Midterm Exam: Oct. 18, in class

- covers material up to and include §4.5 (six questions total)
  - one question each from chapters 1 and 4
  - two questions each from chapters 2 and 3

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- Sample exams will be provided, solutions will be available

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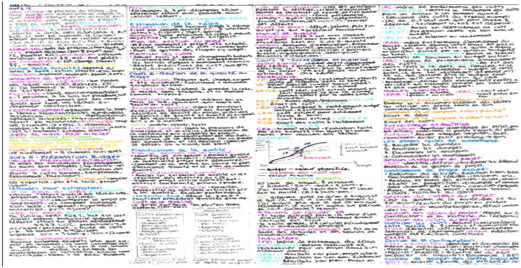
- Sample exams will be provided, solutions will be available
- one page cheat sheet on one side only
- You can skip the exam, but this is NOT encouraged
  - Final worth 50 (as opposed to 30) points if you do skip.
  - If you submit the exam, it WILL count.



Sample exam problem

- How many multiplications and additions are required to determine a sum of the form

$$S \stackrel{def}{=} \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \, \beta_j \tag{1}$$

- Modify the sum in part (1) to an equivalent form that reduces the number of computations.

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- Modify the sum in part (1) to an equivalent form that reduces the number of computations.

SOLUTION:

- it takes $m\,n$ multiplications and $m\,n$ additions in (1).
- Rewrite

$$S \stackrel{def}{=} \left(\sum_{i=1}^{n} \alpha_i\right) \left(\sum_{j=1}^{m} \beta_j\right)$$

it takes 1 multiplication and $m + n$ additions.

Sample exam problem

The following two methods are proposed to compute $7^{1/5}$.
Discuss their orders of convergence, assuming $p_0 = 1$.

1. $p_{n+1} = p_n - \frac{p_n^5 - 7}{5\, p_n^4}$
2. $p_{n+1} = p_n - \frac{p_n^5 - 7}{100}$

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2. $p_{n+1} = p_n - \frac{p_n^5 - 7}{100}$

SOLUTION:

1. Newton's method, quadratic convergence.
2. fixed point iteration, linear convergence.

Sample exam problem

A quadratic spline interpolating function S defined with the nodes $x_0 < x_1 < x_2$ is such that S is a quadratic polynomial on each of the intervals $[x_0, x_1]$ and $[x_1, x_2]$, respectively. Assume that $S(x) \in C^2[x_0, x_2]$. Show that S must be a quadratic polynomial on the entire interval $[x_0, x_2]$.

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SOLUTION: Parameterize S as

$$S(x) = \begin{cases} a_0 + b_0\,(x - x_1) + c_0\,(x - x_1)^2, & \text{if } x \in [x_0, x_1], \\ a_1 + b_1\,(x - x_1) + c_1\,(x - x_1)^2, & \text{if } x \in [x_1, x_2]. \end{cases}$$

The condition that $S(x) \in C^2[x_0, x_2]$ implies that

$$S(x)\Big|_{x=x_1^-} = S(x)\Big|_{x=x_1^+},\; S(x)'\Big|_{x=x_1^-} = S(x)'\Big|_{x=x_1^+},\; S(x)''\Big|_{x=x_1^-} = S(x)''\Big|_{x=x_1^+}$$

This leads to

$$a_0 = a_1, \quad b_0 = b_1, \quad c_0 = c_1$$

§4.3 The Trapezoidal Rule (Review) $n = 1, h = b - a$

$$\textbf{Quadrature} \quad \int_a^b f(x)dx \;=\; \frac{h}{2}\left(f(a) + f(b)\right).$$

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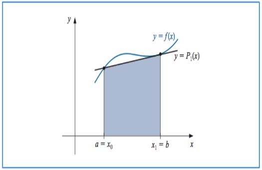
$$\textbf{Quadrature} \quad \int_a^b f(x)dx = \frac{h}{2}\left(f(a) + f(b)\right).$$

$$\begin{aligned}
\textbf{error} &= \int_a^b \frac{1}{2} f''(\xi(x))(x-a)(x-b)dx \\
&= \frac{f''(\xi)}{2}\int_a^b (x-a)(x-b)dx \\
&= -\frac{f''(\xi)}{12}(b-a)^3
\end{aligned}$$



Simpson's Rule: $n = 2$, $x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$, $h = \frac{b-a}{2}$.

- **Quadratic Interpolation**:

$$P_2(x_j) = f(x_j), \quad j = 0, 1, 2.$$

$$\begin{aligned} P_2(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_1)(x-x_0)}{(x_2-x_1)(x_2-x_0)} f(x_2) \\ &\quad + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) \\ f(x) &= P_2(x) + \frac{1}{3!} f^{(4)}(\xi(x))(x-x_0)(x-x_1)(x-x_2) \end{aligned}$$

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$$\int_a^b f(x)dx = \int_a^b P_2(x)dx + \int_a^b \left(\frac{1}{3!} f^{(4)}(\xi(x))(x-x_0)(x-x_1)(x-x_2) \right) dx$$

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Requires special error analysis

Simpson's Rule: $n = 3$, $x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$, $h = \frac{b-a}{2}$.

- **Cubic Interpolation** with $\boxed{\text{double node}}$ in x_1:

$$P_3(x_j) = f(x_j), \quad j = 0, 1, 2; \quad P_3'(x_1) = f'(x_1).$$

$$\begin{aligned} P_3(x) \;\; = \;\; & \frac{(x-x_1)^2(x-x_2)}{(x_0-x_1)^2(x_0-x_2)} f(x_0) + \frac{(x-x_1)^2(x-x_0)}{(x_2-x_1)^2(x_2-x_0)} f(x_2) \\ & + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \left(1 - \frac{(x-x_1)(2x_1-x_0-x_2)}{(x_1-x_0)(x_1-x_2)}\right) f(x_1) \\ & + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f'(x_1). \end{aligned}$$

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- **Interpolation error:**

$$f(x) = P_3(x) + \frac{1}{4!} f^{(4)}(\xi(x))(x-x_0)(x-x_1)^2(x-x_2)$$

Simpson's Rule: $n = 3$, $x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$, $h = \frac{b-a}{2}$.

