FUNCTIONS * RELATIONS
OPERATIONS * OPERATIONS
RELATIONS * FUNCTIONS

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September 9, 2023

Contents

1 Preliminaries

1.1 The language of functions

1.1.1 Mathematical structures

Modern Mathematics is concerned with *mathematical structures*. A "mathematical structure" consists of one or more sets equipped with data of certain type.

This informal initial definition already covers practically all fundamental types of structures that a mathematician encounters on a daily basis.

1.1.2 The concept of a function

An example of a mathematical structure is provided by the familiar concept of a function. A function of n variables consists of

- a list of n sets
$$X_1, \dots, X_n \tag{1}$$
- a set Y
- an assignment
$$x_1, \dots, x_n \longmapsto y \tag{2}$$
that assigns a *single* element y of set Y to *every* list $x_1, \dots, x_n$ such that
$$x_1 \in X_1, \dots, x_n \in X_n . \tag{3}$$

1.1.3 The domain of a function

The list of sets, (1), is called the *domain* of the function. We shall also call it the *source-list* and will refer to n as the *length* of that list.

1.1.4 The antidomain of a function

The set Y is called the *antidomain* of the function. We shall also refer to it as the *target*.

1.1.5 The argument-list and the value of a function

We shall refer to $x_1, \dots, x_n$ satisfying Condition (3) as the *argument-list*. The single element $y \in Y$ that is assigned to it is then called the *value* of the function on that particular argument-list.

If the *name* of the function is, say, f, its value on the list $x_1, \dots, x_n$ is denoted
$$f(x_1, \dots, x_n) \tag{4}$$

1.1.6 The arrow representation of a function

The symbolic representation of a function

$$f : X_1, \dots, X_n \longrightarrow Y \tag{5}$$

at a glance supplies the following information: *the function's name*, often represented by a symbol, its domain, and its target. In (5) the name of the function is 'f', the domain is the list of sets $X_1, \dots, X_n$, and the target is the set denoted Y.

It is often more convenient to place the name of a function above the arrow representating the function

$$X_1, \dots, X_n \xrightarrow{\ f\ } Y \ .$$

1.1.7 Equality of functions

Two functions are declared to be *equal* if

- *their domains are equal,*
- *their targets are equal,*
- *and their assignments are equal.*

In particular, a function

$$V_1, \dots, V_m \xrightarrow{\ f\ } W$$

can be equal to a function

$$X_1, \dots, X_n \xrightarrow{\ g\ } Y$$

only when

$$m = n, \qquad V_1 = X_1, \ \dots, \ V_m = X_m \ , \qquad \text{and} \qquad W = Y.$$

1.1.8 Functions of zero variables

When $n = 0$, the domain of a function is the empty list of sets. The arrow representation of such a function would be thus

$$\xrightarrow{\ f\ } Y \tag{6}$$

There is only one argument list in this case, namely the empty list. The function assigns to it a single element $y \in Y$. In particular,

$$f \quad \longleftrightarrow \quad \text{the value of } f \text{ on the empty argument-list}$$

defines a canonical identification between functions (6) and elements of the target-set Y.

1.1.9 Functions constant in the i-th variable

If the value (4) does not depend on x_i, we say that f is *constant in i-th variable.*

1.1.10

We shall denote the set of all functions (5) by

$$\mathrm{Funct}(X_1, \dots, X_n\,; Y) \tag{7}$$

or

$$Y^{X_1,\dots,X_n}. \tag{8}$$

1.1.11 Lists with omitted entries

Since lists with certain entries having been omitted are frequently encountered in Mathemtics, we have the notation to denote such lists. For example,

$$x_1, \dots, \hat{x}_i, \dots, x_n \tag{9}$$

stands for the list of length $n-1$

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$$

while

$$x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n \tag{10}$$

stands for the list of length $n-2$

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n,$$

and so on.

1.1.12 Freezing a variable in a function of n-variables

For any $1 \leq i \leq n$ and any $a \in X_i$, assignment

$$x_1, \dots, \hat{x}_i, \dots, x_n \longmapsto f(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n)$$

defines a function of $n-1$ variables

$$X_1, \dots, \hat{X}_i, \dots, X_n \longrightarrow Y\,. \tag{11}$$

We shall denote function (11) by $\mathrm{ev}^i_a f$.

1.1.13 The associated evaluation functions of one variable

Assignment

$$x_i \mapsto \mathrm{ev}^i_{x_i} f$$

defines a function of a single variable

$$X_i \longrightarrow \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,; Y) \tag{12}$$

We shall denote function (12) by $\mathrm{ev}^i f$ and call it the i-th *evaluation function* associated with a function f.

1.1.14

Assignment

$$f \longmapsto \mathrm{ev}^i f$$

defines a canonical bijection

$$\mathrm{Funct}(X_1, \dots, X_n\,;\, Y) \longleftrightarrow \mathrm{Funct}(X_i, \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,;\, Y)) \tag{13}$$

whose inverse is given by sending a function

$$\phi \in \mathrm{Funct}(X_i, \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,;\, Y))$$

to the function

$$X_1, \dots, X_n \longrightarrow Y, \qquad x_1, \dots, x_n \longmapsto \big(\phi(x_i)\big)(x_1, \dots, \hat{x}_i, \dots, x_n)\,.$$

1.1.15

Canonical identification (13) in exponential notation (8) acquires the form

$$Y^{X_1,\dots,X_n} \longleftrightarrow \left(Y^{X_1,\dots,\hat{X}_i,\dots,X_n}\right)^{X_i}\,.$$

1.1.16 Surjective functions

A function (5) is said to be surjective if

$$\text{for every } y \in Y \text{ there exists an argument-list } x_1, \dots, x_n \text{ such that } f(x_1, \dots, x_n) = y\,. \tag{14}$$

You are likely to be familiar with an informal expression "a function f is onto" instead of being surjective. I encourage you to use the term surjective.

1.1.17 Injective functions

A function (5) is said to be injective if it has the property

$$\text{if } f(x_1, \dots, x_n) = f(x_1', \dots, x_n'), \text{ for two argument-lists, then the two argument-lists are equal.} \tag{15}$$

You are likely to be familiar with an informal expression "a function f is one-to-one" instead of of being injective.

1.1.18 Bijective functions

A function is said to be bijective if it is both surjective and injective. This terminology is used primarily for functions of a single variable.

1.2 Composition of functions

1.2.1 Postcomposition

Given a function (5) and a function $g : Y \to Y'$, their composition yields the function

$$g \circ f : X_1, \dots, X_n \longrightarrow Y', \qquad x_1, \dots, x_n \longmapsto g(f(x_1, \dots, x_n)). \tag{16}$$

1.2.2

Postcomposition with a function g is itself a function between the function sets

$$g_* : \mathrm{Funct}(X_1, \dots, X_n; Y) \longrightarrow \mathrm{Funct}(X_1, \dots, X_n; Y'), \qquad f \longmapsto g \circ f. \tag{17}$$

1.2.3 Precomposition

Given a function (5) and a function list $h_1, \dots, h_n$,

$$X'_1, \dots, X'_m \xrightarrow{h_1} X_1 \quad , \quad \dots \quad , \quad X'_1, \dots, X'_m \xrightarrow{h_n} X_n \tag{18}$$

their composition yields the function

$$f \circ (h_1, \dots, h_n) : X'_1, \dots, X'_m \longrightarrow Y, \qquad x'_1, \dots, x'_m \longmapsto f(\,h_1(x'_1, \dots, x'_m), \dots, h_n(x'_1, \dots, x'_m)\,). \tag{19}$$

1.2.4

Precomposition with a function list $h_1, \dots, h_n$ is itself a function between the function sets

$$(h_1, \dots, h_n)^* : \mathrm{Funct}(X_1, \dots, X_n\,; Y) \longrightarrow \mathrm{Funct}(X'_1, \dots, X'_m\,; Y), \qquad f \longmapsto f \circ (h_1, \dots, h_n). \tag{20}$$

1.2.5 Invertible functions of a single variable

Composition of functions of a single variable produces a function of a single variable. We say that $f : X \to Y$ is a *left-invertible* function, if there exists a function $g : Y \to X$ such that

$$g \circ f = id_X\,. \tag{21}$$

We say that $f : X \to Y$ is a *right-invertible* function, if there exists a function $g : Y \to X$ such that

$$f \circ g = id_Y\,. \tag{22}$$

Exercise 1 *Show that, if g is a left inverse of f and h is a right inverse of f, then $g = h$.*

1.2.6

We denote that unique left- and right-inverse by f^{-1}.

Exercise 2 *Show that a left-invertible function f is injective and a right-invertible function is surjective.*

In particular, an invertible function is bijective.

Exercise 3 *Show that a bijective function is invertible.*

Lemma 1.1 *Suppose that $f : X \to Y$ is injective. Then there is a natural correspondence between left inverses of f and functions $h : Y \setminus f_* X \longrightarrow X$.*

Proof. The target of a function f is the union of disjoint sets

$$Y' := f_* X \qquad \text{and} \qquad Y'' := Y \setminus f_* X .$$

Exercise 4 *Show that $g : Y \to X$ is a left inverse of f if and only if the restriction of g to Y' is the function*

$$y \longmapsto \textit{ the unique } x \in X \textit{ such that } f(x) = y .$$

Thus, the set of left inverses of f is in bijective correspondence with the set of functions $Y'' \to X$,

$$\text{Left Inverses}(f) \longleftrightarrow \text{Funct}(Y'', X), \qquad g \longmapsto g_{|Y''}.$$

Since the function set $\text{Funct}(Y'', X)$ is not empty as long as either X is not empty or Y'' is empty, we obtain the following two corollaries.

Corollary 1.2 *A function $f : X \to Y$ with $X \neq \emptyset$ is left-invertible if and only if f is injective. A function $f : \emptyset \to Y$ is left invertible if and only $Y = \emptyset$, i.e., if and only if f is bijective.*

Corollary 1.3 *A function $f : X \to Y$ with $X \neq \emptyset$ is bijective if and only if it has a unique left-inverse. That unique left-inverse is also a right-inverse.*

1.2.7 Finite sets

We say that a set is *finite* if every left-invertible function $f : X \to X$ is invertible.

1.2.8 Infinite sets

Accordingly, we say that a set X is infinite, if it admits a left-invertible function $f : X \to X$ that is not right-invertible.

1.2.9 Axiom of Infinity

The so called *Axiom of Infinity* of Set Theory asserts existence of an infinite set.

Existence of an infinite set cannot be proven using the remaining axioms of Set Theory. In fact, the remaining axioms of Set Theory are consistent with the assertion that every set is finite.

We shall prove later that Axiom of Infinity is equivalent to existence of the semiring $(\mathbf{N}, 0, 1, +, \cdot)$ of natural numbers.

1.3 The language of relations

1.3.1 Statements

A *statement* is a well-formed sentence that is either true or false. Any human language whose vocabulary is extended by adding various, previously defined, mathematical terms, is acceptable.

1.3.2 A relation is a function whose values are statements

A *relation* on sets $X_1, \dots, X_n$ is a function of n variables

$$\rho : X_1, \dots, X_n \longrightarrow \text{Statements}, \qquad x_1, \dots, x_n \longmapsto \rho(x_1, \dots, x_n). \tag{23}$$

We say in this case that ρ is an n*-ary* relation. We also say that the relation is *between* elements of sets $X_1, \dots, X_n$.

1.3.3 Nullary, unary, binary, ternary, ... relations

For small values of n, instead of speaking about 0-ary, 1-ary, 2-ary, 3-ary, ..., relations, we speak of *nullary*, *unary*, *binary*, *ternary*, ..., relations.

1.3.4 {nullary relations} $\longleftrightarrow$ {statements}

According to Section 1.1.8, there is a canonical identification between *nullary relations* and *statements*.

1.3.5 Relations *on* a set

When all sets X_i in the domain coincide with a set X, we speak of an n-ary relation *on* X.

The statement $\rho(x_1, \dots, x_n)$ needs not refer to some or even to anyone of the element variables x_i.

1.3.6 Total relations

The statement $\rho(x_1, \dots, x_n)$ may hold for every list of arguments. Such a relation is sometimes referred to as *a total* relation.

1.3.7 Void relations

The statement $\rho(x_1, \dots, x_n)$ may fail for every list of arguments. Such a relation is sometimes referred to as *a void* relation.

1.3.8

Since a nullary relation reduces to a single statement, and since every statement either holds or fails, a nullary relation is either total or void.

1.4 Operations on sets

1.4.1

An n*-ary operation* on a set Y is a function

$$\mu : X_1, \dots, X_n \longrightarrow Y \tag{24}$$

where all the sets $X_1, \dots, X_n$ are equal to Y.

1.4.2 $\{$nullary operations on $Y\} \longleftrightarrow Y$

To declare a nullary operation on a set Y is equivalent to supplying a single element of Y. For this reason, nullary operations on Y are thought of as "distinguished" elements of Y. In particular, there is a canonical bijection between the set of nullary operations on Y and the set Y itself.

