Worksheet 11/22

1.) a.) $x^2+y^2+z^2=a^2$

$\vec{r}(u,v)=\langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle$, $0\le u\le \pi$, $0\le v\le 2\pi$

$\vec{r}_u=\langle a\cos u\cos v,\ a\cos u\sin v,\ -a\sin u\rangle$

$\vec{r}_v=\langle -a\sin u\sin v,\ a\sin u\cos v,\ 0\rangle$

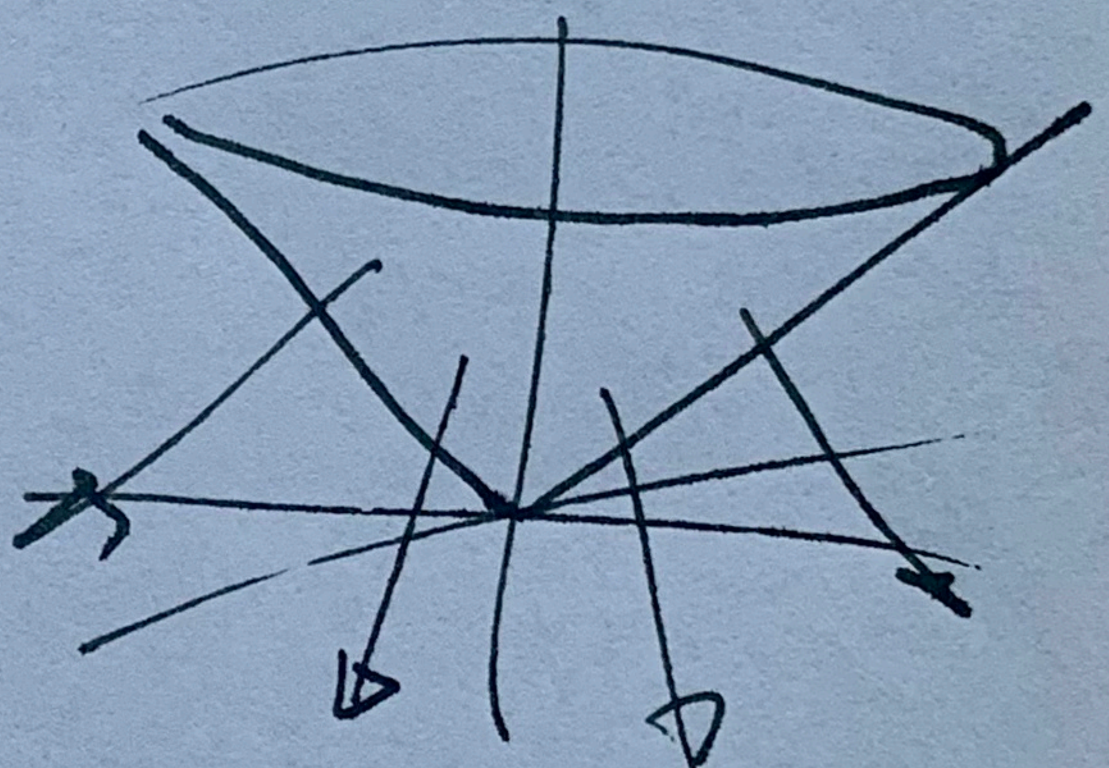
$\vec{r}_u\times\vec{r}_v=a^2\langle \sin^2 u\cos v,\ \sin^2 u\sin v,\ \sin u\cos u\rangle$

$\vec{F}(x,y,z)=\dfrac{c\vec{r}}{|\vec{r}|^3}=\dfrac{c\langle x,y,z\rangle}{|\langle x,y,z\rangle|^3}=\dfrac{c}{a^3}\langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle$

$\vec{F}\cdot(\vec{r}_u\times\vec{r}_v)=c\sin u.$ $\displaystyle\iint_S \vec{F}\cdot d\vec{S}=\iint_D c\sin u\, dA=\int_0^{2\pi}\int_0^{\pi} c\sin u\, du\, dv=4\pi c$

b.) $\vec{E}$ is always parallel to the vectors normal to the surface of the sphere, so $\vec{E}\cdot\vec{n}=\dfrac{c}{|\vec{r}|^2}$. Then $\displaystyle\iint_S \vec{E}\cdot d\vec{S}=\iint_S \frac{c}{|\vec{r}|^2}\, dS=4\pi c.$

2.) $\vec{r}(x,y)=\langle x,y,\sqrt{x^2+y^2}\rangle.$



$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x/\sqrt{x^2+y^2} \\ 0 & 1 & 2y/\sqrt{x^2+y^2} \end{vmatrix}=\left\langle \frac{-x}{\sqrt{x^2+y^2}},\ \frac{-y}{\sqrt{x^2+y^2}},\ 1\right\rangle.$$

