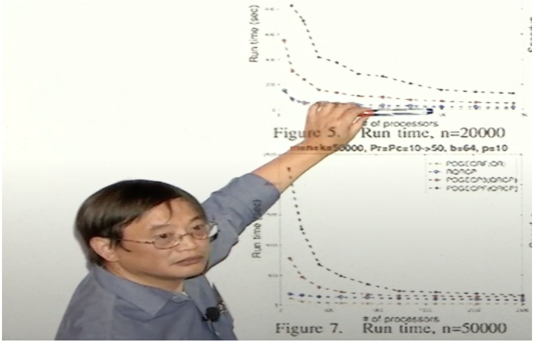
Self Introduction: Prof. Ming Gu, UC Berkeley

- Email: mgu@berkeley.edu
- Office hours: MWF 1:10-2:00PM
  in 861 Evans or by appointment



Self Introduction

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- **teaching:** numerical analysis (over 10 years)

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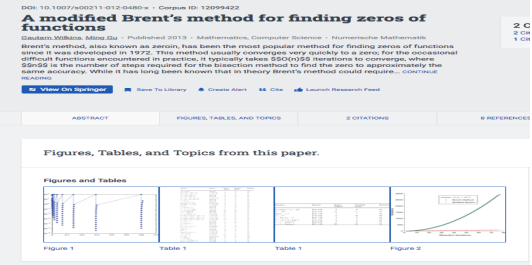
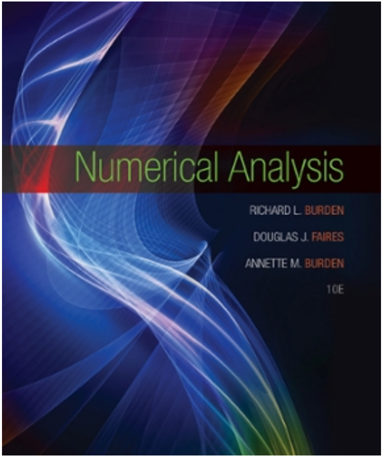
- **student paper:** Gautam was 128A student

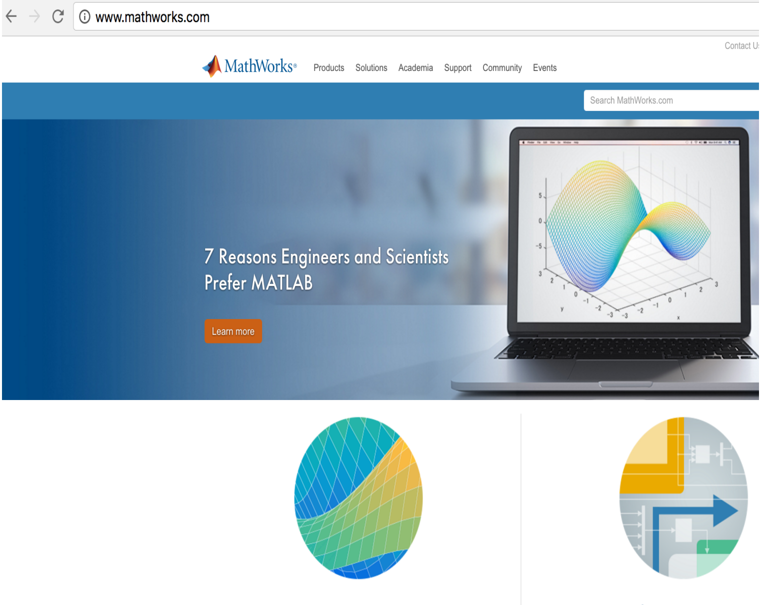


Text Book

- Burden and Faires, **Numerical Analysis**.
  10^{th} edition, required.



Matlab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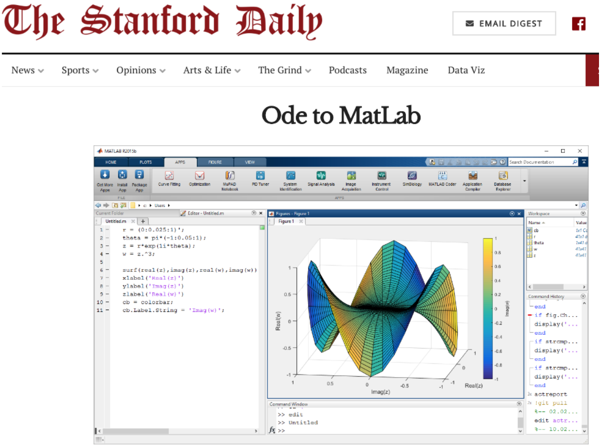
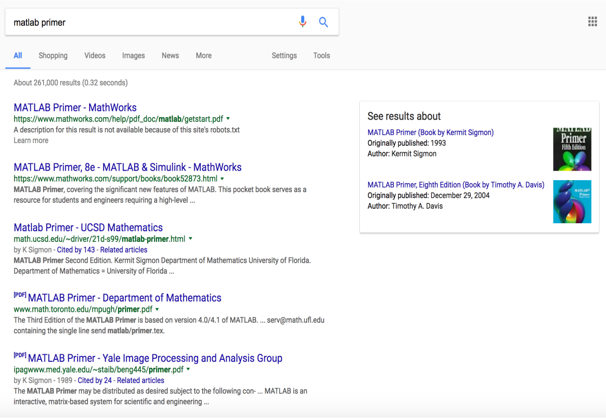
"bittersweet ode to the elusive yet enchanting MATLAB"



