Game Theory Homework #2, 02/02/2021

All the problems are worth 4 points each and will be graded on a 0/1/2/3/4 scale. Due on Wednesday 02/10/2021 before 11:59 pm to be uploaded via Gradescope.

1. Let G be a progressively bounded impartial combinatorial game under normal play. Its Sprague-Grundy function g is defined recursively as follows:

$$g(x) = \text{mex}\{g(y) : x \to y \text{ is a legal move.}\}$$

where mex of a set of numbers, is the minimum excluded value function (smallest non-negative number not in the set). Some examples are as follows:

$$\text{mex}\{0, 1, 3, 4\} = 2, \ \text{mex}\{2, 5, 7\} = 0.$$

Note that the Sprague-Grundy value of any terminal position is $\text{mex}(\emptyset) = 0$.

- Prove that $x \in P$ iff $g(x) = 0$.

  Solution: Define the sets $P' = \{x \in \mathbb{Z}_{\geq 0} : g(x) = 0\}$ and $N' = \{x \in \mathbb{Z}_{\geq 0} : g(x) > 0\}$. Now note that, if $x \in P'$ then we have $g(x) = 0$. This means that $0 \notin \{g(y) : x \to y \text{ is a legal move}\}$, so for every legal move $x \to y$ we have $g(y) > 0$ hence $y \in N'$. In other words, every legal move from a state in P' leads to a state in N'. Next note that, if $x \in N'$, then we have $g(x) > 0$. This means that $0 \in \{g(y) : x \to y \text{ is a legal move}\}$, so there must exist some legal move $x \to y$ with $g(y) = 0$, hence $y \in P'$. In other words, from every state in N', there is some legal move leading to a state in P'. By homework 1 problem 5, we conclude that $P = P'$ and $N = N'$, and this is exactly the desired result.

- Now consider the sum G of two progressively bounded impartial combinatorial game under normal play, G_1 and G_2. Let g, g_1 and g_2 be the respective Sprague-Grundy functions of G, G_1, G_2 respectively. Show that $g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2)$.

  Solution: By homework 1 problem 3, we know that G is progressively bounded. Thus, the game started in position (x_1, x_2) is guaranteed to terminate in at most $B(x_1, x_2)$ turns. We now prove by induction on $n \in \mathbb{Z}_{\geq 0}$ that we have $g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2)$ whenever $B(x_1, x_2) \leq n$.

  For the base case of $n = 0$, suppose that $B(x_1, x_2) = 0$. This implies that (x_1, x_2) is terminal in G, and that x_1 is terminal in G_1 and x_2 is terminal in G_2. This implies $(x_1, x_2) \in P$, so by part 1 of this problem, we have $g(x_1, x_2) = 0$. It also implies $x_1 \in P_1$ and $x_2 \in P_2$, so by part 1 of this problem, we have $g_1(x_1) = g_2(x_2) = 0$. In particular, we see $g(x_1, x_2) = 0 = g_1(x_1) \oplus g_2(x_2)$, so the base case holds.

  For the inductive step, suppose the inductive hypothesis holds and that we have $B(x_1, x_2) \leq n + 1$. Write $A = \{g(y_1, y_2) : (x_1, x_2) \to (y_1, y_2) \text{ is a legal move}\}$. Then note that $B(y_1, y_2) \leq n$ holds whenever $(x_1, x_2) \to (y_1, y_2)$ is a legal move, so the inductive hypothesis applies and shows that we can write $A = \{g_1(y_1) \oplus g_2(y_2) : (x_1, x_2) \to (y_1, y_2) \text{ is a legal move}\}$. Furthermore, recall that legal moves in G consist of picking a coordinate and then moving in that coordinate, so can write $A = \{g_1(x_1) \oplus g_2(y_2) : x_2 \to y_2 \text{ is a legal move}\} \cup \{g_1(y_1) \oplus g_2(x_2) : x_1 \to y_1 \text{ is a legal move}\}$. Now proving $g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2)$ amounts to proving the two statements that $g_1(x_1) \oplus g_2(x_2) \notin A$, and that, if $k < g_1(x_1) \oplus g_2(x_2)$, then $k \in A$.

