Notes on Category Theory

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Contents

1 Preliminaries

1.1 Monomorphisms and epimorphisms

1.1.1

A morphism $\mu\colon d' \to e$ is said to be a *monomorphism* if, for any parallel pair of arrows

$$d \underset{\beta}{\overset{\alpha}{\rightrightarrows}} d' \ , \tag{1}$$

equality

$$\mu \circ \alpha = \mu \circ \beta$$

implies $\alpha = \beta$.

1.1.2

Dually, a morphism $\epsilon\colon c \to d$ is said to be an *epimorphism* if, for any parallel pair (1),

$$\alpha \circ \epsilon = \beta \circ \epsilon$$

implies $\alpha = \beta$.

1.1.3 Arrow notation

Monomorphisms are often represented by arrows $\rightarrowtail$ with a tail while epimorphisms are represented by arrows $\twoheadrightarrow$ with a double arrowhead.

1.1.4 Split monomorphisms

Exercise 1 *Given a morphism α, if there exists a morphism α' such that*

$$\alpha' \circ \alpha = \mathrm{id} \tag{2}$$

then α is a monomorphism.

Such monomorphisms are said to be *split* and any α' satisfying identity (2) is said to be a *left inverse* of α.

1.1.5 Further properties of monomorphisms and epimorphisms

Exercise 2 *Show that, if $\lambda \circ \mu$ is a monomorphism, then μ is a monomorphism. And, if $\lambda \circ \mu$ is an epimorphism, then λ is an epimorphism.*

Exercise 3 *Show that an isomorphism is both a monomorphism and an epimorphism.*

Exercise 4 *Suppose that in the diagram with two triangles, denoted A and B,*

$$\begin{tikzcd}
\bullet \arrow[d, "\alpha"'] & \bullet \arrow[l] \arrow[d, "\beta"] \\
\bullet & \bullet \arrow[l] \arrow[ul, "A", "B"']
\end{tikzcd} \tag{3}$$

the outer square commutes. Show that, if α is a monomorphism and the A triangle commutes, then also the B triangle commutes. Dually, if β is an epimorphism and the B triangle commutes, then the A triangle commutes.

1.1.6

In particular, if α is a monomorphism and β is an epimorphism, then one of the triangles commutes if and only if the other triangle does.

Exercise 5 *Show that, under assumption that α is a monomorphism and β is an epimorphism, the diagonal arrow in diagram (3) is unique.*

Exercise 6 *Suppose that a monomorphism μ factorizes*

$$\mu = \mu' \circ \alpha \tag{4}$$

with μ' being a monomorphism, and μ' factorizes

$$\mu' = \mu \circ \beta.$$

Show that β is the inverse of α.

Exercise 7 *State and prove the corresponding property of epimorphisms.*

1.1.7 Subobjects of a given object

Let us consider the class of monomorphisms with object a as their target. Existence of the double factorization for a pair of monomorphisms, as in Exercise 6, defines an equivalence relation on this class. The corresponding equivalence classes of monomorphisms are referred to as *subobjects* of a.

1.1.8

Existence of a factorization (4) defines a relation between monomorphisms

$$\mu \preceq \mu'$$

with a given target. This relation induces a partial order,

$$m \subseteq m' \qquad (m = [\mu],\ m' = [\mu']),$$

on the class of subobjects of a.

1.1.9 Quotients of a given object

Quotients of an object a are defined as equivalence classes of the dual relation on the class of epimorphisms with object a as their source.

Exercise 8 *Describe the corresponding partial ordering of the class of quotient objects of* a.

1.2 Initial, terminal and zero objects

1.2.1 Initial objects

An object i of a category $\mathcal{C}$ is said to be *initial*, if the set of morphisms from i to any object c consists of a single morphism. We are generally not assuming that all objects in a category are equipped with the identity object, so the single endomorphism

$$\iota\colon i \longrightarrow i \tag{5}$$

is not necessary the identity id_ι.

Exercise 9 *Show that* (5) *is a right identity.*

1.2.2

Note that any morphism ϵ that is a onesided identity is automatically idempotent.

$$\epsilon \circ \epsilon = \epsilon.$$

Exercise 10 *Show that any two initial objects* i *and* i' *whose endomorphisms are the identity morphisms, are isomorphic.*

1.2.3

In particular, in a unital category any two initial objects are isomorphic.

1.2.4 Terminal objects

Terminal objects are defined dually: an object t is said to be *terminal*, if the set of morphisms from any object c to t consists of a single morphism. The single endomorphism of t is a left identity.

1.2.5

Any two terminal objects t and t' whose endomorphisms are the identity morphisms, are isomorphic. In particular, in a unital category any two terminal objects are isomorphic.

1.2.6 Zero objects

An object 0 that is simultaneously an initial and a terminal object is called a *zero* object. The unique endomorphism of a zero object is id_0. In view of this, any two zero objects are always isomorphic.

1.2.7 Zero morphisms

A zero morphism is a morphism that factorizes through a zero object.

Exercise 11 *Let 0 and $0'$ be zero objects and α be a morphism that factorizes through 0. Show that it factorizes also through $0'$.*

Exercise 12 *Show that, for any pair of objects c and c', there exists a unique zero morphism $c \longrightarrow c'$.*

1.2.8

This *unique* zero morphism will be denoted $c \xrightarrow{\ 0\ } c'$. We shall use 0 also as a generic notation for any zero object.

Exercise 13 *If an epimorphism is a zero morphism, then its target is a zero object. Dually, if a monomorphism iz zero, then its source is a zero object.*

1.3 Subcategories and quotient categories

1.3.1 A subcategory of a category

A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is defined by supplying a subclass $\mathcal{C}'_0$ of the class of objects of $\mathcal{C}$ and a subclass $\mathcal{C}'_1$ of the class of arrows of $\mathcal{C}$ such that they are closed with respect to the category operations (source, target, composition of arrows and the identity morphisms, in case of unital subcategories).

1.3.2 Full subcategories

In general,

$$\mathrm{Hom}_{\mathcal{C}'}(a, b) \subseteq \mathrm{Hom}_{\mathcal{C}}(a, b) \tag{6}$$

for any pair of objects in $\mathcal{C}'$. If one has equality in (6) for any such pair, we say that $\mathcal{C}'$ is a *full* subcategory of $\mathcal{C}$.

1.3.3

To specify a full subcategory of $\mathcal{C}$ one needs to specify the subclass of the class of objects of $\mathcal{C}$.

1.3.4 Congruence relations

An equivalence relation $\sim$ on the class of arrows of a category $\mathcal{C}$ which is compatible with the category operations is said to be a *congruence*. More precisely, if $\alpha \sim \alpha'$, then

$$s(\alpha) = s(\alpha') \qquad \text{and} \qquad t(\alpha) = t(\alpha')$$

and, if $\beta \sim \beta'$, then

$$a \circ \beta \sim a' \circ \beta'$$

whenever the corresponding arrows are composable.

1.3.5 The quotient category $\mathcal{C}_{/\sim}$

The class of objects of the quotient category $\mathcal{C}_{/\sim}$ coincides with the class of objects of $\mathcal{C}$. The class of arrows has as its members equivalence classes of arrows in $\mathcal{C}$.

1.4 Natural transformations

1.4.1 id_F

For any functor $F\colon \mathcal{C} \to \mathcal{D}$, the identity transformation is defined as

$$\left(\mathrm{id}_F\right)_c := \mathrm{id}_{Fc} \qquad (c \in \mathrm{Ob}\,\mathcal{C}). \tag{7}$$

Note that id_F exists precisely when the “range” of F has identity morphisms. This is in contrast with the identity functor $\mathrm{id}_{\mathcal{C}}$ that is defined as the identity correspondence both on the class of objects and on the class of arrows of $\mathcal{C}$.

Thus, the identity functor $\mathrm{id}_{\mathcal{C}}$ is defined for any category, irrespective of whether $\mathcal{C}$ has or has not identity morphisms but, for example, the natural transformation $\mathrm{id}_{\mathrm{id}_{\mathcal{C}}}$ is defined precisely when all objects in $\mathcal{C}$ have identity morphisms.

1.4.2 The biaction of functors on natural transformations

Given a natural transformation

$$\begin{tikzcd} \mathcal{D} & \mathcal{C} \arrow[l, "F"'] \arrow[d, phantom, "\Downarrow\phi"] \\ \mathcal{D} & \mathcal{C} \arrow[l, "F'"] \end{tikzcd}$$

and a pair of functors

$$\mathcal{E} \xleftarrow{E} \mathcal{D} \qquad \text{and} \qquad \mathcal{C} \xleftarrow{G} \mathcal{B}$$

let

$$(E\phi)_c := E(\phi_c) \qquad \text{and} \qquad (\phi G)_b := \phi_{Gb}. \tag{8}$$

1.4.3

Note that G in ϕG is "applied" to *objects* in its source category while E in $E\phi$ is "applied" to *morphisms* in its source category.

Exercise 14 *Show that*

$$E\phi := \big((E\phi)_c\big)_{c\in \operatorname{Ob}\mathcal{C}}$$

is a natural transformation from $E\circ F$ *to* $E\circ F'$ *and*

$$\phi G := \big((\phi G)_b\big)_{b\in \operatorname{Ob}\mathcal{B}}$$

is a natural transformation from $F\circ G$ *to* $F'\circ G$.

Exercise 15 *Show that*

$$\mathrm{id}_{\mathcal{D}}\,\phi = \phi = \phi\,\mathrm{id}_{\mathcal{C}}$$

and

$$(E\phi)G = E(\phi G). \tag{9}$$

Exercise 16 *Given composable pairs of functors,*

$$\mathcal{F} \xleftarrow{\quad D \quad} \mathcal{E} \xleftarrow{\quad E \quad} \mathcal{D} \qquad \textit{and} \qquad \mathcal{C} \xleftarrow{\quad G \quad} \mathcal{B} \xleftarrow{\quad H \quad} \mathcal{A}\ ,$$

show that

$$(D\circ E)\phi = D(E\phi) \qquad \textit{and} \qquad \phi(G\circ H) = (\phi G)H, \tag{10}$$

1.5

Identities (10) can be interpreted as meaning that the class of functors acts on the class of natural transformations both on the left and on the right while identity (9) means that the two actions commute. We shall refer to this as the canonical *biaction* of functors on natural transformations.

1.5.1 The "diamond" composition of natural transformations

Consider a pair of functors F and F' from $\mathcal{C}$ to $\mathcal{D}$ and a natural transformation $F \xrightarrow{\phi} F'$. Consider a second pair of functors G and G' from $\mathcal{B}$ to $\mathcal{C}$ and a natural transformation $G \xrightarrow{\psi} G'$.

Exercise 17 *Show that the following "diamond" diagram of natural transformations*

$$\begin{tikzcd} & F' \circ G \arrow[dl, "F'\psi"'] & \\ F' \circ G' & & F \circ G \arrow[ul, "\phi G"'] \arrow[dl, "F\psi"] \\ & F \circ G' \arrow[ul, "\phi G'"] & \end{tikzcd} \tag{11}$$

commutes.

1.5.2

The above diagram can be expressed intrinsically in terms of the pair of natural transformations by utilizing the *source* and *target* correspondences from natural transformations to functors:

$$\begin{tikzcd} & t_\phi \circ s_\psi \arrow[dl, "t_\phi \psi"'] & \\ t_\phi \circ t_\psi & & s_\phi \circ s_\psi \arrow[ul, "\phi s_\psi"'] \arrow[dl, "s_\phi \psi"] \\ & s_\phi \circ t_\psi \arrow[ul, "\phi t_\psi"] & \end{tikzcd} \tag{12}$$

1.5.3

We define $\phi \diamond \psi$ to be the natural transformation from $F \circ G$ to $F' \circ G'$ obtained by composing the natural transformations in diagram (11)

$$\phi \diamond \psi := \phi G' \circ F\psi = F'\psi \circ \phi G \tag{13}$$

or, in notation intrinsic to natural transformations,

$$\phi \diamond \psi := \phi t_\psi \circ s_\phi \psi = t_\phi \psi \circ \phi s_\psi . \tag{14}$$

Exercise 18 *Show that the operation $\diamond$ is associative.* (*Hint. Draw the corresponding diagram consisting of 4 "diamonds" like* (11) *and explain why all 4 commute.*)

1.5.4 The Interchange Identity

Exercise 19 *Given composable pairs of natural transformations*

$$\begin{tikzcd} \mathcal{D} \overset{F}{\longleftarrow} \mathcal{C} \arrow[d, "\phi"] \\ \mathcal{D} \overset{F'}{\longleftarrow} \mathcal{C} \arrow[d, "\phi'"] \\ \mathcal{D} \overset{F''}{\longleftarrow} \mathcal{C} \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} \mathcal{C} \overset{G}{\longleftarrow} \mathcal{B} \arrow[d, "\psi"] \\ \mathcal{C} \overset{G'}{\longleftarrow} \mathcal{B} \arrow[d, "\psi'"] \\ \mathcal{C} \overset{G''}{\longleftarrow} \mathcal{B} \end{tikzcd}$$

show that

$$(\phi' \diamond \psi') \circ (\phi \diamond \psi) = (\phi' \circ \phi) \diamond (\psi' \circ \psi). \tag{15}$$

Identity (15) is known as the *Interchange Identity*.

1.5.5 Notation and terminology

There is no standard notation for this "diamond" composition of natural transformations. It is often referred to as the *horizontal* composition reflecting the habit of drawing natural transformations as vertical arrows (in that case, the original composition of natural transformations occurs in vertical direction).

Exercise 20 *Show that*

$$\phi G = \phi \diamond \mathrm{id}_G$$

when $\mathcal{C}$ (i.e., the target category of G) has identity morphisms, and

$$E\phi = \mathrm{id}_E \diamond \phi$$

when $\mathcal{E}$ (i.e., the target category of E) has identity morphisms.

Under the same hypotheses show that

$$\mathrm{id}_{\mathrm{id}_\mathcal{D}} \diamond \phi = \phi = \phi \diamond \mathrm{id}_{\mathrm{id}_\mathcal{C}} . \tag{16}$$

1.6 The tautological natural transformation

1.6.1 The category of arrows

For any category $\mathcal{C}$, its category of arrows $\operatorname{Arr}\mathcal{C}$ has the class of arrows as its objects and commutative squares φ:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, "\alpha"'] \arrow[d, "\varphi_s"] \\ \bullet \arrow[from=u, "\varphi_t"'] & \bullet \arrow[l, "\alpha'"] \end{tikzcd} \tag{17}$$

as morphisms from α to α'.

1.6.2 The source and the target functors

Exercise 21 *Show that the correspondences*

$$S(\alpha) := s(\alpha) \qquad (\alpha \in \operatorname{Ob}\operatorname{Arr}\mathcal{C})$$

and

$$S(\varphi) = \varphi_s \qquad \left(\varphi \in \operatorname{Hom}_{\operatorname{Arr}\mathcal{C}}(\alpha, \alpha')\right)$$

define a functor from $\operatorname{Arr}\mathcal{C}$ *to* $\mathcal{C}$.

We shall refer to it as the *source* functor.

Exercise 22 *Define, by analogy, the "target" functor* $T\colon \operatorname{Arr}\mathcal{C} \longrightarrow \mathcal{C}$.

1.6.3 The tautological natural transformation

The class of objects of $\operatorname{Arr}\mathcal{C}$ is identical to the class of morphisms of $\mathcal{C}$. Let

$$\tau(\alpha) := \alpha \qquad (\alpha \in \operatorname{Ob}\operatorname{Arr}\mathcal{C})$$

be the identity correspondence between $\operatorname{Ob}\operatorname{Arr}\mathcal{C}$ and $\operatorname{Mor}\mathcal{C}$.

Exercise 23 *Show that* τ *is a natural transformation*

$$\begin{tikzcd} \mathcal{C} & \operatorname{Arr}\mathcal{C} \arrow[l, "S"'] \arrow[d, phantom, "\downarrow\tau"] \\ \mathcal{C} & \operatorname{Arr}\mathcal{C} \arrow[l, "T"] \end{tikzcd} \tag{18}$$

from the source to the target functors.

1.6.4 The universal property of the tautological transformation

Let

$$\begin{tikzcd} \mathcal{D} & \mathcal{C} \arrow[l, "F"'] \arrow[d, "\phi"] \\ \mathcal{D} & \mathcal{C} \arrow[l, "F'"'] \end{tikzcd} \tag{19}$$

be a natural transformation between a pair of functors. By definition it is a correspondence between the class of objects of $\mathcal{C}$ and the class of morphisms of $\mathcal{D}$. We shall extend it by assigning to each morphism $c \xrightarrow{\alpha} c'$ in $\mathcal{C}$, the commutative square

$$\begin{tikzcd} F'c \arrow[d, "F'\alpha"'] & Fc \arrow[l, "\phi_c"'] \arrow[d, "F\alpha"] \\ F'c' & Fc' \arrow[l, "\phi_{c'}"'] \end{tikzcd} \tag{20}$$

Exercise 24 *Show that the correspondences assigning ϕ_c to any object c of $\mathcal{C}$ and the commutative square* (20) *to every morphism α of $\mathcal{C}$, define a functor Φ from $\mathcal{C}$ to* $\operatorname{Arr}\mathcal{D}$ *such that*

$$S \circ \Phi = F \qquad \text{and} \qquad T \circ \Phi = F'.$$

Exercise 25 *Show that*

$$\phi = \tau\Phi. \tag{21}$$

Show that if $\Psi\colon \mathcal{C} \longrightarrow \operatorname{Arr}\mathcal{D}$ is another functor such that $\phi = \tau\Psi$, then $\Phi = \Psi$.

1.7

Identity (21) means that *every* natural transformation (19) is pulled from the tautological transformation on $\operatorname{Arr}\mathcal{D}$ by a unique functor $\mathcal{C} \longrightarrow \operatorname{Arr}\mathcal{D}$.

1.8 The Arr functor

1.8.1

Assignment $\mathcal{C} \longmapsto \operatorname{Arr}\mathcal{C}$ is natural: for any functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$, there is an induced functor

$$\operatorname{Arr} F\colon \operatorname{Arr}\mathcal{C} \longrightarrow \operatorname{Arr}\mathcal{D}.$$

It sends $\alpha \in \operatorname{Mor}\mathcal{C}$ to $F\alpha$ and the commutative square (17) to the commutative square

$$\begin{tikzcd} \bullet \arrow[d, "F\varphi_t"'] & \bullet \arrow[l, "F\alpha"'] \arrow[d, "F\varphi_s"] \\ \bullet & \bullet \arrow[l, "F\alpha'"'] \end{tikzcd}$$

This defines an endomorphism of the category of (small) categories

$$\mathrm{Arr}\colon \mathrm{Cat} \longrightarrow \mathrm{Cat}.$$

Exercise 26 *Show that the correspondences*

$$S\colon \mathcal{C} \longmapsto S_{\mathcal{C}} \qquad \textit{and} \qquad T\colon \mathcal{C} \longmapsto T_{\mathcal{C}} \tag{22}$$

that assign to a category $\mathcal{C}$ its source and target functors define natural transformations $\mathrm{Arr} \longrightarrow \mathrm{id}_{\mathrm{Cat}}$.

1.9 Natural transformations from the $\mathrm{Hom}(c,\ )$-functor

1.9.1 Yoneda natural transformation

Given a unital functor $F\colon \mathcal{C} \longrightarrow$ **Set** from a unital category $\mathcal{C}$ to the category of sets, and given an element $u \in Fc$, let

$$\theta_x\colon \mathrm{Hom}_{\mathcal{C}}(c,x) \longrightarrow Fx, \qquad \chi \longmapsto (F\chi)(u), \qquad \big(\chi \in \mathrm{Hom}_{\mathcal{C}}(c,x)\big) \tag{23}$$

Exercise 27 *Show that family $\theta = (\theta_x)_{x \in \operatorname{Ob}\mathcal{C}}$ is a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(c,\ )$ to F.*

1.9.2

We shall refer to θ as the *Yoneda transformation* associated with $u \in Fc$.

1.9.3 Yoneda correspondence

Noting that $u = \theta_c(\mathrm{id}_c)$, we infer that any natural transformation is a Yoneda transformation $\mathrm{Hom}_{\mathcal{C}}(c,\ ) \longrightarrow F$ for at most one element of Fc. In particular, the correspondence

$$Fc \longrightarrow \operatorname{Nat\,tr}\big(\mathrm{Hom}_{\mathcal{C}}(c,\ ), F\big), \qquad u \longmapsto \theta, \tag{24}$$

is injective.

1.9.4

Given any natural transformation

$$\vartheta\colon \operatorname{Hom}_{\mathcal{C}}(c,\ ) \longrightarrow F \tag{25}$$

and a morphism $c \xrightarrow{\chi} x$, let us consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(c,x) \arrow[r, "\vartheta_x"] & Fx \\
\operatorname{Hom}_{\mathcal{C}}(c,c) \arrow[u, "\chi\circ(\ )"] \arrow[r, "\vartheta_c"'] & Fc \arrow[u, "F\chi"']
\end{tikzcd} . \tag{26}$$

The element $\mathrm{id}_c \in \operatorname{Hom}_{\mathcal{C}}(c,c)$ is sent by $\chi\circ(\ )$ to $\chi \in \operatorname{Hom}_{\mathcal{C}}(c,x)$ and by ϑ_c to

$$u_\vartheta := \vartheta_c(\mathrm{id}_c). \tag{27}$$

Hence

$$\vartheta_x(\chi) = \big(F\chi\big)(u_\vartheta).$$

This demonstrates that ϑ is a Yoneda transformation associated with the element u_ϑ defined in (27). We shall refer to it as the *Yoneda element* of ϑ.

1.9.5

It follows that

> *the* Yoneda correspondence, (24), *between natural tranformations* (25) *and elements of set Fc is bijective.* (28)

It depends naturally on c, F, and also $\mathcal{C}$.

1.9.6 Representable functors

We say that a functor $F\colon \mathcal{C} \longrightarrow$ **Set** is *representable* by an object $c \in \mathcal{C}$, if there exists a natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(c,\ ) \xrightarrow[\sim]{\vartheta} F .$$

1.9.7 Universal property of u_ϑ

Exercise 28 *Show that F is representable by an object $c \in \mathcal{C}$ if and only if there exists an element*

$$u \in Fc$$

such that, for any $x \in \mathcal{C}$ *and any element* $v \in Fx$, *there exists a unique morphism*

$$c \xrightarrow{\chi} x$$

such that

$$v = (F\chi)(u).$$

2 Limits

2.1 Two arrow categories

2.1.1 The category of arrows from F to $\mathcal{D}$

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. The objects of category $F{\to}\mathcal{D}$ are the families of arrows indexed by objects of $\mathcal{C}$

$$\phi = \left(Fc \xrightarrow{\phi_c} d\right)_{c\in \mathrm{Ob}\,\mathcal{C}} \qquad (d \in \mathrm{Ob}\,\mathcal{D}), \tag{29}$$

such that for all arrows in $\mathcal{C}$, the diagram

$$\begin{tikzcd} Fc' \arrow[dr, "\phi_{c'}"] & \\ & d \\ Fc \arrow[uu, "F\gamma"] \arrow[ur, "\phi_c"'] & \end{tikzcd} \qquad \left(\gamma \in \mathrm{Hom}_{\mathcal{C}}(c, c')\right) \tag{30}$$

commutes.

2.1.2

If we think of objects of $F \to \mathcal{D}$ as pairs $\left(d, (\varphi_c)_{c\in\mathrm{Ob}\,\mathcal{C}}\right)$, an object d of category $\mathcal{D}$ *equipped* with a family of morphisms $(\varphi_c)_{c\in\mathrm{Ob}\,\mathcal{C}}$, a natural notion of a morphism between such structures $\partial\colon \phi \longrightarrow \phi'$ is: a morphism between the underlying objects

$$\partial \in \mathrm{Hom}_{\mathcal{D}}(d, d')$$

that *respects* the respective families. Explicitly, this means that family ϕ' is produced from family ϕ by postcomposing it with ∂,

$$\phi'_c = \partial \circ \phi_c \qquad (c \in \mathrm{Ob}\,\mathcal{C}),$$

i.e., the diagrams

$$\begin{tikzcd} & d' \\ Fc \arrow[ur, "\phi'_c"] \arrow[dr, "\phi_c"'] & \\ & d \arrow[uu, dashed, "\partial"'] \end{tikzcd} \tag{31}$$

commute.

2.1.3 Inductive (direct) limits

Initial objects of $F{\to}\mathcal{D}$ are called *inductive* (or, *direct*) *limits* of F.

2.1.4 The category of arrows from $\mathcal{D}$ to F

The objects of category $\mathcal{D}{\to}F$ are the families of arrows

$$\phi = \left(d \xrightarrow{\phi_c} Fc\right)_{c \in \operatorname{Ob}\mathcal{C}} \qquad (d \in \operatorname{Ob}\mathcal{D}), \tag{32}$$

such that for all arrows in $\mathcal{C}$, the diagram

$$\begin{tikzcd} & Fc' \\ d \arrow[ur, "\phi_{c'}"] \arrow[dr, "\phi_c"'] & \\ & Fc \arrow[u, "F\gamma"'] \end{tikzcd} \qquad \left(\gamma \in \operatorname{Hom}_{\mathcal{C}}(c, c')\right) \tag{33}$$

commutes.

2.1.5

Morphisms $\partial\colon \phi \longrightarrow \phi'$ are arrows $\partial \in \operatorname{Hom}_{\mathcal{D}}(d, d')$ such that family ϕ is produced from family ϕ' by precomposing it with ∂,

$$\phi_c = \phi'_c \circ \partial \qquad (c \in \operatorname{Ob}\mathcal{C}),$$

i.e., the diagrams

$$\begin{tikzcd} d' \arrow[dr, "\phi'_c"] & \\ & Fc \\ d \arrow[uu, dashed, "\partial"] \arrow[ur, "\phi_c"'] & \end{tikzcd} \tag{34}$$

commute.

2.1.6 Projective (inverse) limits

Terminal objects of $F{\to}\mathcal{D}$ are called *projective* (or, *inverse*) *limits* of F.

2.1.7

If $\mathcal{D}$ is unital, then both $F{\to}\mathcal{D}$ and $\mathcal{D}{\to}F$ are unital. In this case any two inductive limits are isomorphic by a unique isomorphim. Similarly for projective limits.

2.1.8 Notation and terminology

An inductive limit of F is often denoted $\varinjlim F$ and a projective limit is denoted $\varprojlim F$.

2.1.9 Caveat

In early days Category Theory was used and developed particularly vigorously by algebraic topologists. Many of their habits as well as their terminological jargon left a trace in modern practice. For them *limit* means "projective limit", while *colimit* is used in place of "inductive limit". This terminology was reflected in notation: lim *in place of* $\varprojlim$, and colim *in place of* $\varinjlim$.

2.1.10 Duality between projective and inductive limits

By reversing the direction of arrows both in the source and in the target category, we obtain the *dual* functor $F^\circ : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$,

$$F^\circ := (\ )^{\mathrm{op}} \circ F \circ (\ )^{\mathrm{op}}. \tag{35}$$

Projective limits of F become inductive limits of F° and vice-versa.

2.1.11 Limits in a full subcategory

Suppose that $\mathcal{D}$ is a *full* subcategory of $\mathcal{D}'$. Denote by ι the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{D}'$. Suppose that

$$(Fc \to d)_{c \in \mathrm{Ob}\,\mathcal{C}} \tag{36}$$

is an inductive limit of $\iota \circ F$. If $d \in \mathrm{Ob}\,\mathcal{D}$, then (36) is automatically an initial object of category $F{\to}\mathcal{D}$. And dually, if

$$(d \to Fc)_{c \in \mathrm{Ob}\,\mathcal{C}} \tag{37}$$

is a projective limit of $\iota \circ F$, then (37) is automatically a terminal object of category $\mathcal{D}{\to}F$ provided $d \in \mathrm{Ob}\,\mathcal{D}$. This useful observation is frequently invoked.

2.2 Special cases and examples

2.2.1 Suprema and infima of subcategories

Consider the case when $\mathcal{C}$ is a subcategory of $\mathcal{D}$. An inductive limit of the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{C}$ is a *supremum* of $\mathcal{C}$ in $\mathcal{D}$ while its projective limit is an *infimum* of $\mathcal{C}$ in $\mathcal{D}$. In the case when $\mathcal{D}$ is a partially ordered set, we obtain precisely the *supremum* and the *infimum* as they are defined in theory of partially ordered sets.