https://www.stanforddaily.com/2019/02/14/ode-to-matlab/

- Matlab Primer



- Matlab $\boxed{\text{free download}}$ for UC Berkeley students https://www.mathworks.com/academia/tah-portal/berkeley-731130.html

Math 98: Introduction to MATLAB Programming (1 unit, Pass/No Pass)

- runs 3 weeks, starting 8/24
- two sections (tentative)
  - Tu/Th 9:30-12:30PM (back to back)
- Instructor: Michael Heinz (`michael_heinz@berkeley.edu`)

Class Work

- 13 weekly home work sets,
  up to 1.5 points each, maximum 15 points.
  - homework due on Wednesdays (first one due on Sep. 6)
    submit on gradscope (Entry Code:3J4V26)

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- 6 quizzes;
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  - quiz schedule
    (Sept. 6, Sept. 20, Oct. 4, Nov. 1, Nov. 15, Nov. 29)
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  - every other quiz covers material in the two most recent homework sets

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- Midterm exam (Oct. 18), 20 points.
- Programming Assignment (Due Nov. 22), 20 points.
- Final exam (TBA), 30 points.
  - Final worth 50 points if midterm missing

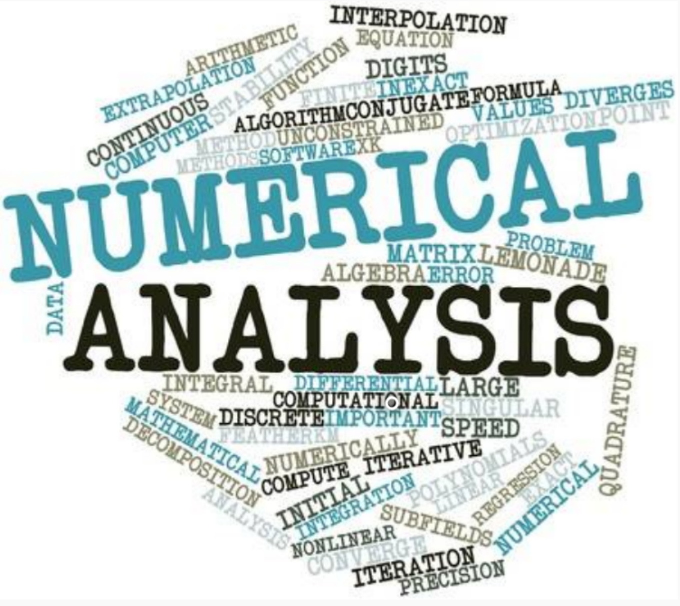
Grade Scale

- **A**- to **A+**: at least 85 points;
- **B**- to **B+**: between 70 and 85 points;
- **C**- to **C+**: between 60 and 70 points;
- **D**: between 55 and 60 points;
- **F**: less than 55 points.

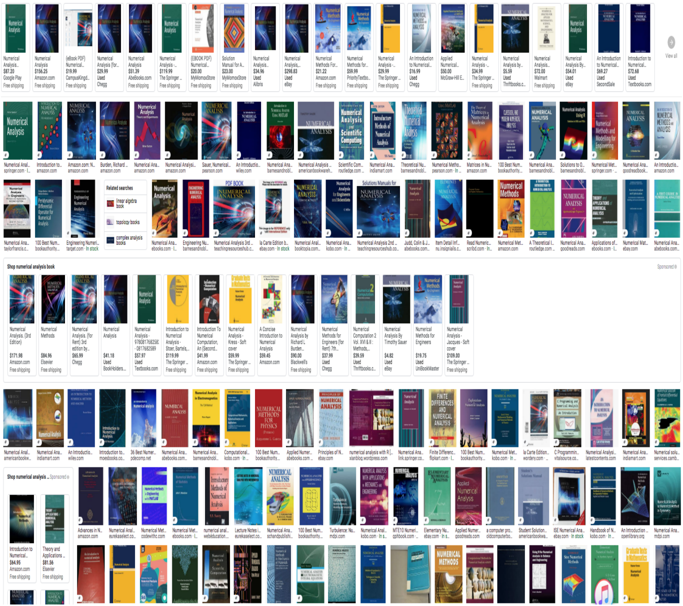
No grade curve: your grade **will** be your own effort

Most people get *A* level or *B* level grades.
More refined grade scales will be discussed later on based on class performance.

