) + C(11, 7)$$

$$= \left( \frac{n!}{r!(n-r)!} \cdot 2 \right) + \frac{n!}{r!(n!-r)!}$$

$$= \left( \frac{7!}{6!(7-6)!} \cdot 2 \right) + \frac{7!}{7!(7!-7)!} \text{ by substitution}$$

$$= \left( \frac{7!}{6! \cdot 1!} \cdot 2 \right) + \frac{7!}{7! \cdot 0!} \text{ by differences of integers}$$

$$= \left( \frac{7!}{6!} \cdot 2 \right) + \frac{7!}{7!} \textbf{ or } \left( \frac{5,040}{720} \cdot 2 \right) + \frac{5,040}{5,040} \text{ by products of integers}$$

$$= (7 \cdot 2) + 1 \text{ by quotients of integers}$$

$$= 14 + 1 = 15 \text{ by products and sums of integers}$$

**Answer:** There are 15 groups without the two uncooperative members.

# Problem 12

*From the set:*

$$\{1, 2, 3, ..., 101\}$$

*How many integer pairs have an even sum?*

Two odd integers or two even integers sum an even integer, by the definition of even integers.

$$2p + 2p = 4p = 2(2p), \text{ where } p \text{ is some integer}$$

$$(2p + 1) + (2p + 1) = 4p + 2 = 2(2p + 1), \text{ where } p \text{ is some integer}$$

There are 101 integers.

$$(101 - 1) + 1 = 101$$

There are 51 odd integers, as the range starts and ends with an even integer: $C(51, 2)$.

$$C(51, 2) = \frac{n!}{r!(n-r)!}$$

$$= \frac{51!}{2!(51-2)!} \text{ by substitution}$$

$$= \frac{51!}{2! \cdot 49!} \text{ by differences of integers}$$

$$= 1,275 \text{ by quotients of integers}$$

There are $1,275$ possible pairs where both integers are odd.

There are 50 even integers: $C(50, 2)$.

$$C(50, 2) = \frac{n!}{r!(n-r)!}$$

$$= \frac{50!}{2!(50-2)!} \text{ by substitution}$$

$$= \frac{50!}{2! \cdot 48!} \text{ by differences of integers}$$

$$= 1,225 \text{ by quotients of integers}$$

There are $1,225$ possible pairs where both integers are even.

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## Problem 12 (Continued)

Use the addition rule to find all possible sums in these conditions.

$$C(51, 2) + C(50, 2) = 1,275 + 1,225 = 2,500 \text{ by sums of integers}$$

**Answer:** There are $2,500$ possible even-integer sums.

# Problem 17b, 17c

*Ten points:*

$$\{A, B, C, D, E, F, G, H, I, J\}$$

*are on a plane, with no three points sharing a straight line.*

*Find the number of:*
*(b) Straight lines not intersecting point A; and*
*(c) Triangles with three of* 10 *points as vertices.*

**Answer (b):**
There are 45 possible straight lines with 10 points and 2 points per line.

$$C(10, 2) = \frac{n!}{r!(n-r)!}$$

$$= \frac{10!}{2!(10! - 2!)} \text{ by substitution}$$

$$= \frac{10!}{2! \cdot 8!} \text{ by differences of integers}$$

$$= 45 \text{ by quotients of integers}$$

Of 45 possible lines, nine intersect point $A$, with nine possible other points and one other point per intersection.

$$C(9, 1) = \frac{n!}{r!(n-r)!}$$

$$= \frac{9!}{1!(9! - 1!)} \text{ by substitution}$$

$$= \frac{9!}{1! \cdot 8!} \text{ by differences of integers}$$

$$= 9 \text{ by quotients of integers}$$

To find the lines not intersecting $A$, use the difference rule.

$$45 - 9 = 36 \text{ by differences of integers}$$

**Answer:** There are 36 lines not intersecting point $A$.

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# Problem 17c (Continued)

Triangles require three points from 10 possible points: $C(10, 3)$.

$$C(10, 3) = \frac{n!}{r!(n-r)!}$$

$$= \frac{10!}{3! \cdot 7!} \text{ by differences of integers}$$

$$= 120 \text{ by quotients of integers}$$

**Answer:** There are 120 possible triangles with three of the points as vertices.

# Problem 13

*Find the number of solutions for the equation.*

$$y_1 + y_2 + y_3 + y_4 = 30, \textit{ where } y_i \geq 2$$

Find the r-combinations with repetition allowed.

