FUNCTIONS * RELATIONS
OPERATIONS * OPERATIONS
RELATIONS * FUNCTIONS

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1 Preliminaries

1.1 The language of functions

1.1.1 Mathematical structures

Modern Mathematics is concerned with *mathematical structures*. A "mathematical structure" consists of one or more sets equipped with data of certain type.

This informal initial definition already covers practically all fundamental types of structures that a mathematician encounters on a daily basis.

1.1.2 The concept of a function

An example of a mathematical structure is provided by the familiar concept of a function. A function of n variables consists of

- a list of n sets
$$X_1, \dots, X_n \tag{1}$$
- a set Y
- an assignment
$$x_1, \dots, x_n \longmapsto y \tag{2}$$
that assigns a *single* element y of set Y to *every* list $x_1, \dots, x_n$ such that
$$x_1 \in X_1, \dots, x_n \in X_n . \tag{3}$$

1.1.3 The domain of a function

The list of sets, (1), is called the *domain* of the function. We shall also call it the *source-list* and will refer to n as the *length* of that list.

1.1.4 The antidomain of a function

The set Y is called the *antidomain* of the function. We shall also refer to it as the *target*.

1.1.5 The argument-list and the value of a function

We shall refer to $x_1, \dots, x_n$ satisfying Condition (3) as the *argument-list*. The single element $y \in Y$ that is assigned to it is then called the *value* of the function on that particular argument-list.

If the *name* of the function is, say, f, its value on the list $x_1, \dots, x_n$ is denoted
$$f(x_1, \dots, x_n) \tag{4}$$

1.1.6 The arrow representation of a function

The symbolic representation of a function

$$f : X_1, \dots, X_n \longrightarrow Y \tag{5}$$

at a glance supplies the following information: *the function's name*, often represented by a symbol, its domain, and its target. In (5) the name of the function is 'f', the domain is the list of sets $X_1, \dots, X_n$, and the target is the set denoted Y.

It is often more convenient to place the name of a function above the arrow representating the function

$$X_1, \dots, X_n \xrightarrow{\quad f \quad} Y \ .$$

1.1.7 Equality of functions

Two functions are declared to be *equal* if

- *their domains are equal,*
- *their targets are equal,*
- *and their assignments are equal.*

In particular, a function

$$V_1, \dots, V_m \xrightarrow{\quad f \quad} W$$

can be equal to a function

$$X_1, \dots, X_n \xrightarrow{\quad g \quad} Y$$

only when

$$m = n, \qquad V_1 = X_1, \ \dots, \ V_m = X_m \ , \qquad \text{and} \qquad W = Y.$$

1.1.8 Functions of zero variables

When $n = 0$, the domain of a function is the empty list of sets. The arrow representation of such a function would be thus

$$\xrightarrow{\quad f \quad} Y \tag{6}$$

There is only one argument list in this case, namely the empty list. The function assigns to it a single element $y \in Y$. In particular,

$$f \quad \longleftrightarrow \quad \text{the value of } f \text{ on the empty argument-list}$$

defines a canonical identification between functions (6) and elements of the target-set Y.

1.1.9 Functions constant in the i-th variable

If the value (4) does not depend on x_i, we say that f is *constant in i-th variable.*

1.1.10

We shall denote the set of all functions (5) by

$$\mathrm{Funct}(X_1, \dots, X_n\,; Y) \tag{7}$$

or

$$Y^{X_1,\dots,X_n}. \tag{8}$$

1.1.11 Lists with omitted entries

Since lists with certain entries having been omitted are frequently encountered in Mathemtics, we have the notation to denote such lists. For example,

$$x_1, \dots, \hat{x}_i, \dots, x_n \tag{9}$$

stands for the list of length $n-1$

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$$

while

$$x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n \tag{10}$$

stands for the list of length $n-2$

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n,$$

and so on.

1.1.12 Freezing a variable in a function of n-variables

For any $1 \leq i \leq n$ and any $a \in X_i$, assignment

$$x_1, \dots, \hat{x}_i, \dots, x_n \;\longmapsto\; f(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n)$$

defines a function of $n-1$ variables

$$X_1, \dots, \hat{X}_i, \dots, X_n \;\longrightarrow\; Y\,. \tag{11}$$

We shall denote function (11) by $\mathrm{ev}_a^i f$.

1.1.13 The associated evaluation functions of one variable

Assignment

$$x_i \;\mapsto\; \mathrm{ev}_{x_i}^i f$$

defines a function of a single variable

$$X_i \;\longrightarrow\; \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,; Y) \tag{12}$$

We shall denote function (12) by $\mathrm{ev}^i f$ and call it the i-th *evaluation function* associated with a function f.

1.1.14 Adjunction correspondence

Assignment

$$f \longmapsto \mathrm{ev}^i f$$

defines a canonical bijection

$$\mathrm{Funct}(X_1, \dots, X_n ; Y) \longleftrightarrow \mathrm{Funct}(X_i, \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n ; Y)) \tag{13}$$

whose inverse is given by sending a function

$$\phi \in \mathrm{Funct}(X_i, \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n ; Y))$$

to the function

$$X_1, \dots, X_n \longrightarrow Y, \qquad x_1, \dots, x_n \longmapsto \big(\phi(x_i)\big)(x_1, \dots, \hat{x}_i, \dots, x_n) .$$

Correspondence (13) is a manifestation of what is perhaps the single most important phenomenon in Modern Mathematics known as a *pair of adjoint functors*. This is not your first encounter with this phenomenon—you encountered it in some fundamental theorems of basic Mathematical curriculum, but it is the first time that you are expressly told about it.

1.1.15

In order to describe the conceptual mechanics behind the concept of *adjoint* functors, one needs to introduce a proper language, the language of *morphisms* and *categories*, cf. Chapters 3 and 4.

1.1.16 Adjunction correspondence in exponential notation

Canonical identification (13) in exponential notation (8) acquires particularly suggestive form

$$Y^{X_1, \dots, X_n} \longleftrightarrow \left(Y^{X_1, \dots, \hat{X}_i, \dots, X_n}\right)^{X_i} . \tag{14}$$

1.1.17 Surjective functions

A function (5) is said to be *surjective* if

$$\text{for every } y \in Y \text{ there exists an argument-list } x_1, \dots, x_n \text{ such that } f(x_1, \dots, x_n) = y . \tag{15}$$

You are likely to be familiar with an informal expression "a function f is onto" instead of being surjective. I encourage you to use the term surjective.

1.1.18 Injective functions

A function (5) is said to be *injective* if it has the property

$$\text{if } f(x_1, \dots, x_n) = f(x'_1, \dots, x'_n) \text{, for two argument-lists, then the two argument-lists are equal.} \tag{16}$$

You are likely to be familiar with an informal expression "a function f is one-to-one" instead of of being injective.

1.1.19 Bijective functions

A function is said to be *bijective* if it is both surjective and injective. This terminology is used primarily for functions of a single variable.

1.2 Composition of functions

1.2.1 Postcomposition

Given a function (5) and a function $g : Y \to Y'$, their composition yields the function

$$g \circ f : X_1, \dots, X_n \longrightarrow Y', \qquad x_1, \dots, x_n \longmapsto g(f(x_1, \dots, x_n)). \tag{17}$$

1.2.2

Postcomposition with a function g is itself a function between the function sets

$$g_{\bullet} : \mathrm{Funct}(X_1, \dots, X_n; Y) \longrightarrow \mathrm{Funct}(X_1, \dots, X_n; Y'), \qquad f \longmapsto g \circ f. \tag{18}$$

1.2.3 Precomposition

Given a function (5) and a function-list $h_1, \dots, h_n$,

$$X'_1, \dots, X'_m \xrightarrow{h_1} X_1 \quad , \quad \dots \quad , \quad X'_1, \dots, X'_m \xrightarrow{h_n} X_n \tag{19}$$

their composition yields the function

$$f \circ (h_1, \dots, h_n) : X'_1, \dots, X'_m \longrightarrow Y, \qquad x'_1, \dots, x'_m \longmapsto f(\, h_1(x'_1, \dots, x'_m), \dots, h_n(x'_1, \dots, x'_m)\,). \tag{20}$$

1.2.4

Precomposition with a function-list $h_1, \dots, h_n$ is itself a function between the function sets

$$(h_1, \dots, h_n)^{\bullet} : \mathrm{Funct}(X_1, \dots, X_n\,; Y) \longrightarrow \mathrm{Funct}(X'_1, \dots, X'_m\,; Y), \qquad f \longmapsto f \circ (h_1, \dots, h_n). \tag{21}$$

1.2.5 Invertible functions of a single variable

Composition of functions of a single variable produces a function of a single variable. We say that $f : \mathrm{X} \to Y$ is a *left-invertible* function, if there exists a function $g : Y \to \mathrm{X}$ such that

$$g \circ f = id_{\mathrm{X}}\,. \tag{22}$$

We say that $f : \mathrm{X} \to Y$ is a *right-invertible* function, if there exists a function $g : Y \to \mathrm{X}$ such that

$$f \circ g = id_Y\,. \tag{23}$$

Exercise 1 *Show that, if* g *is a left-inverse of* f *and* h *is a right-inverse of* f, *then* $g = h$.

1.2.6

We denote that unique left- and right-inverse by f^{-1}.

Exercise 2 *Show that a left-invertible function f is injective and a right-invertible function is surjective.*

In particular, an invertible function is bijective.

Exercise 3 *Show that a bijective function is invertible.*

Lemma 1.1 *Suppose that $f : X \to Y$ is injective. Then there is a natural correspondence between left-inverses of f and functions $h : Y \setminus f_* X \longrightarrow X$.*

Proof. The target of a function f is the union of disjoint sets

$$Y' := f_* X \qquad \text{and} \qquad Y'' := Y \setminus f_* X .$$

Exercise 4 *Show that $g : Y \to X$ is a left-inverse of f if and only if the restriction of g to Y' is the function*

$$y \longmapsto \textit{the unique } x \in X \textit{ such that } f(x) = y .$$

Thus, the set of left-inverses of f is in bijective correspondence with the set of functions $Y'' \to X$,

$$\text{Left Inverses}(f) \longleftrightarrow \text{Funct}(Y'', X), \qquad g \longmapsto g_{|Y''}.$$

□

Since the function set $\text{Funct}(Y'', X)$ is not empty as long as either X is not empty or Y'' is empty, we obtain the following two corollaries.

Corollary 1.2 *A function $f : X \to Y$ with $X \neq \emptyset$ is left-invertible if and only if f is injective. A function $f : \emptyset \to Y$ is left-invertible if and only $Y = \emptyset$, i.e., if and only if f is bijective.*

Corollary 1.3 *A function $f : X \to Y$ with $X \neq \emptyset$ is bijective if and only if it has a unique left-inverse. That unique left-inverse is also a right-inverse.*

1.2.7 Finite sets

We say that a set is *finite* if every left-invertible function $f : X \to X$ is invertible.

1.2.8 Infinite sets

Accordingly, we say that a set X is *infinite*, if it admits a left-invertible function $f : X \to X$ that is not right-invertible.

1.2.9 Axiom of Infinity

The so called *Axiom of Infinity* of Set Theory asserts existence of an infinite set.

Existence of an infinite set cannot be proven using the remaining axioms of Set Theory. In fact, the remaining axioms of Set Theory are consistent with the assertion that every set is finite.

We shall prove later that Axiom of Infinity is equivalent to existence of the *semiring* $(\mathbf{N}, 0, 1, +, \cdot)$ of natural numbers.

1.3 The language of relations

1.3.1 Statements

A *statement* is a well-formed sentence that is either true or false. Any human language whose vocabulary is extended by adding various, previously defined, mathematical terms, is acceptable.

1.3.2 A relation is a function whose values are statements

A *relation* on sets $X_1, \dots, X_n$ is a function of n variables

$$\rho : X_1, \dots, X_n \longrightarrow \text{Statements}, \qquad x_1, \dots, x_n \longmapsto \rho(x_1, \dots, x_n). \tag{24}$$

We say in this case that ρ is an n*-ary* relation. We also say that the relation is *between* elements of sets $X_1, \dots, X_n$.

1.3.3

Statement $\rho(x_1, \dots, x_n)$, i.e., the value of ρ on the argument list $x_1, \dots, x_n$, needs not refer to some or even to anyone of the element variables x_i.

1.3.4 Total relations

Statement $\rho(x_1, \dots, x_n)$ may hold for every list of arguments. Such a relation is sometimes referred to as *a total* relation.

1.3.5 Void relations

Statement $\rho(x_1, \dots, x_n)$ may fail for every list of arguments. Such a relation is sometimes referred to as *a void* relation.

1.3.6 Nullary, unary, binary, ternary, ... relations

For small values of n, instead of speaking about 0-ary, 1-ary, 2-ary, 3-ary, ..., relations, we speak of *nullary, unary, binary, ternary, ...,* relations.

1.3.7 {nullary relations} $\longleftrightarrow$ {statements}

According to Section 1.1.8, there is a canonical identification between *nullary relations* and *statements.*

1.3.8

Since a nullary relation reduces to a single statement, and since every statement either holds or fails, a nullary relation is either total or void.

1.3.9 Relations *on* a set

When all sets X_i in the domain coincide with a set X, we speak of an n-ary relation *on* X.

1.4 Operations on sets

1.4.1

An *n-ary operation* on a set Y is a function

$$\mu : X_1, \dots, X_n \longrightarrow Y \tag{25}$$

where all the sets $X_1, \dots, X_n$ are equal to Y.

1.4.2 $\{$nullary operations on $Y\} \longleftrightarrow Y$

To declare a nullary operation on a set Y is equivalent to supplying a single element of Y. For this reason, nullary operations on Y are thought of as "distinguished" elements of Y. In particular, there is a canonical bijection between the set of nullary operations on Y and the set Y itself.

1.4.3 Induced operations

Given a list of n functions of m variables,

$$f_1, \dots, f_n \in \mathrm{Funct}(X_1, \dots, X_m; Y), \tag{26}$$

let us assign to the argument list

$$x_1, \dots, x_m$$

the list of values

$$f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)$$

and then apply the operation μ. Composite assignment

$$x_1, \dots, x_m \longmapsto f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m) \longmapsto \mu\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big)$$

defines a function $X_1, \dots, X_m \longrightarrow Y$. We shall denote this function by $\mu_\bullet(f_1, \dots, f_n)$.

Assignment

$$f_1, \dots, f_n \longmapsto \mu_\bullet(f_1, \dots, f_n) \tag{27}$$

defines then an n-ary operation $\mu_\bullet$ on the set of functions $\mathrm{Funct}(X_1, \dots, X_m; Y)$. We refer to it as the operation *induced by* μ.

1.4.4

Operations on sets of Y-valued functions induced by operations defined on Y have been playing an essential role in Mathematics since the time when the foundations of Differential and Integral Calculus had been laid down nearly 400 years ago.

1.5 Canonical operations on $\mathscr{P}X$

1.5.1 Canonical operations

A general set X has no distinguished elements, hence it is not equipped with any distinguished nullary operation. Similarly, there are no distinguished binary, ternary, etc., operations on a general set. The identity function

$$\mathrm{id}_X : X \longrightarrow X, \qquad x \longmapsto x, \tag{28}$$

is the only distinguished unary operation.

Certain sets, however, are *naturally* equipped with various operations. We refer to such operations as *canonical*. An example of prime importance is provided by the set of all subsets, $\mathscr{P}X$, of an arbitrary set X. A shorter designation for $\mathscr{P}X$ is the *power-set of* X.

1.5.2 Canonical nullary operations on $\mathscr{P}X$

The power-set of a general nonempty set has exactly two distinguished elements: the empty subset $\emptyset$ and X. In other words, $\mathscr{P}X$ is equipped with exactly two canonical nullary operations.

1.5.3 The complement of a subset

The power-set of a general set has a canonical unary operation

$$\complement : \mathscr{P}X \longrightarrow \mathscr{P}X, \qquad A \longmapsto \complement A := \{x \in X \mid x \notin A\}, \tag{29}$$

that sends a subset $A \subseteq X$ to its *complement*. We shall usually denote the complement of a subset $A \subseteq X$ by A^c and use symbol $\complement$ to denote the complement operation.

1.5.4 Involutions on a set

Note that $\complement^2 := \complement \circ \complement$ is the identity operation. A unary operation $\mu : X \to X$ with this property is called an *involution* (on a set X). The identity operation id_X is a *trivial* involution.

1.5.5 Canonical unary operations on $\mathscr{P}X$

The power-set $\mathscr{P}X$ of a nonempty set is equipped with exactly two unary operations, both of them involutions on $\mathscr{P}X$: the identity operation $\mathrm{id}_{\mathscr{P}X}$ and the complement operation $\complement$.

1.5.6 Canonical binary operations on $\mathscr{P}X$

Union of two sets,

$$A, B \longmapsto A \cup B,$$

intersection of two sets,

$$A, B \longmapsto A \cap B,$$

difference of two sets,

$$A, B \longmapsto A \setminus B,$$

are examples of canonical binary operations on the power-set.

1.5.7

Any one of the above three operations can be expressed in terms of any of the remaining two and of the complement operation. For example, the union and the intersection operations are linked to each other by the following pair of identities

$$A \cap B = \complement(\complement A \cup \complement B) \qquad \text{and} \qquad A \cup B = \complement(\complement A \cap \complement B) \qquad (A, B \subseteq X) \tag{30}$$

called *de Morgan laws*.

Note also the following identities

$$A \cup \complement A = X, \qquad A \cap \complement A = \emptyset \qquad \text{and} \qquad A \setminus B = A \cap \complement B = \complement(\complement A \cup B)^c \qquad (A, B \subseteq X).$$

Exercise 5 *Find the identity expressing* $\cap$ *in terms of* $\setminus$ *and* $\complement$, *and prove it.*

1.6 Operations on Statements

1.6.1 Basic binary operations on sentences

The following table contains the list of basic binary operations on sentences (symbols P and Q stand for arbitrary sentences).

Read:	Symbolic notation	Name
P *and* Q	$P \wedge Q$	Conjunction
P *or* Q	$P \vee Q$	Alternative
if P, *then* Q	$P \Rightarrow Q$	Implication
P *if and only if* Q	$P \Leftrightarrow Q$	Equivalence

1.6.2 Negation

The negated sentence P will be symbolically denoted $\neg P$. In many languages, negating a sentence is performed according to rules that depend on the syntactical structure of that sentence. For this reason, it is difficult or impossible to provide one single reading of the negated sentence $\neg P$. We can circumvent this difficulty by saying, instead, "the negation of P" or "P negated", when we need to refer to $\neg P$.

1.6.3 Validity of the corresponding statements

Assuming that P and Q are statements,

- $P \wedge Q$ holds precisely when P and Q hold;
- $P \vee Q$ holds precisely when P or Q holds;
- $P \Rightarrow Q$ fails if P holds and Q fails, otherwise it holds;
- $P \Leftrightarrow Q$ holds precisely when P and Q both hold or both fail;
- $\neg P$ holds precisely when P fails.

In particular, Conjunction, Alternative, Implication, Equivalence, define binary operations on the set of Statements, while Negation defines a unary operation.

1.6.4 Operations on Statements = Relations on Statements

On the set of statements the concepts of an n-ary operation and of an n-ary relation coincide.

1.6.5 Operations on relations

Any operation on Statements induces the corresponding operations on the sets of relations, $\mathrm{Rel}(X_1, \dots, X_n)$, between elements of sets $X_1, \dots, X_n$.

1.6.6

Thus, given relations $\rho, \sigma \in \mathrm{Rel}(X_1, \dots, X_n)$, we can form the relations $\neg\rho$, $\rho\vee\sigma$, $\rho\wedge\sigma$, $\rho\Rightarrow\sigma$ and $\rho\Leftrightarrow\sigma$. They assign to an argument list $x_1, \dots, x_n$ the statements

$$\neg\rho(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \vee \sigma(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \wedge \sigma(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \Rightarrow \sigma(x_1, \dots, x_n)$$

and, respectively,

$$\rho(x_1, \dots, x_n) \Leftrightarrow \sigma(x_1, \dots, x_n).$$

1.7 Quantification

1.7.1 Universal quantification

Given a relation ρ between elements of sets $X_1, \dots, X_n$, and a subset $A_i \subseteq X_i$, by assigning to a list $x_1, \dots, \hat{x}_i, \dots, x_n$ the statement

$$\textit{for all } x_i \in X_i,\ \rho(x_1, \dots, x_n) \tag{31}$$

we obtain an $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$. Instead of "for all", we can also say "for every" with the same meaning.

Symbolically, statement (31) is represented

$$\forall_{x_i \in A_i} \rho(x_1, \dots, x_n)\,. \tag{32}$$

1.7.2 Universal quantification over a subset

The above construction defines what is called *universal quantification over a subset.* By assigning to a relation $\rho \in \mathrm{Rel}(X_1, \dots, X_n)$ the resulting relation $\forall_{x_i \in A_i} \rho$, we obtain a function

$$\forall_{x_i \in A_i} : \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}\left(X_1, \dots, \hat{X}_i, \dots, X_n\right), \qquad \rho \longmapsto \forall_{x_i \in A_i} \rho\,. \tag{33}$$

1.7.3

At the same time, assignment

$$x_1, \dots, A_i, \dots, x_n \ \longmapsto\ \forall_{x_i \in A_i} \rho(x_1, \dots, x_n) \tag{34}$$

defines an n-ary relation that we shall denote $\forall^i \rho$. Note that

$$\forall^i : \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}\,(X_1, \dots, \mathscr{P}X_i, \dots, X_n)\,, \qquad \rho \longmapsto \forall^i \rho\,, \tag{35}$$

is a *canonically* defined function between relation sets that preserves the number of arguments of ρ and replaces set X_i, the i-th entry in the domain-list, by its power-set $\mathscr{P}X_i$. Superscript i indicates that we are quantifying the relation with respect to the i-th variable.

1.7.4 " Statement S is a special case of statement T "

Suppose $\rho : X \longrightarrow$ Statements is a (unary) relation on a set X. Consider the statements obtained by universal quantification of relation ρ over subsets $A \subseteq X$ and $B \subseteq X$

$$S := \text{“ } \forall_{x \in A}\, \rho(x) \text{ ”} \qquad \text{and} \qquad T := \text{“ } \forall_{x \in B}\, \rho(x) \text{ ”}. \tag{36}$$

Note that, if $A \subseteq B$, then

$$S \Longrightarrow T. \tag{37}$$

If so, we shall say that *statement S is a special case of statement T*.

In general, given two statements S and T, we shall say that S is a special case of T if there exist

$$\text{a unary relation } \rho \text{ on a certain set } X \text{ and subsets } A \subseteq B \subseteq X$$

such that S and T have the form as in (36).