Many Topics in Numerical Analysis



Numerical Analysis, books for every need



Numerical Analysis = Calculus on a computer

First 6 Chapters of Text Book

- **Chapter 1**: Calculus Review, Computer arithmetic
- **Chapter 2**: Solution to non-linear equation $f(x) = 0$.
- **Chapter 3**: Function approximations.
- **Chapter 4**: Numerical differentiation and integration.
- **Chapter 5**: Numerical Initial value ODEs.
  (ordinary differential equations)
- **Chapter 6**: Solution to linear equation $Ax = b$.

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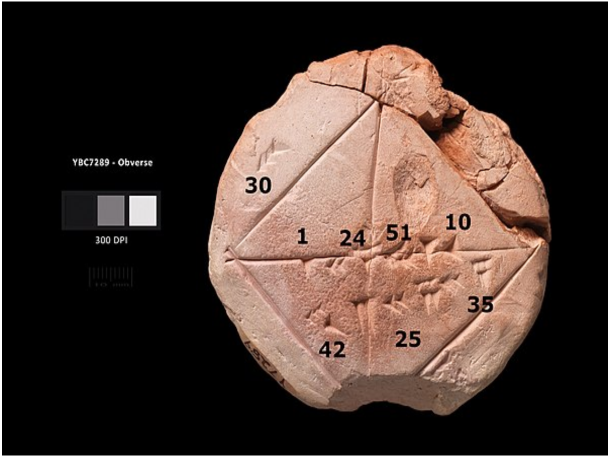
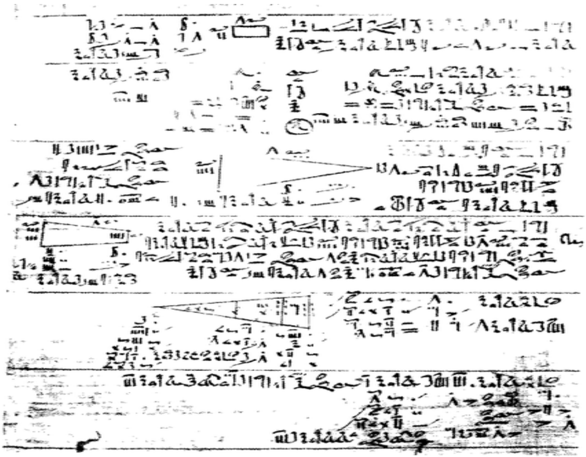
Prerequisite: Calculus/simple `matlab` programming skill

YBC 7289, Babylonian clay tablet, about 3800 years old



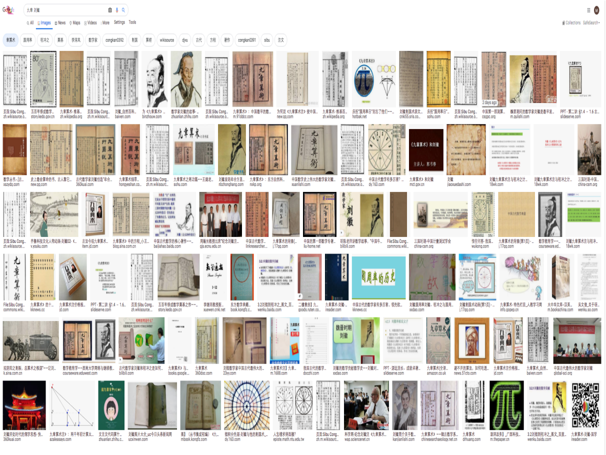
$$\sqrt{2} = \underline{1.41421}35\cdots \approx 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = \underline{1.41421}29\cdots$$

Rhind Papyrus, 3600 years old



Solving linear equations in one unknown

Liu Hui (editor): Nine Chapters of Arithmetic Art ($>$ 2000 years old)



Solving linear system of equations in multiple unknowns,

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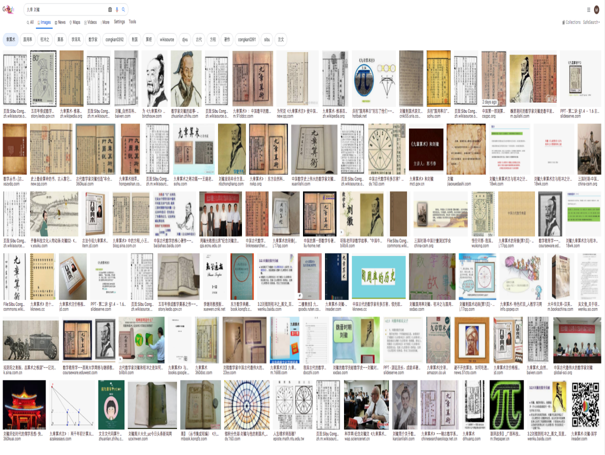
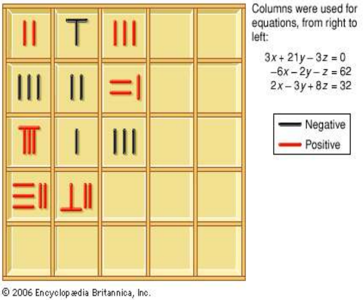


