Game Theory (STAT 155) Midterm Review

This document reviews some of the big-picture ideas and techniques from the class so far. There are fewer details than, say, the homework solutions or the section worksheet solutions, but this should point you in the right direction if you want some further clarification.

- **Combinatorial Games**. Generally speaking, these are games where players alternate turns making moves which alter the state of the game. *Normal play* refers to the protocol where terminal states are losing states.
  - **Impartial Combinatorial Games.** These are games where the set of legal moves from any state does not depend on which player is in that state. A *progressively bounded* impartial combinatorial game is one which from any starting state is guataneed to terminate in finitely many steps. The main theorem from lecture is that any progressively bounded impartial combinatorial game can have its state space X partitioned into $N \cup P$ where from any state in N the current player can force a win, and from any state in P the other player can force a win. Most problems about impartial combinatorial games ask to classify states as either N or P.
    - One useful way to determine the N and P states in a given game is to the following result, which you proved on the first homework: If X is written as a partition $N' \cup P'$ where from any state in N' there is a legal move leading to P' and from any state in P' every legal move leading to N', then $N = N'$ and $P = P'$. Examples: subtraction games with allowed subtractions $\{1, 2, \dots k\}$, HW1P4
    - Another useful way to understand the N and P states in a given game is by computing its Sprague-Grundy function. Recall that the Sprague-Grundy function is defined via $g(x) = \text{mex}\{g(y) : x \to y \text{ is a legal move}\}$. Importantly, you proved on the second homework that $x \in P$ if and only if $g(x) = 0$, so this is a way to determine if states are in N or P. Even more important is that this helps understand the N and P states in a sum of games, since you also proved on the second homework that, if the Sprague-Grundy functions the games G_1 and G_2 are g_1 and g_2, then the Sprague-Grundy function of the game $G_1 + G_2$ is $g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2)$. Examples: Nim, HW2P2
    - Some impartial combinatorial games, in some sense, are just Nim in disguise. Examples: Rims, SW2P3
    - Sometimes one can prove that a certain player has a winning strategy without actually showing what that winning strategy is. The main way to do this is *strategy-stealing*, in which we show that, if one player had a winning strategy, then the other player would be able to utilize it instead. Examples: Chomp, SW1P3, HW1P2
  - **Partisan Combinatorial Games.** These are combinatorial games that are not impartial, that is, the two players either have different legal moves from some state or different terminal positions. Another theorem from lecture is that in a partisan combinatorial games with no ties, one player is guaranteed to have a winning strategy.
    - One principle we saw was that strategy stealing can also work in partisan cominatorial games. That is, if one player had a winning strategy, then the other player could somehow utilize that strategy as well. Often this is used to show that the first player has a winning strategy, since they have an extra turn, and in some games having an extra turn can only make you better off. Examples: recursive majority vote, Hex
- **Zero-Sum Games** These are games where two players simultaneously pick one of their allowed moves, and payoffs are assigned based on the pair of choices, in a way such that one player's gain is the other's loss. We typically encode a zero-sum game in a *payoff matrix* $A = (a_{ij})_{i,j=1}^{m,n} \in \mathbb{R}^{m \times n}$ where a_{ij} represents the payoff to Player I (and the cost to Player II) when Player I plays i and Player II plays j. We saw that introducing randomness into your strategy was an effective way to protect against the other player predicting your moves. For random strategies $x \in \Delta_m$ for Player I and $y \in \Delta_n$ for Player II, we have that $x^{\mathrm{T}} A y$ is the expected payoff to Player I (hence the cost to Player II) under these strategies.
  - **Safety Strategies.** A conservative player will want to optimize for their worst-case scenario. For Player I, this means she wants to maximize her worst case outcome, that is, solve $\arg\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^{\mathrm{T}} A y$. Any strategy that solves this is called a *safety strategy* for Player I. For Player II, this means he wants to minimize his worst-case outcome, that is, solve $\arg\min_{y \in \Delta_n} \max_{x \in \Delta_m} x^{\mathrm{T}} A y$, and this is called a safety strategy for Player II. The main theorem from lecture is that $\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^{\mathrm{T}} A y = \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^{\mathrm{T}} A y$, and this is called the *value* of the game. For two-player zero-sum games, we use the terms *safety strategy* and *optimal strategy* interchangeably.
  - **Finding An Optimal Strategy.** There are several methods that one can use to find an optimal strategy for each player in a two-player zero-sum game. In any given situation, each approach may or not be helpful, but it's good to think about which approaches might work and why. Also note that sometimes we need to use multiple of these approaches in order to solve a single game.
    - One can directly solve the optimization problem $\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^{\mathrm{T}} A y = \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^{\mathrm{T}} A y$. This may not be easy to do in practice, but, in principle, it always works. If at least one of m or n is equal to 2, then one can use the *graphical method* in which we plot the function $x \mapsto \min_{y \in \Delta_n} x^{\mathrm{T}} A y$ and visually find its maximum, or the function $y \mapsto \max_{x \in \Delta_n} x^{\mathrm{T}} A y$ and visually find its minimum. Examples: SW3P1(a), HW2P3