- **Quadrature Rule**

$$\int_a^b P_3(x)dx = f(x_0)\int_a^b \frac{(x-x_1)^2(x-x_2)}{(x_0-x_1)^2(x_0-x_2)}\,dx + f(x_2)\int_a^b \frac{(x-x_1)^2(x-x_0)}{(x_2-x_1)^2(x_2-x_0)}\,dx$$

$$+f(x_1)\int_a^b \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\left(1 - \frac{(x-x_1)(2x_1-x_0-x_2)}{(x_1-x_0)(x_1-x_2)}\right)\,dx$$

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$\boxed{\text{Stroke of luck: } f'(x_1) \text{ does not end up in quadrature}}$

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$$\textbf{Quadrature Error} = \frac{1}{4!}\int_a^b f^{(4)}(\xi(x))(x-x_0)(x-x_1)^2(x-x_2)\,dx$$

$$= \frac{f^{(4)}(\xi)}{4!}\int_a^b (x-x_0)(x-x_1)^2(x-x_2)\,dx = -\frac{f^{(4)}(\xi)}{90}\,h^5$$

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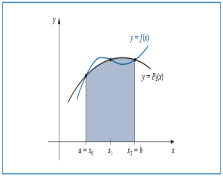

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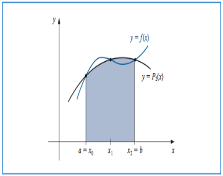

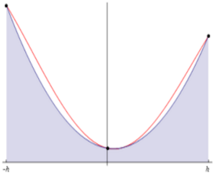

$\boxed{\text{Book wrong (middle): } f'(x_1) \neq P'(x_1)}$

$\boxed{\text{Correct plot (right): } f'(x_1) = P'(x_1)}$

Example: approximate $\int_0^2 f(x)dx$: Simpson wins

	(a)	(b)	(c)	(d)	(e)	(f)
$f(x)$	x^2	x^4	$(x+1)^{-1}$	$\sqrt{1+x^2}$	$\sin x$	e^x
Exact value	2.667	6.400	1.099	2.958	1.416	6.389
Trapezoidal	4.000	16.000	1.333	3.326	0.909	8.389
Simpson's	2.667	6.667	1.111	2.964	1.425	6.421

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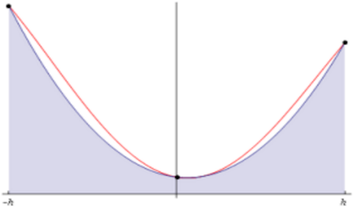
  DoP $= 1$ for Trapezoidal Rule, **DoP** $= 3$ for Simpson.

§4.4 Simpson's Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) - \frac{f^{(4)}(\xi)}{90}\, h^5.$$



$\boxed{\textbf{DoP} = 3:\ f^{(4)}(\xi) = 0 \text{ for } f(x) = 1,\ x,\ x^2,\ x^3}$

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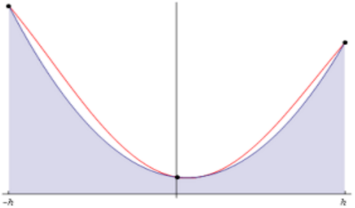

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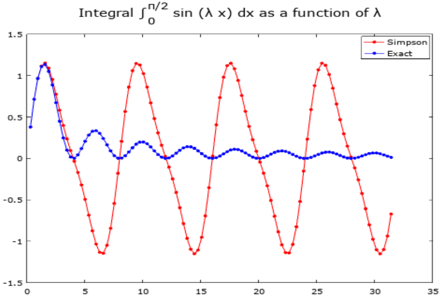
- **Example:**

$$\begin{aligned}\int_0^{\pi/2} \sin\ \lambda\, x\, dx &= -\frac{1}{\lambda}\int_0^{\pi/2} d\ \cos\ \lambda\, x \\ &= \frac{1}{\lambda}\left(1 - \cos\ \lambda\pi/2\right)\end{aligned}$$

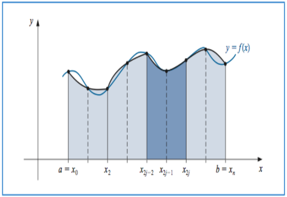
- **Simpson's Rule**

$$\int_0^{\pi/2} \sin\ \lambda\, x\, dx \approx \frac{\pi}{12}\left(4\ \sin\pi\ \lambda/4 + \sin\pi\ \lambda/2\right).$$



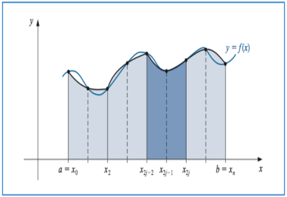
Composite Simpson's Rule ($n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n$)

$$\int_a^b f(x)dx \quad = \quad \sum_{j=1}^{m} \int_{x_{2(j-1)}}^{x_{2j}} f(x)dx$$

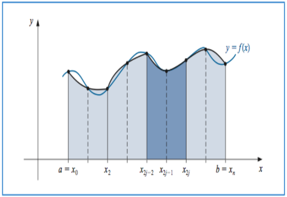


Composite Simpson's Rule ($n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n$)

$$\begin{aligned}
\int_a^b f(x)dx &= \sum_{j=1}^{m} \int_{x_{2(j-1)}}^{x_{2j}} f(x)dx \\
&= \sum_{j=1}^{m} \left(\frac{h}{3} \left(f(x_{2(j-1)}) + 4f(x_{2j-1}) + f(x_{2j}) \right) - \frac{f^{(4)}(\xi_j)}{90}\, h^5 \right)
\end{aligned}$$

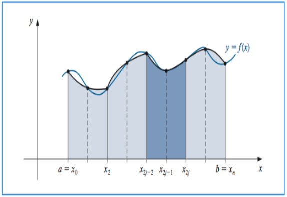


Composite Simpson's Rule ($n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n$)



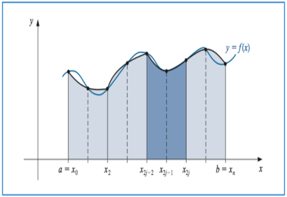
$$
\begin{aligned}
\int_a^b f(x)dx &= \sum_{j=1}^{m} \int_{x_{2(j-1)}}^{x_{2j}} f(x)dx \\
&= \sum_{j=1}^{m} \left(\frac{h}{3} \left(f(x_{2(j-1)}) + 4f(x_{2j-1}) + f(x_{2j}) \right) - \frac{f^{(4)}(\xi_j)}{90} h^5 \right) \\
&= \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b) \right) - \frac{h^5}{90} \sum_{j=1}^{m} f^{(4)}(\xi_j)
\end{aligned}
$$

Composite Simpson's Rule ($n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n$)



$$
\begin{aligned}
\int_a^b f(x)dx &= \sum_{j=1}^{m} \int_{x_{2(j-1)}}^{x_{2j}} f(x)dx \\
&= \sum_{j=1}^{m} \left(\frac{h}{3} \left(f(x_{2(j-1)}) + 4f(x_{2j-1}) + f(x_{2j}) \right) - \frac{f^{(4)}(\xi_j)}{90}\, h^5 \right) \\
&= \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b) \right) - \frac{h^5}{90} \sum_{j=1}^{m} f^{(4)}(\xi_j) \\
&= \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b) \right) - \frac{(b-a)h^4}{180} f^{(4)}(\xi)
\end{aligned}
$$

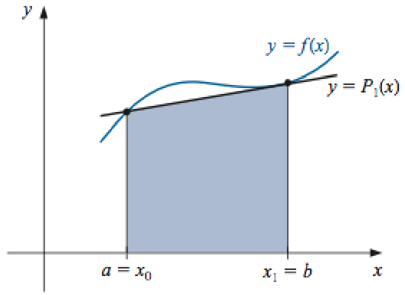
Composite Simpson's Rule ($n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \leq j \leq n$)



$$
\begin{aligned}
\int_a^b f(x)dx &= \sum_{j=1}^{m} \int_{x_{2(j-1)}}^{x_{2j}} f(x)dx \\
&= \sum_{j=1}^{m} \left(\frac{h}{3} \left(f(x_{2(j-1)}) + 4f(x_{2j-1}) + f(x_{2j}) \right) - \frac{f^{(4)}(\xi_j)}{90}\, h^5 \right) \\
&= \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b) \right) - \frac{h^5}{90} \sum_{j=1}^{m} f^{(4)}(\xi_j) \\
&= \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b) \right) - \frac{(b-a)h^{\boxed{4}}}{180} f^{(4)}(\xi)
\end{aligned}
$$

Trapezoidal Rule: $n = 1$, $x_0 = a, x_1 = b$, $h = b - a$.