1.4.3 Induced operations

Given a list of n functions of m variables,

$$f_1, \dots, f_n \in \mathrm{Funct}(X_1, \dots, X_m; Y),$$

let us assign to the argument list

$$x_1, \dots, x_m$$

the list of values

$$f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)$$

and then apply the operation μ. Composite assignment

$$x_1, \dots, x_m \longmapsto f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m) \longmapsto \mu\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big)$$

defines a function $X_1, \dots, X_m \longrightarrow Y$. We shall denote this function by $\mu_*(f_1, \dots, f_n)$.

Assignment

$$f_1, \dots, f_n \longmapsto \mu_*(f_1, \dots, f_n) \tag{25}$$

defines then an n-ary operation μ_* on the set of functions $\mathrm{Funct}(X_1, \dots, X_m; Y)$. We refer to it as the operation *induced by* μ.

1.5 Canonical operations on $\mathscr{P}X$

1.5.1 Canonical operations

A general set X has no distinguished elements, hence it is not equipped with any distinguished nullary operation. Similarly, there are no distinguished binary, ternary, etc., operations on a general set. The identity function

$$\mathrm{id}_X : X \longrightarrow X, \qquad x \longmapsto x, \tag{26}$$

is the only distinguished unary operation.

Certain sets, however, are *naturally* equipped with various operations. We refer to such operations as *canonical*. An example of prime importance is provided by the set of all subsets, $\mathscr{P}X$, of an arbitrary set X. A shorter designation for $\mathscr{P}X$ is the *power-set of* X.

1.5.2 Canonical nullary operations on $\mathscr{P}X$

The power-set of a general nonempty set has exactly two distinguished elements: the empty subset $\emptyset$ and X. In other words, $\mathscr{P}X$ is equipped with exactly two canonical nullary operations.

1.5.3 The complement of a subset

The power-set of a general set has a canonical unary operation

$$\complement : \mathscr{P}X \longrightarrow \mathscr{P}X, \qquad A \longmapsto \complement A := \{x \in X \mid x \notin A\}, \tag{27}$$

that sends a subset $A \subseteq X$ to its *complement*. We shall usually denote the complement of a subset $A \subseteq X$ by A^c and use symbol $\complement$ to denote the complement operation.

1.5.4 Involutions on a set

Note that $\complement^2 := \complement \circ \complement$ is the identity operation. A unary operation $\mu : X \to X$ with this property is called an *involution* (on a set X). The identity operation id_X is a *trivial* involution.

1.5.5 Canonical unary operations on $\mathscr{P}X$

The power-set $\mathscr{P}X$ of a nonempty set is equipped with exactly two unary operations, both of them involutions on $\mathscr{P}X$: the identity operation $\mathrm{id}_{\mathscr{P}X}$ and the complement operation $\complement$.

1.5.6 Canonical binary operations on $\mathscr{P}X$

Union of two sets,

$$A, B \longmapsto A \cup B,$$

intersection of two sets,

$$A, B \longmapsto A \cap B,$$

difference of two sets,

$$A, B \longmapsto A \setminus B,$$

are examples of canonical binary operations on the power-set.

1.5.7

Any one of the above three operations can be expressed in terms of any of the remaining two and of the complement operation. For example, the union and the intersection operations are linked to each other by the following pair of identities

$$A \cap B = (A^c \cup B^c)^c \qquad \text{and} \qquad A \cup B = (A^c \cap B^c)^c \qquad (A, B \subseteq X) \tag{28}$$

called *de Morgan laws*.

Note also the following identities

$$A \cup A^c = X, \qquad A \cap A^c = \emptyset \qquad \text{and} \qquad A \setminus B = A \cap B^c = (A^c \cup B)^c \qquad (A, B \subseteq X).$$

Exercise 5 *Find the identity expressing $\cap$ in terms of $\setminus$ and $\complement$, and prove it.*

1.6 Operations on Statements

1.6.1 Basic binary operations on sentences

The following table contains the list of basic binary operations on sentences (symbols P and Q stand for arbitrary sentences).

Read:	Symbolic notation	Name
P *and* Q	$P \wedge Q$	Conjunction
P *or* Q	$P \vee Q$	Alternative
if P, *then* Q	$P \Rightarrow Q$	Implication
P *if and only if* Q	$P \Leftrightarrow Q$	Equivalence

1.6.2 Negation

The negated sentence P will be symbolically denoted $\neg P$. In many languages, negating a sentence is performed according to rules that depend on the syntactical structure of that sentence. For this reason, it is difficult or impossible to provide one single reading of the negated sentence $\neg P$. We can circumvent this difficulty by saying, instead, "the negation of P" or "P negated", when we need to refer to $\neg P$.

1.6.3 Validity of the corresponding statements

Assuming that P and Q are statements,

- $P \wedge Q$ holds precisely when P and Q hold;
- $P \vee Q$ holds precisely when P or Q holds;
- $P \Rightarrow Q$ fails if P holds and Q fails, otherwise it holds;
- $P \Leftrightarrow Q$ holds precisely when P and Q both hold or both fail;
- $\neg P$ holds precisely when P fails.

In particular, Conjunction, Alternative, Implication, Equivalence, define binary operations on the set of Statements, while Negation defines a unary operation.

1.6.4 Operations on Statements = Relations on Statements

On the set of statements the concepts of an n-ary operation and of an n-ary relation coincide.

1.6.5 Operations on relations

Any operation on Statements induces the corresponding operations on the sets of relations, $\mathrm{Rel}(X_1, \dots, X_n)$, between elements of sets $X_1, \dots, X_n$.

1.6.6

Thus, given relations $\rho, \sigma \in \mathrm{Rel}(X_1, \dots, X_n)$, we can form the relations $\neg\rho$, $\rho \vee \sigma$, $\rho \wedge \sigma$, $\rho \Longrightarrow \sigma$ and $\rho \Leftrightarrow \sigma$. They assign to an argument list $x_1, \dots, x_n$ the statements

$$\neg\rho(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \vee \sigma(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \wedge \sigma(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \Longrightarrow \sigma(x_1, \dots, x_n)$$

and, respectively,

$$\rho(x_1, \dots, x_n) \Leftrightarrow \sigma(x_1, \dots, x_n).$$

1.7 Quantification

1.7.1 Universal quantification

Given a relation ρ between elements of sets $X_1, \dots, X_n$, assigning to a list $x_1, \dots, \hat{x}_i, \dots, x_n$ the statement

$$\textit{for all } x_i \in X_i, \ \rho(x_1, \dots, x_n) \tag{29}$$

defines an $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$. Instead of "for all", we can also say "for every" with the same meaning.

Symbolically, statement (29) is represented

$$\forall_{x_i \in X_i} \rho(x_1, \dots, x_n).$$

1.7.2 " Statement S is a special case of statement T "

Suppose $\rho : X \longrightarrow \text{Statements}$ is a (unary) relation on a set X. Consider the statements obtained by universal quantification of relation ρ over subsets $A \subseteq X$ and $B \subseteq X$

$$S := \text{“ } \forall_{x \in A} \rho(x) \text{ ”} \qquad \text{and} \qquad T := \text{“ } \forall_{x \in B} \rho(x) \text{ ”}. \tag{30}$$

Note that, if $A \subseteq B$, then

$$S \Longrightarrow T. \tag{31}$$

If so, we shall say that *statement S is a special case of statement T*.

In general, given two statements S and T, we shall say that S is a special case of T if there exist

a unary relation ρ on a certain set X and subsets $A \subseteq B \subseteq X$

such that S and T have the form as in (30).

1.7.3 " Statement S trivially implies statement T "

Note that in order to establish implication (31), one deoes not need to know anything about a set X, a relation ρ on X, or subsets A and B. One only needs to know that both statements are obtained by *universal* quantification of the *same* certain unary relation over two subsets $A \subseteq B$ of X.

This is one of those situations when mathematicians are likely to say that a statement S *trivially* implies a statement T.

1.7.4 Existential quantification

Assigning to a list $x_1, \dots, \hat{x}_i, \dots, x_n$ the statement

$$\textit{there exists } x_i \in X_i \textit{ such that } \rho(x_1, \dots, x_n) \tag{32}$$

defines another an $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$.

Symbolically, statement (32) is represented

$$\exists_{x_i \in X_i} \rho(x_1, \dots, x_n).$$

1.7.5

Suppose $\rho : X \longrightarrow \text{Statements}$ is a (unary) relation on a set X. Consider the statements obtained by existential quantification of relation ρ over subsets $A \subseteq X$ and $B \subseteq X$

$$S := \text{“ } \exists_{x \in A} \rho(x) \text{ ”} \qquad \text{and} \qquad T := \text{“ } \exists_{x \in B} \rho(x) \text{ ”}. \tag{33}$$

Note that, if $A \subseteq B$, then

$$T \Longrightarrow S. \tag{34}$$

Also in this case we say that statement T *trivially implies* statement S.

1.7.6

Operations of quantification are frequently iterated. For example, given $i \neq j$,

$$\forall_{x_i \in X_i} \exists_{x_j \in X_j} \rho(x_1, \dots, x_n)$$

denotes the statement:

$$\textit{for all } x_i \in X_i, \textit{ there exists } x_j \in X_j \textit{ such that } \rho(x_1, \dots, x_n).$$

1.7.7

The statement

$$\forall_{\varepsilon \in \mathbf{R}^+} \exists_{i \in \mathbf{N}} \forall_{j \in \mathbf{N}} \; (i \leq j \;\Rightarrow\; |x_j - a| < \varepsilon) \tag{35}$$

describes the fact that a sequence of real numbers (x_n) converges to a point a of the real line. Here, $\mathbf{R}^+$ denotes the set of positive real numbers and $\mathbf{N}$ denotes the set of natural numbers. The statement is about sequences (x_n) of real numbers and points a of the real line. It defines a binary relation between elements of these two sets. The relation is the result of applying one-after-another universal and existential quantification to the statement that has the form of implication

$$i \leq j \;\Rightarrow\; |x_j - a| < \varepsilon\,. \tag{36}$$

Here x_j denotes the j-th term of the sequence (x_n). Statement (36) is a statement about natural numbers i and j, a sequence of real numbers (x_n), a point of the real line a, and a positive real number ε. As such, it is a 5-ary relation. Application of three consecutive quantifications yields the binary relation defined in (35).

What you see here are examples of typical statements encountered in Mathematical Analysis.

Exercise 6 *Let $\rho : X_1, X_2 \longrightarrow \text{Statements}$ be a binary relation. Consider the statements*

$$S := \text{“ } \exists_{x_1 \in A_1} \forall_{x_2 \in A_2}\, \rho(x_1, x_2) \text{ ”} \qquad \textit{and} \qquad T := \text{“ } \exists_{x_1 \in B_1} \forall_{x_2 \in B_2}\, \rho(x_1, x_2) \text{ ”}$$

where A_1 and B_1 are subsets of X_1 while A_2 and B_1 are subsets of X_2. Under what condition on A_1, A_2, B_1 and B_2, statement S implies statement T?

1.8 Comparing relations

1.8.1

Let ρ and σ be two relations between elements of sets $X_1, \dots, X_n$. Let us consider the nullary relation, i.e., the statement

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n} \left(\rho(x_1, \dots, x_n) \Rightarrow \sigma(x_1, \dots, x_n) \right) .$$

Here

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n}$$

is an abbreviation for

$$\forall_{x_1 \in X_1} \cdots \forall_{x_n \in X_n} .$$

1.8.2

We say that ρ is *weaker* than σ, and that σ is *stronger* than ρ, if that statement holds, i.e., if the relation

$$\rho \Rightarrow \sigma \tag{37}$$

is a *total* relation. It is common in this situation to represent this fact symbolically by writing

$$\rho \Longrightarrow \sigma \tag{38}$$

and to say that ρ *implies* σ.

1.8.3 Caveat

Make sure not to confuse (37) with (38). Symbol $\Rightarrow$ in (37) denotes a binary *operation* on the set of relations $\mathrm{Rel}(X_1, \dots, X_n)$ while symbol $\Longrightarrow$ in (38) denotes a binary *relation* on the same set.

1.8.4 Equipotent relations

The terms “weaker” and “stronger” is not an ideal terminology as a relation ρ can be both weaker and stronger than a relation σ. If this happens, we say that the two relations are *equipotent.* This happens, precisely, when the statement

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n} \left(\sigma(x_1, \dots, x_n) \Leftrightarrow \rho(x_1, \dots, x_n) \right)$$

holds, i.e., when

$$\rho \Leftrightarrow \sigma \tag{39}$$

is a *total* relation.

In other words, statement $\sigma(x_1, \dots, x_n)$ holds precisely for the same lists of arguments as statement $\rho(x_1, \dots, x_n)$ does.

It is common in this situation to represent this fact symbolically by writing

$$\rho \Longleftrightarrow \sigma \tag{40}$$

and to say that relations ρ and σ are *equipotent*.

1.8.5 Caveat

Though it is preferable to refer to such relations as equipotent, mathematicians keep saying that such relations are *equivalent*. Since the term "equivalence" is used also as a generic term for a binary relation that is *reflexive*, *symmetric* and *transitive*, cf. Section 3.2.4, I encourage you to use the term "equipotent".

