Problem 1 (10 points)

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Choose a well-known NP-complete problem and give a polynomial-time
many-one reduction from your chosen problem to the Multistage Graph
Simple Path (MSP) problem.
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HAMILTONIAN CYCLE is polynomially transformable to MSP. For a given undirected simple[1] graph $G = \langle V, E \rangle$ of order n, we transform it into a labeled multistage graph $G' = \langle V', E', S, D, L \rangle$ according to the following six steps.

HC TO MULTISTAGE GRAPH $(G = \langle V, E \rangle)$

(1) Let $L = n$, where $n = |V|$
(2) Select vertex v from V. Generate vertices $(v, 0), (v, n) \in V'$. Let $S = (v, 0)$ and $D = (v, n)$
(3) For all $u \in V - \{v\}$, generate $(u, 1)$ of stage 1, $(u, 2)$ of stage 2, …, $(u, L-1)$ of stage $L-1$ in V'
(4) For all $(a, b) \in E, a \neq v$ and $b \neq v$, generate edges from the previous stage to the next stage in E'. These edges are $\langle (a, 1), (b, 2), 2 \rangle, \ldots, (a, L-2), (b, L-1), L-1$ and $\langle (b, 1), (a, 2), 2 \rangle, \ldots, \langle (b, L-2), (a, L-1), L-1 \rangle$. For all $(v, b) \in E$, generate edges from $(v, 0)$ to $(b, 1)$ and generate edges $(b, L-1)$ to (v, n) in E'.
(5) Let $E(u, l) = E' - \{e | e \in E' \text{ and } e \text{ is associated with } (u, 1), \ldots, \text{or } (u, l-1)\}, 1 \leq l \leq L-1$.[2]
(6) Let $E(D) = E(v, n) = E'$

Step 5 is important. The edge set of (u, l) is assigned with a value that permits the appearance of u in stage l and forbids the appearance of u in those stages smaller than l.

Theorem 1.1 G has a Hamiltonian cycle if and only if G' has a simple path from S to D.

Proof If $v - a_1 - a_2 - \cdots - a_n - 1 - v$ is a Hamiltonian cycle of G, $(v, 0) - (a_1, 1) - \cdots - (a_{n-1}, n-1) - (v - n)$ must be a path in G'.

$\Rightarrow$ Since $v, a_1, a_2, \ldots, a_n - 1$ are mutually different, $E(v, n) = E', E(a_i, i) = E' - \{e | e \in E' \text{ and } e \text{ is associated with } (a_i, 1), \ldots, (a_i, i-1)\}, 1 \leq i \leq n-1$, therefore, we have $E(v, n)$ contains $(v, 0) - (a_1, 1) - \cdots - (a_{n-1}, n-1) - (v, n), E(a_i, i)$ contains $(v, 0) - (a_1, 1) - \cdots - (a_i, i)$, where $1 \leq i \leq n-1$. This means $(v, 0) - (a_1, 1) - \cdots - (a_n - 1, n-1) - (v, n)$ is a simple path in G'.

$\Leftarrow$ If $(v, 0) - (a_1, 1) - \cdots - (a_{n-1}, n-1) - (v, n)$ is a simple path in G', $v - a_1 - a_2 - \cdots - a_{n-1} - v$ must be a path which is from v to v in G.

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[1] Nothing in step (4) prevents addition of self-loops. This is an incorrect omission by Jiang. Hence I add the requirement that the graph be simple.

[2] Step (5), as it is written in the paper, includes in $E(u, l)$ all edges of stages which are equal or greater in value than l, regardless the value of u. Furthermore, note the phrase "e is associated with $(u, 1), \ldots, \text{or } (u, l-1)$" in step (5). Because the enumeration of vertices is specified with the word "or", an edge associated with $(u, 1)$ should always be subtracted from E' as $(u, 1)$ does not depend on l. However, I assume Jiang's intended meaning is that an edge associated with $(u, 1)$ should be subtracted only if $u \geq 2$.

Since $(v,0)-(a_1,1)-\cdots-(a_{n-1},n-1)-(v,n)$ is a simple path, we know that $v,a_1,a_2,\ldots,a_{n-1}$ are mutually different. This means that $v-a_1-a_2-\cdots-a_{n-1}-v$ is a Hamiltonian cycle of G. ■

Theorem 1.2 Let G be an undirected graph of order n. The complexity of transforming G into G' is a polynomial function of n.

Proof For each vertex u in G, step 3 will generate $n-1$ vertices in G'. Hence step 3 will generate $(n-1)*(n-1)$ vertices. For each edge in G, step 4 will generate $2*n$ edges in G'. Hence step 4 will generate $2n^3$ edges at most. We can finish step 5 in $O(n^5)$, since each $E(u,l)$ has $2n^3$ edges at most and the number of $E(u,l)$ is no more than n^2. The complexity of the algorithm is $O(n^5)$. ■

Problem 2 (10 points)

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Describe an algorithm for your chosen problem by moving the algorithm
described in this paper for MSP problem to your chosen problem.
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To determine whether a graph $G=\langle V,E\rangle$ has a Hamiltonian cycle, execute the following algorithm.

