

Dynamics on Dilation Tori

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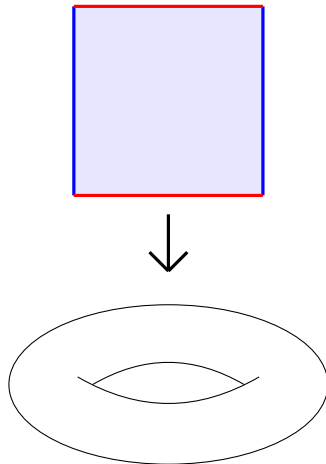
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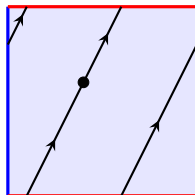
Motivation: The Torus

- Simplest translation surface is a flat torus.
- A square with sides identified by translations.
- What can we study?
 - Topology
 - Geometry
 - Dynamics

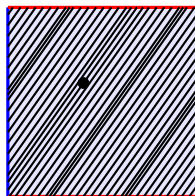


Dynamics on a Torus

- Natural straight-line flow.
- Flows on a torus have fascinating dynamics:
 - Rational slopes:
Periodic flows
 - Irrational slopes:
Minimal (dense) flows



$$m = 2$$

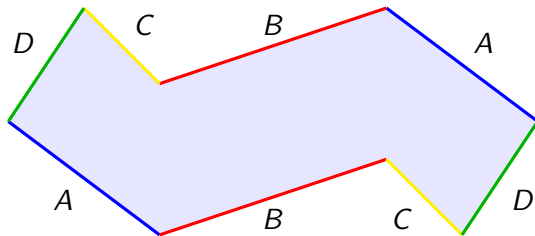


$$m = \sqrt{2}$$

Translation Surfaces: Construction

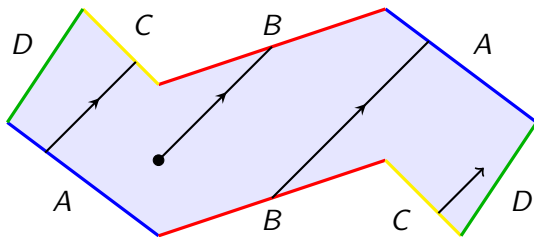
One way to define them:

- Start with a polygon.
- Identify pairs of sides by translations.
- Result: A surface with no curvature, finitely many cone points.



Translation Surfaces: Dynamics

No curvature: There is a natural straight-line flow.



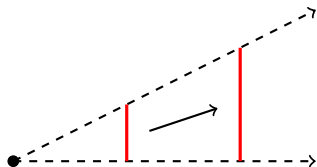
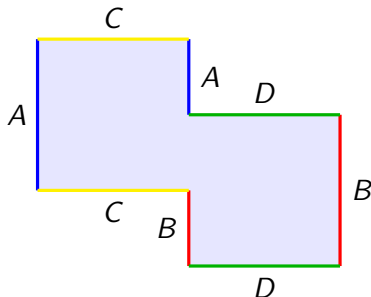
Theorem (S. Kerchhoff, H. Masur, J. Smillie)

For any translation surface S the straight-line flow in almost any direction is minimal.

Dilation Surfaces: Construction

Similarly to translation surfaces:

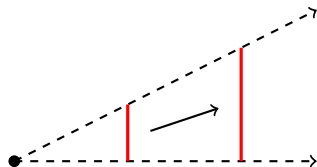
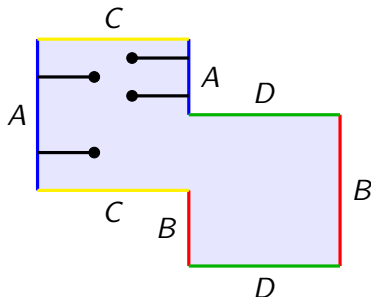
- Start with a polygon.
- Identify pairs of sides by translations or dilations.
- Result: A surface with no local curvature, but nontrivial holonomy (distances change when you go around loops).