1.9 Functions of n variables viewed as $(n+1)$-ary relations

1.9.1

Given sets X_1, ..., X_n and Y, and a function of n variables

$$f : X_1, \dots, X_n \longrightarrow Y, \tag{41}$$

we can associate with it an $(n+1)$-ary relation where statement $\rho(x_1, \dots, x_n, y)$ reads

$$f(x_1, \dots, x_n) = y.$$

1.9.2

The $(n+1)$-ary relation associated to a function has the following property:

$$\text{for every list of elements } x_1 \in X_1, \dots, x_n \in X_n, \text{ there exists a unique } y \in Y, \text{ such that } \rho(x_1, \dots, x_n, y). \tag{42}$$

1.9.3

Given any $(n+1)$-ary relation satisfying property (42), we can define a function (41) where $f(x_1, \dots, x_n)$ is defined to be that unique element $y \in Y$ such that

$$\rho(x_1, \dots, x_n, y).$$

Let us denote this function f_ρ.

Exercise 7 *Show that $f_\sigma = f_\rho$ if and only if σ and ρ are equipotent.*

1.10 Composing relations

1.10.1

Suppose that two relations are given,

$$\text{an } (m+1)\text{-ary relation between elements of sets } X_0, \dots, X_m \ ,$$

denoted σ, and

$$\text{an } (n+1)\text{-ary relation between elements of sets } X_m, \dots, X_{m+n+1} \ ,$$

denoted ρ. Assigning to a list $x_1, \dots, \hat{x}_m, \dots, x_{m+n+1}$ the statement

$$\textit{there exists } x_m \in X_m \textit{ such that } \sigma(x_0, \dots, x_m) \textit{ and } \rho(x_m, \dots, x_{m+n+1}) \tag{43}$$

defines an $(m+n+1)$-ary relation between elements of sets

$$X_1, \dots, \hat{X}_m, \dots, X_{m+n+1} \ .$$

Symbolically, statement (43) is represented

$$\exists_{x_m \in X_m} \ (\sigma(x_0, \dots, x_m) \wedge \rho(x_m, \dots, x_{m+n+1})) \ .$$

1.10.2

We call the relation defined above, the *composite of ρ and σ* and denote it $\rho \circ \sigma$.

1.11 Cartesian product $X_1 \times \cdots \times X_n$

1.11.1

Given a list of sets $X_1, \dots, X_n$, let us form its Cartesian product

$$X_1 \times \cdots \times X_n \ . \tag{44}$$

By definition, its elements are ordered n-tuples $(x_1, \dots, x_n)$ of elements $x_1 \in X_1, \dots, x_n \in X_n$.

1.11.2 The concept of an ordered n-tuple

What is an ordered n-tuple? There is not much difference between lists of length n and ordered n-tuples. When we speak of an ordered n-tuple, we always think of it being a *single* entity, while when we speak of a list of length n, we think of n separate entities.

1.11.3

To illustrate this further, the assignment

$$x, y \mapsto x + y \qquad (x, y \in \mathbf{N})$$

defines a function of 2 variables on the set of natural numbers $\mathbf{N}$, while the assignment

$$(x, y) \mapsto x + y \qquad (x, y \in \mathbf{N})$$

defines a function of a single variable on the Cartesian square $\mathbf{N} \times \mathbf{N}$ of $\mathbf{N}$. The targets of both functions are the same, namely the set of natural numbers.

1.11.4 The equality principle

The principal property built into the concept of an ordered n-tuple is the following equality principle

$$(x_1, \dots, x_m) = (y_1, \dots, y_n)$$

if and only if $m = n$ and $x_i = y_i$ for all $1 \leq i \leq n$.

1.11.5 The standard set-theoretic model of an ordered pair

The actual model of an ordered n-tuple is of little importance. It is possible to prove existence of such a model using only basic set theoretic concepts. For example, the axiom of Set Theory called Axiom of a Pair states that, for any x and y, the set $\{x, y\}$, whose elements are x and y, exists. Thus, $\{x\} = \{x, x\}$ and $\{x, y\}$ exist and therefore also the following set

$$\{\{x\}, \{x, y\}\} \tag{45}$$

exists. This set is a model of an *ordered pair*, i.e., of an ordered a 2-tuple.

Exercise 8 *Show that*

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$$

if and only if $x = x'$ *and* $y = y'$.

1.11.6

If $x \in X$ and $y \in Y$, then (45) is a *family* of subsets of $X \cup Y$, i.e., it is a subset of the power-set of $X \cup Y$,

$$\{\{x\}, \{x, y\}\} \subseteq \mathscr{P}(X \cup Y) .$$

Accordingly, the Cartesian product $X \times Y$ is realized as the appropriate subset of the power-set of the power-set of $X \cup Y$,

$$X \times Y := \{ P \in \mathscr{P}(\mathscr{P}(X \cup Y)) \mid \exists_{x \in X} \exists_{y \in Y} P = \{\{x\}, \{x, y\}\} \} ,$$

which demonstrates its existence.

1.11.7

Having a model of on ordered pair, the ordered pair

$$((x, y), z)$$

becomes a model of an ordered triple and the Cartesian product

$$(X \times Y) \times Z$$

becomes a model of $X \times Y \times Z$. By induction on n, one can construct a model of an ordered n-tuple

$$(x_1, \dots, x_n)$$

and of

$$X_1 \times \dots \times X_n ,$$

There are other, more convenient models.

1.11.8 An ordered n-tuple as a function

A convenient model of an ordered n-tuple $(x_1, \dots, x_n)$ is provided by a function

$$\xi : \{1, \dots, n\} \longrightarrow X_1 \cup \dots \cup X_n \tag{46}$$

whose value at i is, for every $1 \le i \le n$, an element of X_i.

In this model, the Cartesian product $X_1 \times \dots \times X_n$ is represented as a subset of the set of all functions (46).

1.11.9 Universal functions of n-variables

We shall say that a function

$$\tau : X_1, \dots, X_n \longrightarrow T \tag{47}$$

is a *universal* function with the domain list $X_1, \dots, X_n$, if *every* function (5) can be produced from τ by postcomposing τ with a *unique* function $\tilde{f} : T \to Y$,

$$f = \tilde{f} \circ \tau.$$

In that case, the bijective correspondence

$$\mathrm{Funct}(X_1, \dots, X_n; Y) \longleftrightarrow \mathrm{Funct}(T, Y), \qquad f \longleftrightarrow \tilde{f}, \tag{48}$$

identifies the set of Y-valued functions of n-variables, with the domain-list $X_1, \dots, X_n$, with the set of functions of a single variable $T \to Y$.

1.11.10 The canonical function of n-variables $X_1, \dots, X_n \longrightarrow X_1 \times \dots \times X_n$

For every list of sets $X_1, \dots, X_n$, there exists a canonical function of n-variables with that list as its domain. It assigns to an argument list $x_1, \dots, x_n$ the corresponding ordered n-tuple,

$$X_1, \dots, X_n \longrightarrow X_1 \times \dots \times X_n, \qquad x_1, \dots, x_n \longmapsto (x_1, \dots, x_n), \tag{49}$$

1.11.11

The canonical function has the universal property defined in Section 1.11.9. Indeed,

$$f \quad \longmapsto \quad \left(\tilde{f} : X_1 \times \dots \times X_n \to Y, \quad (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)\right)$$

is a bijective correspondence and f is produced by postcomposing function (49) with $\tilde{f}$.

1.11.12 The case of functions of zero variables

When $n = 0$, Cartesian product of the empty list of sets consists of functions from the *empty* set of natural numbers to the union of the empty family of sets. The latter, as we already know, is the empty set. In other words, Cartesian product of the empty list of sets is the set of functions

$$\emptyset^{\emptyset} = \mathrm{Funct}(\emptyset, \emptyset) = \{\mathrm{id}_{\emptyset}\}, \tag{50}$$

and that set has a unique element, namely the identity function associated with the empty set. Exponential notation $\emptyset^{\emptyset}$, cf. (8) is particularly apt in this case. We observe that foundations of Set Theory themeselves are telling us that 0^0 is well defined and equals to 1.

1.11.13 Canonical identification $\mathbf{Op}_0(Y) \longleftrightarrow \mathbf{Funct}(\emptyset^{\emptyset}, Y)$

In particular, nullary operations on a set Y, i.e., Y-valued functions of zero of variables, are canonically identified with functions $\emptyset^{\emptyset} \to Y$.

1.11.14

Every statement containing references to functions of n-variables can be now replaced by an equivalent statement containing references exclusively to functions of a single variable.

This explains why the use of the concept of a function of n-variables has practically disappeared from modern mathematical language. This is also the reason why Cartesian product is today present everywhere where normally one would be mentioning functions of n-variables: Cartesian product

$$X_1 \times \cdots \times X_n$$

is the *target* of the universal function of n-variables (49).

1.11.15 Canonical projections $(\pi_i)_{i \in \{1,\ldots,n\}}$

The Cartesian product is more than just a set, it is a *mathematical structure*, like a relation or a function. One should consider the Cartesian product to consist of a set $X_1 \times \cdots \times X_n$ equipped with a list of functions

$$\pi_1, \ldots, \pi_n , \tag{51}$$

called the *canonical projections*, where π_i is defined as

$$\pi_i : X_1 \times \cdots \times X_n \longrightarrow X_i, \qquad (x_1, \ldots, x_n) \mapsto x_i . \tag{52}$$

Having just set $X_1 \times \cdots \times X_n$ alone would not suffice to recover the list of sets $X_1, \ldots, X_n$. For example, $X_1 \times \cdots \times X_n$ is the empty set whenever at least one set X_i is empty.

1.11.16 *Naturality* of Cartesian product

Cartesian product assigns to a list of sets $X_1, \ldots, X_n$ a single set $X_1 \times \cdots \times X_n$ equipped with the list of functions $\pi_1, \ldots, \pi_n$. A function list

$$X_1 \xrightarrow{f_1} X_1' , \ \ldots\ , \ X_n \xrightarrow{f_n} X_n' , \tag{53}$$

induces a function between the corresponding Cartesian-product sets

$$f_1 \times \cdots \times f_n : X_1 \times \cdots \times X_n \longrightarrow X_1' \times \cdots \times X_n' , \qquad (x_1, \ldots, x_n) \longmapsto (f_1(x_1), \ldots, f_n(x_n)) . \tag{54}$$

Moreover, the assignment

$$f_1, \ldots, f_n \longmapsto f_1 \times \cdots \times f_n$$

commutes with the operations of function composion.

Exercise 9 *Given a function list*

$$X'_1 \xrightarrow{f_1} X''_1 \ , \ \dots \ , \ X'_n \xrightarrow{f_n} X''_n \ ,$$

show that

$$(f'_1 \times \cdots \times f'_n) \circ (f_1 \times \cdots \times f_n) \ = \ (f'_1 \circ f_1, \dots, f'_n \circ f_n) \, .$$

Mathematicians refer to such behavior as *naturality* of the assignment

$$X_1, \dots, X_n \longmapsto X_1 \times \cdots \times X_n \, .$$

1.11.17 The graph of a relation

Given a relation ρ between elements of sets $X_1, \dots, X_n$, the following subset of the Cartesian product,

$$\Gamma_\rho := \{(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n \mid \rho(x_1, \dots, x_n)\} \tag{55}$$

is guaranteed to exist by the axioms of Set Theory. This is the set of those ordered n-tuples for which statement $\rho(x_1, \dots, x_n)$ holds. One calls it the *graph* of ρ.

Exercise 10 *Let ρ and σ be two relations between elements of sets $X_1, \dots, X_n$. Show that ρ is weaker than σ if and only if*

$$\Gamma_\rho \subseteq \Gamma_\sigma \, .$$

1.11.18

In particular, relations ρ and σ are equipotent if and only if their graphs are equal

$$\Gamma_\rho = \Gamma_\sigma \, .$$

1.11.19 Correspondences

The graph of a relation provides another example of a mathematical structure. It involves the list of the following data:

- a list of sets $X_1, \dots, X_n$,
- a subset $C \subseteq X_1 \times \cdots \times X_n$.

Having just the set C alone would not suffice to recover the list of sets $X_1, \dots, X_n$.

A structure of this kind begs for a name. I propose to call it a *correspondence between elements of sets* $X_1, \dots, X_n$ or, an *n-correspondence*, in short.

1.11.20

When all sets X_i are one and the same set X, we shall speak of n-correspondences *on* X.

1.11.21 1-correspondences

In particular, 1-correspondences on X are the same as *subsets* of X.

1.11.22

In practice, we still be denoting a correspondence by the symbol denoting the subset C of $X_1 \times \cdots \times X_n$.

1.11.23 Caveat

In fact, a common practice among mathematicians is to call precisely this structure a *relation*. This approach to the concept of a relation, while being much less intuitive than the 'statements-valued function' approach, it allows one to place theory of relations entirely within the realm of Set Theory. For example, relations with a given domain (1) form a well defined set.

1.11.24

The main advantage of such a restrictive notion of a relation is that it frees a mathematician from any concerns about what is and what is not a *statement* while still being sufficient for studying the whole of Mathematics.