This points upward, so use $\left\langle \dfrac{x}{\sqrt{x^2+y^2}},\ \dfrac{y}{\sqrt{x^2+y^2}},\ -1\right\rangle$.

$$\iint_S \vec{F}\cdot d\vec{S}=\iint_D\left(\frac{-xy}{\sqrt{x^2+y^2}}+\frac{xy}{\sqrt{x^2+y^2}}-(x^2+y^2)\right)dA=-\iint_D (x^2+y^2)\,dA$$

$$=-\int_0^{2\pi}\int_0^1 r^2\, r\, dr\, d\theta=-2\pi\cdot\frac{1}{4}=-\frac{\pi}{2}$$

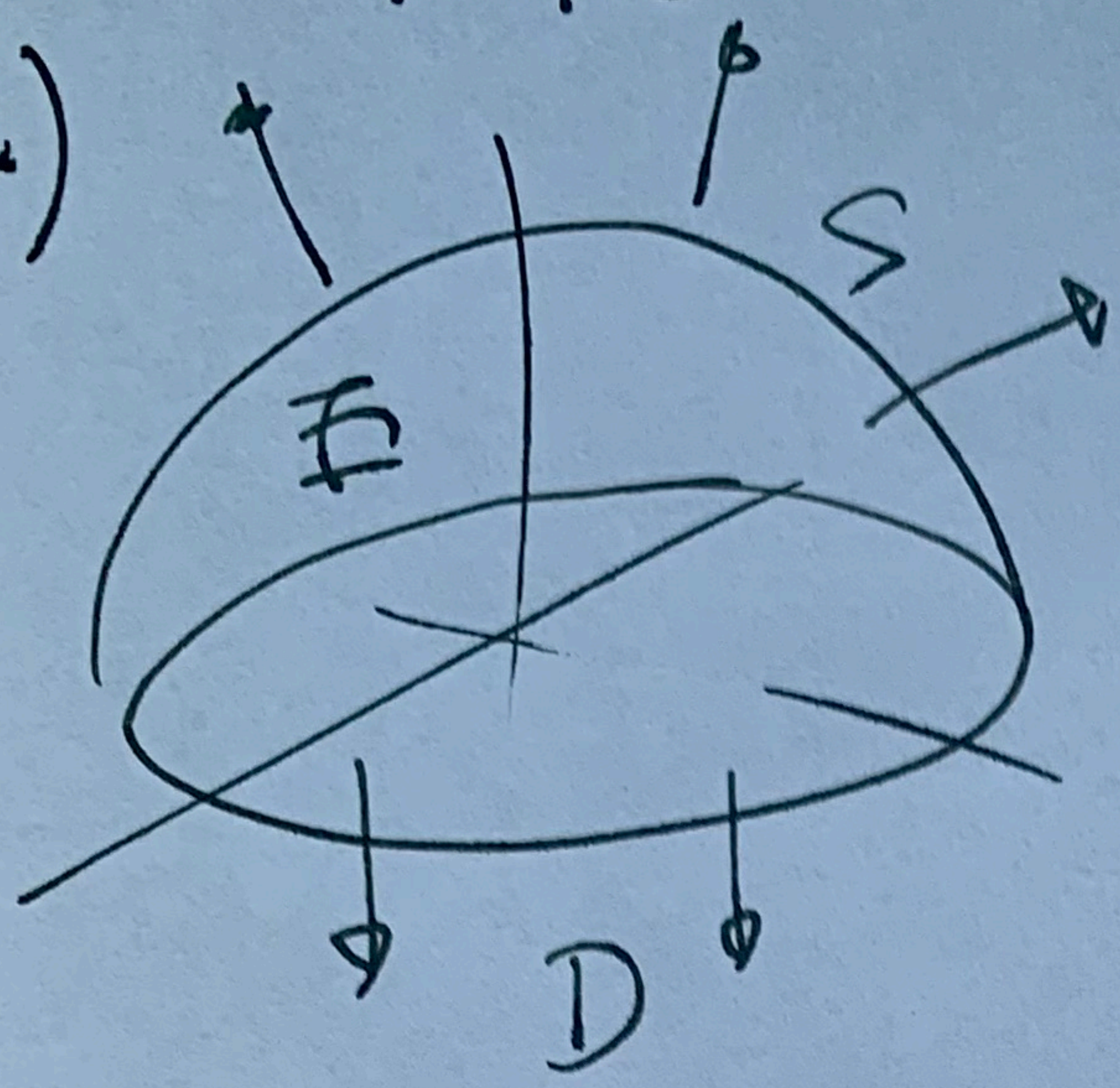
3.) a) False. Let $f=0$ and C a straight line segment

b.) False. Let $g=|x|+|y|$.

c.) False. Not Fubini's Theorem.

Worksheet 11/23

1.)



