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Math128A: Numerical Analysis
Sample Midterm

This is a closed book exam. You are allowed to cite any results, up to Section 4.5 but excluding those in the exercises, from the textbook. Results from anywhere else will be treated the same as your answers, which need to be justified. Completely correct answers given without justification will receive little credit. Partial solutions will get partial credit.

Problem	Maximum Score	Your Score
1	18	
2	16	
3	16	
4	16	
5	16	
6	18	
Total	100	

1. Determine the absolute error, the relative error, and the number of significant digits in the approximation

$$\pi \approx 3.14$$

SOLUTION:

(a) the absolute error is

$$\pi - 3.14 \approx 0.00159\cdots$$

(b) the relative error is

$$\frac{|\pi - 3.14|}{\pi} \approx \frac{0.00159\cdots}{\pi}$$

(c) the relative error is

$$5.0 \times 10^{-4} < \frac{|\pi - 3.14|}{\pi} \approx 5.07 \times 10^{-4}$$

thus the number of significant digits is 3.

2. Let $x_0 < x_1 < x_2$. Find a second degree polynomial $P(x)$ such that

$$P(x_0) = f_0, \quad P(x_1) = f_1, \quad \text{and} \quad P'(x_2) = f_2'.$$

SOLUTION: Write $P(x)$ as

$$P(x) = \alpha + \beta(x - x_0) + \gamma(x - x_0)(x - x_1)$$

The condition

$$P(x_0) = f_0 \quad \text{leads to} \quad \alpha = f_0.$$

The condition

$$P(x_1) = f_1 \quad \text{leads to} \quad \beta = \frac{f_1 - f_0}{x_1 - x_0}.$$

Finally, since

$$P'(x) = \beta + \gamma(2\,x - x_0 - x_1)$$

The condition

$$P'(x_2) = f_2' \quad \text{leads to} \quad \gamma = \frac{f_2' - \beta}{2\,x_2 - x_0 - x_1} \quad \blacksquare$$

3. Show that the cubic equation

$$2x^3 - 6x + 1 = 0 \qquad (1)$$

has a real root in the interval $[0, \quad 1/2]$. Perform one step of Bisection method with this interval.

SOLUTION: Let

$$f(x) = 2x^3 - 6x + 1$$

Since $f(0) = 1 > 0$ and $f(1/2) = -1.75 < 0$, it follows from intermediate value theorem that there must be a $c \in [0, 1/2]$ so that $f(c) = 0$. To perform one step of Bisection method with this interval, let

$$p = \frac{1}{2}(0 + 1/2) = \frac{1}{4}$$

Since

$$f(p) = \frac{2}{4^3} - \frac{6}{4} + 1 < 0,$$

it follows that there is a root in $[0, \quad 1/4]$.

4. Reformulate the above equation (1) as

$$x = \frac{2x^3 + 1}{6}.$$

Define the fixed point iteration (FPI) based on this equation, and show that FPI convergences for any initial guess in $[0, \quad 1/2]$.

SOLUTION: The fixed point iteration is

$$x_{k+1} = \frac{2x_k^3 + 1}{6}$$

for $k = 0, 1, \cdots$, with a given x_0. Define

$$g(x) = \frac{2x^3 + 1}{6}$$

For FPI to converge for any initial guess in $[0, \quad 1/2]$, we need to prove

(a) $g(x) \in [0, \quad 1/2]$ for any $x \in [0, \quad 1/2]$
(b) there exists a $0 < \kappa < 1$ so that $|g'(x)| \le \kappa$ for any $x \in [0, \quad 1/2]$

Indeed, $g'(x) = x^2 \ge 0$ for any $x \in [0, \quad 1/2]$,

(a) therefore $g(x)$ monotonically increases in $[0, \quad 1/2]$, with

$$g(0) = \frac{1}{6} \in [0, \quad 1/2], \quad g(1/2) < \frac{1}{2} \in [0, \quad 1/2]$$

and hence $g(x) \in [0, \quad 1/2]$ for any $x \in [0, \quad 1/2]$
(b) Let $\kappa = \frac{1}{4} < 1$, then for any $x \in [0, \quad 1/2]$,

$$|g'(x)| = x^2 \le \kappa \quad \blacksquare$$

5. Construct the natural cubic spline that approximates

$$f(x) = \frac{\sin x}{x}$$

at the nodes $-1, 0, 1$. Note that at $x = 0$, we define $f(0) = 1$.