Solving linear system of equations in multiple unknowns,
(Gaussian elimination without pivoting, Ch. 6)

Counting sticks for linear equations



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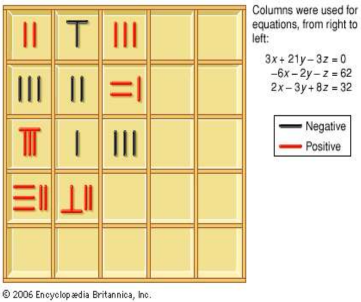
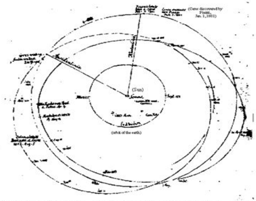


Japanese mathematicians solve linear equations with sticks

Gauss (200 years): Gaussian elimination, Gaussian quadrature

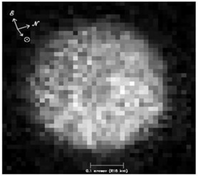
(Left) Carl Friedrich Gauss, considered one of the three greatest mathematicians of all time (along with Archimedes and Sir Isaac Newton). (Right) Gauss at 24, when he computed the orbit of Ceres.



Sketch of the orbits of Ceres and Pallas (nachlaß Gauß, Handb. 4). Courtesy of Universitätsbibliothek Göttingen.

(Left) Gauss' sketch of the orbit of Ceres.

(Right) Image of Ceres from the Hubble telescope.



https://www.youtube.com/watch?v=Y4B6fqNIfAc&ab_channel=TomoNewsUS

Fibonacci's Problem in 1224, with Emperor Frederick II

Solve

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0.$$

Fibonacci's Solution

$$x = 1+22\left(\frac{1}{60}\right)+7\left(\frac{1}{60}\right)^2+42\left(\frac{1}{60}\right)^3+33\left(\frac{1}{60}\right)^4+4\left(\frac{1}{60}\right)^5+40\left(\frac{1}{60}\right)^6.$$

There can only be ONE real solution since

$$f'(x) = 3x^2 + 4x + 10 = 3(x+2/3)^2 + \frac{26}{3} > 0,$$

$f(x)$ is a monotonically increasing function.

Fibonacci's Problem in 1224, with Emperor Frederick II

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$$\begin{aligned} x_{\textbf{Fibon}} &= 1 + 22\left(\frac{1}{60}\right) + 7\left(\frac{1}{60}\right)^2 + 42\left(\frac{1}{60}\right)^3 + 33\left(\frac{1}{60}\right)^4 + 4\left(\frac{1}{60}\right)^5 + \underline{\mathbf{40}}\left(\frac{1}{60}\right)^6 \\ &\approx 1.36880810785\underline{322}, \quad f\left(x_{\textbf{Fibon}}\right) \approx 6.7185 \times 10^{-10} \end{aligned}$$

A better base-60 approximation

$$\begin{aligned} x_{\textbf{Comp}} &= 1 + 22\left(\frac{1}{60}\right) + 7\left(\frac{1}{60}\right)^2 + 42\left(\frac{1}{60}\right)^3 + 33\left(\frac{1}{60}\right)^4 + 4\left(\frac{1}{60}\right)^5 + \underline{\mathbf{39}}\left(\frac{1}{60}\right)^6 \\ &\approx 1.36880810783\underline{179}, \quad f\left(x_{\textbf{Comp}}\right) \approx 2.1976 \times 10^{-10} \end{aligned}$$

Going too far

$$\begin{aligned} x_{\textbf{Far}} &= 1 + 22\left(\frac{1}{60}\right) + 7\left(\frac{1}{60}\right)^2 + 42\left(\frac{1}{60}\right)^3 + 33\left(\frac{1}{60}\right)^4 + 4\left(\frac{1}{60}\right)^5 + \underline{\mathbf{38}}\left(\frac{1}{60}\right)^6 \\ &\approx 1.36880810781\underline{036}, \quad f\left(x_{\textbf{Far}}\right) \approx -2.3239 \times 10^{-10} \end{aligned}$$

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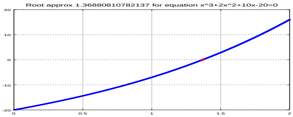
$\boxed{\text{What is the range for the true root?}}$