Indeed, given a correspondence C between elements of sets $X_1, \dots, X_n$, let $\rho_C(x_1, \dots, x_n)$ be the statement

$$(x_1, \dots, x_n) \in C .$$

This defines a relation between elements of sets $X_1, \dots, X_n$.

Exercise 11 *Show that any relation ρ is equipotent to the relation ρ_{Γ_ρ}.*

Exercise 12 *Show that, for any correspondence C, one has $C = \Gamma_{\rho_C}$.*

1.11.25

We shall express the operations on relations, introduced in Sections 1.6.5–1.7, in terms of their graph correspondences. For this we need to introduce some notation.

Exercise 13 *Given a relation ρ, show that*

$$\Gamma_{\neg\rho} = \complement\Gamma_\rho . \tag{56}$$

Exercise 14 *Given relations ρ and σ with the same domain, show that*

$$\Gamma_{\rho\vee\sigma} = \Gamma_\rho \cup \Gamma_\sigma \qquad \textit{and} \qquad \Gamma_{\rho\wedge\sigma} = \Gamma_\rho \cap \Gamma_\sigma . \tag{57}$$

1.11.26

The above two exercises demonstrate that the operations of negation, alternative and conjunction of relations translate into the operations of taking the complement, the union, and the intersection, of correspondences.

Exercise 15 *Given relations ρ and σ with the same domain, show that*

$$\Gamma_{\rho\Rightarrow\sigma} = \complement\Gamma_\rho \cup \Gamma_\sigma . \tag{58}$$

1.11.27 The function-list canonically associated with an n-correspondence

By post-composing the canonical inclusion $\iota : C \hookrightarrow X_1 \times \cdots \times X_n$ with the list of canonical projections $\pi_1, \ldots, \pi_n$, we obtain a list of functions

$$\begin{tikzcd} C \arrow[d, "\partial_1"'] \arrow[d, phantom, shift left=2ex, "\cdots"] \arrow[d, shift left=4ex, "\partial_n"] \\ X_1, \ldots, X_n \end{tikzcd} \tag{59}$$

that is canonically associated with the correspondence. Here $\partial_i := \pi_i \circ \iota$, $1 \leq i \leq n$.

1.11.28 Oriented graphs

When $n = 2$ and X_1 and X_2 are the same set X, a list (59) is called an *oriented graph*. Elements of X are referred to, in this case, as *vertices* and elements of C are referred as *oriented edges*, or *arrows*, of the graph.

1.11.29 2-Correspondences as oriented graphs

In particular, 2-correspondences on a set X can be viewed as oriented graphs with vertices being elements of X, such that no two oriented edges have the same source and the same target.

1.12 Power-set functions induced by a function $f : X \to Y$

1.12.1 The *image-of-a-subset* and the *preimage-of-a-subset* functions f_* and f^*

Given a function $f : X \longrightarrow Y$, there are two associated functions between the power-sets

$$\mathscr{P}X \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathscr{P}Y \ , \tag{60}$$

where the associated *image* function is defined by

$$f_*(A) := \{y \in Y \mid \exists_{x \in X} f(x) = y\} \qquad (A \subseteq X) \tag{61}$$

and the associated *preimage* function is defined by

$$f^*(B) := \{x \in X \mid \exists_{y \in Y} f(x) = y\} \qquad (B \subseteq Y). \tag{62}$$

1.12.2 A comment about notation

What I here denote by $f_*(A)$ and $f^*(B)$ is usually denoted $f(A)$ and $f^{-1}(B)$. This is all right as long as there is no need to consider the assignments

$$A \mapsto f(A) \qquad \text{and} \qquad B \mapsto f^{-1}(B)$$

as functions between the corresponding power-sets. When such a need arises, one needs an appropriate notation to denote the image and the preimage functions associated with f. This is why I adopted the *lower-* and the *upper-star* notation that is universally used in Modern Mathematics to denote all sorts of functions that are naturally associated with a given function.

1.12.3

This has yet another advantage: it often allows us to skip parentheses around the arguments of functions f_* and f^* in the interest of keeping notation as simple as possible, without affecting the intended meaning. Thus, we shall, generally, write f_*A and f^*B instead of $f_*(A)$ and $f^*(B)$.

1.12.4

I will say later why in some cases we mark the associated function by placing $*$ as a *subscript* while in other cases—as a *superscript*.

1.12.5 The characteristic function of a subset

Given a subset $A \subset X$, its *characteristic function* is defined by

$$\chi_A : X \to \mathbf{F}_2, \qquad \chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}, \tag{63}$$

where $\mathbf{F}_2 = \{0, 1\}$ denotes the 2-element field.

Assignment

$$A \longmapsto \chi_A$$

yields a canonical identification

$$\chi : \mathscr{P}X \longleftrightarrow \operatorname{Funct}(X, \mathbf{F}_2). \tag{64}$$

Exercise 16 *Prove that, given a function $f : X \to Y$ and a subset $B \subset Y$, one has*

$$f^*\chi_B = \chi_{f^*B}. \tag{65}$$

In other words, the preimage function $f^* : \mathscr{P}Y \to \mathscr{P}X$ can be viewed also as the precomposition function

$$f^* : \operatorname{Funct}(Y, \mathbf{F}_2) \longrightarrow \operatorname{Funct}(X, \mathbf{F}_2).$$

1.12.6

Identity (65) can be also expressed by saying that the following square diagram of functions

$$\begin{tikzcd} \mathscr{P}X \arrow[r, "\chi"] & \operatorname{Funct}(X, \mathbf{F}_2) \\ \mathscr{P}Y \arrow[u, "f^*"] \arrow[r, "\chi"'] & \operatorname{Funct}(Y, \mathbf{F}_2) \arrow[u, "f^*"'] \end{tikzcd}$$

commutes.

1.12.7

Note how close the definitions of the image and of the preimage are to each other: they are both defined by *existential* quantification of the *same* binary relation

$$X, Y \longrightarrow \text{Statements}, \qquad x, y \longmapsto \rho(x,y) \;:=\; \text{“ } f(x) = y \text{ ”} \tag{66}$$

over the corresponding subsets $A \subseteq X$ and $B \subseteq Y$, respectively. We shall often refer to f_* as the *direct image map* and to f^* as the *inverse image map*.[1]

1.12.8

Note the equality of sets

$$f^*B = \{x \in X \mid f(x) \in B\}. \tag{67}$$

The right-hand-side of (67) is how the inverse image is usually defined. Such a definition, however, obfuscates the fact that f_* and f^* are "twin sisters".

1.12.9 The *conjugate image* function $f_!$

These two concepts or, if you wish, constructions, naturally associated with every function $f : X \longrightarrow Y$, are omnipresent. One encounters them nearly in every mathematical argument involving functions between sets. What remains a very little known fact is that f^* has yet another "sibling"

$$f_! : \mathscr{P}(X) \longrightarrow \mathscr{P}(Y), \qquad A \longmapsto (f_*(A^c))^c, \tag{68}$$

that I propose to call the *conjugate image* function.

The name, "conjugate image" stems from the fact that $f_!$ is the *conjugate* of f_* by the *complement operation*,

$$f_! = \complement \circ f_* \circ \complement. \tag{69}$$

Caveat: the *inner* complement operation is applied to a subset of X whereas the *outer* complement operation is applied to a subset of Y. When fully expanded the value of $f_!$ on a subset A of X equals

$$f_!A = Y \setminus f_*(X \setminus A).$$

Exercise 17 *Let $A \subseteq X$ and $B \subseteq Y$. Show that*

$$A \subseteq f^*B \quad \textit{if and only if} \quad f_*A \subseteq B\,. \tag{70}$$

Exercise 18 *Show that*

$$f^*(B^c) = (f^*B)^c. \tag{71}$$

[1]The term "map" is very frequently used today as an alternative term for "function". This use became established among Mathematical Analysts who preferred to reserve the term "function" for real or complex-valued functions. The word *map* is meant to be an abbreviated form of the word *mapping*, which is a calque from German word *Abbildung*, introduced early in the 20th Century by topologists, writing in German, to denote a function between *spaces*.

1.12.10

Identities (69) and (71) can be expressed by a pair of commutative square diagrams

$$
\begin{tikzcd}
\mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{P}X \arrow[d, "f_!"] \\
\mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] & \mathscr{P}Y
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{P}X \\
\mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] \arrow[u, "f^*"] & \mathscr{P}Y \arrow[u, "f^*"']
\end{tikzcd}
\tag{72}
$$

that can be combined into a single diagram

$$
\begin{tikzcd}
\mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[d, bend right, "f_*"'] \arrow[dr, phantom, "\circlearrowleft\circlearrowright"] & \mathscr{P}X \arrow[d, bend left, "f_!"] \\
\mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] \arrow[u, "f^*"'] & \mathscr{P}Y \arrow[u, "f^*"]
\end{tikzcd}
\tag{73}
$$

in which both squares commute.

1.13 Commutative diagrams: terminology and notation

1.13.1

We say that the diagram *commutes*, or is *commutative*, if the corresponding composite arrows coincide. Commutativity of a diagram whose oriented edges represent functions or function lists, is often signaled by placing a symbol $\circlearrowright$, or its cousins, $\circlearrowleft$, $\circlearrowleft$, or $\circlearrowright$ between two composable paths of arrows, originating in a *common* vertex and terminating in a *common* vertex.

1.13.2

I used two different circle-arrows to make you aware that in the left diagram in (72), the composite arrows have their source at one of the *top* vertices and their target in the diagonally opposite *bottom* vertex. In the right diagram in (72) the roles are reversed: the composite arrows have their source at one of the *bottom* vertices and their target in the diagonally opposite *top* vertex.

Normally, I will be marking commutativity of any (portion of a) diagram by using the circle-arrow symbol that I consider the most appropriate.

Exercise 19 *Show that*

$$f^*B \subseteq A \quad \textit{if and only if} \quad B \subseteq f_!A\,. \tag{74}$$

Exercise 20 *Given an n-ary relation ρ between elements of sets $X_1, \dots, X_n$, let ρ_i be the $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$ defined in Section 1.7.4. Show that*

$$\Gamma_{\rho_i} = (\pi_{\hat{i}})_* \Gamma_\rho \tag{75}$$

where

$$\pi_{\hat{i}} : X_1 \times \cdots \times X_n \longrightarrow X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n \tag{76}$$

removes from an ordered n-tuple its i-th component,

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, \hat{x}_i, \dots, x_n).$$

Exercise 21 *Let $g : Y \to Z$ be a function. Show that*

$$(g \circ f)_* = g_* \circ f_* , \qquad (g \circ f)^* = f^* \circ g^* \qquad \textit{and} \qquad (g \circ f)_! = g_! \circ f_! . \tag{77}$$

Exercise 22 *Show that all three functions*

$$(id_X)_* , \qquad (id_X)^* \qquad \textit{and} \qquad (id_X)_! , \tag{78}$$

are equal to the identity function $\mathrm{id}_{\mathscr{P}X}$ *of the power-set* $\mathscr{P}X$.

An immediate consequence of identities (77) and (78) is that, for every invertible function f, one has

$$(f^{-1})_* = (f_*)^{-1} . \tag{79}$$

Exercise 23 *Show that, for an invertible function f, one has*

$$f^* = (f^{-1})_* .$$

Exercise 24 *Let ρ^i be the $(n-1)$-ary relation defined in Section 1.7.1. Show that*

$$\Gamma_{\rho^i} = (\pi_{\hat{i}})_! \Gamma_\rho . \tag{80}$$

1.14 Families of sets

1.14.1

A *family of sets* is, by definition, a set whose elements are themselves sets. In a restrictive approach to Set Theory every set is requiered to be of this form. It is possible to develop all of Mathematics within such a restrictive framework.