* If a pair i, j is such that a_{ij} is the minimum in its row and the maximum in its column, then it is said to be a *saddle point*, in which case we have that the value of the game is $V = a_{ij}$, that e_i is an optimal strategy for Player I, and that e_j is an optimal strategy for Player II. Examples: SW3P1(b,f)
* If strategies i and i' for Player I are such that $a_{ij} \geq a_{i'j}$ holds for all j, then we say that i' is *dominated by* i. In other words, i' is dominated by i if the payoff to Player I when playing i is at least as large as the payoff to layer I when playing i', no matter what Player II does. Similarly, if strategies j and j' for Player II are such that $a_{ij} \leq a_{ij'}$ holds for all i, then we say that j' is *dominated by* j. We say that a row or column is *strictly dominated* if all the inequalities above are strict. Whenever a strategy is dominated by another strategy, the dominated row or column can be removed from the payoff matrix of the game, and it follows the value of the reduced game is the same as the value of the original game, and also that any optimal strategy in the reduced game corresponds to some optimal strategy in the original game. Examples: SW3P1(d,e,g), SW4P1(c), HW3P2
* If Player I has an optimal strategy that is fully mixed, then Player II's optimal strategy must be such that Player I receives the same expected payoff for either of her moves. (Otherwise, she could move some probability from one move to the other and increase her payoff, so her original strategy would not have been optimal.) This is the principle of *equalization*. If one is interested in finding fully mixed pairs of optimal strategies, then it is a good idea to set up both equalization equations and solve. Examples: SW4P1(a,b), SW4P3, HW3P1
* In some games with a large collection of possible moves for each player, one can notice that certain moves can be grouped together in such a way that moves in the same group are essentially the same. This is usually done by exploiting some *symmetry* of the game. In this way one might be able to reduce a game with a large matrix to one with a smaller matrix which is more reasonable to solve. Examples: submarine salvo, SW4P2,
* Note that it is easier to check whether a pair of strategies is optimal than it is to find a pair of strategies. Thus, a *guess-and-check* method is valid, but sometimes hard to execute in practice. Usually one makes informed guesses by looking at the game. Examples: SW3P1(c), HW3P5

– **Finding All Optimal Strategies.** Sometimes one is interested in finding all optimal strategies for one or both players in a given two-player zero-sum game. There is no general-purpose way to do this, but a useful paradigm is as follows: First, find some optimal strategy and the value of the game. Then, use that information to find the remaining optimal strategies.

  * If you know the value of the game, then you can directly compute which strategies for each player achieve the value. Example: HW3P5
  * Suppose one uses domination to find the initial pair of optimal strategies. Whenever a strict domination occurs, we know that the dominated move cannot be assigned any positive probability under any optimal strategy. Whenever, a non-strict domination occurs, we may lose some optimal strategies in the reduced game. In order to understand which optimal strategies in the original game correspond to a specific optimal strategy in the reduced game, one needs to think about all the ways to distribute probability among these moves in the correct proportion. Example: HW3P2

