EC504 — Homework 1

1. (a)

3.1-4 Is $2^{n+1} = O(2^n)$?

Answer: True. Note that $2^{n+1} = 2 \times 2^n$. We can choose $c \geq 2$ and $n_0 = 0$, such that $0 \leq 2^{n+1} \leq c \times 2^n$ for all $n > n_0$. By definition, $2^{n+1} = O(2^n)$.

Is $2^{2n} = O(2^n)$?

False. Note that $2^{2n} = 2^n \times 2^n = 4^n$. We can find that any c and n_0, such that $0 \leq 2^n = 4^n \leq c \times 2^n$ for all $n \geq n_0$.

3.2-7

For $i = 0$, $\dfrac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \dfrac{1-1}{\sqrt{5}} = 0 = F_0$

For $i = 1$, $\dfrac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \dfrac{1+\sqrt{5} - (1-\sqrt{5})}{2\sqrt{5}}$

$= 1$

$= F_1$

Assume

$$F_{i-1} = \frac{(\phi^{i-1} - \hat{\phi}^{i-1})}{\sqrt{5}}$$ and

$$F_{i-2} = \frac{(\phi^{i-2} - \hat{\phi}^{i-2})}{\sqrt{5}}$$

$$F_i = F_{i-1} + F_{i-2}$$

$$= \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} + \frac{\phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}}$$

$$= \frac{\phi^{i-2}(\phi+1) - \hat{\phi}^{i-2}(\hat{\phi}+1)}{\sqrt{5}}$$

$$= \frac{\phi^{i-2}(\phi+1) - \hat{\phi}^{i-2}(\hat{\phi}+1)}{\sqrt{5}}$$

$$= \frac{\phi^{i-2}\phi^2 - \hat{\phi}^{i-2}\hat{\phi}^2}{\sqrt{5}}$$

$$= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

A	B	O	o	Ω	ω	Θ
$\lg^k n$	n^ϵ	yes	yes	no	no	no
n^k	c^n	yes	yes	no	no	no
$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
2^n	$2^{\frac{n}{2}}$	no	no	yes	yes	no
$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

4.3 - 9

First,

$$T(n) = 3T(\sqrt{n}) + \lg n \quad \text{Let } m = \lg n$$

$$T(2^m) = 3T(2^{\frac{m}{2}}) + m$$

$$S(m) = 3S(m/2) + m$$

Now we guess $S(m) \leq cm^{\lg 3} + dm$

$$S(m) \leq 3(c(\tfrac{m}{2})^{\lg 3} + d(\tfrac{m}{2})) + m$$

$$\leq cm^{\lg 3} + (\tfrac{3}{2}d + 1)m \qquad (d \leq -2)$$

$$\leq cm^{\lg 3} + dm$$

Then we guess $S(m) \geq cm^{\lg 3} + dm$

$$S(m) \geq 3\left(c\left(\frac{m}{2}\right)^{\lg 3} + d\left(\frac{m}{2}\right)\right) + m$$
$$\leq cm^{\lg 3} + \left(\frac{3}{2}d + 1\right)m \quad (d \leq -2)$$
$$\leq cm^{\lg 3} + dm$$

Then we guess $S(m) \geq cm^{\lg 3} + dm$

$$S(m) \geq 3\left(c\left(\frac{m}{2}\right)^{\lg 3} + d\left(\frac{m}{2}\right)\right) + m$$
$$\geq cm^{\lg 3} + \left(\frac{3}{2}d + 1\right)m \quad (d \geq -2)$$
$$\geq cm^{\lg 3} + dm$$

Thus,

$$S(m) = \Theta(m^{\lg 3})$$
$$T(n) = \Theta(\lg^{\lg 3} n)$$

(b)

The rule for ordering is $O(n^n) > O(n!) > O(c^n) > O(n^c) > O(\log n) > O(1)$

$n^2 + 3n\log(n) + 5$ is of $O(n^2)$

$n^2 + n^{-2}$ is of $O(n^2)$

$n^{n^2} + n!$ is of $O(n^{n^2})$

$n^{\frac{1}{n}}$ is of $O(n^{\frac{1}{n}})$

$\ln n$ is of $O(\ln n)$

$\ln(\ln n)$ is of $O(\ln(\ln n))$

$3^{\ln n}$ is of $O(3^{\ln n})$

2^n is of $O(2^n)$

$(1+n)^n$ is of $O(n^n)$

$n^{1+\cos n}$ is of $O(n \cdot n^{\cos n})$

1 is of $O(1)$

n^2+3n+5 is of $O(n^2)$

$n!$ is of $O(n!)$

$\Rightarrow (1+n)^n > n^{n^2}+n > n^{n^2-1} > n^2+3n\log n+5 > n^2+3n+5 >$

$n! > n^2+\frac{1}{n^2} > n^{1+\cos n} > \sum_{k=1}^{\log n} \frac{n^2}{2^k} > 3^{\ln n} > \sum_{k=1}^{n} \frac{1}{k} >$

$\ln(n) > n^{\frac{1}{n}} > \ln(\ln(n)) > (1-\frac{1}{n})^n > \prod_{k=1}^{n}(1-\frac{1}{k^2})$

2.