2.2.2 Initial and terminal objects as limits

A supremum in $\mathcal{D}$ of the empty subcategory $\emptyset$ is an initial object of $\mathcal{D}$. Dually, an infimum of $\emptyset$ in $\mathcal{D}$ is a terminal object of $\mathcal{D}$.

2.2.3 Inductive limit of the $\mathrm{Hom}_{\mathcal{C}}(a,\ )$ functor

Suppose an object $a \in \mathrm{Ob}\,\mathcal{C}$ has a *right* identity ι. Given any object

$$\phi = \big(\mathrm{Hom}_{\mathcal{C}}(a,c) \longrightarrow X\big)_{c \in \mathrm{Ob}\,\mathcal{C}}$$

of the category of arrows from $\mathrm{Hom}_{\mathcal{C}}(a,\ )$ to **Set**, and any $\gamma \in \mathrm{Hom}_{\mathcal{C}}(a,c)$, one has a commutative triangle

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(a,c) \arrow[dr, "\phi_c"] & \\
& X \\
\mathrm{Hom}_{\mathcal{C}}(a,a) \arrow[uu, "\gamma\circ(\ )"] \arrow[ur, "\phi_a"'] &
\end{tikzcd}$$

which shows that $\phi_c(\gamma) = \phi_a(\iota)$. Thus, the maps ϕ_c all have the single element subset $X_1 = \{\phi_a(\iota)\}$ as their target. It follows that, for any single element set $\{*\}$, the unique family of mappings into $\{*\}$

$$\kappa = \big(\mathrm{Hom}_{\mathcal{C}}(a,c) \longrightarrow \{*\}\big)_{c \in \mathrm{Ob}\,\mathcal{C}}$$

is an initial object of the category of arrows from $\mathrm{Hom}_{\mathcal{C}}(a,\ )$ to **Set**, i.e., it is an inductive limit of $\mathrm{Hom}_{\mathcal{C}}(a,\ )$.

2.2.4 Fixed point sets and the sets of orbits

A semigroup $(G,\cdot)$ is the same as a category with a single object $\bullet$. A functor $F\colon G \longrightarrow \mathcal{D}$ is the same as an action of $(G,\cdot)$,

$$\lambda\colon G \longrightarrow \mathrm{End}_{\mathcal{D}}(d),$$

on an object $d = F(\bullet)$ of $\mathcal{D}$.

2.2.5

In the case of $\mathcal{D} = $ **Set**, we speak of G*-sets*. In the case of $\mathcal{D} = k$-**mod**, the category of (left) modules over an associative ring k, we speak of k*-linear representations* of G.

2.2.6

For a G-set X, let

$$X^G := \{x \in X \mid gx = x \text{ for all } g \in G\} \tag{38}$$

(alternatively denotated $\mathrm{Fix}_G X$), and let

$$X_{/G} := X_{/\sim} \tag{39}$$

where $\sim$ is a weakest equivalence relation on X such that

$$x \sim gx \qquad (x \in X,\ g \in G). \tag{40}$$

Exercise 29 *Show that the quotient mapping*

$$X \longrightarrow X_{/G}$$

is an injective limit while the inclusion mapping

$$X^G \hookrightarrow X$$

is a projective limit of the functor $F\colon G \longrightarrow$ **Set**

$$F(\bullet) := X, \qquad Fg := \lambda_g \qquad (g \in G). \tag{41}$$

2.2.7

In the case when X is equipped with a structure of a (left) k-module and the action is by k-linear endomorphisms, X^G is called the *module of G-invariants*, while X_G is defined as $X_{/\sim}$ where $\sim$ is a weakest *k-module congruence* such that (40) holds. In this case X_G is called the *module of G-coinvariants*. These two k-modules supply projective and inductive limits of (41) when the target category of F is the category of k-modules.

2.3 Coproducts and products

2.3.1

Consider a set I as a category with objects being elements of I and the empty class of morphisms. Functors $I \longrightarrow \mathcal{D}$ are the same as I-indexed families of objects

$$(d_i)_{i \in I}. \tag{42}$$

Inductive limits of such functors are called *coproducts*. The corresponding objects that are usually denoted

$$\coprod_{i \in I} d_i$$

are equipped with the arrows

$$\iota_j\colon d_j \longrightarrow \coprod_{i \in I} d_i \tag{43}$$

that are part of their structure.

2.3.2

Projective limits are called *products*. The corresponding objects that are denoted

$$\prod_{i\in I} d_i \qquad \text{or} \qquad \bigtimes_{i\in I} d_i$$

are equipped with the arrows

$$\pi_j\colon \prod_{i\in I} d_i \longrightarrow d_j \tag{44}$$

that are referred as the *product projections*.

2.3.3

Binary coproducts and products are denoted

$$d \sqcup d' \qquad \text{and} \qquad d \times d',$$

respectively. Coproducts and products of a finite family $d_1, \dots, d_n$ are denoted

$$d_1 \sqcup \cdots \sqcup d_n \qquad \text{and} \qquad d_1 \times \cdots \times d_n,$$

respectively.

2.3.4 Coproducts and products in Set

For a family of sets

$$(X_i)_{i\in I}, \tag{45}$$

let

$$X := \bigcup_{i\in I} X_i.$$

Exercise 30 *Show that the set*

$$C := \{(x,i) \in X \times I \mid x \in X_i\}$$

together with the family of embeddings

$$\iota_j\colon X_j \hookrightarrow C, \qquad x \longmapsto (x,j) \qquad (j \in I),$$

is a coproduct of (45) *in the category of sets.*

Exercise 31 *Show that the set*

$$P := \{\mathbf{x}\colon I \longrightarrow X \mid \mathbf{x}(i) \in X_i\} \tag{46}$$

together with the family of evaluation-at-j mappings

$$\pi_j\colon P \longrightarrow X_j, \qquad \mathbf{x} \longmapsto \mathbf{x}(j) \qquad (j \in I),$$

is a product of (45) *in the category of sets.*

2.3.5 Notation and terminology

The *values* of functions $\mathbf{x}\colon I \longrightarrow X$ forming the product are usually written as x_i and are referred to as i-components of $\mathbf{x}$.

2.3.6

An observation that, in the category of sets,

$$\mathrm{Hom}_{\mathbf{Set}}(X,Y) = \prod_{x\in X} Y \tag{47}$$

has numerous consequences for functors $\mathcal{C} \longrightarrow \mathbf{Set}$.

2.3.7

Exercise 32 *Given a family* (42) *of objects in a category* $\mathcal{D}$*, show that there exists a natural bijection*

$$\mathrm{Hom}_{\mathcal{D}}\Big(\coprod_{i\in I} d_i, d'\Big) \longleftrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(d_i, d'). \tag{48}$$

Note that the coproduct on the lef-hand-side of (48) *is formed in* $\mathcal{D}$ *while the product of the* Hom*-sets is formed in the category of sets.*

Exercise 33 *Given a family* (42) *of objects in a category* $\mathcal{D}$*, show that there exists a natural bijection*

$$\mathrm{Hom}_{\mathcal{D}}\Big(d', \prod_{i\in I} d_i\Big) \longleftrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(d', d_i). \tag{49}$$

Note that the product on the lef-hand-side of (49) *is formed in* $\mathcal{D}$ *while the product of the* Hom*-sets is formed in the category of sets.*

2.3.8 Products in the category of ν-ary structures

Given a family $\big((X_i, (\mu_{il})_{l\in L})\big)_{i\in I}$ of ν-ary structures of ν-arity $\nu\colon L \longrightarrow \mathbf{N}$, equip the product of the underlying sets (46) with the product operations

$$\mu_l^{\mathrm{prod}} := \prod_{i\in I} \mu_{il} \qquad (l \in L). \tag{50}$$

In order to view (50) as mappings

$$P \times \cdots \times P \longrightarrow P \qquad (\nu(l) \text{ times}),$$

one needs to identify $P^{\nu(l)}$ with

$$\prod_{i\in I} X_i^{\nu(l)}.$$

If we consider elements of P as functions on I, then each product operation μ_l^{prod} is performed "pointwise", the operation

$$\mu_{il}\colon X_i \times \cdots \times X_i \longrightarrow X_i \qquad (\nu(l) \text{ times})$$

being used at "point" $i \in I$.

2.3.9

Since ν-ary structures on a set X are defined in terms of mappings from products of X to X, the set theoretic product of homomorphisms is a homomorphism and the set theoretic product of structures is a product in the category of ν-ary structures.

2.3.10

The same will be true also for any subcategory of such structures that is closed under formation of product structures. Thus, the set theoretic product of semigroups, monoids, groups, abelian groups, associative rings, k-modules — is a product in the category of, respectively, semigroups, monoids, groups, abelian groups, associative rings, k-modules.

2.3.11

The product of fields (in the category of rings) is not a field, however. In fact, the category of fields lacks even binary products.

Exercise 34 *Show that in the category of fields, a product of fields E and F that have 2 and 3 elements does not exist.*

Exercise 35 *Show that $\mathbf{Q}$ with $\pi_1 = \pi_2 = \mathrm{id}_{\mathbf{Q}}$ is a product of $\mathbf{Q}$ with $\mathbf{Q}$ in the category of fields. What is a product of $\mathbf{Q}$ with any field F?*

2.3.12 Coproducts in the category of commutative monoids

For a family $(M_i)_{i\in I}$ of commutative monoids denote by

$$\bigoplus_{i\in I} M_i := \left\{\mathbf{m} \in \textstyle\prod_{i\in I} M_i \;\middle|\; \operatorname{supp}\mathbf{m} \text{ is finite}\right\} \tag{51}$$

where the *support* of $\mathbf{m}$ is the set of i where $\mathbf{m}$ does not vanish,

$$\operatorname{supp}\mathbf{m} := \{i \in I \mid m_i \neq 0\}. \tag{52}$$

We employ *additive* notation for the binary operation in a commutative monoid, hence the identity element is referred as "zero" and denoted 0. Consider the embeddings

$$\iota_j : M_j \hookrightarrow \bigoplus_{i \in I} M_i, \qquad m \longmapsto \delta_j(m), \tag{53}$$

where

$$\delta_j(m) : I \longrightarrow \bigcup_{i \in I} M_i, \qquad \delta_j(m)(i) := \begin{cases} m & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}$$

is the element of the product whose j-th component equals m and all the other components are zero.

Exercise 36 *Show that (51) equipped with the family of embeddings (53) is a coproduct of $(M_i)_{i \in I}$ in the category of commutative monoids.*

2.3.13 Terminology and notation

We refer to (51) as the *direct sum* of a family $(M_i)_{i \in I}$. This explains why the notation

$$\sum_{i \in I} m_i \tag{54}$$

is used to denote elements of (51) instead of $(m_i)_{i \in I}$ or functional notation. The summation symbol in (54) is employed purely formally. In this notation $\delta_j(m)$ becomes the formal sum (54) for which all m_i but m_j are zero and $m_j = m$.

2.3.14

For $m_1 \in M_{i_1}, \dots, m_n \in M_{i_n}$ we simply write

$$m_1 + \cdots + m_n \tag{55}$$

and consider it as an element of the direct sum. This corresponds to the function $\mathbf{m} : I \longrightarrow \bigcup_{i \in I} M_i$,

$$\mathbf{m}(i) = \begin{cases} m_k & \text{when } i = i_k \\ 0 & \text{otherwise} \end{cases}$$

2.3.15 Coproducts in the category of abelian groups

The category of abelian groups is a full subcategory of the category of commutative monoids and the direct sum (51) is an abelian group if each M_i is an abelian group. In view of the observation made in Section 2.1.11, the direct sum of abelian groups is automatically a coproduct in the category of abelian groups.

2.3.16 Coproducts in the category of k-modules

The direct sum $\bigoplus_{i\in I} M_i$ of a family of k-modules $(M_i)_{i\in I}$ is a k-submodule of the direct product $\prod_{i\in I} M_i$. The same argument as in the case of the category of commutative monoids shows that the direct sum provides a construction of a coproduct in the category of abelian groups, applies also to the category of k-modules.

2.3.17

Similarly for (A, B)-bimodules and the unitary variants of the categories of modules and bimodules.

2.3.18 Coproducts in the category of commutative semigroups

If we express the elements of (51) as formal sums (54), then a small modification allows to describe the coproduct of a family $(M_i)_{i\in I}$ of commutative semigroups as the set of *formal* sums

$$\sum_{i\in J} m_i \tag{56}$$

over all *finite nonempty* subsets $J \subseteq I$. Addition of such sums is performed in an obvious manner:

$$\left(\sum_{i\in J} m_i\right) + \left(\sum_{i\in J'} m'_i\right) := \sum_{i\in J\cup J'} m''_i \tag{57}$$

where

$$m''_i = \begin{cases} m_i + m'_i & \text{if } i \in J \cap J' \\ m_i & \text{if } i \in J \setminus J' \\ m'_i & \text{if } i \in J' \setminus J \end{cases}.$$

2.3.19

As a set such formal sums can be realized as members of the disjoint union of Cartesian products of finite nonenpty subfamilies $(M_i)_{i\in J}$,

$$\coprod_{J\in\mathscr{P}^*_{\mathrm{fin}}(I)} \prod_{i\in J} M_i.$$

where $\mathscr{P}^*_{\mathrm{fin}}(I)$ denotes the set of finite nonempty subsets of I.

2.3.20

For $I = \{1, 2\}$, this becomes

$$M_1 \sqcup M_2 \sqcup M_1{\times}M_2. \tag{58}$$

Exercise 37 *Describe $x+y$ when x and y are in each of the following subsets of* (58)*:*

$$M1, \qquad M_2 \qquad \textit{and} \qquad M_1{\times}M_2.$$

2.4 Coproducts in the category of semigroups

2.4.1 Semigroups of words

For a set X, consider the disjoint union of Cartesian powers of X,

$$WX := X \sqcup X{\times}X \sqcup X{\times}X{\times}X \sqcup \cdots \tag{59}$$

equipped with the concatenation multiplication:

$$(x_1,\dots,x_q)\cdot(x'_1,\dots,x'_r) := (x_1,\dots,x_q,x'_1,\dots,x'_r) \tag{60}$$

Multiplication defined by (60) is associative, and WX equipped with it will be referred as *the semigroup of words on an alphabet X*.

2.4.2

Note that

$$(x_1,\dots,x_q) = x_1\cdots x_q. \tag{61}$$

In particular, every "word" $(x_1,\dots,x_q)$ of length k is a product of k words of length 1 and such a representation is *unique*.

2.4.3

In view of (61) any mapping $f\colon X \longrightarrow M$ into a semigroup M uniquely extends to a homomorphism of semigroups $\tilde{f}\colon WX \longrightarrow M$,

$$\tilde{f}\big((x_1,\dots,x_q)\big) = \tilde{f}(x_1\cdots x_q) = f(x_1)\cdots f(x_q).$$

2.4.4 The tautological epimorphism

For any semigroup M, the identity mapping id_M induces a homomorphism $p = \widetilde{\mathrm{id}}_M$ from WM onto M. We shall refer to it as the *tautological epimorphism* associated with M.

2.4.5 A construction of a coproduct of a family of semigroups

Given a family of semigroups $(M_i)_{i\in I}$, let us form the semigroup of words on the alphabet

$$M^{\sqcup} := \coprod_{i\in I} M_i. \tag{62}$$

Since WM_i is naturally identified with a subsemigroup of $W(M^{\sqcup})$, the *kernel* congruence of the tautological epimorphism $p_i\colon WM_i \twoheadrightarrow M_i$, namely

$$w \sim_i w' \qquad \text{if} \qquad p_i(w) = p_i(w') \qquad (w, w' \in WM_i), \tag{63}$$

can be considered also to be a binary relation on $W(M^{\sqcup})$.

2.4.6 The free product of semigroups

Let $\sim$ be a weakest congruence on $W(M^{\sqcup})$ stronger than each $\sim_i$. The quotient semigroup

$$\mathop{\ast}_{i\in I} M_i := W(M^{\sqcup})_{/\sim} \tag{64}$$

is referred to as the *free product* of the family of semigroups $(M_i)_{i\in I}$.

Exercise 38 *Show that the inclusions of sets $M_j \hookrightarrow M^{\sqcup}$ induce injective homomorphisms of semigroups*

$$M_j \longrightarrow \mathop{\ast}_{i\in I} M_i \qquad (j \in I). \tag{65}$$

2.4.7 The universal property of the free product

A family of semigroup homomorphisms $f_i\colon M_i \longrightarrow N$ gives rise to a single mapping $f^{\sqcup}\colon M^{\sqcup} \longrightarrow N$ and this, in turn, gives rise to a unique homomorphism of semigroups

$$\tilde{f}\colon W(M^{\sqcup}) \longrightarrow N \tag{66}$$

such that its restriction to

$$WM_j \subseteq W(M^{\sqcup})$$

equals $f_j \circ p_j$ where $p_j\colon WM_j \twoheadrightarrow M_j$ is the corresponding tautological epimorphism. Since

$$p_j(w) = p_j(w') \qquad (w, w' \in WM_j)$$

precisely when $w \sim_j w'$,

$$\tilde{f}(w) = \tilde{f}(w') \tag{67}$$

for all such w and w'. Since $\tilde{f}$ is a homomorphism, (67) holds for *any* $w, w' \in W(M^{\sqcup})$ such that $w \sim w'$. Hence $\tilde{f}$ passes to the quotient (64), proving at once that the family (65) is a coproduct of $(M_i)_{i\in I}$ in the category of semigroups.

2.4.8 The semigroup of alternating words

We shall make the construction of the free product more explicit in the case of two semigroups M and N.

Consider the subset of $W(M \sqcup N)$ consisting of "alternating words" $(l_1, \dots, l_q)$, i.e., words such that no two consecutive l_i and l_{i+1} belong to M or N. Let us multiply such words according to the rule

$$(l_1, \dots, l_q) \cdot (l'_1, \dots, l'_r) = \begin{cases} (l_1, \dots, l_q l'_1, \dots, l'_r) & \text{if } l_q, l'_1 \in M \text{ or } l_q, l'_1 \in N \\ (l_1, \dots, l_q, l'_1, \dots, l'_r) & \text{otherwise} \end{cases} \tag{68}$$

Exercise 39 *Inclusions of M and N into the set of alternating words are homomorphisms and therefore induce a homomorphism from $M * N$ to the semigroup of alternating words. Show that it is an isomorphism. (Hint. Show that the semigroup of alternating words together with the inclusions of M and N is a coproduct of semigroups M and N.)*

2.5 Coproducts in the category of monoids

2.5.1

A coproduct of a family of monoids is a *free product of monoids*

$$\mathop{\ast}_{i \in I} M_i \;:=\; W_{\mathrm{un}}\big(M^{\sqcup}\big)_{/\sim}. \tag{69}$$

The argument parallels the argument for semigroups. The only difference is that instead of the semigroup of words WX, we employ the *monoid of words*

$$W_{\mathrm{un}} X \;:=\; \coprod_{q \geq 0} X^q. \tag{70}$$

2.5.2

Note that

$$ww' \;\sim_i\; e,$$

whenever both w and w' belong to the same M_i and their product in M_i equals the identity element e_{M_i}.

2.5.3

It follows that, if each M_i is a group, then the equivalence class of each element in $M^{\sqcup}$ is invertible in (69). But every element in (69) is a product of equivalence classes of elements of $M^{\sqcup}$, hence every element in (69) is invertible.

2.5.4 Coproducts in the category of groups

This demonstrates that the free product of a family of monoids $(M_i)_{i\in I}$ is a group if every member of the family is a group. In particular, a coproduct of a family of groups in the category of monoids is a group. Since **Grp** is a full subcategory of **Mon**, it follows that (69) is also a coproduct in the category of groups.

2.6 Pushouts

2.6.1

Consider the category $\mathbf{2}_{\mathrm{cs}}$ consisting of 2 arrows with the common source

$$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd} \tag{71}$$

Functors $F\colon \mathbf{2}_{\mathrm{cs}} \longrightarrow \mathcal{D}$ are the same as pairs of arrows in $\mathcal{D}$

$$\begin{tikzcd} & d'' \\ d \arrow[ur, "\alpha''"] \arrow[dr, "\alpha'"'] & \\ & d' \end{tikzcd} \tag{72}$$

with a common source. Objects of $F{\rightarrow}\mathcal{D}$ are the same as pairs of morphisms

$$\begin{tikzcd} d'' \arrow[dr, "\xi''"] & \\ & x \\ d' \arrow[ur, "\xi'"'] & \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd}
 & d'' \arrow[dr, "\xi''"] & \\
d \arrow[ur, "\alpha''"] \arrow[dr, "\alpha'"'] & & x \\
 & d' \arrow[ur, "\xi'"'] &
\end{tikzcd}$$

commutes.

2.6.2

An initial object of $F{\to}\mathcal{D}$ is called in this case a *pushout* of diagram (72).

2.6.3 An example: pushouts in Set

For a pair of mappings with the common source

$$\begin{tikzcd}
 & X'' \\
X \arrow[ur, "f''"] \arrow[dr, "f'"'] & \\
 & X'
\end{tikzcd} \tag{73}$$

consider a weakest equivalence relation on the disjoint union $X' \sqcup X''$, such that

$$x' \sim x'' \quad \text{if there exists } x \in X \text{ such that } f'(x) = x' \text{ and } f''(x) = x''. \tag{74}$$

Let $X \overset{X}{\sqcup} X''$ denote the quotient of $X' \sqcup X''$ by $\sim$, and let

$$\begin{tikzcd}
X'' \arrow[dr, "g''"] & \\
 & X \overset{X}{\sqcup} X'' \\
X' \arrow[ur, "g'"'] &
\end{tikzcd} \tag{75}$$

be the mappings obtained by composing the canonical inclusion mappings

$$\iota' : X' \hookrightarrow X \overset{X}{\sqcup} X'' \qquad \text{and} \qquad \iota'' : X'' \hookrightarrow X \overset{X}{\sqcup} X''$$

with the quotient mapping

$$q\colon X' \sqcup X'' \longrightarrow X \overset{X}{\sqcup} X''.$$

Exercise 40 *Show that (75) is a pushout of (73).*

2.7 Pullbacks

2.7.1

Consider the category $\mathbf{2}_{\mathrm{ct}}$ consisting of 2 arrows with the common target

$$\begin{tikzcd} \bullet \arrow[dr] & \\ & \bullet \\ \bullet \arrow[ur] & \end{tikzcd} \tag{76}$$

Functors $F\colon \mathbf{2}_{\mathrm{ct}} \longrightarrow \mathcal{D}$ are the same as pairs of arrows in $\mathcal{D}$

$$\begin{tikzcd} d'' \arrow[dr, "\alpha''"] & \\ & d \\ d' \arrow[ur, "\alpha'"'] & \end{tikzcd} \tag{77}$$

with a common source. Objects of $\mathcal{D}{\to}F$ are the same as pairs of morphisms

$$\begin{tikzcd} & d'' \\ x \arrow[ur, "\xi''"] \arrow[dr, "\xi'"'] & \\ & d' \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} & d'' \arrow[dr, "\alpha''"] & \\ x \arrow[ur, "\xi''"] \arrow[dr, "\xi'"'] & & d \\ & d' \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

commutes.

2.7.2

A terminal object of $\mathcal{D}{\to}F$ is called in this case a *pullback* of diagram (72).

2.7.3 An example: pullbacks in Set

For a pair of mappings with the common target

$$\begin{tikzcd} X'' \arrow[dr, "f''"] & \\ & X \\ X' \arrow[ur, "f'"'] & \end{tikzcd} \tag{78}$$

consider the subset of the Cartesian product

$$X' \times_X X'' := \{(x', x'') \in X' \times X'' \mid f'(x') = f''(x'')\}. \tag{79}$$

This set is called the *fibred product* of X' and X'' *over* X. Let

$$\begin{tikzcd} & X'' \\ X' \times_X X'' \arrow[ur, "p''"] \arrow[dr, "p'"'] & \\ & X' \end{tikzcd} \tag{80}$$

be the mappings obtained by composing the canonical projection mappings

$$\pi' : X' \times X'' \longrightarrow X' \qquad \text{and} \qquad \pi'' : X' \times X'' \longrightarrow X''$$

with the canonical inclusion mapping

$$\iota : X' \times_X X'' \hookrightarrow X' \times X''.$$

Exercise 41 *Show that* (80) *is a pullback of* (78).

2.7.4 Cartesian and co-Cartesian squares

A commutative diagram in a category $\mathcal{D}$

$$\begin{tikzcd} & \bullet \arrow[dr, "\beta''"] & \\ \bullet \arrow[ur, "\alpha''"] \arrow[dr, "\alpha'"'] & & \bullet \\ & \bullet \arrow[ur, "\beta'"'] & \end{tikzcd} \tag{81}$$

is said to be *Cartesian square* if

$$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur, "\alpha''"] \arrow[dr, "\alpha'"'] & \\ & \bullet \end{tikzcd} \tag{82}$$

is a pullback of

$$\begin{tikzcd} \bullet \arrow[dr, "\beta''"] & \\ & \bullet\ , \\ \bullet \arrow[ur, "\beta'"'] & \end{tikzcd} \tag{83}$$

and it is said to be a *co-Cartesian square* if (83) is a pushout of (82).

2.7.5 Pullbacks of arrows

It became a custom to call an arrow α'' to be a *pullback* of an arrow β' if there exist arrows α' and β'' forming a Cartesian square (81). More precisely, α'' is said in this case to be a pullback *of* β' *by* β''.

2.7.6 Pushouts of arrows

Similarly, an arrow β' is said to be a *pushout* of an arrow α'' if there exist arrows α' and β'' forming a co-Cartesian square (81). More precisely, β' is said in this case to be a pushout *of* α'' *by* α'.

Exercise 42 *Show that a pullback of a monomorphism is a monomorphism, and a pushout of an epimorphism is an epimorphism.*

Exercise 43 *For any sets X' and X'', show that*

$$\begin{tikzcd} & X'' \arrow[dr, "\kappa''"] & \\ X' \cap X'' \arrow[ur, "\iota''"] \arrow[dr, "\iota'"'] & & X' \cup X''\ , \\ & X' \arrow[ur, "\kappa'"'] & \end{tikzcd} \tag{84}$$

is a Cartesian and and a co-Cartesian square where $\iota', \iota'', \kappa', \kappa''$ are the canonical inclusion mappings.

2.7.7 Composition of commutative squares

Exercise 44 *Show that if both*

$$
\begin{tikzcd}
\bullet \arrow[d, "\alpha"'] & \bullet \arrow[l, "\beta'"'] \arrow[d, "\alpha'"] \\
\bullet & \bullet \arrow[l, "\beta"]
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\bullet \arrow[d, "\alpha'"'] & \bullet \arrow[l, "\gamma'"'] \arrow[d, "\alpha''"] \\
\bullet & \bullet \arrow[l, "\gamma"]
\end{tikzcd}
\tag{85}
$$

are Cartesian (resp. co-Cartesian), then the square

$$
\begin{tikzcd}
\bullet \arrow[d, "\alpha"'] & \bullet \arrow[l, "\beta\circ\gamma'"'] \arrow[d, "\alpha''"] \\
\bullet & \bullet \arrow[l, "\beta\circ\gamma"]
\end{tikzcd}
\tag{86}
$$

is Cartesian (resp. co-Cartesian).

Exercise 45 *Show that if square* (86) *is Cartesian, then the right square in* (85) *is Cartesian. If, on the other hand, it is co-Cartesian, then the left square in* (85) *is co-Cartesian.*

2.8 Equalizers and coequalizers

2.8.1

Consider the category $\mathbf{2}_{\rightrightarrows}$ consisting of 2 arrows

$$\bullet \rightrightarrows \bullet$$

with the common source and the common target. Functors $F\colon \mathbf{2}_{\rightrightarrows} \longrightarrow \mathcal{D}$ are the same as parallel pairs of arrows in $\mathcal{D}$, cf. (1), Objects of $F{\to}\mathcal{D}$ are the same as morphisms $d' \xrightarrow{\xi'} x$ such that

$$\xi' \circ \alpha = \xi' \circ \beta$$

while objects of $\mathcal{D}{\to}F$ are the same as morphisms $x \xrightarrow{\xi} d$ such that

$$\alpha \circ \xi = \beta \circ \xi.$$

2.8.2

An initial object of $F{\to}\mathcal{D}$ is called in this case a *coequalizer* of a *parallel pair* of arrows (1), while a terminal object of $\mathcal{D}{\to}F$ is called an *equalizer* of (1).