Let D be the disk of radius 1 and $S \cup D$ is the boundary of the enclosed volume E. D is oriented downwards. By divergence thm,

$$\iint_{S\cup D} \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E (x^2+y^2+z^2)\, dV$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 (\rho^2)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = 2\pi\left(\int_0^{\pi/2} \sin\phi\, d\phi\right)\left(\int_0^1 \rho^4\, d\rho\right) = \frac{2\pi}{5}$$

The normal vector for D is $-\vec{k}$.

$$\iint_D \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(-\vec{k})\, dS = \iint_D -y^2\, dA = \int_0^{2\pi}\int_0^1 -r^2 \sin^2\theta \, r\, dr\, d\theta$$

$$= \left(\int_0^1 -r^3\, dr\right)\left(\int_0^{2\pi} \sin^2\theta\, d\theta\right) = \left(-\frac{1}{4}\right)\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = -\frac{\pi}{4}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_D \vec{F}\cdot d\vec{S} = \frac{2\pi}{5} - \left(\frac{\pi}{4}\right) = \frac{13\pi}{20}$$

2.)a.) $\vec{r} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

$\vec{r}_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle$, $\vec{r}_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle$

$\vec{r}_\phi \times \vec{r}_\theta = \langle -\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, -\sin\phi\cos\phi \rangle$, $|\vec{r}_\phi \times \vec{r}_\theta| = \sin\phi$

$$\iint_S z^{888}\, dS = \int_0^{2\pi}\int_0^{\pi} (\cos\phi)^{888} \sin\phi \, d\phi \, d\theta = 2\pi \int_{-1}^{1} u^{888}\, du = \frac{4\pi}{889}$$

$u = -\cos\phi$

b.) Easy normal vector: $\vec{n} = \langle x, y, z \rangle$. $\vec{F} = \langle 0, 0, z^{887} \rangle$, $\vec{F} \cdot \vec{n} = z^{888}$.

$$\iint_S \vec{F}\, d\vec{S} = \iiint_B \operatorname{div}\vec{F}\, dV = \iiint_B 887 z^{886}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} 887(\rho\cos\phi)^{886} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} 887(\cos\phi)^{886} \sin\phi \, \rho^{888}\, d\rho\, d\phi\, d\theta = \frac{4\pi}{889}$$

3.)a.) True. Flux $= \iiint_E \operatorname{div} F\, dV = \iiint_E dV = \text{vol}(E)$.

b.) True. $\nabla \cdot (\text{curl}(\nabla g)) = \nabla \cdot (\nabla g)$

$$0 = \Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}$$

c.) False. Let $\vec{G} = \vec{F} + \langle 1, 1, 1 \rangle$.

Worksheet 11/28

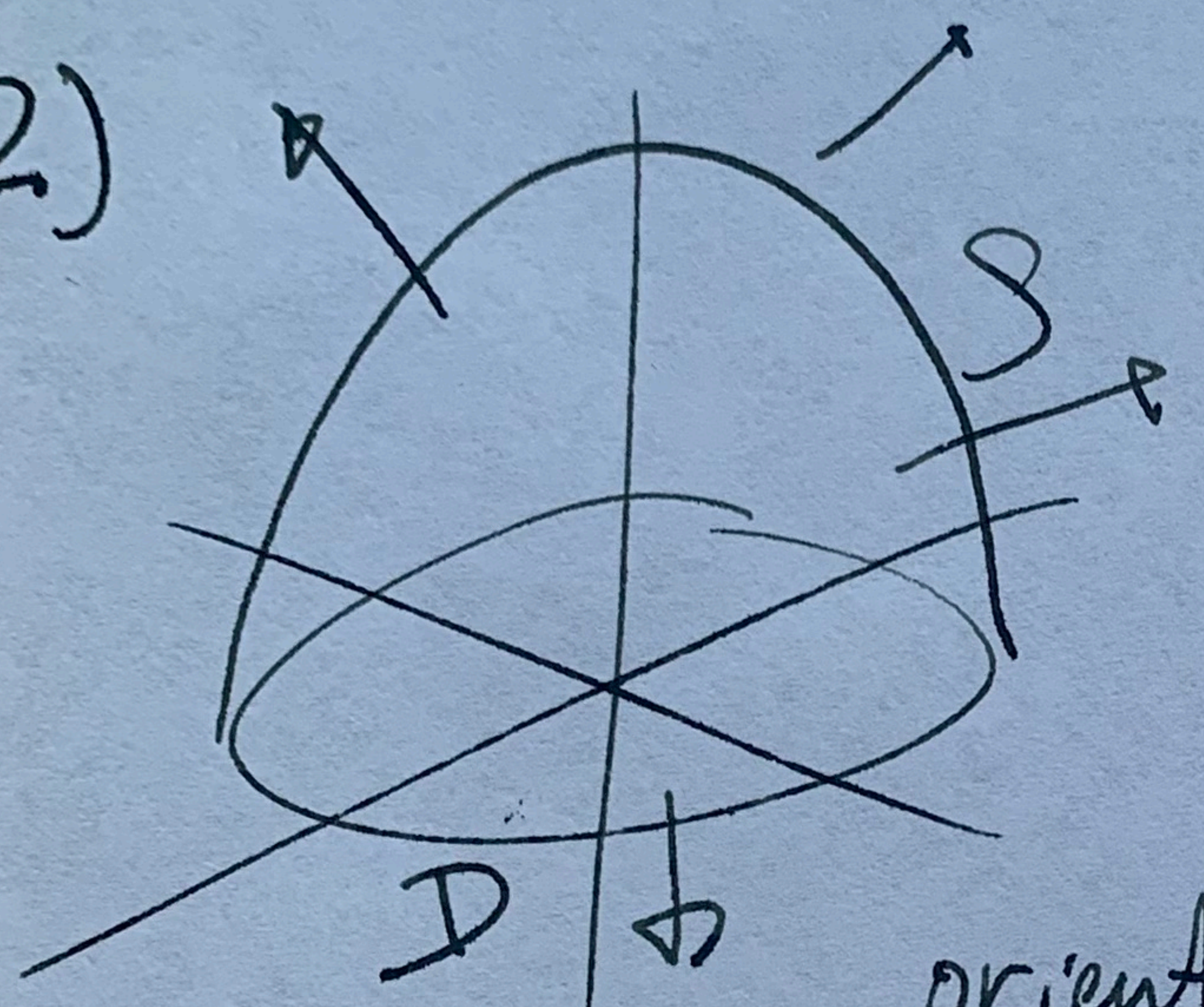
1.) a.) $\nabla\cdot(f\nabla g) = \nabla f\cdot\nabla g + f\Delta g$. By divergence theorem,