SOLUTION: Parameterize S as

$$S(x) = \begin{cases} a_0 + b_0\,x + c_0\,x^2 + d_0\,x^3, & \text{if } x \in [-1,\, 0], \\ a_1 + b_1\,x + c_1\,x^2 + d_1\,x^3, & \text{if } x \in [0,\, 1]. \end{cases}$$

The condition that $S(x) \in C^2[-1,\, 1]$ implies that

$$S(x)\,|_{x=0^-} = S(x)\,|_{x=0^+}\,,\ S(x)'\,|_{x=0^-} = S(x)'\,|_{x=0^+}\,,\ S(x)''\,|_{x=0^-} = S(x)''\,|_{x=0^+}$$

which leads to

$$a_0 = a_1 = f(0) = 1,\ b_0 = b_1,\ c_0 = c_1$$

Additionally, the natural cubic spline condition implies that

$$\begin{aligned} S(x)''\,|_{x=-1^+} &= 2\,c_0 - 6\,d_0 = 0 \\ S(x)''\,|_{x=1^-} &= 2\,c_0 + 6\,d_1 = 0 \end{aligned}$$

which implies

$$d_0 = \frac{c_0}{3}, \quad d_1 = -\frac{c_0}{3}$$

Putting it all together,

$$S(x) = \begin{cases} 1 + b_0\,x + c_0\,x^2 + \frac{c_0}{3}\,x^3, & \text{if } x \in [-1,\, 0], \\ 1 + b_0\,x + c_0\,x^2 - \frac{c_0}{3}\,x^3, & \text{if } x \in [0,\, 1]. \end{cases}$$

Finally, the conditions

$$S(-1) = S(1) = \sin 1$$

leads to conditions

$$\begin{aligned} 1 - b_0 + c_0 - \frac{c_0}{3} &= \sin 1 \\ 1 + b_0 + c_0 - \frac{c_0}{3} &= \sin 1 \end{aligned}$$

which has solution

$$b_0 = 0, \quad c_0 = -\frac{3}{2}\,(1 - \sin 1)$$

and

$$S(x) = 1 - \frac{3\,x^2}{2}\,(1 - \sin 1)\left(1 - \frac{|x|}{3}\right)$$

6. Suppose that

$$L = \lim_{h\to 0} f(h) \quad \text{and} \quad L - f(h) = c_6 h^6 + c_9 h^9 + \cdots .$$

Find a combination of $f(h)$ and $f(h/2)$ with an $O(h^9)$ error estimate of L.

SOLUTION: The expansion on h should work for all tiny h, therefore

$$\begin{aligned} L - f(h) &= c_6 h^6 + c_9 h^9 + \cdots \\ L - f(h/2) &= c_6 (h/2)^6 + c_9 (h/2)^9 + \cdots \end{aligned}$$

Multiply the second equation by 2^6 and then take a difference with the first to eliminate the h^6 terms,

$$\left(2^6 - 1\right) L - \left(2^6\, f(h/2) - f(h)\right) = -\left(1 - 2^{-3}\right)\, c_9\, h^9 + \cdots$$

therefore,

$$L - \left(f(h/2) + \frac{f(h/2) - f(h)}{2^6 - 1}\right) = -\frac{1 - 2^{-3}}{2^6 - 1}\, c_9\, h^9 + \cdots$$

The left hand side gives a combination of $f(h)$ and $f(h/2)$ with an $O(h^9)$ error estimate of L.