Exercise 46 *Show that a coequalizer of a parallel pair is an epimorphism while an equalizer is a monomorphism.*

Exercise 47 *If ϵ is a coequalizer of a parallel pair, then in any commutative square*

$$
\begin{tikzcd}
\bullet \arrow[d, tail, "\mu"'] & \bullet \arrow[l] \arrow[d, two heads, "\epsilon"] \\
\bullet & \bullet \arrow[l]
\end{tikzcd}
$$

with μ being a monomorphism, there exists an arrow

$$
\begin{tikzcd}
\bullet & \\
& \bullet \arrow[ul, dashed]
\end{tikzcd}
$$

such that both triangles in the diagram

$$
\begin{tikzcd}
\bullet \arrow[d, tail, "\mu"'] & \bullet \arrow[l] \arrow[d, two heads, "\epsilon"] \\
\bullet & \bullet \arrow[l] \arrow[ul, dashed]
\end{tikzcd}
$$

commute.

2.8.3 Strong epimorphisms

Epimorphisms possessing the property described in Exercise 47 are said to be *strong*.

Exercise 48 *State the dual property for monomorphisms and prove that every equalizer is a strong monomorphism.*

Exercise 49 *Show that composition of strong epimorphisms produces a strong epimorphism.*

Exercise 50 *Suppose that $\alpha \circ \beta$ is a strong epimorphism. Show that β is a strong epimorphism.*

2.8.4

A mononomorphism which is also an epimorphism does not need to possess an inverse. It does, however, if it is strong.

Exercise 51 *Show that a morphism is an isomorphism if and only if it is a monomorphism and a strong epimorphism.*

Exercise 52 *If ϵ is a strong epimorphism and μ is a monomorphism such that $\mu \circ \epsilon$ is an epimorphism, then μ is an isomorphism.*

2.8.5 Extremal epimorphisms

Epimorphisms satisfying the property in Exercise 52 are said to be *extremal*. In other words, every strong epimorphism is extremal.

2.8.6 Category of factorizations of an arrow

Given an arrow α, consider the category $\mathrm{Fact}(\alpha)$ whose objects are representations of α as $\beta \circ \gamma$ and morphisms from a factorization $\beta \circ \gamma$ to a factorization $\beta' \circ \gamma'$ are arrows ϕ such that

$$\beta = \beta' \circ \phi \qquad \text{and} \qquad \phi \circ \gamma = \gamma.$$

Morphisms correspond to commutative diagrams

$$\begin{tikzcd} \bullet \arrow[d, equal] & \bullet \arrow[l, "\beta"'] \arrow[d, "\phi"] & \bullet \arrow[l, "\gamma"'] \arrow[d, equal] \\ \bullet & \bullet \arrow[l, "\beta'"] & \bullet \arrow[l, "\gamma'"] \end{tikzcd}$$

2.8.7 Mono-strong epi factorizations

If α can be represented as $\mu \circ \epsilon$, where μ is a monomorphism and ϵ is a strong epimorphism, we say that α admits a *mono-strong epi factorization*.

Exercise 53 *Show that there exists a unique morphism v from a mono-strong epi factorization $\alpha = \mu \circ \epsilon$ to any mono-epi factorization $\alpha = \mu' \circ \epsilon'$.*

2.8.8

In other words, mono-strong epi factorizations are *initial* objects of the subcategory $\mathrm{Mono\text{-}epi}(\alpha)$ of mono-epi factorizations of α.

2.8.9

Dually, strong mono-epi factorizations are terminal objects of $\mathrm{Mono\text{-}epi}(\alpha)$. This follows from the fact that a mono-epi factorization $\alpha = \mu \circ \epsilon$ in the opposite category becomes

$$\alpha^{\mathrm{op}} = (\mu \circ \epsilon)^{\mathrm{op}} = \epsilon^{\mathrm{op}} \circ \mu^{\mathrm{op}}$$

with ϵ^{op} being a monomorphism and μ^{op} being an epimorphism. In particular, the concept of a mono-epi factorization is self dual. Note that ϵ^{op} is a strong monomorphism if and only if ϵ is a strong epimorphism.

2.9 Kernels and cokernels

2.9.1

In a category *with zero* an equalizer of the parallel pair

$$d' \overset{\alpha}{\underset{0}{\leftleftarrows}} d \tag{87}$$

is called a *kernel* of α, while its coequalizer is called a *cokernel* of α.

2.9.2

The following exercise provides a simple yet surprisingly useful criterion for a morphism to be zero.

Exercise 54 *In a commutative diagram*

$$\begin{tikzcd}
& \bullet \arrow[d, "\xi"'] & \\
\bullet & \bullet \arrow[l, "\alpha"'] \arrow[d, "\alpha'"'] & \bullet \arrow[l, "\beta"'] \arrow[dl, tail] \\
& \bullet &
\end{tikzcd}$$

one has

$$\xi = 0 \qquad \textit{if and only if} \qquad \alpha \circ \xi = 0 \quad \textit{and} \quad \alpha' \circ \xi = 0\,.$$

2.9.3 Zero morphisms as kernels and cokernels

Exercise 55 *If a kernel of α is a zero morphism, then its source is a zero object,*

$$d \longleftarrow 0\ .$$

Dually, if a cokernel of α is a zero morphism, then its target is a zero object,

$$0 \longleftarrow d'\ .$$

Exercise 56 *Show that a cokernel of the composite arrow $\alpha \circ \beta$ is a cokernel of α if β is an epimorphism and, vice-versa, a cokernel of α is cokernel of $\alpha \circ \beta$.*

2.9.4

Dually, a kernel of $\alpha \circ \beta$ is a kernel of β if α is a monomorphism and, vice-versa, a kernel of β is a kernel of $\alpha \circ \beta$.

Exercise 57 *Given a composable pair of arrows*

$$\begin{tikzcd} \bullet & \bullet \arrow[l, "\alpha"'] & \bullet \arrow[l, "\beta"'] \end{tikzcd}$$

show that β is a kernel of α if and only if the square

$$\begin{tikzcd} 0 \arrow[d] & \bullet \arrow[l] \arrow[d, "\beta"] \\ \bullet & \bullet \arrow[l, "\alpha"] \end{tikzcd} \tag{88}$$

is Cartesian.

Exercise 58 *Dually, show that α is a cokernel of β if and only if square* (88) *is co-Cartesian.*

2.9.5 An example: the category of monomorphisms

Let $\operatorname{Mono}\mathcal{C}$ denote the *category of monomorphisms* in $\mathcal{C}$, i.e., the full subacategory of the category of arrows $\operatorname{Arr}\mathcal{C}$, whose objects are monomorphisms in $\mathcal{C}$.

Exercise 59 *Show that a morphism ϕ from λ to μ,*

$$\begin{tikzcd} \bullet \arrow[d, tail, "\mu"'] & \bullet \arrow[l, dotted, "\phi_s"'] \arrow[d, tail, "\lambda"] \\ \bullet & \bullet \arrow[l, dotted, "\phi_t"] \end{tikzcd} \tag{89}$$

is a mono, respectively, an epimorphism, in $\operatorname{Mono}\mathcal{C}$*, if and only if ϕ_t is a mono, respectively, an epimorphism, in $\mathcal{C}$.*

Exercise 60 *Suppose that $\mathcal{C}$ is a category with zero object. Describe zero objects in* $\operatorname{Mono}\mathcal{C}$ *and show that a morphism ι from κ to λ,*

$$\begin{tikzcd} \bullet \arrow[d, tail, "\lambda"'] & \bullet \arrow[l, dotted, "\iota_s"'] \arrow[d, tail, "\kappa"] \\ \bullet & \bullet \arrow[l, dotted, "\iota_t"] \end{tikzcd} \tag{90}$$

is a kernel of ϕ, cf. (89)*, in* $\operatorname{Mono}\mathcal{C}$ *if and only if ϕ_t is a kernel of ϕ_t in $\mathcal{C}$ and* (90) *is a Cartesian square.*

2.9.6 Kernels versus cokernels

Exercise 61 *Show that if α is a cokernel of β, and γ is a kernel of α, then α is a cokernel of γ.*

2.9.7 An image of a morphism

If a kernel of a cokernel of β exists, it is said to be an *image* of β. Exercise 61 says that a cokernel of β is automatically a cokernel of an image of βo.

2.9.8 A coimage of a morphism

Dually, if β is a kernel of α, and ∂ is a cokernel of β, then ∂ is said to be a *coimage* of α and β is a kernel of a∂, i.e., a kernel of α is automatically a kernel of a coimage of α.

2.9.9 Functoriality of a kernel

Given a commutative diagram

$$\begin{tikzcd} a' \arrow[d, "\phi"'] & c' \arrow[l, "\alpha'"'] \arrow[d, "\phi'"] & b' \arrow[l, "\beta'"'] \\ a & c \arrow[l, "\alpha"'] & b \arrow[l, "\beta"'] \end{tikzcd} , \tag{91}$$

with β being a kernel of α and β' being a kernel of α', there exists a unique arrow ϕ'' such that the diagram

$$\begin{tikzcd} a' \arrow[d, "\phi"'] & c' \arrow[l, "\alpha'"'] \arrow[d, "\phi'"] & b' \arrow[l, "\beta'"'] \arrow[d, "\phi''"] \\ a & c \arrow[l, "\alpha"'] & b \arrow[l, "\beta"'] \end{tikzcd}$$

commutes.

2.9.10 A kernel of a pullback

Given a diagram with a Cartesian square

$$\begin{tikzcd} a' \arrow[d, "\phi"'] & c' \arrow[l, "\alpha'"'] \arrow[d, "\phi'"] & \\ a & c \arrow[l, "\alpha"'] & b \arrow[l, "\beta"'] \end{tikzcd} \tag{92}$$

and β being a kernel of α, there exists a unique arrow $c' \xleftarrow{\beta'} b$ such that

$$0 = \alpha \circ \beta' \qquad \text{and} \qquad \beta = \phi' \circ \beta'.$$

2.9.11

If an arrow $x \xrightarrow{\xi} c'$ satisfies $\alpha' \circ \xi = 0$, then also

$$\alpha \circ (\phi' \circ \xi) = \phi \circ (\alpha' \circ \xi) = 0.$$

Recalling that β is a kernel of α, we obtain a factorization of $\phi' \circ \xi$,

$$\phi' \circ \xi = \beta \circ \tilde{\xi}$$

for some $\tilde{\xi}$. Note that

$$\alpha' \circ (\beta' \circ \tilde{\xi}) = 0 = \alpha' \circ \xi \qquad \text{and} \qquad \phi' \circ (\beta' \circ \tilde{\xi}) = \beta \circ \tilde{\xi} = \phi' \circ \xi.$$

The universal property of pullback implies that

$$\beta' \circ \tilde{\xi} = \xi. \tag{93}$$

Uniqueness of $\tilde{\xi}$ satisfying (93) follows if we notice that β' is a monomorphism while $\beta' \circ \tilde{\xi}$ is given.

2.9.12

We established that the unique arrow β' that makes the diagram

$$\begin{tikzcd}
a' \arrow[d, "\phi"'] & c' \arrow[l, "\alpha'"'] \arrow[d, "\phi'"] & b \arrow[l, "\beta'"'] \arrow[d, equal] \\
a & c \arrow[l, "\alpha"] & b \arrow[l, "\beta"]
\end{tikzcd} \tag{94}$$

commute, is a kernel of the pulled-back arrow α'.

Exercise 62 *State and prove the corresponding property of a cokernel of a pushout.*

2.10 Kernel–Cartesian and cokernel–co-Cartesian lemmata

2.10.1 Kernel–Cartesian lemmata

Consider a morphism of composable pairs

$$
\begin{tikzcd}
\bullet \arrow[d, "\phi''"'] & \bullet \arrow[l, "\alpha"'] \arrow[d, "\phi"'] & \bullet \arrow[l, "\beta"'] \arrow[d, "\phi'"] \\
\bullet & \bullet \arrow[l, "\alpha'"] & \bullet \arrow[l, "\beta'"]
\end{tikzcd}
\tag{95}
$$

Lemma 2.1 *If*

- *β' is a kernel of α' and*
- *the $\beta\phi$-square is Cartesian,*

then β is a kernel of α.

Proof. If $\alpha \circ \xi = 0$, then

$$\alpha' \circ (\phi \circ \xi) = \phi'' \circ \alpha \circ \xi = 0,$$

hence $\phi \circ \xi$ uniquely factorizes

$$\phi \circ \xi = \beta' \circ \xi'$$

through β',

$$
\begin{tikzcd}
& \bullet \arrow[ddl, "0"'] \arrow[d, "\xi"'] \arrow[ddr, dashed, "\xi'"] & \\
& \bullet \arrow[d, "\phi"'] & \\
\bullet & \bullet \arrow[l, "\alpha'"] & \bullet \arrow[l, "\beta'"]
\end{tikzcd}
$$

and, by the universal property of a Cartesian square, there exists a unique arrow $\tilde{\xi}$ such that

$$\xi = \beta \circ \tilde{\xi} \tag{96}$$

and

$$\xi' = \phi' \circ \tilde{\xi}. \tag{97}$$

If $\tilde{\xi}$ is *any* arrow that satisfies identity (96), then

$$\beta' \circ \phi' \circ \tilde{\xi} = \phi \circ \beta \circ \tilde{\xi} = \phi \circ \xi = \beta' \circ \times'$$

which implies identity (97) since β' is a monomorphism. □

Lemma 2.2 *If*

- β *is a kernel of* α ,
- $\alpha' \circ \beta' = 0$, *and*
- β' *and* ϕ'' *are monomorphisms,*

then the $\beta\phi$ *-square is Cartesian.*

Exercise 63 *Prove Lemma 2.2.*

Corollary 2.3 *If*

- β' *is a kernel of* α' *and*
- ϕ'' *is a monomorphism,*

then

$$\text{the } \beta\phi\text{-square is Cartesian.} \quad \Longleftrightarrow \quad \beta \text{ is a kernel of } \alpha \tag{98}$$

□

2.10.2 Cokernel–co-Cartesian lemmata

The following are the dual versions of Lemmata 2.1, 2.2, and of Corollary ??.

Lemma 2.4 *If*

- α *is a cokernel of* β *and*
- *the* $\alpha\phi$ *-square is co-Cartesian,*

then α' *is a cokernel of* β' . □

Lemma 2.5 *If*

- α' *is a cokernel of* β' ,
- $\alpha \circ \beta = 0$, *and*
- α *and* ϕ' *are epimorphisms,*

then the $\alpha\phi$ *-square is co-Cartesian.* □

Corollary 2.6 *If*

- α *is a cokernel of* β *and*
- ϕ' *is a epimorphism,*

then

$$\text{the } \alpha\phi\text{-square is co-Cartesian.} \quad \Longleftrightarrow \quad \alpha' \text{ is a cokernel of } \beta' \tag{99}$$

□

2.10.3

Diagramatically the above lemmata and corollaries can be represented as

$$
\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \arrow[dr, phantom, "\mathrm{Car}"] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l, "\ker"]
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l, "\ker"'] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}
$$

Lemma 2.1

$$
\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \arrow[dr, phantom, "\mathrm{Car}"] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}
\quad \Longleftarrow \quad
\begin{tikzcd}
\bullet \arrow[d, tail] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l, "\ker"'] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[ll, bend left, tail, "0"]
\end{tikzcd}
$$

Lemma 2.2

$$
\begin{tikzcd}
\bullet \arrow[d] \arrow[dr, phantom, "\text{co-Car}"] & \bullet \arrow[l, "\mathrm{coker}"'] \arrow[d] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l, "\mathrm{coker}"] & \bullet \arrow[l]
\end{tikzcd}
$$

Lemma 2.4

$$
\begin{tikzcd}
\bullet \arrow[d] \arrow[dr, phantom, "\text{co-Car}"] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}
\quad \Longleftarrow \quad
\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l, two heads] \arrow[d] & \bullet \arrow[l] \arrow[d, two heads] \arrow[ll, bend right, "0"'] \\
\bullet & \bullet \arrow[l, "\mathrm{coker}"] & \bullet \arrow[l]
\end{tikzcd}
$$

Lemma 2.5

$$
\begin{tikzcd}
\bullet \arrow[d, tail] & \bullet \arrow[l] \arrow[d] \arrow[dr, phantom, "\mathrm{Car}"] & \bullet \arrow[l] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l, "\ker"]
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
\bullet \arrow[d, tail] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l, "\ker"'] \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l, "\ker"]
\end{tikzcd}
$$

Corollary 2.3

$$
\begin{tikzcd}
\bullet \arrow[d] \arrow[dr, phantom, "\text{co-Car}"] & \bullet \arrow[l, "\mathrm{coker}"'] \arrow[d] & \bullet \arrow[l] \arrow[d, two heads] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l, "\mathrm{coker}"'] \arrow[d] & \bullet \arrow[l] \arrow[d, two heads] \\
\bullet & \bullet \arrow[l, "\mathrm{coker}"] & \bullet \arrow[l]
\end{tikzcd}
$$

Corollary 2.6

3 Graded categories and categories of graded objects

3.1 The category of X-graded objects

3.1.1

Given a set X, an X-*graded* object of a category $\mathcal{C}$ is, by definition, an X-indexed family of objects $(c_x)_{x\in X}$ of $\mathcal{C}$. Families of morphisms

$$\left(c_x \xrightarrow{\alpha_x} c'_x \right)_{x\in X}$$

are natural morphisms between X-graded objects.

3.2 G-graded categories

3.2.1

Let G be a semigroup. A G-graded category $\mathcal{C}$ consists of a class $\operatorname{Ob}\mathcal{C}$ and a graded class $(\operatorname{Arr}_g \mathcal{C})_{g\in G}$ equipped with the source and and target correspondences

$$s_g\colon \operatorname{Arr}_g \mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{C} \qquad \text{and} \qquad t_g\colon \operatorname{Arr}_g \mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{C}$$

and the associative composition correspondences

$$\operatorname{Arr}_g \mathcal{C} \times_{\operatorname{Ob}\mathcal{C}} \operatorname{Arr}_{g'} \mathcal{C} \xrightarrow{\quad\circ\quad} \operatorname{Arr}_{gg'} \mathcal{C}$$

where $\operatorname{Arr}_g \mathcal{C} \times_{\operatorname{Ob}\mathcal{C}} \operatorname{Arr}_{g'}$ denotes the class of composable pairs of arrows

$$\operatorname{Arr}_g \mathcal{C} \times_{\operatorname{Ob}\mathcal{C}} \operatorname{Arr}_{g'} := \{(\alpha, \alpha') \mid \alpha \in \operatorname{Arr}_g \mathcal{C},\ \alpha' \in \operatorname{Arr}_{g'} \mathcal{C} \text{ and } s(\alpha) = t(\alpha')\}.$$

3.2.2 Terminology

If α is a member of Arr_g, we say that α is a morphism *of degree* g. Morphisms with the source c and the target c' form a G-graded set

$$\left(\operatorname{Hom}_{\mathcal{C}}(c, c')_g\right)_{g\in G}.$$

3.2.3

For the singleton group $G = \{e\}$ we obtain a usual definition of a category. The concept of a graded category *generalizes* and enriches the concept of a category. Per se, a graded category *is not* a category.

3.2.4

The concept of unital graded category requires G to be a monoid, the identity morphisms being of degree e where e is the neutral element of the monoid.

3.2.5 The graded category of objects graded by a G set

Let X be a (left) G-set (tacitly assumed to be associative). For any $g \in G$ and a pair of X-graded objects

$$\mathbf{c} = (c_x)_{x \in X} \qquad \text{and} \qquad \mathbf{c}' = (c'_x)_{x \in X}$$

let us denote by

$$\mathrm{Hom}(\mathbf{c}, \mathbf{c}')_g$$

the set of X-indexed families

$$\alpha = \left(c_x \xrightarrow{\alpha_x} c'_{gx} \right)_{x \in X}$$

of morphisms in $\mathcal{C}$.

3.2.6

Composition of two such families α and α' is executed according to the rule

$$(\alpha \circ \alpha')_x := \alpha_{gx} \circ \alpha'_x \qquad (x \in X).$$

Exercise 64 *Show that the composition defined above is associative.*

3.2.7

The resulting structure is perhaps the most common example of a G-graded category.

3.2.8 The graded category of G-graded objects

In the special case of $X = G$ with the left action of G by left multiplication, we obtain the graded category of G-graded objects.

4 Reflections

4.1 Two “arrow” categories

4.1.1

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and d let be an object of the target category $\mathcal{D}$.

4.1.2 Category $d{\to}F\mathcal{C}$

The objects of category $d{\to}F\mathcal{C}$ are pairs $\left(c, d \xrightarrow{\partial} Fc\right)$ and morphisms

$$\left(c, d \xrightarrow{\partial} Fc\right) \longrightarrow \left(c, d \xrightarrow{\partial'} Fc'\right)$$

are the arrows $c \xrightarrow{\alpha} c'$ in the source category $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} & Fc' \\ d \arrow[ur, "\partial'"] \arrow[dr, "\partial"'] & \\ & Fc \arrow[uu, dashed, "F\alpha"'] \end{tikzcd} \tag{100}$$

4.1.3 A reflection of d along F

An initial object of category $d{\to}F\mathcal{C}$ will be called a *reflection* of d *along* functor F.

4.1.4 Category $F\mathcal{C}{\to}d$

The objects of category $F\mathcal{C}{\to}d$ are pairs $\left(c, Fc \xrightarrow{\partial} d\right)$ and morphisms

$$\left(c, Fc \xrightarrow{\partial} d\right) \longrightarrow \left(c, Fc' \xrightarrow{\partial'} d\right)$$

are the arrows $c \xrightarrow{\alpha} c'$ in the source category $\mathcal{C}$ such that the mirror reflection of diagram (100) commutes

$$\begin{tikzcd} Fc' \arrow[dr, "\partial'"] & \\ & d \\ Fc \arrow[uu, dashed, "F\alpha"] \arrow[ur, "\partial"'] & \end{tikzcd} \tag{101}$$

4.1.5

Note that the objects of the sibling categories $d{\to}F\mathcal{C}$ and $F\mathcal{C}{\to}d$ are objects of category $\mathcal{C}$ *equipped* with data of a certain type. Morphisms in such categories are defined to be the morphisms in $\mathcal{C}$ that are *compatible* with the data. The data in this case consist of a morphism in category $\mathcal{D}$ from d to Fc and, respectively, from Fc to d.

4.1.6 A coreflection of d along F

A terminal object of category $F\mathcal{C}{\to}d$ will be called a *coreflection* of d *along* functor F.

4.1.7 Terminology

More appropriate would be perhaps to talk of the *source* and the *target* reflections instead of *reflections* and *coreflections*, depending on whether d is the source of arrows into $F\mathcal{C}$ or the target of arrows from $F\mathcal{C}$.

4.1.8 Transitivity of reflections

Given a pair of composable functors

$$\mathcal{B} \xrightarrow{G} \mathcal{C} \xrightarrow{F} \mathcal{D} \ ,$$

suppose

$$(\bar{d}, d \xrightarrow{\bar{\delta}} F\bar{d})$$

is a reflection of $d \in \operatorname{Ob}\mathcal{D}$ along F and

$$(\bar{\bar{d}}, \bar{d} \xrightarrow{\bar{\bar{\delta}}} G\bar{\bar{d}})$$

is a reflection of $\bar{d}$ along G.

Exercise 65 *Show that*

$$\left(\bar{\bar{d}},\ d \xrightarrow{\quad F\bar{\bar{\delta}}\circ\bar{\delta}\quad} (F\circ G)\bar{\bar{d}}\right)$$

is a reflection of d along $F\circ G$.

4.2 Automatic naturality of reflections

4.2.1

Given two objects d and d' of $\mathcal{D}$, and their reflections

$$\left(\bar{d}, d \xrightarrow{\bar{\delta}} F\bar{d}\right) \qquad \text{and} \qquad \left(\bar{d}', d' \xrightarrow{\bar{\delta}'} F\bar{d}'\right),$$

for any morphism $d \xrightarrow{\beta} d'$, there exists a unique morphism $\bar{d} \xrightarrow{\bar{\beta}} \bar{d}'$ such that the diagram

$$\begin{tikzcd} d' \arrow[r, "\bar{\delta}'"] & F\bar{d}' \\ d \arrow[u, "\beta"] \arrow[r, "\bar{\delta}"'] & F\bar{d} \arrow[u, dashed, "F\bar{\beta}"'] \end{tikzcd} \tag{102}$$

commutes ($\bar{\delta}' \circ \beta \colon d \longrightarrow F\bar{d}'$ uniquely factorizes through $\bar{\delta}$).

4.2.2

Given another morphism $d' \xrightarrow{\beta'} d''$ and a reflection

$$\left(\bar{d}', d' \xrightarrow{\bar{\delta}'} F\bar{d}'\right),$$

we obtain the morphism $\bar{d} \xrightarrow{\bar{\beta}} \bar{d}'$ such that

$$\begin{tikzcd} d'' \arrow[r, "\bar{\delta}''"] & F\bar{d}'' \\ d' \arrow[u, "\beta'"] \arrow[r, "\bar{\delta}'"'] & F\bar{d}' \arrow[u, dashed, "F\bar{\beta}'"'] \end{tikzcd} \tag{103}$$

commutes. It follows that the diagram

$$\begin{tikzcd} d'' \arrow[r, "\bar{\delta}''"] & F\bar{d}'' \\ d \arrow[u, "\beta' \circ \beta"] \arrow[r, "\bar{\delta}"'] & F\bar{d} \arrow[u, dashed, "F\bar{\beta}' \circ F\bar{\beta}"'] \end{tikzcd} \tag{104}$$

does that as well. But

$$F\bar{\beta}' \circ F\bar{\beta} = F(\bar{\beta}' \circ \bar{\beta}).$$

Uniqueness of an arrow $\overline{\beta' \circ \beta} \colon \bar{d} \longrightarrow \bar{d}''$ such that $F\overline{\beta' \circ \beta}$ closes up (104) to a commutative diagram means that

$$\overline{\beta' \circ \beta} = \bar{\beta}' \circ \bar{\beta}.$$

4.2.3 The associated natural transformation $\eta : (\mathcal{D}' \hookrightarrow \mathcal{D}) \longrightarrow F \circ G$

Denote by $\mathcal{D}'$ the full subcategory of $\mathcal{D}$ consisting of objects d that have a reflection along F. We demonstrated that *any* assignment of a reflection

$$d \longmapsto (\bar{d}, d \xrightarrow{\bar{\delta}} F\bar{d}) \qquad (d \in \operatorname{Ob} \mathcal{D}'),$$

to every object of $\mathcal{D}'$ produces *in a unique manner* a functor

$$G\colon \mathcal{D}' \longrightarrow \mathcal{C} \qquad \text{where} \qquad Gd := \bar{d} \qquad \text{and} \qquad G\beta := \bar{\beta},$$

equipped with a natural transformation

$$\eta\colon (\mathcal{D}' \hookrightarrow \mathcal{D}) \longrightarrow F \circ G \qquad \text{where} \qquad \eta_d := \bar{\delta},$$

from the inclusion functor $\mathcal{D}' \hookrightarrow \mathcal{D}$ to $F \circ G$.

4.2.4

We shall refer to (G, η) as a *left adjoint* pair for F, while G will be referred to as a *left adjoint* to functor F. It is essential to understand, however, that whenever we talk of a *left adjoint* functor we mean a functor *equipped* with natural transformation η.

4.2.5 Terminological comments

Normally one speaks of left adjoint functors under the hypothesis that $\mathcal{D}' = \mathcal{D}$, i.e., assuming that *every* object of $\mathcal{D}$ admits a reflection along F. We say, in this case that functor F *admits a left adjoint.*

In literature you will encounter only the case when the two categories and the functor between them are assumed to be unital.