$$\iint_{\partial D} f\nabla g\, d\vec{S} = \iiint_D \nabla f\cdot\nabla g\, dV + \iiint_D f\Delta g\, dV$$

$$\iiint_D f\Delta g\, dV = \iint_{\partial D} f\nabla g\cdot d\vec{S} - \iiint_D \nabla f\cdot\nabla g\, dV$$

b.) let $f=1$.

2.)



let D be the disk of radius 1, and $S\cup D$ is the boundary of the enclosed volume E. D is oriented downwards. By divergence theorem,

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = 0 \quad \text{since}$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(y^3 z) + \frac{\partial}{\partial y}(x^3 z) + \frac{\partial}{\partial z}(1+e^{x^2+y^2}) = 0.$$

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_D \vec{F}\cdot d\vec{S} = -\iint_D \langle y^3 z, x^3 z, 1+e^{x^2+y^2}\rangle\cdot\langle 0,0,-1\rangle\, dS$$

$$= \iint_{\{x^2+y^2\le 1\}} (1+e^{x^2+y^2})\, dx\, dy = \int_0^1\int_0^{2\pi} (1+e^{r^2})\, r\, dr\, d\theta = e\pi$$

3.) a.) $\mathbb{R}^3 \setminus \{0,0,0\}$.

b.) $\nabla \cdot \vec{F} = 0 \Rightarrow \iint_{\partial Q} \vec{F} \cdot d\vec{S} = \iiint_{Q} \nabla \cdot \vec{F} \, dV = 0$

c.) The outward unit normal vector : $\vec{n} = \dfrac{\langle x, y, z \rangle}{r}$.

$$\Rightarrow \vec{F} = \frac{\langle x, y, z \rangle}{r^3}.$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_S \frac{1}{R^2} \, dS$$

$$= \frac{\text{Area of } S}{R^2} = 4\pi.$$

Worksheet 11/30

1.) If $\vec{F} = \nabla \times \vec{G}$, 2 ways to do this:

By Stokes' Theorem, $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{G} \cdot d\vec{S} = \oint \vec{G} \cdot d\vec{r} = 0$, or
(S has no boundary)

By Divergence Theorem, $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{G} \cdot d\vec{S} = \iiint_Q \operatorname{div}(\nabla \times \vec{G})\, dV$

(Q is the volume S encloses)

$= \iiint_Q 0\, dV = 0.$

Either way, this is false from 3.c. from 11/28, so $\vec{F} \neq \nabla \times \vec{G}$.

