1. Homework 1 Solutions

Exercise 1.2.1

(a) Let's assume for contradiction $\sqrt{3}$ is a rational number. Then there exist integers p and q where p and q has no common divisor and $\frac{p}{q} = \sqrt{3}$. It follows that $p^2 = 3q^2$ and thus p^2 is divisible by 3. But if an integer p^2 is divisible by 3 then so is p (To be shown below). In other words $9k^2 = 3q^2$ or equivalently $3k^2 = q^2$ for some integer k. That is to say q is divisible by 3 as well, which leads to contradiction since we assumed p and q has no common divisor.

Lemma 1.1. *$\forall p \in \mathbb{Z}$, $3 \mid p^2 \implies 3 \mid p$*

Proof. We observe that any integer p can be written as $3k+2, 3k+1$ or $3k$ for some $k \in \mathbb{Z}$. If $p = 3k+2$, then $p^2 = 9k^2 + 12k + 4$ which is not divisible by 3. In other words, p cannot be of the form $3k+2$. A similar argument shows p cannot be of the form $3k+1$ either, therefore it has to be in form $3k = p$ for some integer k. Hence it must be divisible by 3. □

A similar argument will work for $\sqrt{6}$ as well, because if $p^2 = 6q^2$ then $6 \mid p^2$. This also implies p^2 is even and so is p. That is, there is an integer k such that $p = 2k$ and thus $4k^2 = 6q^2$ or equivalently $2k^2 = 3q^2$. Therefore q must be even as well and we get a contradiction.

(b) If we follow the same argument, we will obtain $p^2 = 4q^2$ which does not imply $4 \mid p$. Thus the argument fails.

Exercise 1.2.5 (De Morgan's Laws)

(a) Let x be an element in $(A \cap B)^c$. Then x is outside of $(A \cap B)$ meaning that either $x \notin A$ or $x \notin B$. Thus $x \in A^c$ or $x \in B^c$ which implies that $x \in A^c \cup B^c$ by definition of union. Since x is an arbitrary element in $(A \cap B)^c$, we conclude that $(A \cap B)^c \subset A^c \cup B^c$.

(b) Let x be an element of $A^c \cup B^c$. By definition, $x \in A^c$ or $x \in B^c$ or equivalently $x \notin A$ or $x \notin B^c$. This implies x is cannot be an element of $A \cap B$, that is, $x \in (A \cap B)^c$. As before, since x was an arbitrary choice, we obtain $(A \cap B)^c \supset A^c \cup B^c$. Combining with the result from part (a) we obtain $(A \cap B)^c = A^c \cup B^c$

(c) Similar to the previous parts, one can prove the statement by dividing into into two parts.
   1. For every $x \in (A \cup B)^c$, we have $x \notin (A \cup B) \implies x \notin A$ and $x \notin B \implies x \in A^c$ and $x \in B^c \implies x \in A^c \cap B^c$. Thus $(A \cup B)^c \subset A^c \cap B^c$.
   2. For every $x \in A^c \cap B^c$, we have $x \in A^c$ and $x \in B^c \implies x \notin A$ and $x \notin B \implies x \notin (A \cup B) \implies x \in (A \cup B)^c$. Thus $(A \cup B)^c \subset A^c \cap B^c$.

   Combining two inclusions above proves that $(A \cup B)^c = A^c \cap B^c$ as desired.

Exercise 1.2.7

(a) Let $f(x) = x^2$, $A = [0, 2]$ and $B = [1, 4]$. Clearly, $A \cap B = [1, 2]$,$A \cup B = [0, 4]$,$f(A) = [0, 4]$ and $f(B) = [1, 16]$. Then, $f(A \cup B) = [1, 4] = f(A) \cap f(B)$ and $f(A \cup B) = [0, 16] = f(A) \cup f(B)$.

(b) Consider the sets $A = [-1, 0]$ and $B = [0, 1]$. $f(A \cap B) = 0$, but $f(A) \cap f(B) = [0, 1]$.

(c) Let A, B any two subsets of $\mathbb{R}$. To show inclusion, we start with an arbitrary element y in $g(A \cap B)$. Then, there exists an $x \in A \cap B$ such that $g(x) = y$. Since $x \in A \cap B$, $x \in A$ and $x \in B$ which imply $y \in g(A)$ and $y \in g(B)$, respectively. Thus $y \in g(A) \cap g(B)$. As y was arbitrary in $g(A \cap B)$, we conclude that $g(A \cap B) \subseteq g(A) \cap g(B)$.