Fibonacci's Cubic Root is correct to 11 digits

```
>> format long e;
>> h = [1 2 10 -20];
>>  r = roots(h)

r =

     -1.684404053910685e+00 + 3.431331350197691e+00i
     -1.684404053910685e+00 - 3.431331350197691e+00i
      1.368808107821373e+00

>> Fibonacci = (((((40/60+4)/60+33)/60+42)/60+7)/60+22)/60+1

Fibonacci =

     1.368808107853224e+00

>>  r(3)-Fibonacci

ans =

    -3.185118835347112e-11

>> Better = ((((((31/60+38)/60+4)/60+33)/60+42)/60+7)/60+22)/60+1

Better =

     1.368808107821430e+00

>> r(3)-Better

ans =

    -5.795364188543317e-14
```

<u>One</u> real root, <u>two</u> complex roots



Roots of a Random Quintic Polynomial: No closed form formula

```
>>  format short g
>>  hrand = randn(1,6)

hrand =

      -1.3499       3.0349       0.7254    -0.063055      0.71474     -0.20497

>> rrand = roots(hrand)

rrand =

       2.4872
     -0.70735
        0.105 +    0.56831i
        0.105 -    0.56831i
       0.2584
```

$\boxed{\text{Numerical Methods can find roots for any polynomial}}$

Simple Numerical Integration (I)

- Some integrals are easy to integrate by hand

$$I_1 = \int_0^1 \sqrt{1+x}\, dx = \frac{2}{3}\left(2^{\frac{3}{2}} - 1\right).$$

- But others are harder

$$I_2 = \int_0^1 \sqrt{1+x^{1.1}}\, dx = ?$$

$\boxed{\text{Numerical Methods attempt to numerically evaluate ANY integral}}$

Numerical Integration for $\int_{-1}^{1} \sqrt{1+|x|^\alpha}\, d\, x, \quad \alpha = 1, 2$: (6 correct digits)

```
>> format long g
>> I2 = quad(@(x) sqrt(1+x.^2), -1 ,1 )

I2 =

         2.29558701441275

>> I1 = quad(@(x) sqrt(1+x), -1 ,1 )

I1 =

         1.88561089016424

>> I1 - (4/3)*sqrt(2)

ans =

    -7.19299988971578e-06
```

Numerical Integration for $\int_{-1}^{1} \sqrt{1+|x|^{\alpha}}\, d\,x, \quad \alpha = 1, 1.1$: (12 correct digits)

```
>> I1 = quad(@(x) sqrt(1+x.^(1.0)),-1,1,1e-12)
I1 =  1.88561808316413
>>
>> I1 - 4/3*sqrt(2)
ans =   -8.88178419700125e-16
>>
>> I2 = quad(@(x) sqrt(1+abs(x).^(1.1)),-1,1,1e-12)
I2 =  2.41785196129704
```

Simple Numerical Integration: can't go overboard in accuracy

```
>> [I1,fcnt1] = quad(@(x) sqrt(1+x), -1 ,1, 1e-4);disp([I1-(4/3)*sqrt(2),fcnt1])
  -4.5143e-04   2.1000e+01

>> [I1,fcnt1] = quad(@(x) sqrt(1+x), -1 ,1, 1e-8);disp([I1-(4/3)*sqrt(2),fcnt1])
  -4.0728e-08   1.1300e+02

>> [I1,fcnt1] = quad(@(x) sqrt(1+x), -1 ,1, 1e-12);disp([I1-(4/3)*sqrt(2),fcnt1])
  -3.5500e-12   6.9700e+02

>> [I1,fcnt1] = quad(@(x) sqrt(1+x), -1 ,1, 1e-16);disp([I1-(4/3)*sqrt(2),fcnt1])
  -4.4409e-16   4.3930e+03

>> [I1,fcnt1] = quad(@(x) sqrt(1+x), -1 ,1, 1e-20);disp([I1-(4/3)*sqrt(2),fcnt1])
Warning: Maximum function count exceeded; singularity likely.
> In quad at 106
  -5.6018e-06   1.0017e+04
```

