MATH 135: SET THEORY
FINAL EXAM

Name: ____ **SID:** ___________________________

Instructions

(1) **Very important:** Write your name and SID on **both sides** of each page. This exam will be scanned and then graded from a computer screen. It is very likely that individual pages may appear in the file out of order and will thus not be credited correctly if the name is missing.
(2) Write your answers in the space provided. If you find that you need additional space, you may use an additional blank piece of paper provided to you. Write your name, SID and problem number on each side of each additional page. On the page where the problem is set, write "Additional work to be found on attached sheets."
(3) When you are finished, stack your papers in order, and fasten them with the paper clip. Do **not** use a staple.
(4) Each of the regular questions is worth ten points. Each extra credit question at the end of the exam is worth an additional three points.

Date: 17 December 2014.

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1. For each of the following questions give a *formal definition* of the following terms. You may write your definition using other defined terms, but these must also be defined. Note that in particular you may need to define the terms "ordered pair", "relation", "empty set", "union", "function", *et cetera*.

a. If R and y are sets, then $y = \operatorname{ran} R$ if and only if ______________.

b. The set n is a natural number if and only if ______________.

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2. With the following questions you are asked to state precisely and formally some of the axioms of Zermelo-Frankel set theory which we have introduced. As with the definitions, you may use defined terms but those terms must be defined formally.

a. The Subset Axiom Schema

b. The Regularity Axiom

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3. Without using the Axiom of Choice, show that if I is finite and $H : I \to A$ is a function for which $(\forall i \in I) H(i) \neq \varnothing$, then $\prod_{i \in I} H(i) \neq \varnothing$.

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4. Let X be a finite set and $<$ a total order on X. **Prove** that $(X, <)$ is well-ordered.

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5. Let R be a symmetric, reflexive relation with $\mathrm{fld}(R) = X$. Define $g : \omega \to \mathcal{P}(X \times X)$ by $g(0) = \mathrm{id}_X$ and $g(n^+) = R \circ g(n)$ for $n \in \omega$. Define $E := \bigcup \mathrm{ran}(f)$. **Show** that

$$E = \bigcap \{F \subseteq X \times X \ : \ F \text{ is an equivalence relation on } X\} \ .$$

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6. Let R, S, and T be three sets. Show that $R \circ (S \circ T) = (R \circ S) \circ T$.

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7. Give a direct proof relative to the other axioms of set theory that the well-ordering principle implies that for every onto function $f : A \to B$ there is a right inverse $g : B \to A$.

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8. Let $(X, <)$ be a well-ordered set and $Y \subseteq X$ be a subset of X. Suppose that α is the $\in$-image of X and β is the $\in$-image of $(Y, <)$ (where we restrict $<$ to Y).

Show that

$$\beta \underline{\in} \alpha \ .$$

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Extra credit: Show that if $G : I \to A$ and $H : I \to A$ are two functions with the same domain for which $(\forall i \in I)\, G(i) \prec H(i)$, then

$$\bigcup_{i \in I} G(i) \prec \prod_{i \in I} H(i)$$

[This is sometimes written more suggestively as $\mu_i < \lambda_i$ for all $i \in I$ implies $\sum \mu_i < \prod \lambda_i$.]

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Extra credit: Show that for any countable ordinal α there is a set A of real numbers and an order isomorphism $f : (\alpha, \in_\alpha) \to (A, <)$ where $<$ is the restriction of the usual ordering on $\mathbb{R}$ to A.

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Extra credit: Find, with proof, all ordinals α for which $\alpha \cdot \omega = \omega \cdot \alpha$.

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