Exercise 66 *Show that the mapping*

$$\operatorname{Hom}_{\mathcal{C}}(Gd, c) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(d, Fc) \qquad (c \in \operatorname{Ob} \mathcal{C},\ d \in \operatorname{Ob} \mathcal{D}') \tag{105}$$

sending $Gd \xrightarrow{\alpha} c$ to $F\alpha \circ \bar{\delta}$ is a bijection.

Exercise 67 *Show that the morphism $Gd \xrightarrow{\iota} Gd$ that corresponds to $d \xrightarrow{\eta_d} FGd$ is a right identity if functor F is injective on morphisms.*

Exercise 68 *Show that bijections* (105) *are natural in* d *and* c*, i.e., given morphisms* $d \xrightarrow{\beta} d'$ *and* $c \xrightarrow{\alpha} c'$*, the following diagrams*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Gd', c) \arrow[r, leftrightarrow] \arrow[d, "(\ )\circ G\beta"'] & \mathrm{Hom}_{\mathcal{D}}(d', Fc) \arrow[d, "(\ )\circ\beta"] \\
\mathrm{Hom}_{\mathcal{C}}(Gd, c) \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathcal{D}}(d, Fc)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Gd, c) \arrow[r, leftrightarrow] \arrow[d, "\alpha\circ(\ )"'] & \mathrm{Hom}_{\mathcal{D}}(d, Fc) \arrow[d, "F\alpha\circ(\ )"] \\
\mathrm{Hom}_{\mathcal{C}}(Gd, c') \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathcal{D}}(d, Fc')
\end{tikzcd}$$

commute.

Exercise 69 *Show that if* $\mathcal{C}$*,* $\mathcal{D}$ *and* F *are unital, then* $G(\mathrm{id}_d) = \mathrm{id}_{Gd}$.

4.3 Automatic naturality of coreflections

4.3.1

Given two objects d and d' of $\mathcal{D}$, and their coreflections

$$\left(\underline{d}, F\underline{d} \xrightarrow{\underline{\partial}} d\right) \qquad \text{and} \qquad \left(\underline{d}', F\underline{d}' \xrightarrow{\underline{\partial}'} d'\right),$$

for any morphism $d \xrightarrow{\beta} d'$, there exists a unique morphism $\underline{d} \xrightarrow{\underline{\beta}} \underline{d}'$ such that

$$\begin{tikzcd}
F\underline{d}' \arrow[r, "\underline{\partial}'"] & d' \\
F\underline{d} \arrow[u, dashed, "F\underline{\beta}"] \arrow[r, "\underline{\partial}"'] & d \arrow[u, "\beta"']
\end{tikzcd} \tag{106}$$

($\underline{\partial}' \circ \beta \colon d \longrightarrow F\underline{d}'$ uniquely factorizes through $\underline{\partial}$).

4.3.2

Given another morphism $d' \xrightarrow{\beta'} d''$ and a coreflection

$$\left(\underline{d}', F\underline{d}' \xrightarrow{\underline{\partial}'} d'\right),$$

we obtain the morphism $\underline{d} \xrightarrow{\underline{\beta}} \underline{d}'$ such that

$$\begin{tikzcd} F\underline{d}'' \arrow[r, "\underline{\partial}''"] & d'' \\ F\underline{d}' \arrow[u, dashed, "F\underline{\beta}'"] \arrow[r, "\underline{\partial}'"'] & d' \arrow[u, "\beta'"'] \end{tikzcd} \tag{107}$$

commutes. In particular,

$$\begin{tikzcd} F\underline{d}'' \arrow[r, "\underline{\partial}''"] & d'' \\ F\underline{d} \arrow[u, "F\underline{\beta}' \circ F\underline{\beta}"] \arrow[r, "\underline{\partial}"'] & d \arrow[u, "\beta' \circ \beta"'] \end{tikzcd} \tag{108}$$

commutes. Uniqueness of the arrow

$$\underline{\beta' \circ \beta} \colon \underline{d} \longrightarrow \underline{d}''$$

making diagram (106) commutative means that,

$$\underline{\beta' \circ \beta} = \underline{\beta}' \circ \underline{\beta}.$$

4.3.3 The associated natural transformation $\varepsilon : F \circ G \longrightarrow (\mathcal{D}' \hookrightarrow \mathcal{D})$

Denote by $\mathcal{D}''$ the full subcategory of $\mathcal{D}$ consisting of objects d that have a coreflection along F. We demonstrated that *any* assignment of a coreflection

$$d \longmapsto (\underline{d}, F\underline{d} \xrightarrow{\underline{\partial}} d) \qquad (d \in \operatorname{Ob} \mathcal{D}''),$$

to every object of $\mathcal{D}''$ produces *in a unique manner* a functor

$$G \colon \mathcal{D}'' \longrightarrow \mathcal{C} \qquad \text{where} \qquad Gd := \underline{d} \qquad \text{and} \qquad G\beta := \underline{\beta},$$

equipped with a natural transformation

$$\epsilon \colon F \circ G \longrightarrow (\mathcal{D}'' \hookrightarrow \mathcal{D}) \qquad \text{where} \qquad \epsilon_d := \underline{\partial},$$

from $F \circ G$ to the inclusion functor $\mathcal{D}'' \hookrightarrow \mathcal{D}$.

4.3.4

We shall refer to (G, ϵ) as a *right adjoint* pair for F, while G will be referred to as a *right adjoint* to functor F. It is essential to understand, however, that whenever we talk of a *right adjoint* functor we mean a functor *equipped* with natural transformation ϵ.

4.3.5 Terminological comments

Normally one speaks of right adjoint functors under the hypothesis that $\mathcal{D}'' = \mathcal{D}$, i.e., assuming that *every* object of $\mathcal{D}$ admits a coreflection along F. We say, in this case that functor F *admits a right adjoint.*

In literature you will encounter only the case when the two categories and the functor between them are assumed to be unital.

Exercise 70 *Show that the mapping*

$$\mathrm{Hom}_{\mathcal{C}}(c, Gd) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Fc, d) \qquad (c \in \mathrm{Ob}\,\mathcal{C},\ d \in \mathrm{Ob}\,\mathcal{D}'') \tag{109}$$

sending $c \xrightarrow{\alpha} Gd$ *to* $\underline{\partial} \circ F\alpha$ *is a bijection.*

Exercise 71 *Show that the morphism* $Gd \xrightarrow{\iota} Gd$ *that corresponds to* $FGd \xrightarrow{\epsilon_d} d$ *is a left identity.*

Exercise 72 *Show that bijections* (109) *are natural in* d *and* c*, i.e., given morphisms* $d \xrightarrow{\beta} d'$ *and* $c \xrightarrow{\alpha} c'$*, the following diagrams*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c, Gd) \arrow[r, leftrightarrow] \arrow[d, "G\beta\circ(\ )"'] & \mathrm{Hom}_{\mathcal{D}}(Fc, d) \arrow[d, "\beta\circ(\ )"] \\
\mathrm{Hom}_{\mathcal{C}}(c, Gd') \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathcal{D}}(Fc, d')
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c', Gd) \arrow[r, leftrightarrow] \arrow[d, "(\ )\circ\alpha"'] & \mathrm{Hom}_{\mathcal{D}}(Fc', d) \arrow[d, "(\ )\circ F\alpha"] \\
\mathrm{Hom}_{\mathcal{C}}(c, Gd) \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathcal{D}}(Fc, d)
\end{tikzcd}$$

commute.

Exercise 73 *Show that if* F *is a unital functor (which automatically means that its source and its target are categories with the identity morphisms), then* $G(\mathrm{id}_d) = \mathrm{id}_{Gd}$.

4.3.6 Left–right adjoint duality

Note that

$$(d{\to}F\mathcal{C})^{\mathrm{op}} = F^{\circ}\mathcal{C}^{\mathrm{op}}{\to}d^{\mathrm{op}} \tag{110}$$

where $F^{\circ}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is the *dual* functor, cf. (35). In particular, (G, η) is a left adjoint pair for F if and only if $(G^{\circ}, \eta^{\mathrm{op}})$ is a right adjoint pair for F°.

4.4 Adjoint pairs of unital functors

4.4.1

Below we assume that

$$F\colon \mathcal{C} \to \mathcal{D} \qquad \text{and} \qquad G\colon \mathcal{D} \to \mathcal{C} \tag{111}$$

is a pair of *unital* functors between *unital* categories.

4.4.2

Suppose that bijections

$$\operatorname{Hom}_{\mathcal{C}}(Gd, c) \xleftrightarrow{\phi_{dc}} \operatorname{Hom}_{\mathcal{D}}(d, Fc) \qquad (c \in \operatorname{Ob}\mathcal{C},\ d \in \operatorname{Ob}\mathcal{D}) \tag{112}$$

are given that are *natural* in both d and c.

Exercise 74 *Let* $d \xrightarrow{\eta_d} FGd$ *be a morphism corresponding under* (112) *to* id_{Gd}. *Show that* $\eta = (\eta_d)_{d \in \operatorname{Ob}\mathcal{D}}$ *is a natural transformation* $\mathrm{id}_{\mathcal{D}} \to F \circ G$ *and* (G, η) *is a left adjoint pair to* F.

Exercise 75 *Let* $GFc \xrightarrow{\epsilon_c} c$ *be a morphism corresponding under* (112) *to* id_{Fc}. *Show that* $\epsilon = (\epsilon_c)_{c \in \operatorname{Ob}\mathcal{C}}$ *is a natural transformation* $G \circ F \to \mathrm{id}_{\mathcal{C}}$ *and* (F, ϵ) *is a right adjoint pair to* G.

Exercise 76 *Show that*

$$\phi_{dc}(\alpha) = F\alpha \circ \eta_d \qquad (\alpha \in \operatorname{Hom}_{\mathcal{C}}(Gd, c))$$

and

$$\phi_{dc}^{-1}(\beta) = \epsilon_c \circ G\beta \qquad (\beta \in \operatorname{Hom}_{\mathcal{D}}(d, Fc)).$$

(*A hint to all three exercises: utilize naturality of* ϕ_{dc} *in* d *and* c.)

4.4.3

In other words, we have the following identities

$$\alpha = \epsilon_c \circ GF\alpha \circ G\eta_d \qquad (\alpha \in \operatorname{Hom}_{\mathcal{C}}(Gd, c)) \tag{113}$$

and

$$\beta = F\epsilon_c \circ FG\beta \circ \eta_d \qquad (\beta \in \operatorname{Hom}_{\mathcal{D}}(d, Fc)). \tag{114}$$

4.4.4

It follows that in the unital case, with the target of one functor being the source of the other functor, and vice-versa, the natural transformations

$$\mathrm{id}_{\mathcal{D}} \xrightarrow{\eta} F \circ G \qquad \text{and} \qquad G \circ F \xrightarrow{\epsilon} \mathrm{id}_{\mathcal{C}} \tag{115}$$

are already encoded in the structure of natural bijections (112). They are referred to as the *unit* and, respectively, the *counit of adjunction*. In that case, it is sufficient to talk about *pairs of adjoint functors* in which G is a left adjoint of F while F becomes automaticaly a right adjoint of G.

Exercise 77 *Suppose both G and G' are left adjoint to F. Show that there exists a unique natural transformation $G \xrightarrow{\phi} C'$ such that the diagram commutes*

$$\begin{tikzcd} & F \circ G' \\ d \arrow[ur, "\eta'"] \arrow[dr, "\eta"'] & \\ & F \circ G \arrow[uu, dashed, "F\phi"'] \end{tikzcd} \tag{116}$$

Deduce that, for a unital functor between unital categories, any two left adjoints are isomorphic by a unique isomorphism of functors compatible with the corresponding units of adjunction.

4.4.5

Dually, for a unital functor F between unital categories, any two right adjoints are isomorphic by a unique isomorphism of functors compatible with the corresponding counits of adjunction

$$\begin{tikzcd} G' \circ F \arrow[dr, "\epsilon'"] & \\ & d \\ G \circ F \arrow[uu, dashed, "\phi F"] \arrow[ur, "\epsilon"'] & \end{tikzcd} .$$

4.4.6

The perfect symmetry between left and right adjoint functors in the unital case is to some extent affected by the fact that in modern Mathematics one often is presented with a single functor. The existence and construction of its left and right adjoints are then the question and the task that are addressed.

Exercise 78 *Show that*

$$F\epsilon \circ \eta F = \mathrm{id}_F \qquad \textit{and} \qquad \epsilon G \circ G\eta = \mathrm{id}_G . \tag{117}$$

Exercise 79 *Suppose F and G is a pair of unital functors* (111) *between unital categories, equipped with a pair of natural transformations* (115) *satisfying the pair of identities* (117)*. Show that G is left adjoint to F and F is right adjoint to G.*

4.5 Reflections and projective limits

4.5.1

Suppose that

$$\lambda = \left(l \xrightarrow{\lambda_b} Gb\right)_{b \in \operatorname{Ob} \mathcal{B}}$$

is a projective limit of a functor $G\colon \mathcal{B} \longrightarrow \mathcal{C}$. An object in the category $\mathcal{D}{\to}FG$ is a family of arrows

$$\xi = \left(x \xrightarrow{\xi_b} FGb\right)_{b \in \operatorname{Ob} \mathcal{B}}$$

such that for all arrows in $\mathcal{C}$, the diagram

$$\begin{tikzcd} & FGb' \\ x \arrow[ur, "\xi_{b'}"] \arrow[dr, "\xi_b"'] & \\ & FGb \arrow[uu, "FG\beta"'] \end{tikzcd} \qquad \left(\beta \in \operatorname{Hom}_{\mathcal{C}}(b, b')\right) \tag{118}$$

commutes. If x has a reflection along F,

$$(\bar{x}, x \xrightarrow{\chi} F\bar{x}),$$

then, for each ξ_b, there exists a unique morphism $\bar{x} \xrightarrow{\bar{\xi}_b} Gb$ such that the diagram

$$\begin{tikzcd} & FGb \\ x \arrow[ur, "\xi_b"] \arrow[dr, "\chi"'] & \\ & F\bar{x} \arrow[uu, dashed, "F\bar{\xi}_b"'] \end{tikzcd}$$

commutes.

Exercise 80 *Show that the diagrams*

$$\begin{tikzcd} & Gb' \\ \bar{x} \arrow[ur, "\bar{\xi}_{b'}"] \arrow[dr, "\bar{\xi}_b"'] & \\ & Gb \arrow[uu, "G\beta"'] \end{tikzcd} \qquad \left(\beta \in \operatorname{Hom}_{\mathcal{C}}(b, b')\right)$$

commute.

4.5.2

Thus,

$$\bar{\xi} = \left(\bar{x} \xrightarrow{\bar{\xi}_b} Gb\right)_{b \in \operatorname{Ob} \mathcal{B}} \tag{119}$$

is an object of the category of arrows $\mathcal{C} {\rightarrow} G$, and therefore there exists a unique morphism $\bar{\xi} \xrightarrow{\gamma} \lambda$. In particular, $F\gamma \circ \chi$ is a morphism in the category of arrows $\mathcal{D} {\rightarrow} FG$ from ξ to $F\lambda$.

Exercise 81 *Show that any morphism from ξ to $F\lambda$ in the category of arrows $\mathcal{D} {\rightarrow} FG$ can be represented as $F\alpha \circ \chi$ for some morphism α from $\bar{\xi}$ to λ in the category of arrows $\mathcal{C} {\rightarrow} G$.*

4.5.3

Since λ is terminal in $\mathcal{C} {\rightarrow} G$, we infer that $\alpha = \gamma$. In particular, $F\gamma \circ \chi$ is a *unique* morphism from ξ to $F\lambda$ in the category of arrows $\mathcal{D} {\rightarrow} FG$.

4.5.4

If *every* object $x \in \operatorname{Ob} \mathcal{D}$ has a reflection along F, then $F\lambda$ is a terminal object in $\mathcal{D} {\rightarrow} FG$, i.e., $F\lambda$ is a projective limit of $F \circ G$.

4.5.5

This fundamental property is usually stated as:

functors that admit left adjoints preserve all projective limits. (120)

4.5.6

Dually,

functors that admit right adjoints preserve all inductive limits. (121)

5 Embedding functors

5.1 Functors adjoint to the inclusion functors

5.1.1 A reflection of an object in a subcategory

Let $\mathcal{C}$ be a subcategory of a category $\mathcal{D}$. A reflection of an object $d \in \operatorname{Ob}\mathcal{D}$ along the inclusion functor $\iota : \mathcal{C} \hookrightarrow \mathcal{D}$ is usually referred to as a reflection of d *in* subcategory $\mathcal{C}$.

5.1.2 Reflective and coreflective subcategories

If every object of $\mathcal{D}$ has a reflection in $\mathcal{C}$, we say that $\mathcal{C}$ is a *reflective* subcategory. Similarly defined are *coreflective* subcategories. The corresponding left and, respectively, right adjoint functors $\mathcal{D} \longrightarrow \mathcal{C}$ are often of great importance and there is a multitude of examples in all areas of Mathematics.

5.1.3

Let $(\bar{d}, \bar{\partial})$ be a reflection in $\mathcal{C}$ of an object $d \in \operatorname{Ob}\mathcal{D}$. Before providing any examples of reflective and coreflective categories, we shall examine the case when $d \in \mathcal{C}$. In this case both d and $\bar{d}$ belong to $\mathcal{C}$ but they need to be equal and, even if they are equal, $\bar{\partial}$ needs not to be an identity endomorphism.

Exercise 82 *Suppose that $d \in \mathcal{C}$. Show that*

$$\bar{\partial} \in \operatorname{Mor}_{\mathcal{C}} \quad \Leftrightarrow \quad \forall_{c \in \operatorname{Ob}\mathcal{C}} \ \operatorname{Hom}_{\mathcal{C}}(d, c) = \operatorname{Hom}_{\mathcal{D}}(d, c). \tag{122}$$

5.1.4

In particular, if a *full* subcategory $\mathcal{C} \subseteq \mathcal{D}$ is reflective and every object c of subcategory $\mathcal{C}$ admits a right identity endomorphism, then any reflection assignment

$$d \longmapsto (\bar{d}, \bar{\partial}) \qquad (d \in \operatorname{Ob}\mathcal{D})$$

such that $\bar{\partial}$ is a right identity endomorphism of d whenever $d \in \operatorname{Ob}\mathcal{C}$, gives rise to a unique left adjoint functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ of the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{D}$ that is also its left inverse,

$$G \circ (\mathcal{C} \hookrightarrow \mathcal{D}) = \mathrm{id}_{\mathcal{C}}$$

Exercise 83 *State the analogs of Exercise 82 and Observation 5.1.4 for coreflective subcategories.*

5.2 First examples

5.2.1 Unitalization of a binary structure

Given a binary structure $(M,\cdot)$, let $\widetilde{M}$ be the set

$$\widetilde{M} := \{e\} \sqcup M \tag{123}$$

equipped with the multiplication that extends multiplication on M by

$$e \cdot \tilde{m} = \tilde{m} \cdot e = \tilde{m} \qquad (\tilde{m} \in \widetilde{M}). \tag{124}$$

Exercise 84 *Show that the correspondence $M \longmapsto \widetilde{M}$ gives rise to a functor*

$$\mathbf{Bin} \longrightarrow \mathbf{Bin}_{\mathrm{un}} \tag{125}$$

from the category of binary structures $\mathbf{Bin}$ to the category of unital binary structures $\mathbf{Bin}_{\mathrm{un}}$, and show that this functor is left adjoint to the inclusion functor $\mathbf{Bin}_{\mathrm{un}} \hookrightarrow \mathbf{Bin}$.

5.2.2

Note that $\widetilde{M}$ is a monoid when M is a semigroup. Restriction of the unitalization functor (125) to semigroups defines a left adjoint functor to the inclusion of the category of monoids into the category of semigroups.

Exercise 85 *Show that inclusion $\mathbf{Mon} \hookrightarrow \mathbf{Sgr}$ has no right adjoint functor. (Hint. Compare* $\mathrm{Hom}_{\mathbf{Mon}}(M,\ )$ *and* $\mathrm{Hom}_{\mathbf{Sgr}}(M,\ )$*, for example, when M has a single element.)*

5.2.3 The category of groups as a subcategory of the category of monoids

Exercise 86 *Show that the correspondence*

$$M \longmapsto M^* := \{m \in M \mid m \text{ is invertible}\}$$

gives rise to a functor $\mathbf{Mon} \longrightarrow \mathbf{Grp}$, and show that this functor is right adjoint to the inclusion functor $\mathbf{Grp} \hookrightarrow \mathbf{Mon}$.

5.2.4 The group completion functor

A left adjoint functor to $\mathbf{Grp} \hookrightarrow \mathbf{Mon}$ is called a *group completion* functor. A reflection of a monoid M in the category of groups can be constructed as the quotient of a coproduct of M and M^{op} in the category of monoids

$$M \sqcup_{\mathbf{Mon}} M^{\mathrm{op}} \tag{126}$$

(which is realized by the free product of monoids $M *_{\mathrm{un}} M^{\mathrm{op}}$) by a weakest congruence $\sim$ such that

$$mm^{\mathrm{op}} = e \qquad \text{and} \qquad m^{\mathrm{op}}m = e \qquad (m \in M),$$

where e is the identity element in (126). Let us denote $\left(M \sqcup_{\mathbf{Mon}} M^{\mathrm{op}}\right)_{/\sim}$ by $G(M)$.

Exercise 87 *Show that the inverse in the monoid $G(M)$ of the equivalence class of a word*

$$w = l_1 \cdots l_q$$

is the class of the word

$$w' := l'_q \cdots l'_1$$

where

$$l' := \begin{cases} l^{\mathrm{op}} & \text{if } l \in M \\ l & \text{if } l \in M^{\mathrm{op}} \end{cases}.$$

5.2.5

Thus, $G(M)$ is a group. Any homomorphism of monoids $f\colon M \longrightarrow G$ induces a homomorphism

$$M^{\mathrm{op}} \xrightarrow{(\ )^{-1} \circ f \circ (\ )^{\mathrm{op}}} G \tag{127}$$

and the two together give rise to a unique homomorphism of groups

$$G(M) \longrightarrow G$$

whose restriction to $M \subseteq G(M)$ equals f and whose restriction to $M^{\mathrm{op}} \subseteq G(M)$ equals (127).

5.3 Subcategories of categories of ν-ary structures

5.3.1

Let $\mathcal{A}$ be a certain category of of ν-ary structures. By definition, this means that $\mathcal{A}$ is a subcategory of *all* such structures and their homomorphisms ν-**alg str**.

5.3.2 Identities

An identity in a ν-ary structure is a *formal* equality

$$w(t_1, \ldots, t_n) = w'(t_1, \ldots, t_n) \tag{128}$$

where both w and w' are expressions obtained by *formally* applying a finite number of times operations of a ν-ary structure to symbols $t_1, \ldots, t_n$.

For example,

$$t_1(t_2 + t_3) = t_1t_2 + t_1t_3$$

is an identity involving three symbols and two binary operations (addition and multiplication). It expresses left distributivity of multiplication with respect to addition.

5.3.3 A subcategory defined by a set of identities

We say that a structure $A \in \mathrm{Ob}\,\mathcal{A}$ satisfies identity (128) if substitution of any n elements $a_1, \dots, a_n$ under symbols $t_1, \dots, t_n$ produces an equality in A. Let $\mathcal{I}$ be a set of identities like (128) and let ${}^{\mathcal{I}}\!\mathcal{A}$ denote the *full* subcategory of $\mathcal{A}$, consisting of those structures $A \in \mathrm{Ob}\,\mathcal{A}$ which satisfy all identities from set $\mathcal{I}$.

5.3.4 The congruence $\sim_{\mathcal{I}}$ associated with $\mathcal{I}$

Let $\sim_{\mathcal{I}}$ be a weakest congruence on a astructure $A \in \mathrm{Ob}\,\mathcal{A}$ such that

$$w(a_1, \dots, a_n) \sim_{\mathcal{I}} w'(a_1, \dots, a_n)$$

for all $a_1, \dots, a_n \in A$. The quotient structure $A_{/\sim_{\mathcal{I}}}$ satisfies all identities from $\mathcal{I}$ and any homomorphism $A \longrightarrow A'$ into any structure from ${}^{\mathcal{I}}\!\mathcal{A}$ uniquely factorizes through the quotient homomorphism $A \twoheadrightarrow A_{/\sim_{\mathcal{I}}}$. It follows that the assignment $A \longmapsto A_{/\sim_{\mathcal{I}}}$ gives rise to a functor $\mathcal{A} \longrightarrow {}^{\mathcal{I}}\!\mathcal{A}$ that is left adjoint to the inclusion functor ${}^{\mathcal{I}}\!\mathcal{A} \hookrightarrow \mathcal{A}$,

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "(\ )_{/\sim_{\mathcal{I}}}"] & {}^{\mathcal{I}}\!\mathcal{A} \arrow[l, bend left, "\text{inclusion}"] \end{tikzcd} . \tag{129}$$

5.3.5 Commutativization of a binary structure

There are numerous important examples of the situation described above. For example, the functor sending a structure A to its *reflection* in the category of commutative binary structures

$$\mathbf{Bin} \longrightarrow \mathbf{Bin}_{\mathrm{co}}, \qquad A \longmapsto A^{\mathrm{co}}. \tag{130}$$

Its restriction to the the category of groups, yields a functor $\mathbf{Grp} \longrightarrow \mathbf{Ab}$, called *abelianization*. It sends a group to the quotient by its commutator subgroup

$$G \longmapsto G^{\mathrm{ab}} := G/[G,G] \tag{131}$$

and is left adjoint to the inclusion functor $\mathbf{Grp} \hookrightarrow \mathbf{Ab}$.

Exercise 88 *Show that* (131) *is a reflection of a group G in the category of abelian groups.*

5.3.6 Associativization of a binary structure

A functor sending a binary structure to its reflection in the category of semigroups

$$A \longmapsto A^{\mathrm{as}} \tag{132}$$

is left adjoint to the inclusion functor **Sgr** $\hookrightarrow$ **Bin**.

5.3.7

Above we saw that the category of groups is a reflective and a coreflective subcategory of the category of monoids. Note that **Grp** is a full subcategory of **Mon** but is not defined by any set of identities involving the operations of multiplication or the 0-ary operation of identity on a monoid.

5.3.8

In contrast, full subcategories of algebraic structures defined by sets of identities are generally only reflective. For example, G^{ab} is the largest abelian quotient group of G but there is no similar *largest* abelian subgroup in G, except when G is abelian itself. For this reason, inclusion **Ab** $\hookrightarrow$ **Grp** has no right adjoint functor.

5.4 The diagonal functor $\Delta\colon \mathcal{C} \longrightarrow \mathcal{C}^I$

5.4.1

Given a unital category $\mathcal{C}$ and a small category I, the diagonal embedding functor $\Delta\colon \mathcal{C} \longrightarrow \mathcal{C}^I$ of $\mathcal{C}$ into the category of I-diagrams $\mathcal{C}^I$, is defined as follows. One assigns to each $c \in \operatorname{Ob}\mathcal{C}$ the *constant* I-diagram, i.e, a functor $I \longrightarrow \mathcal{C}$,

$$\Delta i := c, \qquad \Delta\iota := \mathrm{id}_c \qquad (i \in \operatorname{Ob} I,\ \iota \in \operatorname{Mor} I). \tag{133}$$

To each morphisms $c \xrightarrow{\alpha} c'$ one assigns the *constant* natural transformation $\Delta\alpha$,

$$(\Delta\alpha)_i := \alpha \qquad (i \in \operatorname{Ob} I). \tag{134}$$

Exercise 89 *Show that correspondences* (133)–(134) *define a unital functor*

$$\Delta\colon \mathcal{C} \longrightarrow \mathcal{C}^I. \tag{135}$$

Show that Δ *embeds* $\mathcal{C}$ *onto the full subcategory of* $\mathcal{C}^I$ *provided* I *is nonempty.*

5.4.2 Inductive limits as reflections along Δ

Reflections of a diagram $D \in \operatorname{Ob}\mathcal{C}^I$ along Δ are the same as *inductive* limits of D.

5.4.3 Projective limits as coreflections along Δ

Coreflections of a diagram $D \in \operatorname{Ob} \mathcal{C}^I$ along Δ are the same as *projective* limits of D.