1.14.2 Notation

A general practice is to denote *elements* of sets by lower case Latin alphabet letters:

$$a, b, c, d, e, f, g, h, i, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, x, z,$$

and to denote *sets* by capital letters:

$$A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, X, Z.$$

1.14.3 Families of sets

A set whose elements are sets is often referred to as a *family of sets*. We shall denote families of sets by capital calligraphic letters:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}, \mathscr{K}, \mathscr{L}, \mathscr{M}, \mathscr{O}, \mathscr{P}, \mathscr{Q}, \mathscr{R}, \mathscr{S}, \mathscr{T}, \mathscr{U}, \mathscr{V}, \mathscr{W}, \mathscr{X}, \mathscr{Y}, \mathscr{Z}.$$

1.14.4 The union of a family of subsets of a set

Given a family of subsets $\mathscr{A}$ of a set X, the *union of* $\mathscr{A}$ is the set

$$\bigcup \mathscr{A} \ := \ \{x \in X \mid \exists_{A \in \mathscr{A}}\, x \in A\}\,. \tag{81}$$

The existence of such a set is guaranteed by the axioms of Set Theory. It is the *smallest* subset of X *containing* each member set $A \in \mathscr{A}$. An alternative notation:

$$\bigcup_{x \in \mathscr{A}} A\,. \tag{82}$$

1.14.5 The intersection of a family of subsets of a set

The set

$$\bigcap \mathscr{A} \ := \ \{x \in X \mid \forall_{A \in \mathscr{A}}\, x \in A\} \tag{83}$$

is called the *intersection* of (family) $\mathscr{A}$. It is the *greatest* subset of X *contained in* each member set $A \in \mathscr{A}$. An alternative notation

$$\bigcap_{x \in X} A\,. \tag{84}$$

1.14.6

Union and intersection define two canonical functions

$$\mathscr{P}X \underset{\bigcap}{\overset{\bigcup}{\leftleftarrows}} \mathscr{P}\mathscr{P}X\,. \tag{85}$$

Exercise 25 *Let $\mathscr{A} \subseteq \mathscr{B}$ (we say, in this case, that $\mathscr{A}$ is a* subfamily *of $\mathscr{B}$. Show that*

$$\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B} \qquad \textit{and} \qquad \bigcap \mathscr{A} \supseteq \bigcap \mathscr{B}. \tag{86}$$

1.14.7 Union and intersection of the *empty* family of subsets

If $\mathscr{A} = \{A\}$ consists of a single set A, then

$$\bigcup \mathscr{A} = A = \bigcap \mathscr{A}.$$

Since the empty family $\emptyset$ of subsets of X is contained in every family of subsets, in particular in the singleton family $\{\emptyset\}$, the union of the empty family is contained in set $\emptyset$,

$$\bigcup \emptyset \subseteq \bigcup \{\emptyset\} = \emptyset,$$

hence it is the empty set.

Since the empty family $\emptyset$ of subsets of X is contained in the singleton family $\{X\}$, the intersection of the empty family of subsets of X contains set X,

$$\bigcap \emptyset \supseteq \bigcap \{X\} = X,$$

hence it equals X.

1.14.8

The above argument demonstrates that the union of the empty family of subsets of X is the empty set independently of what set X is.

On the other hand, the intersection of the empty family of subsets of X equals X, hence it *does* depend on X.

1.14.9 Selectors of a family

A function $\xi : \mathscr{X} \longrightarrow \bigcup \mathscr{X}$ satisfying the property

$$\forall_{X \in \mathscr{X}}\ \xi(X) \in X \tag{87}$$

is called a *selector* or a *choice function* of family $\mathscr{X}$.

1.14.10 A comment about the use of the quantifier notation

Mathematicians, unless they are logicians or axiomatic-set-theorists, prefer to limit the use of the quantifier symbols in their formulae to those rare occasions when their use clarifies, not obfuscates, the meaning. The reason is partly a reflection of their habits, partly is related to the physiology of human brain perception of abstract symbolic expressions. The defining property of a selector (87) can be also written as:

$$\xi(X) \in X \quad \textit{for every} \quad X \in \mathscr{X}. \tag{88}$$

or, more tersely,

$$\xi(X) \in X \qquad (X \in \mathscr{X}). \tag{89}$$

Each expression (87)–(89) carries exactly the same meaning and can be read in the same way. From now on you will be frequently exposed to notation (89) that eliminates the need to use quantifier symbols in phrases involving only universal quantifiers.

1.14.11 Axiom of Choice

For obvious reasons, no selector exists if family $\mathscr{X}$ contains the empty set $\emptyset$. It is not obvious, however, that a selector exists *for every* family of nonempty sets. *Axiom of Choice* states just that. That statment was proven to be independent of other axioms of Set Theory. Some mathematicians do not accept it automatically while all mathematicians are, generally, cautious when they are forced to use it. Much of Mathematics can be developed without assuming its validity.

1.14.12 The product of a family of sets

The set of all selectors of family $\mathscr{X}$ forms the set

$$\prod \mathscr{X}, \qquad \text{alternately denoted} \qquad \prod_{X \in \mathscr{X}} X, \tag{90}$$

which is called the *product* of (family) $\mathscr{X}$.

1.14.13

Axiom of Choice says :

$$\textit{The product of a family of nonempty sets is nonempty.} \tag{91}$$

1.14.14 An equivalent form of Axiom of Choice

$$\textit{Every surjective function } f : \mathrm{X} \to Y \textit{ is right-invertible.} \tag{92}$$

1.14.15 Independence of Axiom of Choice

It was established long ago that Axiom of Choice is consistent with the remaining axioms of Set Theory. This means that if there are contradictory statements in Mathematics provable with the aid of Axiom of Choice, then there are contradictory statements provable without Axiom of Choice.

It took much longer to resolve the open question whether Axiom of Choice is, or is not, a consequence of the remaining axioms of Set Theory. This was finally resolved by a brilliant mathematician, Paul Cohen, whose demonstrated strength was in Harmonic and Functional Analysis, not in Set Theory or Mathematical Logic. He proved that Axiom of Choice is *not* a consequence of axioms of Set Theory. Statements in Mathematics that are consistent but not provable are said to be *independent* of axioms of Set Theory.

1.15 Canonical functions between the sets-of-families

1.15.1

As we saw in Sections 1.12.1 and 1.12.9, every function $f : X \to Y$ induces three functions between the corresponding power-sets

$$\begin{tikzcd} \mathscr{P}Y \arrow[d, "f^*"] \\ \mathscr{P}X \arrow[u, bend left, "f_*"] \arrow[u, bend right, "f_!"'] \end{tikzcd} \;. \tag{93}$$

Families of subsets of X are elements of the power-set-of-the-power-set $\mathscr{P}\mathscr{P}X$ and similary for families of subsets of Y. In particular, each of the three functions in diagram (93) induces three functions between the corresponding sets of families of subsets:

$$\begin{matrix} (f_*)_* & (f_*)^* & (f_*)_! \\ (f^*)_* & (f^*)^* & (f^*)_! \\ (f_!)_* & (f_!)^* & (f_!)_! \, . \end{matrix} \tag{94}$$

One can omit parentheses provided one carefully observes the spacing that distinguishes between, e.g., $f_*{}^*$ and $f^*{}_*$.

$$\begin{matrix} f_{**} & f_*{}^* & f_{*!} \\ f^*{}_* & f^{**} & f^*{}_! \\ f_{!*} & f_!{}^* & f_{!!} \end{matrix} . \tag{95}$$

Exercise 26 *Find all functions in diagram (95) that are functions from $\mathscr{P}\mathscr{P}X$ to $\mathscr{P}\mathscr{P}Y$.*

1.15.2

Of these nine canonical functions between sets of families of subsets, four play an important role in Topology, Measure Theory, Mathematical Analysis, where families of subsets are essential objects of study.

1.15.3

Let $\mathscr{A} \subset \mathscr{P}X$ be a family of subsets of X, let $\mathscr{B} \subset \mathscr{P}Y$ be a family of subsets of Y.

Exercise 27 *Show that*

$$f_*\left(\bigcup \mathscr{A}\right) = \bigcup f_{**}\mathscr{A} \qquad \textit{and} \qquad f^*\left(\bigcup \mathscr{B}\right) = \bigcup f^*{}_*\mathscr{B} \tag{96}$$

and express each identity in the form of a commutative diagram.

Exercise 28 *Show that*

$$f^*\left(\bigcap \mathscr{B}\right) = \bigcap f^*{}_*\mathscr{B} \qquad \textit{and} \qquad f_!\left(\bigcap \mathscr{A}\right) = \bigcap f_{!*}\mathscr{A} \tag{97}$$

and express each identity in the form of a commutative diagram.[2]

Exercise 29 *Show that*

$$f_*\left(\bigcap \mathscr{A}\right) \subseteq \bigcap f_{**}\mathscr{A} \qquad \textit{and} \qquad f_!\left(\bigcup \mathscr{A}\right) \subseteq \bigcup f_{!*}\mathscr{A} . \tag{98}$$

In general, $\subseteq$ cannot be replaced by $=$ in (98).

1.16 Indexed families of sets

1.16.1

An indexed family of sets $(X_i)_{i \in I}$ is, by definition, a function from a certain set I to the power-set of a certain set U,

$$I \longrightarrow \mathscr{P}(U), \qquad i \mapsto X_i .$$

The standard notation for the value at $i \in I$ is X_i. The set I is referred to as the *indexing set*.

[2] A hint for both exercises: recall that $\bigcup$ and $\bigcap$ define certain canonical functions, cf. (85).

1.16.2 The union and the intersection of an indexed family

Let us denote by $\mathscr{X}$ the *image* of this function in $\mathscr{P}(U)$. It is a family of sets. The union and the intersection of $\mathscr{X}$ are called, respectively, the *union* and the *intersection* of $(X_i)_{i\in I}$, and denoted

$$\bigcup_{i\in I} X_i \qquad \text{and} \qquad \bigcap_{i\in I} X_i \,.$$

Explicitly,

$$\bigcup_{i\in I} X_i \;:=\; \{x \mid \exists_{i\in I}\, x \in X_i\} \tag{99}$$

and

$$\bigcap_{i\in I} X_i \;:=\; \{x \mid \forall_{i\in I}\, x \in X_i\} \,. \tag{100}$$

1.16.3

When the indexing set I is empty, the comments made about the union and the intersection of an empty family of subsets apply, cf. 1.14.8.

1.16.4 Selectors of an indexed family

Functions

$$I \longrightarrow \bigcup_{i\in I} X_i \,, \qquad i \mapsto x_i \,, \tag{101}$$

satisfying

$$x_i \in X_i \qquad (i \in I) \,,$$

could be called *selectors* of indexed family $(X_i)_{i\in I}$. They are more frequently called I*-tuples* because in the case

$$I = \{1, \dots, n\} \,,$$

they correspond to ordered n-tuples of elements of $\bigcup_{i\in I} X_i$.

1.16.5 "Tuple" notation

Standard notation for an I-tuple is $(x_i)_{i\in I}$. The subscript $i \in I$ is usually omitted when the indexind set is understood from the context.

1.16.6 The product of an indexed family of sets

Predictably, the set of all I-tuples of $(X_i)_{i\in I}$ is called the *product* of $(X_i)_{i\in I}$ and is denoted

$$\prod_{i\in I} X_i \,. \tag{102}$$

1.16.7

For $I = \{1, 2\}$, the product is naturally identified with the Cartesian product

$$X_1 \times X_2 \, ,$$

and, for $I = \{1, \dots, n\}$, it provides the most convenient model of the Cartesian product

$$X_1 \times \cdots \times X_n \, .$$

1.16.8 Canonical projections (π_J)

Restricting a function (101) to a subset $J \subseteq I$ defines a function

$$\pi_J : \prod_{i \in I} X_i \longrightarrow \prod_{i \in J} X_i \, , \tag{103}$$

called the *canonical projection* (associated with a subset J of the indexing set. We have encountered these functions in Section 1.11.15 where $I = \{1, \dots n\}$ and $J = \{i\}$.

1.16.9 Notation

In the interest of keeping notation simple, when, e.g., $J = \{2, 5, 7\}$, we write

$$\pi_{2,5,7} \quad \text{instead} \quad \pi_{\{2,5,7\}}$$

or, even, as

$$\pi_{257}$$

when it it is clear from the context that the elements of J are natural numbers less than 10.

A general rule is to separate the items in a list of subscripts or superscripts by commas when notation is, otherwise, ambiguous, and to omit the commas when no ambiguity arises.

1.16.10 Composition of correspondences

Given correspondences

$$C \subseteq X_0 \times \cdots \times X_{m+1} \qquad \text{and} \qquad D \subseteq X_{m+1} \times \cdots \times X_{m+n+1} \, ,$$

their preimages under the canonical projections

$$\pi^*_{0,\dots,m+1} C \qquad \text{and} \qquad \pi^*_{m+1,\dots,m+n+1} D$$

are correspondences between elements of sets

$$X_0, \dots, X_{m+n+1} \, .$$

In particular, we can form their intersection

$$\pi^*_{0,\dots,m+1} C \; \cap \; \pi^*_{m+1,\dots,m+n+1} D$$

and project it into $X_0 \times \cdots \times \hat{X}_{m+1} \times \cdots \times X_{m+n+1}$,

$$(\pi_{\widehat{m+1}})_* (\pi^*_{0,\dots,m+1} C \; \cap \; \pi^*_{m+1,\dots,m+n+1} D) \, , \tag{104}$$

where

$$\pi_{\widehat{m+1}} \; = \; \pi_{0,\dots,\widehat{m+1},\dots,m+n+1} \, .$$

We shall denote (104) by $C \circ D$.

1.16.11

Explicitly, $C \circ D$ consists of $(m+n+1)$-tuples

$$(x_0, \dots, \hat{x}_{m+1}, \dots, x_{m+n+1})$$

for which there exists $x_{m+1} \in X_{m+1}$ such that

$$(x_0, \dots, x_{m+1}) \in C \qquad \text{and} \qquad (x_{m+1}, \dots, x_{m+n+1}) \in D\,.$$

1.16.12

It follows that for $C = \Gamma_\rho$ and $D = \Gamma_\sigma$, one has

$$\Gamma_{\rho\circ\sigma} = \Gamma_\rho \circ \Gamma_\sigma\,. \tag{105}$$

2 The language of mathematical structures

2.1 Mathematical structures

2.1.1 The concept of a mathematical structure

A list of sets

$$X_1, \dots, X_n$$

equipped with some 'data' is what a mathematical structure is. As such, a mathematical structure can be thought of as an ordered pair

$$(X_1, \dots, X_n\,;\ \text{'data'})$$

2.1.2

This sinple concept became a focal point of modern Mathematics because it allows to view many apparently distant phenomena as manifestations of the same general laws.