(d)

Lemma 1.3. *Let A, B any two subsets of $\mathbb{R}$ and any function $g : \mathbb{R} \mapsto \mathbb{R}$, $g(A \cup B) = g(A) \cup g(B)$.*

Proof. Since this is equality of two sets, as usual we need to show two inclusions

(1) For any $y \in g(A \cup B)$, there exists an x in A or B such that $y = g(x)$. It follows that $y = g(x) \in g(A)$ or $y = g(x) \in g(B)$, thus $y = g(A) \cup g(B)$ i.e $g(A \cup B) \subseteq g(A) \cup g(B)$.

(2) For any $y \in g(A) \cup g(B)$, $y \in g(A)$ or $y \in g(B)$. Then, there exists an x in A or B such that $y = g(x)$ meaning that $y \in g(A \cup B)$. Therefore, $g(A) \cup g(B) \subseteq g(A \cup B)$.

By combining two parts above, we prove the statement of lemma 1.3. □

Exercise 1.2.12

(a) We want to prove that $y_n > -6$ for all $n \in \mathbb{N}$ by induction:

(1) Induction basis: y_1 is indeed greater than -6

(2) Induction hypothesis: Assume for a natural number n, $y_n > -6$. Then $2y_n > -12$, and $(2y_n - 6)/3 > -18/3$ which implies $y_{n+1} > -6$. With this, we showed that if y_n is greater than -6, so is y_{n+1}.

So we conclude by induction that $y_n > -6$ for all $n \in \mathbb{N}$.

(b) Let us demonstrate that y_n is a decreasing sequence.

(1) Induction basis: $y_2 = 2$ and $y_1 = 6$, hence $y_2 - y_1 < 0$

(2) Induction hypothesis: Assume for a natural number n it is true that $y_n > y_{n+1}$. Then,

$$y_n > y_{n+1} \implies 2y_n > 2y_{n+1} \implies 2y_n - 6 > 2y_{n+1} - 6 \implies y_{n+1} = (2y_n - 6)/3 > (2y_{n+1} - 6)/3 = y_{n+2}$$

Thus, by induction

Since this holds for any n and induction basis holds, we obtain that $y_{n+1} - y_n < 0$ i.e $y_{n+1} < y_n$ for all $n \in \mathbb{N}$.

Exercise 1.2.13

(a) We want to show that $(A_1 \cup A_2 \cup ...A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$ by induction

(1) Induction base:$(A_1 \cup A_2) = A_1^c \cap A_2^c$ follows from Exercise 1.2.5,c

(2) Induction hypothesis: if $(A_1 \cup A_2 \cup ...A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$ for an natural number n, then $(A_1 \cup A_2 \cup ...A_{n+1})^c = A_1^c \cap A_2^c \cap ... \cap A_{n+1}^c$.

In order to show this, Assume $(A_1 \cup A_2 \cup ...A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$ and observe that:

$$
\begin{aligned}
(A_1 \cup A_2 \cup ...A_{n+1})^c &= ((A_1 \cup A_2 \cup ...A_n) \cup A_{n+1})^c && \text{by the associative law} \\
(A_1 \cup A_2 \cup ...A_{n+1})^c &= (A_1 \cup A_2 \cup ...A_n)^c \cap A_{n+1}^c && \text{by induction basis} \\
&= A_1^c \cap A_2^c \cap ... \cap A_n^c \cap A_{n+1}^c && \text{by our assumption}
\end{aligned}
$$

Thus, induction hypothesis holds and we proved the statement by induction.

(b) Consider the sets $B_i = (0, \frac{1}{i})$ for $i \in \mathbb{N}$. $\bigcap_{i=1}^n B_i = (0, \frac{1}{n})$ whereas $\bigcap_{i=1}^\infty B_i = \emptyset$.

(c) We want to show that

Lemma 1.4.

$$\left(\bigcup_{i=1}^\infty A_i \right)^c = \bigcap_{i=1}^\infty A_i^c$$

We already observed that, infinite case cannot be proven by induction. Instead we will take an even more direct approach and show two sides of inclusion:

Proof. Let $x \in \left(\bigcup_{i=1}^\infty A_i \right)^c$, i.e $x \notin \bigcup_{i=1}^\infty A_i$ meaning that x cannot be in A_i for any $i \in \mathbb{N}$. This implies $x \in A_i^c$ for all i and thus $x \in \bigcap_{i=1}^\infty A_i^c$. That is, because x was an arbitrary choice, $\left(\bigcup_{i=1}^\infty A_i \right)^c \subseteq \bigcap_{i=1}^\infty A_i^c$.

On the other hand, for any x in $\bigcap_{i=1}^\infty A_i^c$, $x \in A_i^c$, i.e. $x \notin A_i$ for all i. This implies x is not an element of $\bigcup_{i=1}^\infty A_i$ which allows us to show that $\bigcap_{i=1}^\infty A_i^c \subseteq \left(\bigcup_{i=1}^\infty A_i \right)^c$. $\square$