Game Theory (STAT 155) Homework #3, 02/10/2021

All the problems are worth 4 points each and will be graded on a 0/1/2/3/4 scale. Due on Wednesday 02/17/2021 before 11:59 pm to be uploaded via Gradescope.

1. Player II is moving an important item in one of three cars, labelled $1, 2$ and 3. Player I will drop a bomb on one of the cars of his choosing. He has no chance of destroying the item if he bombs the wrong car. If he chooses the right car, then his probability of destroying the item depends on that car. The probabilities for cars 1,2 and 3 are equal to $3/4$, $1/4$ and $1/2$. Write the 3×3 payoff matrix for the game, and find some optimal winning strategies for each of the players.

   Solution: First note that the payoff matrix for this game is

$$\begin{pmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

   Now let's use the principle of equalization to guess that an optimal strategy (x_1, x_2, x_3) for Player I might be such that it satisfies $\frac{3}{4}x_1 = \frac{1}{4}x_2 = \frac{1}{2}x_3$. Combining this with $x_1 + x_2 + x_3 = 1$ leads us to $x_* = (\frac{2}{11}, \frac{6}{11}, \frac{3}{11})$. Under this strategy, Player I guarantees himself (note that Player I uses he/him/his in this example) a payoff of at least $\frac{3}{22}$, the value of the game is $V \geq \frac{3}{22}$.

   The symmetry of the game suggests that an optimal strategy for Player II is also $y_* = (\frac{2}{11}, \frac{6}{11}, \frac{3}{11})$. In particular, Player II guarantees a loss of at most $\frac{3}{22}$ under this winning strategy.

   Therefore, the value of the game is $V = \frac{3}{22}$ and a pair of optimal strategies is $x_* = y_* = (\frac{2}{11}, \frac{6}{11}, \frac{3}{11})$.

2. Find the value and all safety strategies for the game

$$\begin{pmatrix} 9 & 6 & 7 \\ 3 & 0 & 1 \\ 4 & 16 & 12 \end{pmatrix} \tag{1}$$

   Solution: Let A denote the matrix in (1). Then note that the row 2 is strictly dominated by row 1 (also by row 3), so we can reduce the game to the matrix

$$\begin{pmatrix} 9 & 6 & 7 \\ 4 & 16 & 12 \end{pmatrix}. \tag{2}$$

   which we call A'. Moreover, since the domination was strict, we know that no optimal strategy for Player I assigns any positive probability to row 2. In this reduced game (also in the original game), we see that $\frac{1}{3}$ of column 1 plus $\frac{2}{3}$ of column 2 equals columns 3. This means we that our game has the same value as the reduced game

$$\begin{pmatrix} 9 & 6 \\ 4 & 16 \end{pmatrix}, \tag{3}$$

   which we call A''. Performing a straightforward calculation shows that the only optimal strategies in A'' are $(\frac{4}{5}, \frac{1}{5})$ and $(\frac{2}{3}, \frac{1}{3})$. In particular, the game has a value of 8. For Player I, this implies that her only optimal strategy in A is $x_* = (\frac{4}{5}, 0, \frac{1}{5})$.

   For Player II, we need to think about how to "transfer some of the probability" between column 3 and columns 1 and 2 in the right proportion. To do this, we first claim that, if $y = (y_1, y_2, y_3) \in \Delta_3$ is an optimal strategy for A', then $y' = (y_1 + \frac{1}{3}y_3, y_1 + \frac{2}{3}y_3) \in \Delta_2$ is an optimal strategy for A''. To see this, define

$$C = \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{pmatrix}$$

   and note that the map $y \mapsto y'$ is just the map $y \mapsto Cy$. Moreover, we have $A''C = A'$. Therefore, we have

$$\max_{x \in \Delta_2} x^{\mathrm{T}} A' y = \max_{x \in \Delta_2} x^{\mathrm{T}} A'' C y = \max_{x \in \Delta_2} x^{\mathrm{T}} A'' y',$$

   so y being optimal for A' implies that y' is optimal for A''.

   Now we know that the only possible optimal strategies for Player II are $y \in \Delta_3$ that satisfy $Cy = (\frac{2}{3}, \frac{1}{3})^{\mathrm{T}}$. This is just an inhomogeneous system of linear equations, which we can easily solve. A particular solution is seen to be $y_0 = (\frac{2}{3}, \frac{1}{3}, 0)$, and all solutions to the homogeneous system $Cy = 0$ are exactly tu for $t \in \mathbb{R}$ where $u = (-1, -2, 3)^{\mathrm{T}}$. Therefore, all solutions to $Cy = (\frac{2}{3}, \frac{1}{3})^{\mathrm{T}}$ are $y_t = y_0 + tu = (\frac{2}{3} - t, \frac{1}{3} - 2t, 3t)$, provided that $y_t \in \Delta_3$.

   Finally, we check that these candidate strategies are in fact optimal. Since $A'u = 0$, we have

$$x^{\mathrm{T}} A' y_t = x^{\mathrm{T}} A' (y_0 + tu) = x^{\mathrm{T}} A' y_0 + t x^{\mathrm{T}} A' u = x^{\mathrm{T}} A' y_0.$$

hence

$$\max_{x\in\Delta_2} x^{\mathrm{T}}A'y_t = \max_{x\in\Delta_2} x^{\mathrm{T}}A'y_0.$$

Therefore, y_0 being optimal implies that y_t is optimal. Of course, we can only allow t which are such that the vector y_t is a valid probability vector. It is clear that the sum of the entries of y_t is 1 for any t, but we need to restrict t to be such that all the entries of y_t are non-negative and bounded by 1.