1.7.5 " Statement S trivially implies statement T "

Note that in order to establish implication (37), one deoes not need to know anything about a set X, a relation ρ on X, or subsets A and B. One only needs to know that both statements are obtained by *universal* quantification of the *same* certain unary relation over two subsets $A \subseteq B$ of X.

This is one of those situations when mathematicians are likely to say that a statement S *trivially* implies a statement T.

1.7.6 Existential quantification

Assigning to a list $x_1, \dots, \hat{x}_i, \dots, x_n$ the statement

$$\textit{there exists } x_i \in X_i \textit{ such that } \rho(x_1, \dots, x_n) \tag{38}$$

defines another an $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$.

Symbolically, statement (38) is represented

$$\exists_{x_i \in X_i}\, \rho(x_1, \dots, x_n).$$

Exercise 6 *Formulate the definitions of functions $\exists_{x_i \in A_i}$ and $\exists^i$ in analogy with the definitions given in Sections 1.7.2 and 1.7.3.*

1.7.7

Suppose $\rho : X \longrightarrow$ Statements is a (unary) relation on a set X. Consider the statements obtained by existential quantification of relation ρ over subsets $A \subseteq X$ and $B \subseteq X$

$$S := \text{“ } \exists_{x \in A}\, \rho(x) \text{ ”} \qquad \text{and} \qquad T := \text{“ } \exists_{x \in B}\, \rho(x) \text{ ”}. \tag{39}$$

Note that, if $A \subseteq B$, then

$$T \Longrightarrow S. \tag{40}$$

Also in this case we say that statement T *trivially implies* statement S.

1.7.8 The direct image function f_*

Operations of quantification are frequently iterated. An example of this is present in the definition of the *direct image* function associated with an arbitrary function (5).

$$f_* : \mathscr{P}X_1, \dots, \mathscr{P}X_n \longrightarrow \mathscr{P}Y \tag{41}$$

where

$$A_1, \dots, A_n \longmapsto f_*(A_1, \dots, A_n) := \left\{ y \in Y \mid \exists_{x_1 \in X_1} \cdots \exists_{x_n \in X_n} f(x_1, \dots, x_n) = y \right\} . \tag{42}$$

Changing the order of iterated quantifiers *of the same type* produces relations that are equipotent. This allows to use compressed notation like

$$\exists_{x_1 \in X_1, \dots, x_n \in X_n}$$

in place of

$$\exists_{x_1 \in X_1} \cdots \exists_{x_n \in X_n}$$

in the definition of f_*,

$$A_1, \dots, A_n \longmapsto f_*(A_1, \dots, A_n) := \left\{ y \in Y \mid \exists_{x_1 \in X_1, \dots, x_n \in X_n} f(x_1, \dots, x_n) = y \right\} . \tag{43}$$

1.7.9 Caveat

Changing the order in which a universal and an existential quantifier are applied has usually a drammatic effect, however. Thus, given $i \neq j$,

$$\forall_{x_i \in X_i} \exists_{x_j \in X_j} \rho(x_1, \dots, x_n) \tag{44}$$

denotes the statement:

$$\textit{for all } x_i \in X_i, \textit{ there exists } x_j \in X_j \textit{ such that } \rho(x_1, \dots, x_n) \tag{45}$$

while

$$\exists_{x_j \in X_j} \forall_{x_i \in X_i} \rho(x_1, \dots, x_n) \tag{46}$$

denotes the nonequivalent statement:

$$\textit{there exists } x_j \in X_j \textit{ such that, for all } x_i \in X_i,\ \rho(x_1, \dots, x_n) . \tag{47}$$

In particular, relations (44) and (46) are almost never equipotent.

1.7.10

Relations with several levels of quantifications are frequently encountered in important definitions and constructions. Processing with understanding such relations can pose a serious challenge to a beginner and is one of the reasons why Mathematics is considered to be difficult.

For example, the statement

$$\forall_{\varepsilon \in \mathbf{R}^+} \exists_{i \in \mathbf{N}} \forall_{j \in \mathbf{N}} \left(i \leq j \implies |x_j - a| < \varepsilon \right) \tag{48}$$

describes the fact that a sequence of real numbers (x_n) converges to a point a of the real line. Here, $\mathbf{R}^+$ denotes the set of positive real numbers and $\mathbf{N}$ denotes the set of natural numbers. The statement is about sequences (x_n) of real numbers and points a of the real line. It defines a binary relation between elements of these two sets. The relation is the result of applying one-after-another universal and existential quantification to the statement that has the form of implication

$$i \leq j \;\Rightarrow\; |x_j - a| < \varepsilon \,. \tag{49}$$

Here x_j denotes the j-th term of the sequence (x_n). Statement (49) is a statement about natural numbers i and j, a sequence of real numbers (x_n), a point of the real line a, and a positive real number ε. As such, it is a 5-ary relation. Application of three consecutive quantifications yields the binary relation defined in (48).

What you see here is a typical example of statements encountered in Mathematical Analysis.

Exercise 7 *Let $\rho : X_1, X_2 \longrightarrow$ Statements be a binary relation. Consider the statements*

$$S := \text{“ } \exists_{x_1 \in A_1} \forall_{x_2 \in A_2} \rho(x_1, x_2) \text{ ”} \qquad \textit{and} \qquad T := \text{“ } \exists_{x_1 \in B_1} \forall_{x_2 \in B_2} \rho(x_1, x_2) \text{ ”}$$

where A_1 and B_1 are subsets of X_1 while A_2 and B_1 are subsets of X_2. Under what condition on A_1, A_2, B_1 and B_2, statement S implies statement T?

1.8 Binary relations *on* a set: a vocabulary of terms

1.8.1

Binary relations on a set X call for a special attention in view of the central role they play in every area of Mathematics.

1.8.2 Infix notation

In view of the fact that binary relations have been used by mathematicians long before the concept of a general relation was formulated and are still the most frequently encountered type of relation, special notation has been employed when binary relations are mentioned. The symbolic expression

$$x_1 \rho x_2$$

has the meaning:

Statement $\rho(x_1, x_2)$ holds.

1.8.3 Tilde notation

More likely, however, you will see expressions like

$$x_1 \sim x_2 \,, \tag{50}$$

since symbol $\sim$ and its variants have been adopted as a generic symbol denoting a binary relation. The meaning of (50) is:

the binary relation in question, denoted $\sim$, holds for elements $x_1 \in X_1$ and $x_2 \in X_2$.

The difference between the *functional* notation and the *tilde* notation, when talking about binary relations, is similar to the difference between *direct speech* and *indirect speech*: compare the statements

$$3 < 5$$

and

$$\textit{inequality } 3 < 5 \textit{ holds}.$$

1.8.4 Various types of binary relations on a set

A binary relation ρ on a set X is said to be:

reflexive if

$$\forall_{x \in X}\, \rho(x, x) \tag{51}$$

symmetric if

$$\forall_{x,y \in X} \left(\rho(x, y) \Rightarrow \rho(y, x) \right) \tag{52}$$

antisymmetric if

$$\forall_{x,y \in X} \left(\rho(x, y) \Rightarrow \neg\rho(y, x) \right) \tag{53}$$

weakly antisymmetric if

$$\forall_{x,y \in X} \left(\rho(x, y) \wedge \neg\rho(y, x) \;\Rightarrow\; x = y \right) \tag{54}$$

transitive if

$$\forall_{x,y,z \in X} \left(\rho(x, y) \wedge \rho(y, z) \;\Rightarrow\; \rho(x, z) \right) \tag{55}$$

1.8.5

Of all the properties that a binary relation ρ on a set X may have, by far the most important is its *transitivity*.

1.8.6 Preorder relations

A transitive and reflexive relation is called a *preorder* or a *quasiorder*.

1.8.7 Equivalence relations

A symmetric preorder is called an *equivalence relation*. The set

$$[x]_\rho := \{y \in X \mid \rho(x, y)\} \tag{56}$$

of elements ρ*-related* to x is then called the *equivalence class of an element* $x \in X$. Since ρ is symmetric, $[x]_\rho = [y]_\rho$ precisely when $\rho(x, y)$.

Exercise 8 *Show that, for any* $x, y \in X$,

$$[x]_\rho \cap [y]_\rho \neq \emptyset,$$

if and only if,

$$[x]_\rho = [y]_\rho.$$

1.8.8 The set of equivalence classes of an equivalence relation

The set of equivalence classes of an equivalence relation ρ

$$\left\{C \subseteq X \mid \exists_{x\in X}\ C = [x]_\rho\right\} \tag{57}$$

makes a frequent appearance in every area of modern Mathematics. It has been one of the Mathematics most important constructs.

1.8.9

Set (57) is an example of a *family of subsets of* X and it appears, for example, in the construction of a *quotient of a set* X *by a binary relation on* X. For this reason, set (57) is often denoted $X_{/\rho}$ and the surjective function

$$[\ ]_\rho : X \longrightarrow X_{/\rho}, \qquad x \longmapsto [x]_\rho, \tag{58}$$

is called the *canonical quotient map*. We revisit this construction in Chapter **??** devoted to the concept of a *quotient structure*.

1.8.10 A remark about terminology: a *map*, a *mapping*

The term *map* is very frequently employed today as an alternative term for *function*. This use became established among Mathematical Analysts who preferred to reserve the term 'function' for real- or complex-valued functions. The word *map* is an abbreviated form of the word *mapping*, which is a calque from German word *Abbildung*, introduced early in the 20th Century by topologists, writing in German, to denote functions between spaces of real or complex-valued functions, and between more general spaces.

1.8.11 The equivalence relation canonically associated with a preorder

Suppose that ρ is a preorder relation on a set X.

Exercise 9 *Show that the conjunction of* ρ *and its opposite relation* ρ^{op}

$$\rho \wedge \rho^{\mathrm{op}} : X, X \longrightarrow \text{Statements}, \qquad x, y \longmapsto \text{“}\ \rho(x,y) \wedge \rho(y,x)\ \text{”}, \tag{59}$$

is an equivalence relation on X.

We shall refer to a pair of elements satisfying $\rho \wedge \rho^{\mathrm{op}}$ as ρ*-equivalent*.

1.8.12 Order relations

A weakly antisymmetric preorder is called an *order relation*. A preorder is an order relation precisely when (59) is the *weakest equivalence relation* on X, i.e., when $\rho \wedge \rho^{\mathrm{op}}$ is equipotent with the equality relation $=$.

1.8.13 Sharp order relations

An antisymmetric transitive relation is called a *sharp-order relation*.

1.8.14 Preordered sets

A set X equipped with a preorder relation will be called a *preordered set.* We shall use the generic symbol $\precsim$ to denote the preorder relation. When using the term 'preordered set', remember that it is not a set, it is a *mathematical structure*: a *set equipped with a binary relation* $(X, \precsim)$.

1.8.15 Ordered sets

When $\precsim$ is weakly-antisymmetric, i.e., when $\precsim$ is an order relation, we shall be using generic symbol $\preccurlyeq$ to denote it and we shall refer to a set X equipped with an order relation, $(X, \preccurlyeq)$, as an *ordered set.*

1.8.16 Comments about terminology and notation

To emphasize that elements of an ordered set are not necessarily *comparable,* the adverb "partially" is often placed in front of "ordered". Those who insisted on using the term "partially ordered set" soon began to abbreviate it in typed texts as "p. o. sets." When the abbreviation dots got lost, a monstrous term "poset" was born. *Do not use that term.*

1.8.17 Linearly ordered sets

Ordered sets whose elements are *comparable,* i.e., satisfy the condition

$$\forall_{x,y\in X}\ x\preccurlyeq y \vee y\preccurlyeq x\,, \tag{60}$$

are called *linearly,* or *totally,* ordered.

Naturally defined linear orders are scarce, unlike (partial) orders.

1.8.18 Well-ordered sets

Even scarcer are *well-ordering* relations, i.e., order relations for which every *nonempty* subset $A \subseteq X$ has the smallest element. A prime example of a well-ordered set is the set of natural numbers $\mathbf{N}$, cf. Section 3.3.11.

1.8.19 $|A| = |B|$

We say that subsets A and B of a set X *have the same cardinality* if there exists a bijection $f : A \to B$. One expresses this by writing

$$|A| = |B|\,. \tag{61}$$

Assignment

$$A, B \longmapsto \text{“}\ |A| = |B|\ \text{”} \tag{62}$$

defines a binary relation on the power-set of X.

Exercise 10 *Show that* (62) *is an equivalence relation.*

1.8.20 Caveat

Note that we do *not* define the *cardinality* of a set X. We only define a binary relation *between* subsets of $\mathscr{P}X$.

1.8.21 $|A| \leq |B|$

Let us define the binary relation on $\mathscr{P}X$

$$A, B \longmapsto \text{“ } |A| \leq |B| \text{ ”} \tag{63}$$

by replacing the word 'bijection' in the definition of (61) by 'injective'. In other words, we mean by $|A| \leq |B|$ that there exists an *injective* function $A \rightarrowtail B$.

Exercise 11 *Show that* (63) *is a preorder relation on* $\mathscr{P}X$.

It is a nontrivial fact, established early in development of Set Theory, that existence of injective functions $A \rightarrowtail B$ and $B \rightarrowtail A$ implies existence of a bijection. We state it here without proof.

Theorem 1.4 (Cantor,Bernstein,Schröder) *For any sets* A *and* B,

$$|A| \leq |B| \wedge |B| \leq |A| \iff |A| = |B|. \tag{64}$$

□

1.8.22 $|A| = \mathfrak{c}$

We write

$$|A| = \mathfrak{c} \tag{65}$$

and say that a set A has the *cardinality of continuum* if

$$|A| = |\mathbf{R}|. \tag{66}$$

The following exercise is an application of Theorem 1.4.

Exercise 12 *Let* $A \subseteq \mathbf{R}$ *be a subset of the real line that contains an interval* (a, b). *Show that* $|A| = \mathfrak{c}$.

1.8.23 'Continuum Hypothesis'

The statement

$$\forall_{A \subseteq \mathbf{R}} \; |A| < \infty \vee |A| = \aleph_0 \vee |A| = \mathfrak{c}. \tag{67}$$

is known as *Continuum Hypothesis*. It was conjectured to be a theorem of Set Theory. All attempts to prove it were futile. Kurt Gödel proved that (67) was consistent with Axioms of Set Theory. In the early 1960-ties, Paul Cohen proved that its negation was also consistent with Axioms of Set Theory. Theorems of Gödel and Cohen together mean that assertion (67) cannot be proved or disproved. Such assertions are known as being *undecidable*.

1.8.24 Various approaches to the concept of the 'size' of a set

It is natural to *define* $|A|$ as the equivalence class of A with respect to the same-cardinality relation on $\mathscr{P}X$.

1.8.25 $|A| < \infty$ or $|A| = \infty$

When A is an infinite set, it is common to write

$$|A| = \infty. \tag{68}$$

Symbol ∞ here has no independent meaning. One should consider whole Expression (68) as saying that A *is an infinite set*. Accordingly,

$$|A| < \infty. \tag{69}$$

expresses the fact that set A is finite.

1.8.26 $|A| = n$

For a finite set, the expression

$$|A| = n. \tag{70}$$

means that there exists a bijection between A and the interval

$$\mathbf{n} := \{0, \dots, n-1\}$$

of the set of natural numbers $\mathbf{N}$. Element $n \in \mathbf{N}$ is then called the *number of elements of* A. Since $\mathbf{N}$ is equipped with a canonical linear order, a bijection $\mathbf{n} \leftrightarrow A$ is the same as linearly-ordering set A,

$$a_1 < \dots < a_n.$$

where $n \in \mathbf{N}$ is a certain natural number that This corresponds to 'counting' the elements of A.

1.8.27 $|A| = \aleph_0$

We say that a set A is *countably infinite* or, an *infinite countable* set, if there exists a bijection between A and the set of natural numbers $\mathbf{N}$. In this case it is common to write

$$|A| = \aleph_0 \tag{71}$$

and to say that A *has the cardinality aleph zero*. Such sets are the departure point for developing *theory of cardinal numbers* within the scope of theory of well-ordered sets. These advanced topics are covered in a course on Set Theory.

1.8.28 A canonical ordered-set structure on the power-set $\mathscr{P}X$ of a set X

The set of all subsets of a given set X is guaranteed to exist by one of the Axioms of Set Theory. Informally referred to as the *power-set of* X, it is denoted $\mathscr{P}X$. Inclusion of subsets,

$$\subseteq : \mathscr{P}X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad A, B \longmapsto \text{“}A \subseteq B\text{”}. \tag{72}$$

is a *canonical* order relation on $\mathscr{P}X$. Note that the inclusion relation is defined in terms of the *membership relation*

$$\in : X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad x, A \longmapsto \text{“ } x \in y \text{ ”}, \tag{73}$$

by

$$A \subseteq B \;:=\; \text{“ } \forall_{x \in A}\, x \in B \text{ ”}. \tag{74}$$

Both $(\mathscr{P}X, \subseteq)$ and the *opposite* ordered set, $(\mathscr{P}X, \supseteq)$, play a central role in Mathematics.

1.9 Induced relations

1.9.1

Given a list of n functions of m variables (26) and an n-ary relation $\rho \in \mathrm{Rel}_n Y$, the composite $\rho_\bullet(f_1, \dots, f_n)$ is an m-ary relation

$$\rho_\bullet(f_1, \dots, f_n) : X_1, \dots, X_m \longrightarrow \text{Statements}, \qquad x_1, \dots, x_m \longmapsto \rho\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big). \quad (75)$$

Universal quantification over $x_1 \in X_1, \dots, x_m \in X_m$ transforms (75) into a statement. If we assign thas statement to function-list $f_1, \dots, f_n$,

$$f_1, \dots, f_n \longmapsto \forall_{x_1 \in X_1, \dots, x_m \in X_m}\ \rho\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big), \quad (76)$$

we obtain an n-ary relation on the set of functions $\mathrm{Funct}(X_1, \dots, X_m; Y)$. Here

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n}$$

is an abbreviation for

$$\forall_{x_1 \in X_1} \cdots \forall_{x_n \in X_n}.$$

We shall denote relation (76) by $\hat{\rho}$ and will refer to it as the *induced relation* or as the relation *induced by* ρ.

Exercise 13 *Show that the relation induced by a preorder is a preorder.*

1.9.2 Induced relations on $\mathrm{Rel}(X_1, \dots, X_n)$

Recalling that, on the set of statements, there is no difference between relations and operations, we observe that any binary operation on the set of statements induces a binary relation on the set of relations $\mathrm{Rel}(X_1, \dots, X_n)$.

1.9.3 The equipotence relation on $\mathrm{Rel}(X_1, \dots, X_n)$

We shall denote by $\Longleftrightarrow$ the relation induced by Equivalence $\Leftrightarrow$, an *operation* on the set of statemnets. We shall refer to this induced relation as the *equipotence relation*. Following an old habit, mathematicians refer to equipotent relations as *equivalent*. This is one of many uses of the term *equivalent* by mathematicians. One should remember that an *equivalence relation* is a generic term for a binary relation on any set that is *reflexive,symmetric, and transitive*, cf. Section 1.8.7.

1.9.4 Caveat

One must be careful to distinguish, $\rho \Leftrightarrow \sigma$, the relation obtained by applying the induced binary *operation*

$$\mathrm{Rel}(X_1, \dots, X_n), \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}(X_1, \dots, X_n), \qquad \rho, \sigma \longmapsto \rho \Leftrightarrow \sigma, \quad (77)$$

to the pair ρ, σ, from $\rho \Longleftrightarrow \sigma$. The latter is a single *statement*, namely

$$\rho \Longleftrightarrow \sigma \quad := \quad \text{“}\ \forall_{x_1 \in X_1, \dots, x_n \in X_n}\ \rho(x_1, \dots, x_n) \Leftrightarrow \sigma(x_1, \dots, x_n)\ \text{”}.$$

The former, $\rho \Leftrightarrow \sigma$, is a function

$$X_1, \dots, X_n \longrightarrow \text{Statements}, \qquad x_1, \dots, x_n \longmapsto \text{“ } \rho(x_1, \dots, x_n) \Leftrightarrow \sigma(x_1, \dots, x_n) \text{ ”} . \tag{78}$$

This distinction disappears when the domain-list $X_1, \dots, X_n$ is empty, i.e., when ρ and σ are *statements*.

1.9.5 Equipotence classes of statements

By definition, there are just two equipotence classes of statements

$$\top := \{S \in \text{Statements} \mid S \text{ holds}\} \qquad \text{and} \qquad \bot := \{S \in \text{Statements} \mid S \text{ does not hold}\} . \tag{79}$$

There is a canonical bijective correspondence

$$\left\{ \begin{array}{c} \text{Equipotence classes of relations} \\ \rho : X_1, \dots, X_n \longrightarrow \text{Statements} \end{array} \right\} \longleftrightarrow \text{Funct}\,(X_1, \dots, X_n\,;\, \{\top, \bot\}) . \tag{80}$$

1.9.6 The implication relation on $\mathbf{Rel}(X_1, \dots, X_n)$

We shall denote by $\Longrightarrow$ the relation on $\text{Rel}(X_1, \dots, X_n)$ that is induced by Conditional $\Rightarrow$, an *operation* on the set of statements. We shall refer to this induced relation as the *implication relation.*

1.9.7 Caveat

A warning similar to the one issued in 1.9.4 is due: do not confuse $\rho \Rightarrow \sigma$ with $\rho \Longrightarrow \sigma$.

1.9.8 The canonical preorder on $\mathbf{Rel}(X_1, \dots, X_n)$

Since Conditional $\Rightarrow$ is a *preorder* relation on the set of statements, the induced relation $\Longrightarrow$ is a preorder relation. It is not only a canonical preorder on the set of relations, it is, in fact, a vital part of any reasoning process. Any form of rigorous reasoning employs the implication relation.

1.9.9 Terminology: *implies, is weaker than, is stronger than*

Given two relations $\rho, \sigma \in \text{Rel}(X_1, \dots, X_n)$ such that

$$\rho \Longrightarrow \sigma , \tag{81}$$

we shall say that ρ *implies* σ or that ρ is *weaker* than σ. In that case we shall σ is *stronger* than ρ.

The terms “weaker” and “stronger” is not an ideal terminology: ρ is both weaker and stronger than σ precisely when ρ and $_s$ are *equipotent*, not equal.

1.9.10

The implication preorder induces a canonical order relation on the set of equipotence classes of relations and canonical correspondence (80) *identifies* that set with the set of $\{\top, \perp\}$-valued functions equipped with the order relation induced by the order relation on $\{\top, \perp\}$ such that $\top$ is *greater* than $\perp$.

Lemma 1.5 *Any transitive relation on the set of statements that is stronger than* $\Leftrightarrow$ *is equipotent to*

$$\Leftrightarrow, \quad \Rightarrow, \quad \Leftarrow, \tag{82}$$

or is a total relation.