2.1.3

Functions, operations, relations, are obvious examples of mathematical structures.

2.1.4 Structures of functional type

Sets X equipped with a family $\mathscr{O} \subset \mathrm{Funct}(X, \mathbf{R})$ of real-valued functions on X,

$$(X, \mathscr{O}),$$

are a backbone of Analysis. Think, for example, of a subset X of Euclidean space $\mathbf{R}^n$ and $\mathscr{O}$ being the set of all infinitely differentiable functions on X.

2.1.5 Structures of topological type

Sets X equipped with a family $\mathscr{A} \subset \mathscr{P}X$ of subsets

$$(X, \mathscr{A})$$

are the central objects in Topology, Geometry, Measure Theory, Combinatorics.

2.1.6 Example: topological spaces

A set X equipped with a family of subsets $\mathscr{T} \subset \mathscr{P}X$ closed under formation of *finite* intersections and arbitrary unions is called a *topological space*.

2.1.7 Example: measurable spaces

A set X equipped with a family of subsets $\mathscr{M} \subset \mathscr{P}X$ closed under formation of *countable* intersections and unions, and under the complement operation, c. 1.5.3, is called a *measurable space*.

2.2 Algebraic structures

2.2.1

Sets X equipped with an indexed family $(\mu_i)_{i\in I}$ of operations on X are called *algebraic structures*. Groups, rings, fields, vector spaces, etc., are all examples of algebraic structures.

2.2.2 Example: groups

A group is an algebraic structure

$$(X\,;\,\mu_0,\mu_1,\mu_2)$$

where μ_2 is a binary operation on X,

$$X,X \longrightarrow X, \qquad x,y \mapsto xy\,,$$

referred to as *the multiplication*, μ_0 is a nullary operation on X,

$$\longrightarrow X, \qquad \langle\text{the empty list}\rangle \longmapsto e\,,$$

referred to as *the identity* element, and μ_1 is a unary operation on X,

$$X \longrightarrow X, \qquad x \mapsto \bar{x}\,,$$

that assigns to an element $x \in X$ its inverse. This family of 3 operations is required to satisfy 3 properties: *Associativity*, (two-sided) *Identity Element Property*, and (two-sided) *Inverse-Element Property*.

2.2.3 Associativity: $\forall_{x,y,z\in X}\,(xy)z = x(yz)\,.$

Associativity of a binary operation μ_2 is equivalent to commutativity of the diagram

$$\begin{tikzcd}
& X,X,X \arrow[dl, "\mu_2"'] \arrow[dr, "\mu_2"] \arrow[dd, phantom, "\circlearrowleft"] & \\
X,X \arrow[dr, "\mu_2"'] & & X,X \arrow[dl, "\mu_2"] \\
& X &
\end{tikzcd}\;. \tag{106}$$

2.2.4 Identity Element Property: $\forall_{x\in X}\, ex = x = xe\,.$

The left and the right equalities in the Identity Element Property are equivalently described as commutativity of the left and, respectively, right triangles in the diagram

$$\begin{tikzcd}
& X \arrow[dl, "\mu_0"'] \arrow[dr, "\mu_0"] \arrow[dd, equal] & \\
X,X \arrow[dr, "\mu_2"'] \arrow[r, phantom, "\circlearrowleft"] & {} & X,X \arrow[dl, "\mu_2"] \arrow[l, phantom, "\circlearrowright"] \\
& X &
\end{tikzcd}\;. \tag{107}$$

2.2.5 Inverse-Element Property: $\forall_{x \in X}\, \bar{x}x = e = x\bar{x}$.

The left and the right equalities in the Inverse-Element Property are equivalently described as commutativity of the left and, respectively, right triangles in the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_1"] \arrow[dd, "e_X"] & \\
X, X \arrow[dr, "\mu_2"'] & & X, X \arrow[dl, "\mu_2"] \\
 & X &
\end{tikzcd} \tag{108}$$

where e_X denotes the *constant* function

$$X \longrightarrow X, \qquad x \longmapsto e \qquad (x \in X).$$

Note that $e_X : X \to X$ is the composite function

$$X \longrightarrow \emptyset^{\emptyset} \xrightarrow{\tilde{e}} X$$

where $X \longrightarrow \emptyset^{\emptyset}$ is the unique function from X to the singleton set $\emptyset^{\emptyset}$ and $\tilde{e}$ is the function of a single variable canonically corresponding to the function of zero variables $e : \longrightarrow X$, cf. Section 1.11.13.

2.2.6 Example: monoids

If we remove from the definition of a group unary operation μ_1 and the Inverse Element Property, we obtain the definition of a *monoid*.

2.2.7 The canonical monoid structure on $\mathrm{Op}_1(X)$

Composition $\circ$ is a canonical binary operation on the set of all unary operations $\mathrm{Op}_1(X)$ on an arbitrary set X. The identity operation id_X is a distinguished element of $\mathrm{Op}_1(X)$. Composition of functions is associative and id_X is an identity element for the operation of composition.

Thus, $(\mathrm{Op}_1(X), \mathrm{id}_X, \circ)$ is a monoid and $\mathrm{Op}_1(X)$ provides an example of a set that is equipped with a canonical structure of a monoid.

2.2.8 Example: semigroups

If we remove from the definition of a group unary operation μ_1, nullary operation μ_0, and the two properties in which these two operations occur, we are left with a set equipped with a single binary operation that satisfies the Associativity Property. Such a structure is called a *semigroup*.

2.2.9

Semigroups, monoids, groups, are encountered everywhere where mathematical considerations are involved.

2.3 Relational structures

2.3.1

Sets X equipped with an indexed family $(\rho_i)_{i\in I}$ of relations on X are called *relational structures*. Such structures are encountered in all areas of Mathematics and especially so in Mathematical Logic and in Incidence Geomtry.

2.3.2

Particularly important are *binary relational structures*, i.e., sets equipped with a single binary relation. We discuss them in Chapter 3 devoted to binary relations.

2.4 Morphisms

2.4.1 Interactions between mathematical structures

If mathematical structures are *objects* of mathematical theories, studying a given structure is nearly always executed by observing how that structure *interacts* with other structures of the same type. Binary interactions between structures are expressed in the language of *morphisms*.

2.4.2 The concept of a morphism

A *morphism*

$$(X, \text{data}) \longrightarrow (X', \text{data}') \tag{109}$$

is most commonly understood to be a function between the *underlying sets*

$$f : X \longrightarrow X'$$

that *respects* the corresponding data. It is assumed that the data must be of the same type. The term 'respects' can be replaced by: 'is compatible with'. The meaning of this term is nearly always natural for each type of data. We shall illustrate this for some types of mathematical structures mentioned above.

2.4.3 The arrow notation

Morphisms are represented graphically as arrows. Every arrow has its source and its target, each being a structure of the same type. They are referred to as the *source* and the *target* of a morphism.

2.4.4 Morphisms between algebraic structures

Suppose that a set X is equipped with an n-ary operation μ and a set X' is equipped with an n-ary operation μ'. We say that a function $f : X \to X'$ is *compatible* with the operations if

$$\forall_{x_1,\dots,x_n\in X}\; f\big(\mu(x_1, \dots, x_n)\big) = \mu\big(f(x_1), \dots, f(x_n)\big)\,. \tag{110}$$

Algebraists refer to such functions as *homomorphisms*.

2.4.5

The definition of a morphism between sets equipped with an n-ary operation can be also expressed as commutativity of the following square diagram

$$\begin{tikzcd} X', \dots, X' \arrow[r, "\mu'"] \arrow[dr, phantom, "\circlearrowleft"] & X' \\ X, \dots, X \arrow[u, "f \uparrow \cdots f"] \arrow[r, "\mu"] & X \arrow[u, "f"] \end{tikzcd} . \tag{111}$$

2.4.6

The above definition can be easily extended to general algebraic structures. A morphism

$$(X, (\mu_i)_{i\in I}) \longrightarrow (X', (\mu_i')_{i\in I})$$

is a function $f : X \to X'$ such that it is a homomorphism

$$(X, \mu_i) \longrightarrow (X', \mu_i')$$

for *each* $i \in I$. Notice that μ_i and μ_i' must have the same 'arity' for every $i \in I$.

The concept of a homomorphism provides the most natural definition of a morphism between algebraic structures.

2.4.7 Morphisms between n-ary relations

Consider two n-ary relations

$$\rho : X_1, \dots, X_n \longrightarrow \text{Statements} \qquad \text{and} \qquad \rho' : X_1', \dots, X_n' \longrightarrow \text{Statements}\,.$$

A natural definition is to declare a function list (53) a *morphism from* ρ *to* ρ' if

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n}\ \rho(x_1, \dots, x_n) \Rightarrow \rho'\big(f_1(x_1), \dots, f_n(x_n)\big)\,. \tag{112}$$

2.4.8

Condition (112) is equivalently stated as

$$\rho \Longrightarrow (f_1, \dots, f_n)^* \rho' \tag{113}$$

where $\Longrightarrow$ denotes the *implication* relation on the set $\mathrm{Rel}_n(X)$ of n-ary relations on a set X.

Exercise 30 *Show that* $f_1, \dots, f_n$ *is a morphism from* ρ *to* ρ' *if and only if*

$$\Gamma_\rho \subseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'}. \tag{114}$$

2.4.9 Morphisms between relational structures

When

$$X_1 = \cdots = X_n = X\,, \qquad X_1' = \cdots = X_n' = X' \qquad \text{and} \qquad f_1 = \cdots = f_n = f\,,$$

we obtain the definition of a *morphism from a relational structure* (X, ρ) *to a relational structure* (X', ρ').

2.4.10

Condition (112) can be also expressed in the form of the diagram

$$
\begin{tikzcd}
X'_1,\dots,X'_n \arrow[dr, "\rho'"] & \\
 & \text{Statements} \\
X_1,\dots,X_n \arrow[uu, "f_1 \cdots f_n"] \arrow[ur, "\rho"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
\;. \tag{115}
$$

2.4.11 Faithful morphisms between n-ary relations

By replacing *Implicationi* $\Rightarrow$ in Condition (112) by *Equivalence* $\Leftrightarrow$, we obtain a stronger condition

$$
\forall_{x_1\in X_1,\dots,x_n\in X_n}\ \rho(x_1,\dots,x_n) \Leftrightarrow \rho'\big(f_1(x_1),\dots,f_n(x_n)\big)\,. \tag{116}
$$

We shall say in this case that the function list $f_1,\dots,f_n$ is a *faithful* morphism from ρ to ρ'.

2.4.12 Morphisms between structures of functional type

Suppose that a set X is equipped with a family of functions $\mathcal{O} \subset \mathrm{Funct}(X,\mathbf{R})$ and a set X' is equipped with a family of functions $\mathcal{O}' \subset \mathrm{Funct}(X',\mathbf{R})$. We say that a function $f : X \to X'$ is a morphism if

$$
\forall_{\phi'\in\mathcal{O}'}\ f^*\phi' = \phi'\circ f \in \mathcal{O}\,. \tag{117}
$$

2.4.13

An equivalent form of condition (117) is

$$
(f^*)_*\mathcal{O}' \subset \mathcal{O}\,. \tag{118}
$$

Equivalently, $f : X \to X'$ is a morphism if the diagram

$$
\begin{tikzcd}
\mathcal{O} \arrow[d, hook] & \mathcal{O}' \arrow[d, hook] \\
\mathrm{Funct}(X,\mathbf{R}) & \mathrm{Funct}(X',\mathbf{R}) \arrow[l, "f^*"']
\end{tikzcd}
$$

admits a completion to a commutative square diagram

$$
\begin{tikzcd}
\mathcal{O} \arrow[d, hook] & \mathcal{O}' \arrow[l] \arrow[d, hook] \\
\mathrm{Funct}(X,\mathbf{R}) & \mathrm{Funct}(X',\mathbf{R}) \arrow[l, "f^*"']
\end{tikzcd}
$$

2.4.14 Morphisms between structures of topological type

Suppose that a set X is equipped with a family of subsets $\mathscr{A} \subset \mathscr{P}X$ and a set X' is equipped with a family of subsets $\mathscr{A}' \subset \mathscr{P}X'$. We say that a function $f : X \to X'$ is a morphism if the preimage under f of every member of family $\mathscr{A}'$ is a member of $\mathscr{A}$,

$$\forall_{A' \in \mathscr{A}'}\, f^*A' \in \mathscr{A}\,. \tag{119}$$

2.4.15

An equivalent form of condition (119) is

$$(f^*)_*\mathscr{A}' \subset \mathscr{A}\,. \tag{120}$$

Notice the similarity to condition (118).

