Things that we have covered since midterm 1

① Rouché theorem, open mapping, maximum principle

② Fourier transformation. Relationship between analyticity of $f(z)$, and properties of $\hat{f}(\xi)$

- $f(z)$ is defined for $\{|\mathrm{Im}(z)| < a\}$
  $\Leftrightarrow$ $\hat{f}(\xi)$ decays like $e^{-2\pi a|\xi|}$

(Paley-Wiener)
- $f(z)$ is entire and has growth bounded by
  $|f(z)| < e^{2\pi M|z|}$
  $\hat{f}(\xi)$ vanishes for $\xi \in \mathbb{R}$ and $|\xi| > M$.

③ Entire function and infinite product.

- Order of growth and distribution of zero.

- Jensen's formula

$$\sum \log\left|\frac{a_n}{R}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

a_n roots of $f(z)$ in $D_R(0)$

- If order of growth $\leq \rho$, then
  $\#\{\text{roots of } f(z) \text{ in } D_R(0)\} \leq C\cdot R^{\rho}$

- <u>infinite product</u>:
  - Weierstrass Formula: construct function with prescribed roots.
  - Hadamard Factorization: for functions of finite order growth.

(did not prove in class)

④ Γ- function

- integral presentation, analytic continuation.
- relation with sine function.

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Sample Midterm 2 Questions:

(1) if $f_0(z)$ is a polynomial with roots $z=1, z=2$

and $f_1(z)$ ———————— with roots $z=3, z=4$.

and if we define $f_t(z) = (1-t)\cdot f_0(z) + t\, f_1(z)$,

then $f_t(z)$ will have two roots, moving from

$\{1,2\}$ continuously to $\{3,4\}$, true or false?

False. $f_0(z) = (z-1)(z-2)$, $f_1(z) = (z-3)(z-4)$, $\Rightarrow f_{\frac{1}{2}}(z) = az+b$

only 1 root.

(2). If $f(z) = \dfrac{(z-a_1)\cdots(z-a_n)}{(z-b_1)\cdots(z-b_n)}$, with a_i, b_j

all distinct ($a_i \neq b_j \ \forall i,j$), then f defines a

holomorphic function : $\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ (i.e. a

rational function), what is $f(\infty) = ?$ 1

what is $f^{-1}(\infty) = ?$ $\{b_1, \cdots, b_n\}$

(3) Let $f(z) = z^2$, $\Omega = \{\operatorname{Im} z > 0\}$.

is $f(\Omega)$ an open set? What is $f(\Omega)$?

yes, open $= \mathbb{C} \setminus \mathbb{R}_{\geq 0}$

$a>0$

(4). Let $f(z) = \dfrac{1}{z^2+a^2}$, what is $\hat{f}(\xi)$?

$\dfrac{\pi}{a} e^{-2\pi a|\xi|}$

(5) Let $f(z) = e^{-z^4}$, what can we say

about $\hat{f}(\xi)$? Does it exist? Yes

Is $\hat{f}(\xi)$ a hol'c function in ξ ? Yes.

- For $\xi \in \mathbb{R}$

$$\hat{f}(\xi) := \int_{-\infty}^{+\infty} e^{-x^4} \cdot e^{-i2\pi x \cdot \xi} \, dx$$

$$|\hat{f}(\xi)| \leq \int_{-\infty}^{+\infty} e^{-x^4} \, dx < \infty$$

so $\hat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$.

- For any $\xi \in \mathbb{C}$, say $\xi = a+ib$ $\quad a, b \in \mathbb{R}$, then

$$|\hat{f}(\xi)| \leq \int_{-\infty}^{+\infty} e^{-x^4} \cdot e^{-2\pi \, \mathrm{Re}\,(i \cdot (a+bi) \cdot x)} \, dx$$

$$= \int_{-\infty}^{+\infty} e^{-x^4} \, e^{2\pi b x} \, dx < \infty$$

So $\hat{f}(\xi)$ exists for all $\xi \in \mathbb{C}$.