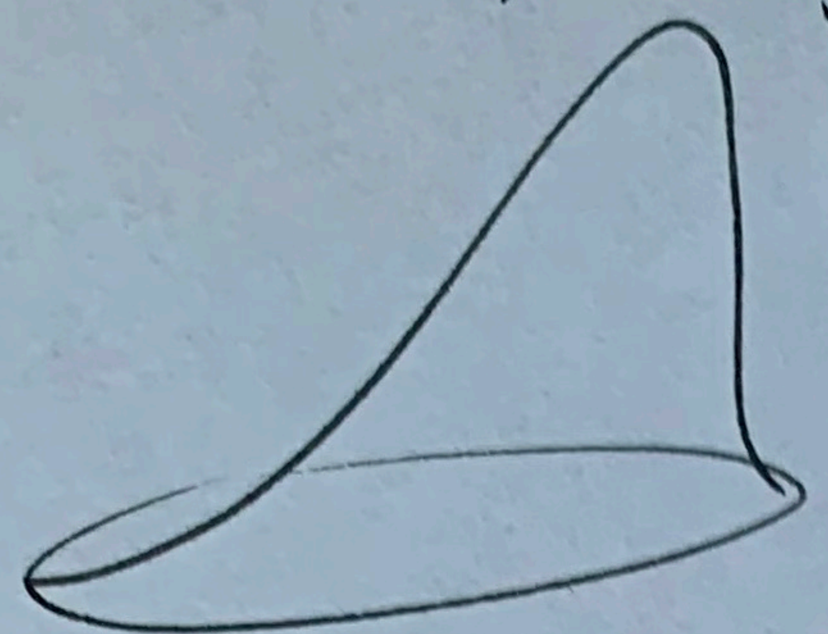
2.) a.) True. Curl free: $\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla f$ for some f.
Divergence free: $\nabla \cdot \vec{F} = \nabla \cdot \nabla f = \Delta f = 0$.

b.) True. Let $\vec{F} = \langle z, x, y \rangle$.

c.) False. $\nabla \cdot (\nabla \times \vec{F}) = 0$ for any $\vec{F}$, but $\nabla \cdot \langle x, y, z \rangle = 3$.

d.) False. Consider $S_1 = \{(x,y,z) \mid x^2+y^2+z^2=1,\ z \geq 0\}$, the upper hemisphere of a sphere of radius 1, and $S_2 = \{(x,y,0) \mid x^2+y^2 \leq 1\}$, the disk of radius 1, both with upward pointing normal vectors. Let S be the solid enclosed by S_1 & S_2. Let $\vec{F} = \langle x, 0, 0 \rangle$. By divergence theorem,
$\iint_{S_1} \vec{F} \cdot d\vec{S} - \iint_{S_2} \vec{F} \cdot d\vec{S} = \iiint_S \operatorname{div} \vec{F}\, dV \neq 0.$

3.) $\nabla \times \vec{F} = 0$, so $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \vec{0} \cdot d\vec{S} = 0$

4.)  a weird surface. Let $\vec{G} = \langle G_1, G_2, 0 \rangle$.

If $\vec{F} = \nabla \times \vec{G}$, then $\langle 2x, 2z-2x, 2x-2z \rangle = \langle -\frac{\partial G_2}{\partial z}, \frac{\partial G_1}{\partial z}, \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \rangle$.

$-\frac{\partial G_2}{\partial z} = 2x \Rightarrow G_2 = -2xz + M(x,y)$

$\frac{\partial G_1}{\partial z} = 2z - 2x \Rightarrow G_1 = z^2 - 2xz + N(x,y)$

$\Rightarrow \frac{\partial}{\partial x}(-2xz + M(x,y)) - \frac{\partial}{\partial y}(z^2 - 2xz + N(x,y))$

$= -2z + \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 2x - 2z$

$\Rightarrow \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 2x$. Choose any M & N that works.

Let $N = 0$. Then $\frac{\partial M}{\partial x} = 2x \Rightarrow M = x^2$.

$\Rightarrow \vec{G} = \langle z^2 - 2xz, x^2 - 2xz, 0 \rangle$

Parameterize the boundary of S: $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \le t \le 2\pi$