Proof. Transitivity of ρ means that

$$\forall_{P,Q,R \in \mathrm{Statements}} \quad \rho(P,Q) \wedge \rho(Q,R) \Rightarrow \rho(P,R)\,.$$

It follows that a transitive relation stronger than $\Leftrightarrow$ has the properties

$$\forall_{P,P',Q \in \mathrm{Statements}} \quad P \Leftrightarrow P' \wedge \rho(P,Q) \Rightarrow \rho(P',Q)$$

and

$$\forall_{P,Q,Q' \in \mathrm{Statements}} \quad \rho(P,Q) \wedge Q \Leftrightarrow Q' \Rightarrow \rho(P,Q')$$

which, in view of *symmetry* of relation $\Leftrightarrow$, imply the stronger properties

$$\forall_{P,P',Q \in \mathrm{Statements}} \quad P \Leftrightarrow P' \Rightarrow \big(\rho(P,Q) \Leftrightarrow \rho(P',Q)\big) \tag{83}$$

and

$$\forall_{P,Q,Q' \in \mathrm{Statements}} \quad Q \Leftrightarrow Q' \Rightarrow \big(\rho(P,Q) \Leftrightarrow \rho(P,Q')\big)\,. \tag{84}$$

It follows that ρ defines a binary operation $\cdot_\rho \in \mathrm{Op}_2\{\top, \perp\}$,

$$T_1 \cdot_\rho T_2 := \begin{cases} \top & \text{if } \rho(P_1, P_2) \text{ for any } P_1 \in T_1 \text{ and } P_2 \in T_2 \\ \perp & \text{otherwise} \end{cases}$$

where $T_1, T_2 \in \{\top, \perp\}$.

Since ρ is stronger than $\Leftrightarrow$, one has $T_1 \cdot_\rho T_2 = \top$ whenever $T_1 = T_2$. This leaves four possibilities

$\top\cdot\perp=\perp\cdot\top=\perp$ In this case ρ is equipotent to $\Leftrightarrow$.

$\top\cdot\perp=\perp\cdot\top=\top$ In this case ρ is a total relation ,

$\top\cdot\perp=\perp \wedge \perp\cdot\top=\top$ In this case ρ is equipotent to $\Rightarrow$.

$\top\cdot\perp=\top \wedge \perp\cdot\top=\perp$ In this case ρ is equipotent to $\Leftarrow$.

□

1.10 Functions of n variables viewed as $(n+1)$-ary relations

1.10.1

Given sets $X_1, \dots, X_n$ and Y, and a function of n variables

$$f : X_1, \dots, X_n \longrightarrow Y, \tag{85}$$

we can associate with it an $(n+1)$-ary relation

$$\rho_f : X_1, \dots, X_n, Y \longrightarrow \text{Statements}, \qquad x_1, \dots, x_n, y \longmapsto \text{“ } f(x_1, \dots, x_n) = y \text{ ”}. \tag{86}$$

Functions corresponding to reflexive relations

1.10.2

The $(n+1)$-ary relation ρ_f has the following property:

$$\textit{for every list of elements } x_1 \in X_1, \dots, x_n \in X_n\textit{, there exists a unique } y \in Y\textit{, such that } \rho(x_1, \dots, x_n, y). \tag{87}$$

1.10.3

Given any $(n+1)$-ary relation satisfying property (87), we can define a function (85) where $f(x_1, \dots, x_n)$ is defined to be that unique element $y \in Y$ such that

$$\rho(x_1, \dots, x_n, y).$$

Let us denote this function f_ρ.

Exercise 14 *Show that $f_\rho = f_\sigma$ if and only if ρ and σ are equipotent.*

1.11 Composing relations

1.11.1

Suppose that two relations are given,

$$\text{an } (m+1)\text{-ary relation between elements of sets } X_0, \dots, X_m ,$$

denoted σ, and

$$\text{an } (n+1)\text{-ary relation between elements of sets } X_m, \dots, X_{m+n+1} ,$$

denoted ρ. Assigning to a list $x_1, \dots, \hat{x}_m, \dots, x_{m+n+1}$ the statement

$$\textit{there exists } x_m \in X_m \textit{ such that } \sigma(x_0, \dots, x_m) \textit{ and } \rho(x_m, \dots, x_{m+n+1}) \tag{88}$$

defines an $(m+n+1)$-ary relation between elements of sets

$$X_1, \dots, \hat{X}_m, \dots, X_{m+n+1} .$$

Symbolically, statement (88) is represented

$$\exists_{x_m \in X_m} \left(\sigma(x_0, \dots, x_m) \wedge \rho(x_m, \dots, x_{m+n+1})\right) .$$

1.11.2

We call the relation defined above, the *composite of ρ and σ* and denote it $\rho \circ \sigma$.

1.12 Cartesian product $X_1 \times \cdots \times X_n$

1.12.1

Given a list of sets $X_1, \dots, X_n$, let us form its Cartesian product

$$X_1 \times \cdots \times X_n \,. \tag{89}$$

By definition, its elements are ordered n-tuples $(x_1, \dots, x_n)$ of elements $x_1 \in X_1, \dots, x_n \in X_n$.

1.12.2 The concept of an ordered n-tuple

What is an ordered n-tuple? There is not much difference between lists of length n and ordered n-tuples. When we speak of an ordered n-tuple, we always think of it being a *single* entity, while when we speak of a list of length n, we think of n separate entities.

1.12.3

To illustrate this further, the assignment

$$x, y \mapsto x + y \qquad (x, y \in \mathbf{N})$$

defines a function of 2 variables on the set of natural numbers $\mathbf{N}$, while the assignment

$$(x, y) \mapsto x + y \qquad (x, y \in \mathbf{N})$$

defines a function of a single variable on the Cartesian square $\mathbf{N} \times \mathbf{N}$ of $\mathbf{N}$. The targets of both functions are the same, namely the set of natural numbers.

1.12.4 The equality principle

The principal property built into the concept of an ordered n-tuple is the following equality principle

$$(x_1, \dots, x_m) = (y_1, \dots, y_n)$$

if and only if $m = n$ and $x_i = y_i$ for all $1 \leq i \leq n$.

1.12.5 The standard set-theoretic model of an ordered pair

The actual model of an ordered n-tuple is of little importance. It is possible to prove existence of such a model using only basic set theoretic concepts. For example, the axiom of Set Theory called Axiom of a Pair states that, for any x and y, the set $\{x, y\}$, whose elements are x and y, exists. Thus, $\{x\} = \{x, x\}$ and $\{x, y\}$ exist and therefore also the following set

$$\{\{x\}, \{x, y\}\} \tag{90}$$

exists. This set is a model of an *ordered pair*, i.e., of an ordered a 2-tuple.

Exercise 15 *Show that*

$$\big\{\{x\},\{x,y\}\big\} = \big\{\{x'\},\{x',y'\}\big\}$$

if and only if $x = x'$ *and* $y = y'$.

1.12.6

If $x \in X$ and $y \in Y$, then (90) is a *family* of subsets of $X \cup Y$, i.e., it is a subset of the power-set of $X \cup Y$,

$$\big\{\{x\},\{x,y\}\big\} \subseteq \mathscr{P}(X \cup Y)\ .$$

Accordingly, the Cartesian product $X \times Y$ is realized as the appropriate subset of the power-set of the power-set of $X \cup Y$,

$$X \times Y := \big\{\ P \in \mathscr{P}(\mathscr{P}(X \cup Y))\ \mid\ \exists_{x\in X}\, \exists_{y\in Y}\, P = \{\{x\},\{x,y\}\}\ \big\}\,,$$

which demonstrates its existence.

1.12.7

Having a model of on ordered pair, the ordered pair

$$\big((x,y),z\big)$$

becomes a model of an ordered triple and the Cartesian product

$$(X \times Y) \times Z$$

becomes a model of $X \times Y \times Z$. By induction on n, one can construct a model of an ordered n-tuple

$$(x_1,\dots,x_n)$$

and of

$$X_1 \times \cdots \times X_n\ ,$$

There are other, more convenient models.

1.12.8 An ordered n-tuple as a function

A convenient model of an ordered n-tuple $(x_1,\dots,x_n)$ is provided by a function

$$\xi : \{1,\dots,n\} \longrightarrow X_1 \cup \cdots \cup X_n \tag{91}$$

whose value at i is, for every $1 \le i \le n$, an element of X_i.

In this model, the Cartesian product $X_1 \times \cdots \times X_n$ is represented as a subset of the set of all functions (91).

1.12.9 Universal functions of n-variables

We shall say that a function

$$\tau : X_1, \dots, X_n \longrightarrow T \tag{92}$$

is a *universal* function with the domain-list $X_1, \dots, X_n$, if *every* function (5) can be produced from τ by postcomposing τ with a *unique* function $\tilde{f} : T \to Y$,

$$f = \tilde{f} \circ \tau.$$

In that case, the bijective correspondence

$$\mathrm{Funct}(X_1, \dots, X_n; Y) \longleftrightarrow \mathrm{Funct}(T, Y), \qquad f \longleftrightarrow \tilde{f}, \tag{93}$$

identifies the set of Y-valued functions of n-variables, with the domain-list $X_1, \dots, X_n$, with the set of functions of a single variable $T \to Y$.

1.12.10 The canonical function of n-variables $X_1, \dots, X_n \longrightarrow X_1 \times \cdots \times X_n$

For every list of sets $X_1, \dots, X_n$, there exists a canonical function of n-variables with that list as its domain. It assigns to an argument list $x_1, \dots, x_n$ the corresponding ordered n-tuple,

$$X_1, \dots, X_n \longrightarrow X_1 \times \cdots \times X_n, \qquad x_1, \dots, x_n \longmapsto (x_1, \dots, x_n), \tag{94}$$

1.12.11

The canonical function has the universal property defined in Section 1.12.9. Indeed,

$$f \quad \longmapsto \quad \big(\tilde{f} : X_1 \times \cdots \times X_n \to Y, \quad (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)\big)$$

is a bijective correspondence and f is produced by postcomposing function (94) with $\tilde{f}$.

1.12.12 The case of functions of zero variables

When $n = 0$, Cartesian product of the empty list of sets consists of functions from the *empty* set of natural numbers to the union of the empty family of sets. The latter, as we already know, is the empty set. In other words, Cartesian product of the empty list of sets is the set of functions

$$\emptyset^{\emptyset} = \mathrm{Funct}(\emptyset, \emptyset) = \{\mathrm{id}_{\emptyset}\}, \tag{95}$$

and that set has a unique element, namely the identity function associated with the empty set. Exponential notation $\emptyset^{\emptyset}$, cf. (8) is particularly apt in this case. We observe that foundations of Set Theory themeselves are telling us that 0^0 is well defined and equals to 1.

1.12.13 Canonical identification $\mathrm{Op}_0(Y) \longleftrightarrow \mathrm{Funct}(\emptyset^{\emptyset}, Y)$

In particular, nullary operations on a set Y, i.e., Y-valued functions of zero of variables, are canonically identified with functions $\emptyset^{\emptyset} \to Y$.

1.12.14

Every statement containing references to functions of n-variables can be now replaced by an equivalent statement containing references exclusively to functions of a single variable.

This explains why the use of the concept of a function of n-variables has practically disappeared from modern mathematical language. This is also the reason why Cartesian product is today present everywhere where normally one would be mentioning functions of n-variables: Cartesian product

$$X_1 \times \cdots \times X_n$$

is the *target* of the universal function of n-variables (94).

1.12.15 Canonical projections $(\pi_i)_{i\in\{1,\dots,n\}}$

The Cartesian product is more than just a set, it is a *mathematical structure*, like a relation or a function. One should consider the Cartesian product to consist of a set $X_1 \times \cdots \times X_n$ equipped with a list of functions

$$\pi_1, \dots, \pi_n , \tag{96}$$

called the *canonical projections*, where π_i is defined as

$$\pi_i : X_1 \times \cdots \times X_n \longrightarrow X_i, \qquad (x_1, \dots, x_n) \mapsto x_i . \tag{97}$$

Having just set $X_1 \times \cdots \times X_n$ alone would not suffice to recover the list of sets $X_1, \dots, X_n$. For example, $X_1 \times \cdots \times X_n$ is the empty set whenever at least one set X_i is empty.

1.12.16 *Naturality* of Cartesian product

Cartesian product assigns to a list of sets $X_1, \dots, X_n$ a single set $X_1 \times \cdots \times X_n$ equipped with the list of functions $\pi_1, \dots, \pi_n$. A function-list

$$X_1 \xrightarrow{f_1} X'_1 , \ \dots \ , \ X_n \xrightarrow{f_n} X'_n , \tag{98}$$

induces a function between the corresponding Cartesian-product sets

$$f_1 \times \cdots \times f_n : X_1 \times \cdots \times X_n \longrightarrow X'_1 \times \cdots \times X'_n , \qquad (x_1, \dots, x_n) \longmapsto (f_1(x_1), \dots, f_n(x_n)) . \tag{99}$$

Moreover, the assignment

$$f_1, \dots, f_n \longmapsto f_1 \times \cdots \times f_n$$

commutes with the operations of function composion.

Exercise 16 *Given a function-list*

$$X'_1 \xrightarrow{f'_1} X''_1 , \ \dots \ , \ X'_n \xrightarrow{f'_n} X''_n ,$$

show that

$$(f'_1 \times \cdots \times f'_n) \circ (f_1 \times \cdots \times f_n) = (f'_1 \circ f_1, \dots, f'_n \circ f_n) .$$

Mathematicians refer to such behavior as *naturality* of the assignment

$$X_1, \dots, X_n \longmapsto X_1 \times \cdots \times X_n .$$

1.12.17 The graph of a relation

Given a relation ρ between elements of sets $X_1, \dots, X_n$, the following subset of the Cartesian product,

$$\Gamma_\rho := \{(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n \mid \rho(x_1, \dots, x_n)\} \tag{100}$$

is guaranteed to exist by the axioms of Set Theory. This is the set of those ordered n-tuples for which statement $\rho(x_1, \dots, x_n)$ holds. One calls it the *graph* of ρ.

Exercise 17 *Let ρ and σ be two relations between elements of sets $X_1, \dots, X_n$. Show that ρ is weaker than σ if and only if*

$$\Gamma_\rho \subseteq \Gamma_\sigma .$$

1.12.18

In particular, relations ρ and σ are equipotent if and only if their graphs are equal

$$\Gamma_\rho = \Gamma_\sigma .$$

1.12.19 Correspondences

The graph of a relation provides another example of a mathematical structure. It involves the list of the following data:

- a list of sets $X_1, \dots, X_n$,
- a subset $C \subseteq X_1 \times \cdots \times X_n$.

Having just the set C alone would not suffice to recover the list of sets $X_1, \dots, X_n$.

A structure of this kind begs for a name. I propose to call it a *correspondence between elements of sets* $X_1, \dots, X_n$ or, an *n-correspondence*, in short.

1.12.20

When all sets X_i are one and the same set X, we shall speak of n-correspondences *on* X.

1.12.21 1-correspondences

In particular, 1-correspondences on X are the same as *subsets* of X.

1.12.22

In practice, we still be denoting a correspondence by the symbol denoting the subset C of $X_1 \times \cdots \times X_n$.

1.12.23 Caveat

In fact, a common practice among mathematicians is to call precisely this structure a *relation*. This approach to the concept of a relation, while being much less intuitive than the 'statements-valued function' approach, it allows one to place theory of relations entirely within the realm of Set Theory. For example, relations with a given domain (1) form a well defined set.

1.12.24

The main advantage of such a restrictive notion of a relation is that it frees a mathematician from any concerns about what is and what is not a *statement* while still being sufficient for studying the whole of Mathematics.

Indeed, given a correspondence C between elements of sets $X_1, \dots, X_n$, let $\rho_C(x_1, \dots, x_n)$ be the statement

$$(x_1, \dots, x_n) \in C .$$

This defines a relation between elements of sets $X_1, \dots, X_n$.

Exercise 18 *Show that any relation ρ is equipotent to the relation ρ_{Γ_ρ}.*

Exercise 19 *Show that, for any correspondence C, one has $C = \Gamma_{\rho_C}$.*

1.12.25

We shall express the operations on relations, introduced in Sections 1.6.5–1.7, in terms of their graph correspondences. For this we need to introduce some notation.

Exercise 20 *Given a relation ρ, show that*

$$\Gamma_{\neg\rho} = \complement\Gamma_\rho . \tag{101}$$

Exercise 21 *Given relations ρ and σ with the same domain, show that*

$$\Gamma_{\rho\vee\sigma} = \Gamma_\rho \cup \Gamma_\sigma \qquad \textit{and} \qquad \Gamma_{\rho\wedge\sigma} = \Gamma_\rho \cap \Gamma_\sigma . \tag{102}$$

1.12.26

The above two exercises demonstrate that the operations of negation, alternative and conjunction of relations translate into the operations of taking the complement, the union, and the intersection, of correspondences.

Exercise 22 *Given relations ρ and σ with the same domain, show that*

$$\Gamma_{\rho\Rightarrow\sigma} = \complement\Gamma_\rho \cup \Gamma_\sigma . \tag{103}$$

1.12.27 The function-list canonically associated with an n-correspondence

By post-composing the canonical inclusion $\iota : C \hookrightarrow X_1 \times \cdots \times X_n$ with the list of canonical projections $\pi_1, \dots, \pi_n$, we obtain a list of functions

$$\begin{array}{c} C \\ {\scriptstyle \partial_1}\big\downarrow \cdots \big\downarrow{\scriptstyle \partial_n} \\ X_1, \dots, X_n \end{array} \tag{104}$$

that is canonically associated with the correspondence. Here $\partial_i := \pi_i \circ \iota$, $1 \le i \le n$.

1.12.28 Oriented graphs

When $n = 2$ and X_1 and X_2 are the same set X, a list (104) is called an *oriented graph*. Elements of X are referred to, in this case, as *vertices* and elements of C are referred as *oriented edges*, or *arrows*, of the graph.

1.12.29 2-Correspondences as oriented graphs

In particular, 2-correspondences on a set X can be viewed as oriented graphs with vertices being elements of X, such that no two oriented edges have the same source and the same target.

1.13 The language of diagrams

1.13.1

Situations involving several functions are frequently expressed and analyzed in the language of oriented graphs, represented visually as diagrams drawn on a blackboard, or on a page. Arrows in a diagram represent functions. Vertices represent their domains and targets. *Oriented paths* in such graphs represent composable lists of functions.

1.13.2 Commutative diagrams

When composition of two paths with the same origin and the same terminus produces the same result, we call such a diagram *commutative*. Most common examples of commutative diagrams have a form of a *commutative sqaure*,

$$\begin{tikzcd} X \arrow[d, "\alpha"'] & S \arrow[l, "\beta"'] \arrow[d, "\delta"] \arrow[dl, phantom, "\circlearrowleft"] \\ T & Y \arrow[l, "\gamma"] \end{tikzcd} . \tag{105}$$

Commutativity of square diagram (105) expresses the equality

$$\alpha \circ \beta = \gamma \circ \delta .$$

1.13.3

Commutativity of a diagram is often signaled by placing a symbol $\circlearrowleft$, or its cousins: $\curvearrowleft$, $\circlearrowright$, or $\circlearrowleft$, between two composable paths of arrows originating and terminating in a common vertex.

1.13.4

Diagrams are employed not only to illustrate situations that can be discussed without introducing diagrams. It has been long observed that employing diagrams can greatly clarify and enhance analysis of complex scenarios. We shall illustrate it here by considering one example. Later in these notes you will see many more appearances of commutative diagrams.

1.13.5 An example

Consider a commutative diagram

$$\begin{tikzcd} Z_1 & Y_1 \arrow[l, "\varphi_1"'] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 \arrow[from=1-1, "\alpha"'] & Y_2 \arrow[l, "\varphi_2"] \arrow[u, phantom, "\circlearrowleft"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \tag{106}$$

We do not know whether it is possible to complete diagram (106) to a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} , \tag{107}$$

we observe, however, that diagram (106) defines in a canonical manner a binary relation between elements of Y_1 and Y_2,

$$\rho : Y_1, Y_2 \longrightarrow \text{Statements}, \qquad y_1, y_2 \longmapsto \text{“} \; \exists_{x_1 \in X_1} \, y_1 = \chi_1(x_1) \wedge y_2 = (\chi_2 \circ \gamma)(x_1) \; \text{”} . \tag{108}$$

1.13.6

It is clear that

$$\forall_{y_1 \in Y_1} \exists_{y_2 \in Y_2} \rho(y_1, y_2)$$

if and only if function χ_1 is surjective.

1.13.7

Let $y_2, y_2' \in Y_2$ be two elements in relation with a given element $y_1 \in Y_1$. Then, there are elements $x_1, x_1' \in X_1$ such that

$$y_1 = \chi_1(x_1) = \chi_1(x_1'), \qquad y_2 = (\chi_2 \circ \gamma)(x_1) \qquad \text{and} \qquad y_2' = (\chi_2 \circ \gamma)(x_1') .$$

By combining this with commutativity of diagram (106) we obtain a chain of equalities

$$\varphi_2(y_2) = (\varphi_2 \circ \chi_2 \circ \gamma)(x_1) = (\alpha \circ \varphi_1)(\chi_1(x_1)) = (\alpha \circ \varphi_1)(\chi_1(x_1')) = (\varphi_2 \circ \chi_2 \circ \gamma)(x_1') = \varphi_2(y_2') .$$

If φ_2 is injective, then $y_2 = y_2'$ and relation (108) defines a function

$$\beta : Y_1 \longrightarrow Y_2, \qquad y_1 \longmapsto \text{the unique } y_2 \in Y_2 \text{ such that } \rho(y_1, y_2) .$$

1.13.8 Diagram chasing

The method we used to construct relation (108) and then to prove that under suitable hypotheses (108) defines a function, is referred to as *diagram chasing*.

1.13.9

Let us represent surjective functions by two-headed arrows $\twoheadrightarrow$ and injective functions by tailed arrows $\rightarrowtail$. We established the following fact.

Lemma 1.6 *Every commutative diagram*

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, phantom, "\circlearrowleft"] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\
Z_2 & Y_2 \arrow[l, tail, "\varphi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd} . \tag{109}$$

admits a completion to a commutative diagram (107)*. Moreover, function* β *that makes diagram* (107) *commutative is unique.* □

Exercise 23 *Prove uniqueness of* β*.*

1.13.10

Consider now an arbitrary commutative diagram

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] & X_1 \arrow[l, "\chi_1"'] \\
Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd} . \tag{110}$$

If arrow χ_2 admits a right-inverse $\xi : Y_2 \to X_2$, then

$$\gamma := \xi \circ \beta \circ \chi_1$$

obviously makes the diagram

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\
Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd}$$

commute.

1.13.11

We can sum our discussion up in the following lemma.