5.4.4 Example: $\mathbf{Set} \hookrightarrow G\text{-}\mathbf{set}$

When I is a single object category,

$$\operatorname{Ob} I = \{\bullet\}, \qquad \operatorname{Mor} I = \operatorname{End}(\bullet) = G,$$

where G is a semigroup, the diagonal functor becomes the embedding

$$\Delta\colon \mathbf{Set} \hookrightarrow G\text{-}\mathbf{set} \tag{136}$$

of the category of sets onto a full subcategory of G-sets with trivial action.

Exercise 90 *Show that assigning to a G-set its set of orbits*

$$X \longmapsto X_{/G}, \tag{137}$$

cf. (39) *defines a functor* $G\text{-}\mathbf{set} \longrightarrow \mathbf{Set}$. *Then, directly from definition, prove that the orbit-set functor is left adjoint to* (136). *Show that assigning to a G-set its set of fixed points*

$$X \longmapsto X^G, \tag{138}$$

cf. (38)*, defines a functor* $G\text{-}\mathbf{set} \longrightarrow \mathbf{Set}$. *Then, directly from definition, prove that the fixed-point functor is right adjoint to* (136).

5.4.5 Generalization: $G'\text{-}\mathbf{set} \hookrightarrow G\text{-}\mathbf{set}$

Let $\phi\colon G \longrightarrow G'$ be an epimorphism of groups. Denote its kernel by N. Any G'-set can be considered as a G-set on which the subgroup N acts trivially. This defines an embedding of the category of G'-sets onto the full subcategory of G-sets with trivial action of N

$$G'\text{-}\mathbf{set} \hookrightarrow G\text{-}\mathbf{set}. \tag{139}$$

5.4.6

For a G-set X, the normal subgroup $N \subseteq G$ acts trivially on the set of fixed points X^N and on the set of orbits $X_{/N}$. Thus, the action of G on these two sets induces the corresponding actions of the quotient group $G/N \simeq G'$ and assignments

$$X \longmapsto X_{/N} \tag{140}$$

and

$$X \longmapsto X^N \tag{141}$$

define functors $G\text{-}\mathbf{set} \longrightarrow G'\text{-}\mathbf{set}$.

Exercise 91 *Show that* (140) *is left adjoint to* (139) *while* (141) *is right adjoint.*

6 Forgetful functors

6.1 The forgetful functor $\mathbf{Sgr}_{\mathrm{co}} \longrightarrow \mathbf{Set}$

6.1.1 The free commutative semigroup functor

Consider the functor

$$\mathbf{Sgr}_{\mathrm{co}} \xrightarrow{\;|\ |\;} \mathbf{Set}$$

that sends a commutative semigroup $(M, +)$ to the underlying set M, *forgetting* the binary operation. This functor has a *left* adjoint that sends a set X to the coproduct of the constant family of semigroups $(\mathbf{Z}_+)_{x \in X}$. We shall denote that coproduct $\mathbf{Z}_+ X$. We can think of members of $\mathbf{Z}_+ X$ as being formal linear combinations

$$\sum_{x \in A} l_x x \qquad (l \in \mathbf{Z}_+) \tag{142}$$

over all *finite nonempty* subsets $A \subseteq X$. In particular, $\mathbf{Z}_+ \emptyset$ is the empty semigroup.

6.1.2

Assigning to an element $x \in X$ the sum (142) with $A = \{x\}$ and $l_x = 1$, embeds X into $\mathbf{Z}_+ X$. Any mapping into a commutative semigroup $f\colon X \longrightarrow M$ uniquely extends to a homomorphism of commutative semigroups

$$\mathbf{Z}_+ X \longrightarrow M, \qquad \sum_{x \in A} l_x x \longmapsto \sum_{x \in A} l_x f(x),$$

demonstrating that $X \hookrightarrow \mathbf{Z}_+ X$ is a reflection of set X along the forgetful functor. In paticular, $\mathbf{Z}_+(\ )$ is left adjoint to the forgetful functor $|\ |$,

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathbf{Z}_+(\ )"] & \mathbf{Sgr}_{\mathrm{co}} \arrow[l, bend left, "|\ |"] \end{tikzcd}$$

6.1.3 Free commutative semigroups

Commutative semigroups isomorphic to $\mathbf{Z}_+ X$ for some set X are referred to as *free*. We shall now provide an explicit realization of $\mathbf{Z}_+ X$ as the semigroup of *symmetric words* on alphabet X.

6.1.4 Symmetric powers of a set

The symmetric q-th power $\Sigma^q X$ of a set X is defined as the set of orbits of the action of the permutation group Σ_q of $1, \dots, q$ on the q-th Cartesian power of X. Thus, elements of $\Sigma^q X$ are equivalence classes of the equivalence relation

$$(x_1, \dots, x_q) \sim \left(x_{\sigma(1)}, \dots, x_{\sigma(q)}\right).$$

Note that $\Sigma^0 X = X^0$ and $\Sigma^1 X = X$.

6.1.5 The symmetric semigroup of words

For a set X, consider the disjoint union of symmetric powers of X,

$$\Sigma X := X \sqcup \Sigma^2 X \sqcup \Sigma^3 X \sqcup \cdots \tag{143}$$

equipped with the multiplication of orbits of the permutation groups induced by concatenation of their representatives:

$$\overline{(x_1, \dots, x_q)} \cdot \overline{(x'_1, \dots, x'_r)} := \overline{(x_1, \dots, x_q, x'_1, \dots, x'_r)}. \tag{144}$$

Exercise 92 *Show that multiplication* (144) *is well defined and is associative.*

Exercise 93 *Show that assigning to a q-tuple its Σ^q-orbit,*

$$(x_1, \dots, x_q) \longmapsto \overline{(x_1, \dots, x_q)},$$

defines a homomorphism of semigroups $WX \longrightarrow \Sigma X$ which is a commutativization of the free semigroup WX.

6.1.6

We shall refer to ΣX equipped with multiplication (144) as *the symmetric semigroup of words on an alphabet X*. It is isomorphic to $\mathbf{Z}_+ X$ with

$$\overline{(x_1, \dots, x_q)}$$

corresponding to the formal linear combination

$$\sum_{x \in A} l_x x$$

where

$$A := \{x_1, \dots, x_q\} \qquad \text{and} \qquad l_x := \left|\{1 \le i \le q \mid x_i = x\}\right|.$$

6.2 The forgetful functor $\mathbf{Mon}_{\mathrm{co}} \longrightarrow \mathbf{Set}$

6.2.1 The free commutative monoid functor

Consider the functor

$$\mathbf{Mon}_{\mathrm{co}} \xrightarrow{\;|\;|\;} \mathbf{Set}$$

that sends a commutative monoid $(M, +)$ to the underlying set M, *forgetting* the binary operation. This functor has a *left* adjoint that sends a set X to the coproduct of the constant family of monoids $(\mathbf{N})_{x \in X}$. We shall denote that coproduct $\mathbf{N}X$. It is relaized as the direct sum

$$\mathbf{N}X = \bigoplus_{x \in X} \mathbf{N}.$$

We can think of members of $\mathbf{N}X$ as being formal linear combinations

$$\sum_{x \in X} l_x x \qquad (l \in \mathbf{N}) \tag{145}$$

with only finitely many $l_x \neq 0$. In particular, $\mathbf{N}\emptyset$ is the *zero* monoid, consisting of a single element.

6.2.2

Like for commutative semigroups, assigning to an element $x \in X$ the sum (145) with $A = \{x\}$ and $l_x = 1$, embeds X into $\mathbf{N}X$. Any mapping into a commutative monoid $f \colon X \longrightarrow M$ uniquely extends to a homomorphism of commutative monoids

$$\mathbf{N}X \longrightarrow M, \qquad \sum_{x \in X} l_x x \longmapsto \sum_{x \in X} l_x f(x),$$

demonstrating that $X \hookrightarrow \mathbf{N}X$ is a reflection of set X along the forgetful functor. In paticular, $\mathbf{N}(\ )$ is left adjoint to the forgetful functor $|\ |$,

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathbf{N}(\ )"] & \mathbf{Mon}_{\mathrm{co}} \arrow[l, bend left, "|\ |"] \end{tikzcd}$$

6.2.3 Free commutative monoids

Commutative monoids isomorphic to $\mathbf{N}X$ for some set X are referred to as *free*.

6.3 The forgetful functor $\mathbf{Ab} \longrightarrow \mathbf{Set}$

6.3.1 The free abelian group functor

Replacing everywhere the monoid $\mathbf{N}$ by the group $\mathbf{Z}$, we obtain the *free abelian group* functor,

$$X \longmapsto \mathbf{Z}X = \bigoplus_{x \in X} \mathbf{Z} \qquad (X \in \operatorname{Ob} \mathbf{Set}).$$

We can think of members of $\mathbf{Z}X$ as being formal linear combinations

$$\sum_{x \in X} l_x x \qquad (l \in \mathbf{Z}) \tag{146}$$

with only finitely many $l_x \neq 0$.

Exercise 94 *Show that* $\mathbf{Z}(\ )$ *is left adjoint to the forgetful functor* $\mathbf{Ab} \longrightarrow \mathbf{Set}$,

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathbf{Z}(\ )"] & \mathbf{Ab} \arrow[l, bend left, "|\ |"] \end{tikzcd}$$

6.3.2 Free abelian groups

Abelian groups isomorphic to $\mathbf{Z}X$ for some set X are referred to as *free*.

Exercise 95 *Find a left adjoint functor to the forgetful functor* $k\text{-}\mathbf{mod} \longrightarrow \mathbf{Set}$,

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "?"] & k\text{-}\mathbf{mod} \arrow[l, bend left, "|\ |"] \end{tikzcd}$$

and define free k*-modules.*

Exercise 96 *Let* k *be a unital ring. Find a left adjoint functor to the forgetful functor from the category of unitary* k *modules to the category of sets,*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "?"] & k\text{-}\mathbf{mod}_{\mathrm{un}} \arrow[l, bend left, "|\ |"] \end{tikzcd}$$

and define free k*-modules.*

6.4 The forgetful functor A-set $\longrightarrow$ Set

6.4.1 The category of A-sets (A is a set)

Let A be a set. Sets equipped with a family of self-mappings

$$\left(X \xrightarrow{L_a} X\right)_{a \in A}$$

will be referred as *A-sets*. They are precisely the ν-ary structures with

$$\nu\colon A \longrightarrow \mathbf{N}, \qquad \nu(a) = 1 \qquad (a \in A). \tag{147}$$

6.4.2

We shall denote $L_a(x)$ by ax. Morphisms $X \xrightarrow{f} X'$ are *equivariant* mappings, i.e., mappings satisfying

$$f(ax) = af(x) \qquad (a \in A,\ x \in X).$$

Exercise 97 *Show that a coproduct of a family of* A*-sets* $(X_i)_{\in I}$ *in the category of sets,*

$$\coprod_{i \in I} X_i,$$

is also a coproduct in A-**set**.

6.4.3 A-sets of words

For a set X, consider the disjoint union of the Cartesian products,

$$W(A;X) \ := \ X \sqcup A{\times}X \sqcup A{\times}A{\times}X \sqcup \cdots \tag{148}$$

equipped with the action of A,

$$a(a_1,\dots,a_q,x) := (a,a_1,\dots,a_q,x) \qquad (k \geq 0). \tag{149}$$

We shall refer to it as *the A-set of words with coefficients in X*.

6.4.4

Given any mapping $f\colon X \longrightarrow Y$ into an A-set Y, the formula

$$\tilde{f}\big((a_1,\dots,a_q,x)\big) := a_1(\cdots(a_q f(x))\dots) \tag{150}$$

defines a mapping $W(A;X) \longrightarrow Y$.

Exercise 98 *Show that* (150) *is equivariant. Show that if $g\colon W(A;X) \longrightarrow Y$ is an equivariant mapping whose restriction to X equals f, then $g = \tilde{f}$.*

6.4.5 The free A-set functor

Thus, inclusion

$$X \hookrightarrow W(A;X)$$

is a reflection of a set X in the category of A-sets and assignment

$$X \longmapsto W(A;X)$$

gives rise to a functor **Set** $\longrightarrow$ A-**set** that is left adjoint to the forgetful functor A-**set** $\longrightarrow$ **Set**

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "W(A;\ )"] & A\text{-}\mathbf{set} \arrow[l, bend left, "|\ |"] \end{tikzcd}$$

6.4.6 The category of A-sets (A a semigroup)

Let A be a semigroup. *Associative* A-sets, i.e., A-sets satisfying the identity

$$(aa')x = a(a'x) \qquad (a,a' \in A,\ x \in X), \tag{151}$$

form a full subcategory of the category of all A-sets. We shall denote it A_{sgr}-**set** (when there is no danger of confusing it with A-set, we shall drop subscript "sgr").

Exercise 99 *Show that a coproduct of a family of associative A-sets $(X_i)_{\in I}$ in the category of sets,*

$$\coprod_{i\in I} X_i,$$

is also a coproduct in A_{sgr}**-set**.

Exercise 100 *Show that the formulae*

$$ax := (a,x) \qquad \text{and} \qquad a(a',x) := (aa',x)$$

define an associative action of a semigroup A on the set

$$X \sqcup A{\times}X. \tag{152}$$

Show that any mapping $f\colon X \longrightarrow Y$ into any associative A-set Y extends to a unique equivariant mapping

$$X \sqcup A{\times}X \xrightarrow{\tilde f} Y \ .$$

6.4.7

Thus, inclusion

$$X \hookrightarrow X \sqcup A{\times}X$$

is a reflection of a set X in the category of associative A-sets and the assignment

$$X \longmapsto X \sqcup A{\times}X$$

gives rise to a functor **Set** $\longrightarrow$ A_{sgr}**-set** that is left adjoint to the forgetful functor A_{sgr}**-set** $\longrightarrow$ **Set**

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "(\ )\sqcup A\times(\ )"] & A_{\mathrm{sgr}}\text{-}\mathbf{set} \arrow[l, bend left, "|\ |"] \end{tikzcd}$$

6.4.8

Since A**-set** is the category of ν-ary structures, cf. (147), and A_{sgr}**-set** is the full subcategory of A**-set** defined by identity (151), it is a reflective subcategory of A**-set**.

Exercise 101 *Find*[1] *an equivariant mapping*

$$W(A;X) \longrightarrow X \sqcup A{\times}X \tag{153}$$

that is a reflection of the A-set $W(A;X)$ in the category of associative A-sets.

[1]It goes without saying that whenever you are asked to *find* any object satisfying a certain property, you must prove that the object you "found" has indeed that property.

6.4.9 The category of A-sets (A a binary structure)

The definition of an associative A-set requires that identity (151) is satisfied. It does not require that the binary structure A is itself associative. Thus, we could consider the category A_{bin}**-set** of A-sets satisfying identity (151) for any binary structure A.

Exercise 102 *Let $\phi\colon A \longrightarrow B$ be a homomorphism of binary structures. Given a B-set Y, let $\phi^{\bullet}Y$ be the same set equipped with the induced action by A,*

$$ay \;:=\; \phi(a)y \qquad (a \in A,\, y \in Y).$$

Show that the correspondence

$$Y \;\longmapsto\; \phi^{\bullet}Y \qquad (Y \in \operatorname{Ob} B_{\text{bin}}\textbf{-set})$$

gives rise to a functor $\phi^{\bullet}\colon B_{\text{bin}}\textbf{-set} \longrightarrow A_{\text{bin}}\textbf{-set}$.

Exercise 103 *Show that $\phi^{\bullet}$ is an isomorphism of categories when ϕ is a reflection of a binary structure A in the category of semigroups.*

6.4.10

In other words, A_{bin}**-set** is canonically *isomorphic* to the category $(A^{\text{as}})_{\text{bin}}$**-set** of associative sets over the semigroup A^{as}. In particular, the forgetful functor $A_{\text{bin}}\textbf{-set} \longrightarrow \textbf{Set}$ has as its left adjoint the functor that sends a set X to the A-set

$$X \sqcup A^{\text{as}}{\times}X$$

where A^{as} is canonically an A-set via the associativization homomorphism $A \twoheadrightarrow A^{\text{as}}$. Note that A-set (152) is not associative unless A is itself associative.

6.4.11 The category of A-sets (A a monoid)

Let A be a monoid. *Associative* and *unitary* A-sets, i.e., A-sets satisfying identity (151) and the identity

$$ex = x \qquad (x \in X), \tag{154}$$

where e is the identity of A, form a full subcategory of the category of all A-sets. We shall denote it A_{mon}**-set** (when there is no danger of confusing it with A-set, we shall drop subscript "mon").

Exercise 104 *Show that a coproduct of a family of associative and unitary A-sets $(X_i)_{\in I}$ in the category of sets,*

$$\coprod_{i\in I} X_i,$$

*is also a coproduct in A_{mon}**-set**.*

Exercise 105 *Show that the formulae*

$$a(a', x) := (aa', x)$$

define an associative action of a semigroup A on the set

$$A{\times}X.$$

Show that, for any mapping $f\colon X \longrightarrow Y$ into any associative unitary A-set Y, there exists a unique equivariant mapping

$$A{\times}X \xrightarrow{\tilde f} Y$$

such that $\tilde f(e, x) = f(x)$.

6.4.12

Thus, inclusion

$$X \hookrightarrow A{\times}X, \qquad x \longmapsto (e, x),$$

is a reflection of a set X in the category of associative A-sets and the assignment

$$X \longmapsto A{\times}X$$

gives rise to a functor **Set** $\longrightarrow$ A_{mon}**-set** that is left adjoint to the forgetful functor A_{mon}**-set** $\longrightarrow$ **Set**

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "A\times(\ )"] & A_{\mathrm{mon}}\text{-}\mathbf{set} \arrow[l, bend left, "|\ |"] \end{tikzcd}$$

6.4.13

Since A_{mon}**-set** is the full subcategory of A_{sgr}**-set** defined by identity (154), it is a reflective subcategory of A_{sgr}**-set**.

Exercise 106 *Find an equivariant mapping*

$$X \sqcup A{\times}X \longrightarrow A{\times}X \tag{155}$$

that is a reflection of A-set (152) in the category of associative and unitary A-sets.

6.4.14

Any associative A-set X is automatically an associative and unitary $\tilde{A}$-set where $\tilde{A}$ denotes the unitalization of A. In particular, the two categories

$$A_{\mathrm{sgr}}\text{-}\mathbf{set} \qquad \text{and} \qquad \tilde{A}_{\mathrm{mon}}\text{-}\mathbf{set}$$

are *isomorphic*. Note that

$$X \sqcup A{\times}X = \tilde{A} \times X,$$

i.e, *free* objects in A_{sgr}-**set** are free objects of A_{mon}-**set**.

6.4.15

Any A-set X is automatically an associative WA-set where WA denotes the semigroup of words on the alphabet A. In particular, the three categories

$$A\text{-}\mathbf{set}, \qquad (WA)_{\mathrm{sgr}}\text{-}\mathbf{set} \qquad \text{and} \qquad (W_{\mathrm{un}}A)_{\mathrm{mon}}\text{-}\mathbf{set}$$

are *isomorphic*. Note that

$$W(A;X) = X \sqcup WA{\times}X = W_{\mathrm{un}}A \times X,$$

i.e, *free* objects in A-**set** are free objects of A_{sgr}-**set** as well as free objects of A_{mon}-**set**.

6.5 General functors $F\colon \mathcal{C} \longrightarrow$ **Set**

6.5.1

If a set $X = \{*\}$ has a reflection along $F\colon \mathcal{C} \longrightarrow$ **Set**,

$$(\bar{X}, X \xrightarrow{\bar{\eth}} F\bar{X}),$$

then the set of mappings $X \longrightarrow Fc$ is in a natural one-to-one correspondence with the set of morphisms $\bar{X} \longrightarrow c$. For a single element X, mappings $X \longrightarrow Fc$ are in a natural one-to-one correspondence with elements of Fc. The composition of these two correspondences yields an isomorphism of functors $F \simeq \mathrm{Hom}_{\mathcal{C}}(\bar{X}, \ )$ where X is any single element set.

6.5.2

In other words, if a single element set has a reflection along $F\colon \mathcal{C} \longrightarrow$ **Set**, then F is *representable*. Moreover, it is representable by any reflection of such a set.

6.5.3 Reflections along $\mathrm{Hom}_{\mathcal{C}}(a,\ )$

Given an object $\alpha \in \mathcal{C}$ and a set X, objects of the category of arrows from X to $\mathrm{Hom}_{\mathcal{C}}(a,\mathcal{C})$ are pairs

$$\left(c, X \xrightarrow{\partial} \mathrm{Hom}_{\mathcal{C}}(a,c)\right),$$

i.e., an object $c \in \mathcal{C}$ and a family $(\alpha_x)_{x\in X}$ of morphisms $a \longrightarrow c$ indexed by X.

Exercise 107 *Show that*

$$\left(\bar{X}, X \xrightarrow{\bar{\partial}} \mathrm{Hom}_{\mathcal{C}}(a,\bar{X})\right),$$

is an initial object of the category of arrows from X to $\mathrm{Hom}_{\mathcal{C}}(a,\mathcal{C})$ if and only if the X-indexed family of arrows defined by by $\bar{\partial}\colon X \longrightarrow \mathrm{Hom}_{\mathcal{C}}(a,\bar{X})$ is a coproduct

$$\coprod_{x\in X} a \tag{156}$$

of the constant family $(a)_{x\in X}$ in $\mathcal{C}$.

6.5.4

It follows that every set X has a reflection in category $\mathcal{C}$ along functor $\mathrm{Hom}_{\mathcal{C}}(a,\ )$ if and only if all coproducts (156) exist. In particular, the correspondence

$$X \longmapsto \coprod_{x\in X} a \qquad (X \in \mathrm{Ob}\,\mathbf{Set})$$

gives rise to a functor $\mathbf{Set} \longrightarrow \mathcal{C}$ that is left adjoint to $\mathrm{Hom}_{\mathcal{C}}(a,\ )$,

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\coprod_{x\in(\ )} a"] & \mathcal{C} \arrow[l, bend left, "\mathrm{Hom}_{\mathcal{C}}(a,\ )"] \end{tikzcd}. \tag{157}$$

6.5.5

All "free structure" functors we examined are of this type. Indeed, the free structures generated by a set X are coproducts of X-indexed families of the corresponding structure generated by a single element. These were: the semigroup

$$t^{\mathbf{Z}_+} \qquad \text{(or } \mathbf{Z}_+ t \text{ in additive notation)}, \tag{158}$$

the monoid

$$t^{\mathbf{N}} \qquad \text{(or } \mathbf{N}t \text{ in additive notation)}, \tag{159}$$

and the group

$$t^{\mathbf{Z}} \qquad \text{(or } \mathbf{Z}t \text{ in additive notation)} \tag{160}$$

— all freely generated by $\{t\}$. Note that (158)–(160) are also free *power-associative*: binary structure, binary structure with identity and, respectively, loop. Their coproducts in the categories of such structures describe free objects in those categories.

6.5.6

One can further extend this list by the nonunital and, respectively, unital power-associative semirings,

$$\mathbf{Z}_+[t]t = \mathbf{Z}_+ t^{\mathbf{Z}_+} \qquad \text{and} \qquad \mathbf{Z}_+[t] = \mathbf{Z}_+ t^{\mathbf{N}},$$

by the nonunital and, respectively, unital power-associative semirings-with-zero,

$$\mathbf{N}_+[t]t = \mathbf{N} t^{\mathbf{Z}_+} \qquad \text{and} \qquad \mathbf{N}_+[t] = \mathbf{N} t^{\mathbf{N}},$$

by the nonunital and, respectively, unital power-associative rings

$$\mathbf{Z}[t]t = \mathbf{Z} t^{\mathbf{Z}_+} \qquad \text{and} \qquad \mathbf{Z}[t] = \mathbf{Z} t^{\mathbf{N}}$$

— all freely generated by $\{t\}$. They are realized as the ring of of polynomials with integral coefficients in symbolic variable t, and its sub-(semi)-rings of polynomials without constant terms, with non-negative or, finally, with positive coeeficients.

Their coproducts in the corresponding categories of associative or only power-associative rings or semirings, are the "free" objects in those categories.

6.5.7

Free A-sets are coproducts

$$\coprod_{x \in X} W_{\mathrm{un}} A,$$

free associative A-sets are coproducts

$$\coprod_{x \in X} \tilde{A}$$

and, finally, free associative A-sets are coproducts

$$\coprod_{x \in X} A.$$

In this case, the coproducts calculated in a subcategory are automatically coproducts in a larger category but the free A-sets generated by a single element set are different.

7 Tensor product

7.1 Pairings and the Hom-functor

7.1.1 Pairings in the category of sets

Let M, N and P be sets. We shall refer to mappings of two variables

$$M, N \xrightarrow{\ \phi\ } P \tag{161}$$

as *pairings* from M and N to P.

7.1.2 Induced mappings

Any pairing (161) induces two mappings

$$\lambda\colon M \longrightarrow \mathrm{Hom}_{\mathbf{Set}}(N,P), \qquad m \longmapsto \lambda_m,$$

$$\rho\colon N \longrightarrow \mathrm{Hom}_{\mathbf{Set}}(M,P), \qquad n \longmapsto \rho_n,$$

where

$$\lambda_m(n) := \phi(m,n) =: \rho_n(m) \qquad (m \in M,\ n \in N).$$

Exercise 108 *Show that the correspondence*

$$\phi \longmapsto \lambda \tag{162}$$

defines a bijection

$$\mathrm{Map}(M,N;P) \longleftrightarrow \mathrm{Hom}_{\mathbf{Set}}\big(M, \mathrm{Hom}_{\mathbf{Set}}(N,P)\big). \tag{163}$$

Exercise 109 *What bijection does the correspondence*

$$\phi \longmapsto \rho \tag{164}$$

induce?

7.1.3 Naturality in P

Postcomposing a pairing (161) with a mapping

$$h\colon P \longrightarrow P' \tag{165}$$

produces another pairing

$$M, N \xrightarrow{\ h\circ\phi\ } P'$$

and the diagram

$$
\begin{tikzcd}
\mathrm{Map}(M,N;P') \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathbf{Set}}\big(M,\mathrm{Hom}_{\mathbf{Set}}(N,P')\big) \\
\mathrm{Map}(M,N;P) \arrow[r, leftrightarrow] \arrow[u, "h\circ(\ )"] & \mathrm{Hom}_{\mathbf{Set}}\big(M,\mathrm{Hom}_{\mathbf{Set}}(N,P)\big) \arrow[u, "\big(h\circ()\big)\circ(\ )"']
\end{tikzcd}
\tag{166}
$$

whose rows are bijections (163) and columns are induced by postcomposition with (165). Commutativity of (166) is referred to as *naturality in* P.

7.1.4 Naturality in M

Next, $\circ_1$-precomposing a pairing (161) with a mapping

$$
f\colon M' \longrightarrow M \tag{167}
$$

produces the pairing

$$
M',N \xrightarrow{\phi\circ_1 f} P
$$

and the diagram

$$
\begin{tikzcd}
\mathrm{Map}(M',N;P) \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathbf{Set}}\big(M',\mathrm{Hom}_{\mathbf{Set}}(N,P)\big) \\
\mathrm{Map}(M,N;P) \arrow[r, leftrightarrow] \arrow[u, "(\ )\circ_1 f"] & \mathrm{Hom}_{\mathbf{Set}}\big(M,\mathrm{Hom}_{\mathbf{Set}}(N,P)\big) \arrow[u, "(\ )\circ_1 f"']
\end{tikzcd}
\tag{168}
$$

whose rows are bijections (163) and columns are induced by $\circ_1$-precomposition with (167). Commutativity of (168) is referred to as *naturality in* M.

7.1.5 Naturality in N

Finally, $\circ_2$-precomposing a pairing (161) with a mapping

$$
g\colon N' \longrightarrow N \tag{169}
$$

produces the pairing

$$
M,N' \xrightarrow{\phi\circ_2 g} P
$$

and the corresponding diagrams

$$\begin{tikzcd} \mathrm{Map}(M,N';P) \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathbf{Set}}\big(M,\mathrm{Hom}_{\mathbf{Set}}(N',P)\big) \\ \mathrm{Map}(M,N;P) \arrow[r, leftrightarrow] \arrow[u, "h\circ(\ )"] & \mathrm{Hom}_{\mathbf{Set}}\big(M,\mathrm{Hom}_{\mathbf{Set}}(N,P)\big) \arrow[u, "\big(()\circ_2 g\big)\circ(\ )"'] \end{tikzcd} \tag{170}$$

whose rows are bijections (163) and columns are induced by postcomposition with (165). Commutativity of (170) is referred to as *naturality in* N.