  For the first statement, assume for the sake of contradiction that $g_1(x_1) \oplus g_2(x_2) \in A$. This means there is either some legal move $x_1 \to y_1$ in G_1 such that $g_1(x_1) \oplus g_2(x_2) = g_1(y_1) \oplus g_2(x_2)$ or some legal move $x_2 \to y_2$ in G_2 such that $g_1(x_1) \oplus g_2(x_2) = g_1(x_1) \oplus g_2(y_2)$. The argument is identical in either case, so let's assume that the former holds. Then note that $g_1(x_1) \oplus g_2(x_2) = g_1(y_1) \oplus g_2(x_2)$ implies $g_1(x_1) = g_1(y_1)$. However, this is a contradiction since $g_1(y_1) \in \{g_1(\widetilde{y}_1) : x_1 \to \widetilde{y}_1 \text{ is a legal move}\}$. (In other words, the Sprague-Grundy function must change after making any legal move.)

  For the second statement, suppose that $k < g_1(x_1) \oplus g_2(x_2)$. This means that, when we write k and $g_1(x_1) \oplus g_2(x_2)$ both in binary, there is some index $i \in \mathbb{Z}_{\geq 0}$ such that k has a zero in bit i and $g_1(x_1) \oplus g_2(x_2)$ has a 1 in bit i. In particular, exactly one of $g_1(x_1)$ or $g_2(x_2)$ must have a 1 in bit i and the other a 0 in bit i; suppose these are $g_1(x_1)$ and $g_2(x_2)$, respectively, since the argument is identical in either case. Then there is some $k_1 < g_1(x_1)$ such that $k_1 \oplus g_2(x_2) = k$. By the definition of Sprague-Grundy function, this means there is a legal move $x_1 \to y_1$ in G_1 such that $g_1(y_1) = k_1$. In particular, the legal move $(x_1, x_2) \to (y_1, x_2)$ in G has $g(y_1, x_2) = g_1(y_1) \oplus g_2(x_2) = k_1 \oplus g_2(x_2) = k$ by the inductive hypothesis, so $k \in A$.

2. Find the Sprague-Grundy function for the Nim game $(n_1, n_2, \dots n_k)$.

   Solution: First, we claim that the Sprague-Grundy function for the game of Nim with one pile is just $g(n) = n$. To see this, we prove by induction on $n \in \mathbb{Z}_{\geq 0}$ that we have $\{g(y) : n+1 \to y \text{ is a legal move}\} = \{0, 1, \dots n\}$. The base case is just $\{g(y) : 1 \to y \text{ is a legal move}\} = \{0\}$, which follows by part 1 of problem 1 of this homework along with $0 \in P$. Now assume the inductive hypothesis for n, and note that this impies $g(n+1) = \{g(y) : n+1 \to y \text{ is a legal move}\} = n+1$, hence $\{g(y) : n+2 \to y \text{ is a legal move}\} = \{0, 1, \dots n+1\}$. This proves the claim.

   Next, we use part 2 of problem 1 of this homework to get that the Sprague-Grundy function for Nim with k piles is just $g(n_1, n_2, \dots n_k) = n_1 \oplus n_2 \oplus \cdots \oplus n_k$. (Technically speaking, the result of part 2 of problem 1 only shows how to find the Sprague-Grundy function of the sum of two games. So, to find the Sprague-Grundy function of the sum of k games, we should use induction on $k \in \mathbb{Z}_{\geq 1}$. But the details of this are not so important.) This result means that the characterization of P states in terms of Sprague-Grundy functions generalizes the characterization of P states in Nim.

   Consider a game of Nim with four piles, of sizes $9, 10, 11, 12$.

   - Is this position a win for the next player or the previous player (assuming optimal play)? Describe all the winning first moves in case of the former.

     Solution: By the first part of this problem (or, by the characterization of P states we already learned about Nim), we just need to find the value of $9 \oplus 10 \oplus 11 \oplus 12$. So, let's convert these numbers to binary and take their bitwise sum modulo two:

9	1001
10	1010
11	1011
12	1100
	0100

     This calculation shows that $9 \oplus 10 \oplus 11 \oplus 12 = 4 \neq 0$, so $(9, 10, 11, 12)$ is an N position. A winning move for the first player is to make the Nim sum of the numbers of chips in the remaining piles equal to zero. In particular, we can subtract 4 chips from the last pile, which moves the state of the game to $(9, 10, 11, 8)$. For this new position we calculate

9	1001
10	1010
11	1011
8	1000
	0000

     so $9 \oplus 10 \oplus 11 \oplus 8 = 0$. This proves that $(9, 10, 11, 8)$ is a P position, hence this is a winning move for the first player.