Lemma 1.7 *Consider a diagram*

$$\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] & X_1 \arrow[l, two heads, "\chi_1"'] \\
Z_2 & Y_2 \arrow[l, tail, "\varphi_2"] & X_2 \arrow[l, two heads, "\chi_2"]
\end{tikzcd} \tag{111}$$

The following three properties of diagram (111) *are equivalent:*

(a) *it admits a completion to a commutative diagram* (106)*;*

(b) *it admits a completion to a commutative diagram* (110)*.*

(c) *it admits a completion to a commutative diagram*

$$
\begin{tikzcd}
Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dl, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"]
\end{tikzcd} . \tag{112}
$$

Implications (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) rely on Axiom of Choice, cf. Section 1.15.15, or one has to add the hypothesis to the effect that χ_2 has a right-inverse (such functions are said to be *split surjections*).

1.13.12 $\sim$-commutative diagrams

A $\sim$*-commutative diagram* is a slight yet a very significant generalization of a commutative diagram, cf. 1.13.2. Whole areas of advanced modern Mathematics and Mathematical Physics are devoted to studying phenomena expressed in the language of $\sim$-commutative diagrams.

Commutativity of a diagram means that two composable paths of arrows (representating functions between sets), that have a common source and a common target, are equal. If that common target, call it T, is equipped with a binary relation $\sim$, then *equality* may be replaced by the condition that the corresponding composite functions satisfy the relation induced by $\sim$ on the set of T-valued functions.

Since $\sim$ is not necessarily symmetric, one needs to indicate which composite function appears as the *left* argument and which appears as the *right* argument of the relation in question.

This can be represented in a diagram by placing a small arrow (ideally, a bent arrow) near the common target of two composable paths of arrows, as is shown in the following simple example. A square-shaped diagram

$$
\begin{tikzcd}
X \arrow[d, "\upsilon"'] & S \arrow[l, "\varphi"'] \arrow[d, "\psi"] \arrow[dl, phantom, "\overset{\sim}{\curvearrowleft}"] \\
T & Y \arrow[l, "\chi"]
\end{tikzcd}
$$

expresses the statement

$$\forall_{s \in S}\, \chi(\psi(s)) \sim \upsilon(\varphi(s)) ,$$

i.e., the composite arrow $\chi \circ \psi$ is in relation, induced by $\sim$, with the composite arrow $\upsilon \circ \varphi$.

When the binary relation on the target is clear from the context, the label ($\sim$ here) may be omitted.

1.14 Power-set functions induced by a function $f : X \to Y$

1.14.1 The *image-of-a-subset* and the *preimage-of-a-subset* functions f_* and f^*

Given a function $f : X \longrightarrow Y$, there are two associated functions between the power-sets

$$\mathscr{P}X \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathscr{P}Y \ , \tag{113}$$

where the associated *image* function is defined by

$$f_*(A) := \{y \in Y \mid \exists_{x \in A}\, f(x) = y\} \qquad (A \subseteq X) \tag{114}$$

and the associated *preimage* function is defined by

$$f^*(B) := \{x \in X \mid \exists_{y \in B}\, f(x) = y\} \qquad (B \subseteq Y). \tag{115}$$

1.14.2

Function (114) is a single-variable case of the direct-image function f_* introduced in Section 1.7.8 and associated with an arbitrary function f of n variables.

1.14.3 A comment about notation

What I here denote by $f_*(A)$ and $f^*(B)$ is usually denoted $f(A)$ and $f^{-1}(B)$. This is all right as long as there is no need to consider the assignments

$$A \mapsto f(A) \qquad \text{and} \qquad B \mapsto f^{-1}(B)$$

as functions between the corresponding power-sets. When such a need arises, one needs an appropriate notation to denote the image and the preimage functions associated with f. This is why I adopted the *lower-* and the *upper-star* notation that is universally used in Modern Mathematics to denote all sorts of functions that are naturally associated with a given function.

1.14.4

This has yet another advantage: it often allows us to skip parentheses around the arguments of functions f_* and f^* in the interest of keeping notation as simple as possible, without affecting the intended meaning. Thus, we shall, generally, write f_*A and f^*B instead of $f_*(A)$ and $f^*(B)$.

1.14.5

I will say later why in some cases we mark the associated function by placing $*$ as a *subscript* while in other cases—as a *superscript*.

1.14.6 The *fiber* of a function $f : X \to Y$ at $y \in Y$

The preimage f^*B of a singleton subset $B = \{y\}$ is referred to as the *fiber of f at y*. It is usually denoted $f^{-1}y$ or $f^{-1}(y)$. We shall denote it $f^*\{y\}$.

1.14.7 Caveat

One must be careful not to confuse notation $f^{-1}(y)$, when it is used to denote the *fiber* of f at y, with notation $f^{-1}(y)$ used to denote the *value* of the *inverse* function. The inverse function, denoted f^{-1}, is defined only when f is invertible. In that case, the fiber of f at $y \in Y$ is given by

$$f^*\{y\} = \{f^{-1}(y)\}.$$

1.14.8 The characteristic function of a subset

Given a subset $A \subset X$, its *characteristic function* is defined by

$$\chi_A : X \to \mathbf{F}_2, \qquad \chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}, \tag{116}$$

where $\mathbf{F}_2 = \{0,1\}$ denotes the 2-element field.

Assignment

$$A \longmapsto \chi_A$$

yields a canonical identification

$$\chi : \mathscr{P}X \longleftrightarrow \mathrm{Funct}(X, \mathbf{F}_2). \tag{117}$$

Exercise 24 *Prove that, given a function $f : X \to Y$ and a subset $B \subset Y$, one has*

$$f^\bullet \chi_B = \chi_{f^*B}. \tag{118}$$

In other words, the preimage function $f^* : \mathscr{P}Y \to \mathscr{P}X$ can be viewed also as the precomposition function

$$f^\bullet : \mathrm{Funct}(Y, \mathbf{F}_2) \longrightarrow \mathrm{Funct}(X, \mathbf{F}_2).$$

1.14.9

Identity (118) can be also expressed by saying that the following square diagram of functions

$$\begin{tikzcd} \mathscr{P}X \arrow[r, "\chi"] & \mathrm{Funct}(X, \mathbf{F}_2) \\ \mathscr{P}Y \arrow[u, "f^*"] \arrow[r, "\chi"'] & \mathrm{Funct}(Y, \mathbf{F}_2) \arrow[u, "f^\bullet"'] \end{tikzcd}$$

commutes.

1.14.10

Note how close the definitions of the image and of the preimage are to each other: they are both defined by *existential* quantification of the *same* binary relation

$$X, Y \longrightarrow \text{Statements}, \qquad x, y \longmapsto \rho(x,y) := \text{“ } f(x) = y \text{ ”} \tag{119}$$

over the corresponding subsets $A \subseteq X$ and $B \subseteq Y$, respectively. We often refer to f_* as the *direct image map* and to f^* as the *inverse image map.*

1.14.11 Comments about the usual "definitions" of the image and the preimage functions.

The image function is usually "defined" by

$$f_*A := \{f(x) \mid x \in A\}\,. \tag{120}$$

This should be considered only as an *informal definition* since it violates the requirement that brace notation we use to define a subset of Y *must* be of the form

$$\{y \in Y \mid \rho(y)\}$$

where ρ is a unary relation on Y. Additionally, note the equality of sets

$$f^*B = \{x \in X \mid f(x) \in B\}\,. \tag{121}$$

The right-hand-side of (121) is how the inverse image is usually defined. Such a definition, however, obfuscates the fact that f^* is a "twin sister of f_*".

1.14.12 The *conjugate image* function $f_!$

These two concepts or, if you wish, constructions, naturally associated with every function $f : X \longrightarrow Y$, are omnipresent. One encounters them nearly in every mathematical argument involving functions between sets. What remains a very little known fact is that f^* has yet another "sibling,"

$$f_! : \mathscr{P}(X) \longrightarrow \mathscr{P}(Y), \qquad A \longmapsto \complement f_*(\complement A)\,, \tag{122}$$

that I propose to call the *conjugate image* function.

The name, "conjugate image" stems from the fact that $f_!$ is the *conjugate* of f_* by the *complement operation*,

$$f_! = \complement \circ f_* \circ \complement. \tag{123}$$

Caveat: the *inner* complement operation is applied to a subset of X whereas the *outer* complement operation is applied to a subset of Y. When fully expanded the value of $f_!$ on a subset A of X equals

$$f_!A = Y \setminus f_*(X \setminus A).$$

Exercise 25 *Show that*

$$f_!A = \{y \in Y \mid \forall_{x\in X}\, f(x){=}y \Rightarrow x{\in}A\}\,. \tag{124}$$

Exercise 26 *Let $A \subseteq X$ and $B \subseteq Y$. Show that*

$$A \subseteq f^*B \quad \textit{if and only if} \quad f_*A \subseteq B\,. \tag{125}$$

1.14.13

Exercise 26 expresses the fact that f_*, f^* form what in the language of ordered sets is known as a *Galois connection*, cf. Section **??**.

Exercise 27 *Show that*

$$f^* \circ f_* = \mathrm{id}_{\mathscr{P}X} \quad \textit{if and only if} \quad f \ \textit{is injective}. \tag{126}$$

Exercise 28 *Show that*

$$f_* \circ f^* = \mathrm{id}_{\mathscr{P}Y} \quad \textit{if and only if} \quad f \ \textit{is surjective}. \tag{127}$$

Exercise 29 *Show that*

$$f^*(\complement B) = \complement(f^* B). \tag{128}$$

1.14.14

Identities (123) and (128) can be expressed by a pair of commutative square diagrams

$$\begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{P}X \arrow[d, "f_!"] \\ \mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] & \mathscr{P}Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{P}X \\ \mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] \arrow[u, "f^*"] & \mathscr{P}Y \arrow[u, "f^*"'] \end{tikzcd} \tag{129}$$

that can be combined into a single diagram

$$\begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[d, bend right, "f_*"'] \arrow[dr, phantom, "\circlearrowleft\circlearrowright"] & \mathscr{P}X \arrow[d, bend left, "f_!"] \\ \mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] \arrow[u, "f^*"'] & \mathscr{P}Y \arrow[u, "f^*"] \end{tikzcd} \tag{130}$$

in which both squares commute.

1.14.15

I used two different circle-arrows to make you aware that in the left diagram in (129), the composite arrows have their source at one of the *top* vertices and their target in the diagonally opposite *bottom* vertex. In the right diagram in (129) the roles are reversed: the composite arrows have their source at one of the *bottom* vertices and their target in the diagonally opposite *top* vertex.

Normally, I will be marking commutativity of any (portion of a) diagram by using the circle-arrow symbol that I consider the most appropriate.

Exercise 30 *Show that*

$$f^* B \subseteq A \quad \textit{if and only if} \quad B \subseteq f_! A\,. \tag{131}$$

1.14.16

Exercise 30 expresses the fact that $f^*, f_!$ form what in the language of ordered sets is known as a *Galois connection.*

Exercise 31 *Given an* n *-ary relation* ρ *between elements of sets* $X_1, \dots, X_n$*, let* ρ_i *be the* $(n-1)$ *-ary relation between elements of sets* $X_1, \dots, \hat{X}_i, \dots, X_n$ *defined in Section 1.7.6. Show that*

$$\Gamma_{\rho_i} = (\pi_{\hat{i}})_* \Gamma_\rho \tag{132}$$

where

$$\pi_{\hat{i}} : X_1 \times \dots \times X_n \longrightarrow X_1 \times \dots \times \hat{X}_i \times \dots \times X_n \tag{133}$$

removes from an ordered n *-tuple its* i *-th component,*

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, \hat{x}_i, \dots, x_n).$$

Exercise 32 *Let* $g : Y \to Z$ *be a function. Show that*

$$(g \circ f)_* = g_* \circ f_*, \qquad (g \circ f)^* = f^* \circ g^* \qquad \textit{and} \qquad (g \circ f)_! = g_! \circ f_!. \tag{134}$$

Exercise 33 *Show that all three functions*

$$(id_X)_*, \qquad (id_X)^* \qquad \textit{and} \qquad (id_X)_!, \tag{135}$$

are equal to the identity function $\mathrm{id}_{\mathscr{P}X}$ *of power-set* $\mathscr{P}X$.

An immediate consequence of identities (134) and (135) is that, for every invertible function f, one has

$$(f^{-1})_* = (f_*)^{-1}. \tag{136}$$

Exercise 34 *Show that, for an invertible function* f*, one has*

$$f^* = (f^{-1})_*.$$

Exercise 35 *Let* ρ^i *be the* $(n-1)$ *-ary relation defined in Section 1.7.1. Show that*

$$\Gamma_{\rho^i} = (\pi_{\hat{i}})_! \Gamma_\rho. \tag{137}$$

1.14.17 Pull-back of a relation

Given a function-list (98), we refer to the associated precomposition functions

$$(f_1, \dots, f_n)^\bullet : \mathrm{Rel}\left(X'_1, \dots, X'_n\right) \longrightarrow \mathrm{Rel}(X_1, \dots, X_n), \qquad \rho' \longmapsto (f_1, \dots, f_n)^\bullet \rho'. \tag{138}$$

as the *pull-back* functions.

Exercise 36 *Show that*

$$\Gamma_{(f_1, \dots, f_n)^\bullet \rho} = (f_1 \times \dots \times f_n)^* \Gamma_{\rho'}. \tag{139}$$

Exercise 37 *Let* $f : X \to X'$ *be a function and* ρ' *be a binary relation on* X'*. Which properties of* ρ'*, from the list given in Section 1.8.4, are inherited by* $(f, f)^\bullet \rho'$ *?*

1.14.18 Push-forward of a relation

We define the *push-forward* functions

$$(f_1, \dots, f_n)_\sharp : \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}\left(X'_1, \dots, X'_n\right), \qquad \rho \longmapsto (f_1, \dots, f_n)_\sharp \rho, \tag{140}$$

where $(f_1, \dots, f_n)_\sharp \rho$ is the relation

$$x'_1, \dots, x'_n \longmapsto \text{“ } \exists_{x_1 \in X_1, \dots, x_n \in X_n} \ \rho(x_1, \dots, x_n) \wedge f_1(x_1) = x'_1 \wedge \dots \wedge f_n(x_n) = x'_n \text{ ”}. \tag{141}$$

Exercise 38 *Show that*

$$\Gamma_{(f_1, \dots, f_n)_\sharp \rho} = (f_1 \times \dots \times f_n)_* \Gamma_\rho. \tag{142}$$

1.14.19

The analog of Identity (142),

$$\Gamma_{(f_1, \dots, f_n)_\natural \rho} = (f_1 \times \dots \times f_n)_! \Gamma_\rho, \tag{143}$$

exists also for the *conjugate* direct image function and *conjugate push-forward* functions

$$(f_1, \dots, f_n)_\natural : \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}\left(X'_1, \dots, X'_n\right), \qquad \rho \longmapsto (f_1, \dots, f_n)_\natural \rho, \tag{144}$$

where $(f_1, \dots, f_n)_\natural \rho$ is the relation

$$x'_1, \dots, x'_n \longmapsto \text{“ } \forall_{x_1 \in X_1, \dots, x_n \in X_n} \ \left(f_1(x_1) = x'_1 \wedge \dots \wedge f_n(x_n) = x'_n \right) \Rightarrow \rho(x_1, \dots, x_n) \text{ ”}. \tag{145}$$

Exercise 39 *Prove Identity* (143).

1.15 Families of sets

1.15.1

A *family of sets* is, by definition, a set whose elements are themselves sets. In a restrictive approach to Set Theory every set is requiered to be of this form. It is possible to develop all of Mathematics within such a restrictive framework.

1.15.2 Notation

A general practice is to denote *elements* of sets by lower case Latin alphabet letters:

$$a, b, c, d, e, f, g, h, i, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, x, z,$$

and to denote *sets* by capital letters:

$$A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, X, Z.$$

1.15.3 Boldface notation

Certain particularly common sets are denoted by upright boldface letters in order to make them stand out wherever they appear. Thus, $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, became standard notation for the sets of natural numbers, of integers, of rational numbers, of real numbers and, respectively, of complex numbers.

A bad habit that infected publishing practice like a noxious virus and that **should not be followed**, is to use in printed texts in place of those boldface letters their "blackboard" equivalents: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

1.15.4 Families of sets

A set whose elements are sets is often referred to as a *family of sets*. We shall denote families of sets by capital calligraphic letters:

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}, \mathscr{K}, \mathscr{L}, \mathscr{M}, \mathscr{O}, \mathscr{P}, \mathscr{Q}, \mathscr{R}, \mathscr{S}, \mathscr{T}, \mathscr{U}, \mathscr{V}, \mathscr{W}, \mathscr{X}, \mathscr{Y}, \mathscr{Z}.$$

1.15.5 The union of a family of subsets of a set

Given a family of subsets $\mathscr{A}$ of a set X, the *union of* $\mathscr{A}$ is the set

$$\bigcup \mathscr{A} \;:=\; \{x \in X \mid \exists_{A \in \mathscr{A}}\, x \in A\}\,. \tag{146}$$

The existence of such a set is guaranteed by the axioms of Set Theory. It is the *smallest* subset of X *containing* each member set $A \in \mathscr{A}$. An alternative notation:

$$\bigcup_{x \in \mathscr{A}} A\,. \tag{147}$$

1.15.6 The intersection of a family of subsets of a set

The set

$$\bigcap \mathscr{A} \;:=\; \{x \in X \mid \forall_{A \in \mathscr{A}}\, x \in A\} \tag{148}$$

is called the *intersection* of (family) $\mathscr{A}$. It is the *greatest* subset of X *contained in* each member set $A \in \mathscr{A}$. An alternative notation

$$\bigcap_{x \in X} A\,. \tag{149}$$

1.15.7

Union and intersection define two canonical functions

$$\mathscr{P}X \overset{\cup}{\underset{\cap}{\longleftarrow}} \mathscr{P}\mathscr{P}X\,. \tag{150}$$

Exercise 40 *Let* $\mathscr{A} \subseteq \mathscr{B}$ *(we say, in this case, that* $\mathscr{A}$ *is a* subfamily *of* $\mathscr{B}$*. Show that*

$$\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B} \qquad \text{and} \qquad \bigcap \mathscr{A} \supseteq \bigcap \mathscr{B}. \tag{151}$$

1.15.8 Union and intersection of the *empty* family of subsets

If $\mathcal{A} = \{A\}$ consists of a single set A, then

$$\bigcup \mathcal{A} = A = \bigcap \mathcal{A}.$$

Since the empty family $\emptyset$ of subsets of X is contained in every family of subsets, in particular in the singleton family $\{\emptyset\}$, the union of the empty family is contained in set $\emptyset$,

$$\bigcup \emptyset \subseteq \bigcup \{\emptyset\} = \emptyset,$$

hence it is the empty set.

Since the empty family $\emptyset$ of subsets of X is contained in the singleton family $\{X\}$, the intersection of the empty family of subsets of X contains set X,

$$\bigcap \emptyset \supseteq \bigcap \{X\} = X,$$

hence it equals X.

1.15.9

The above argument demonstrates that the union of the empty family of subsets of X is the empty set independently of what set X is.

On the other hand, the intersection of the empty family of subsets of X equals X, hence it *does* depend on X.

1.15.10 Selectors of a family

A function $\xi : \mathcal{X} \longrightarrow \bigcup \mathcal{X}$ satisfying the property

$$\forall_{X \in \mathcal{X}}\ \xi(X) \in X \tag{152}$$

is called a *selector* or a *choice function* of family $\mathcal{X}$.

1.15.11 A comment about the use of the quantifier notation

Mathematicians, unless they are logicians or axiomatic-set-theorists, prefer to limit the use of the quantifier symbols in their formulae to those rare occasions when their use clarifies, not obfuscates, the meaning. The reason is partly a reflection of their habits, partly is related to the physiology of human brain perception of abstract symbolic expressions. The defining property of a selector (152) can be also written as:

$$\xi(X) \in X \quad \textit{for every} \quad X \in \mathcal{X}. \tag{153}$$

or, more tersely,

$$\xi(X) \in X \qquad (X \in \mathcal{X}). \tag{154}$$

Each expression (152)–(154) carries exactly the same meaning and can be read in the same way. From now on you will be frequently exposed to notation (154) that eliminates the need to use quantifier symbols in phrases involving only universal quantifiers.

1.15.12 Axiom of Choice

For obvious reasons, no selector exists if family $\mathscr{X}$ contains the empty set $\emptyset$. It is not obvious, however, that a selector exists *for every* family of nonempty sets. *Axiom of Choice* states just that. That statment was proven to be independent of other axioms of Set Theory. Some mathematicians do not accept it automatically while all mathematicians are, generally, cautious when they are forced to use it. Much of Mathematics can be developed without assuming its validity.

1.15.13 The product of a family of sets

The set of all selectors of family $\mathscr{X}$ forms the set

$$\prod \mathscr{X}, \qquad \text{alternately denoted} \qquad \prod_{X \in \mathscr{X}} X, \tag{155}$$

which is called the *product* of (family) $\mathscr{X}$.

1.15.14

Axiom of Choice says:

The product of a family of nonempty sets is nonempty. (156)

1.15.15 An equivalent form of Axiom of Choice

Every surjective function $f : \mathrm{X} \to Y$ *is right-invertible.* (157)

1.15.16 Independence of Axiom of Choice

It was established long ago that Axiom of Choice is consistent with the remaining axioms of Set Theory. This means that if there are contradictory statements in Mathematics provable with the aid of Axiom of Choice, then there are contradictory statements provable without Axiom of Choice.

It took much longer to resolve the open question whether Axiom of Choice is, or is not, a consequence of the remaining axioms of Set Theory. This was finally resolved by a brilliant mathematician, Paul Cohen, whose demonstrated strength was in Harmonic and Functional Analysis, not in Set Theory or Mathematical Logic. He proved that Axiom of Choice is *not* a consequence of axioms of Set Theory. Statements in Mathematics that are consistent but not provable are said to be *independent* of axioms of Set Theory.

1.16 Canonical functions between the sets-of-families

1.16.1

As we saw in Sections 1.14.1 and 1.14.12, every function $f : X \to Y$ induces three functions between the corresponding power-sets

$$
\begin{tikzcd}
\mathscr{P}Y \arrow[d, "f^*"] \\
\mathscr{P}X \arrow[u, bend left=60, "f_*"] \arrow[u, bend right=60, "f_!"']
\end{tikzcd}
\quad . \tag{158}
$$

Families of subsets of X are elements of the power-set-of-the-power-set $\mathscr{P}\mathscr{P}X$ and similary for families of subsets of Y. In particular, each of the three functions in diagram (158) induces three functions between the corresponding sets of families of subsets:

$$
\begin{array}{ccc}
(f_*)_* & (f_*)^* & (f_*)_! \\
(f^*)_* & (f^*)^* & (f^*)_! \\
(f_!)_* & (f_!)^* & (f_!)_! \, .
\end{array} \tag{159}
$$

One can omit parentheses provided one carefully observes the spacing that distinguishes between, e.g., $f_*{}^*$ and $f^*{}_*$.

$$
\begin{array}{ccc}
f_{**} & f_*{}^* & f_{*!} \\
f^*{}_* & f^{**} & f^*{}_! \\
f_{!*} & f_!{}^* & f_{!!}
\end{array} \quad . \tag{160}
$$

Exercise 41 *Find all functions in diagram* (160) *that are functions from $\mathscr{P}\mathscr{P}X$ to $\mathscr{P}\mathscr{P}Y$.*

1.16.2

Of these nine canonical functions between sets of families of subsets, four play an important role in Topology, Measure Theory, Mathematical Analysis, where families of subsets are essential objects of study.