Combining this all, we have shown that the value of the game A is 8, that the only optimal strategy for Player I is $x_* = (\frac{4}{5}, \frac{1}{5})^{\mathrm{T}}$, and that all the optimal strategies for Player II are

$$\left\{ \left(\frac{2}{3} - t, \frac{1}{3} - 2t, 3t \right)^{\mathrm{T}} : 0 \le t \le \frac{1}{6} \right\}.$$

3. Show that in Submarine Salvo the submarine has an optimal strategy where all choices containing a corner and a clockwise adjacent site are excluded.

   Solution: Consider the strategy for the submarine where he picks corner uniformly at random (so each is chosen with probability $\frac{1}{4}$), then places the submarine at that corner and the counter-clockwise adjacent site. Of course, this strategy never places the submarine in a corner and clockwise adjacent site. Also, we know from lecture that the value of Submarine Salvo is $1/4$, so, in order to show that this strategy is optimal for the submarine, we just need to show that his loss is guaranteed to be no more than $1/4$. To do this, let's check cases:

   - If the bomber plays in any corner tile, then the probability that submarine also chose that corner is $1/4$, so the loss for the submarine is $1/4$.
   - If the bomber plays in any midside tile, then the probability that that tile was counter-clockwise of the submarine's selected corner is $1/4$, so the loss for the submarine is $1/4$.
   - If the bomber plays in the middle tile, then there is no outcome in which the submarine is hit, so the loss for the submarine is zero.

   In any case, the worst-case loss for the submarine is $1/4$, so this strategy is optimal.

4. Given that $p = (52/143, 50/143, 41/143)$ is optimal for Player I in the game with the following matrix, what is the value?

$$\begin{pmatrix} 0 & 5 & -2 \\ -3 & 0 & 4 \\ 6 & -4 & 0 \end{pmatrix}$$

   Solution: The value of the game is the amount of payoff that Player I can guarantee herself, regardless of the move of Player II. So if $p \in \Delta_3$ is optimal for Player I, then the value V of the game is the smallest entry of the row vector $p^{\mathrm{T}}A$. We can easily compute

$$p^{\mathrm{T}}A = \left(\frac{96}{143}, \frac{96}{143}, \frac{96}{143} \right)$$

   so the value of the game is $V = \frac{96}{143}$.

5. Consider a zero-sum game with the following payoff matrix;

$$\begin{pmatrix} 12 & 7 & 4 & 3 & 2 & 1 & 12 \\ 1 & 2 & 3 & 4 & 6 & 11 & 11 \end{pmatrix}$$

   Find all possible optimal strategies x_* and y_* for the first player and the second player, respectively, and the value of game.

   Solution: Let the matrix be denoted by A. Note that, except for the last column, the two rows look fairly similar except in reverse order. So it seems reasonable that the strategy $x_* = (\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$ might be optimal for Player I. What is her worst-case payoff under this strategy? Note that we have

$$x_*^{\mathrm{T}}A = \begin{pmatrix} \frac{13}{2} & \frac{9}{2} & \frac{7}{2} & \frac{7}{2} & 4 & 6 & \frac{23}{2} \end{pmatrix}$$

   and that all of these entries are at least as large as $\frac{7}{2}$. So, the value V of the game satisfies $V \ge \frac{7}{2}$. We can conclude $V = \frac{7}{2}$ if we find a strategy y_* for Player II such this his loss is guaranteed to be no more than $\frac{7}{2}$. By noticing the 2×2 submatrix consisting of 3's and 4's, it seems reasonable that $y_* = (0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)^{\mathrm{T}}$ is such a strategy. Indeed, compute:

$$Ay_* = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Since these entries are all bounded above by $\frac{7}{2}$, we see that this strategy guarantees Player II a loss of no more than this. Thus, $V \leq \frac{7}{2}$ and hence $V = \frac{7}{2}$.

We have shown that the value of the game is $V = \frac{7}{2}$ and that $x_* = (\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}, y_* = (0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)^{\mathrm{T}}$ is a pair of optimal strategies. Now let us show that there are no other optimal strategies.

For Player I, assume that $x_t = (\frac{1}{2} - t, \frac{1}{2} + t)$ were optimal for some $t \neq 0$. If $t > 0$, then the third entry of $x_t^{\mathrm{T}} A$ is strictly smaller than $\frac{7}{2}$, and, if $t < 0$, then the fourth entry of $x_t^{\mathrm{T}} A$ is strictly smaller than $\frac{7}{2}$. In either case, this means that Player II has a (pure) strategy which gives Player I a payout that is strictly less than the value of the game, hence this strategy cannot be optimal.

For Player II, suppose that $y \in \Delta_7$ is optimal. Since x_* is the only optimal solution for Player I, this means we must have

$$x_*^{\mathrm{T}} A y = \frac{13}{2} y_1 + \frac{9}{2} y_2 + \frac{7}{2} y_3 + \frac{7}{2} y_4 + 4 y_5 + 6 y_6 + \frac{23}{2} y_7 \leq \frac{7}{2}.$$

However, this can only be satisfied if $y_i = 0$ for $i \in \{1, 2, 5, 6, 7\}$. Therefore the only optimal strategies for Player II are of the form $y_t = (0, 0, t, 1-t, 0, 0, 0)^{\mathrm{T}}$. Using the principle of equalization, since Player I has a fully mixed optimal strategy, shows that the only optimal strategy for Player II is y_*.