Exercise 110 *Show that diagrams* (166)*,* (168) *and* (170) *commute.*

7.1.6

Similarly, the correspondence

$$\phi \longmapsto \rho$$

defines a bijection

$$\mathrm{Map}(M,N;P) \longleftrightarrow \mathrm{Hom}_{\mathbf{Set}}\big(N,\mathrm{Hom}_{\mathbf{Set}}(M,P)\big)$$

natural in M, N and P.

7.1.7

Postcomposing or precomposing with morphisms of the category of sets is an obvious way to "generate" pairings. In fact, there exists a *universal* pairing[2]

$$M,N \xrightarrow{\ \upsilon\ } T \tag{171}$$

such that any pairing (161) can be produced from (171) by postcomposing with a *unique* mapping $h\colon T \longrightarrow P$. We shall realize the universal pairing (171) as an *initial* object in the appropriate category of pairings.

7.1.8 The category $\mathrm{Bimap}(M,N)$

The objects are pairings

$$M,N \xrightarrow{\ \phi\ } X \tag{172}$$

[2] υ is the letter *upsilon*, it precedes ϕ in the Greek alphabet and is also the first letter of the word *universal*.

with arbitrary sets X as targets. The morphisms

$$\left(\ M,N \xrightarrow{\ \phi\ } X\ \right) \longrightarrow \left(\ M,N \xrightarrow{\ \phi'\ } X'\ \right) \tag{173}$$

are mappings $h\colon X \longrightarrow X'$ such that

$$\begin{tikzcd} & X' \\ M,N \arrow[ur, "\phi'"] \arrow[dr, "\phi"'] & \\ & X \arrow[uu, dashed, "h"'] \end{tikzcd} \tag{174}$$

commutes.

7.1.9

An *initial* object (171) in $\mathrm{Bimap}(M,N)$ is called a *tensor product of* M *and* N. Since $\mathrm{Bimap}(M,N)$ is a unital category, any two initial objects are isomorphic by a unique isomorphism.

7.1.10 The functor Bimap_{MN}

Exercise 111 *Show that the correspondences*

$$X \longmapsto \mathrm{Map}(M,N;X) \qquad (X \in \mathrm{Ob}\,\mathbf{Set}),$$

and

$$h \longmapsto h \circ (\ ) \qquad (h \in \mathrm{Hom}_{\mathbf{Set}}(X,X'))$$

define a functor that will be denoted

$$\mathrm{Bimap}_{MN}\colon \mathbf{Set} \longrightarrow \mathbf{Set}.$$

Exercise 112 *Show that functor* Bimap_{MN} *is representable by a set* T *if and only if there exists a pairing* (171) *that is an initial object of category* $\mathrm{Bimap}(M,N)$.

7.1.11 Automatic naturality of tensor product

Given two pairs of sets M,N and M',N', and their tensor products

$$M,N \xrightarrow{\ v\ } T \qquad \text{and} \qquad M',N' \xrightarrow{\ v'\ } T'\ ,$$

for any pair of mappings

$$M \xrightarrow{f} M' \qquad \text{and} \qquad N \xrightarrow{g} N', \tag{175}$$

there exists a unique mapping $T \longrightarrow T'$, such that the diagram

$$\begin{tikzcd} M',N' \arrow[r, "v'"] & T' \\ M,N \arrow[u, "f,g"] \arrow[r, "v"'] & T \arrow[u, dashed] \end{tikzcd} \tag{176}$$

commutes. Indeed, $v' \circ (f,g) \colon M,N \longrightarrow T'$ uniquely factorizes through v. Let us denote this mapping by $T(f,g)$.

7.1.12

Given a third pair of sets (M'', N'') and their tensor product

$$M'', N' \xrightarrow{v''} T'' \quad ,$$

and a pair of mappings

$$M' \xrightarrow{f'} M'' \qquad \text{and} \qquad N' \xrightarrow{g'} N'',$$

we similarly obtain a unique mapping $T(f', g') \colon T' \longrightarrow T''$ for which the diagram

$$\begin{tikzcd} M'',N'' \arrow[r, "v''"] & T'' \\ M',N' \arrow[u, "f',g'"] \arrow[r, "v'"'] & T' \arrow[u, dashed, "T(f',g')"'] \end{tikzcd} \tag{177}$$

commutes. It follows that the diagram

$$\begin{tikzcd} M'',N'' \arrow[r, "v''"] & T'' \\ M,N \arrow[u, "(f',g')\circ(f,g)"] \arrow[r, "v"'] & T' \arrow[u, dashed, "T(f',g')\circ T(f,g)"'] \end{tikzcd} \tag{178}$$

does that as well. But

$$(f', g') \circ (f, g) \;=\; f' \circ f,\, g' \circ g.$$

Uniqueness of an arrow $T \longrightarrow T''$ closing up (178) to a commutative diagram thus implies that the following two mappings are equal

$$T(f' \circ f, g' \circ g) = T(f', g') \circ T(f, g).$$

7.1.13 Tensor product functors

We shall demonstrate shortly that a tensor product of any pair of sets indeed exists. As we observed above, *any* assignment of a tensor product (i.e, an initial object of category $\mathrm{Bimap}(M,N))$ to each pair of sets M and N gives rise in a unique manner to a *bifunctor*, i.e., a functor of two variables

$$\mathbf{Set}, \mathbf{Set} \xrightarrow{T} \mathbf{Set} \quad ,$$

for which those pairings

$$M,N \xrightarrow{v_{MN}} T(M,N) \qquad (M,N \in \mathrm{Ob}\,\mathbf{Set}), \tag{179}$$

are *natural* in M and N.

7.2 Naturality

Naturality here means that for any pair of mappings (175), the diagram

$$\begin{tikzcd} M',N' \arrow[r, "v_{M'N'}"] & T(M',N') \\ M,N \arrow[u, "f{,}g"] \arrow[r, "v_{MN}"'] & T(M,N) \arrow[u, "T(f{,}g)"'] \end{tikzcd} \tag{180}$$

commutes.

7.2.1

The correspondence between such assignments and the corresponding bifunctors equipped with universal pairings (179) that are natural in M and N, is bijective.

7.2.2 Existence of a tensor product

Consider the pairing

$$v_{MN}\colon M,N \longrightarrow M\times N, \qquad v_{MN}(m,n) := (m,n), \tag{181}$$

with the target being *the Cartesian product* of M and N, i.e., the set of *ordered pairs* of elements of M and N. The existence of the ordered pair is guaranteed by axioms of Set Theory. We shall refer to (181) as the *tautological pairing*.

It assigns to arguments $m \in M$ and $n \in N$ the ordered pair

$$(m,n) \in M \times N.$$

Note that the parentheses in "(m,n)" form a *part* of the standard notation for the ordered pair. On the other hand, the parentheses in "$v_{MN}(m,n)$" are present only to *delimit* the list of arguments to v_{MN}. They are entirely dispensable and are employed, like in many other mathematical formulae, to make the corresponding symbolic expressions easier to parse for a human eye.

7.2.3

As we see, the Cartesian product of M and N serves a double purpose. It provides a *binary product* of M and N in the category of sets, i.e., a projective limit of the functor $\mathbf{0}_2 \longrightarrow \mathbf{Set}$,

$$\bullet \longmapsto M, \qquad \bullet' \longmapsto N,$$

from the category $\mathbf{0}_2$ that has two objects $\bullet$ and $\bullet'$, and no morphisms.

It also *represents* mappings of two variables as mappings of a single variable, i.e., as morphisms of the category of sets (note that mappings of two variables themselves are *not* morphisms in $\mathbf{Set}$).

7.2.4

More precisely, the functor Bimap_{MN} is representable by the Cartesian product $M \times N$, and isomorphisms of functors

$$\mathrm{Hom}_{\mathbf{Set}}(M \times N, \;) \simeq \mathrm{Bimap}_{MN} \tag{182}$$

are in bijective correspondence with those pairings

$$M, N \xrightarrow{\;v\;} M \times N$$

that are initial objects of category $\mathrm{Bimap}(M,N)$. The latter serve as the *Yoneda elements* of functor isomorphisms (182), cf. Section 1.9.4. Thus, the bijection between isomorphisms (182) and initial objects of category $\mathrm{Bimap}(M,N)$ is a restriction of the bijective correspondence between natural transformations

$$\mathrm{Hom}_{\mathbf{Set}}(M \times N, \;) \longrightarrow \mathrm{Bimap}_{MN}$$

and elements of $\mathrm{Bimap}_{MN}(M \times N)$.

7.2.5

One such, *canonical*, isomorphism of functors (182) is induced by the tautological pairing, cf. (181).

Exercise 113 *Show that the correspondences*

$$M \longmapsto M \times N \qquad \textit{and} \qquad P \longmapsto \mathrm{Hom}_{\mathbf{Set}}(N, P) \qquad (M, P \in \mathrm{Ob}\,\mathbf{Set}), \tag{183}$$

give rise to functors

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "(\ )\times N"] & \mathbf{Set} \arrow[l, bend left, "\mathrm{Hom}_{\mathbf{Set}}(N,\ )"] \end{tikzcd} \tag{184}$$

and show that $(\ )\times N$ *is left adjoint to* $\mathrm{Hom}_{\mathbf{Set}}(N,\ )$.

7.2.6

Noting that

$$M \times N = \coprod_{n \in N} M,$$

we observe that the pair of adjoint functors (184) is an instance of a general case (157) examined before.

7.3 q-ary mappings

7.3.1 Terminology

We shall refer to mappings of q variables as *q-ary mappings*.

7.3.2 Ternary tensor product

By considering initial objects in the corresponding categories of *ternary* mappings $\mathrm{Map}_3(M, N, P)$,

$$M, N, P \xrightarrow{\ \phi\ } X\ ,$$

one can define in a similar manner *ternary* tensor product functors

$$,\quad \mathbf{Set}, \mathbf{Set}, \mathbf{Set} \xrightarrow{\ T\ } \mathbf{Set}$$

equipped with universal ternary mappings

$$M, N, P \xrightarrow{\ v_{MPQ}\ } T(M, N, P) \tag{185}$$

that are natural in M, N and P.

7.3.3 Naturality

Naturality here means that, for any mappings,

$$M \xrightarrow{f} M, \qquad N \xrightarrow{g} N' \qquad \text{and} \qquad P \xrightarrow{h} P', \tag{186}$$

the diagram

$$\begin{tikzcd}
M', N', P' \arrow[r, "v_{M'N'P'}"] & T(M', N', P') \\
M, N, P \arrow[u, "{f,g,h}"] \arrow[r, "v_{MNP}"'] & T(M, N, P) \arrow[u, "{T(f,g,h)}"']
\end{tikzcd} \tag{187}$$

commutes.

7.3.4

Ternary tensor product functors are again unique up to a unique isomorphism of functors *equipped with natural universal ternary mappings.*

7.3.5 "Associativity" of binary tensor product

Two iterated binary tensor products provide the corresponding triple tensor product functors:

$$M, N, P \longmapsto T\big(T(M,N), P\big) \qquad v_{MN|P} := v_{T(M,N),P} \circ_1 v_{MN}, \tag{188}$$

and

$$M, N, P \longmapsto T\big(M, T(N,P)\big), \qquad v_{M|NP} := v_{M,T(M,N)} \circ_2 v_{NP}. \tag{189}$$

For example, for the *tautological pairings* (181), the corresponding triple tensor product functor (188) becomes

$$M, N, P \longmapsto (M \times N) \times P \qquad v_{MN|P}(m,n,p) := \big((m,n),p\big)$$

while (189) becomes

$$M, N, P \longmapsto M \times (N \times P) \qquad v_{M|NP}(m,n,p) := \big(m,(n,p)\big).$$

7.3.6

Uniqueness of a triple tensor product functor *up to a unique isomorphim compatible with universal triadditive mappings* (185), means that these iterated binary tensor product functors are isomorphic via such *unique* isomorphism. This is known as *associativity* of binary tensor product. Note, that this is not the *strict* associativity in the sense of *equality* of functors. But tensor product itself is defined up to such a unique isomorphism.

7.3.7

Above we encountered a situation that is very common in modern Mathematics: associativity

up to an isomorphism of a certain kind.

In the case of tensor product, an isomorphism compatible with the data that our functors are equipped with is *unique*. In this situation, one can proceed, essentially, as if the corresponding functors were all *equal*.

7.3.8

The case of q-ary mappings is handled similarly. A standard model for the universal q-ary mapping

$$M_1, \dots, M_q \xrightarrow{\phi} X \tag{190}$$

is provided by the *tautological q-ary mapping*

$$M_1, \dots, M_q \xrightarrow{v^{\mathrm{taut}}} M_1 \times \cdots \times M_q \tag{191}$$

where

$$v^{\mathrm{taut}}(m_1, \dots, m_q) := (m_1, \dots, m_q).$$

7.3.9

Here any model can be used for the *ordered q-tuple* the most common being a mapping

$$\{1, \dots, q\} \xrightarrow{f} M_1 \cup \cdots \cup M_q$$

such that

$$f(i) \in M_i \qquad (1 \leq i \leq q).$$

7.3.10 Caveat

The habit of subconsciously identifying mappings of q-variables $m_1, \dots, m_q$ with mappings of a single variable, realized as the ordered q-tuple

$$(m_1, \dots, m_q) \in M_1 \times \cdots \times M_q,$$

is so deeply ingrained in modern mathematical notation and terminology that one loses from sight the fact that mappings of q variables form an independent concept, similar to $q + 1$-ary relations being a different concept from binary relations.

The habit of *omiting* the parentheses when writing the value of a function

$$f\colon M_1 \times \cdots \times M_q \longrightarrow N$$

as

$$f(m_1, \dots, m_q) \qquad \text{instead of} \qquad f\big((m_1, \dots, m_q)\big)$$

removes even further any distinction between the two concepts.

7.3.11

Since tensor product of q sets is realized by Cartesian product, there is no need to employ separate terminology. This is also the resaon why one normally does not hear about tensor products *of sets*. Understanding, however, the multiple roles Cartesian product plays in the category of sets helps greatly to comprehend the concept of tensor product in general as well as in concrete cases, like the categories of semigroups, monoids, abelian groups, and, more generally, G-sets, semimodules, modules, bimodules, etc.

7.4 Tensor product of commutative semigroups

7.4.1 Biadditive pairings

We shall refer to the binary operation in the category $\mathbf{Sgr}_{\mathrm{co}}$ of commutative semigroups as *addition* and denote it accordingly by employing $+$ symbol. Pairings in $\mathbf{Sgr}_{\mathrm{co}}$ are *biadditive pairings*, i.e., binary mappings (161) which in each argument are morphisms in $\mathbf{Sgr}_{\mathrm{co}}$, and that means additivity.

7.4.2 Notation

For any element m in a commutative semigroup M let

$$am = ma := \underbrace{m + \cdots + m}_{a \text{ times}} \qquad (a \in \mathbf{Z}_+) \tag{192}$$

7.4.3

The sets of morphisms $\mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(M, N)$ are naturally equipped with addition

$$(f+g)(m) := f(m) + g(m) \qquad (m \in M).$$

Exercise 114 *Show that the composition pairings*

$$\mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(M,N),\ \mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(N,P) \xrightarrow{\ \circ\ } \mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(M,P)$$

are themselves biadditive.

7.4.4

Even though $M \times N$ has a canonical commutative semigroup structure, the mapping

$$\bar{\phi}\colon M \times N \longrightarrow P$$

representing a biadditive pairing (161) is not additive because

$$\begin{aligned}\bar{\phi}\big((m,n)+(m',n')\big) = \bar{\phi}\big((m+m',n+n')\big) = \phi(m+m',n+n')\\ \neq \phi(m,n)+\phi(m',n') = \bar{\phi}\big((m,n)\big)+\bar{\phi}\big((m',n')\big).\end{aligned} \tag{193}$$

In other words, tensor product of commutative semigroups performed in the category of sets does not produce morphisms of $\mathbf{Sgr}_{\mathrm{co}}$. The problem of existence—for a given pair of commutative semigroups—of a *universal biadditive pairing* is, nevertheless, handled exactly the same way as before: a tensor product of *commutative semigroups* M and N is defined as an initial object of the corresponding category $\mathrm{Biadd}(M,N)$ of biadditive pairings whose sources are M and N.

7.4.5 The category $\mathrm{Biadd}(M,N)$

The definition of $\mathrm{Biadd}(M,N)$ is completely analogous to $\mathrm{Bimap}(M,N)$. In place of sets one considers commutative semigroups, in place of mappings – additive mappings, in place of pairings – biadditive pairings. Thus, the objects are biadditive pairings (172) with arbitrary commutative semigroups X as targets. The morphisms (173) are are additive mappings $h\colon X \longrightarrow X'$ such that diagram (174) commutes.

7.4.6

An *initial* object $M \times N \xrightarrow{v} T$ in $\mathrm{Biadd}(M,N)$ is called a *tensor product* of *commutative semigroups* M and N. Since $\mathrm{Biadd}(M,N)$ is a unital category, any two initial objects are isomorphic by a unique isomorphism.

7.4.7 The functor Biadd_{MN}

The correspondences

$$X \longmapsto \mathrm{Map}(M,N;X) \qquad (X \in \mathrm{Ob}\,\mathbf{Sgr}_{\mathrm{co}}),$$

and

$$h \longmapsto h \circ (\ ) \qquad (h \in \mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(X,X')$$

define a functor that will be denoted

$$\mathrm{Biadd}_{MN}\colon \mathbf{Sgr}_{\mathrm{co}} \longrightarrow \mathbf{Sgr}_{\mathrm{co}}.$$

Exercise 115 *Show that functor* Biadd_{MN} *is representable by a commutative semigroup* T *if and only if there exists a biadditive pairing* (171) *that is an initial object of category* $\mathrm{Biadd}(M,N)$.

7.4.8 Tensor product functors

Tensor product in the category of commutative semigroups enjoys the same automatic naturality properties as in the case of the category of sets and the same argument demonstrates that.

We shall demonstrate shortly that a tensor product of any pair of commutative semigroups indeed exists. Thus, the functors

$$\mathbf{Sgr}_{\mathrm{co}}, \mathbf{Sgr}_{\mathrm{co}} \xrightarrow{T} \mathbf{Sgr}_{\mathrm{co}}$$

equipped with biadditive pairings

$$M, N \xrightarrow{v_{MN}} T(M,N) \tag{194}$$

such that, for any pair of homomorphisms (175), the diagram

$$\begin{tikzcd} M', N' \arrow[r, "v_{M'N'}"] & T(M',N') \\ M, N \arrow[u, "f{,}g"] \arrow[r, "v_{MN}"'] & T(M,N) \arrow[u, "T(f{,}g)"'] \end{tikzcd} \tag{195}$$

commutes, are in one-to-one correspondence with assignments of a tensor product (i.e, an initial object of category $\mathrm{Biadd}(M,N)$) to each pair of commutative semigroups M and N.

7.4.9 Tensor product notation

A tensor product functor is generally denoted $\otimes$ and the value of the corresponding universal "bimorphism"

$$M, N \longrightarrow M \otimes N \ , \tag{196}$$

on $m \in M$ and $n \in N$ is denoted $m \otimes n$. The morphism that functor $\otimes$ assigns to a pair of morphisms (175), is denoted

$$f \otimes g \colon M \otimes N \longrightarrow M' \otimes N'.$$

This notational practice is applied nearly in all situations when one encounters the concept of tensor product. The category of sets (and, as we shall see soon, the categories of G-sets) are rare exceptions. In those categories another, earlier introduced structure, fulfills the purpose of tensor product. Variants to this notational practice are marked by placing various subscripts or, in case of topological tensor products, "ornaments" like $\widehat{\otimes}$.

Exercise 116 *Show that*

$$\phi(ma, n) = \phi(m, an) \qquad (m \in M,\ n \in N,\ a \in \mathbf{Z}_+)$$

for any biadditive pairing (161)*. In particular,*

$$ma \otimes n = m \otimes an \qquad (m \in M,\ n \in N,\ a \in \mathbf{Z}_+). \tag{197}$$

7.4.10 Divisible elements

An element $m \in M$ of a semigroup is said to be *q-divisible* if for every power q^d of q, $d \geq 1$, there exists an element $l \in M$ such that $m = l^{q^d}$. If M is commutative, this condition in additive notation becomes $m = q^d l$.

7.4.11 Divisible semigroups

A semigroup is q-divisible if every element is q-divisible.

Exercise 117 *Show that $M \otimes N$ is q-divisible if either M or N is divisible.*

7.4.12 Elements of finite order

An element $n \in N$ of a monoid has *finite order*, if there exists an integer $q > 0$ such that $n^q = 1$. If N is commutative, this condition in additive notation becomes $qn = 0$.

Exercise 118 *Show that $m \otimes n = 0$ in $M \otimes N$ for any q-divisible element m and any element n such that $q^d n = 0$ for some positive integer d.*

7.4.13

As a corollary we obtain that

$$\mu_\infty \otimes \mu_\infty = 0 \qquad \text{and} \qquad \mu_{p^\infty} \otimes \mu_{p^\infty} = 0$$

where μ_∞ is the multiplicative group of complex roots of unity, and μ_{p^∞} is the subgroup of roots of order being a power of prime p.

7.4.14

A corollary of the previous observation is that the abelian group

$$\mathbf{C}^* \otimes \mathbf{C}^* \tag{198}$$

has no elements of finite order since every element of the multiplicative group of complex numbers is q-divisible for any positive integer q. In other words, (198) is a uniquely divisible abelian group, i.e.. is a vector space over the field of rational numbers $\mathbf{Q}$. The quotient of this group by a weakest congruence $\sim$ such that

$$w \otimes z \sim z \otimes w \qquad \text{and} \qquad z \otimes (1-z) \sim 1 \otimes 1$$

is isomorphic to $K_2(\mathbf{C})$, the 2nd algebraic K-group of the field of complex numbers by a celebrated theorem of Matsumoto. This is one of the earliest and still a fundamental result of Algebraic K-Theory. Note that here we preserve usual multiplicative notation for multiplication in the field of complex numbers.

7.4.15

Suppose that both M and N are monoids and elements $m \in M$ and $n \in N$ satisfy

$$am = 0_M \qquad \text{and} \qquad bn = 0_N$$

for some positive integers a and b. Note that

$$m \otimes 0_N = m \otimes (a0_N) = (am) \otimes 0_N = 0_M \otimes 0_N$$

and, similarly,

$$0_M \otimes n = 0_M \otimes 0_N.$$

7.4.16

The *greatest common divisor* d of a and b can be represented as their linear combination with integral coefficients a' and b',

$$d = aa' + bb'.$$

Since $a, c, d > 0$, one of the factors a', b' is positive, another one—negative. Without loss of generality, suppose

$$a > 0, \qquad b < 0.$$

Then

$$(aa')m = a'(am) = a'0_M = 0_M \quad \text{and} \quad (-bb')n = -(b')(bn) = -(b')0_N = 0_N,$$

and

$$\begin{aligned} d(m \otimes n) &= m \otimes (dn) + m \otimes 0_N = m \otimes (dn) + m \otimes (-bb')n \\ &= m \otimes (d - bb')n = m \otimes (aa')n = (aa')m \otimes n \\ &= 0_M \otimes 0_N. \end{aligned}$$

In particular,

$$m \otimes m = 0_M \otimes 0_N$$

if a and b are relatively prime.

7.4.17 Tensor product of cyclic groups

The smallest positive integer a such that $am = 0_M$ is called the *order* of an element m in a monoid M. The submonoid m generates

$$\{0_M, m, 2m, 3m, \dots, (a-1)m\}$$

is a *cyclic group* of order a. Tensor product of two finite cyclic groups C_a and C_b of orders a and b is generated by a single element, namely $g \otimes h$, where g and h are the corresponding generators of order a and, respectively, b. As we saw in Section 7.4.16, the order of $g \otimes h$ is at most $d = \gcd(a, b)$.

Exercise 119 *Show that the pairing*

$$C_a, C_b \xrightarrow{\phi} \mathbf{Z}/d\mathbf{Z}, \qquad \phi(ig, jh) := ij \mod d, \tag{199}$$

where $i, j \in \mathbf{N}$, is well defined, is biadditive and surjective.

7.4.18

In particular, pairing (199) induces a surjective homomorphisms of $C_a \otimes C_b$ onto the cyclic group $\mathbf{Z}/d\mathbf{Z}$. Since $C_a \otimes C_b$ has no more than d elements, this must be an isomorphism. We demonstrated that the tensor product of finite cyclic groups $C_a \otimes C_b$ is a cyclic group of order $d = \gcd(a, b)$.

7.4.19 Semilattices

A commutative semigroup M is a *semilattice* if every element in M is an idempotent. Recall that

$$m \preceq m' \qquad \text{if} \qquad m + m' = m'$$

defines an order relation on M such that the binary operation becomes

$$m + m' = \sup\{m, m'\}.$$

Note that in a semilattice a sink is the greatest element $\max M$.. In particular, a semilattice has no more than a single sink. The identity element of addition, denoted 0, is the smallest element $\min M$.

Exercise 120 *Show that $M \otimes N$ is a semilattice if one of the two semigroups is a semilattice.*

Exercise 121 *Show that the* first-component *pairing*

$$M, N \xrightarrow{\pi_1} M, \qquad \pi_1(m, n) = m, \tag{200}$$

is biadditive if M is a semilattice. In particular, there exists a surjective homomorphism of commutative semigroups $M \otimes N \twoheadrightarrow M$.

7.4.20 Tensor product of a semilattice with an abelian group

Exercise 122 *Suppose that N is an abelian group and $z \in M$ is an idempotent, i.e., $z + z = z$. Show that*

$$z \otimes n = z \otimes 0_N \qquad (n \in N). \tag{201}$$

(*Hint. This is less obvious than it seems.*)

If M is a semilattice while N is an abelian group, then, according to Exercise 122, the tensor product $M \otimes N$ is additively generated by $m \otimes 0_N$, in view of the fact that every element in a semilattice is idempotent. The first-component pairing (200) is surjective and at the same time biadditive, according to Exercise 121, hence it induces a surjective homomorphism

$$M \otimes N \longrightarrow\!\!\!\!\rightarrow M \ . \tag{202}$$

This shows that all elements $m \otimes 0_N$ are different. Since

$$m \otimes 0_N + m' \otimes 0_N = (m + m') \otimes N,$$

we conclude that (202) is an isomorphism of semigroups.

7.4.21 Tensor product of two semilattices

Let us consider a special case when both M and N are semilattices. If

$$m \preceq m' \qquad \text{and} \qquad n \preceq n',$$

then

$$m + m' = m' \qquad \text{and} \qquad n + n' = n',$$

and therefore

$$\begin{aligned} (m + m') \otimes (n + n') &= m \otimes n + m' \otimes n + m \otimes n' + m' \otimes n' \\ &= m' \otimes n'. \end{aligned} \tag{203}$$

Adding $m \otimes n$ to the left side of (203) does not change it, since

$$m \otimes n + m \otimes n = (m + m) \otimes n = m \otimes n.$$

Hence

$$m \otimes n + m' \otimes n' = m' \otimes n'.$$

In particular,

$$m \otimes n \preceq m' \otimes n'. \tag{204}$$

7.4.22 Example: $\{0,1\} \otimes \{0,1\}$

Let $M = \{0,1\}$ be the simplest nontrivial semilattice, with $0 < 1$. Thus, 1 is a *sink* and 0 is the identity element of the additively written binary operation.

The tensor product $M \otimes M$ is additively generated by

$$0 \otimes 0, \quad 0 \otimes 1, \quad 1 \otimes 0 \quad \text{and} \quad 1 \otimes 1.$$

The results of Section 7.4.21 show that

$$0 \otimes 0 = \min M \otimes M, \qquad 1 \otimes 1 = \max M \otimes M,$$

while $0 \otimes 1 + 1 \otimes 0$ is greater or equal than both $0 \otimes 1$ and $1 \otimes 0$. This almost completely determines the structure of $M \otimes M$. It implies, for example, that the set

$$\left\{0 \otimes 0,\ 0 \otimes 1,\ 1 \otimes 0,\ 0 \otimes 1 + 1 \otimes 0,\ 1 \otimes 1\right\} \tag{205}$$

is closed under addition, hence equals $M \otimes M$. It remains to show that these elements are all different.