1.16.3

Let $\mathscr{A} \subset \mathscr{P}X$ be a family of subsets of X, let $\mathscr{B} \subset \mathscr{P}Y$ be a family of subsets of Y.

Exercise 42 *Show that*

$$
f_*\left(\bigcup \mathscr{A}\right) = \bigcup f_{**}\mathscr{A} \qquad \textit{and} \qquad f^*\left(\bigcup \mathscr{B}\right) = \bigcup f^*{}_*\mathscr{B} \tag{161}
$$

and express each identity in the form of a commutative diagram.

Exercise 43 *Show that*

$$f^*\left(\bigcap \mathscr{B}\right) = \bigcap f^*{}_*\mathscr{B} \qquad \textit{and} \qquad f_!\left(\bigcap \mathscr{A}\right) = \bigcap f_{!*}\mathscr{A} \tag{162}$$

and express each identity in the form of a commutative diagram.[1]

Exercise 44 *Show that*

$$f_*\left(\bigcap \mathscr{A}\right) \subseteq \bigcap f_{**}\mathscr{A} \qquad \textit{and} \qquad f_!\left(\bigcup \mathscr{A}\right) \supseteq \bigcup f_{!*}\mathscr{A} \,. \tag{163}$$

In general, $\subseteq$ cannot be replaced by $=$ in (163).

1.17 Indexed families of sets

1.17.1

An indexed family of sets $(X_i)_{i\in I}$ is, by definition, a function from a certain set I to the power-set of a certain set U,

$$I \longrightarrow \mathscr{P}(U)\,, \qquad i \mapsto X_i\,.$$

The standard notation for the value at $i \in I$ is X_i. The set I is referred to as the *indexing set*.

1.17.2 The union and the intersection of an indexed family

Let us denote by $\mathscr{X}$ the *image* of this function in $\mathscr{P}(U)$. It is a family of sets. The union and the intersection of $\mathscr{X}$ are called, respectively, the *union* and the *intersection* of $(X_i)_{i\in I}$, and denoted

$$\bigcup_{i\in I} X_i \qquad \text{and} \qquad \bigcap_{i\in I} X_i\,.$$

Explicitly,

$$\bigcup_{i\in I} X_i \;:=\; \{x \mid \exists_{i\in I}\; x \in X_i\} \tag{164}$$

and

$$\bigcap_{i\in I} X_i \;:=\; \{x \mid \forall_{i\in I}\; x \in X_i\}\,. \tag{165}$$

1.17.3

When the indexing set I is empty, the comments made about the union and the intersection of an empty family of subsets apply, cf. 1.15.9.

[1] A hint for both exercises: recall that $\bigcup$ and $\bigcap$ define certain canonical functions, cf. (150).

1.17.4 Selectors of an indexed family

Functions

$$I \longrightarrow \bigcup_{i\in I} X_i\,, \qquad i \mapsto x_i\,, \tag{166}$$

satisfying

$$x_i \in X_i \qquad (i \in I)\,,$$

could be called *selectors* of indexed family $(X_i)_{i\in I}$. They are more frequently called I*-tuples* because in the case

$$I = \{1, \dots, n\}\,,$$

they correspond to ordered n-tuples of elements of $\bigcup_{i\in I} X_i$.

1.17.5 "Tuple" notation

Standard notation for an I-tuple is $(x_i)_{i\in I}$. The subscript $i \in I$ is usually omitted when the indexind set is understood from the context.

1.17.6 The product of an indexed family of sets

Predictably, the set of all I-tuples of $(X_i)_{i\in I}$ is called the *product* of $(X_i)_{i\in I}$ and is denoted

$$\prod_{i\in I} X_i\,. \tag{167}$$

1.17.7

For $I = \{1, 2\}$, the product is naturally identified with the Cartesian product

$$X_1 \times X_2\,,$$

and, for $I = \{1, \dots, n\}$, it provides the most convenient model of the Cartesian product

$$X_1 \times \cdots \times X_n\,.$$

1.17.8 Canonical projections (π_J)

Restricting a function (166) to a subset $J \subseteq I$ defines a function

$$\pi_J : \prod_{i\in I} X_i \longrightarrow \prod_{i\in J} X_i\,, \tag{168}$$

called the *canonical projection* (associated with a subset J of the indexing set. We have encountered these functions in Section 1.12.15 where $I = \{1, \dots n\}$ and $J = \{i\}$.

1.17.9 Notation

In the interest of keeping notation simple, when, e.g., $J = \{2, 5, 7\}$, we write

$$\pi_{2,5,7} \quad \text{instead} \quad \pi_{\{2,5,7\}}$$

or, even, as

$$\pi_{257}$$

when it it is clear from the context that the elements of J are natural numbers less than 10.

A general rule is to separate the items in a list of subscripts or superscripts by commas when notation is, otherwise, ambiguous, and to omit the commas when no ambiguity arises.

1.17.10 Composition of correspondences

Given correspondences

$$C \subseteq X_0 \times \cdots \times X_{m+1} \qquad \text{and} \qquad D \subseteq X_{m+1} \times \cdots \times X_{m+n+1} \, ,$$

their preimages under the canonical projections

$$\pi^*_{0,\dots,m+1} C \qquad \text{and} \qquad \pi^*_{m+1,\dots,m+n+1} D$$

are correspondences between elements of sets

$$X_0, \dots, X_{m+n+1} \, .$$

In particular, we can form their intersection

$$\pi^*_{0,\dots,m+1} C \, \cap \, \pi^*_{m+1,\dots,m+n+1} D$$

and project it into $X_0 \times \cdots \times \hat{X}_{m+1} \times \cdots \times X_{m+n+1}$,

$$(\pi_{\widehat{m+1}})_* (\pi^*_{0,\dots,m+1} C \, \cap \, \pi^*_{m+1,\dots,m+n+1} D) \, , \tag{169}$$

where

$$\pi_{\widehat{m+1}} \; = \; \pi_{0,\dots,\widehat{m+1},\dots,m+n+1} \, .$$

We shall denote (169) by $C \circ D$.

1.17.11

Explicitly, $C \circ D$ consists of $(m + n + 1)$-tuples

$$(x_0, \dots, \hat{x}_{m+1}, \dots, x_{m+n+1})$$

for which there exists $x_{m+1} \in X_{m+1}$ such that

$$(x_0, \dots, x_{m+1}) \in C \qquad \text{and} \qquad (x_{m+1}, \dots, x_{m+n+1}) \in D \, .$$

1.17.12

It follows that for $C = \Gamma_\rho$ and $D = \Gamma_\sigma$, one has

$$\Gamma_{\rho \circ \sigma} = \Gamma_\rho \circ \Gamma_\sigma \, . \tag{170}$$

2 The language of mathematical structures

2.1 Mathematical structures

2.1.1 The concept of a mathematical structure

A list of sets

$$X_1, \ldots, X_n \tag{171}$$

equipped with some 'data' is what a mathematical structure is. As such, a mathematical structure can be thought of as an ordered pair

$$(X_1, \ldots, X_n\,;\ \text{'data'}) \tag{172}$$

2.1.2

This sinple concept became a focal point of modern Mathematics because it allows to view many apparently distant phenomena as manifestations of the same general laws.

2.1.3

Functions, operations, relations, are obvious examples of mathematical structures.

2.1.4 Structures of functional type

Sets X equipped with a family $\mathcal{O} \subset \operatorname{Funct}(X, \mathbf{R})$ of real-valued functions on X,

$$(X, \mathcal{O}),$$

are a backbone of Analysis. Think, for example, of a subset X of Euclidean space $\mathbf{R}^n$ and $\mathcal{O}$ being the set of all infinitely differentiable functions on X.

2.1.5 Structures of topological type

Sets X equipped with a family $\mathcal{A} \subset \mathcal{P}X$ of subsets

$$(X, \mathcal{A})$$

are the central objects in Topology, Geometry, Measure Theory, Combinatorics.

2.1.6 Example: topological spaces

A set X equipped with a family of subsets $\mathcal{T} \subset \mathcal{P}X$ closed under formation of *finite* intersections and arbitrary unions is called a *topological space*. Members of $\mathcal{T}$ are referred to as *open subsets*.

2.1.7 Example: measurable spaces

A set X equipped with a family of subsets $\mathcal{M} \subset \mathcal{P}X$ closed under formation of *countable* intersections and under the complement operation $\complement$, cf. Section 1.5.3, is called a *measurable space*. Members of $\mathcal{M}$ are referred to as *measurable subsets*.

2.2 Algebraic structures

2.2.1 $(X, (\mu_i)_{i\in I})$

A set X equipped with an indexed family $(\mu_i)_{i\in I}$ of operations on X is called an *algebraic structure*. Groups, rings, fields, vector spaces, etc., are all examples of algebraic structures.

2.2.2 The *signature* of an algebraic structure

The function

$$\nu : I \longrightarrow \mathbf{N}, \qquad i \longmapsto \nu(i) := \text{the arity of operation } \mu_i \tag{173}$$

is called the *signature* of algebraic structure $\big(X, (\mu_i)_{i\in I}\big)$.

2.2.3 The associated algebraic structure on the power-set

The power-set of X, equipped with the family of direct-image operation $(\mu_i)_*$, cf. Section 1.7.8, forms an algebraic structure

$$\big(\mathscr{P}X, ((\mu_i)_*)_{i\in I}\big)$$

of the same signature.

2.2.4

When the family of operations is finite, we prefer to employ the *list-of-operations* notation

$$(X\,;\, li\mu n)\,.$$

2.2.5 Example: a binary structure

A *binary structure* consists of a set X equipped with a single binary operation on X,

$$(X; \mu_2)\,.$$

Here the list has length 1. Here, I chose the subscript $_2$ to signal that the arity of that single operation is 2.

2.2.6 Multiplicative notation: xy

Traditionally, the generic term for a binary operation has been *multiplication*, and the value $\mu_2(x,y)$ is written as xy or, by using *infix* notation, as

$$x\cdot y\,, \quad x*y\,, \quad \text{et caetera.}$$

2.2.7 Multiplicative notation: AB, aB, Ab

Similarly, the result of the direct-image operation

$$(\mu_2)_*(A,B)$$

applied to a pair of subsets $A,B\subseteq X$, is denoted AB or, when using infix notation, as

$$A\cdot B\,, \quad A*B\,, \quad \text{et caetera.}$$

2.2.8 Cosets of a subset

We skip braces when one of the sets is a singleton set. Thus, sets $\{a\}B$ are generally denoted

$$aB \qquad (a \in A) \tag{174}$$

and sets $A\{b\}$ are denoted

$$Ab \qquad (b \in A)\,. \tag{175}$$

Sets (175) form a family of *right cosets of A* while sets (174) form a family of *left cosets of B*.

2.2.9 Coset ternary relations

Consider the following two relations

$$\rho_r : X, \mathscr{P}X, X \longrightarrow \text{Statements}\,, \qquad x, A, y \longmapsto \text{“ } Ax \ni y \text{ ”}\,, \tag{176}$$

and

$$\rho_l : X, \mathscr{P}X, X \longrightarrow \text{Statements}\,, \qquad x, A, y \longmapsto \text{“ } xA \ni y \text{ ”}\,, \tag{177}$$

2.2.10 *A*-divisor relations

By freezing the subset variable we obtain the corresponding two *A-divisor* binary relations on X,

$$_A| \; : X, X \longrightarrow \text{Statements}\,, \qquad x, y \longmapsto x \,_A|\, y := \text{“ } Ax \ni y \text{ ”}\,, \tag{178}$$

and

$$|_A : X, X \longrightarrow \text{Statements}\,, \qquad x, y \longmapsto x \,|_A\, y := \text{“ } xA \ni y \text{ ”}\,. \tag{179}$$

We can read $x \,_A|\, y$ as

x is a right A-divisor of y

which means that

$$\exists_{a \in A}\; ax = y\,.$$

Similarly, we can read $x \,_A|\, y$ as

x is a left A-divisor of y

which means that

$$\exists_{a \in A}\; xa = y\,.$$

2.2.11 The *opposite* binary structure

By flipping the arguments in a binary operation we obtain another binary operation on X

$$x, y \longmapsto \mu_2^{\mathrm{op}}(x, y) := \mu_2(y, x) \qquad (x, y \in X)\,. \tag{180}$$

The binary structure (X, μ_2^{op}) is referred to as the *opposite of* (X, μ_2^{op}) and is often denoted $X, \mu_2)^{\mathrm{op}}$.

When using generic multiplicative notation it is highly advisable to mark elements of X considered as elements of the opposite binary structure, with the $^{\mathrm{op}}$ tag. Then Definition (180) of the opposite operation becomes

$$x^{\mathrm{op}} y^{\mathrm{op}} := (yx)^{\mathrm{op}} \qquad (x, y \in X)\,. \tag{181}$$

2.2.12 Left- and Right-Cancellation Properties

If

$$\forall_{x,y,z\in X}\ xy=xz \implies y=z \tag{182}$$

we say that $(X,\cdot)$ satisfies *Left Cancellation Property* or is *left-cancellative*.

If

$$\forall_{x,y,z\in X}\ xz=yz \implies x=y \tag{183}$$

we say that $(X,\cdot)$ satisfies *Right Cancellation Property* or is *right-cancellative*.

2.2.13 Left- and right-identity elements

An element $e \in X$ is said to be a *left-identity* if the following identity is satisfied

$$\forall_{x\in X}\ ex = x \tag{184}$$

and is said to be a *right-identity* if the identity

$$\forall_{x\in X}\ xe = x \tag{185}$$

holds.

Note that e^{op} is a right-identity in the opposite structure precisely when e is a left-identity, and vice-versa.

Exercise 45 *Let e be a left-identity in $(X,\cdot)$. Show that $\{e\}$ is a left-identity in $(\mathscr{P}X,\cdot_*)$.*

2.2.14

A binary structure may admit none, one, or many left- or right-identity elements. For example, for the operation

$$X,X \longrightarrow X, \qquad x_1,x_2 \longmapsto x_1,$$

that discards the second entry from the argument list, *every* element is a right-identity, and none is a left-identity as long as X is not a singleton set.

2.2.15 Unital binary structures

If a binary structure admits at least one left-identity, say $e \in X$, and at least one right-identity, say $e' \in X$, then they coincide in view of the double equality

$$e = ee' = e'.$$

In this case we refer to that unique double-sided identity as *the identity element* of a binary structure. If we consider the identity element e as a distinguished element of X, i.e., as a *nullary* operation, μ_0, then the algebraic structure $(X;\mu_0,\mu_2)$ is referred to as a *unital binary structure*.

2.2.16

The defining pair of Identities (184) and (184) is equivalently described as commutativity of the left and, respectively, right triangles in the diagram

$$
\begin{tikzcd}
& X \arrow[dl, Rightarrow, "\mu_0"'] \arrow[dr, Rightarrow, "\mu_0"] \arrow[dd, equal] & \\
X,X \arrow[dr, "\mu_2"'] \arrow[r, phantom, "\circlearrowleft"] & {} \arrow[r, phantom, "\circlearrowright"] & X,X \arrow[dl, "\mu_2"] \\
& X &
\end{tikzcd}
\qquad . \tag{186}
$$

2.2.17 Left and right-inverses of an element

If elements $x, y \in X$ satisfy equality

$$xy = e\,,$$

where $e \in X$ is the identity, then x is said to be a *left-inverse of* y and y is said to be a *right-inverse* of x. In this case we also say that x is a right-invertible, while y is a *left-invertible* element.

Note that in the opposite structure x^{op} is a right-inverse while y^{op} is a left-inverse.

2.2.18 Pointed sets

A set equipped just with a nullary operation (X, μ_0) is frequently encountered in Topology where it would be called a *pointed set*, and the preferred notation would be (X, x_0).

2.2.19 Idempotents

A *square* of any element $x \in X$ in a binary structure is defined to be

$$x^2 := x \cdot x\,.$$

Elements $e \in X$ such that $e^2 = e$ are called *idempotents*.

2.2.20

An attempt to define x^3 requires performing two multiplications and produces two outcomes

$$x \cdot x^2 \qquad \text{and} \qquad x^2 \cdot x\,.$$

2.2.21

An attempt to define x^4 requires performing three multiplications and produces five outcomes

$$\begin{tikzcd}
 & x^2 \cdot x^2 \arrow[dl, dash] \arrow[dr, dash] & \\
x \cdot (x \cdot x^2) \arrow[d, dash] & & (x^2 \cdot x) \cdot x \arrow[d, dash] \\
x \cdot (x^2 \cdot x)) \arrow[rr, dash] & & (x \cdot x^2) \cdot x
\end{tikzcd} \tag{187}$$

2.2.22 Power associative binary structures

A binary structure is said to be *power associative* if, for every positive integer n, the product of n copies of an arbitrary element, calculated by applying $n-1$ times multiplication, produces one and the same result regardless of how we group the arguments.

At this point it is necessary to mention that construction of the sequences of powers of an element in a binary structure is accomplished by *Recursive Definition*. Existence of recursively defined sequences is one of the key features of of the set of natural numbers $\mathbf{N}$, more precisely, of its being *well-ordered*, cf. Section 3.3.11 below.

2.2.23 Semigroups

A single most important property of a binary operation on a set is known as *associativity* and is expressed in the form of the identity

$$\forall_{x,y,z\in X}\ (xy)z = x(yz) \tag{188}$$

or, alternatively, as commutativity of the diagram

$$\begin{tikzcd}
 & X, X, X \arrow[dl, Rrightarrow, "\mu_2"'] \arrow[dr, Rrightarrow, "\mu_2"] & \\
X, X \arrow[dr, "\mu_2"'] \arrow[rr, phantom, "\circlearrowleft"] & & X, X \arrow[dl, "\mu_2"] \\
 & X &
\end{tikzcd} . \tag{189}$$

An associative binary structure is called a *semigroup*.

Exercise 46 *Show that $(\mathscr{P}X, \cdot_*)$ is a semigroup if $(X, \cdot)$ is a semigroup.*

2.2.24

Associativity of a binary operation, of course, implies its power-associativity, but no vice-versa.

In multiplicative notation, powers of an element x are written x^n. In additive notation x^n becomes nx. Here $n \in \mathbf{N}\setminus\{0\}$.

2.2.25 Commutative binary structures

A second important property of a binary operation on a set is known as *commutativity* and is expressed in the form of the identity

$$\forall_{x,y\in X}\, xy = yx\,. \tag{190}$$

A binary structure $(X,\cdot)$ is commuttaive precisely when

$$\mu_2 = \mu_2^{\mathrm{op}}\,,$$

i.e., when $(X,\cdot)$ coincides with the opposite structure

$$(X,\cdot) = (X,\cdot)^{\mathrm{op}}\,.$$

Exercise 47 *Show that* $(\mathscr{P}X,\cdot_*)$ *is a commuttaive binary structure if* $(X,\cdot)$ *is commuttaive.*

2.2.26 Terminology: *abelian* groups

We encounter commutative semigroups, monoids, semirings, rings, etc. Commutative *groups*, however, are traditionally called *abelian groups*. This tradition predates introduction of general algebraic structures.

2.2.27 Unital semigroups, i.e., monoids

Unital semigroups, i.e., unital binary structures $(X\,;\mu_0,\mu_2)$ with an associative multiplication are called *monoids*. They appear in nearly every aspect of Modern Mathematics.

Exercise 48 *Show that* $\left(\mathscr{P}X,\{\mu_0\},(\mu_2)_*\right)$ *is a monoid if* (X,μ_0,μ_2) *is a monoid.*

2.2.28 Left- and right-invertible elements in a monoid

If

$$xx' = e \qquad \text{and} \qquad yy' = e\,,$$

then the short calculation

$$(xy)(y'x') = x((yy')x') = x(ex') = xx' = e$$

demonstrates that the subset of X formed by all right-invertible elements is closed under multiplication. Let us denote it X^{inv}. Similarly, the subset of X formed by all left-invertible elements is closed under multiplication. Let us denote it ${}^{\mathrm{inv}}X$.

2.2.29 Invertible elements

If

$$xx' = e = x''x\,,$$

then the short calculation that makes use of associativity of multiplication,

$$x' = ex' = (x''x)x' = x''(xx') = x''e = x''\,,$$

demonstrates that, in a monoid, any element that admits a left and and a right-inverse, has precisely a single left and a single right inverse, and they necessary coincide. That unique double inverse of an element x is usually denoted x^{-1} and is referred to as *the inverse of* x.

The subset of invertible elements of a monoid is sometimes denoted X^*, at other times it may be denoted $G(X)$. We established above that

$$X^* = {}^{\mathrm{inv}}X \cap X^{\mathrm{inv}}.$$

2.2.30

Assignment

$$X \longrightarrow X, \qquad x \longmapsto x^{-1},$$

defines a unary operation on X^* and the set of invertible elements X^* equipped with the identity element, the inverse-element operation, and multiplication, is known as the *group of invertible elements of* $(X; \mu_0, \mu_2)$.

2.2.31 Groups

A *group* is an algebraic structure

$$(X; \mu_0, \mu_1, \mu_2)$$

such that $(X; \mu_0, \mu_2)$ is a monoid and μ_1 is a unary operation that sends an arbitrary element $x \in X$ to its inverse element, x^{-1}.

2.2.32

The defining pair of identities $x^{-1}x = e = xx^{-1}$ is equivalently described as commutativity of the left and, respectively, of the right triangle in the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\mu_1"'] \arrow[dl, equal, shift left=2] \arrow[dd, "e_X"] \arrow[dr, "\mu_1"] \arrow[dr, equal, shift right=2] & \\
X, X \arrow[dr, "\mu_2"'] \arrow[r, phantom, "\circlearrowleft"] & {} \arrow[r, phantom, "\circlearrowright"] & X, X \arrow[dl, "\mu_2"] \\
 & X &
\end{tikzcd} \tag{191}$$

where e_X denotes the *constant* function

$$X \longrightarrow X, \qquad x \longmapsto e \qquad (x \in X).$$

Note that $e_X : X \to X$ is the composite function

$$X \longrightarrow \emptyset^{\emptyset} \xrightarrow{\ \tilde{e}\ } X$$

where $X \longrightarrow \emptyset^{\emptyset}$ is the unique function from X to the singleton set $\emptyset^{\emptyset}$ and $\tilde{e}$ is the function of a single variable canonically corresponding to the function of zero variables $e : \longrightarrow X$, cf. Section 1.12.13.