   - Consider the same initial position, but suppose that each player is allowed to remove at most 9 chips in a single move (Other rules of Nim remain in force.) Is this an N or P position?

     Solution: Let G' denote this game of modified Nim for one pile, and let g' denote its Sprague-Grundy function. We can compute $g'(n)$ by hand for $n \leq 12$, as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$g'(n)$	0	1	2	3	4	5	6	7	8	9	0	1	2

     (From this table, one might guess that $g'(n) = n \bmod 10$, but we don't need to prove this for the result.) Now to understand $(9, 10, 11, 12)$, we just need to compute the Sprague-Grundy function of each number, convert these results into binary, and find their sum modulo two:

$g'(9)$	9	1001
$g'(10)$	0	0000
$g'(11)$	1	0001
$g'(12)$	2	0010
		1010

     Hence, $g'(9) \oplus g'(10) \oplus g'(11) \oplus g'(12) = 10 \neq 0$, so $(9, 10, 11, 12)$ is an N position.

3. Consider the following two-person zero-sum game. Both players simultaneously call out one of the numbers 2, 3. Player I wins if the sum of the numbers called is odd and player II wins if their sum is even. The loser pays the winner the product of the two numbers called (in dollars). Find the payoff matrix, the value of the game, and an optimal strategy for each player.

   Solution: We can make the payoff matrix for this game, where. for Player I, the top row represents calling 2 and the bottom row representing calling 3, and, for Player II, the left column represents calling 2 and the right column represents calling 3:

$$\begin{pmatrix} -4 & 6 \\ 6 & -9 \end{pmatrix} \tag{1}$$

   On the one hand, suppose that Player I calls 2 with probability x and calls 3 with probability $1 - x$. Then her payoff is at least $\min\{-4x + 6(1 - x), 6x - 9(1 - x)\}$. This is maximized at $x = \frac{3}{5}$, achieveing a value of zero. On the other hand, suppose that Player II calls 2 with probability y and calls 3 with probability $1 - y$. Then his payoff is at most $\max\{-4x + 6(1 - x), 6x - 9(1 - x)\}$, which is minimized at $x = \frac{3}{5}$, achieveing a value of zero. This shows that the value of the game is zero, and that either player's optimal strategy is to call 2 with probability $\frac{3}{5}$ and to call 3 with probability $\frac{2}{5}$.

4. Show that for any 2×2 payoff matrix there exists either a pair of optimal strategies that are both pure or are both fully mixed. Show that this can fail for a 3×3 matrix.

Solution: Since in any two-person zero-sum game there exists a pair of optimal strategies, we must at least one of

(a) there exists a pair of pure optimal strategies,

(b) there exists a pair of mixed but not pure optimal strategies, or

(c) there exists a pair of optimal strategies where one player's strategy is pure and the other's is mixed but not pure.

(Note that, for a player with 2 moves, a mixed strategy is fully mixed if and only if it is not pure.) Our goal is to show that, in a two-person zero-sum game corresponding to a 2×2 payoff matrix, the case (c) above cannot hold alone. So, we show that, if (c) holds, then (a) must also hold. To do this, write an arbitrary payoff matrix as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2}$$

Without loss of generality, we can assume that Player I's optimal strategy is to purely play the top row, and that Player II's optimal strategy is to play the left column with probability y and the right column with probability $1 - y$, where $y \in (0, 1)$.

First, we note that Player II's strategy is fully mixed, so the principle of equalization tells us that a and b must be equal (and also equal to the value of the game). So we reduce the game to

$$\begin{pmatrix} a & a \\ c & d \end{pmatrix}. \tag{3}$$

Moreover, we cannot have both $a < c$ and $a < d$, because then the bottom row would strictly dominate the top row for Player I, hence the given strategy would not be optimal. So there are two cases to consider: If $a \geq c$, then the top-left cell is a saddle point, hence Player II has a pure optimal strategy in $y = 1$. Otherwise we have $a \geq d$, so the top-right is a saddle point, hence Player II has a pure optimal strategy in $y = 0$. This completes the proof.