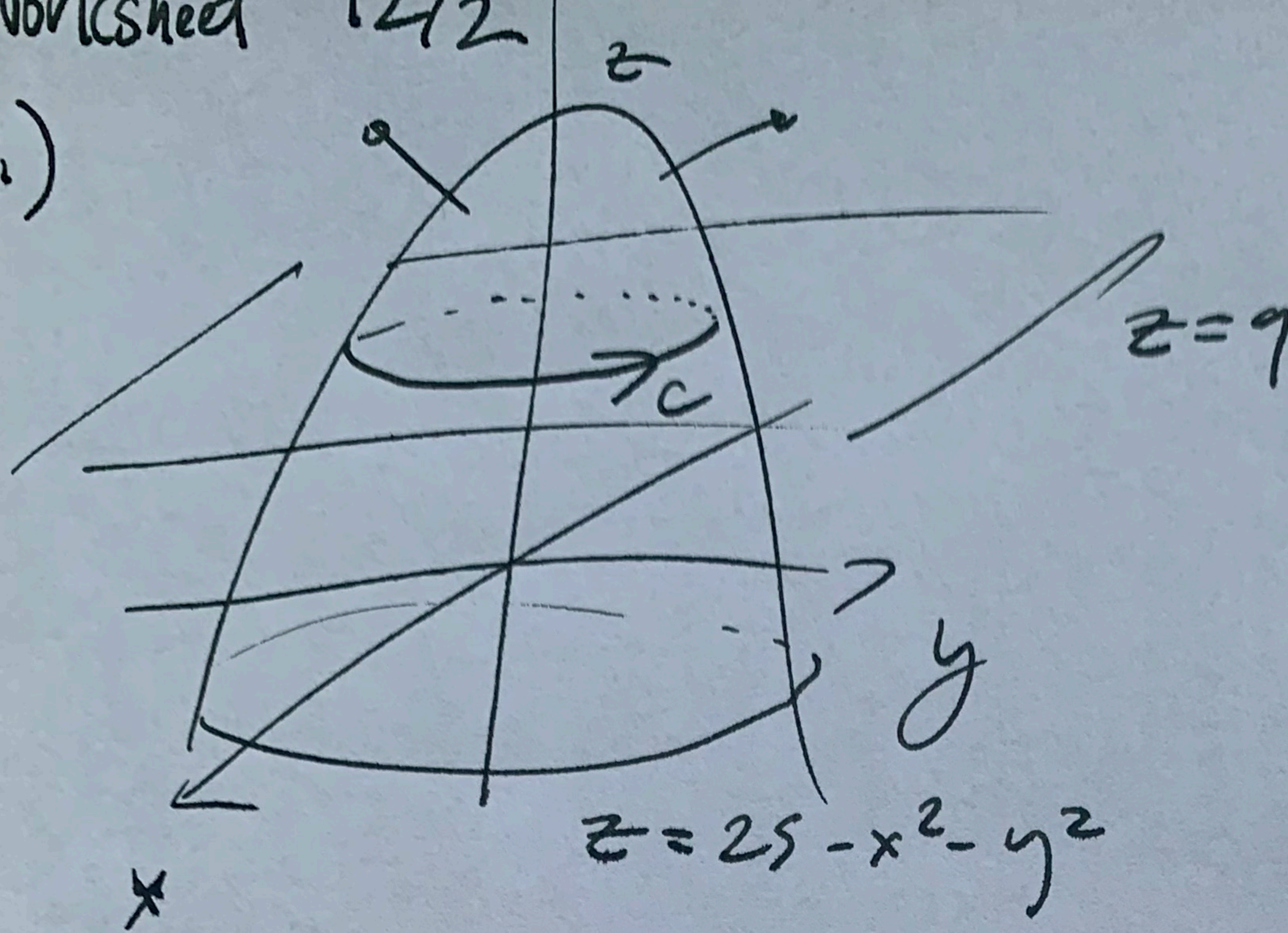
$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\nabla \times \vec{G}) \cdot d\vec{S} = \oint_C \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \vec{G}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$

$= \int_0^{2\pi} (\cos^2 t)(\cos t)\, dt = \int_0^{2\pi} (1 - \sin^2 t)\cos t\, dt = \sin t - \frac{\sin^3 t}{3} \Big|_0^{2\pi}$

$= 0.$

Worksheet 12/2

1.)



$$9 = z = 25 - x^2 - y^2 \implies x^2 + y^2 = 16$$

$$\vec{r}(t) = \langle 4\cos t, 4\sin t, 9 \rangle, \quad 0 \le t \le 2\pi$$

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle 4\cos t + 9, 4\cos t + 4\sin t, 64\cos^3 t \rangle \cdot \langle -4\sin t, 4\cos t, 0 \rangle dt$$

$$= \int_0^{2\pi} (-36\sin t + 16\cos^2 t)\, dt = \left(36\cos t + 8t + 4\sin 2t\right)\Big|_0^{2\pi} = 16\pi$$

2.) a.) False: Example: $\vec{F} = \langle 0, 0, x \rangle$, $\vec{G} = \langle 1, 0, 0 \rangle$.