- **General-Sum Games** These are games where two or more players simultaneously pick one of their allowed moves, and each player has payoffs that depend on all of the chosen moves. Two-player general-sum games are typically encoded in a pair of payoff matrices $A = (a_{ij})_{i,j=1}^{m,n}$ and $B = (b_{ij})_{i,j=1}^{m,n}$ in $\mathbb{R}^{m\times n}$, where a_{ij} and b_{ij} represent the payoffs to Player I and Player II, respectively, when Player I plays i and Player II plays j. As in the case of zero-sum games, we allow players to choose strategies which incorporate randomness. When Player I plays $x \in \Delta_m$ and Player II plays $y \in \Delta_n$, the expected payoff to Player I is $x^{\mathrm{T}}Ay$ and the expected payoff to Player II is $x^{\mathrm{T}}By$.

  – **Safety Strategies.** As in the case of zero-sum games, a conservative player will want to optimize for their worst-case scenario. For Player I, this means she wants to solve $\arg\max_{x\in\Delta_m}\min_{y\in\Delta_n} x^{\mathrm{T}}Ay$, and for Player II, this means he wants to solve $\arg\max_{y\in\Delta_n}\min_{x\in\Delta_m} x^{\mathrm{T}}By$. Any strategy with this property is called a *safety strategy*. Unlike in zero-sum games, general-sum games may not have a well-defined value.

  – **Nash Equilibria.** A pair of strategies in which neither player can increase their payoff by changing their strategy while the other player fixes their strategy is called a *Nash equilibrium*. More formally, a pair $(x, y) \in \Delta_m \times \Delta_n$ is a Nash equilibrium if for any $x' \in \Delta_m$ we have $(x')^{\mathrm{T}}Ay \leq x^{\mathrm{T}}Ay$ and for any $y' \in \Delta_n$ we have $x^{\mathrm{T}}By' \leq x^{\mathrm{T}}By$. The main theorem from lecture is that every general-sum has some Nash equilibrium. General-sum games can have many Nash equilibria, and the expected payoff to each player can be different across different Nash equilibria.

  – **Finding A Nash Equilibrium.** There are several methods that one can use to find a Nash equilibirum for each player in a general-sum game. As before, these approaches do not always yield a Nash equilibrium, and sometimes you may need to use multiple approaches in order to analyze a single game.

    * Nash equilibria in which both players have pure strategies can be found by directly analyzing the matrices for the game. This is similar to looking for saddle points in a zero-sum game. Examples: SW5P1(a,b), HW4P1
    * If strategies i and i' for Player I are such that $a_{ij} \geq a_{i'j}$ holds for all j, then we say that i' is *dominated by* i. Similarly, if strategies j and j' for Player II are such that $b_{ij} \geq b_{ij'}$ holds for all i, then we say that j' is

dominated by j. We say that a row or column is *strictly dominated* if all the inequalities above are strict. As in the case of zero-sum games, if a general-sum game has a row or column that is dominated, then we can remove it from the matrices and the reduced game will still contain a Nash equilibrium. If a row or column is strictly dominated, then there is no Nash equilibrium which assigns positive probability to that move.

  * In Nash equilibria in which at least one player's strategy is mixed, we must have that, for all move which are assigned positive probability, the player's expected payoff under those moves is the same. (Otherwise, that player could strictly increase their payoff by shifting some of the probability to the strategy with the higher payoff.) This is again a principle of *equalization*. Examples: SW5P1(a,c), HW4P1, HW4P2

- **Finding All Nash Equilibria.** In order to find all Nash equilibria in a general-sum game, one must usually consider all the possible cases of whether strategies are pure, mixed, or fully mixed. In a 2×2 game, there are only 4 cases (pure-pure, pure-fully-mixed, fully-mixed-pure, and fully-mixed-fully-mixed) so this is not so much work. Examples: HW4P1, HW4P2

- **Symmetric Nash equilibria.** Consider a general-sum game, possibly with more than two players, such that each player has the same payoff matrix. Such games are called *symmetric general-sum games*. The main result from lecture about these games is that any symmetric general-sum games has a symmetric Nash equilibria. However, symmetric general-sum games can also have asymmetric Nash equilibria. In a symmetric general-sum game, one can find all symmetric Nash equilibria by simply considering the possible cases of the moves which are assigned probability. Examples: SW5P2, HW4P4