Each $y_i$ must be at least 2, and there are four $y_i$ variables.

$$30 - (2 \cdot 4) = 30 - 8 = 22$$

$$\begin{aligned} C(r+n-1, r) &= C(22+4-1, 22) = C(25, 22) \\ &= \frac{25!}{22!(25-22)!} \text{ by substitution} \\ &= \frac{25!}{22! \cdot 3!} \text{ by differences of integers} \\ &= 2,300 \text{ by quotients of integers} \end{aligned}$$

**Answer:** There are $2,300$ possible solutions.

# Problem 18

*In a collection of pennies, nickels, dimes, and quarters, find the number of 30-coin collections possible when there are:*
*(a)* 30 *of each coin;*
*(b)* 15 *quarters,* 30 *of each other coin;*
*(c)* 20 *dimes,* 30 *of each other coin; and*
*(d)* 15 *quarters,* 20 *dimes, and* 30 *of each other coin.*

**Answer(a):**
Find the r-combinations with repetition allowed.

$$C(30+4-1,30) = C(30+3,30) = C(33,30)$$
$$= \frac{33!}{30!(33-30)!} \text{ by substitution}$$
$$= \frac{33!}{30! \cdot 3!} \text{ by differences of integers}$$
$$= 5,456 \text{ by differences of integers}$$

**Answer (a):** There are 5,456 30-coin collections possible.

**Answer(b):**
Use the difference rule.

$$\text{[collections with at most 15 quarters]}$$
$$= \text{[all collections]} - \text{[collections with at least 16 quarters]}$$

There are 5,456 30-coin collections possible: $C(33,30)$.

$$C(33,30) - C(14+4-1,14) = C(33,30) - C(17,14)$$
$$= 5,456 - \frac{17!}{14!(17-14)!} \text{ by substitution}$$
$$= 5,456 - \frac{17!}{14! \cdot 3!} \text{ by differences of integers}$$
$$= 5,456 - 680 \text{ by quotients of integers}$$
$$= 4,776 \text{ by differences of integers}$$

**Answer (b):** There are 4,776 possible 30-coin collections with at most 15 quarters.

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# Problem 18 (Continued)

**Answer(c):**
Use the difference rule.

$$[\text{collections with at most 20 dimes}]$$

$$= [\text{all collections}] - [\text{collections with at least 21 dimes}]$$

There are $5,456$ 30-coin collections possible: $C(33,30)$.

$$C(33,30) - C(9+4-1,9) = C(33,30) - C(12,9)$$

$$= 5,456 - \frac{12!}{9!(12-9)!} \text{ by substitution}$$

$$= 5,456 - \frac{12!}{9! \cdot 3!} \text{ by differences of integers}$$

$$= 5,456 - 220 \text{ by quotients of integers}$$

$$= 5,236 \text{ by differences of integers}$$

**Answer (c):** There are $5,236$ possible 30-coin collections with at most 20 dimes.

**Answer(d):**
Use the difference rule.

$$[\text{collections with at most 15 quarters and 20 dimes}]$$

$$= [\text{all collections}] - [\text{number of disallowed combinations}]$$

Find the number of disallowed combinations.

$$[\text{number of disallowed combinations}]$$

$$= [\text{collections with at least 16 quarters } \textbf{or} \text{ at least 21 dimes}]$$