2.2.33 Caveat

The algebraic structure $\left(\mathscr{P}X\,;\ \{\mu_0\}, (\mu_1)_*, (\mu_2)_*\right)$ associated with a group $(X\,;\ \mu_0, \mu_1, \mu_2)$ is *not* a group :

$$\{\mu_0\} \subseteq A \cdot A^{-1} \quad \text{but} \quad \{\mu_0\} \neq A \cdot A^{-1}$$

if A has at least two elements.

2.2.34 The canonical monoid structure on $\mathrm{Op}_1(X)$

Composition $\circ$ is a canonical binary operation on the set of all unary operations $\mathrm{Op}_1(X)$ on an arbitrary set X. The identity operation id_X is a distinguished element of $\mathrm{Op}_1(X)$. Composition of functions is associative and id_X is an identity element for the operation of composition.

Thus, $(\mathrm{Op}_1(X), \mathrm{id}_X, \circ)$ is a monoid and $\mathrm{Op}_1(X)$ provides an example of a set that is equipped with a canonical structure of a monoid.

2.2.35 Fixed points of a unary operation

Given an operation $\tau \in \mathrm{Op}_1(X)$ and an element of $x \in X$, we say that x is a *fixed point of* τ if

$$\tau(x) = x\,.$$

The set of fixed points of τ is often denoted

$$X^{\tau}\,.$$

2.2.36 A *retraction* of a set onto its subset

An idempotent τ in monoid $(\mathrm{Op}_1(X), \mathrm{id}_X, \circ)$ is called a *retraction*. If $Y = \tau_* X$ is the image of τ, then we often say that τ is a *retraction of a set X onto its subset Y*.

A unary operation τ is a retarction if and only if its image is contained in its set of fixed points,

$$\tau_* X \subseteq X^{\tau}\,.$$

2.2.37 The permutation group of a set

Invertible unary operations on a set X are, traditionally, called *permutations* (of elements of X). They form the group of permutations, denoted

$$\Sigma_X\,, \quad S_X\,, \quad \mathrm{Per}\,X\,, \quad \text{or} \quad \mathrm{Sym}\,X\,.$$

It is one of the most important groups in Mathemamtics.

2.2.38 Actions of sets on other sets

A set X, equipped with a family of unary operations $(\lambda_a)_{a \in A}$ indexed by a set A, is referred to as a set equipped with an *action of set A*. A short designation for this structure is an *A-set*.

An action of a set A on a set X is the same as a function

$$\lambda : A \longrightarrow \mathrm{Op}_1(X)\,. \tag{192}$$

We shall use, in general, notation (X, λ) to denote A-sets where λ is a function (192).

2.2.39 Standard multiplicative notation

The value of operation λ_a on an element $x \in X$ is frequently denoted ax.

2.2.40 Example: the *left* and the *right regular* actions of a semigroup

Given an element $a \in X$ of a a binary structure $(X, \cdot)$, left and, respectively, right multiplication by a define two actions of X on set X,

$$L_a : X \longrightarrow X, \qquad x \longmapsto L_a(x) := ax, \tag{193}$$

and

$$R_a : X \longrightarrow X, \qquad x \longmapsto R_a(x) := xa. \tag{194}$$

Multiplication in $(X, \cdot)$ is associative if and only if unary operations L_a and R_b commute with each other,

$$\forall_{a,b \in X}\ L_a R_b = R_b L_a. \tag{195}$$

2.2.41 Example: the *adjoint* action of the group of invertible elements of a monoid

Given an invertible element $g \in X^*$ of a monoid $(X, e, \cdot)$, the formula

$$\mathrm{ad}_g : X \longrightarrow X, \qquad x \longmapsto \mathrm{ad}_g(x) := gxg^{-1}, \tag{196}$$

defines an action of X^* on set X.

2.2.42 The *conjugacy* class of an element

For any element $x \in X$ and an invertible element $g \in X^*$, the element $\mathrm{ad}_g(x)$ is called the *conjugate of x by g* and is frequently denoted ${}^g x$.

The set of all conjugates of an element $x \in X$,

$$\left\{ y \in X \mid \exists_{g \in X^*}\ y = {}^g x \right\}, \tag{197}$$

is called the *conjugacy class of x*.

Exercise 49 *Show that, for any $g, h \in X^*$, one has*

$$\mathrm{ad}_{gh} = \mathrm{ad}_g \circ \mathrm{ad}_h, \qquad \mathrm{ad}_e = \mathrm{id}_X \qquad \textit{and} \qquad \mathrm{ad}_{g^{-1}} = \left(\mathrm{ad}_g\right)^{-1}. \tag{198}$$

2.2.43 *Normal* subsets

A subset $A \subseteq X$ is said to be *normal* if, for every invertible element $g \in X^*$,

$${}^g A = A,$$

i.e., A is a fixed point of operations

$$(\mathrm{ad}_g)_* : \mathcal{P}X \longrightarrow \mathcal{P}X, \qquad A \longmapsto {}^g A := gAg^{-1}, \qquad (g \in X^*). \tag{199}$$

Exercise 50 *Show that A is a normal subset if and only if it is closed under operations* (199).

Solution. Note that

$$^{g}A \subseteq A \;\Leftrightarrow\; A = {}^{g^{-1}}({}^{g}A) \subseteq {}^{g^{-1}}A\,.$$

Since the inverse-element operation is bijective, we have

$$\left(\forall_{g\in G}\ {}^{g}A \subseteq A\right) \;\Leftrightarrow\; \left(\forall_{g\in G}\ A \subseteq {}^{g}A\right).$$

□

Exercise 51 *Let $A \subseteq X$ be a subset of a monoid $(X, e, \cdot)$. Show that, for every invertible element $g \in X^*$,*

$$\forall_{a,b,g\in G}\ (ag)\,|_A\,(bg) \;\Leftrightarrow\; a\,|_{{}^{g}A}\,b\,.$$

Exercise 52 *Let $A \subseteq G$ be a subset of a group G and $g \in G$. Show that ${}^{g}A$ is a subgroup if and only if A is a subgroup.*

2.2.44

Semigroups, monoids, groups, are encountered everywhere where mathematical considerations are involved.

2.2.45

Algebraic structures involving two binary operations lead to algebraic structures known as semirings and rings. They will be introduced and discussed later.

2.3 Relational structures

2.3.1

Sets X equipped with an indexed family $(\rho_i)_{i\in I}$ of relations on X are called *relational structures*. Such structures are encountered in all areas of Mathematics and especially so in Mathematical Logic and in Incidence Geometry.

2.3.2 Binary relational structures

Particularly important are *binary relational structures*, i.e., sets equipped with a single binary relation. We discuss them in Chapter ?? devoted to binary relations.

2.3.3 Example: (pre)ordered sets

(Pre)ordered sets, introduced in Sections 1.8.14–1.8.15, are examples of binary relational structures.

2.4 Substructures

2.4.1

For every type of a mathematical structure there is a naturally defined notion of a *substructure*.

2.4.2

For a list of sets (171) equipped with no 'data', its *substructure* is the same as a list of subsets, i.e., a list

$$Y_1, \dots, Y_n \tag{200}$$

where

$$Y_1 \subseteq X_1 \,, \ \dots \ , \ Y_n \subseteq X_n \,.$$

2.4.3

For a list of sets equipped with 'data' *of a given type*, (172), its *substructure* is a list of subsets, (200), equipped with the *restriction* of the 'data' to Y. The requirement that that restricted data is of the same type may impose a constraint on a subset Y.

2.4.4 Subfunctions

We shall illustrate this general concept when the mathematical structure considered is a function f from a set X to a set X'. A *subfunction* of f is a function g from a *subset* Y of the source of f to a *subset* Y' of the target of f such that

$$\forall_{x \in Y}\, g(x) = f(x)\,. \tag{201}$$

Exercise 53 *Show that the constraint on the list of subsets X', Y' expressed by condition* (201) *is equivalent to the condition*

$$f_* X' \subseteq Y'\,. \tag{202}$$

Condition (202) is equivalently expressed by saying that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ Y \arrow[u, hook] & Y' \arrow[u, hook] \end{tikzcd}$$

admits a completion to a commutative square diagram

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ Y \arrow[u, hook] \arrow[r] & Y' \arrow[u, hook] \end{tikzcd}\,.$$

2.4.5 Subfunctions of n variables

The above definition of a subfunction has an obvious extension to the case of a function of n variables

$$f : X_1, \dots, X_n \longrightarrow X'\,. \tag{203}$$

Exercise 54 *Formulate the definition of a subfunction of* (203) *that, for $n = 1$, yields the original definition of a subfunction of a function of a sungle variable.*

2.4.6 Suboperations

When all sets $X_1, \ldots, X_n, X'$ are equal and coincide with a set X, i.e., when a function is an algebraic operation on a set X,

$$\mu : X_1, \ldots, X_n \longrightarrow X, \tag{204}$$

a *suboperation* of operation (204) is a subfunction of (204) such that all subsets $Y_1, \ldots, Y_n, Y'$ are equal to a certain subset $Y \subseteq X$.

2.4.7 Algebraic substructures

For a set X equipped with a family of operations $(\mu_i)_{i \in I}$, its substructures consist of subsets $Y \subsetneq X$ that are *closed* under each operation μ_i, i.e., such that the following diagram

$$\begin{tikzcd}
X, \ldots, X \arrow[r, "\mu_i"] & X \\
Y, \ldots, Y \arrow[u, hook, "\cdots"'] & Y \arrow[u, hook]
\end{tikzcd} \tag{205}$$

admits completion to the commutative diagram

$$\begin{tikzcd}
X, \ldots, X \arrow[r, "\mu_i"] & X \\
Y, \ldots, Y \arrow[u, hook, "\cdots"'] \arrow[r, dashed, "\bar{\mu}_i"'] \arrow[ur, phantom, "\circlearrowright"] & Y \arrow[u, hook]
\end{tikzcd} . \tag{206}$$

Note that function $\bar{\mu}_i$ is unique when it exists, and its values coincide with the corresponding values of μ_i,

$$\forall_{x_1, \ldots, x_n \in X} \ \bar{\mu}_i(x_1, \ldots, x_n) = \mu_i(x_1, \ldots, x_n) .$$

We refer to μ_i as the operation *induced by* μ_i on Y.

2.4.8

Note that $\bar{\mu}_i$ is *not* the *restriction* of μ_i to Y. The restriction of a function has a smaller domain and the *same* target. In particular, the restriction of μ_i produces a function

$$Y_1, \ldots, Y_n \longrightarrow X,$$

not an operation on a subset Y.

2.4.9

If a subset Y is closed under *every* operation μ_i, then $\left(Y, (\bar{\mu}_i)_{i \in I}\right)$ is called a *substructure* of a structure $\left(X, (\mu_i)_{i \in I}\right)$.

This is how we define *subgroups, submonoids, subsemigroups, subrings, vector subspaces*, etc.

2.4.10 The ordered set of substructures $\mathrm{Substr}\left(X, (\mu_i)_{i\in I}\right)$

Let us denote by $\mathrm{Substr}\left(X, (\mu_i)_{i\in I}\right)$ the set of substructures of $\left(X, (\mu_i)_{i\in I}\right)$, i..e, the set of subsets of X that are closed under every operation μ_i. It is ordered by inclusion.

Exercise 55 *Let $\mathscr{Y} \subsetneq \mathrm{Substr}\left(X, (\mu_i)_{i\in I}\right)$ be a family of substructures of $\left(X, (\mu_i)_{i\in I}\right)$. Show that the intersection of memebers of $\mathscr{Y}$ is a substructure.*

2.4.11

The union of a family of substructures is not a substructure, in general. For example, the union of two vector subspaces $V' \cup V''$ of a vector space V is closed under addition of vectors if and only if either $V' \subseteq V''$ or $V'' \subseteq V'$.

Exercise 56 *Suppose that $V' \cup V''$ is closed under addition of vectors. Show that either $V' \subseteq V''$ or $V'' \subseteq V'$.*

Solution. Suppose that $V'' \nsubseteq V'$. Let $v' \in V'$ and $v'' \in V'' \setminus V'$, and assume that $v' + v'' \in V' \cup V''$. If $v' + v'' \in V'$, then
$$v'' = (v' + v'') - v' \in V' .$$
Since $v'' \notin V'$, we deduce that
$$v' + v'' \in (V' \cup V'') \setminus V' \subseteq V'' .$$
It follows that
$$v' = (v' + v'') - v'' \in V'' ,$$
i.e., $V' \subseteq V''$. □

2.4.12 Locally filtered families of subsets

We shall say that a family of subsets $\mathscr{Y} \subseteq \mathscr{P}X$ is *locally filtered* if, for every *finite subset*
$$F \subseteq \bigcup \mathscr{Y} ,$$
there exists a member $Y \in \mathscr{Y}$, such that
$$F \subseteq Y .$$

Exercise 57 *Let $\mathscr{Y} \subsetneq \mathrm{Substr}\left(X, (\mu_i)_{i\in I}\right)$ be a locally filtered family of substructures of $\left(X, (\mu_i)_{i\in I}\right)$. Show that the union of members of $\mathscr{Y}$ is a substructure.*

2.4.13 The substructure $\langle A \rangle$ generated by a subset $A \subseteq X$

Given a subset $A \subseteq X$, the intersection of the family
$$\mathscr{Y}_A := \left\{ Y \in \mathrm{Substr}\left(X, (\mu_i)_{i\in I}\right) \mid Y \supseteq A \right\}$$
is the smallest substructure containing subset A. We shall denote it $\langle A \rangle$ and call it the substructure *generated by A*.

Exercise 58 *Show that*
$$\forall_{A,B\subseteq X}\ A\subseteq B \Rightarrow \langle A\rangle \subseteq \langle B\rangle . \tag{207}$$

Exercise 59 *Show that*
$$\forall_{A\subseteq X}\ \langle A\rangle = \langle\langle A\rangle\rangle . \tag{208}$$

2.4.14 Invariant subsets

Subsets $Y \subseteq X$ closed under a *unary* operation $\tau : X \to X$ are frequently encountered outside of Algebra, for example in Theory of Group Actions, Theory of Dynamical Systems, Topology, Operator Theory. Such sets are said to be *invariant* or, more precisely, τ-invariant. In the language of diagrams invariance of a subset Y is expressed by saying that

$$\begin{tikzcd} X \arrow[r, "\tau"] & X \\ Y \arrow[u, hook] & Y \arrow[u, hook] \end{tikzcd} \tag{209}$$

admits completion to the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\tau"] & X \\ Y \arrow[u, hook] \arrow[r, dashed, "\tilde{\tau}"'] \arrow[ur, phantom, "\circlearrowleft"] & Y \arrow[u, hook] \end{tikzcd} . \tag{210}$$

Invariance of a subset $Y \subseteq X$ is expressed in terms of the *direct* image function by the condition

$$\tau_* Y \subseteq Y \tag{211}$$

or, in terms of the *inverse* image function, by the equivalent condition

$$Y \subseteq \tau^* Y . \tag{212}$$

2.4.15 Coinvariant subsets

If we reverse the relation in Condition (212),

$$Y \supseteq \tau^* Y , \tag{213}$$

then we obtain a *dual* condition that is equivalent to

$$\tau_! Y \supseteq Y . \tag{214}$$

We shall say in such case that Y is a *coinvariant* subset or, more precisely, a *τ-coinvariant* subset.

2.5 Subgroups

2.5.1

A subset $H \subseteq G$ of a group G that is closed under the idenity operation,

$$H \ni e ,$$

under the inverse-element operation,

$$H^{-1} \subseteq H$$

and under multiplication,

$$H \cdot H \subseteq H ,$$

is called a *subgroup of G*. Each of these properties of a subset $H \subseteq G$ admits an equivalent characterization in terms of the corresponding divisor relations, cf. Section 2.2.10.

Exercise 60 *Show that H-divisor relation $|_H$ is reflexive if and only if $H \ni e$.*

Exercise 61 *Show that*

$$(|_H)^{\mathrm{op}} \;\Leftrightarrow\; |_{H^{-1}} \;.$$

Exercise 62 *Show that relation $|_H$ is symmetric if and only if $H = H^{-1}$.*

Exercise 63 *Show that*

$$|_H \circ |_H \Leftrightarrow |_{H \cdot H} \;.$$

Exercise 64 *Show that relation $|_H$ is transitive if and only if $H \cdot H \subseteq H$.*

2.5.2

It follows that relation $|_H$ is an equivalence relation precisely when H is a subgroup of G, and the same for the other H-divisor relation ${}_H|$.

2.5.3

Note that

$$\forall_{x,y \in G} \; x\,{}_H| \, y \;\Leftrightarrow\; x^{-1} \,|_{H^{-1}}\, y^{-1} . \tag{215}$$

Subset H is closed under each of the group operations if and only if the set of inverses H^{-1} is closed under the same operation.

Moreover, a binary relation ρ on G is reflexive, symmetric, or transitive, precisely when the relation

$$x, y \;\longmapsto\; \rho\left(x^{-1}, y^{-1}\right) \qquad (x, y \in G)$$

is reflexive, symmetric or, respectively, transitive. It follows that in Exercises 60–64 we could replace the left H-divisor relation $|_H$ by the right H-divisor relation ${}_H|$.

2.5.4

The equivalence class of an element $x \in G$ for relation $|_H$ coincides with the left coset xH. Right multiplication by x, defines a function

$$R_x : H \longrightarrow Hx$$

whose inverse is right multiplication by x^{-1}.

Similarly, left multiplication by x, defines a function

$$L_x : H \longrightarrow xH$$

whose inverse is left multiplication by x^{-1}.

In particular, right and left cosets of H have the same cardinality as H.

2.5.5 Terminology: the *order* of a group G

In Group Theory the number of elements of a *finite* group G is denoted $|G|$ and called the *order of* G. Accordingly, the groups whose underlying sets are infinite are referred to as groups of *infinite order*.

2.5.6 Terminology: the *order* of an element $g \in G$

The *order of an element* $g \in G$ is defined as the order of the subgroup $\langle g \rangle \subseteq G$ generated by element g and is denoted $|g|$. Groups generated by a single element are said to be *cyclic*.

2.5.7 The *index* of a subgroup $H \subseteq G$

The cardinality of the set of right cosets ${}_{H\backslash}G$ coincides with the cardinality of the set of left cosets $G_{/H}$. When those sets are finite, the number of right cosets of H in G, which coincides with the number of left cosets, is called the *index of a subgroup* H *in* G and is denoted $|G : H|$.

2.5.8

When the sets of cosets are infinite, we say that H *is a subgroup of infinite index*.

2.5.9

Infinite groups may have finite subgroups and may also have infinite subgroups of finite index.

2.5.10

Since G is the union of disjoint right cosets and each coset of H has the same cardinality as H, the number of elements in a finite group G is the number of elements in H multiplied by the number of cosets. This simple counting argument was discovered more than 200 years ago and bears the name of franco-italian mathematician Lagrange.

Theorem 2.1 (Lagrange) *For any subgroup H of a finite group G, one has the*

$$|G| = \sum_{C \in G/H} |C| = \sum_{C \in G/H} |H| = |G : H| \cdot |H| . \tag{216}$$

□

Corollary 2.2 *The order of any subgroup H of a finite group G divides the order of G.*

□

Corollary 2.3 *The order of any element g of a finite group G divides the order of G.*

□

Corollary 2.4 *A group whose order is prime has no nontrivial proper subgroups. The order of any element $g \neq e$ in such a group equals $|G|$, hence*

$$\langle g \rangle = G .$$

In particular, a group of prime order is cyclic.

□

Exercise 65 *Show that a finite subgroup H in an infinite group G has infinite index.*

Exercise 66 *Show that a subgroup H of finite index in an infinite group G is infinite.*

3 Morphisms

3.1 Interactions between mathematical structures

3.1.1

If mathematical structures are *objects* of mathematical theories, studying a given structure is nearly always executed by observing how that structure *interacts* with other structures of the same type. Binary interactions between structures are expressed in the language of *morphisms*.

3.1.2 The concept of a *concrete* morphism

A *morphism*

$$f : (X, \mathrm{data}) \longrightarrow (X', \mathrm{data}') \tag{217}$$

is most commonly understood to be a function between the *underlying sets*

$$f : X \longrightarrow X'$$

that *respects* the corresponding data. We refer to such morphisms as being *concrete morphisms*.

3.1.3

It is assumed that the data must be of the same type. The term 'respects' can be replaced by : 'is compatible with'. The meaning of this term is nearly always natural for each type of data. We shall illustrate this for some types of mathematical structures mentioned above.

The crucial expectation when introducing a suitable concept of a morphism between sets equipped with data is that the composite $g \circ f$ of two composable morphisms

$$(X, \mathrm{data}) \xrightarrow{\ f\ } (X', \mathrm{data}') \xrightarrow{\ g\ } (X'', \mathrm{data}'')$$

is again a morphism.

Additionally, it is expected that the identity operation id_X is an endomorphism of

$$(X, \mathrm{data}) \tag{218}$$

irrespective of what type of data we may consider.

3.1.4 Terminology : an *endo*morphism

When $(X, \mathrm{data}) = (X', \mathrm{data}')$ a morphism (217) is referred to as an *endomorphism of* (X, data)..

3.1.5 The monoid of endomorphisms $\mathrm{End}(X, \mathrm{data})$

In agreement with the requirements spelled out in Section 3.1.3, the set of endomorphisms of (218) is equipped with a canonical monoid structure

$$\big(\mathrm{End}(X, \mathrm{data}), \mathrm{id}_X, \circ\big)\,. \tag{219}$$

This is the most important source of monoids in Mathematics and its applications.

3.1.6 Terminology: an *iso*morphism

When there exists a morphism
$$g : (X', \mathrm{data}') \longrightarrow (X, \mathrm{data}) \tag{220}$$
such that
$$f \circ g = id_{X'} \qquad \text{and} \qquad g \circ f = \mathrm{id}_X\,, \tag{221}$$
we say that f is an *isomorphism between* $(X, data)$ and (X', data'). A pair of morphisms satisfying pair of equalities (221) is said to be *inverse to each other*.

3.1.7 Terminology: an *auto*morphism

An endomorphism of (X, data) that is an isomorphism is said to be an *automorphism of* $(X, data)$.

3.1.8 The group of automorphisms $\mathrm{Aut}(X, \mathrm{data})$

Automorphisms form a subset of the monoid $\mathrm{End}(X, \mathrm{data})$ that is closed under composition and formation of inverses. It is therefore naturally equipped with a structure of a group. This is the most important source of groups in Mathematics and its applications.