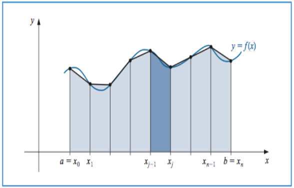
$$\int_a^b f(x)dx = \frac{h}{2}\left(f(x_0) + f(x_1)\right) - \frac{f''(\xi)}{12}h^3.$$



Degree of precision $= 1$: $f''(\xi) = 0$ for $f(x) = 1, x$

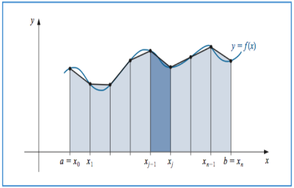
Composite Trapezoidal Rule ($x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n$)

$$\int_a^b f(x)dx \;=\; \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx$$



Composite Trapezoidal Rule ($x_j = a + j\,h,\ h = \frac{b-a}{n},\ 0 \le j \le n$)

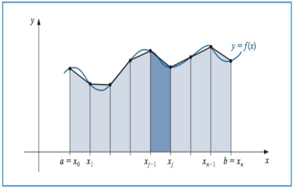
$$\int_a^b f(x)dx = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx$$



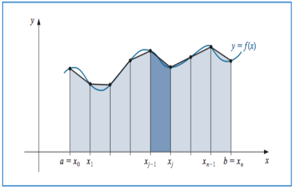
$$= \sum_{j=1}^{n} \left(\frac{h}{2}\left(f(x_{j-1}) + f(x_j)\right) - \frac{f''(\xi_j)}{12}\, h^3 \right)$$

Composite Trapezoidal Rule ($x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n$)

$$
\begin{aligned}
\int_a^b f(x)dx &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx \\
&= \sum_{j=1}^{n} \left(\frac{h}{2} \left(f(x_{j-1}) + f(x_j)\right) - \frac{f''(\xi_j)}{12} h^3 \right) \\
&= \frac{h}{2} \left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b) \right) - \frac{h^3}{12} \sum_{j=1}^{n} f''(\xi_j)
\end{aligned}
$$



Composite Trapezoidal Rule ($x_j = a + j\,h,\ h = \frac{b-a}{n},\ 0 \le j \le n$)



$$
\begin{aligned}
\int_a^b f(x)dx &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx \\
&= \sum_{j=1}^{n} \left(\frac{h}{2} \left(f(x_{j-1}) + f(x_j) \right) - \frac{f''(\xi_j)}{12} h^3 \right) \\
&= \frac{h}{2} \left(f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right) - \frac{h^3}{12} \sum_{j=1}^{n} f''(\xi_j) \\
&= \frac{h}{2} \left(f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right) - \frac{(b-a)h^{\boxed{2}}}{12} f''(\xi)
\end{aligned}
$$

For the same work, Composite Simpson yields twice as many correct digits.

Composite Simpson's Rule, example

Determine values of h for an approximation error $\leq \epsilon = 10^{-5}$ when approximating $\int_0^\pi \mathbf{sin}(x)\, dx$ with Composite Simpson.

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Solution:

$$|f^{(4)}(\mu)| = |\mathbf{sin}(\mu)| \leq 1, \quad |\mathbf{Error}| = \left|\frac{\pi\, h^4}{180} f^{(4)}(\mu)\right| \leq \frac{\pi^5}{180 n^4}.$$

Choosing

$$\frac{\pi^5}{180 n^4} \leq \epsilon, \quad \text{leading to} \quad n \geq \pi \left(\frac{\pi}{180\epsilon}\right)^{\frac{1}{4}} \approx 20.3.$$

or $h = \frac{\pi}{2m}$ with $m \geq 11$.

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or $h = \frac{\pi}{2m}$ with $m \geq 11$. For $n = 2m = 22$,

$$\begin{aligned} 2 &= \int_0^\pi \mathbf{sin}(x)\,dx \approx \frac{\pi}{3 \times 22}\left(2\sum_{j=1}^{10} \mathbf{sin}\left(\frac{j\pi}{11}\right) + 4\sum_{j=1}^{11} \mathbf{sin}\left(\frac{(2j-1)\pi}{22}\right)\right) \\ &\approx 2.0000046. \end{aligned}$$

Composite Simpson's Rule, example

Determine values of h for an approximation error $\le \epsilon = 10^{-5}$ when approximating $\int_0^\pi \mathbf{sin}(x)\,dx$ with Composite Simpson.

Solution:

$$|f^{(4)}(\mu)| = |\mathbf{sin}(\mu)| \le 1, \quad |\mathbf{Error}| = \left|\frac{\pi\, h^4}{180} f^{(4)}(\mu)\right| \le \frac{\pi^5}{180 n^4}.$$

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$$\left(\text{Trapezoidal:} \quad \int_0^\pi \mathbf{sin}(x)\,dx \approx \frac{\pi}{2 \times 22}\left(2\sum_{j=1}^{21} \mathbf{sin}\left(\frac{j\pi}{22}\right)\right) \approx 1.9966\right)$$

Composite Simpson's Rule: Round-Off Error Stability $\left(n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n\right)$

$$
\begin{aligned}
\int_a^b f(x)dx \quad \approx \quad & \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\right) \\
\stackrel{def}{=} \quad & \mathcal{I}(f).
\end{aligned}
$$

Composite Simpson's Rule: Round-Off Error Stability $\left(n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n\right)$

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\right)\\ &\stackrel{def}{=} \mathcal{I}(f).\end{aligned}$$

Assume round-off error model:

$$f(x_i) = \widehat{f}(x_i) + e_i, \quad |e_i| \le \epsilon, \quad i = 0, 1, \cdots, n.$$

Composite Simpson's Rule: Round-Off Error Stability ($n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \leq j \leq n$)

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\right) \stackrel{def}{=} \mathcal{I}(f).$$