Trajectory with ODE

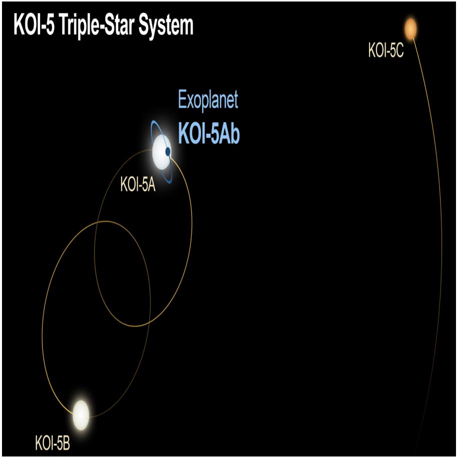
Problem to solve: find function $y(t)$ so that

$$y'(t) = f(y(t), t), \quad y(t_0) = y_0$$

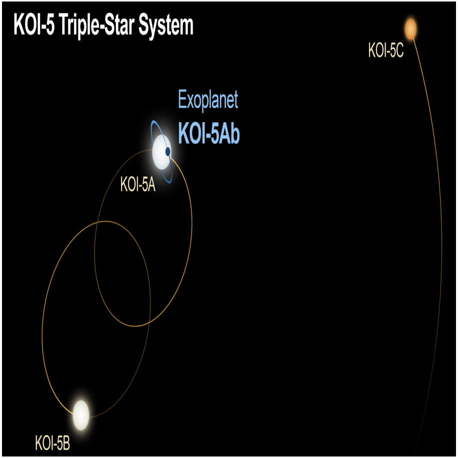
$y(t)$ could be trajectory of a flying bullet, y_0 is initial position.

trajectory follows ODE, aiming = initial conditions

Planet orbit with 3 Suns



Planet orbit with 3 Suns



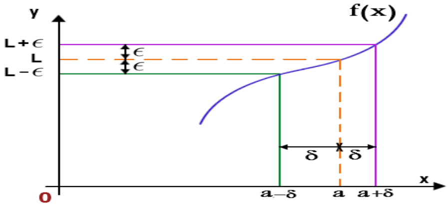
Another planet with 3 suns

https://www.nasa.gov/feature/goddard/2016/newly-discovered-planet-has-3-suns

Teaching Philosophy

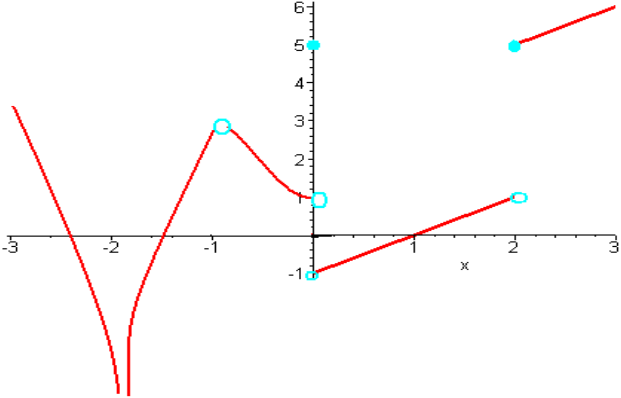
"I expect you all to be independent, innovative, critical thinkers who will do exactly as I say!"

§1.1 Calculus Review: Limit/Continuity

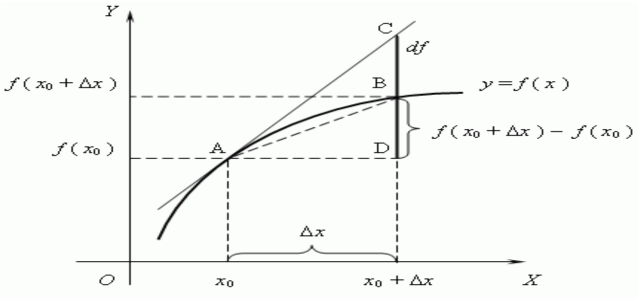


$$\lim_{x\to a} f(x) = f(a) = L$$

Continuity vs. Discontinuity

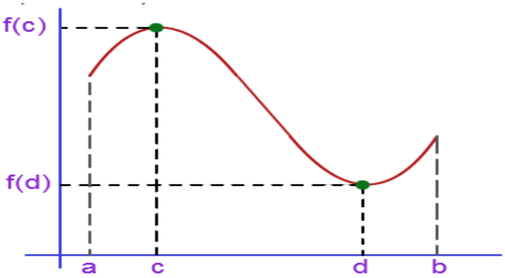


Def: Differentiability



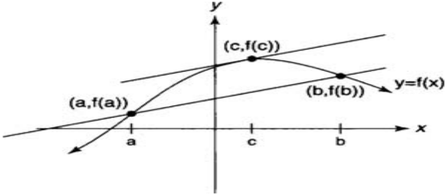
$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Extreme Value Theorem



- Maximum $f(c)$ and minimum $f(d)$ attainable in $[a, b]$ if $f(x)$ continuous.
- For very high dimensions, this means data analysis, artificial intelligence, etc

Mean Value Theorem

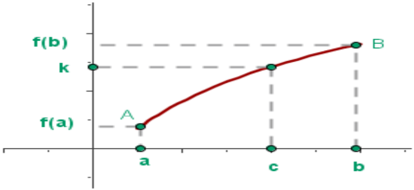


- If $f(x)$ continuous, then c exists in $[a, b]$ so

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Basis of much of theoretical analysis.

