UC Santa Cruz
Physics 115
Spring 2018

Exercise Set #2
due online Wednesday, April 18 at 10:10 AM

Please refer to the instructions from Exercise Set #1 about how to code and submit your work.

1. **Numerical differentiation**
   Consider the function $f(x) = xe^x$.
   (a) Using the forward-difference and central-difference algorithms, calculate approximations to $f''(2)$, with $h = 0.5, 0.45, \ldots, 0.05$.
   (b) Verify the order of the errors by plotting log error vs. log h. You can find the exponent by fitting the log-log plot. (Why?)

2. **Calculations with the relativistic Breit-Wigner function**
   The relativistic Breit-Wigner function is a probability function used to model resonances (unstable particles) in particle physics. It provides the probability that a particle of nominal mass M and resonance width Γ may be measured with a different mass energy E. The definition of the Breit-Wigner probability function is

$$f(E) = \frac{k}{(E^2 - M^2)^2 + M^2\Gamma^2},$$

   where the scaling factor k is

$$k = \frac{2\sqrt{2}M\Gamma\gamma}{\pi\sqrt{M^2+\gamma}},$$

   and $\gamma = \sqrt{M^2(M^2+\Gamma^2)}$.
   (a) Plot the Breit-Wigner distribution for a hypothetical particle having $M = 90\,\mathrm{GeV}$ and $\Gamma = 10\,\mathrm{GeV}$. A reasonable range of energies might be $[0, 180]$ GeV.
   (b) Write a function that integrates an arbitrary list of x and y data points using the trapezoidal rule. Use your function to integrate the Breit-Wigner distribution. To check your answer, integrate the same data using a pre-existing integration package/function (for instance, numpy's "`trapz`" function).
   (c) Write a similar function utilizing Simpson's rule. Note that this technique requires an odd number of data points! Should your function be given an even number of points, print an error message to the user. If your distribution from part (a) contains an even number of points, simply remove the last point. Check for consistency with your previous answer – they should be fairly similar.

(d) In the experiment, we would like to keep all of the produced particles within $\pm 3\Gamma$ of the B-W peak to investigate the particle's properties. What fraction of the particles will we keep?

In fact, the Z boson has $M = 91.2\,\mathrm{GeV}$ and $\Gamma = 2.5\,\mathrm{GeV}$, so this is a very realistic example.

3. **Romberg integration**
Use Romberg integration to calculate

$$\int_0^{\pi/2} \frac{d\theta}{1+\cos\theta}$$

to 8 decimal places of accuracy, as determined from a relative error check. Print out the successive Romberg estimates as a way of showing your work. (Producing a structured printout may be the hard part!)

4. **Diffraction at a Knife's Edge**
The study of physical optics introduced the idea of diffraction of light around a straightedge. The intensity of light varies as we move away from the edge, and we see maximum and minimum intensity lines. The intensity is given by

$$I = \frac{I_0}{2}\left\{[C(v)+0.5]^2 + [S(v)+0.5]^2\right\},$$

where I_0 is the intensity of the incident light, v is proportional to the distance from the straightedge, and $C(v)$ and $S(v)$ are the Fresnel integrals

$$C(v) = \int_0^v \cos(\pi w^2/2)dw$$

and

$$S(v) = \int_0^v \sin(\pi w^2/2)dw.$$

(a) Evaluate I/I_0 as a function of v.

(b) Plot the results.

5. **The Simple Pendulum**
Consider a simple pendulum of length ℓ. It oscillates with a maximum angle from the vertical of θ_m. We know from introductory physics that if θ_m is small the pendulum undergoes simple harmonic motion with period $T_0 = 2\pi\sqrt{\ell/g}$ where g is the acceleration due to gravity. In the rest of this exercise, we investigate how the period changes when the amplitude is no longer small.

(a) Use conservation of energy, with

$$E = \frac{1}{2}m\left(\ell\dot{\theta}\right)^2 + mg\ell(1-\cos\theta),$$

to show that the period of oscillation $T(\theta_m)$ is related to the period for small oscillations T_0 through the following integral:

$$\frac{T(\theta_m)}{T_0} = \frac{\sqrt{2}}{\pi}\int_0^{\theta_m}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_m}}.$$

(b) The integral in the previous part contains a divergence when $\theta \to \theta_m$. Show that we can rewrite the integral as

$$\frac{T(\theta_m)}{T_0} = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\psi}{\sqrt{1-\sin^2(\theta_m/2)\sin^2(\psi)}}$$

(The key substitutions are $\cos\theta = 1-2\sin^2(\theta/2)$ and $\sin\psi = \sin(\theta/2)/\sin(\theta_m/2)$.) We have removed the $\theta \to \theta_m$ singularity, but we still have a singularity at $\theta_m = \pi$.

(c) The final integral in the previous part is called a "complete elliptic integral of the first kind." Evaluate $T(\theta_m)/T_0$ numerically for $\theta_m = 0.1, 0.2, \pi/4, \pi/2$, and $3\pi/4$.

(d) For what values of the amplitude does the period vary more than 1% from the simple $T_0 = 2\pi\sqrt{\ell/g}$?