(a) For $T(n) = c_1 n + c_2 n \log_2(n)$

when $n=1$, $T(1) = c_1$

when $n=2$, $T(2) = 2c_1 + 2c_2$

when $n=4$, $T(4) = 4c_1 + 8c_2$

For $T(n) = 2T(\frac{n}{2}) + n$

$2c_1 + 2c_2 = 2c_1 + 2$, $c_2 = 1$

(b)

Given $T(n) = c_1 n^r + c_2 n^k$

$$T\left(\frac{n}{b}\right) = c_1 \left(\frac{n}{b}\right)^r + c_2 \left(\frac{n}{b}\right)^k$$

$$T\left(\frac{n}{b}\right) = \frac{c_1}{b^r} n^r + \frac{c_2}{b^k} n^k$$

$$b^r T\left(\frac{n}{b}\right) = \frac{b^r c_1}{b^r} n^r + \frac{b^r c_2}{b^k} n^k$$

$$a T\left(\frac{n}{b}\right) = c_1 n^r + c_2 b^{r-k} n^k, \quad \because b^r = a$$

$$a T\left(\frac{n}{b}\right) + n^k = c_1 n^r + n^k + c_2 b^{r-k} n^k$$

$$T(n) = c_1 n^r + n^k (1 + c_2 b^{r-k})$$

$\because T(n) = c_1 n^r + c_2 n^k$

$$c_1 n^r + c_2 n^k = c_1 n^r + n^k (1 + c_2 b^{r-k})$$

$$(c_2 - 1 - c_2 b^{r-k}) = 0, \quad n^k \neq 0$$

$$c_2 (1 - b^{r-k}) = 1$$

$$c_2 = \frac{1}{1 - b^{r-k}}$$

if $r \neq k$, $T(n) = c_1 n^r + \frac{1}{1 - b^{r-k}} n^k$

$c_1 \in R$, $T(n) = b^r T(\frac{n}{b}) + n^k$

if $r = k$, c_2 doesn't exist.

$$T(n) = c_1 n^r + c_2 n^k$$
$$= b^r T(\frac{n}{b}) + n^k$$

Let $k = r$ then

$$T(n) = b^r T(\frac{n}{b}) + n^r$$

Let $T(n) = c_1 n^r + c_2 n^r \log_2(n)$

$$T(n) = b^r T(\frac{n}{b}) + n^r$$

$$T(\frac{n}{b}) = c_1 (\frac{n}{b})^r + c_2 (\frac{n}{b})^r \log_2(\frac{n}{b})$$
$$= \frac{c_1}{b^r} n^r + \frac{c_2}{b^r} n^r \log_2(\frac{n}{b})$$

$$b^r T(\frac{n}{b}) = c_1 n^r + c_2 n^r \log_2(\frac{n}{b})$$
$$= c_1 n^r + c_2 n^r (\log_2 n - \log_2 b)$$
$$= c_1 n^r + c_2 n^r \log_2 n - c_2 n^r \log_2 b$$
$$= T(n) - c_2 n^k \log_2 b$$

$$\Rightarrow \quad b^r T\left(\frac{n}{b}\right) + c_2 n^r \log_2 b = T(n)$$

$$\Rightarrow \quad b^r T\left(\frac{a}{b}\right) + c_2 n^r \log_2 b = b^r T\left(\frac{n}{b}\right) + n^r$$

$$\Rightarrow \quad n^r (c_2 \log_2 b - 1) = 0$$

$$\Rightarrow \quad c_2 \log_2 b = 1, \quad \because n^r \neq 0$$

$$\underline{c_2 = \frac{1}{\log_2 b}}$$

for $r = k$, $T(n) = c_1 n^r + \dfrac{1}{\log_2 b} n^r \log_2(n)$, for $c_1 \in R$

for $r \neq k$, $T(n) = c_1 n^r + \dfrac{1}{1 - b^{r-k}} n^k$, $c_1 \in R$