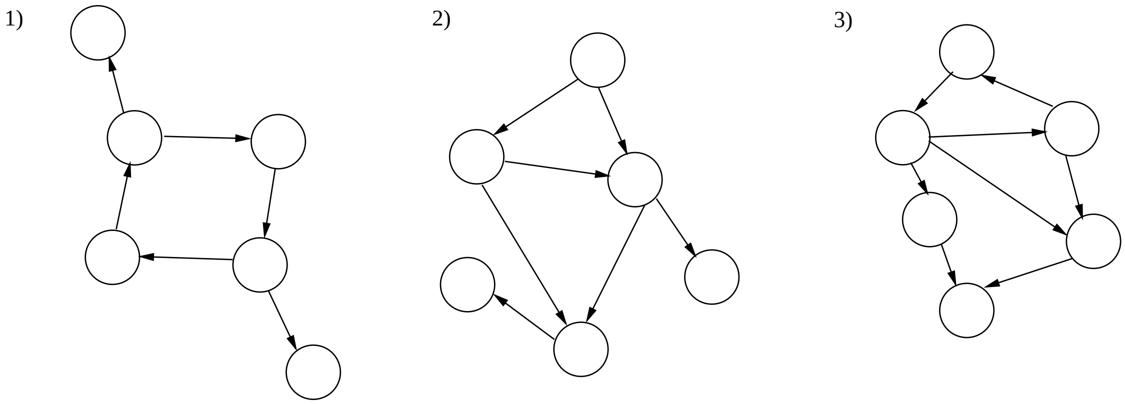
Game Theory (STAT 155) Worksheet #1, 01/25/2021

1. Two people play a game on a directed graph. There is a stone in one of the vertices of the graph. Players move the stone across directed edges in turns. A player who is unable to move loses. For each of the graphs on the figure find the sets **N** and **P**. For which of the graphs the game is progressively bounded?



2. Find the set of P-positions for the subtraction games with subtraction sets
   (a) $S = \{1, 3, 5\}$;
   (b) $S = \{2, 3\}$;
   (c) $S = \{2, 7, 8\}$;
   (d) Who wins each of these games if play starts at 100 chips, the first player or the second?

3. Consider the following game: the initial position is number 4. Two players make moves in turns: at each turn the player increases the current number by one of its divisors, but not by the number itself (e.g. one can increase 12 by 1, 2, 3, 4 or 6, but not by 12). The player who makes the number larger than 1000 loses. Who wins in this game, the first or the second player?

4. This is not a problem to solve, but rather an example of a *partisan* combinatorial game; we'll study these more in detail in class this week. A partisan game is a combinatorial game in which the players either have different win conditions, or in which there is some step at which the players have different sets of possible moves. (So, partisan combinatorial games are just combinatorial games that are not impartial.)

   (Hex) Hex is played on a rhombus-shaped board tiled with hexagons (see the picture below). Each player is assigned a color, either blue or yellow, and two opposing sides of the board. The players take turns coloring in empty hexagons. The goal for each player is to link their two sides of the board with a chain of hexagons in their own color. Thus, the terminal positions of Hex are the full or partial colorings of the board that have a chain crossing.

   There are two important observations to make about Hex.

   - Any fully-colored board has a either blue chain crossing or a yellow chain crossing, but not both. Thus the game cannot end in a tie.
   - Having extra tiles can only help you. In other words, if the game ends early and the yellow player wins with tiles $\{x_1, \ldots x_n\}$, then any configuration in which the yellow player has tiles $\{x_1, \ldots x_n, x_{n+1}\}$ is also a winning position.

   It turns out that these observations tell us that the first player has a winning strategy.

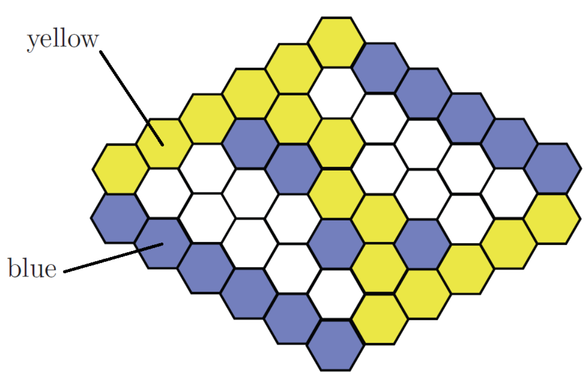


FIGURE 1.9. A completed game of Hex with a yellow chain crossing.