2.4.16

Condition (120) can be expressed by saying that the diagram

$$\begin{tikzcd} \mathscr{A} \arrow[d, hook] & \mathscr{A}' \arrow[d, hook] \\ \mathscr{P}Y & \mathscr{P}Y' \arrow[l, "f^*"'] \end{tikzcd}$$

admits a completion to a commutative square diagram

$$\begin{tikzcd} \mathscr{A} \arrow[d, hook] & \mathscr{A}' \arrow[l] \arrow[d, hook] \\ \mathscr{P}Y & \mathscr{P}Y' \arrow[l, "f^*"'] \end{tikzcd}$$

2.4.17

Another condition that can be interpreted as saying that f respects distinguished families of subsets reads

$$\forall_{A \in \mathscr{A}}\, f_*A \in \mathscr{A}' \tag{121}$$

or, equivalently,

$$(f_*)_*\mathscr{A} \subset \mathscr{A}'\,. \tag{122}$$

Either condition can serve as a definition of a morphism between structures of topological type. It is however the former, (119), that plays a fundamental role in Topology and Measure Theory, not the latter, (121).

2.5 The language of categories

2.5.1

Whatever definition of a morphism between mathematical structures one adopts, it always has the following features

- any morphism α has a *source* and a *target* that are mathematical structures of the same type
- if the source $s(\alpha)$ of a morphism α coincides with the target $t(\beta)$ of a morphism β, then their composition $\alpha \circ \beta$ is defined and

$$t(\alpha \circ \beta) = t(\alpha) \qquad \text{and} \qquad s(\alpha \circ \beta) = s(\beta)$$

- composition of morphisms is associative, i.e.,

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

  for any composable list α, β, γ of morphisms

2.5.2

The above observations led to the introduction of the concept of a *category*. In a nutshell, a *category* $\mathscr{C}$ consists of two classes, a class $\mathscr{C}_0$ of *objects* and a class $\mathscr{C}_1$ of *morphisms*, equipped with an associative operation of composition of morphisms. You will be introduced to the language of categories gradually.

2.5.3

Various classes of mathematical structures equipped with appropriate classes of morphisms form natural categories. Studying the category of groups is what Group Theory does. Studying the category of rings is what Ring Theory does. Algebraic geometers study the category of *algebraic varieties* and the bigger category of *algebraic schemes*. Topologists study the category of *topological spaces*, and so on.

2.5.4

Every mathematical theory can be expressed in a categorical language. This usually provides an added degree of clarity to a theory and yields insights that are otherwise lost.

2.5.5 Endomorphisms

Morphisms whose source and target coincide with an object c are referred as *endomorphisms* of object c.

2.5.6 The semigroup of endomorphisms

Equipped with composition as its binary operation, the set of endomorphisms of an object c of any category becomes a semigroup, denoted

$$\mathrm{End}_{\mathscr{C}}\, c\,. \tag{123}$$

The semigroups of endomorphisms of various mathematical structures play a fundamental role in nearly every area of Mathematics and Mathematical Pysics.

2.5.7 The monoid of endomorphisms

In many categories objects have a distinguished endomorphism, referred to as the *identity endomorphism*, and denoted id_c or, sometimes, 1_c, which serves as the Identity Element for the binary operation of composition. In this case,

$$(\mathrm{End}_{\mathscr{C}}\, c, \mathrm{id}_c, \circ)$$

is a monoid. For example, the monoid of unary operations $\mathrm{Op}_1(X)$ on a set X is precisely the monoid of endomorphsisms of X viewed as an object of the category of sets.

3 Binary relations

3.1 Preliminaries

3.1.1 Canonical identification $\mathbf{Rel}(X,Y) \longleftrightarrow \mathbf{Rel}(Y,X)$

Given a binary relation

$$\rho : X, Y \longrightarrow \text{Statements}, \tag{124}$$

the *opposite* relation is defined by flipping the 2 arguments

$$\rho^{\mathrm{op}} : Y, X \longrightarrow \text{Statements}, \qquad y, x \longmapsto \rho(x,y). \tag{125}$$

Note that $(\rho^{\mathrm{op}})^{\mathrm{op}} = \rho$. In particular, assignment

$$\rho \longmapsto \rho^{\mathrm{op}}$$

defines a canonical identification of the sets of binary relations

$$\mathrm{Rel}(X,Y) \longleftrightarrow \mathrm{Rel}(Y,X).$$

3.1.2 A canonical involution on $\mathbf{Rel}_2(X)$

When $X = Y$, $(\ )^{\mathrm{op}}$ is a (canonical) involution on the set of binary relations on X.

3.1.3 The infix notation

In view of the fact that binary relations have been used by mathematicians long before the concept of a general relation was formulated and are still the most frequently encountered type of relation, special notation has been used when talking about binary relations. The symbolic expression

$$x_1 \rho x_2$$

has the meaning:

$$\text{Statement } \rho(x_1, x_2) \text{ holds.}$$

3.1.4 The $\sim$ notation

More likely, however, you will see

$$x_1 \sim x_2, \tag{126}$$

since the symbol $\sim$ and its variants have been adopted as a generic symbol denoting a binary relation. The meaning of (126) is:

the binary relation, denoted $\sim$, holds for elements $x_1 \in X_1$ and $x_2 \in X_2$.

The difference between the *functional* notation and the *tilde* notation, when talking about binary relations, is similar to the difference between *direct speech* and *indirect speech*: compare the statements

$$3 < 5$$

and

inequality $3 < 5$ holds.

3.2 Binary relations *on* a set: a vocabulary of terms

3.2.1 Various types of binary relations on a set

A binary relation ρ on a set X is said to be :

reflexive if

$$\forall_{x\in X}\, \rho(x,x) \tag{127}$$

symmetric if

$$\forall_{x,y\in X}\, \big(\, \rho(x,y) \Rightarrow \rho(y,x)\, \big) \tag{128}$$

antisymmetric if

$$\forall_{x,y\in X}\, \big(\, \rho(x,y) \Rightarrow \neg\rho(y,x)\, \big) \tag{129}$$

weakly antisymmetric if

$$\forall_{x,y\in X}\, \big(\, \rho(x,y) \wedge \neg\rho(y,x) \;\Rightarrow\; x = y\, \big) \tag{130}$$

transitive if

$$\forall_{x,y,z\in X}\, \big(\, \rho(x,y) \wedge \rho(y,z) \;\Rightarrow\; \rho(x,z)\, \big) \tag{131}$$

3.2.2

Of all the properties that a binary relation ρ on a set X may have, by far the most important is its *transitivity*.

3.2.3 Preorder relations

A transitive and reflexive relation is called a *preorder* or a *quasiorder*.

3.2.4 Equivalence relations

A symmetric preorder that is called an *equivalence relation*.

3.2.5 Order relations

A weakly antisymmetric preorder is called an *order relation*.

3.2.6 Sharp order relations

An antisymmetric transitive relation is called a *sharp-order relation*.

3.2.7 Ordered sets

A set X equipped with an order relation will be called an *ordered set*. We shall use the generic symbol $\preccurlyeq$ to denote the order relation. When using the term "ordered set", remember that it is not a set, it is a binary relational structure $(X, \preccurlyeq)$.

3.2.8 Terminology

To emphasize that elements of an ordered set are not necessarily *comparable*, the adverb "partially" is often placed in front of "ordered".

3.2.9 Linearly ordered sets

Ordered sets whose elements are *comparable*, i.e., satisfy the condition

$$\forall_{x,y\in X}\; x \preccurlyeq y \vee y \preccurlyeq x, \tag{132}$$

are called *linearly*, or *totally*, ordered.

Naturally defined linear orders are scarce, unlike (partial) orders.

3.2.10 A canonical ordered-set structure on $\mathscr{P}X$

The set-containment relation equips the power-set of any set with a canonical ordered-set structure

$$\subseteq : \mathscr{P}X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad A, B \longmapsto \text{"}A \subseteq B\text{"}. \tag{133}$$

It proves to be one of the central structures of Mathematics.

3.3 Functions naturally associated with a binary relation

3.3.1 The *set-of-relatives* functions

Given a binary relation (124), we have 2 associated with it evaluation functions

$$\mathrm{ev}^1\rho : X \longrightarrow \mathrm{Rel}_1(Y) \qquad \text{and} \qquad \mathrm{ev}^2\rho : Y \longrightarrow \mathrm{Rel}_1(X)$$

cf. (12). By composing them with the graph functions

$$\mathrm{Rel}_1(Y) \xrightarrow{\;\Gamma\;} \mathscr{P}Y \qquad \text{and} \qquad \mathrm{Rel}_1(X) \xrightarrow{\;\Gamma\;} \mathscr{P}X\;,$$

we obtain a pair of functions

$$X \longrightarrow \mathscr{P}Y, \qquad x \longmapsto [x\rangle_\rho := \{y \in Y \mid \rho(x,y)\}, \tag{134}$$

and, respectively,

$$Y \longrightarrow \mathscr{P}X, \qquad y \longmapsto {}_\rho\langle y] := \{x \in X \mid \rho(x,y)\}. \tag{135}$$

When the context allows that, we shall simplify notation by omitting the subscript denoting the relation. We shall refer to $[x\rangle$ as the set of *right relatives* of $x \in X$, and to $\langle y]$ as the set of *left relatives* of $y \in Y$.

Accordingly, we shall refer to (134) as the *right-relatives* function, and to (135) as the *left-relatives* function.

3.3.2

Assignments

$$\begin{tikzcd} & \rho \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ [\ \rangle_\rho & & {}_\rho\langle\ ] \end{tikzcd} \tag{136}$$

define functions

$$\begin{tikzcd} & \mathrm{Rel}(X,Y) \arrow[dl] \arrow[dr] & \\ \mathrm{Funct}(X,\mathscr{P}Y) & & \mathrm{Funct}(Y,\mathscr{P}X) \end{tikzcd} \tag{137}$$

Exercise 31 *Show that*

$${}_{\rho^{\mathrm{op}}}\langle\ ] = [\ \rangle_\rho \qquad \textit{and} \qquad [\ \rangle_{\rho^{\mathrm{op}}} = {}_\rho\langle\ ].$$

Exercise 32 *Show that function* $\mathrm{Rel}(X,Y) \longrightarrow \mathrm{Funct}(X,\mathscr{P}Y)$ *in* (137) *is surjective.*

Exercise 33 *Show that, for any* $\rho,\sigma \in \mathrm{Rel}(X,Y)$, *one has*

$${}_\rho\langle\ ] = {}_\sigma\langle\ ] \qquad \textit{if and only if} \qquad \rho \Leftrightarrow \sigma.$$

3.3.3

It follows that the left- and the right-relatives functions induce a canonical identification of the set of *equipotence* classes of binary relations between elements of sets X and Y, with the sets of functions $\mathrm{Funct}(X,\mathscr{P}Y)$ and, respectively, $\mathrm{Funct}(Y,\mathscr{P}X)$.

3.3.4 The preorders on X and Y naturally associated with $\rho \in \mathrm{Rel}(X,Y)$

Let us denote by $\precsim$ the preorder on Y induced by the containment relation on $\mathscr{P}X$,

$$y \precsim y' \quad \text{if} \quad \langle y] \subseteq \langle y'] \qquad (y,y' \in Y). \tag{138}$$

Let us denote by $\succsim$ the preorder on X induced by the containment relation on $\mathscr{P}Y$;

$$x \succsim x' \quad \text{if} \quad [x\rangle \subseteq [x'\rangle \qquad (x,x' \in X). \tag{139}$$

3.3.5

Suppose $X = Y$. In that case all three relations, ρ, $\precsim$ and $\succsim$, are elements of the same set $\mathrm{Rel}_2(X)$ which is preordered by the $\Longrightarrow$ relation. This leads to the following natural questions that I am stating as exercises.

Exercise 34 *Characterize binary relations* $\rho \in \mathrm{Rel}_2(X)$ *such that* $\rho \Longrightarrow \precsim$.[3]

Exercise 35 *Characterize binary relations* $\rho \in \mathrm{Rel}_2(X)$ *such that* $\precsim \Longrightarrow \rho$.

Exercise 36 *State the analogs of the above two exercises for* $\succsim$ *instead of* $\precsim$.

[3] *Characterize* means: 1° Find a property of ρ, that can be stated directly in terms of ρ and is as simple as possible, that holds precisely when $\rho \Longrightarrow \precsim$; 2° then prove that.

3.4 A pair of canonical functions $U : \mathscr{P}X \leftrightarrows \mathscr{P}Y : L$.

3.4.1 $U : \mathscr{P}X \longrightarrow \mathscr{P}Y$

Given a subset $A \subseteq X$ we obtain a family $([x\rangle)_{x\in A}$ of subsets of Y whose intersection,

$$UA := \bigcap_{x\in A} [x\rangle = \{y \in Y \mid \forall_{x\in A}\, \rho(x,y)\}, \tag{140}$$

consists of those elements of Y that are right relatives of *every* element of A. In theory of ordered sets UA is called the *set of upper bounds of* A.