Assume round-off error model:

$$f(x_i) = \widehat{f}(x_i) + e_i, \quad |e_i| \leq \epsilon, \quad i = 0, 1, \cdots, n.$$

$$\mathcal{I}(f) = \mathcal{I}(\widehat{f}) + \frac{h}{3}\left(e_0 + 2\sum_{j=1}^{m-1} e_{2j} + 4\sum_{j=1}^{m} e_{2j-1} + e_n\right).$$

$$\begin{aligned} |\mathcal{I}(f) - \mathcal{I}(\widehat{f})| &\leq \frac{h}{3}\left(|e_0| + 2\sum_{j=1}^{m-1} |e_{2j}| + 4\sum_{j=1}^{m} |e_{2j-1}| + |e_n|\right) \\ &\leq h\,n\,\epsilon = (b-a)\,\epsilon \quad \text{(numerically stable!!!)} \end{aligned}$$

§4.5 Recursive Composite Trapezoidal: with $n = 2^k,\ h_j = \frac{b-a}{2^j}$.

$$\int_a^b f(x)dx \quad = \quad \frac{h_k}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right) - \frac{(b-a)h_k^2}{12} f''(\mu)$$

§4.5 Recursive Composite Trapezoidal: with $n = 2^k,\ h_j = \frac{b-a}{2^j}$.

$$\begin{aligned}
\int_a^b f(x)d\,x &= \frac{h_k}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right) - \frac{(b-a)h_k^2}{12} f''(\mu) \\
&\underline{\underline{\ \textit{book}\ }}\ \frac{h_k}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right) + \sum_{j=1}^{\infty} K_j h_k^{2j}. \\
&\underline{\underline{\ \textit{def}\ }}\ \mathbf{R}_{k,1} + \sum_{j=1}^{\infty} K_j h_k^{2j}.
\end{aligned}$$

Recursive Composite Trapezoidal: with $n = 2^k$, $h_j = \frac{b-a}{2^j}$.

$$\int_a^b f(x)dx = \mathbf{R}_{k,1} + \sum_{j=1}^{\infty} K_j h_k^{2j}, \quad | \quad \mathbf{R}_{k,1} \stackrel{def}{=} \frac{h_k}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right).$$

Recursive Composite Trapezoidal: with $n = 2^k$, $h_j = \frac{b-a}{2^j}$.

$$\int_a^b f(x)d\,x = \mathbf{R}_{k,1} + \sum_{j=1}^{\infty} K_j h_k^{2j}, \quad | \quad \mathbf{R}_{k,1} \stackrel{def}{=} \frac{h_k}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right).$$

$$\mathbf{R}_{1,1} \quad = \quad \frac{h_0}{2}\left(f(a) + f(b)\right) = \frac{b-a}{2}\left(f(a) + f(b)\right) \quad \left(\stackrel{def}{=} \mathcal{N}_1\left(h_0\right)\right),$$

Recursive Composite Trapezoidal: with $n = 2^k$, $h_j = \frac{b-a}{2^j}$.

$$\int_a^b f(x)d\,x = \mathbf{R}_{k,1} + \sum_{j=1}^{\infty} K_j h_k^{2j}, \quad | \quad \mathbf{R}_{k,1} \overset{def}{=} \frac{h_k}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right).$$

$$\begin{aligned}
\mathbf{R}_{1,1} &= \frac{h_0}{2}\left(f(a) + f(b)\right) = \frac{b-a}{2}\left(f(a) + f(b)\right) \quad \left(\overset{def}{=} \mathcal{N}_1\left(h_0\right)\right), \\
\mathbf{R}_{2,1} &= \frac{h_1}{2}\left(f(a) + f(b) + 2\,f(a + h_1)\right) \\
&= \frac{1}{2}\left(\mathbf{R}_{1,1} + h_0\,f(a + h_1)\right), \quad \left(\overset{def}{=} \mathcal{N}_1\left(\frac{h_0}{2}\right)\right)
\end{aligned}$$

Recursive Composite Trapezoidal: with $n = 2^k$, $h_j = \frac{b-a}{2^j}$.

$$\int_a^b f(x)d\,x = \mathbf{R}_{k,1} + \sum_{j=1}^{\infty} K_j h_k^{2j}, \quad | \quad \mathbf{R}_{k,1} \stackrel{def}{=} \frac{h_k}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right).$$

$$\begin{aligned}
\mathbf{R}_{1,1} &= \frac{h_0}{2}\left(f(a)+f(b)\right) = \frac{b-a}{2}\left(f(a)+f(b)\right) \quad \left(\stackrel{def}{=} \mathcal{N}_1\left(h_0\right)\right), \\
\mathbf{R}_{2,1} &= \frac{h_1}{2}\left(f(a)+f(b)+2\,f(a+h_1)\right) \\
&= \frac{1}{2}\left(\mathbf{R}_{1,1} + h_0\,f(a+h_1)\right), \quad \left(\stackrel{def}{=} \mathcal{N}_1\left(\frac{h_0}{2}\right)\right) \\
\vdots \quad & \quad \vdots \\
\mathbf{R}_{k,1} &= \frac{1}{2}\left(\mathbf{R}_{k-1,1} + h_{k-2}\sum_{j=1}^{2^{k-2}} f(a+(2j-1)h_{k-1})\right)
\end{aligned}$$

Recursive Composite Trapezoidal: with $n = 2^k$, $h_j = \frac{b-a}{2^j}$.

$$\int_a^b f(x)\,dx = \mathbf{R}_{k,1} + \sum_{j=1}^{\infty} K_j h_k^{2j}, \quad | \quad \mathbf{R}_{k,1} \stackrel{def}{=} \frac{h_k}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right).$$

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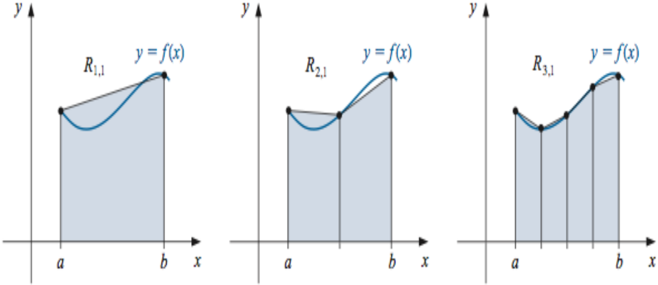
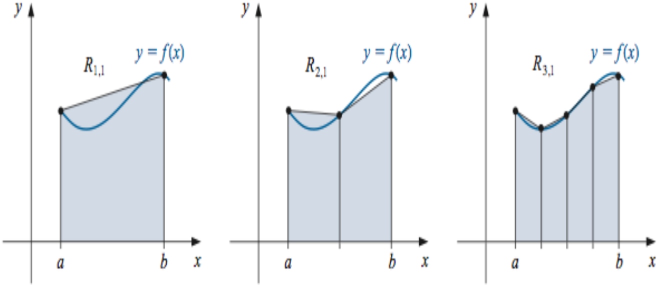
\mathbf{R}_{1,1} &= \frac{h_0}{2}\left(f(a) + f(b)\right) = \frac{b-a}{2}\left(f(a) + f(b)\right) \quad \left(\stackrel{def}{=} \mathcal{N}_1\left(h_0\right)\right), \\
\mathbf{R}_{2,1} &= \frac{h_1}{2}\left(f(a) + f(b) + 2\,f(a + h_1)\right) \\
&= \frac{1}{2}\left(\mathbf{R}_{1,1} + h_0\,f(a + h_1)\right), \quad \left(\stackrel{def}{=} \mathcal{N}_1\left(\frac{h_0}{2}\right)\right) \\
\vdots \quad & \quad \vdots \\
\mathbf{R}_{k,1} &= \frac{1}{2}\left(\mathbf{R}_{k-1,1} + h_{k-2}\sum_{j=1}^{2^{k-2}} f(a + (2j-1)h_{k-1})\right) \quad \left(\stackrel{def}{=} \mathcal{N}_1\left(\frac{h_0}{2^{k-1}}\right)\right),
\end{aligned}$$



