Game Theory Homework #2, 02/02/2021

All the problems are worth 4 points each and will be graded on a 0/1/2/3/4 scale. Due on Wednesday 02/10/2021 before 11:59 pm to be uploaded via bCourses.

1. Let G be a progressively bounded impartial combinatorial game under normal play. Its Sprague-Grundy function g is defined recursively as follows:
$$g(x) = \text{mex}\{g(y) : x \to y \text{ is a legal move.}\}$$
where mex of a set of numbers, is the minimum excluded value function (smallest non-negative number not in the set). Some examples are as follows:
$$\text{mex}\{0, 1, 3, 4\} = 2, \ \text{mex}\{2, 5, 7\} = 0.$$
Note that the Sprague-Grundy value of any terminal position is $\text{mex}(\emptyset) = 0$.
   - Prove that $x \in P$ iff $g(x) = 0$.
   - Now consider the sum G of two progressively bounded impartial combinatorial game under normal play, G_1 and G_2. Let g, g_1 and g_2 be the respective Sprague-Grundy functions of G, G_1, G_2 respectively. Show that $g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2)$.

2. Find the Sprague-Grundy function for the Nim game $(n_1, n_2, \ldots n_k)$.
Consider a game of Nim with four piles, of sizes $9, 10, 11, 12$.
   - Is this position a win for the next player or the previous player (assuming optimal play)? Describe all the winning first moves in case of the former.
   - Consider the same initial position, but suppose that each player is allowed to remove at most 9 chips in a single move (Other rules of Nim remain in force.) Is this an N or P position?

3. Consider the following two-person zero-sum game. Both players simultaneously call out one of the numbers 2, 3. Player I wins if the sum of the numbers called is odd and player II wins if their sum is even. The loser pays the winner the product of the two numbers called (in dollars). Find the payoff matrix, the value of the game, and an optimal strategy for each player.

4. Show that for any 2×2 payoff matrix there exists either a pair of optimal strategies that are both pure or are both fully mixed. Show that this can fail for a 3×3 matrix.

5. Consider an $n \times n$ payoff matrix which is anti-symmetric i.e. $a_{i,j} = -a_{j,i}$.. What is the value of such a game? Given an optimal strategy x_* for Player I find an optimal strategy for Player II.