3.4.2 $\mathscr{P}X \longleftarrow \mathscr{P}Y : L$

Given a subset $B \subseteq Y$ we obtain a family $(\langle y])_{y\in B}$ of subsets of X whose intersection,

$$LB := \bigcap_{y\in B} \langle y] = \{x \in X \mid \forall_{y\in B}\, \rho(x,y)\}, \tag{141}$$

consists of those elements of X that are left relatives of *every* element of B. In theory of ordered sets LB is called the *set of lower bounds of* B.

Exercise 37 *Show that U is a morphism of ordered sets $(\mathscr{P}X, \subseteq) \to (\mathscr{P}Y, \supseteq)$ and L is a morphism of ordered sets $(\mathscr{P}Y, \supseteq) \to (\mathscr{P}X, \subseteq)$.*

In the following exercises, A denotes an arbitrary subset of X and Y denonets an arbitrary subset of Y.

Exercise 38 *Show that*

$$A \subseteq LB \quad \textit{if and only if} \quad UA \supseteq B\,. \tag{142}$$

Exercise 39 *Show that*

$$A \subseteq LUA \qquad \textit{and} \qquad ULB \supseteq B. \tag{143}$$

Exercise 40 *Show that*

$$LULB = LB \qquad \textit{and} \qquad UA = ULUA. \tag{144}$$

Exercise 41 *Consider the* membership *relation*

$$\in\, : X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad x, A \longmapsto \text{“}x \in A\text{''}. \tag{145}$$

Determine UA and $L\mathscr{A}$ for $A \subseteq X$ and $\mathscr{A} \subseteq \mathscr{P}X$.

Exercise 42 *Consider the set-containment relation* (133)*. Determine $U\mathscr{A}$ and $L\mathscr{B}$ for $\mathscr{A} \subseteq \mathscr{P}X$ and $\mathscr{B} \subseteq \mathscr{P}X$.*

3.5 Morphisms between binary relations

3.5.1

A binary relation provides an example of a mathematical structure where, beyond the “obvious” definition of a morphism, there is another natural yet much less obvious one.

3.5.2 Morphisms between binary relations

The "obvious definition" was given in Section 2.4.7: a morphism from a binary relation

$$\rho : X, Y \longrightarrow \text{Statements}$$

to a binary relation

$$\rho' : X', Y' \longrightarrow \text{Statements}$$

consists of a pair of functions $f : X \to X'$ and $g : Y \to Y'$ such that

$$\begin{tikzcd} X', Y' \arrow[dr, "\rho'"] & \\ & \text{Statements} \\ X, Y \arrow[uu, "f", "g"'] \arrow[ur, "\rho"'] \arrow[ur, phantom, shift left=4ex, "\Uparrow"] & \end{tikzcd} . \tag{146}$$

3.6 Bimorphisms between binary relations

3.6.1

We shall say that a pair of functions

$$f : X \to X' \qquad \text{and} \qquad Y \longleftarrow Y' : g \tag{147}$$

is a *bimorphism from* ρ *to* ρ' if relation $(\mathrm{id}_X, g)^* \rho$ implies relation $(f, \mathrm{id}_Y)^* \rho'$,

$$(\mathrm{id}_X, g)^* \rho \Longrightarrow (f, \mathrm{id}_Y)^* \rho' . \tag{148}$$

We shall represent Condition (148) by the following diagram

$$\begin{tikzcd} X', Y' \arrow[dr, "\rho'"] & \\ X, Y' \arrow[u, "f", "\|"'] \arrow[d, "\|", "g"'] \arrow[r, phantom, "\Uparrow"] & \text{Statements} \\ X, Y \arrow[ur, "\rho"'] & \end{tikzcd} . \tag{149}$$

Exercise 43 *Suppose* f, g *is a bimorphism from* ρ *to* ρ' *and* f', g' *is a bimorphism from* ρ' *to* ρ''. *Show that* $f' \circ f, g \circ g'$ *is a bimorphism from* ρ *to* ρ''.

Lemma 3.1 *The following conditions are equivalent:*

(a) *A pair of functions* (147) *is a bimorphism.*

(b) $\forall_{y' \in Y'}\ f^* \langle y'] \supseteq \langle g(y')]$.

(c) $\forall_{x \in X}\ [f(x)\rangle \supseteq g^*[x\rangle$.

3.6.2 ~-commutative diagrams

A ~-commutative diagram is a slight but very significant generalization of a commutative diagram, cf. 1.13. Whole areas of advanced modern Mathematics and Mathematical Physics are devoted to studying phenomena expressed in the language of ~-commutative diagrams.

Commutativity of a diagram means that two composable paths of arrows (representating functions between sets), that have a common source and a common target, are equal. If that common target, call it T, is equipped with a binary relation ~, then equality may be replaced by the condition that the corresponding composite functions satisfy the relation induced by ~ on the set of T-valued functions.

Since ~ is not necessarily symmetric, one needs to indicate which of the composite functions appears as the *left* argument and which appears as the *right* argument of the relation in question.

This can be represented in a diagram by placing a small arrow (ideally, a bent double arrow) near the common target of two composable paths of arrows, as is shown in the following simple example. A square-shaped diagram

$$\begin{tikzcd} R \arrow[d, "\upsilon"'] \arrow[dr, phantom, "\sim"] & S \arrow[l, "\varphi"'] \arrow[d, "\psi"] \\ T & U \arrow[l, "\chi"] \end{tikzcd}$$

expresses the statement

$$\forall_{s\in S}\, \chi(\psi(s)) \sim \upsilon(\varphi(s))\,,$$

i.e., the composite arrow $\chi\circ\psi$ is in relation, induced by ~, with the composite arrow $\upsilon\circ\varphi$.

We shall usually omit the label (~ here) when the binary relation on the target is clear from the context.

Exercise 44 *Express each identity in* (98) *in the form of a ~-commutative diagram for an appropriate relation.*

3.6.3

For the *equality* relation =, the class of =-comutative diagrams coincides with the class of commutative diagrams.

3.6.4

We have encountered $\Longrightarrow$-commutative but, generally, not commutative, diagrams in the definition of a morphism between relations, cf. (115) and (146), and in the definition of a bimorphisms between binary relations, cf. (149).

3.6.5

Condition (b) of Lemma 3.1 says that the diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] \arrow[dr, phantom, "\subseteq"] & Y' \arrow[l, "\langle\,]"'] \arrow[d, "g"] \\ \mathscr{P}X & Y \arrow[l, "\langle\,]"] \end{tikzcd} \tag{150}$$

is $\subseteq$-commutative, while Condition (c) says that the diagram

$$\begin{tikzcd} X' \arrow[r, "{[\ \rangle}"] & \mathscr{P}Y' \\ X \arrow[u, "f"] \arrow[r, "{[\ \rangle}"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g^*"'] \end{tikzcd} \tag{151}$$

is $\supseteq$-commutative.

Proof of Lemma 3.1. For every $x \in X$ and $y' \in Y'$, one has the following two sequences of equivalent statements

$$[f(x)\rangle \ni y' \iff \rho'(f(x), y') \iff f(x) \in \langle y'] \iff x \in f^*\langle y']$$

and

$$g^*[x\rangle \ni y' \iff [x\rangle \ni g(y') \iff \rho(x, g(y')) \iff x \in \langle g(y')] .$$

If any of the statements in the bottom sequence implies any of the statements in the top sequence, then any other statement at the bottom implies any statement at the top.

The first statement in the formulation of Lemma 3.1 says that

$$\forall_{x \in X} \left(\forall_{y' \in Y'}\ \rho(x, g(y')) \Rightarrow \rho'(f(x), y') \right)$$

which is equivalent to the statement

$$\forall_{y' \in Y'} \left(\forall_{x \in X}\ \rho(x, g(y')) \Rightarrow \rho'(f(x), y') \right) .$$

Condition (b) of Lemma 3.1 is equivalent to the statement

$$\forall_{x \in X} \left(\forall_{y' \in Y'}\ g^*[x\rangle \ni y' \Rightarrow [f(x)\rangle \ni y' \right) .$$

Finally, Condition (c) of Lemma 3.1 is equivalent to the statement

$$\forall_{y' \in Y'} \left(\forall_{x \in X}\ x \in \langle g(y')] \Rightarrow x \in f^*\langle y'] \right) .$$

Thus, each of the conditions in Lemma 3.1 implies the remaining two.

3.6.6

Note that the proof uses the fact that interchanging the order in which universal quantification is applied produces equivalent statements. The same holds for existential quantification of relations. This is an analogue of Fubini's Theorem stating that (under a mild integrability hypothesis) interchanging the order of integration in evaluation of an iterated double integral produces the same result.

Beware that interchanging the order in which universal and existential quantification are applied produces statements that are rarely equivalent.

Exercise 45 *Show that if f, g is a bimorphism from ρ to ρ', then g, f is a bimorphism from $(\rho')^{\mathrm{op}}$ to ρ^{op}.*

Later we shall see that this means that the category of binary relations and bimorphisms is equipped with a canonical $*$*-category* structure, i.e., it is a category with a canonical *anti-involution*. The exact meaning of these terms will be explained later. Here it is sufficient to say that $*$*-structures* play a fundamental role in Mathematics and, especially, in Mathematical Physics.

Corollary 3.2 *The following conditions are equivalent:*

(a) *A pair of functions* (147) *is a bimorphism.*

(b') $\forall_{B'\subseteq Y'}\ f^*LB' \supseteq L(g_*B')$.

(c') $\forall_{A\subseteq X}\ U(f_*A) \supseteq g^*UA$.

Proof. Condition (b) of Lemma 3.1 implies that

$$\forall_{y'\in B'}\ f^*\langle y'] \supseteq \langle g(y')]$$

which, in turn, implies that

$$\bigcap_{y'\in B'} f^*\langle y'] \supseteq \bigcap_{y'\in B'} \langle g(y')] . \tag{152}$$

The left-hand-side of (152) equals, in view of identity (97),

$$f^*\left(\bigcap_{y'\in B'}\langle y']\right)$$

while the right-hand-side equals

$$\bigcap_{x\in g_*B'}\langle x] = L(g_*B') .$$

This demonstrates that statement (b) implies statement (b').

Exercise 46 *Write down two proofs of the equivalence*

$$\text{statement (c)} \quad \Longleftrightarrow \quad \text{statement (c')},$$

an explicit proof and a proof that deduces this from the already proven equivalence of statements (b) and (b').

The reverse implications (b') $\Longrightarrow$ (b) and (c') $\Longrightarrow$ (c) hold because

$$g_*\{y'\} = \{g(y')\}, \quad f_*\{x\} = \{f(x)\}, \quad L\{y\} = \langle y], \ \& \ U\{x\} = [x\rangle ,$$

hence statement (b) is a *special case* of statement (b') and statement (c) is a *special case* of statement (c'), cf. Section 1.7.2.

3.7 Galois connections

3.7.1 Faithful bimorphisms

By replacing, in the definition of a bimorphism, Implication $\Rightarrow$ by Equivalence $\Leftrightarrow$, we obtain the definition of a *faithful bimorphism*. Like for faithful morphisms, composition of faithful bimorphisms produces a faithful bimorphism. This follows immediately from the transitivity of the Equivalence relation on the set of statements.

The following characterization of faithful bimorphisms is an immediate corollary of Lemma 3.1.

Corollary 3.3 *The following conditions are equivalent:*

(a) *A pair of functions* (147) *is a faithful bimorphism.*

(b) $\forall_{y'\in Y'}\ f^*\langle y'] = \langle g(y')]\,.$

(c) $\forall_{x\in X}\ [f(x)\rangle = g^*[x\rangle\,.$

3.7.2

Condition (b') in Corollary 3.3 says that the square diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowleft"] & Y' \arrow[l, "\langle\ ]"'] \arrow[d, "g"] \\ \mathscr{P}X & Y, \arrow[l, "\langle\ ]"] \end{tikzcd} \tag{153}$$

is commutative, while Condition (c') says that the square diagram

$$\begin{tikzcd} X' \arrow[r, "[\ \rangle"] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{P}Y' \\ X \arrow[u, "f"] \arrow[r, "[\ \rangle"'] & \mathscr{P}Y \arrow[u, "g^*"'] \end{tikzcd} \tag{154}$$

is commutative.

Exercise 47 *State the analogue of Corollary 3.2 for faithful bimorphisms and represent two out of the three conditions as commutativity of appropriate diagrams.*

Exercise 48 *Show that, if f, g is a faithful bimorphism, then f is a morphism $(X, \succsim) \longrightarrow (X', \succsim)$ of preordered sets and, likewise, g is a morphism of preordered sets $(Y', \precsim) \longrightarrow (Y, \precsim)$. Here $\succsim$ and $\precsim$ are the corresponding canonical preorders associated with the binary relations ρ and ρ', cf. Section 3.3.4.*

3.7.3 Terminology

Faithful bimorphisms between ordered sets, i.e., when $X = Y$, $X' = Y'$, and the binary relations ρ and ρ' are order relations, are known as *Galois connections*.

Exercise 49 *Peruse these notes from the beginning up to this point and identify all Galois connections that you can find.*

3.7.4

Galois connections point towards one of the pillars of Modern Mathematics, the concept of a *pair of adjoint functors*. For this reason, I am tempted to extend the usage of the term *Galois connection* to arbitrary faithful bimorphisms.