3.1.9 The arrow notation

Morphisms are represented graphically as arrows. Every arrow has its source and its target, each being a structure of the same type. They are referred to as the *source* and the *target* of a morphism.

3.2 Morphisms between algebraic structures

3.2.1 Homomorphisms

Suppose that a set X is equipped with an n-ary operation μ and a set X' is equipped with an n-ary operation μ'. We say that a function $f : X \to X'$ is *compatible* with the operations if
$$\forall_{x_1,\dots,x_n \in X}\; f\big(\mu(x_1, \dots, x_n)\big) = \mu\big(f(x_1), \dots, f(x_n)\big)\,. \tag{222}$$
Algebraists refer to such functions as *homomorphisms*.

3.2.2

The definition of a morphism between sets equipped with an n-ary operation can be also expressed as commutativity of the following square diagram
$$\begin{tikzcd} X', \dots, X' \arrow[r, "\mu'"] & X' \\ X, \dots, X \arrow[u, "f \,\cdots\, f"] \arrow[r, "\mu"] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[u, "f"'] \end{tikzcd} \,. \tag{223}$$

3.2.3

The above definition can be easily extended to general algebraic structures. A morphism

$$(X, (\mu_i)_{i\in I}) \longrightarrow (X', (\mu'_i)_{i\in I})$$

is a function $f : X \to X'$ such that it is a homomorphism

$$(X, \mu_i) \longrightarrow (X', \mu'_i)$$

for *each* $i \in I$. Notice that μ_i and μ'_i must have the same 'arity' for every $i \in I$.

The concept of a homomorphism provides the most natural definition of a morphism between algebraic structures.

3.2.4 Example: morphisms between pointed sets

A morphism from a pointed set (X, x_0) to a pointed set (X', x'_0) is, by definition, a function $f : X \to X'$ such that

$$f(x_0) = x'_0 . \tag{224}$$

3.2.5 Example: morphisms between A-sets

A morphism from an A-set (X, λ) to an A-set (X', λ'), cf. Section 2.2.38, is, by definition, a function $f : X \to X'$ such that

$$\forall_{a\in A}\ f \circ \lambda_a = \lambda'_a \circ f \tag{225}$$

or, equivalently, in multiplicative notation,

$$\forall_{a\in A, x\in X}\ f(ax) = af(x) . \tag{226}$$

3.2.6

Condition (225) can be expressed as commutativity of square diagrams

$$\begin{tikzcd} X' \arrow[r, "\lambda'_a"] & X' \\ X \arrow[u, "f"] \arrow[r, "\lambda_a"'] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[u, "f"'] \end{tikzcd} \tag{227}$$

for all $a \in A$.

3.2.7 *Anti*homomorphisms between binary structures

Homomorphisms between binary structures

$$(A, \cdot)^{\mathrm{op}} \longrightarrow (A', \cdot') \qquad \text{or} \qquad (A, \cdot) \longrightarrow (A', \cdot')^{\mathrm{op}}$$

are the same as *antihomomorphisms* $(A, \cdot) \longrightarrow (A', \cdot')$, i.e., functions $f : A \to A'$ that satisfy the condition

$$\forall_{a,b\in A}\ f(ab) = f(b)f(a) . \tag{228}$$

3.2.8 Actions of binary structures $(A, \cdot)$ on sets

The set of unary operations $\mathrm{Op}_1(X)$ of any set X is canonically equipped with a structure of a monoid. When a set A, equipped with a binary operation $\cdot$, is acting on a set X, it is usually assumed that the action function λ in (192) is a homomorphism of binary algebraic structures, i.e., that

$$\forall_{a,b\in A}\ \lambda_{a\cdot b} = \lambda_a \circ \lambda_b\,. \tag{229}$$

Condition (229) is equivalently expressed as the identity that closely resembles Associativity

$$\forall_{a,b\in A}\ \forall_{x\in X}\ (a\cdot b)x = a(bx)\,. \tag{230}$$

3.2.9

If the same generic multiplicative notation is used for the binary operation in A and for the action of A on X, then the requirement that λ be a homomorphism takes the form of the identity

$$\forall_{a,b\in A}\ \forall_{x\in X}\ (ab)x = a(bx) \tag{231}$$

that is indistinguishable from Associativity. And for a good reason: Associativity of a binary algebraic structure $(A, \cdot)$ expresses the fact that the structure acts on set A by left-multiplication.

Exercise 67 *Show that a binary algebraic structure $(A, \cdot)$ is associative if and only if the left-multiplication function, cf.* (193),

$$L : A \longrightarrow \mathrm{Op}_1(A)\,, \qquad a \longmapsto L_a\,, \tag{232}$$

is a homomorphism of binary algebraic structures.

3.2.10 Right actions

What we described above is also known as a *left* action of a binary structure $(A, \cdot)$. A *right action* is an *antihomomorphism*

$$\varrho : A \longrightarrow \mathrm{Op}_1(X)\,. \tag{233}$$

Exercise 68 *Show that a binary algebraic structure $(A, \cdot)$ is associative if and only if the right-multiplication function, cf.* (194),

$$R : A \longrightarrow \mathrm{Op}_1(A)\,, \qquad a \longmapsto R_a\,, \tag{234}$$

is an antihomomorphism of binary algebraic structures.

3.2.11

Generic multiplicative notation for right actions places an element $a \in A$ that acts on $x \in X$ on the *right*

$$\varrho_a(x) = xa\,. \tag{235}$$

This is where the terms *left* and *right* action come from.

The property of ϱ being an antihomomorphism then again has the form of the familiar associativity condition

$$\forall_{a,b\in A}\ \forall_{x\in X}\ x(ab) = (xa)b\,. \tag{236}$$

3.2.12

The left and the right regular actions of a semigroup on itself, introduced in Section 2.2.40 are particularly important in Group Theory and in Theory of Group Actions.

3.3 Semirings

3.3.1 Sets equipped with two binary operations

Suppose a set X is equipped with two binary operations, denoted $*$ and $\cdot$, respectively.

3.3.2 Left Distributivity Property

If the operations of left multiplication by a,

$$L_a \in \mathrm{Op}_1 X \qquad (a \in X),$$

cf. (193), act on X as *endomorphisms of binary structure* $(X, *)$, i.e., if

$$L_a \in \mathrm{End}(X, *) \qquad (a \in X),$$

we say that operation $\cdot$ *left-distributes over operation* $*$. Left Distributivity of $\cdot$ over $*$ is equivalent to the following identity

$$\forall_{a,x,y \in X} \quad a \cdot (x * y) = a \cdot x * a \cdot y\,. \tag{237}$$

3.3.3 Right Distributivity Property

If the operations of right multiplication by a,

$$R_a \in \mathrm{Op}_1 X \qquad (a \in X),$$

cf. (194), act on X as *endomorphisms of binary structure* $(X, *)$, i.e., if

$$R_a \in \mathrm{End}(X, *) \qquad (a \in X),$$

we say that operation $\cdot$ *right-distributes over operation* $*$. Right Distributivity of $\cdot$ over $*$ is equivalent to the following identity

$$\forall_{a,x,y \in X} \quad (x * y) \cdot a = x \cdot a * y \cdot a\,. \tag{238}$$

3.3.4 Commutative semigroups

The binary operation in a commutative semigroup is often referred to as *addition* and $+$ is the generic symbol for such an operation.

3.3.5 Semirings

Suppose a commuttive semigroup $(S, +)$ is equipped with a secondary operation $\cdot$, referred to as *multiplication*, that is both left and right distributive over addition. We call

$$(S, +, \cdot)$$

a *semiring*. We say that a semiring is *associative*, *commutative*, *unital*, if the multiplicative binary structure $(S, \cdot)$ is associative, commutative or, respectly, unital.

3.3.6 0 and 1 in a semiring

The identity element of the additive semigroup $(S, +)$ is referred to as the *zero* element, if it exists, and is denoted 0.

The identity element of the multiplicative binary structure $(S, \cdot)$ is denoted 1, when it exists, and is simply referred to as the *identity element* or the *unit element* (of the semiring).

3.3.7

In general, $s \cdot 0$ may not equal 0. This is so, however, if the additive semigroup $(S, +)$ is *cancellative*, cf. Section (2.2.12).

Exercise 69 *Show that in a semiring-with-zero*

$$\forall_{s \in S} \ 0 \cdot s = 0 = s \cdot 0 \tag{239}$$

if addition is cancellative.

3.3.8 Rings

When the additive semigroup of a semiring is a group, we say that a semiring is a ring.

3.3.9 The ordered unital semiring-with-zero of natural numbers $(\mathbf{N}, 0, 1, +, \cdot, \leq)$

A principal example of a semiring is provided by the set of natural numbers equipped with the standard addition and multiplication operations. Its existence is equivalent to existence of an infinite set. We prove that and we establish some of its key features by studying *twisted sets*, i.e., unary algebraic structures (X, μ_1). We do this in separate sets of notes.

3.3.10

One such feature is that the additive semigroup $(\mathbf{N}, +)$ and the multiplicative semigroup $(\mathbf{N} \setminus \{0\}, \cdot)$ are *cancellative*, cf. Section 2.2.12.

3.3.11

Another feature is presence of the order relation that can be expressed entirely in terms of the operation of addition

$$\forall_{m,n \in \mathbf{N}} \ (m \leq n \ \Leftrightarrow \ \exists_{l \in \mathbf{N}} \ l + m = n). \tag{240}$$

and that has the following properties:

(i) natural number l in (240) is unique and is denoted $n - m$;

(ii) $\forall_{n \in \mathbf{N}} \ 0 \leq n < 1 \Rightarrow 0 = n$;

(iii) $\forall_{m,n \in \mathbf{N}} \ m < n \Leftrightarrow m + 1 \leq n$;

(iv) $(\mathbf{N}, \leq)$ is a well-ordered set, cf. Section 1.8.18;

(v) $(\mathbf{N}, +, \leq)$ and $(\mathbf{N}, \cdot, \leq)$ are *ordered semigroups*, i.e.,

$$\forall_{m,n,m',n' \in \mathbf{N}} \quad m \leq m' \wedge n \leq n' \implies m+n \leq m'+n' \wedge mn \leq m'n' .$$

The following lemma is frequently used.

Lemma 3.1 (Euclid) *For every $m \in \mathbf{N}$ and $n \in \mathbf{N} \setminus \{0\}$, there exist unique $q, r \in \mathbf{N}$ such that*

$$m = qn + r \qquad \text{and} \qquad 0 \leq r < n . \tag{241}$$

Proof. Consider the set

$$E := \{ l \in \mathbf{N} \mid m < ln \} .$$

Since $1 \leq n$, one has $m \cdot 1 \leq mn$; hence $mn \in E$ and E is not empty. Let k be its smallest element. Since every element of E is greater than 0, there exists $q := k - 1 \in \mathbf{N}$ and

$$qn \leq m < qn + n .$$

Equivalently, $r := m - qn$ satisfies the double inequality

$$0 \leq r < n .$$

If q and r satisfy Equality (241), then $q + 1$ is the smallest element of set E; hence, representation (241) of m is unique. □

3.4 Morphisms between n-ary relations

3.4.1

A general approach to binary interactions between n-ary relations with the same domain-list consists of using a binary relation $\sim$ on $\mathrm{Rel}(X_1, \ldots, X_n)$ to verify, for given $\rho, \rho' \in \mathrm{Rel}(X_1, \ldots, X_n)$, whether $\rho \sim \rho'$ or not.

3.4.2

Given two n-ary relations whose domain-lists are arbitrary and not necessarily equal

$$\rho : X_1, \ldots, X_n \longrightarrow \text{Statements} \qquad \text{and} \qquad \rho' : X'_1, \ldots, X'_n \longrightarrow \text{Statements} ,$$

we may use a function-list (98) to *pull-back* ρ' to the domain-list of ρ and then *declare* that function-list a *$\sim$-morphism from ρ to ρ'* if

$$\rho \sim (f_1, \ldots, f_n)^{\bullet} \rho' . \tag{242}$$

Note that the identity-list

$$\mathrm{id}_{X_1}, \ldots, \mathrm{id}_{X_n}$$

is a $\sim$-morphism precisely when $\rho \sim \rho'$.

3.4.3

This approach requires that every domain-list $X_1, \dots, X_n$ has been equipped with a binary relation $\sim$.

There is a canonical way to equip sets $\mathrm{Rel}(X_1, \dots, X_n)$ with binary relations that are induced by a single binary relation on the common target of all relations, the set of statements.

3.4.4 Definition of a $\sim$-morphism

Given a binary relation $\sim$ on the set of statements, we *declare* a function-list $f_1, \dots, f_n$ to be a *$\sim$-morphism from ρ to ρ'* if condition (242) holds for the relation induced by $\sim$ on $\mathrm{Rel}(X_1, \dots, X_n)$.

3.4.5

Note that we use the same symbol $\sim$ to denote the original relation on the set of statements, and the induced relation on $\mathrm{Rel}(X_1, \dots, X_n)$. This is a common practice and rarely leads to confusion if used with care. The actual meaning is usually clear from the context.

This practice is analogous to using the same symbol $+$ for addition of real numbers, as well as the *induced* operation of addition of real-valued functions.

3.4.6 $\Rightarrow$-morphisms, $\Leftarrow$-morphisms, $\Leftrightarrow$-morphisms

An essential feature of the language of morphisms is the expectation that morphisms can be *composed* and that the composition law is associative. This requirement narrows the choice of the binary relations $\sim$ on the set of statements to transitive relations.

Recall that any transitive relation on the set of statements that is stronger than the *equipotence* relation $\Leftrightarrow$, is necessarily equipotent to $\Leftrightarrow$, $\Rightarrow$, $\Leftarrow$, or is a total relation, cf. Lemma 1.5. The first three are, in practice, the only choices for $\sim$ that lead to a nontrivial notion of a morphism between relations.

Note that a function-list (98) is a $\Leftrightarrow$-morphism if and only if it is at once a $\Rightarrow$-morphism and a $\Leftarrow$-morphism.

3.4.7

Composition of two $\sim$-morphisms, where $\sim$ is one of those three relations $\Rightarrow$, $\Leftarrow$, or $\Leftrightarrow$, and both morphisms have the same type, is again a $\sim$-morphism and of the same type.

3.4.8

The definition of a $\sim$-morphism for each of those three choices of $\sim$ is expressed by means of the corresponding diagram

($\Rightarrow$-**morphism**)

$$\begin{tikzcd} X'_1, \dots, X'_n \arrow[dr, "\rho'"] & \\ & \text{Statements} \\ X_1, \dots, X_n \arrow[uu, "f_1 \ \cdots \ f_n"] \arrow[ur, "\rho"'] \arrow[ur, phantom, shift left=6, "\Uparrow"] & \end{tikzcd} \tag{243}$$

$$(\Leftarrow\text{-}\mathbf{morphism}) \qquad \begin{tikzcd} X'_1,\dots,X'_n \arrow[dr, "\rho'"] & \\ {} \arrow[r, phantom, "\Downarrow"] & \text{Statements} \\ X_1,\dots,X_n \arrow[uu, "f_1 \,\cdots\, f_n"] \arrow[ur, "\rho"'] & \end{tikzcd} \tag{244}$$

$$(\Leftrightarrow\text{-}\mathbf{morphism}) \qquad \begin{tikzcd} X'_1,\dots,X'_n \arrow[dr, "\rho'"] & \\ {} \arrow[r, phantom, "\Updownarrow"] & \text{Statements} \\ X_1,\dots,X_n \arrow[uu, "f_1 \,\cdots\, f_n"] \arrow[ur, "\rho"'] & \end{tikzcd} \tag{245}$$

Exercise 70 *Show that $f_1, \dots, f_n$ is a $\sim$-morphism from ρ to ρ' if and only if*

$$\Gamma_\rho \subseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'} \,, \qquad \Gamma_\rho \supseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'} \qquad \textit{or, respectively,} \qquad \Gamma_\rho = (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'} \,, \tag{246}$$

depending on whether $\sim$ is $\Rightarrow$, $\Leftarrow$, or $\Leftrightarrow$.

3.4.9 Characterization of $\Rightarrow$-morphisms

Recall that

$$(f_1 \times \cdots \times f_n)_* \Gamma_\rho \subseteq \Gamma_{\rho'} \;\Leftrightarrow\; \Gamma_\rho \subseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'} \,. \tag{247}$$

It follows from (246)–(247) that the following conditions are equivalent.

(a) $f_1, \dots, f_n$ is a $\Rightarrow$-morphism from ρ to ρ'.

(b) ρ is *weaker* than the pull-back of ρ' by $f_1, \dots, f_n$, cf. Section 1.14.17.

(c) ρ' is *stronger* than the push-forward of ρ by $f_1, \dots, f_n$, cf. Section 1.14.18.

3.4.10 Characterization of $\Leftarrow$-morphisms

There is a similar characterization of $\Leftarrow$-morphims. It is based on the middle part of (246) and on the equivalence

$$(f_1 \times \cdots \times f_n)_! \Gamma_\rho \supseteq \Gamma_{\rho'} \;\Leftrightarrow\; \Gamma_\rho \supseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'} \,. \tag{248}$$

The following conditions are equivalent.

(a) $f_1, \dots, f_n$ is a $\Leftarrow$-morphism from ρ to ρ'.

(b) ρ is *stronger* than the pull-back of ρ' by $f_1, \dots, f_n$.

(c) ρ' is *weaker* than the conjugate push-forward of ρ by $f_1, \dots, f_n$, cf. Section 1.14.19.

3.4.11 Terminology

In practice, $\Rightarrow$-morphisms are usually referred simply as *morphisms* (between relations), $\Leftrightarrow$-morphisms are frequently referred to as *strict morphisms,* while $\Leftarrow$-morphisms rarely make their appearance in actual arguments built by mathematicians.

3.4.12

Functions $\forall_{x_i \in A_i}$ and $\forall^i$, cf. (33) and (35), defined by *universal* quantification, are $\Rightarrow$-morphisms of binary relational structures if we equip the sets of relations with the *implication* relation $\Longrightarrow$.

This fact, known since at least the times of Aristotle, is hardly ever mentioned, yet it is constantly used when reasoning is based on rules of Logic.

Exercise 71 *Is $\forall_{x_i \in A_i}$ or $\forall^i$ a $\Leftrightarrow$-morphism?*

Exercise 72 *Is $\exists_{x_i \in A_i}$ or $\exists^i$ a $\sim$-morphism for $\sim$ being $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$?*

3.4.13 Morphisms between relational structures

When

$$X_1 = \cdots = X_n = X\,, \qquad X'_1 = \cdots = X'_n = X' \qquad \text{and} \qquad f_1 = \cdots = f_n = f\,,$$

we shall be denoting the pulled-back relation $(f, \dots, f)^{\bullet}\rho'$ by $f^{\bullet}\rho'$.

We say that $f : X \to X'$ is a *$\sim$-morphism from a relational structure (X, ρ) to a relational structure (X', ρ')* if

$$\rho \sim f^{\bullet}\rho'\,.$$

3.5 The ordered $*$-monoid of 2-correspondences $(\mathscr{P}(X{\times}X); \Delta_X, \tau_*, \circ\,; \subseteq)$

3.5.1

If $C \subseteq X{\times}X$ and $D \subseteq X{\times}X$, then their composition $C \circ D$, defined in Section 1.17.10,

$$C \circ D = \big\{(x,y) \in X{\times}X \mid \exists_{z\in X}\ (x,z){\in}C \wedge (z,y){\in}D\big\}\,, \tag{249}$$

is contained in $X{\times}X$. In particular, the set of 2-correspondences on a set X, equipped with the composition operation, is a semigroup.

3.5.2 (Pre)ordered binary algebraic structures

Let $(B, \cdot\,, \precsim)$ be a set equipped with a binary algebraic operation and a preorder relation $\precsim$. We say that $(B, \cdot\,, \precsim)$ is a preordered binary algebraic structure if

$$\forall_{a,a',b,b'\in B}\quad a{\precsim}a' \wedge b{\precsim}b' \Rightarrow ab \precsim a'b'\,. \tag{250}$$

Exercise 73 *Show that, if $C \subseteq C'$ and $D \subseteq D'$, then*

$$C{\circ}D \subseteq C'{\circ}D'\,, \tag{251}$$

i.e., $(\mathscr{P}(X{\times}X), \circ\,, \subseteq)$ is an ordered semigroup.

3.5.3 The diagonal subsets $\Delta_n(X) \subset X^n$

The subset

$$\Delta_n(X) := \{(x_1, \dots, x_n) \in X \times \cdots \times X \,|\, x_1 = \cdots = x_n\} \tag{252}$$

is referred to as the *n-diagonal*.

3.5.4 The diagonal function $\Delta : X \longrightarrow X \times X$ and its image Δ_X

The 2-diagonal set is usually denoted Δ_X. It coincides with the image of the *diagonal function*

$$\Delta : X \longrightarrow X \times X, \qquad x \longmapsto (x, x). \tag{253}$$

Exercise 74 *Show that,*

$$\forall_{C \subseteq X \times X}\ \Delta_X \circ C = C = C \circ \Delta_X, \tag{254}$$

i.e., Δ_X is an identity element for $\circ$. In particular, $(\mathscr{P}(X\times X), \Delta_X, \circ, \subseteq)$ is an ordered monoid.

3.5.5 The graph homomorphism $\Gamma : (\mathrm{Op}_1 X, \mathrm{id}_X, \circ) \longrightarrow (\mathscr{P}(X\times X), \Delta_X, \circ)$

The graph-of-a-function correspondence

$$f \longmapsto \Gamma_f := \{(x, y) \in X\times X \mid f(x) = y\}$$

is an injective homomorphism of monoids

$$(\mathrm{Op}_1 X, \mathrm{id}_X, \circ) \longrightarrow (\mathscr{P}(X\times X), \Delta_X, \circ).$$

It identifies the monoid of unary operations on X with a submonoid of 2-correspondences on X.

3.5.6 Antiinvolutions

Let $(B, \cdot)$ be a binary algebraic structure. An operation $\alpha : B \longrightarrow B$ is said to be an *antiinvolution* if it satisfies the identities

$$\alpha \circ \alpha = \mathrm{id}_B \qquad \text{and} \qquad \forall_{a,b \in B}\ \alpha(ab) = \alpha(b)\alpha(a). \tag{255}$$

3.5.7 $*$-binary structures

A binary structure equipped with an antiinvolution, $(B, \alpha, \cdot)$, is called a $*$-binary structure.