We shall consider a number of surjective homomorphisms

$$M \otimes M \twoheadrightarrow M \tag{206}$$

that will distinguish these elements. Thus, the first-component pairing (200) induces a homomorphism (206) that sends $0 \otimes 1$ to 0 while it sends $1 \otimes 0$ to 1. This shows that

$$0 \otimes 1 \ \neq \ 1 \otimes 0$$

and, since $1 \otimes 0 \ \preceq \ 0 \otimes 1 + 1 \otimes 0$, also

$$0 \otimes 1 \ \neq \ 0 \otimes 1 + 1 \otimes 0, \qquad \text{i.e.,} \qquad 0 \otimes 1 \ \prec \ 0 \otimes 1 + 1 \otimes 0.$$

By considering the homomorphism (206) induced by the second-component pairing

$$\pi_2(m, m') := m',$$

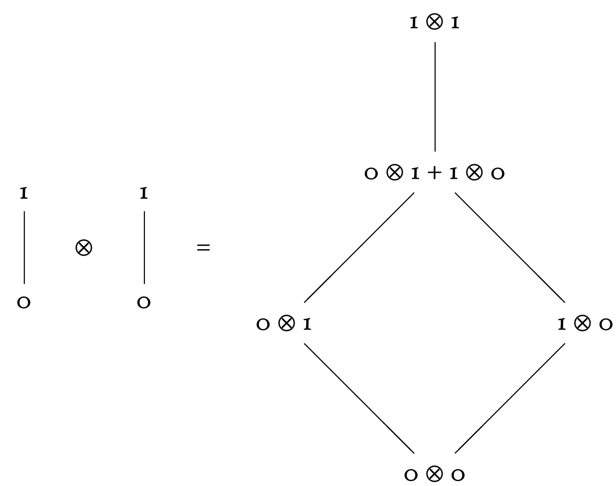
we show that

$$1 \otimes 0 \ \prec \ 0 \otimes 1 + 1 \otimes 0.$$

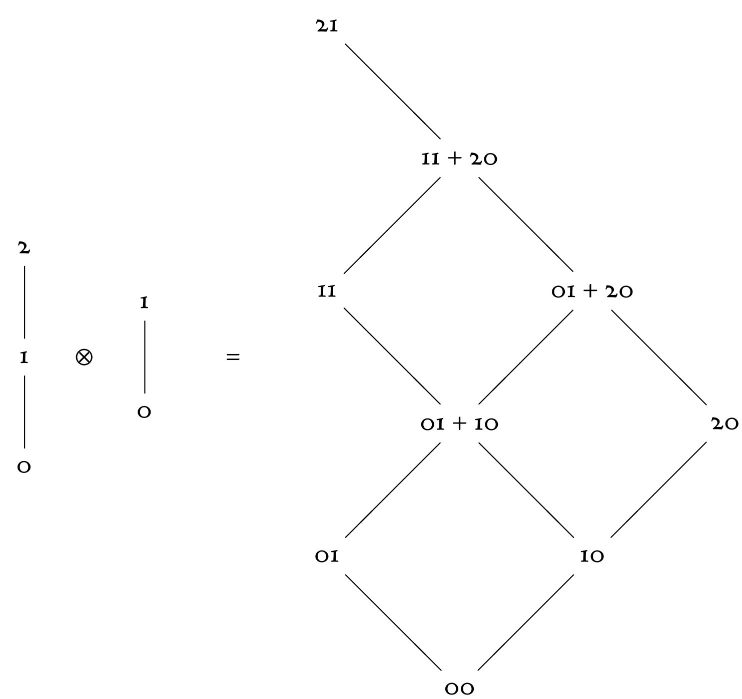
Finally, the pairing

$$\phi(m, m') := \begin{cases} 1 & \text{if } m = m' = 1 \\ 0 & \text{otherwise} \end{cases}$$

(check that it is biadditive *!*) induces a homomorphism (206) that sends $1 \otimes 1$ to 1 and every other element of (205) to 0. Thus all 5 elements of (205) are indeed different. What we demonstrated can be represented in terms of the *Hasse* diagrams of the corresponding lattices as:



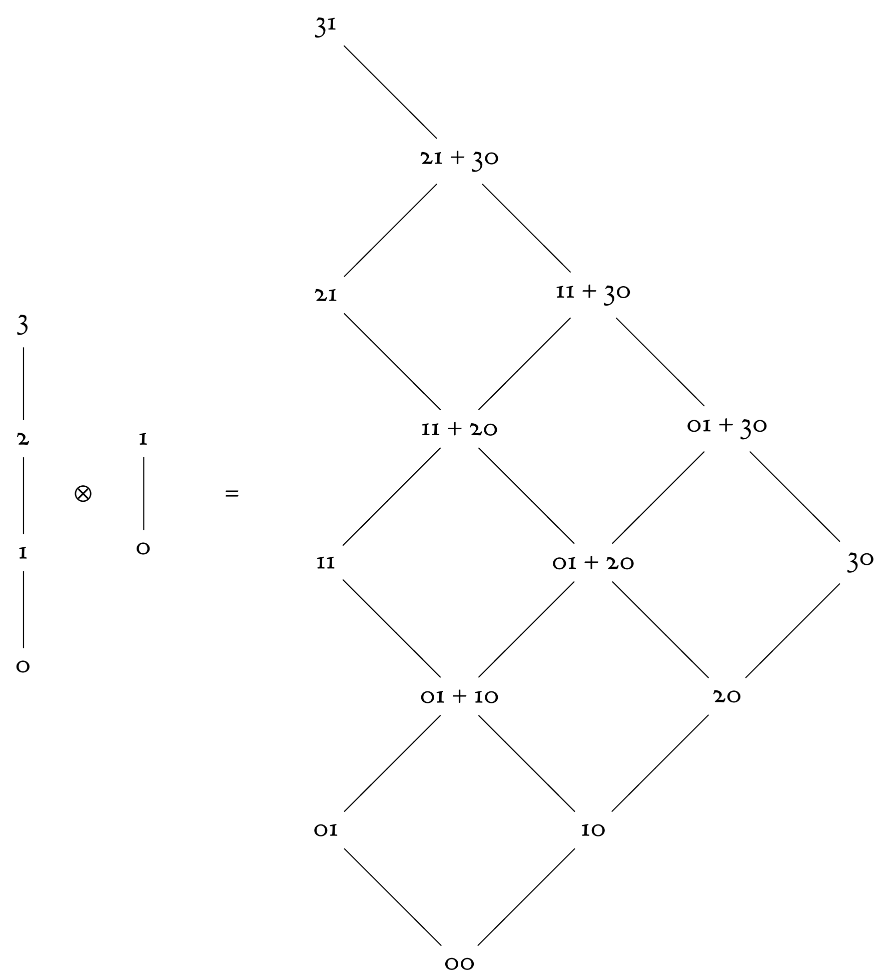
Exercise 123 *Show that the tensor product of linearly ordered sets with 3 and 2 elements is the following lattice with 9 elements*



To simplify notation we abbreviate

$$2 \otimes 1 \quad \text{to} \quad 21, \qquad 1 \otimes 1 + 2 \otimes 0 \quad \text{to} \quad 11 + 20, \qquad \text{etc.}$$

Exercise 124 *Show that the tensor product of linearly ordered sets with 4 and 2 elements is the following lattice with 14 elements*



7.4.23

One can show that the tensor product of a linearly ordered set with q elements by $\{0, 1\}$,

$$\begin{array}{ccc} q-1 & & \\ | & & 1 \\ \vdots & \otimes & | \\ | & & 0 \\ 0 & & \end{array}$$

is a lattice with

$$\frac{q(q+3)}{2} \text{ elements,}$$

of which $2q$ are *rank 1 tensors*

$$i \otimes 0 \qquad \text{and} \qquad i \otimes 1,$$

and $\frac{q(q-1)}{2}$ are *rank 1 tensors*

$$i \otimes 1 + j \otimes 0 \qquad (1 \leq i < j \leq q).$$

Exercise 125 *Draw the Hasse diagram of the tensor product of the linearly ordered set with 5 elements* $\{0, 1, 2, 3, 4\}$ *and* $\{0, 1\}$.

7.4.24

Note that we were able to answer a number of questions about tensor product of commutative semigroups without even having a single explicit *construction* of tensor product at our disposal. This is how such questions should be handled. A construction that is coming is, in fact, highly nonexplicit, and is hardly ever used in actual questions involving tensor products — the universal properties of tensor product is what is employed instead.

7.4.25 A comment on constructing morphisms $M \otimes N \longrightarrow P$

We determined the structure of $M \otimes M$, where $M = \{0, 1\}$, by constructing a number of homomorphisms into a specific semigroup (in our case it was M itself). Each such morphism was constructed by defining a biadditive pairing $M, M \xrightarrow{\phi} M$. This is the *only* admissible way of defining such homomorphisms. Attempts to define homomorphisms

$$M \otimes N \longrightarrow P$$

directly on generating elements $m \otimes n$ is not admissible in view of the fact that even though such elements generate $M \otimes N$ but they are usually subject to intricate relations. Presence and nature of such relations between tensors is known to be connected to some of the most profound phenomena in Geometry and Mathematics in general.

Exercise 126 *Show that $M \otimes N$ is a group when both M and N are abelian groups.*

7.4.26 A construction of a tensor product

Consider the *free* commutative semigroup $F(M \times N)$ with basis $M \times N$. Its elements are *formal* linear combinations

$$\sum_{(m,n)\in S} l_{mn}(m,n) \qquad (l_{mn} \in \mathbf{Z}_+),$$

where S is a *nonempty* subset of $M \times N$. Elements of $M \times N$ correspond to the sums with

$$S = \{(m,n)\} \qquad \text{and} \qquad l_{mn} = 1.$$

Any *mapping* (172) into any commutative semigroup X uniquely extends to a homomorphism

$$\tilde{\phi}\colon F(M \times N) \longrightarrow X \tag{207}$$

by the formula

$$\tilde{\phi}\Big(\sum l_{mn}(m,n)\Big) := \sum l_{mn}\beta(m,n).$$

7.4.27

Consider a weakest congruence $\sim$ on the free semigroup $F(M \times N)$ such that

$$(m+m',n) \sim (m,n)+(m',n) \quad \text{and} \quad (m,n+n') \sim (m,n)+(m,n'), \tag{208}$$

and set

$$T(M,N) := F(M \times N)_{/\sim}. \tag{209}$$

7.4.28

Denote the equivalence class of (m,n) by $\overline{(m,n)}$. By design, the pairing

$$\upsilon_{MN}\colon M \times N \longrightarrow T(M,N), \qquad (m,n) \longmapsto \overline{(m,n)} \tag{210}$$

is biadditive and homomorphism (207) uniquely factorizes through congruence $\sim$ which demonstrates that (210) is an initial object of $\mathrm{Biadd}(M,N)$.

7.4.29

Notice that the construction given above is functorial in M and N.

Exercise 127 *Show that correspondence (162) defines an isomorphism of comutative semigroups*

$$\mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(M \otimes N, P) \;\simeq\; \mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}\big(M, \mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(N, P)\big).$$

7.4.30

We obtain a pair of functors

$$\begin{tikzcd} \mathbf{Sgr}_{\mathrm{co}} \arrow[r, bend left, "(\ )\otimes N"] & \mathbf{Sgr}_{\mathrm{co}} \arrow[l, bend left, "\mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(N,\ )"] \end{tikzcd} . \tag{211}$$

Exercise 128 *Show that* $(\ )\otimes N$ *is left adjoint to* $\mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(N,\ )$.

7.5 q-ary tensor product of commutative semigroups

7.5.1 Ternary tensor product

By replacing biadditive mappings by *triadditive* mappings

$$M, N, P \longrightarrow X$$

one can similarly define tensor product functors

$$\mathbf{Sgr}_{\mathrm{co}}, \mathbf{Sgr}_{\mathrm{co}}, \mathbf{Sgr}_{\mathrm{co}} \xrightarrow{\ T\ } \mathbf{Sgr}_{\mathrm{co}}$$

equipped with universal triadditive mappings

$$M, N, P \xrightarrow{\ v_{MPQ}\ } T(M, N, P)\ . \tag{212}$$

7.5.2

Ternary tensor product functors are again unique up to a unique isomorphism of functors *equipped with natural universal triadditive ternary mappings.*

7.5.3 "Associativity" of binary tensor product

Two iterated binary tensor products provide the corresponding triple tensor product functors:

$$M, N, P \longmapsto (M \otimes N) \otimes P, \qquad v_{MN,P}\colon (m, n, p) \longmapsto (m \otimes n) \otimes p,$$

and

$$M, N, P \longmapsto M \otimes (N \otimes P), \qquad v_{M,NP}\colon (m, n, p) \longmapsto m \otimes (n \otimes p).$$

Uniqueness of a triple tensor product functor *up to a unique isomorphim compatible with universal triadditive mappings* (212), means that these two iterated binary tensor product functors are isomorphic via such *unique* isomorphism, exactly like we saw it before in the case of the category of sets. In particular, binary tensor product of commutative semigroups is "associative" in the same sense as was explained in that case.

7.5.4

The case of multiadditive mappings is handled similarly and the same comments apply as in the case of the category of sets.

Exercise 129 *Let $M_1, \dots, M_q$ be a sequence of commutative semigroups. Provide a correct definition of a q-additive q-ary mapping* (190).

7.5.5 Terminology and notation

We shall abbreviate "q-additive q-ary mapping" to *q-additive mapping* or *additive q-ary mapping*. A universal q-additive mapping $v_{M_1 \dots M_q}$ is denoted

$$M_1, \dots, M_q \longrightarrow M_1 \otimes \cdots \otimes M_q \tag{213}$$

with

$$v_{M_1 \dots M_q}(m_1, \dots, m_q) \;:=\; m_1 \otimes \cdots \otimes m_q.$$

7.6 Tensor product of semilattices

7.6.1

We observed that a tensor product of semilattices in the category of semigroups is a semilattice, cf. Exercise 120. But there is also a tensor product in the category of semilattices, namely an initial object in the category of *bimorphisms* of semilattices,

$$\mathrm{Bihom}_{\mathbf{SLt}}(M, N), \tag{214}$$

i.e., parings (172) that are homomorphisms of semilattices *in each argument*.

7.6.2 The Idem functor

The image of any homomorphism of a lattice M into a commutative semigroup X has its image contained in the set of idempotents

$$\mathrm{Idem}\, X := \{x \in X \mid 2x = x\}. \tag{215}$$

Note that (215) is a subsemigroup of X. Thus

$$\mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(M, X) \;=\; \mathrm{Hom}_{\mathbf{SLt}}(M, \mathrm{Idem}\, X)$$

which shows that assignment $X \longmapsto \mathrm{Idem}\, X$ gives rise to a functor

$$\mathbf{Sgr}_{\mathrm{co}} \longrightarrow \mathbf{SLt}$$

that is *right adjoint* to the inclusion functor

$$\mathbf{SLt} \hookrightarrow \mathbf{Sgr}_{\mathrm{co}}$$

that embeds the category of semilattices onto a full subcategory of commutative monoids which means that

$$\mathrm{Hom}_{\mathbf{SLt}}(M, N) = \mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}(M, N)$$

for any semilattices M and N.

In other words, **SLt** is a full *coreflective* subcategory of $\mathbf{Sgr}_{\mathrm{co}}$.

7.6.3

Since **SLt** is a *full* subcategory of $\mathbf{Sgr}_{\mathrm{co}}$, if a tensor product $M \otimes N$ of semilattices in the category of semigroups happens to be a semilattice, then an initial object in category $\mathrm{Biadd}(M, N)$ is also an object in category $\mathrm{Bihom}_{\mathbf{SLt}}(M, N)$, and therefore also an initial object of $\mathrm{Bihom}_{\mathbf{SLt}}(M, N)$. In other words, for semilattices,

$$M \otimes_{\mathbf{Sgr}_{\mathrm{co}}} N = M \otimes_{\mathbf{SLt}} M. \tag{216}$$

7.6.4 $\mathscr{P}^*_{\mathbf{fin}}(X \times Y) = \mathscr{P}^*_{\mathbf{fin}}(X) \otimes \mathscr{P}^*_{\mathbf{fin}}(Y)$

Let $\mathscr{P}^*_{\mathrm{fin}} X$ denote the set of nonempty finite subsets of X. Equipped with the operation of union $\cup$ it is a semilattice *freely* generated by one-element subsets: each nonempty finite subset $A \subseteq \mathscr{P}_{\mathrm{fin}}(X)$ is represented as the union of distinct one-element sets,

$$A = \{x_1\} \cup \cdots \cup \{x_q\}, \tag{217}$$

and such a representation is unique, with only those $\{x\}$ contributing to representation (217) being over all distinct elements of A.

7.6.5

In particular, each homomorphism of semilattices

$$\varphi\colon \mathscr{P}^*_{\mathrm{fin}}(X) \longrightarrow L \tag{218}$$

is determined by its restriction to the set of one-element subsets $\mathscr{P}_1(X)$, and any mapping $f\colon \mathscr{P}_1(X) \longrightarrow L$ has a unique extension to a morphism (218) in the category $\mathbf{SLt}_{\mathrm{un}}$ of semilattices,

$$\varphi_f(A) := \sum_{x\in A} f(\{x\})$$

In other words, there is a natural one-to-one correspondence

$$\mathrm{Hom}_{\mathbf{Sgr}_{\mathrm{co}}}\big(\mathscr{P}^*_{\mathrm{fin}}(X), L\big) = \mathrm{Hom}_{\mathbf{SLt}}\big(\mathscr{P}^*_{\mathrm{fin}}(X), L\big) \longleftrightarrow \mathrm{Hom}_{\mathbf{Set}}\big(\mathscr{P}_1(X), L\big) \quad .$$

7.6.6

Similarly, there is a natural one-to-one correspondence

$$\mathrm{Bihom}_{\mathbf{SLt}}\big(\mathscr{P}^*_{\mathrm{fin}}(X), \mathscr{P}^*_{\mathrm{fin}}(Y); L\big) \longleftrightarrow \mathrm{Hom}_{\mathbf{Set}}\big(\mathscr{P}_1(X\times Y), L\big) \quad .$$

A mapping $f\colon \mathscr{P}_1(X\times Y) \longrightarrow L$ has a unique extension to a biadditive mapping

$$\phi_f(A,B) := \sum_{\substack{x\in A\\ y\in B}} f\big(\{x\}\times\{y\}\big) \tag{219}$$

It follows that

$$\mathscr{P}^*_{\mathrm{fin}}(X), \mathscr{P}^*_{\mathrm{fin}}(Y) \xrightarrow{\ \times\ } \mathscr{P}^*_{\mathrm{fin}}(X\times Y) \quad , \qquad A, B \longmapsto A\times B, \tag{220}$$

is an initial object in the category of *bimorphisms* of semilattices

$$\mathrm{Bihom}_{\mathbf{SLt}}\big(\mathscr{P}^*_{\mathrm{fin}}(X), \mathscr{P}^*_{\mathrm{fin}}(Y)\big),$$

7.6.7

Note that pairing (220) extends to

$$\mathscr{P}^*(X), \mathscr{P}^*(Y) \xrightarrow{\ \times\ } \mathscr{P}^*(X\times Y) \quad , \qquad A, B \longmapsto A\times B. \tag{221}$$

This is still a tensor product of $\mathscr{P}^*(X)$ and $\mathscr{P}^*(Y)$ if at least one of the sets X or Y is finite. When both are infinite, then (221) induces only an embedding

$$\mathscr{P}^*(X)\otimes\mathscr{P}^*(Y) \hookrightarrow \mathscr{P}^*(X\times Y) \quad .$$

7.7 Tensor product of commutative monoids

7.7.1

In the category of *commutative monoids* pairings and, more generally, q-ary mappings (190), are expected to be monoid homomorphisms in each argument. This means that besides additivity also

$$\phi(m_1, \dots, m_q) = 0_X \tag{222}$$

is expected if any one $m_1, \dots, m_q$ is the identity element of the corresponding monoid (the latter in additive notation is denoted "0"). The category of commutative monoids $\mathbf{Mon}_{\mathrm{co}}$ is *not* a full subcategory of $\mathbf{Sgr}_{\mathrm{co}}$, so (222) has to be postulated.

7.7.2

With this caveat, tensor product of commutative monoids is defined exactly like for commutative semigroups, as an initial object of the corresponding category $\mathrm{Biadd}_{\mathrm{un}}(M, N)$, of biadditive *and unital in each argument* pairings (172).

7.7.3 Tensor product of unital semilattices

The argument of Section 7.6.2 shows also that unital semilattices form a full coreflective subcategory of the category of commutative monoids with the Idem functor being a right adjoint to the inclusion functor ι,

$$\begin{tikzcd} \mathbf{SLt}_{\mathrm{un}} \arrow[r, bend left, "\iota"] & \mathbf{Mon}_{\mathrm{co}} \arrow[l, bend left, "\mathrm{Idem}"] \end{tikzcd}.$$

7.7.4

Noting that a tensor product of unital semilattices is a unital semilattice, we conclude that

$$M \otimes_{\mathbf{Mon}_{\mathrm{co}}} N = M \otimes_{\mathbf{SLt}_{\mathrm{un}}} N \tag{223}$$

for any unital semilattices M and N.

7.7.5 $\mathscr{P}_{\mathbf{fin}}(X \times Y) = \mathscr{P}_{\mathbf{fin}}(X) \otimes_{\mathbf{mon}} \mathscr{P}_{\mathbf{fin}}(Y)$

The semilattice $(\mathscr{P}_{\mathrm{fin}}(X), \cup)$ of all finite subsets of a set X is *freely* generated by one-element subsets in the category of *unital* semilattices.

This is seen by inspecting formulae (218) and (219) in which one needs now to take into account also the cases when A or B are $\emptyset$. This is done by setting

$$\varphi(\emptyset) = 0 \qquad \text{and} \qquad \phi(\emptyset, B) = \phi(A, \emptyset) = 0.$$

This adaptation of the corresponding argument for $\mathscr{P}^*_{\text{fin}}$ shows that

$$\mathscr{P}_{\text{fin}}(X \times Y) = \mathscr{P}_{\text{fin}}(X) \otimes \mathscr{P}_{\text{fin}}(Y)$$

in the category of *unital* semilattices and therefore also in the category of commutative monoids.

7.7.6 Construction of a tensor product of monoids

In the construction of its existence one still employs the free commutative *semigroup* $F\langle M \times N\rangle$ and replaces congruence $\sim$ by a weakest congruence $\sim_{\text{mon}}$ on the free semigroup $F(M \times N)$ such that it satisfies both

$$(m + m', n) \sim_{\text{mon}} (m, n) + (m', n) \quad \text{and} \quad (m, n + n') \sim_{\text{mon}} (m, n) + (m, n'), \tag{224}$$

and

$$(0_M, n) \sim_{\text{mon}} (0_M, 0_N) \sim_{\text{mon}} (m, 0_N). \tag{225}$$

Then

$$T_{\text{mon}}(M, N) := F(M \times N)_{/\sim_{\text{mon}}} \tag{226}$$

provides a monoid that is a target of a universal biadditive pairing such that

$$v(0_M, n) = 0 = v(m, 0_N).$$

Note that the $0_{T_{\text{mon}}(M,N)}$ is the equivalence class of $(0_M, 0_N)$.

7.7.7

An alternative approach is to utilize tensor product in the ambient category $\mathbf{Sgr}_{\text{co}}$ and modify it by *enforcing* unitality of both the universal pairing and the induced morphisms $M \otimes N \longrightarrow P$.

So, for commutative monoids M and N, let

$$M \otimes_{\text{mon}} N := M \otimes N_{/\sim_0}$$

where $M \otimes N$ denotes a tensor product in the category of commutative semigroups and $\sim_0$ is a weakest monoid congruence such that

$$0_M \otimes n \sim_0 0_M \otimes 0_N \qquad \text{and} \qquad m \otimes 0_M \sim_0 0_M \otimes 0_N.$$

Exercise 130 *Given two monoids, show that the composition of the universal biadditive pairing* (196) *with the quotient mapping* $M \otimes N \twoheadrightarrow M \otimes_{\text{mon}} N$ *is an initial object of category* $\text{Biadd}_{\text{un}}(M, N)$.

7.7.8

We obtain a pair of functors

$$\begin{tikzcd}
\mathbf{Mon}_{\mathrm{co}} \arrow[r, bend left, "(\ )\otimes_{\mathrm{mon}} N"] & \mathbf{Mon}_{\mathrm{co}} \arrow[l, bend left, "\mathrm{Hom}_{\mathbf{Mon}_{\mathrm{co}}}(N,\ )"]
\end{tikzcd}$$

7.7.9 Notation

We shall employ notation $M \otimes_{\mathrm{mon}} N$ and $m \otimes_{\mathrm{mon}} n$ in order to indicate that we form a tensor product in the category of monoids. When there is no danger of confusion, the subscript $_{\mathrm{mon}}$ may dropped.

Exercise 131 *Show that* $\{0,1\} \otimes_{\mathrm{mon}} \{0,1\}$ *is isomorphic to* $\{0,1\}$ *.*

Exercise 132 *Show that*

$$\begin{tikzcd}
 & & & & & 2\otimes 2 \arrow[d, dash] & \\
2 \arrow[d, dash] & & 2 \arrow[d, dash] & & & 1\otimes 2 + 2\otimes 1 \arrow[dl, dash] \arrow[dr, dash] & \\
1 \arrow[d, dash] & \otimes_{\mathrm{mon}} & 1 \arrow[d, dash] & = & 1\otimes 2 \arrow[dr, dash] & & 2\otimes 1 \arrow[dl, dash] \\
0 & & 0 & & & 1\otimes 1 \arrow[d, dash] & \\
 & & & & & 0\otimes 0 &
\end{tikzcd}$$

7.7.10 Abelian groups

Exercise 133 *Show that*

$$M \otimes_{\mathrm{mon}} N \sim M \otimes N$$

when both M and N are abelian groups, i.e., show that tensor product of abelian groups formed in the category of commutative semigroups is also a tensor product of those groups in the category of commutative monoids.

Exercise 134 *Show that $M \otimes_{\text{mon}} N$ is an abelian group if one of the two monoids is a group.*

7.7.11

This is in stark contrast with tensor product of an abelian group with a monoid in $\mathbf{Sgr}_{\text{co}}$, see Exercise 7.4.20. In fact, tensor product of an abelian group N and a semilattice with identity M in the category of commutative monoids vanishes. Thus

$$M \otimes N \simeq M \qquad \text{and} \qquad M \otimes_{\text{mon}} N = 0.$$

Exercise 135 *Show that $m \otimes_{\text{mon}} n = 0_M \otimes 0_N$ when M is an abelian group and N is a semilattice with identity.*

7.7.12

Since the category of abelian groups $\mathbf{Ab}$ is a full subcategory of the category of commutative semigroups $\mathbf{Sgr}_{\text{co}}$, the pair of adjoint functors (211) restricts to the pair of similarly adjoint functors on $\mathbf{Ab}$.

$$\begin{tikzcd} \mathbf{Ab} \arrow[r, bend left, "(\ )\otimes N"] & \mathbf{Ab} \arrow[l, bend left, "\mathrm{Hom}_{\mathbf{Ab}}(N,\ )"] \end{tikzcd}. \tag{227}$$

7.7.13

In view of Exercise 134, we also obtain a pair of adjoint functors

$$\begin{tikzcd} \mathbf{Mon}_{\text{co}} \arrow[r, bend left, "(\ )\otimes_{\text{mon}} N"] & \mathbf{Ab} \arrow[l, bend left, "\mathrm{Hom}_{\mathbf{Mon}_{\text{co}}}(N,\ )"] \end{tikzcd}$$

for any abelian group N.

Exercise 136 *Show that for any group G,*

$$G \longrightarrow \mathrm{Hom}_{\mathbf{Mon}_{\text{co}}}(\mathbf{Z}, G), \qquad g \longmapsto h_g(h_g(l) := g^l),$$

is an isomorphism of groups.

7.7.14

In particular, $\mathrm{Hom}_{\mathbf{Mon}_{\text{co}}}(\mathbf{Z},\ )$ is isomorphic to the inclusion functor

$$\mathbf{Ab} \hookrightarrow \mathbf{Mon}_{\text{co}} \tag{228}$$

and we deduce that $(\ )\otimes_{\text{mon}} \mathbf{Z}$ is left adjoint to (228).