Dilation Surfaces: Construction

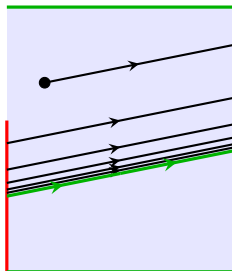
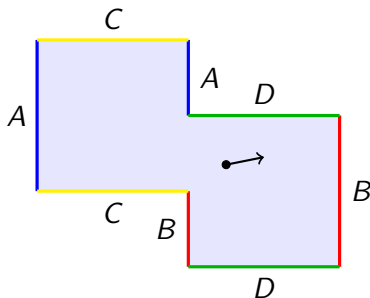
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- Start with a polygon.
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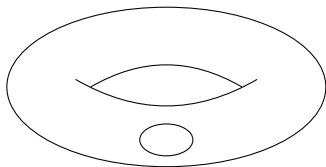
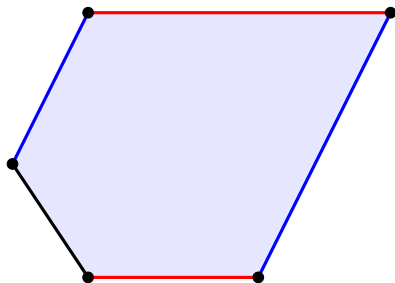
Dilation Surfaces: Dynamics

- There is still a natural straight-line flow.
- But now the dynamics are different!
- Here: The flow converges to a periodic orbit.



Dilation Tori with a Single Boundary Component

- A simple class of dilation surfaces.
- One piece of boundary, one cone point.
- Folds together into a torus with a hole.

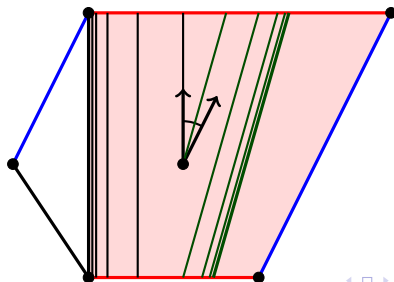


Dilation Tori: Dynamics

Consider the straight line flow in directions which point away from the boundary. We find the following result:

Theorem (H.-Wang)

Let S be a dilation torus with single boundary component. The straight-line flow in almost any direction converges to a periodic orbit in S . The remaining directions form a Cantor set.

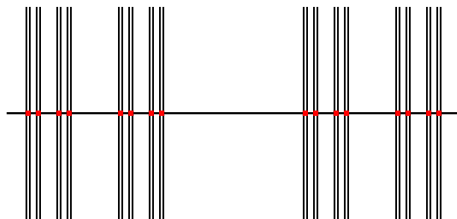
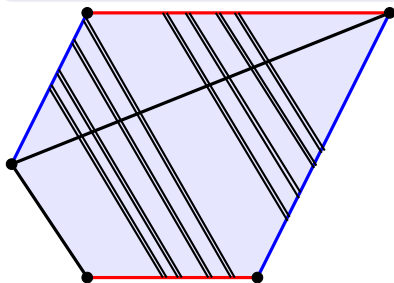


Dilation Tori: Cantor Sets

What happens to the remaining Cantor set of directions?

Theorem (H.-Wang)

Let C be the Cantor set of directions in S whose flow does not converge to a periodic orbit. For all but countably many directions in C , the flow accumulates to a set whose cross section is a measure zero Cantor set.



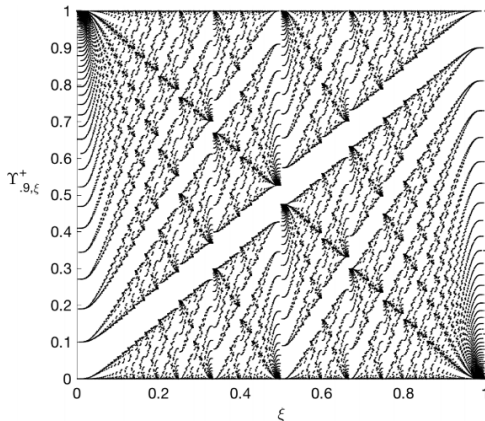
Next Steps

- Dilation surfaces have rich dynamical behavior.
- Torus represents translation surface dynamics.
- Do dilation tori represent dilation surface dynamics?

Conjecture (S. Ghazouani)

For any dilation surface S which is not a translation surface, the straight-line flow in almost any direction converges to some periodic orbit.

Dilation Tori: Limit Sets



ξ parameterizes direction. Plot by Bowman and Sanderson.

Questions

Thank you for listening.

Any questions or comments are welcome.