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Math128A: **Numerical Analysis** Midterm

This is a closed book exam. You are allowed to cite any results, up to Section 4.5 but excluding those in the exercises, from the textbook. Results from anywhere else will be treated the same as your answers, which need to be justified. Completely correct answers given without justification will receive little credit. Partial results and partial solutions will get partial credit.

Problem	Maximum Score	Your Score
1	18	
2	16	
3	16	
4	16	
5	16	
6	18	
Total	100	

Your Name: ______________________________

Your SID: ______________________________

1. (a) Show that the equation $(x-2)^2 - \mathbf{ln}\, x = 0$ has at least one solution in the interval $[1,\, 2]$.

   (b) Find $\mathbf{max}_{0 \le x \le 1} |f(x)|$ for $f(x) = e^x - 2\,x$

SOLUTION:

(a) Let $g(x) = (x-2)^2 - \mathbf{ln}\, x$, we have $g(1) = 1 > 0$ and $g(2) = -\mathbf{ln}\, 2 < 0$. This implies there must be a $c \in (1,\, 2)$ such that $g(c) = 0$.

(b) The derivative $f'(x) = e^x - 2$, with a unique root $\mathbf{ln}\, 2$ in $(0,\, 1)$. Thus

$$\mathbf{max}_{0 \le x \le 1} |f(x)| = \mathbf{max}\, \{|f(0)|,\, |f(\mathbf{ln}\, 2)|,\, |f(1)|\} = \mathbf{max}\, \{1,\, 2 - 2\,\mathbf{ln}\, 2,\, e - 2\} = 1$$

2. Let $f(x) = x^2(x-1)$. Assume that Newton's method converges to a root of $f(x)$

   (a) What is the order of convergence if Newton's method converges to the root $x = 1$? Why?

   (b) What is the order of convergence if Newton's method converges to the root $x = 0$? Why?

SOLUTION:

(a) The root $x = 1$ is simple, therefore order of convergence is 2.

(b) The root $x = 0$ is a double root, therefore order of convergence is 1.

3. Let $f(x) = x^2 - 2$. Perform one step of Secant iteration with $p_0 = 0$, $p_1 = 1$. SOLUTION: First

$$f(p_1) = -1, \quad f(p_0) = -2$$

Therefore,

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 1 - \frac{(-1) * (1 - 0)}{(-1) - (-2)} = 2$$

4. Construct a quadratic function $P(x)$ that satisfies the following conditions

$$P(-1) = 0, \quad P(0) = 1, \quad P(1) = 2$$

SOLUTION:

$$P(x) = 1 \cdot \frac{(x-(-1)) * (x-1)}{(0-(-1)) * (0-1)} + 2 \cdot \frac{(x-(-1)) * (x-0)}{(1-(-1)) * (1-0)} = x + 1$$

$x_0 = -1$	$f[x_0]$		
		$f[x_0,\, x_1]$	
$x_1 = 0$	$f[x_1]$		$f[x_0,\, x_1, x_2] = 2$
		$f[x_1,\, x_2] = 1$	
$x_2 = 1$	$f[x_2] = 0$		

5. Given the above divided difference table
   (a) Determine $f[x_0,\, x_1]$
   (b) Determine $f[x_0]$ and $f[x_1]$

SOLUTION:

(a) Since

$$2 = f[x_0,\, x_1, x_2] = \frac{f[x_0,\, x_1] - f[x_1,\, x_2]}{x_0 - x_2} = \frac{f[x_0,\, x_1] - 1}{(-1) - 1},$$

we have $f[x_0,\, x_1] = -2 * 2 + 1 = -3$

(b) Since

$$1 = f[x_1,\, x_2] = \frac{f[x_1] - f[x_2]}{x_1 - x_2} = \frac{f[x_1] - 0}{0 - 1},$$

we have $f[x_1] = -1$. Since

$$-3 = f[x_0,\, x_1] = \frac{f[x_0] - f[x_1]}{x_0 - x_1} = \frac{f[x_0] - (-1)}{(-1) - 0},$$

we have $f[x_0] = 3 - 1 = 2$

6. Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_0^1 f(x)\,dx = c_0\, f(0) + c_1\, f(x_1)$$

has the highest possible degree of precision. SOLUTION: There are

three parameters to determine, and we will try to make the formula exact for $f(x) = 1,\ x,\ x^2$, leading to equations

$$\begin{aligned} 1 = \int_0^1 1\,dx &= c_0 + c_1 \\ \frac{1}{2} = \int_0^1 x\,dx &= c_0 * 0 + c_1 * x_1 \\ \frac{1}{3} = \int_0^1 x^2\,dx &= c_0 * 0 + c_1 * x_1^2 \end{aligned}$$

Taking division with the last two equations,

$$x_1 = \frac{c_1 * x_1^2}{c_1 * x_1} = \frac{2}{3}$$

The second equation now implies

$$c_1 = \frac{1}{2\,x_1} = \frac{3}{4}$$

and therefore $c_0 = \frac{1}{4}$ from the first equation.