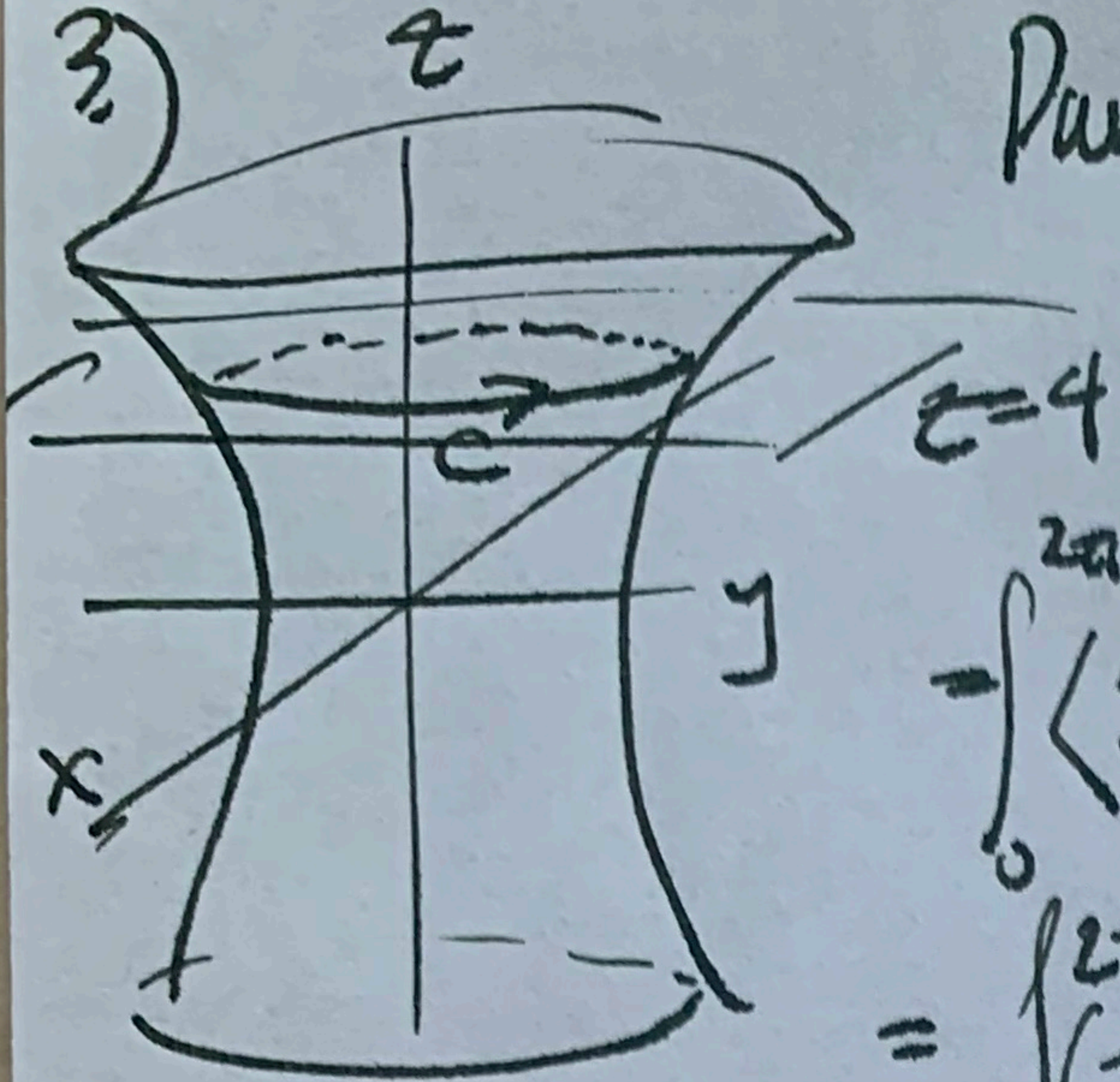
$\vec{F} \times \vec{G} = \langle 0, x, 0 \rangle$ is not conservative.

b.) True: $\nabla \times (f \nabla f) = (\nabla f) \times (\nabla f) + f \nabla \times (\nabla f) = 0$

c.) False: $\vec{F} = \nabla \phi$. $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F}\, dV = \iiint_E \nabla \cdot \nabla \phi\, dV = \iiint_E \Delta \phi\, dV$.

Let $\phi = x^2$. Then $\iiint_E \Delta \phi\, dV = 2 \iiint_E dV = 2\,\text{volume}(E)$.

3)



Parameterize C: $\vec{r}(t) = \langle 5\cos t, 5\sin t, 4 \rangle$

$$\oint \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle 5\cos t \cdot 5\sin t, 5\sin t \cdot 4, 5\cos t \cdot 4 \rangle \cdot \langle -5\sin t, 5\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} (-125\sin^2 t\cos t + 100\sin t\cos t)\, dt = 0$$

Alternatively, you could do Stokes': $\vec{r}(x,y) = \langle x, y, 4 \rangle$, $\vec{n} = \vec{k}$