(Be careful not to misinterpret this result as saying "there cannot exist a pair of optimal strategies where one player's strategy is pure and the other's is mixed but not pure". Indeed, in the game with $a \neq c = d$, Player II has both pure and fully mixed optimal strategies.)

To see that this can fail for a 3×3 matrix, it suffices to to find a game such that there is no pair of pure optimal strategies, nor a pair of fully mixed optimal strategies. We claim that the following satisfies this criterion:

$$\begin{pmatrix} 3 & 3 & 3 \\ 4 & 4 & 0 \\ 0 & 4 & 4 \end{pmatrix}. \tag{4}$$

To see this, we first show that the value of the game is 3. Indeed, if Player I plays the pure strategy of the topmost row, then she guarantees a payoff of at least 3. Also, if Player II plays the mixed strategy of playing the leftmost column and the rightmost column with equal probability $\frac{1}{2}$, then he guarantees himself a loss of no more than 3. So, the value of the game is 3.

To see that there is no pair of pure optimal strategies, simply note that the only cells with 3's are not saddle points. To see that there is no pair of fully mixed optimal strategies, note that if $x_* = (1-x_2-x_3, x_2, x_3)$ and $y_* = (y_1, 1-y_1-y_3, y_3)$ were optimal and fully mixed, then the principle of equalization would tell us that we have $x_* = (1, 0, 0)$ and $y_* = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, a contradiction.

5. Consider an $n \times n$ payoff matrix which is anti-symmetric i.e. $a_{i,j} = -a_{j,i}$.. What is the value of such a game? Given an optimal strategy x_* for Player I find an optimal strategy for Player II.

Solution: Let A be the $n \times n$ payoff matrix for this game, and note that the anti-symmetry condition is nothing more than $A^{\mathrm{T}} = -A$. Now recall that we can write the payoff of a pair of strategies $x, y \in \Delta_n$ as the product $x^{\mathrm{T}} Ay$. Also, recall that we can pushing a minus sign through a maximum flips it to a minimum, and vice versa. Therefore, we can compute that, if the value of the game is V, then we have

$$\begin{aligned} V &= \max_{x \in \Delta_n} \min_{y \in \Delta_n} x^{\mathrm{T}} Ay \\ &= -\min_{x \in \Delta_n} \max_{y \in \Delta_n} x^{\mathrm{T}} (-A) y \\ &= -\min_{x \in \Delta_n} \max_{y \in \Delta_n} x^{\mathrm{T}} A^{\mathrm{T}} y \\ &= -\min_{x \in \Delta_n} \max_{y \in \Delta_n} y^{\mathrm{T}} Ax \\ &= -\min_{y \in \Delta_n} \max_{x \in \Delta_n} x^{\mathrm{T}} Ay = -V. \end{aligned} \tag{5}$$

Of course $V = -V$ implies $V = 0$, so the value of the game is zero.

Finally, we claim that, given an optimal strategy x_* for Player I, an optimal strategy for Player II is just $y_* = x_*$. To see this, note that x_* being optimal for Player I means

$$\min_{y \in \Delta_n} x_*^{\mathrm{T}} Ay = 0 \tag{6}$$

Multipling both sides by -1 and then doing a similar calculation as above yields

$$\begin{aligned} 0 &= -\min_{y \in \Delta_n} x_*^{\mathrm{T}} Ay \\ &= \max_{y \in \Delta_n} x_*^{\mathrm{T}} (-A) y \\ &= \max_{y \in \Delta_n} x_*^{\mathrm{T}} A^{\mathrm{T}} y \\ &= \max_{y \in \Delta_n} y^{\mathrm{T}} A^{\mathrm{T}} x_* \\ &= \max_{x \in \Delta_n} x^{\mathrm{T}} A^{\mathrm{T}} y_*, \end{aligned} \tag{7}$$

and this means that $y_* = x_*$ is also optimal for Player II.