y
$R_{1,1}$
$y = f(x)$
a
b
x
y
$R_{2,1}$
$y = f(x)$
a
b
x
y
$R_{3,1}$
$y = f(x)$
a
b
x



Romberg Extrapolation (= even term Richardson Extrapolation)

$$
\begin{array}{llll}
O(h_k^2) & O(h_k^4) & O(h_k^6) & O(h_k^8) \\
\hline
R_{1,1} \searrow & & & \\
\quad \rightarrow & & & \\
R_{2,1} \searrow & R_{2,2} \searrow & & \\
\quad \rightarrow & \quad \rightarrow & & \\
R_{3,1} \searrow & R_{3,2} \searrow & R_{3,3} \searrow & \\
R_{4,1} \rightarrow & R_{4,2} \rightarrow & R_{4,3} \rightarrow & R_{4,4}
\end{array}
$$

Romberg Extrapolation for $\int_0^{\pi} \sin(x)\, dx,\ n = 1, 2, 2^2, 2^3, 2^4, 2^5.$

$$
\begin{aligned}
R_{1,1} &= \frac{\pi}{2}\left(\sin(0) + \sin(\pi)\right) = 0, \\
R_{2,1} &= \frac{1}{2}\left(R_{1,1} + \pi \sin(\frac{\pi}{2})\right) = 1.57079633, \\
R_{3,1} &= \frac{1}{2}\left(R_{2,1} + \frac{\pi}{2}\sum_{j=1}^{2} \sin(\frac{(2j-1)\pi}{4})\right) = 1.89611890, \\
R_{4,1} &= \frac{1}{2}\left(R_{3,1} + \frac{\pi}{4}\sum_{j=1}^{4} \sin(\frac{(2j-1)\pi}{8})\right) = 1.97423160, \\
R_{5,1} &= \frac{1}{2}\left(R_{4,1} + \frac{\pi}{8}\sum_{j=1}^{8} \sin(\frac{(2j-1)\pi}{16})\right) = 1.99357034, \\
R_{6,1} &= \frac{1}{2}\left(R_{5,1} + \frac{\pi}{16}\sum_{j=1}^{2^4} \sin(\frac{(2j-1)\pi}{32})\right) = 1.99839336.
\end{aligned}
$$

Romberg Extrapolation, $\int_0^\pi \mathbf{sin}(x)\,dx = 2$

0					
1.57079633	2.09439511				
1.89611890	2.00455976	1.99857073			
1.97423160	2.00026917	1.99998313	2.00000555		
1.99357034	2.00001659	1.99999975	2.00000001	1.9999999	
1.99839336	2.00000103	2.00000000	2.00000000	2.0000000	2.0000000

33 FUNCTION EVALUATIONS USED IN THE TABLE.

Recursive Composite Simpson:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\right)$$
$$- \frac{(b-a)h^4}{12} f^{(4)}(\mu)$$

Recursive Composite Simpson:

$$\begin{aligned}
\int_a^b f(x)dx \quad &\approx \quad \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\right) \\
&\quad -\frac{(b-a)h^4}{12} f^{(4)}(\mu) \\
&\overset{exists}{=\!=\!=} \quad \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\right) \\
&\quad + \sum_{j=\boxed{2}}^{\infty} K_j h^{2j}. \\
&\overset{def}{=\!=\!=} \quad \mathbf{R}_{k,1} + \sum_{j=2}^{\infty} K_j h^{2j}, \quad \text{for} \quad n = 2^k.
\end{aligned}$$

Recursive Composite Simpson: with $h_k = (b-a)/2^{k-1}$.

$$\int_a^b f(x)dx \approx \quad \mathbf{R}_{k,1} + \sum_{j=2}^{\infty} K_j h^{2j}, \quad \text{for} \quad n = 2^k.$$

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$$\int_a^b f(x)dx \approx \quad \mathbf{R}_{k,1} + \sum_{j=2}^{\infty} K_j h^{2j}, \quad \text{for} \quad n = 2^k.$$

$$\mathbf{R}_{1,1} \quad = \quad \frac{b-a}{6}\left(f(a) + 4\mathbf{S}_1 + f(b)\right), \quad \mathbf{S}_1 = f((a+b)/2),$$

Recursive Composite Simpson: with $h_k = (b-a)/2^{k-1}$.

$$\int_a^b f(x)dx \approx \quad \mathbf{R}_{k,1} + \sum_{j=2}^{\infty} K_j h^{2j}, \;\; \text{for} \;\; n = 2^k.$$

$$\begin{aligned}
\mathbf{R}_{1,1} &= \frac{b-a}{6}\left(f(a) + 4\mathbf{S}_1 + f(b)\right), \quad \mathbf{S}_1 = f((a+b)/2), \\
\vdots \quad & \quad \vdots \\
\mathbf{T}_k &= \sum_{j=1}^{2^{k-1}} f(a + (2j-1)h_k), \\
\mathbf{R}_{k,1} &= \frac{h_k}{3}\left(f(a) + 2\mathbf{S}_{k-1} + 4\mathbf{T}_k + f(b)\right), \\
\mathbf{S}_k &= \mathbf{S}_{k-1} + \mathbf{T}_k, \; k = 2, \cdots, \mathbf{log}_2 n.
\end{aligned}$$

Romberg Extrapolation Table for Simpson Rule

$$
\begin{array}{llll}
O(h_k^4) & O(h_k^6) & O(h_k^8) & O(h_k^{10}) \\
\hline
R_{1,1} \searrow & & & \\
\quad\to & & & \\
R_{2,1} \searrow & R_{2,2} \searrow & & \\
\quad\to & \quad\to & & \\
R_{3,1} \searrow & R_{3,2} \searrow & R_{3,3} \searrow & \\
R_{4,1} \to & R_{4,2} \to & R_{4,3} \to & R_{4,4}
\end{array}
$$

Special even term Richardson Extrapolation

Tricks of the Trade for $\int_a^b f(x)dx$

- Composite Simpson/Trapezoidal rules:
  - Adding more EQUI-SPACED points.