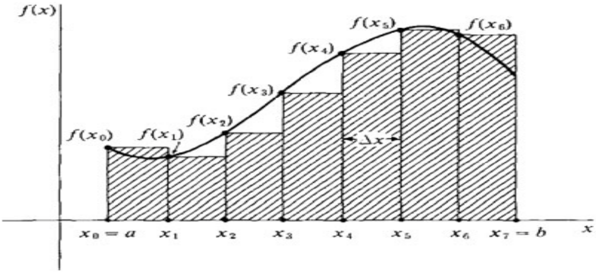
Intermediate Value Theorem



- If $f(x)$ continuous, then c exists in $[a, b]$ so $f(c) = k$ for any k between $f(a)$ and $f(b)$.
- Basis of methods for solving $f(x) = 0$.

Calculus says c exists; we will find c to within a TOLERANCE.

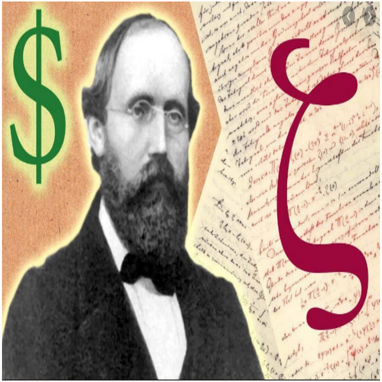
Riemann Sum

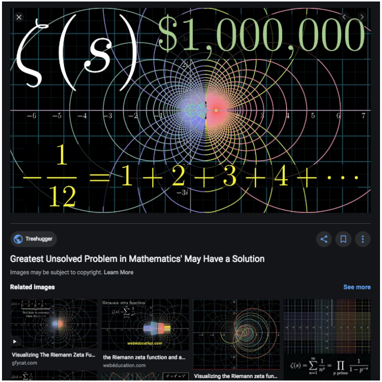


- $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k).$$

Calculus shows limit exists; we find it to within a TOLERANCE.

Riemann, 1826 – 1866





Machine Precision

- Computer numbers (floating point numbers) are a **finite** subset of rational numbers.
- There is a smallest positive computer number ϵ so that

$$1 + \epsilon > 1$$

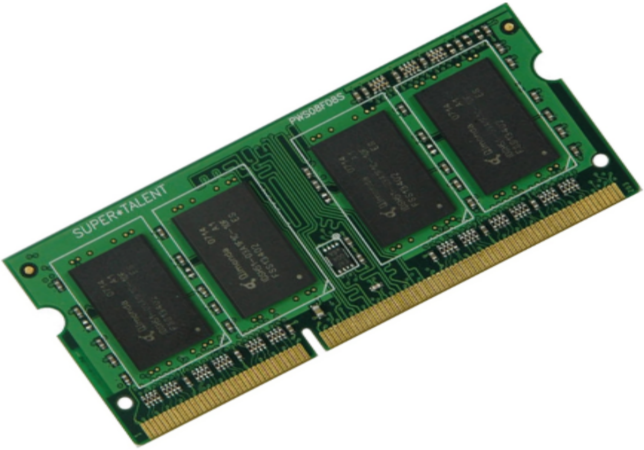
§1.2 Machine Precision

```
>>  eps

ans =

     2.220446049250313e-16

>>  x = 1 + eps; disp([eps, (x-1)/eps])
     2.220446049250313e-16     1.000000000000000e+00

>>  delta = 0.75*eps

delta =

     1.665334536937735e-16

>>  x = 1 + delta; disp([delta, (x - 1)/delta])
     1.665334536937735e-16     1.333333333333333e+00

>> Delta = 0.5*eps

Delta =

     1.110223024625157e-16

>>  x=1+Delta;disp([Delta, (x - 1)/Delta])
     1.110223024625157e-16                          0
```

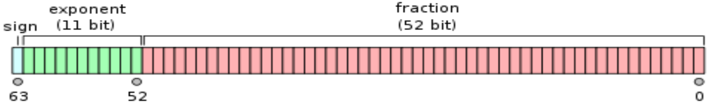
Overflow

```
>>  x=2^1023
 x=2^1023

x =

  8.9885e+307

>> x = 2* x
x = 2* x

x =

   Inf

>> y = 2^(-1023)
y = 2^(-1023)

y =

  1.1125e-308

>> y = y/(2^51)
y = y/(2^51)

y =

  4.9407e-324

>> y = y / 2
y = y / 2

y =

     0
```

Apple Memory Chip: Huge but finite capacity



IEEE 754 Double Precision Format



Every floating point number fits within 64 bits.