7.7.15

A left adjoint to (228) is employed in definition of the K-*functor* of a topological space and of an associative ring. The former leads to *Topological*, the latter — to *Algebraic K-Theory*. Both are among the most fundamental and most difficult subjects in modern Mathematics. Both are related to some of modern Mathematics' greatest achievements.

7.8 A-sets

7.8.1

Let A be a binary structure. The objects of the category **A-set** are sets M equipped with the *left* action of A, i.e., a homomorphism of A into the multiplicative monoid

$$L\colon A \longrightarrow \mathrm{End}_{\mathbf{Set}}(M)^{\times}, \qquad a \longmapsto L_a$$

where $L_a(m)$ is denoted am. Property of being a homomorphism of binary structures is expressed by the identity

$$(aa')m = a(a'm) \qquad (a, a' \in A,\ m \in M).$$

7.8.2

Equivariant mappings between A-sets, i.e., mappings $f\colon M \longrightarrow N$ such that

$$f(am) = af(n) \qquad (a \in A,\ m \in M), \tag{229}$$

are the morphisms in the category of A-sets.

7.8.3 Right A-sets

Sets M equipped with the *right* action of A, i.e., an anti-homomorphism of A into the multiplicative monoid

$$R\colon A \longrightarrow \mathrm{End}_{\mathbf{Set}}(M)^{\times}, \qquad a \longmapsto L_a$$

where $R_a(m)$ is denoted ma. Property of being an anti-homomorphism of binary structures is expressed by the identity

$$m(aa') = (ma)a' \qquad (a, a' \in A,\ m \in M).$$

7.8.4

Right equivariant mappings, i.e., mapping satisfying

$$f(ma) = f(m)a \qquad (a \in A,\ m \in M),$$

are morphisms and the category of right A-sets is denoted **set-A**.

7.8.5

Note that A is an A-set, via left multiplication, if and only if the multiplication in A is associative. The same for right multiplication.

7.8.6

Since each binary structure acts on M via its image in the monoid $\mathrm{End}_{\mathbf{Set}} M$, and that substructure of the multiplicative structure of $\mathrm{End}_{\mathbf{Set}} M$ is associative, one can restrict attention to actions by semigroups.

7.8.7

Structure A acts naturally both on the left and on the right on the set of *all* mappings $M \longrightarrow N$,

$$(a'f)(m) := a'\big(f(m)\big) \qquad \text{and} \qquad (fa'')(m) := f(a''m). \tag{230}$$

Exercise 137 *Show that $a'f$ and fa'' are equivariant if a' and a'' belong to the center $a' \in Z(A)$. Show that in that case*

$$a'f = fa'$$

7.8.8

It follows that

$$\mathrm{Hom}_{A\text{-}\mathbf{set}}(M, N) \tag{231}$$

is an A-set itself if A is commutative. Tensor product of A-sets M and N is the defined as an initial object in the category

$$\mathrm{Bihom}_{A\text{-}\mathbf{set}}(M, N)$$

of *biequivariant* pairings (172), exactly like in previously considered categories. The corresponding A-set

$$M \otimes_{A\text{-}\mathbf{set}} N$$

is realized as the *balanced product*,

$$M \times_A N := M \times N / \sim,$$

and the latter is defined as the quotient of $M \times N$ by the equivalence relation

$$(am, n) \sim (m, an).$$

7.9 (A,B)-bisets

7.9.1

The lack of an appropriate A-action on (231) is overcome when one realizes that each of the two A-ctions on (231) are induced by another action on M and on N, not necessarily by the same structure as A, which *commute* with the actions of A on M and N. This leads us to the notion of an (A,B)-set.

7.9.2

Let A and B be binary structures. A set M equipped with a *left* action of A and a *right* action of B such that they commute, i.e., the identity

$$(am)b = a(mb) \qquad (a \in A,\ b \in B) \tag{232}$$

holds, will be called an (A,B)*-set.* We shall say in this case that M is equiped with an (A,B)*-biaction.*

7.9.3 The categories of (A,B)-bisets

Sets equipped with an (A,B)-biaction form naturally categories denoted $A\text{-}\mathbf{set}\text{-}B$. The morphisms are mappings that are simultaneously A and B-equivariant.

7.9.4 The induced biset structure on $\mathrm{Hom}_{A\text{-}\mathbf{set}}(M,N)$

Exercise 138 *Let M be an (A,B)-set and N be an (A,C)-set. Show that*

$$(bf)(m) := \big(f(mb)\big) \qquad \text{and} \qquad (fc)(m) := \big(f(m)c\big) \qquad (b \in B,\ c \in C), \tag{233}$$

defines a (B,C)-biaction on (231).

7.9.5

When $A = B = C$ is commutative and the left and right actions of A on M coincide, and similarly for N, we speak of *symmetric A-bisets.* In this case the two A-actions (233) are nothing but the actions introduced in (230) that make $\mathrm{Hom}_{A\text{-}\mathbf{set}}(M,N)$ a symmetric A-bimodule.

7.9.6 Terminology

In general, *A-bisets* are (A,B)-bisets with $A = B$. In other words, the sets equipped with two commuting actions of A, one left, one right.

7.9.7

In absence of commutativity, one has to *postulate* separate right actions on both M and N in order to have (231) equipped with an action by A. This calls for a left and a right action on each set given separately from eacher subject to the constraint that they commute with each other. In such circumstances there is no reason to limit oneself to actions on both sides by a single structure.

7.9.8

The concept of a biset allows one to extend the notion of a tensor product to sets equipped with actions by noncommutative structures. This is done by an appropriate refinement of the notion of an q-ary morphism.

7.9.9

Let $A_0, \dots, A_q$ be a sequence of binary structures and $M_1, \dots, M_q$ be a sequence of bisets, with M_i being an (A_{i-1}, A_i)-biset, $1 \leq i \leq q$.

7.9.10

We say that a q-ary mapping (190) whose sources are (A_{i-1}, A_i)-bisets and the target is an (A_0, A_q)-set X is *balanced* if

$$\phi(a_0 m_1, m_2, \dots, m_q) = a_0 \phi(m_1, m_2, \dots, m_q), \tag{234}$$

$$\phi(m_1, \dots, m_i a_i, m_{i+1}, \dots, m_q) = \phi(m_1, \dots, m_i, a_i m_{i+1}, \dots, m_q) \qquad (1 \leq i < q), \tag{235}$$

and

$$\phi(m_1, m_2, \dots, m_q a_q) = \phi(m_1, m_2, \dots, m_q) a_q, \tag{236}$$

for all $a_i \in A_i$ and $m_i \in M_i$.

7.9.11 Balanced product

Let $\sim$ be a weakest equivalence relation on

$$M_1 \times \dots \times M_q \tag{237}$$

such that

$$(m_1, \dots, m_i a_i, m_{i+1}, \dots, m_q) \sim (m_1, \dots, m_i, a_i m_{i+1}, \dots, m_q) \qquad (1 \leq i < q)$$

for all $a_i \in A_i$ and $m_i \in M_i$.

7.9.12

Denote by

$$M_1 \times_{A_1} \cdots \times_{A_{q-1}} M_q \tag{238}$$

the quotient of (237) by $\sim$. We shall call it the *balanced product* of $M_1, \dots, M_q$.

Exercise 139 *Show that*

$$a_0 \overline{(m_1, \dots, m_q)} := \overline{(a_0 m_1, \dots, m_q)} \tag{239}$$

and

$$\overline{(m_1, \dots, m_q)} a_q := \overline{(m_1, \dots, m_q a_q)} \tag{240}$$

are well defined.

Exercise 140 *Show that the composition of the quotient mapping with the tautological q-ary mapping* (191),

$$M_1, \dots, M_q \xrightarrow{\ v^{\text{taut}}\ } M_1 \times \dots \times M_q \longrightarrow M_1 \times_{A_1} \cdots \times_{A_{q-1}} M_q \tag{241}$$

provides an initial object in the category $\operatorname{Bal\,Map}_q(M_1, \dots, M_q)$ *whose objects are balanced q-ary mappings* (190) *and morphisms are morphisms of (A_0, A_q)-bisets $h\colon X \longrightarrow X'$ such that*

$$\begin{tikzcd} & X' \\ M_1, \dots, M_q \arrow[ur, "\phi'"] \arrow[dr, "\phi"'] & \\ & X \arrow[uu, dashed, "h"'] \end{tikzcd} \tag{242}$$

commutes.

7.9.13 Special case: symmetric A-bisets

When $A_0, \dots, A_q$ are all equal to a commutative structure A, and $M_1, \dots, M_q$ are all symmetric A-bisets, then balanced q-ary mappings (190) are just q-ary morphisms of A-sets. In particular, balanced product

$$M_1 \times_A \cdots \times_A M_q$$

is a tensor product

$$M_1 \otimes_{A\text{-set}} \cdots \otimes_{A\text{-set}} M_q$$

in the category of A-sets, introduced in Section 7.8.8.

7.10

The q-ary correspondence

$$M_1, \dots, M_q \longmapsto M_1 \times_{A_1} \cdots \times_{A_{q-1}} M_q$$

gives rise to a q-ary functor

$$A_0\text{-}\mathbf{set}\text{-}A_1, \ \dots, \ A_{q-1}\text{-}\mathbf{set}\text{-}A_q \longrightarrow A_0\text{-}\mathbf{set}\text{-}A_q$$

7.10.1 Balanced pairings and the Hom-functor

Exercise 141 *For any (A,B)-set M, a (B,C)-set N and an (A,C)-set P, show that the correspondence (162) defines a bijection*

$$\mathrm{Hom}_{A\text{-}\mathbf{set}\text{-}C}(M \times_B N, P) \longleftrightarrow \mathrm{Hom}_{A\text{-}\mathbf{set}\text{-}B}\big(M, \mathrm{Hom}_{\mathbf{set}\text{-}C}(N,P)\big)$$

that is natural in M, N and P.

Exercise 142 *Show that the correspondences*

$$M \longmapsto M \times_B N \qquad (M \in \mathrm{Ob}\, A\text{-}\mathbf{set}\text{-}B), \tag{243}$$

and

$$P \longmapsto \mathrm{Hom}_{\mathbf{set}\text{-}C}(N,P) \qquad (P \in \mathrm{Ob}\, A\text{-}\mathbf{set}\text{-}C), \tag{244}$$

give rise to functors

$$\begin{tikzcd} A\text{-}\mathbf{set}\text{-}B \arrow[r, bend left, "(\ )\times_B N"] & A\text{-}\mathbf{set}\text{-}C \arrow[l, bend left, "\mathrm{Hom}_{\mathbf{set}\text{-}C}(N,\ )"] \end{tikzcd}$$

and show that $(\ )\times_B N$ is left adjoint to $\mathrm{Hom}_{\mathbf{set}\text{-}C}(N,\ )$.

Exercise 143 *For any (A,B)-set M, a (B,C)-set N and an (A,C)-set P, show that the correspondence (164) defines a bijection*

$$\mathrm{Hom}_{A\text{-}\mathbf{set}\text{-}C}(M \times_B N, P) \longleftrightarrow \mathrm{Hom}_{B\text{-}\mathbf{set}\text{-}C}\big(N, \mathrm{Hom}_{A\text{-}\mathbf{set}}(M,P)\big)$$

that is natural in M, N and P.

Exercise 144 *Show that the correspondences*

$$N \longmapsto M \times_B N \qquad (N \in \mathrm{Ob}\, B\text{-}\mathbf{set}\text{-}C), \tag{245}$$

and

$$P \longmapsto \mathrm{Hom}_{A\text{-}\mathbf{set}}(M,P) \qquad (P \in \mathrm{Ob}\, A\text{-}\mathbf{set}\text{-}C), \tag{246}$$

give rise to functors

$$
\begin{tikzcd}
B\text{-}\mathbf{set}\text{-}C \arrow[r, bend left, "M\times_B(\ )"] & A\text{-}\mathbf{set}\text{-}C \arrow[l, bend left, "\mathrm{Hom}_{A\text{-}\mathbf{set}}(M,\ )"]
\end{tikzcd}
$$

and show that $M \times (\ )$ *is left adjoint to* $\mathrm{Hom}_{A\text{-}\mathbf{set}}(M,\ )$.

7.10.2 The functor $f^{\bullet}\colon A\text{-}\mathbf{set}\text{-}C \longrightarrow A\text{-}\mathbf{set}\text{-}B$

A homomorphism of binary structures $f\colon B \longrightarrow C$ induces the functor

$$f^{\bullet}\colon A\text{-}\mathbf{set}\text{-}C \longrightarrow A\text{-}\mathbf{set}\text{-}B$$

which is identical on the underlying sets and on morphisms, preserves the left action of A and replaces the right action of C by the action of B:

$$pb := pf(b) \qquad (p \in P,\ b \in B).$$

7.10.3 The functor $f_{\bullet}\colon A\text{-}\mathbf{set}\text{-}B \longrightarrow A\text{-}\mathbf{set}\text{-}C$

When C is a semigroup, then C is naturally a (B,C)-set, with $bc := f(b)c$ and cc' being the multiplication in C. Balanced product with C over B provides the functor $f_{\bullet} = (\ ) \times_B C$.

7.10.4

The two functors

$$
\begin{tikzcd}
A\text{-}\mathbf{set}\text{-}B \arrow[r, bend left, "f_{\bullet}"] & A\text{-}\mathbf{set}\text{-}C \arrow[l, bend left, "f^{\bullet}"]
\end{tikzcd}
$$

are not adjoint, in general. If C is a monoid, then $M \times_B C$ is automatically a *unitary* C-set, and the resulting pair of functors

$$
\begin{tikzcd}
A\text{-}\mathbf{set}\text{-}B \arrow[r, bend left, "f_{\bullet}"] & A\text{-}\mathbf{set}\text{-}C \arrow[l, bend left, "f^{\bullet}"]
\end{tikzcd}
$$

is indeed adjoint with $f_{\bullet}$ being left adjoint to $f^{\bullet}$.

7.11 Semimodules

7.11.1 Semirings

Binary structures in the category of commutative semigroups are called (binary) *semirings*. Semigroups in the category of commutative semigroups are *associative semirings*.

7.11.2 Semiring $\mathrm{End}_{\mathbf{Sgr}_{\mathrm{co}}} M$

In view of biadditivity of composition in $\mathbf{Sgr}_{\mathrm{co}}$, cf. (114), the monoid of endomorphisms of any commutative semigroup M is an associative and unital semiring.

7.11.3 Semimodules

If A is a semiring, then a biadditive pairing

$$A, M \longrightarrow M, \qquad (a, m) \longmapsto am,$$

is said to be a (left) *A-semimodule* structure on a commutative semigroup M if it is simultaneously an action of the multiplicative structure of A, i.e., if

$$(aa')m = a(a'm) \qquad (a, a' \in A,\ m \in M).$$

7.11.4

An A-semimodule structure on M is the same as a homomorphism of semirings

$$L\colon A \longrightarrow \mathrm{End}_{\mathbf{Sgr}_{\mathrm{co}}} M, \qquad a \longmapsto L_a \qquad \big(L_a(m) := am\big).$$

Denote by $\bar{A} = L(A)$ the homomorphic image of A in $\mathrm{End}_{\mathbf{Sgr}_{\mathrm{co}}} M$. The action of A on M is entirely determined by the action of the associative semiring $\bar{A}$, hence the notion of a semimodule reduces essentially to the case when *the semiring of coefficients* A is associative.

7.11.5 Right semimodules

Right semimodules are defined analogously, with the multiplicative structure of A acting on a commutative semigroup M on the right.

A right A-semimodule structure on M is the same as an anati-homomorphism of semirings

$$R\colon A \longrightarrow \mathrm{End}_{\mathbf{Sgr}_{\mathrm{co}}} M, \qquad a \longmapsto R_a \qquad \big(R_a(m) := ma\big).$$

7.11.6 (A, B)-semimodules

Let A and B be semirings. A commutative semigroup M equipped with a left A-semimodule structure and a right B-semimodule structure is said to be an (A, B)-*bisemimodule*, if the two structures commute, i.e., if (232) holds.

7.11.7 Terminology

We speak of A-bisemimodules when $A = B$, and of *symmetric* A-bisemimodules when A is commutative and the two semimodule structures coincide.

7.11.8 Tensor product of semibimodules

Tensor product

$$M_1 \otimes_{A_1} \cdots \otimes_{A_q} M_q$$

of semibimodules is defined as an initial object in the category

$$\mathrm{Bal\,Add}_q(M_1, \dots, M_q) \tag{247}$$

whose objects are balanced q-additive mappings (190) and morphisms are morphisms of (A_0, A_q)-bisemimodules $h\colon X \longrightarrow X'$ such that diagram (242) commutes.

7.11.9 A construction of a tensor product of semibimodules

On a tensor product

$$M_1 \otimes \cdots \otimes M_q \tag{248}$$

of the underlying commutative semigroups, let $\sim$ be a weakest semigroup congruence such that

$$m_1 \otimes \cdots \otimes m_i a_i \otimes m_{i+1} \otimes \cdots \otimes m_q \;\sim\; m_1 \otimes \cdots \otimes m_i \otimes a_i m_{i+1} \otimes \cdots \otimes m_q \tag{249}$$

for any $1 \leq i < q$ and any $a_i \in A_i$.

7.11.10

Any q-additive mapping (190) factors uniquely through a universal q-additive pairing (213) and then, in view of being balanced, it further factors uniquely through the canonical quotient mapping

$$M_1 \otimes \cdots \otimes M_q \longrightarrow\!\!\!\!\rightarrow \left(M_1 \otimes \cdots \otimes M_q\right)_\sim \tag{250}$$

thus demonstrating that the composition of a universal q-additive pairing (213) with an (A_0, A_q)-semibimodule morphism (250) is an initial object of (247).

7.12 Semimodules and semibimodules with zero

7.12.1

Here one assumes that all semirings and semimodules have 0 and that

$$0_A m = 0_M \qquad (m \in M).$$

7.12.2 Tensor product of semibimodules with zero

Tensor product

$$M_1 \otimes_{A_1} \cdots \otimes_{A_q} M_q \tag{251}$$

of semibimodules with zero is defined as an initial object in the category

$$\mathrm{Bal\,Hom}_q(M_1, \dots, M_q) \tag{252}$$

whose objects are balanced q-additive mappings (190) sending zero to zero, and morphisms are morphisms of (A_0, A_q)-bisemimodules with zero $h\colon X \longrightarrow X'$ such that diagram (242) commutes.

7.12.3

A construction of a tensor product is a simple modification of the construction of Section 7.11.9, with the tensor product (248) replaced by the tensor product in the category of commutative monoids

$$M_1 \otimes_{\mathrm{mon}} \cdots \otimes_{\mathrm{mon}} M_q. \tag{253}$$

7.12.4 Tensor product of bimodules

When $A_0, \dots, A_q$ are associative rings and $M_1, \dots, M_q$ are the corresponding bimodules, then their tensor product (251) in the category of semibimodules is automatically an (A_0, A_q)-bimodule. Since the category of bimodules is a full subcategory of the category of semibimodules with zero, (251) is also a tensor product in the category of bimodules.

7.13 k-semialgebras

7.13.1 Binary k-semialgebras

A bisemimodule A over an associative semiring k, equipped with a bilinear multiplication

$$A, A \longrightarrow A$$

7.14 k-algebras and unitalization

7.14.1 Binary k-algebras

Let k be an associative ring. A k-bimodule A equipped with a bilinear multiplication distributive with respect to addition

$$A \times A \longrightarrow A$$

is called a *k-algebra.* A k-algebra structure on a k-bimodule is the same as a k-bimodule homomorphism

$$A \otimes_k A \longrightarrow A.$$

7.14.2 The ground ring

In this situation, k is referred to as *the ground ring* of an algebra A.

7.14.3

When k is a *unital* ring, the bimodule A is expected to be *unitary*, i.e., $1 \in k$ is supposed to act on the left and on the right as id_A. Let us denote by k-**alg** the corresponding category of associative k-algebras. It contains the full subcategory of unital k-algebras $k\text{-}\mathbf{alg}_{\mathrm{un}}$.

7.14.4

A homomorphism of k-algebras $f\colon A \longrightarrow A'$ is, by definition, a homomorphism of binary ring structures and of the underlying k-bimodule structures.

7.14.5

A homomorphism of unital k-algebras is supposed to send 1_A to $1_{A'}$.

Exercise 145 *Let A and A' be unital k-algebras over a unital ground ring. Show that a unital ring homomorphism $f\colon A \longrightarrow A'$ is automatically a k-bimodule homomorphism, hence f is a homomorphism of unital k-algebras.*

7.14.6

In particular, for unital homomorphisms, the classes of ring and of k-algebra homomorphisms coincide.

7.14.7 The unitalization functor

For any $A \in k$-**alg** consider the k-algebra

$$\tilde{A}_k := k \ltimes A. \tag{254}$$

The additive group of $\tilde{A}_k$ is $k \times A$ with multiplication given by

$$(c, a) \cdot (c', a') := (cc', ca' + ac'). \tag{255}$$

The inclusion

$$k \hookrightarrow \tilde{A}_k, \qquad c \longmapsto (c, 0), \tag{256}$$

is a homomorphism of unital k-algebras, the k-bimodule structure of $\tilde{A}$ is realized as multiplication by elements of the embedded copy of k.

Exercise 146 *Show that*

$$\tilde{f}_k \colon \tilde{A}_k \longrightarrow \tilde{A}'_k, \qquad \tilde{f}_k\big((c,a)\big) := \big(c, f(a)\big) \tag{257}$$

is a homomorphism of unital k-algebras.

7.14.8

Since one has clearly

$$\widetilde{f \circ g}_k = \tilde{f}_k \circ \tilde{g}_k \qquad \text{and} \qquad \widetilde{\mathrm{id}_A} = \mathrm{id}_{\tilde{A}_k},$$

the correspondences

$$A \longmapsto \tilde{A}_k \qquad \text{and} \qquad f \longmapsto \tilde{f}_k,$$

define a unital functor from k-**alg** to k-$\mathbf{alg}_{\mathrm{un}}$.

Exercise 147 *Show that the unitalization functor $U_k \colon k$-**alg** $\longrightarrow k$-$\mathbf{alg}_{\mathrm{un}}$ is left adjoint to the inclusion functor $\iota \colon k$-$\mathbf{alg}_{\mathrm{un}} \longrightarrow k$-**alg**.*

7.14.9 Symmetric bimodules

A right k-module structure on A is the same as a left k^{op}-module structure. When $k = k^{\mathrm{op}}$, i.e., when k is commutative, we say that A is a *symmetric* k-bimodule if the left and the right k-module structures coincide. The concept of a symmetric k-bimodule thus reduces to the concept of a k-module.

7.14.10 Z-algebras

An abelian group A is already equipped with a structure of a $\mathbf{Z}$-module,

$$na := \begin{cases} \underbrace{a + \cdots + a}_{q \text{ times}} & n > 0 \\ -(\underbrace{a + \cdots + a}_{-n \text{ times}}) & n < 0 \\ 0 & n = 0 \end{cases}$$

and this is the only unitary $\mathbf{Z}$-module structure on A. This is equivalent to a simple observation that for any unital ring R, there is only one unital homomorphism $\mathbf{Z} \longrightarrow R$.

Exercise 148 *Let R be any binary ring. For any idempotent $e \in R$,*

$$f_e\colon \mathbf{Z} \longrightarrow R, \qquad n \longmapsto ne,$$

defines a homomorphism of nonunital binary rings. Show that the correspondence, $e \longmapsto f_e$, defines a bijection

$$\{\textit{idempotents in } R\} \longleftrightarrow \left\{\begin{matrix}\textit{binary ring homomorphisms}\\ \mathbf{Z} \longrightarrow R\end{matrix}\right\}$$

7.14.11

In particular, every unitary $\mathbf{Z}$-bimodule is symmetric and any unital ring R has only one unital $\mathbf{Z}$-algebra structure.

Exercise 149 *Show tat the image of $\mathbf{Z}$ in R is contained in the* center *of R*

$$Z(R) := \{c \in R \mid [c, r] = 0 \textit{ for all } r \in R\}. \tag{258}$$

7.14.12

If we apply this observation to the ring $\mathrm{End}_{\mathbf{Ab}} M$ of endomorphisms of an abelian group M, we conclude that any left or right R-module structure on M commutes with the unique unitary $\mathbf{Z}$-module structure. In particular, a left R-module is the same as an $(R, \mathbf{Z})$-bimodule and a right R-module is the same as an $(\mathbf{Z}, R)$-module.

7.14.13

Traditionally, k-algebras over *commutative* ground rings are expected to be k-modules, i.e., *symmeric* bimodules.

7.15 Tensor algebra

7.15.1 The tensor algebra functor

For any k-bimodule M, we define its *tensor algebra* by

$$TM := \bigoplus_{q>0} M^{\otimes_k q} \tag{259}$$

where

$$M^{\otimes_k q} := M \otimes_k \cdots \otimes_k M \qquad (q \text{ times}). \tag{260}$$

7.15.2

The multipliction

$$M^{\otimes_k p} \times M^{\otimes_k q} \longrightarrow M^{\otimes_k (p+q)}$$

sends $(m_1 \otimes \cdots \otimes m_p, m_1' \otimes \cdots \otimes m_q')$ to

$$m_1 \otimes \cdots \otimes m_p \otimes m_1' \otimes \cdots \otimes m_q'.$$

Its associativity is automatic in view how we define q-ary tensor products of bimodules.

7.15.3

This defines a functor

$$T\colon k\text{-bimod} \longrightarrow k\text{-}\mathbf{alg}, \qquad M \longmapsto TM. \tag{261}$$

7.15.4

Let A be a k-algebra. To provide a bimodule mapping $TM \longrightarrow A$ is equivalent to providing a sequence of q-linear mappings

$$\alpha_q\colon M^q \longrightarrow A \qquad (q > 0).$$

Such a sequence corresponds to a homomorphism of k-algebras if and only if, for all $m_1, \dots, m_{p+q} \in M$ and all $p, q > 0$, one has

$$\alpha_{p+q}(m_1, \dots, m_{p+q}) = \alpha_p(m_1, \dots, a_p)\alpha_q(m_{p+1}, \dots, m_{p+q}). \tag{262}$$

In view of associativity of multiplication in A, identities (262) are equivalent to the identities

$$\alpha_q(m_1, \dots, m_q) = \alpha_1(m_1) \cdots \alpha_q(m_q). \tag{263}$$

In particular, a k-algebra homomorphism $TM \longrightarrow A$ is uniquely determined by its degree 1 component $\alpha_1\colon M \longrightarrow A$. Vice-versa, any bimodule homomorphism $f\colon M \longrightarrow A$ extends to a k-algebra homomorphism $TM \longrightarrow A$ by defining α_q via (263).

7.15.5

The bijective correspondence

$$\mathrm{Hom}_{k\text{-}\mathbf{alg}}(TM, A) \longleftrightarrow \mathrm{Hom}_{k\text{-bimod}}(M, A)$$

is natural both in M and A. This proves that the tensor algebra functor is a left adjoint to the forgetful functor $F\colon k\text{-}\mathbf{alg} \longrightarrow k\text{-bimod}$.

7.15.6 The unital version

On the category k-bimod$_{\mathrm{un}}$ of unitary k-bimodules over a unital ground ring k, the correspondence

$$M \longmapsto T_{\mathrm{un}}M := \bigoplus_{q \geq 0} M^{\otimes_k q} \tag{264}$$

gives rise to a functor

$$k\text{-bimod}_{\mathrm{un}} \longrightarrow k\text{-}\mathbf{alg}_{\mathrm{un}}$$

which is left adjoint to the forgetful functor

$$F_{\mathrm{un}} \colon k\text{-}\mathbf{alg}_{\mathrm{un}} \longrightarrow k\text{-bimod}_{\mathrm{un}}$$

from the category of unital k-algebras to the category of unitary k-bimodules.

7.15.7

Note that $M^{\otimes_k 0} = k$ for any, even zero, k-bimodule. In particular, $T_{\mathrm{un}} 0 = k$.

7.15.8

Note that $T_{\mathrm{un}}M$ is the unitalization of TM.