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- Next trick: $\boxed{\text{Adaptive quadratures:}}$
  - Adding more points ONLY WHEN NECESSARY.

$\boxed{\texttt{quad}\text{ function of }\texttt{matlab}\text{: combines all three tricks.}}$

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$\boxed{\text{Crying baby Principle}}$

in adaptive algorithms

ADD MORE POINTS IN REGIONS OF INADEQUATE ACCURACY

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$\boxed{\textbf{crying babies get more candies}}$

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Crying baby Principle

in adaptive algorithms

ADD MORE POINTS IN REGIONS OF INADEQUATE ACCURACY

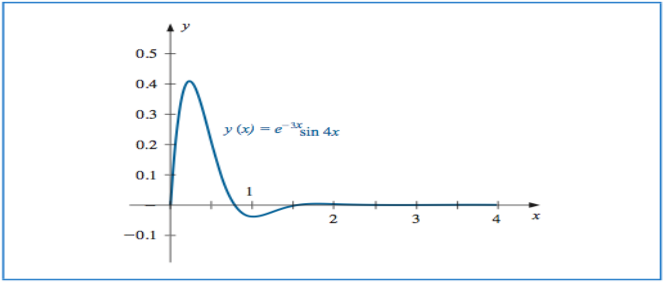
crying babies get more candies

presumed innocence: accuracy adequate unless detected otherwise

§4.6 Adaptive Quadrature Methods: step-size matters

$$y(x) = \mathbf{e}^{-3x}\mathbf{sin}\,4x.$$

- Oscillation for small x; nearly 0 for larger x.
  - Mechanical engineering
    (spring and shock absorber systems)
  - Electrical engineering
    (circuit simulations)



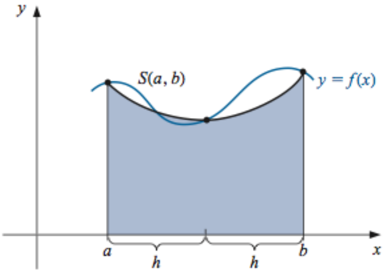
- $y(x)$ behaves differently for small x than for large x.

Adaptive Quadrature (I)

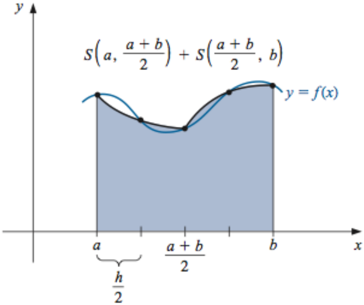
- $$\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} f^{(4)}(\xi), \quad \xi \in (a,b),$$

where $\quad S(a,b) = \frac{h}{3}\left(f(a) + 4f(a+h) + f(b)\right), \quad h = \frac{b-a}{2}.$

Simpson on $[a, b]$



Composite Simpson



Adaptive Quadrature (II)

- $$\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} f^{(4)}(\xi), \quad \xi \in (a,b)$$

- $$\begin{aligned} \int_a^b f(x)dx &= \int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx \\ &= S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b) \\ &\quad - \frac{(h/2)^5}{90} f^{(4)}(\xi_1) - \frac{(h/2)^5}{90} f^{(4)}(\xi_2) \\ = S(a, \frac{a+b}{2}) &+ S(\frac{a+b}{2}, b) - \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\widehat{\xi}), \end{aligned}$$

  where

$$\xi_1 \in (a, \frac{a+b}{2}), \quad \xi_2 \in (\frac{a+b}{2}, b), \quad \widehat{\xi} \in (a,b).$$

Adaptive Quadrature (II)

- $$\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} f^{(4)}(\xi), \quad \xi \in (a,b)$$

- $$\begin{aligned}\int_a^b f(x)dx &= \int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx \\ &= S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b) \\ &\quad - \frac{(h/2)^5}{90} f^{(4)}(\xi_1) - \frac{(h/2)^5}{90} f^{(4)}(\xi_2) \\ &= S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b) - \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\widehat{\xi}),\end{aligned}$$

  where

  $$\xi_1 \in (a, \frac{a+b}{2}), \quad \xi_2 \in (\frac{a+b}{2}, b), \quad \widehat{\xi} \in (a,b).$$

$$\frac{h^5}{90} f^{(4)}(\xi) \overset{hopefully}{\approx} \frac{h^5}{90} f^{(4)}(\widehat{\xi}).$$

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  where

  $$\xi_1 \in (a, \frac{a+b}{2}), \quad \xi_2 \in (\frac{a+b}{2}, b), \quad \widehat{\xi} \in (a,b).$$

$$\frac{h^5}{90} f^{(4)}(\xi) \overset{hopefully}{\approx} \frac{h^5}{90} f^{(4)}(\widehat{\xi}).$$

$$\left(\frac{h^5}{90}\right) f^{(4)}(\widehat{\xi}) \approx \frac{16}{15}\left(S(a,b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b)\right)$$

Adaptive Quadrature (II)

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$$= S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b) - \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\widehat{\xi}),$$

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$$\xi_1 \in (a, \frac{a+b}{2}), \quad \xi_2 \in (\frac{a+b}{2}, b), \quad \widehat{\xi} \in (a,b).$$

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$$\left|\int_a^b f(x)dx - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b)\right| = \left|\frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\widehat{\xi})\right|$$
$$\approx \frac{1}{15}\left|S(a,b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b)\right|.$$

Adaptive Quadrature (III)

For a given tolerance τ,

- $$\text{if} \quad \frac{1}{15}\left|S(a,b) - S(a,\frac{a+b}{2}) - S(\frac{a+b}{2},b)\right| \le \tau,$$

  then $S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b)$ is sufficiently accurate approximation to $\int_a^b f(x)dx$;

- otherwise recursively develop quadratures on $(a, \frac{a+b}{2})$ and $(\frac{a+b}{2}, b)$, respectively.