Round-off Errors and Floating Point Arithmetic

- **Binary Machine Numbers:** any double precision non-zero *floating point number* has form

$$x = (-1)^s \, 2^{c-1023} \, (1+f), \quad \text{with 64 bits.}$$

  - s = SIGN BIT: 0 for $x > 0$ and 1 for $x < 0$.
  - c = CHARACTERISTIC, with 11 bits:

$$c = c_1 \cdot 2^{10} + c_2 \cdot 2^9 + c_3 \cdot 2^8 + c_4 \cdot 2^7 + c_5 \cdot 2^6 + c_6 \cdot 2^5 + c_7 \cdot 2^4 + c_8 \cdot 2^3 + c_9 \cdot 2^2 + c_{10} \cdot 2^1 + c_{11} \cdot 2^0,$$

    with each $c_j = 0$ or 1.
  - f = MANTISSA with 52 bits

$$f = f_1 \cdot \left(\frac{1}{2}\right) + \cdots + f_{52} \cdot \left(\frac{1}{2}\right)^{52} = \sum_{j=1}^{52} f_j \cdot \left(\frac{1}{2}\right)^j, \quad \text{each } f_j = 0 \text{ or } 1.$$

  - **Floating Point**: Binary point always comes after 1, independent of c.
- $\boxed{\text{Special cases for special numbers}}$

Round-off Errors and Floating Point Arithmetic

- **Binary Machine Numbers:** Example binary string

  0 10000000011 1011100100010000000000000000000000000000000000000000

- $s = 0$, $c = (10000000011)_2 = 1024 + 2 + 1 = 1027$, and

$$f = 1\cdot\left(\frac{1}{2}\right)^1 + 1\cdot\left(\frac{1}{2}\right)^3 + 1\cdot\left(\frac{1}{2}\right)^4 + 1\cdot\left(\frac{1}{2}\right)^5 + 1\cdot\left(\frac{1}{2}\right)^8 + 1\cdot\left(\frac{1}{2}\right)^{12}.$$

$$(-1)^s 2^{c-1023}(1+f) = (-1)^0 \cdot 2^{1027-1023}\left(1+\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{256}+\frac{1}{4096}\right)\right)$$

$$= 27.56640625.$$

$\boxed{\text{binary representation: ineffective for humans, magical for machines}}$

Street Numbers in Binary? (City of machines)

Round-off Errors and Floating Point Arithmetic

- k-**digit <u>Decimal</u> Machine Numbers:**

$$x = \pm 0.d_1 d_2 \cdots d_k \times 10^n, \quad \text{where} \quad 1 \le d_1 \le 9, \quad 0 \le d_i \le 9, i \ge 2.$$

- Any positive real number

$$\begin{aligned} y &= 0.d_1 d_2 \cdots d_k d_{k+1} d_{k+2} \cdots \times 10^n, \\ &\approx 0.d_1 d_2 \cdots d_k \times 10^n \stackrel{def}{=} fl(y) \quad (\textbf{chopping}) \\ &\approx 0.\delta_1 \delta_2 \cdots \delta_k \times 10^n \stackrel{def}{=} fl(y) \quad (\textbf{rounding}), \end{aligned}$$

where

$$\textbf{rounding} = \textbf{chopping on} \quad y + 5 \times 10^{n-(k+1)}.$$

- If $d_{k+1} < 5$: **rounding = chopping.**
- If $d_{k+1} \ge 5$: cut off d_{k+1} and below, then add 1 to d_k.

Round-off Errors and Floating Point Arithmetic

- **5-digit Decimal Machine Numbers for π:**

$$
\begin{aligned}
\pi &= 0.314159265\cdots \times 10^1 \\
&\approx 0.31415 \times 10^1 = 3.1415 \quad (\textbf{chopping}) \\
&\approx (0.31415 + 0.00001) \times 10^1 = 3.1416 \quad (\textbf{rounding}).
\end{aligned}
$$

Absolute error vs. relative error

Suppose that p^* is an approximation to $p \neq 0$.

- **absolute error** $= |p - p^*|$,
- **relative error** $= \frac{|p-p^*|}{|p|}$.

Example

- **absolute errors:**

$$|\pi - 3.1415| \approx 9 \times 10^{-5}, \quad |\pi - 3.1416| \approx 7 \times 10^{-6}.$$

- **relative errors:**

$$\frac{|\pi - 3.1415|}{\pi} \approx 3 \times 10^{-5}, \quad \frac{|\pi - 3.1416|}{\pi} \approx 2 \times 10^{-6}.$$

Cool $200,000 wager by LeSean McCoy, 2017

Cool $\$200,000$ wager by LeSean McCoy, 2017

Cool $\$200,000$ wager by LeSean McCoy, 2017

- Wager: Warriors to win NBA Finals
- McCoy made $\$6M$ in 2017. $\dfrac{\textbf{wager}}{\textbf{salary}} \approx 3\%$
- If lost, wager would be a **huge** absolute error, but **small** relative error, to his salary. He won $\$62,500$