Problem 3

Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$c_0 = 2, c_1 = 2, c_2 = 6$
$c_k = 3c_{k-3}$, *for every integer* $k \geq 3$

Prove c_n is even for each integer $n \geq 0$.

Proof:
Suppose the property $P(n)$ is the sentence "c_n is even." We must prove $P(n)$ is true for each integer $k \geq 3$.

<u>1. Base Cases:</u> First, we must show $P(0)$, $P(1)$, and $P(2)$ are true.

$P(0) = c_0 = 2$, by substitution
$P(1) = c_1 = 2$, by substitution
$P(2) = c_2 = 6$, by substitution

Integers 2 and 6 are even, so $P(0)$, $P(1)$, and $P(2)$ are true.

<u>2. Inductive Hypothesis:</u>
Let k be any integer where $k \geq 1$. Suppose c_i is even for each integer i with $0 \leq i \leq k$.

We must show c_{k+1} is even, where $c_{k+1} = 3c_{k-2}$.

<u>3. Inductive Step:</u>
Now, $k-2$ is even because $k-2 < k$.

$c_{k-2} = 2p$, by the definition of even
$c_{k+1} = 3(2p)$

As c_{k+1} will always be a product of 3 and an even integer, c_{k+1} is an even integer by the definition of even integers and the closure property of integers.

Problem 8a

Suppose $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$h_0 = 1, h_1 = 2, h_2 = 3$
$h_k = h_{k-1} + h_{k-2} + h_{k-3}$, *for each integer* $k \geq 3$

Statement:
(a) Prove $h_n \leq 3^n$, for every integer $n \geq 0$

Proof:
Suppose the property $P(n)$ is the sentence "$h_n \leq 3^n$." We must prove $P(n)$ is true for each integer $k \geq 3$.

1.Base Cases: First, we must show $P(0)$, $P(1)$, and $P(2)$ are true.

$P(0) = h_n \leq 3^n = h_0 \leq 3^0$, by substitution
$P(0) = 1 \leq 1$, by substitution and exponentiation

$P(1) = h_n \leq 3^n = h_1 \leq 3^1$, by substitution
$P(1) = 2 \leq 3$, by substitution and exponentiation

$P(2) = h_n \leq 3^n = h_2 \leq 3^2$, by substitution
$P(3) = 3 \leq 9$, by substitution and exponentiation

So, $P(0)$, $P(1)$, and $P(2)$ are true.

2. Inductive Hypothesis: Let k be any integer where $k \geq 2$. Suppose $P(i) = h_i \leq 3^i$ for all i with $0 \leq i \leq k$.

We must show $P(k+1)$ is true, where $h_{k+1} \leq 3^{k+1}$.

Problem 8a (Continued)

3. Inductive Step:
We know the following.

$$h_{k+1} = h_k + h_{k-1} + h_{k-2}$$

$$h_k + h_{k-1} + h_{k-2} \leq 3^k + 3^{k-1} + 3^{k-2}$$

$$3^k + 3^{k-1} + 3^{k-2} = \frac{3^{k+1}}{3^1} + \frac{3^{k+1}}{3^2} + \frac{3^{k+1}}{3^3}, \text{ by the exponent quotient rule}$$

$$= \frac{3^{k+1}}{3} + \frac{3^{k+1}}{9} + \frac{3^{k+1}}{27}, \text{ by exponentiation}$$

$$= 3^{k+1}(\frac{1}{3} + \frac{1}{9} + \frac{1}{27}), \text{ by factoring } 3^{k+1}$$

$$= 3^{k+1}(\frac{1 \cdot 9 \cdot 27}{3 \cdot 9 \cdot 27} + \frac{1 \cdot 3 \cdot 27}{9 \cdot 3 \cdot 27} + \frac{1 \cdot 3 \cdot 9}{27 \cdot 3 \cdot 9}), \text{ by products of denominators}$$

$$= 3^{k+1}(\frac{9 + 3 + 1}{27}), \text{ by canceling, combining like terms}$$

$$3^{k+1}(\frac{13}{27}), \text{ by addition}$$

Now, $3^{k+1}(\frac{13}{27}) \leq 3^{k+1}$. So, $P(k+1)$ is true, and the statement is true for every integer $n \geq 0$.

Problem 9

Define a sequence $a_1, a_2, a_3, \ldots$ *as:*

$a_1 = 1, a_2 = 3$
$a_k = a_{k-1} + a_{k-2}$, *for each integer* $k \geq 3$

Prove $a_n \leq (\frac{7}{4}^n)$ *for every integer* $n \geq 1$.

