Math 110, Spring 2023.

Homework 4, due February 18.

Prob 1. Let $V = \mathcal{P}_2(\mathbb{R})$, $W = \mathbb{R}$. Are the maps

$$T : f \mapsto f(2), \quad S : f \mapsto \int_0^1 f(x)dx$$

in $\mathcal{L}(V, W)$? Are they linearly independent?

Prob 2. Suppose V is a nonzero finite-dimensional vector space and W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Prob 3. Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{range} S \subset \operatorname{null} T.$$

Prove that $(ST)^2 = 0$.

Prob 4. Suppose $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ is defined by the formula $(Tf)(x) = 2xf''(x) - f'(x)$. Check that $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ and find a basis for the null space and a basis for the range of T.

Prob 5. Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.