HAMILTONIAN CYCLE VIA Z-H ALGORITHM $(G=\langle V,E\rangle)$

1 Execute HC TO MULTISTAGE GRAPH $(G=\langle V,E\rangle)$ as defined in Problem 1. Output is of the form $G'=\langle V',E',S,D,L\rangle$.

2 Execute only the first three steps of Z-H ALGORITHM $(G'=\langle V',E',S,D,L\rangle)$ as specified below.

  2.1 If $\mathrm{Comp}\big(E(D),D,R(E)\big)=\emptyset$, then $G=\langle V,E\rangle$ has no Hamiltonian cycle.

  2.2 Else $G=\langle V,E\rangle$ has at least one Hamiltonian cycle.

Z-H ALGORITHM $(G=\langle V,E,S,D,L\rangle)$

1 For all $e\in E$, we use operator $\mathrm{Init}(R(e))$ to generate $R(e)$ directly.

2 **for** $l=1$ to $L-1$ **do**

  2.1 For all $\langle u,v,l\rangle$ of stage l, call $\mathrm{Change}(R(u,v,l))$ to modify $R(u,v,l)$

  2.2 For all v of stage l, $E(v)\leftarrow \mathrm{Comp}(E(v),v,R(E))$

  2.3 For all $\langle a,b,k\rangle\in E, k\le l$, execute the following two steps:

    2.3.1 $R(a,b,k)[k+1{:}\,l]\leftarrow \bigcup_{v\in V_l}\big[R(a,b,k)\cap \mathrm{Comp}\big(E(v),v,R(E)\big)\big]_b^v$

    2.3.2 $R(a,b,k)\leftarrow [R(a,b,k)]_b^D$

3 Repeat step 2 until no $R(u,v,l)$ in $R(E)=\{R(e)|e\in E\}$ will change any more.

4 **Return** $\mathrm{Comp}\big(E(D),D,R(E)\big)$.

Init, Change, Comp, $[ES]_u^v$, and $R(u,v,l)$ are all as defined in the paper. According to the paper's definition of $ES[i{:}\,j]$, $R(a,b,k)[k+1{:}\,l]$ in the above algorithm represents all edges of $R(a,b,k)$ from stage $k+1$ to stage l. The meaning of step 2.3.1 is that a part of $R(a,b,k)$ is replaced by $\bigcup_{v\in V_l}\big[R(a,b,k)\cap \mathrm{Comp}\big(E(v),v,R(E)\big)\big]_b^v$.

Problem 3 (10 points)

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Explain why algorithm obtained in (2) does not run in polynomial-
time.
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Theorem 3.1 The time complexity of the paper's Operator 1 $[ES]_u^v$ is, for a DTM, exponential in the number of nodes of the Hamiltonian cycle problem input to the HAMILTONIAN CYCLE VIA Z-H ALGORITHM.

Proof Suppose $ES \subseteq E$ and $u, v \in V$. We define $[ES]_u^v =$ $\{e | e \in ES, e$ is on a path $u - \cdots - v,$ and all the edges on $u - \cdots - v$ are contained in $ES\}$.

Let n equal the number of vertices we input to HAMILTONIAN CYCLE VIA Z-H ALGORITHM from my response to Problem 2, i.e. the graph for which we would like to determine whether a Hamiltonian cycle exists.

Observe the fanin to $v \leq n - 1$. The same is true for the fanout from u. Let the stage of u be k, and let the stage of v be l. There are $n - 1$ vertices in each stage between k and l. Each such vertex has fanout $\leq n - 2$ assuming a simple graph for the Hamiltonian cycle problem.

There are $(l - k - 1)(n - 1) + 1$ vertices that may have out-edges in $[ES]_u^v$. If $u = S$ and $v = D$, then $l - k = n$ and $[ES]_u^v$ has $(n - 1)^2 + 1$ vertices with out-edges in $[ES]_u^v$. Then the maximum number of edges in ES is $(n - 1)^2(n - 2) + (n - 1)$, which is $O(n^3)$.

For each $e \in ES$, we must determine whether e is on a path $u - \cdots - v$. This is easily accomplished in time linear in the length of said path, which itself is no longer than n. However, we have no oracle to provide us said path. Therefore, we must list all simple paths so we can verify an arbitrary edge is in fact on at least one path.

To build a path, we may start at vertex v at stage k and proceed against the direction of any of at most $n - 1$ in-edges to v, thereby arriving at some vertex at stage $k - 1$. We then repeatedly arrive at a vertex and proceed against the direction of one of its in-edges until we either arrive at u or fail to arrive at u. If we arrive at u, we did so in time linear in the length of the path, which is no longer than n.

Recall the branching factor of all but S and D is at most $n - 2$, and the branching factor of S is at most $n - 1$. Further recall the worst case time complexity of depth-first search is exponential in the depth of the graph. Here, the worst case of DFS is what we require to enumerate and store all separate paths if the branching factor is maximum. Because the depth of the graph may be as high as n, listing all paths is exponential in the number of nodes of the original Hamiltonian circuit problem. ■