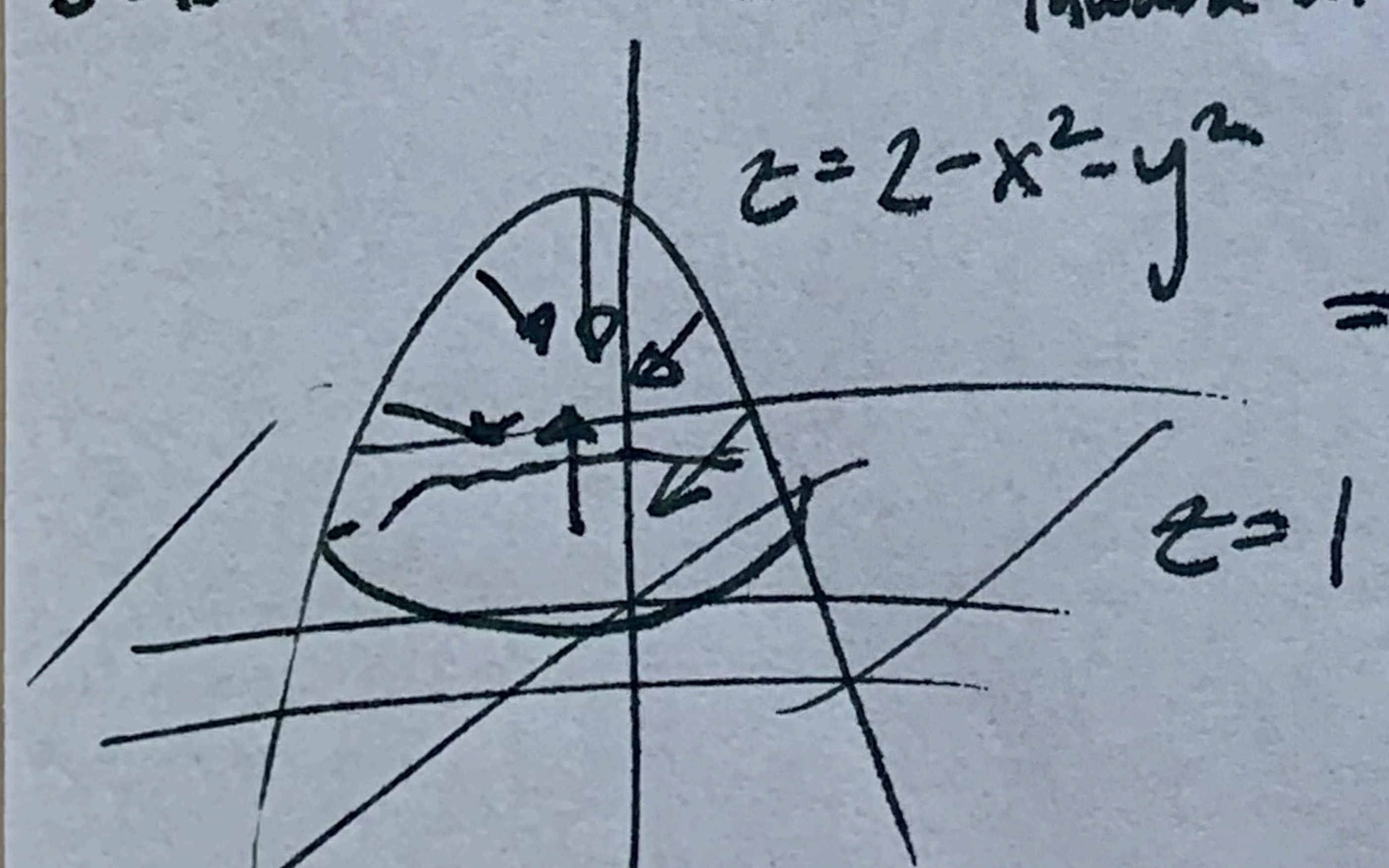
$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \langle -y, -4, -x \rangle \cdot \langle 0,0,1 \rangle \, dx\,dy = \iint_D -x \, dx\,dy$$

$$= \int_0^{2\pi} \int_0^{3} -r\cos\theta \; r\,dr\,d\theta = 0$$

4.) Close the surface: Let D = disk of radius 1 in $z=1$.

$$\iint_{S \cup D} \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\, dV = -\iiint_E 1 \, dx\,dy\,dz = -\int_0^{2\pi}\int_0^1\int_1^{2-r^2} r\,dz\,dr\,d\theta$$

inward on E



$$= -2\pi \int_0^1 (2 - r^2 - 1)\, r\,dr = -2\pi \left(\frac{r^2}{2} - \frac{r^4}{4} \right)\Big|_0^1 = -\frac{\pi}{2}$$

Disk is oriented up, so $\vec{n} = \langle 0,0,1 \rangle$

$$\iint_D \vec{F} \cdot d\vec{S} = \iint_D \langle \tan^{-1}(y^2), \ln(x^2+1), 1 \rangle \cdot \langle 0,0,1 \rangle \, dA = \int_0^{2\pi}\int_0^1 1 \, r\,dr\,d\theta = \pi$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\, dV - \iint_D \vec{F} \cdot d\vec{S} = -\frac{\pi}{2} - \pi = -\frac{3\pi}{2}$$