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AdaptQuad$(f,[a,b],\tau)$ for computing $\int_a^b f(x)\,dx$

- **compute** $S(a,b)$, $S(a,\frac{a+b}{2})$, $S(\frac{a+b}{2},b)$,
- **if**

  $$\frac{1}{15}\left|S(a,b) - S(a,\frac{a+b}{2}) - S(\frac{a+b}{2},b)\right| \le \tau,$$

  return $S(a,\frac{a+b}{2}) + S(\frac{a+b}{2},b)$.
- **else return**

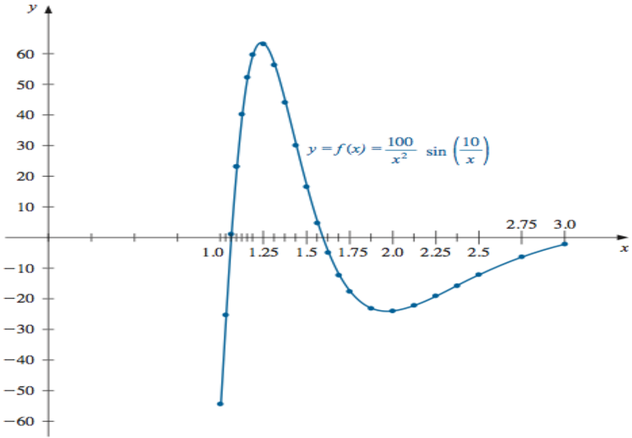
$$\textbf{AdaptQuad}(f,[a,\frac{a+b}{2}],\tau/2) + \textbf{AdaptQuad}(f,[\frac{a+b}{2},b],\tau/2)$$

Adaptive Simpson, example

- Integral $\int_1^3 f(x)\,dx$,

$$f(x) = \frac{100}{x^2}\mathbf{sin}\left(\frac{10}{x}\right).$$

- Tolerance $\tau = 10^{-4}$.



function quad($f, [a, b], \tau$) of `matlab`

For a given tolerance τ,

- **composite Simpson**: $S(a, b), S(a, \frac{a+b}{2})$ and $S(\frac{a+b}{2}, b)$.
- **Romberg extrapolation**:

$$\mathcal{Q}_1 = S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b), \quad \mathcal{Q} = \mathcal{Q}_1 + \frac{1}{15}\left(\mathcal{Q}_1 - S(a, b)\right).$$

- **if**

$$|\mathcal{Q} - \mathcal{Q}_1| \leq \tau,$$

  return $\mathcal{Q}$

function $\mathtt{quad}(f, [a, b], \tau)$ of `matlab`

For a given tolerance τ,

- **composite Simpson**: $S(a, b), S(a, \frac{a+b}{2})$ and $S(\frac{a+b}{2}, b)$.
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$$\mathcal{Q}_1 = S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b), \quad \mathcal{Q} = \mathcal{Q}_1 + \frac{1}{15}\left(\mathcal{Q}_1 - S(a, b)\right).$$

- **if**

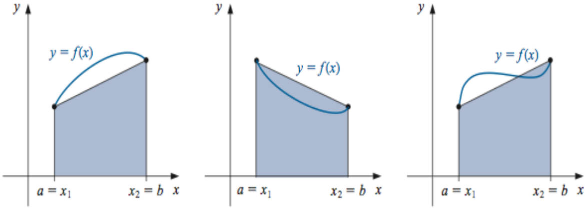
$$|\mathcal{Q} - \mathcal{Q}_1| \leq \tau,$$

  return $\mathcal{Q}$ (**Adaptive Simpson** returns $\mathcal{Q}_1$)
- **else return**

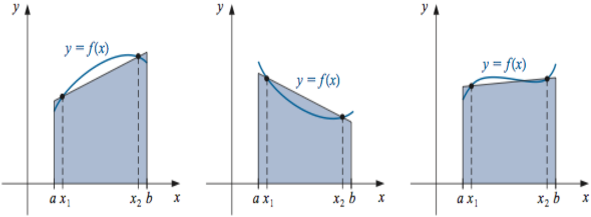
$$\mathtt{quad}(f, [a, \frac{a+b}{2}], \tau/2) + \mathtt{quad}(f, [\frac{a+b}{2}, b], \tau/2).$$

§4.7 Gaussian Quadrature (I)

- Trapezoidal nodes $x_1 = a, x_2 = b$ unlikely best choices.



- Likely better node choices.



Gaussian Quadrature (II)

- Given $n > 0$, choose both distinct nodes $x_1, \cdots, x_n \in [-1, 1]$ and weights $c_1, \cdots, c_n$, so quadrature

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{j=1}^{n} c_j f(x_j), \tag{1}$$

gives the greatest degree of precision (**DoP**).

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gives the greatest degree of precision (**DoP**).

- $2n$ total number of parameters in quadrature, could choose $2n$ monomials in equation (1):

$$f(x) = 1, x, x^2, \cdots, x^{2n-1} \quad (2)$$

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- $2n$ total number of parameters in quadrature, could choose $2n$ monomials in equation (1):
$$f(x) = 1, x, x^2, \cdots, x^{2n-1} \qquad (2)$$
- directly solving equation (1) with all $f(x)$ in (2) very hard.

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- directly solving equation (1) with all $f(x)$ in (2) very hard.

The world of quadratures is not a democratic society:
Efficiency more important than chaos
nodes matter, weights matter

Gaussian Quadrature, $n = 2$ (I)

- Consider Gaussian quadrature

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2).$$

Gaussian Quadrature, $n = 2$ (I)

- Consider Gaussian quadrature

$$\int_{-1}^{1} f(x)\, dx \approx c_1 f(x_1) + c_2 f(x_2).$$

- Choose parameters c_1, c_2 and $x_1 < x_2$ so that Gaussian quadrature is exact for $f(x) = 1, x, x^2, x^3$:

$$\int_{-1}^{1} f(x)\, dx = c_1 f(x_1) + c_2 f(x_2),$$

Gaussian Quadrature, $n = 2$ (I)

- Consider Gaussian quadrature

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2).$$

- Choose parameters c_1, c_2 and $x_1 < x_2$ so that Gaussian quadrature is exact for $f(x) = 1, x, x^2, x^3$:

$$\int_{-1}^{1} f(x)\,dx = c_1 f(x_1) + c_2 f(x_2), \quad \text{or}$$

$$\begin{aligned} 2 &= \int_{-1}^{1} 1\,dx = c_1 + c_2, & 0 &= \int_{-1}^{1} x\,dx = c_1\,x_1 + c_2\,x_2, \\ \frac{2}{3} &= \int_{-1}^{1} x^2\,dx = c_1\,x_1^2 + c_2\,x_2^2, & 0 &= \int_{-1}^{1} x^3\,dx = c_1\,x_1^3 + c_2\,x_2^3. \end{aligned}$$

Gaussian Quadrature, $n = 2$ (II)

- $x_1 < x_2$,
$$c_1\, x_1 = -c_2\, x_2, \quad c_1\, x_1^3 = -c_2\, x_2^3,$$
implying $x_1^2 = x_2^2$. Thus $x_1 = -x_2$ and $c_1 = c_2$.

Gaussian Quadrature, $n = 2$ (II)

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implying $x_1^2 = x_2^2$. Thus $x_1 = -x_2$ and $c_1 = c_2$.
- $$c_1 + c_2 = 2, \quad c_1\, x_1^2 + c_2\, x_2^2 = \frac{2}{3},$$
which implies $c_1 = c_2 = 1$, $x_2 = \frac{1}{\sqrt{3}}$.
- Gaussian quadrature for $n = 2$
$$\int_{-1}^{1} f(x)\, dx \approx f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}),$$
- exact for $f(x) = 1, x, x^2, x^3$, but not for $f(x) = x^4$.