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## Problem 18 (Continued)

Let:
$Q$ = the set of collections with at least 16 quarters
$D$ = the set of collections with at least 21 dimes
$(Q \cup D)$ = the set of collections with at least 16 quarters or 21 dimes
$(Q \cap D)$ = the set of collections with at least 16 quarters and 21 dimes

By the inclusion-exclusion formula, the intersection is the total minus the union.

$$(Q \cup D) = Q + D - (Q \cap D)$$

$$(Q \cup D) + (Q \cap D) = Q + D \text{ by transposition}$$

$$(Q \cap D) = Q + D - (Q \cup D) \text{ by transposition}$$

The intersection set $(Q \cap D)$ isn't possible, as $16 + 21 > 30$.

So, the answer is:

$$\begin{aligned}
&[\text{collections with at most 15 quarters and 20 dimes}]\\
&= [\text{all collections}] - ([\text{collections with at least 16 quarters}] + [\text{collections with at least 21 dimes}])\\
&= C(33, 30) - (C(14 + 4 - 1, 14) + C(9 + 4 - 1, 9))\\
&= C(33, 30) - (C(17, 14) + C(12, 9))\\
&= 5,456 - (680 + 220) \text{ by substitution}\\
&= 4,556 \text{ by sums and differences of integers}
\end{aligned}$$

**Answer (d):** There are $4,556$ possible collections, with at most 15 quarters and 20 dimes.

# Problem 20

*A camera shop has at most:*
 10 *A76 batteries*
 30 *of seven other types*

*Find how many ways:*
 *(a)* 30 *batteries are distributed among eight types; and*
 *(b)* 30 *batteries are distributed among eight types with at most six D303*

**Answer (a):**
Use the difference rule.

$$\begin{aligned}
&\text{[collections with at most 10 A76 batteries]}\\
&= \text{[all collections]} - \text{[collections with at least 20 A76 batteries]}\\
&= C(30+8-1,30) - C(10+8-1,10)\\
&= C(37,30) - C(17,10)\\
&= \frac{37!}{30!(37-30)!} - \frac{17!}{10!(17-10)!} \text{ by substitution}\\
&= \frac{37!}{30!\cdot 7!} - \frac{17!}{10!\cdot 7!} \text{ by differences of integers}\\
&= 10,295,472 - 19,448 \text{ by quotients of integers}\\
&= 10,276,024 \text{ by differences of integers}
\end{aligned}$$

**Answer(a):** There are $10,276,024$ possible distributions.

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# Problem 20 (Continued)

**Answer (b):**
Use the difference rule.

$$[\text{collections with at most 10 A76 and 6 D303 batteries}]$$
$$= [\text{all collections}] - [\text{unallowed combinations}]$$

To find the unallowed combinations:

$$[\text{unallowed combinations}]$$
$$= [\text{combinations with at least 20 AT6 and 24 D303 batteries}]$$

This isn't possible: $20 + 24 > 30$.

So, the answer is:

$$[\text{combinations with at most 10 A76 and 6 D303 batteries}]$$
$$= [\text{all collections}] - ([\text{combinations with at least 20 A76 batteries}] + [\text{combinations with at least 24 D303 batteries}])$$
$$= 10,295,472 - (C(10 + 8 - 1, 10) + C(6 + 8 - 1, 6))$$
$$= 10,295,472 - (19,448 + C(13, 6)) \text{ by substitution}$$
$$= 10,295,472 - \left(19,448 + \frac{13!}{6!(13-6)!}\right) \text{ by substitution}$$
$$= 10,295,472 - \left(19,448 + \frac{13!}{6! \cdot 7!}\right) \text{ by differences of integers}$$
$$= 10,295,472 - (19,448 + 1,716) \text{ by quotients of integers}$$
$$= 10,274,308 \text{ by sums and differences of integers}$$

**Answer(b):** There are $10,274,308$ possible distributions.