- Finally, to show $\hat{f}(\xi)$ is a hol'c function, we need to show the following integral vanishes, $\forall$ T triangle in $\mathbb{C}$

$$\int_T \hat{f}(\xi) \, d\xi = \int_T \int_{-\infty}^{+\infty} e^{-x^4} \, e^{-2\pi i \cdot x \cdot \xi} \, dx \, d\xi .$$

since $\int_T \int_{-\infty}^{+\infty} |e^{-x^4} \, e^{-2\pi i x \xi}| \, dx \, d\xi$

$$\leq \int_T \int_{-\infty}^{+\infty} e^{-x^4} \cdot e^{2\pi |x| \cdot M} \, dx \, d\xi$$

where $M = \sup_{\xi \in T} |\xi|$.

$\leq \infty$

Hence, we may apply Fubini's thm and switch the order of integral

$$\int_{-\infty}^{+\infty} e^{-x^4} \cdot \int_T e^{-2\pi i x \cdot \xi} \, d\xi \, dx = 0$$

By Moirera thm, $\hat{f}(\xi)$ is hol'c. in $\mathbb{C}$.

(6) For $|q| < 1$, consider the infinite product

$$\varphi_q(z) = \prod_{n=1}^{\infty} (1 - q^n z)$$

- is the product convergent ?
- what's the zero and poles for $\varphi_q(z)$?
- what's the order of growth for $\varphi_q(z)$?

- If $\sum |a_n| < \infty$, then $\prod_{n=1}^{\infty} (1 + a_n) < \infty$. (Prop from Stein)

Since $\sum_{n=1}^{\infty} |q^n z| = |z| \sum_{n=1}^{\infty} |q|^n = |qz| \frac{1}{1-|q|} < \infty$

we have $\prod (1 - q^n z)$ convergent

- $\varphi_q(z)$ is convergent $\forall z \in \mathbb{C}$, thus there is no pole.

If $z = \frac{1}{q^n}$ for some $n = 1, 2, \cdots$, then $\varphi_q(z) = 0$.

For all other z, $\varphi_q(z) \neq 0$.

This one is too hard for the exam. ignore it

∘ The order of growth of $\varphi_q(z)$ is 0. To show this, we show that

$$|\varphi_q(z)| \le A e^{P(\log|z|)} \qquad \forall z \in \mathbb{C},$$

where $P(u)$ is a degree 2 polynomial.

Note that

$$\varphi_q\left(\frac{z}{q}\right) = \prod_{n=1}^{\infty}\left(1 - q^n \frac{z}{q}\right) = (1-z)\,\varphi_q(z)$$

Let $\Omega_1 = \{1 \le |z| \le q^{-1}\}$, $\Omega_2 = \{q^{-1} \le |z| \le q^{-2}\} \cdots$,

Let $M_n = \sup_{z \in \Omega_n} |\varphi_q(z)|$. Then

$$M_{n+1} = \sup_{z \in \Omega_{n+1}} |\varphi_q(z)| = \sup_{z \in \Omega_n} |\varphi_q(q^{-1} \cdot z)|$$

$$= \sup_{z \in \Omega_n} |(1-z)\varphi_q(z)| \le M_n \cdot \sup_{z \in \Omega_n} |1-z|$$

$$\le M_n \cdot (1 + q^{-n}).$$

$$\therefore \quad M_n \le M_1 \prod_{k=1}^{n-1}(1 + q^{-k}) \le M_1 \prod_{k=1}^{n-1}\left[2 \cdot (q^{-1})^k\right]$$

$$\le M_1 \cdot 2^n \cdot (q^{-1})^{n^2} = M_1\, e^{\log(q^{-1}) \cdot n^2 + \log 2 \cdot n}$$

Since $n \leq (\log|z| / \log|q^{-1}| + 1)$, we have

$$|\varphi_q(z)| \leq A \cdot e^{P(\log|z|)} \quad \forall z \in \mathbb{C}.$$

where P is some degree 2 polynomial.

Note that, $\forall \varepsilon > 0$, there exist $N_\varepsilon > 0$. s.t. $\forall u > N_\varepsilon$

$$P(u) < e^{\varepsilon u}$$

Thus, for $|z|$ large enough (i.e. $|z| > e^{N_\varepsilon}$), we have.

$$e^{P(\log|z|)} \leq e^{|z|^\varepsilon}$$

Hence we have order of growth of $\varphi_q(z) \leq \varepsilon \quad \forall \varepsilon > 0$.

Thus, order of growth of $\varphi_q(z) = 0$.