Gaussian Quadrature (III)

- Given $n > 0$, choose both distinct nodes $x_1, \cdots, x_n \in [-1, 1]$ and weights $c_1, \cdots, c_n$, so quadrature
$$\int_{-1}^{1} f(x)\, dx \approx \sum_{j=1}^{n} c_j f(x_j), \tag{2}$$
gives the greatest degree of precision (**DoP**).

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- $2n$ parameters, allowing exact quadrature for $2n$ monomials

$$\int_{-1}^{1} f(x)\, dx = \sum_{j=1}^{n} c_j f(x_j), \quad \text{for} \quad f(x) = 1, x, x^2, \cdots, x^{2n-1} \tag{3}$$

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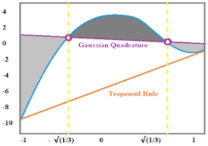
$$\int_{-1}^{1} f(x)\, dx = \sum_{j=1}^{n} c_j f(x_j), \quad \text{for} \quad f(x) = 1, x, x^2, \cdots, x^{2n-1} \tag{3}$$

- directly solving (2) can be very hard. But for $n = 2$,

$$\int_{-1}^{1} f(x)\, dx \approx f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$$

$\boxed{\text{exact}}$ for $f(x) = 1, x, x^2, x^3$,
$\boxed{\text{not}}$ for $f(x) = x^4$.



$\boxed{\text{exact}}$ for cubic

Gaussian Quadrature by Legendre polynomials

Gaussian Quadrature by Legendre polynomials: $P_0(x) = 1, P_1(x) = x$,

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad \text{for } n \geq 1.$$

- $P_n(x)$ has degree n, always with n $\boxed{\text{distinct}}$ roots $x_1, x_2, \cdots, x_n \in (-1, \ 1)$

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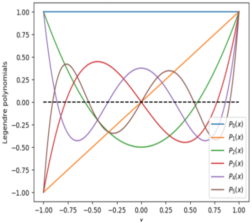
- $P_n(x)$ has degree n, <u>always</u> with n $\boxed{\text{distinct}}$ roots $x_1, x_2, \cdots, x_n \in (-1, \ 1)$
- MUTUALLY ORTHOGONAL:

$$\int_{-1}^{1} P_n(x)\, P_j(x)\, dx = 0, \quad \text{for} \quad j < n$$



Gaussian Quadrature by Legendre polynomials

Gaussian Quadrature by Legendre polynomials: $P_0(x) = 1, P_1(x) = x$,

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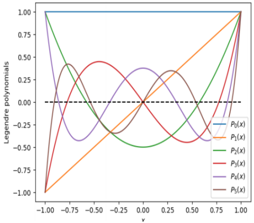

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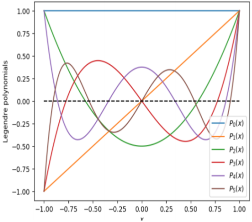

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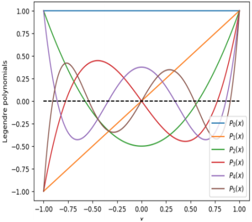

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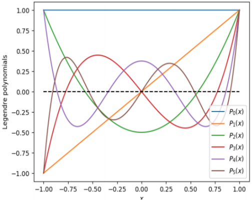

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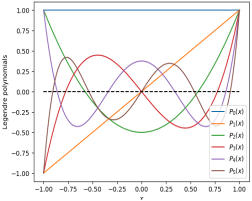

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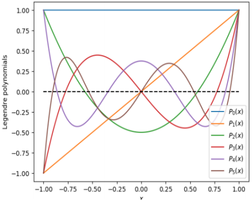

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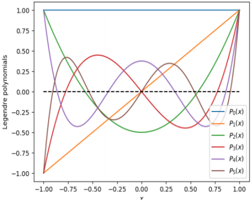

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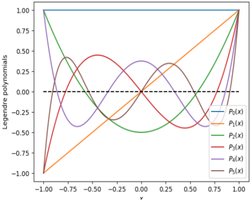

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Hermite Interpolation on roots of Legendre polynomials

- Given Legendre roots $x_1, x_2, \cdots, x_n$ with

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- Interpolating polynomial $H(x)$ of degree $\leq 2n-1$ satisfies

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- (recall) **Thm**: For each $x \in [a, b]$, there exists $\xi(x) \in (a, b)$,

$$f(x) = H(x) + R(x), \quad R(x) = \frac{f^{(2n)}(\xi(x))}{(2n)!}(x-x_1)^2(x-x_2)^2 \cdots (x-x_n)^2.$$

Hermite Interpolation, with Gaussian Quadrature

$$\int_{-1}^{1} f(x)\,dx \qquad = \qquad \int_{-1}^{1} H(x)\,dx + \int_{-1}^{1} R(x)\,dx$$

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\end{aligned}
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Gaussian Quadrature Error Estimate, `matlab` code

$$\mathbf{R} \;=\; \frac{f^{(2n)}(\xi)}{(2n)!} \int_{-1}^{1} (x - x_1)^2 (x - x_2)^2 \cdots (x - x_n)^2 \, dx$$

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Rapid convergence for smooth functions

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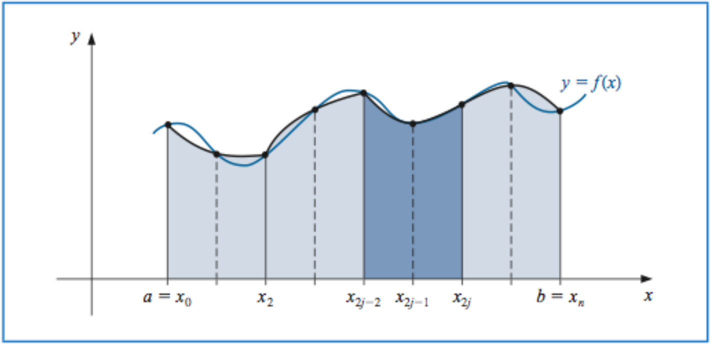
$\boxed{\text{Rapid convergence for smooth functions}}$

```
function [c,x] = Legendre(n)
%
b      = transpose((1:n-1));
b      = b./sqrt((2*b-1).*(2*b+1));
B      = diag(b,1)+diag(b,-1);
[Q,D]  = eig(B);
x      = diag(D);
x(abs(x)<1e-15) = 0;
c      = 2*transpose(Q(1,:).^2);
```

In contrast: Composite Simpson's Rule
$(n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n)$

$$\int_a^b f(x)dx = \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\right) - \frac{(b-a)h^4}{180} f^{(4)}(\mu)$$



Simpson/Trapezoidal vs. Gaussian Quadratures

Simpson/Trapezoidal:

- Composite rules:
  - Adding more EQUI-SPACED points.
- Romberg extrapolation:
  - Obtaining higher order rules from lower order rules.
- Adaptive quadratures:
  - Adding more points ONLY WHEN NECESSARY.

Gaussian Quadrature:

- points different for different n.

Gaussian Quadrature good for given n,
Simpson good for given tolerance.