MATH 135: SET THEORY
AUTUMN 1999
FINAL EXAMINATION

1. Complete the following definitions. Unless explicitly stated to the contrary, if you refer to a word introduced in this class, you must define that word as well.

a The empty set, $\varnothing$, is ______________.

b An *injective function* $f : X \to Y$ from the set X to the set Y is ______________. (You need not define *ordered pair* or *Cartesian product*, but you do need to define *function*.

c A *Cauchy sequence of rational numbers* is ______________. (You need not define $\mathbb{Q}$ or any of its arithmetic or order theoretic structure.)

d An *ordinal* is ______________.

e A *cardinal* is ______________.

f A relation R on a set X is *transitive* if ______________.

g By definition, if $n, m \in \omega$ are natural numbers, then $n < m$ if and only if ______________.

h A *choice function* for a set X is ______________.

2. State the following axioms in the formal language. If you wish, you may state the axioms in mathematical English also. Defined terms are acceptable in the formal version.

a Comprehension (or Subset) Axiom

b Axiom of Choice (in the form about right inverses)

c Replacement Axiom

3.

a Show that there exists a unique operation on ordinals, called ordinal addition, satisfying $\alpha + 0 = \alpha$, $\alpha + \beta^+ = (\alpha + \beta)^+$, and $\alpha + \lambda = \bigcup\{\alpha + \beta \mid \beta \in \lambda\}$ for limit ordinals λ. (Recall that a limit ordinal is a non-zero ordinal which is not of the form γ^+.)

b Show that for any two ordinals α and β, $|\alpha + \beta| = |\alpha| + |\beta|$.

c Compute $\omega + 1$ and $1 + \omega$.

4. For $X \subseteq \mathbb{R}$, let $\mathcal{C}(X, \mathbb{R}) := \{f : X \to \mathbb{R} \mid f \text{ is continuous }\}$. Calculate $|\mathcal{C}(\mathbb{R}, \mathbb{R})||$. (Hint: Compare $\mathcal{C}(\mathbb{R}, \mathbb{R})$ with $\mathcal{C}(\mathbb{Q}, .\mathcal{R})$.)

5. We showed in class that for every set X there is some ordinal α so that there is no injection $\psi : X \to \alpha$. Show that the least such α is a cardinal, and, in fact, is the least cardinal greater than $|X|$. (This requires the Axiom of Choice.)

6. Without using any other form of the Axiom of Choice, prove that the well-ordering theorem implies the version of the Axiom of Choice given in question **2b**.

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7. Prove or disprove: If α is a countable ordinal, then there is a set of real numbers $X \subseteq \mathbb{R}$ which with the ordering induced by $\mathbb{R}$ is order isomorphic to (α, ϵ).

8. Define a relation E on $\mathbb{R}$ by $\langle x, y\rangle \in E \Leftrightarrow x - y \in \mathbb{Q}$.

a Show that E is an equivalence relation.
b Calculate $|\mathbb{R}/E|$.
c Prove or disprove: there is a function $F : \mathbb{R}/E \to \mathbb{R}/E$ satisfying $F([x]_E) = [x^2]_E$ for $x \in \mathbb{R}$.
d Prove or disprove: There is a set of real number $A \subseteq \mathbb{R}$ such that $|A \cap C| = 1$ for each $C \in \mathbb{R}/E$.