Proof:
Suppose the property $P(n)$ is $a_n \leq (\frac{7}{4}^n)$. We must prove $P(n)$ is true for each integer $n \geq 1$

1. Base Cases: First, we must how $P(1)$ and $P(2)$ are true.

$P(1) = a_n \leq (\frac{7}{4}^n) = a_1 \leq (\frac{7}{4}^1) = 1 \leq (\frac{7}{4})$, by substitution and exponentiation
$P(2) = a_n \leq (\frac{7}{4}^n) = a_2 \leq (\frac{7}{4}^2) = 3 \leq (\frac{49}{16})$, by substitution and exponentiation

So, $P(1)$ and $P(2)$ are true.

2. Inductive Hypothesis: Let k be any integer where $k \geq 2$. Suppose $P(i) = a_i \leq (\frac{7}{4}^i)$ for all i with $1 \leq i \leq k$.

We must show $P(k+1)$ is true, where $a_{k+1} \leq (\frac{7}{4}^{k+1})$

Problem 9 (Continued)

3. Inductive Step:
We know the following.

$$a_{k+1} = a_k + a_{k-1}$$

$$a_k + a_{k-1} \leq \frac{7}{4}^k + \frac{7}{4}^{k-1}$$

$$\frac{7}{4}^k + \frac{7}{4}^{k-1} = \frac{7}{4}^{k+1} + \frac{7}{4}^{k+1}\text{, by inductive hypothesis substitution}$$

$$= \frac{\frac{7}{4}^{k+1}}{\frac{7}{4}^1} + \frac{\frac{7}{4}^{k+1}}{\frac{7}{4}^2}\text{, by the exponent quotient rule}$$

$$= \frac{\frac{7}{4}^{k+1}}{\frac{7}{4}} + \frac{\frac{7}{4}^{k+1}}{\frac{49}{16}}\text{, by exponentiation}$$

$$= \frac{7}{4}^{k+1}(\frac{1}{\frac{7}{4}} + \frac{1}{\frac{49}{16}})\text{, by factoring } \frac{7}{4}^{k+1}$$

$$= \frac{7}{4}^{k+1}(\frac{1 \cdot \frac{49}{16}}{\frac{7}{4} \cdot \frac{49}{16}} + \frac{1 \cdot \frac{7}{4}}{\frac{49}{16} \cdot \frac{7}{4}})\text{, by product of common denominator}$$

$$= \frac{7}{4}^{k+1}(\frac{\frac{7}{4} + 1}{\frac{49}{16}})\text{, by canceling, combining like terms}$$

$$= \frac{7}{4}^{k+1}(\frac{\frac{7}{4} + \frac{1 \cdot 4}{1 \cdot 4}}{\frac{49}{16}})\text{, by product of denominator}$$

$$= \frac{7}{4}^{k+1}(\frac{\frac{7+4}{4}}{\frac{49}{16}}) = \frac{7}{4}^{k+1}(\frac{\frac{11}{4}}{\frac{49}{16}})\text{, by canceling, combining like terms, then addition}$$

$$= \frac{7}{4}^{k+1}(\frac{11}{4} \cdot \frac{16}{49}) = \frac{7}{4}^{k+1}(\frac{176}{196}) = \frac{7}{4}^{k+1}(\frac{44}{49})\text{, by division, simplification of fractions}$$

Now, $\frac{7}{4}^{k+1}(\frac{44}{49}) < \frac{7}{4}^{k+1}$. So, $P(k+1)$ is true, and the statement is true for every integer $n \geq 1$.

Canvas Problem

*Use strong induction to prove the statement without using **Proposition 5.3.1**.*

Statement:
A postage of n cents can be formed using just five- and seven-cent stamps, for $n \geq 24$.

Proof:
The property $P(n)$ is the sentence "A postage of n cents can be formed using just five- and seven-cent stamps."

We must prove $P(n)$ is true for each integer $n \geq 24$.

1. Base Cases: First, we must prove $P(24), P(25), P(26), P(27)$, and $P(28)$ are true.

$P(24) = 5p + 7q = 5(2) + 7(2) = 10 + 14 = 24$
$P(25) = 5p + 7q = 5(5) + 7(0) = 25 + 0 = 25$
$P(26) = 5p + 7q = 5(1) + 7(3) = 5 + 21 = 26$
$P(27) = 5p + 7q = 5(4) + 7(1) = 20 + 7 = 27$
$P(28) = 5p + 7q = 5(0) + 7(4) = 0 + 28 = 28$

So, $P(24), P(25), P(26), P(27)$, and $P(28)$ are true.

2. Inductive Hypothesis:
Let k be any integer where $k \geq 28$. Suppose $P(i)$ is true for $24 \leq i \leq k$.

$k = 5p + 5q$

We must prove $P(k + 1)$ is true, where $k + 1$ cents can be formed using five- and seven-cent stamps.

3. Inductive Step:
Removing one five-cent stamp for $k + 1$ cents is $(k + 1) - 5$ cents.

We know $(k + 1) - 5$ is true from our inductive hypothesis of k, since $(k + 1) - 5 = k - 4$, and $k - 4 < k$.