Exercise 75 *Let $e \in B$ be a left-identity element for $(B, \cdot)$. Show that $\alpha(e)$ is a right-identity for $(B, \cdot)$.*

Solution. For any $a \in B$, one has

$$a\alpha(e) = (\alpha(\alpha(a))\alpha(e) = \alpha(e\alpha(a)) = \alpha(\alpha(a)) = a.$$

□

Exercise 76 *Let $e \in B$ be a right-identity element for $(B, \cdot)$. Show that $\alpha(e)$ is a left-identity for $(B, \cdot)$.*

In Section 2.2.15 we observed that, if a binary structure $(B, \cdot)$ admits both a left and a right-identity, then they are equal. It follows that if a $*$-binary structure contains a one-sided identity element e, then this element is necessarily a two-sided identity and e is fixed by the antiinvolution

$$\alpha(e) = e. \tag{256}$$

3.5.8 The flip operation on $X{\times}X$

Let us denote by τ the operation on set $X{\times}X$ that transposes the factors in $X{\times}X$,

$$\tau(x_1, x_2) := (x_2, x_1)\,. \tag{257}$$

Exercise 77 *Show that τ_* is an antiinvolution on the monoid of 2-correspondences $(\mathscr{P}(X{\times}X), \Delta_X, \circ)$, i.e.,*

$$\tau_*(C{\circ}D) = \tau_* D \circ \tau_* C\,. \tag{258}$$

In particular, $(\mathscr{P}(X{\times}X); \Delta_X, \tau_*, \circ\,; \subseteq)$ is an ordered $*$-monoid.

3.5.9

In Mathematics and, especially, in Mathematical Physics, $*$-structures play an important role. We encounter $*$-semigroups, $*$-monoids, $*$-groups, and $*$-rings, i..e, rings equipped with an antiinvolution for addition and multiplication.

3.5.10

The ring of $n{\times}n$-matrices equipped with matrix transposition is an example of a $*$-ring that you are familiar with. Another example is provided by the field of complex numbers $\mathbf{C}$ equipped with complex conjugation.

Theory of $*$-rings of linear operators has been one of the most active areas of Mathematics during the last 80 years. One of its multiple applications to Mathematical Physics has been Constructive Quantum Field Theory.

3.5.11 The preordered $*$-structure of binary relations $(\mathbf{Rel}_2 X; =, (\ )^{\mathrm{op}}, \circ\,; \Rightarrow)$

The set of binary relations on a set X is canonically equipped with a preordered $*$-structure. Implication $\Rightarrow$ is the preorder, composition of relations is the (nonassociative) binary operation and the opposite-relation operation is the antiinvolution.

3.5.12 The graph homomorphism $\Gamma : (\mathbf{Rel}_2 X; =, (\ )^{\mathrm{op}}, \circ\,; \Rightarrow) \to (\mathscr{P}(X{\times}X); \Delta_X, \tau_*, \circ\,; \subseteq)$

Exercise 78 *Show that the graph of the equality relation $=$ on X equals Δ_X.*

Exercise 79 *Show that, for any binary relation ρ on X, one has*

$$\Gamma_{\rho^{\mathrm{op}}} = \tau_* \Gamma_\rho\,. \tag{259}$$

Exercise 80 *Show that, for any binary relations ρ and σ on X, one has*

$$\Gamma_{\rho\circ\sigma} = \Gamma_\rho \circ \Gamma_\sigma\,. \tag{260}$$

By combining Exercises 17, 78, 79, and 80, we conclude that the graph function

$$\Gamma : \mathrm{Rel}_2 X \longrightarrow \mathscr{P}(X{\times}X)$$

is a surjective homomorphism of preordered $*$-structures.

3.5.13 Graph characterizations of various types of binary relations

Each of the important properties of a binary relation ρ on a set X admits a characterization in terms of the graph Γ_ρ of ρ.

3.5.14 Subidempotent correspondences

Let us denote by $\mathscr{P}^{\mathrm{subid}}(X{\times}X)$ the set

$$\{C \subseteq X{\times}X \| C \circ C \subseteq C\} \tag{261}$$

of *subidempotent* correspondences.

Exercise 81 *Show that ρ is transitive if and only if*

$$\Gamma_\rho \circ \Gamma_\rho \subseteq \Gamma_\rho\,, \tag{262}$$

i.e., Γ_ρ is a subidempotent in ordered semigroup $(\mathscr{P}(X{\times}X), \circ, \subseteq)$.

Exercise 82 *Let $\mathscr{C} \subseteq \mathscr{P}^{\mathrm{subid}}(X{\times}X)$. Show that*

$$\bigcap \mathscr{C} \in \mathscr{P}^{\mathrm{subid}}(X{\times}X)\,,$$

i.e., the family of subidempotent correspondences $\mathscr{P}^{\mathrm{subid}}(X{\times}X)$ is closed under intersection of arbitrary subfamilies.

3.5.15

Given $C \subseteq X{\times}X$, the family

$$\mathscr{C}_C^{\mathrm{subid}} := \left\{D \in \mathscr{P}^{\mathrm{subid}}(X{\times}X) \,|\, D \supseteq C\right\} \tag{263}$$

contains $X{\times}X$, hence is not nempty. According to Exercise 82,

$$C^{\mathrm{subid}} := \bigcap \mathscr{C}_C^{\mathrm{subid}} \tag{264}$$

is the smallest subidempotent correspondence containing C.

3.5.16 A weakest transitive relation stronger than ρ

Suppose that $\rho \Rightarrow \sigma$ and σ is a transitive relation. This is equivalent to

$$\Gamma_\rho \subseteq \Gamma_\sigma \qquad \text{and} \qquad \Gamma_\sigma \in \mathscr{P}^{\mathrm{subid}}(X{\times}X)\,.$$

Then

$$\Gamma_\rho \subseteq \left(\Gamma_\rho\right)^{\mathrm{subid}} \subseteq \Gamma_\sigma\,,$$

which is equivalent to

$$\rho \Rightarrow \rho' \Rightarrow \sigma$$

where ρ' is any relation with graph $\left(\Gamma_\rho\right)$.

In particular, we established that, for any relation $\rho \in \mathrm{Rel}_2 X$, there exists a *weakest transitive relation stronger than ρ*. Its graph is the smallest subidempotent correspondence containing Γ_ρ.

3.5.17 A weakest reflexive relation stronger than ρ

Exercise 83 *Show that ρ is reflexive if and only if*

$$\Gamma_\rho \supseteq \Delta_X . \tag{265}$$

Exercise 84 *Show that, for a reflexive relation ρ, one has*

$$\Gamma_\rho \subseteq \Gamma_\rho \circ \Gamma_\rho . \tag{266}$$

Exercise 85 *Let $\rho \in \mathrm{Rel}_2 X$. Show that*

$$\textit{if } \rho \Rightarrow \sigma \textit{ and } \sigma \textit{ is reflexive, then } \rho \Rightarrow \rho\vee= \;\Rightarrow \sigma . \tag{267}$$

Here $\rho\vee=$ denotes the alternative of ρ and the equality relation on X.

In other words, for any relation $\rho \in \mathrm{Rel}_2 X$, relation $\rho\vee=$ is a *weakest reflexive relation stronger than* ρ.

Exercise 86 *Show that ρ is a preorder if and only if*

$$\Gamma_\rho \circ \Gamma_\rho = \Gamma_\rho , \tag{268}$$

i.e., Γ_ρ is an idempotent in semigroup $(\mathscr{P}(X\times X), \circ)$.

3.5.18 A weakest preorder stronger than ρ

Since the intersection of any family of correspondences containing Δ_X contains Δ_X, the intersection of any family of idempotents, i.e., subidempotents containing Δ_X, is an idempotent.

Thus,

$$\bigcap\{D \subseteq X\times X \mid D\circ D = D \text{ and } D \supseteq C\} \tag{269}$$

is the smallest idempotent correspondence containing C.

Exercise 87 *Let $\rho \in \mathrm{Rel}_2 X$. Show that there exists a preorder ρ' satisfying the following universal property:*

$$\textit{if } \rho \Rightarrow \sigma \textit{ and } \sigma \textit{ is a preorder, then } \rho \Rightarrow \rho' \Rightarrow \sigma . \tag{270}$$

In other words, for any relation $\rho \in \mathrm{Rel}_2 X$, there exists a *weakest preorder stronger than* ρ.

3.5.19 A weakest symmetric relation stronger than ρ

Exercise 88 *Show that ρ is symmetric if and only if its graph Γ_ρ is τ-invariant, i.e.,*

$$\Gamma_\rho \subseteq \tau_* \Gamma_\rho . \tag{271}$$

Exercise 89 *Show that* (271) *implies (and therefore is equivalent to) the stronger condition*

$$\Gamma_\rho = \tau_* \Gamma_\rho . \tag{272}$$

In other words, a relation ρ is symmetric if and only if its graph is a fixed point of τ_*.

Exercise 90 *Let* $\rho \in \mathrm{Rel}_2 X$.

$$\textit{if } \rho \Longrightarrow \sigma \textit{ and } \sigma \textit{ is symmetric, then } \rho \Longrightarrow \rho \vee \rho^{\mathrm{op}} \Longrightarrow \sigma\,. \tag{273}$$

In other words, for any relation $\rho \in \mathrm{Rel}_2 X$, relation $\rho \vee \rho^{\mathrm{op}}$ is a *weakest symmetric relation stronger than* ρ.

Exercise 91 *Show that* ρ *is an equivalence relation if and only if* Γ_ρ *is a* τ*-invariant idempotent in* $(\mathscr{P}(X \times X), \Delta_X, \circ)$.

3.5.20 A weakest equivalence relation stronger than ρ

Intersection of any family of τ-invariant subsets of $\mathscr{P}(X \times X)$ is τ-invariant. Thus,

$$\bigcap \{D \subseteq X \times X \mid D \circ D = D,\ D \subseteq \tau_* D \text{ and } D \supseteq C\} \tag{274}$$

is the smallest τ-invariant idempotent correspondence containing C.

Exercise 92 *Let* $\rho \in \mathrm{Rel}_2 X$. *Show that there exists an equivalence relation* ρ' *satisfying the following universal property:*

$$\textit{if } \rho \Longrightarrow \sigma \textit{ and } \sigma \textit{ is an equivalence relation, then } \rho \Longrightarrow \rho' \Longrightarrow \sigma\,. \tag{275}$$

In other words, for any relation $\rho \in \mathrm{Rel}_2 X$, there exists a *weakest equivalence relation stronger than* ρ.

3.5.21

Exercise 93 *Show that* ρ *is antisymmetric if and only if*

$$\Gamma_\rho \cap \tau_* \Gamma_\rho = \emptyset\,. \tag{276}$$

Exercise 94 *Show that* ρ *is weakly antisymmetric if and only if*

$$\Gamma_\rho \cap \tau_* \Gamma_\rho \subseteq \Delta_X\,. \tag{277}$$

Exercise 95 *Show that* ρ *is an order relation if and only if* Γ_ρ *is an idempotent in* $(\mathscr{P}(X \times X), \Delta_X, \circ)$ *and*

$$\Gamma_\rho \cap \tau_* \Gamma_\rho = \Delta_X\,.$$

3.6 Morphisms between structures of functional type

3.6.1

Suppose that a set X is equipped with a family of functions

$$\mathscr{O} \subset \mathrm{Funct}(X, \mathbf{R})$$

and a set X' is equipped with a family of functions

$$\mathscr{O}' \subset \mathrm{Funct}(X', \mathbf{R})\,.$$

We say that a function $f : X \to X'$ is a morphism if, for every $\phi' \in \mathscr{O}'$, the composite function $f^\bullet \phi' = \phi' \circ f$ belongs to $\mathscr{O}$,

$$\forall_{\phi' \in \mathscr{O}'}\ f^\bullet \phi' \in \mathscr{O}\,. \tag{278}$$

3.6.2

An equivalent form of condition (278) is

$$(f^\bullet)_* \mathscr{O}' \subset \mathscr{O}\,. \tag{279}$$

This, in turn, can be expressed in the language of diagrams: a function $f : X \to X'$ is a morphism if the diagram

$$\begin{tikzcd}
\mathscr{O} \arrow[d, hook] & \mathscr{O}' \arrow[d, hook] \\
\mathrm{Funct}(X, \mathbf{R}) & \mathrm{Funct}(X', \mathbf{R}) \arrow[l, "f^\bullet"']
\end{tikzcd}$$

admits a completion to a commutative square diagram

$$\begin{tikzcd}
\mathscr{O} \arrow[d, hook] & \mathscr{O}' \arrow[l] \arrow[d, hook] \\
\mathrm{Funct}(X, \mathbf{R}) & \mathrm{Funct}(X', \mathbf{R}) \arrow[l, "f^\bullet"']
\end{tikzcd}$$

3.7 Morphisms between structures of topological type

3.7.1

Suppose that a set X is equipped with a family of subsets $\mathscr{A} \subset \mathscr{P}X$ and a set X' is equipped with a family of subsets $\mathscr{A}' \subset \mathscr{P}X'$. We say that a function $f : X \to X'$ is a morphism if the preimage under f of every member of family $\mathscr{A}'$ is a member of $\mathscr{A}$,

$$\forall_{A' \in \mathscr{A}'}\; f^\bullet A' \in \mathscr{A}\,. \tag{280}$$

3.7.2

An equivalent form of condition (280) is

$$(f^\bullet)_* \mathscr{A}' \subset \mathscr{A}\,. \tag{281}$$

Notice the similarity to condition (279).

3.7.3

Condition (281) can be expressed by saying that the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[d, hook] & \mathscr{A}' \arrow[d, hook] \\
\mathscr{P}Y & \mathscr{P}Y' \arrow[l, "f^*"']
\end{tikzcd}$$

admits a completion to a commutative square diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[d, hook] & \mathscr{A}' \arrow[l] \arrow[d, hook] \\
\mathscr{P}Y & \mathscr{P}Y' \arrow[l, "f^*"']
\end{tikzcd}$$

3.7.4 Continuous functions

When $\mathscr{A}$ and $\mathscr{A}'$ have the meaning of being the families of *open subsets* in a topological spaces, i.e., when $(X, \mathscr{A})$ and $(X', \mathscr{A}')$ are topological spaces, cf. Section 2.1.6, we obtain the definition of a morphism between topological spaces. This is precisely how a continuous function is defined.

3.7.5 Measurable functions

When $\mathscr{A}$ and $\mathscr{A}'$ have the meaning of being the families of *measurable subsets* in a measurable spaces, i.e., when $(X, \mathscr{A})$ and $(X', \mathscr{A}')$ are measurable spaces, cf. Section 2.1.7, we obtain the definition of a morphism between measurable spaces. This is precisely how a measurable function is defined.

3.7.6

Another condition that can be interpreted as saying that f respects distinguished families of subsets reads

$$\forall_{A \in \mathscr{A}} \, f_* A \in \mathscr{A}' \tag{282}$$

or, equivalently,

$$(f_*)_* \mathscr{A} \subset \mathscr{A}' \,. \tag{283}$$

Either condition can serve as a definition of a morphism between structures of topological type. It is however the former, (280), that plays a fundamental role in Topology and Measure Theory, not the latter, (282).

4 The language of categories

4.1 The concept of a category

4.1.1

Whatever definition of a morphism between mathematical structures one adopts, it always has the following features

- any morphism α has a *source* and a *target* that are mathematical structures of the same type
- if the source $s(\alpha)$ of a morphism α coincides with the target $t(\beta)$ of a morphism β, then their composition $\alpha \circ \beta$ is defined and

$$t(\alpha \circ \beta) = t(\alpha) \qquad \text{and} \qquad s(\alpha \circ \beta) = s(\beta)$$

- composition of morphisms is associative, i.e.,

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

  for any composable list α, β, γ of morphisms

4.1.2

The above observations led to the introduction of the concept of a *category*. In a nutshell, a *category* $\mathcal{C}$ consists of two classes, a class $\mathcal{C}_0$ of *objects* and a class $\mathcal{C}_1$ of *morphisms*, equipped with an associative operation of composition of morphisms.

4.1.3

Various classes of mathematical structures equipped with appropriate classes of morphisms form natural categories. Studying the category of groups is what Group Theory does. Studying the category of rings is what Ring Theory does. Algebraic geometers study the category of *algebraic varieties* and the bigger category of *algebraic schemes*. Topologists study the category of *topological spaces*, and so on.

4.1.4

Every mathematical theory can be expressed in a categorical language. This usually provides an added degree of clarity to a theory and yields insights that are otherwise lost.

4.2 Basic vocabulary

4.2.1 Epimorphisms

A morphism α is said to be a *epimorphism* if, for any diagram

$$\bullet \underset{\psi}{\overset{\phi}{\leftleftarrows}} \bullet \xleftarrow{\ \alpha\ } \bullet\ , \tag{284}$$

equality $\phi \circ \alpha = \psi \circ \alpha$ implies $\phi = \psi$.

Exercise 96 *Show that the composite of two epimorphisms is an epimorphism.*

Exercise 97 *Show that if $\alpha \circ \beta$ is an epimorphism, then α is an epimorphism.*

Exercise 98 *Show that a function f is an epimorphism in the category of sets if and only if f is surjctive.*

4.2.2 Monomorphisms

A morphism α is said to be a *monomorphism* if, for any diagram

$$\bullet \xleftarrow{\alpha} \bullet \underset{\psi}{\overset{\phi}{\leftleftarrows}} \bullet \,, \tag{285}$$

quality $\alpha \circ \phi = \alpha \circ \psi$ implies $\phi = \psi$.

Exercise 99 *Show that the composite of two monomorphisms is a monomorphism.*

Exercise 100 *Show that if $\alpha \circ \beta$ is a monomorphism, then β is an monomorphism.*

Exercise 101 *Show that a function f is a monomorphism in the category of sets if and only if f is injective.*

4.2.3 Initial objects

An object i is said to be *initial* if, for every object c, there exists a unique morphism $i \to c$.

4.2.4 Terminal objects

An object t is said to be *termial* if, for every object c, there exists a unique morphism $c \to t$.

4.3 Endomorphisms

4.3.1

Morphisms whose source and target coincide with an object c are referred as *endomorphisms* of object c.

4.3.2 The identity endomorphism

Let ι be an endomorpism of an object c such that

$$\forall_\alpha \; s(\alpha){=}c \Longrightarrow \alpha{\circ}\iota = \alpha$$

and

$$\forall_\beta \; t(\beta){=}c \Longrightarrow \iota{\circ}\beta = \beta \,.$$

We shall call it an *identity endomorphism*.

Exercise 102 *Suppose that ι and ι' are identity endomorphisms of an object c. Show that $\iota = \iota'$.*

It follows that, if c admits an identity endomorphism, then it is unique. We denote this unique identity endomorphism id_c or 1_c.

4.3.3 Unital categories

A category is said to be *unital* if every object admits an identity endomorphism.

4.3.4 A right-inverse of a morphism

We say that β is a *right-inverse* of a morphism α if

$$s(\alpha) = t(\beta)\,, \qquad t(\alpha) = s(\beta) \qquad \text{and} \qquad \alpha \circ \beta = \mathrm{id}_{t(\alpha)}\ .$$

Existence of the identity endomorphism $\mathrm{id}_{t(\alpha)}$ is a necessary condition for α to be right-invertible.

4.3.5 Split epimorphisms

Exercise 103 *Show that a right-invertible morphism is an epimorphism.*

On this account, we call a right-invertible morphism a *split epimorphism*. A *splitting* of a epimorphism is, by definition, any of its right-inverses.

Exercise 104 *Show that the composite of two split epimorphisms is a split epimorphism.*

4.3.6 A left-inverse of a morphism

We say that β is a *left-inverse* of a morphism α if

$$s(\alpha) = t(\beta)\,, \qquad t(\alpha) = s(\beta) \qquad \text{and} \qquad \beta \circ \alpha = \mathrm{id}_{s(\alpha)}\ .$$

Existence of the identity endomorphism $\mathrm{id}_{s(\alpha)}$ is a necessary condition for α to be left-invertible.

4.3.7 Split monomorphisms

Exercise 105 *Show that a left-invertible morphism is a monomorphism.*

On this account, we call a left-invertible morphism a *split monomorphism*. A *splitting* of a monomorphism is, by definition, any of its left-inverses.

4.3.8 The inverse of a morphism

Exercise 106 *Let β be a right-inverse of α and β' be a left-inverse of α. Show that $\beta = \beta'$.*

Solution. In view of associativity of composition of morphisms, one has

$$\beta = \mathrm{id}_{s(\alpha)} \circ \beta = (\beta' \circ \alpha) \circ \beta = \beta' \circ (\alpha \circ \beta) = \beta' \circ \mathrm{id}_{t(\alpha)} = \beta'\,.$$

□

It follows that existence of a (two-sided) inverse of α is equivalent to existence of a right and of a left inverse. Moreover, a two-sided inverse is unique when it exists. We denote it α^{-1}.

4.3.9 Isomorphisms

An invertible morphism, i.e., a morphism that admits an inverse, is called an *isomorphism*. Objects c and d are said to be *isomorphic* if there exists an isomorphism $c \to d$. Symbolically, this is expressed by $c \simeq d$.

4.3.10

According to Exercise 106 an isomorphism is morphism that is, at once, a split epimorphism and a split monomorphism.

4.3.11 Arrow notation

We signal that a morphism $\alpha : c \to d$ is a monomorphism, an epimorphism, or an isomorphism, by employing the following arrow notation

monomorphism	$\alpha : c \rightarrowtail d$	(286)
epimorphism	$\alpha : c \twoheadrightarrow d$	(287)
isomorphism	$\alpha : c \xrightarrow{\sim} d\,.$	(288)

4.3.12 The semigroup of endomorphisms

Equipped with composition as its binary operation, the set of endomorphisms of an object c of any category becomes a semigroup, denoted

$$\mathrm{End}_{\mathcal{C}}\, c\,. \tag{289}$$

The semigroups of endomorphisms of various mathematical structures play a fundamental role in nearly every area of Mathematics and Mathematical Pysics.

4.3.13 The monoid of endomorphisms

If object c admits an identity endomorphism, then

$$(\mathrm{End}_{\mathcal{C}}\, c, \mathrm{id}_c, \circ)$$

is a monoid. For example, the monoid of unary operations $\mathrm{Op}_1(X)$ on a set X is precisely the monoid of endomorphsisms of X viewed as an object of the category of sets.

4.3.14 The group of autorphisms

An invertible endomorphism of c is called an *automorphism*. The set $\mathrm{Aut}_{\mathcal{C}}\, c$ of automorphisms of c contains id_c and is closed under the operations of composition and passing to the inverse element. It coincides with the group of invertible elements in the monoid $\mathrm{End}_{\mathcal{C}}\, c$ of endomorphisms of c.

4.3.15 An action of a set A on an object of a category

If A is a set and c is an object of a category $\mathscr{C}$, we have a ready definition of an *action of A on c* if we notice that $\mathrm{Op}_1(X)$ in (192) coincides with the monoid of endomorphisms of X in the category of sets. Thus, an action of a set A on an object c is defined to be a function

$$L : A \longrightarrow \mathrm{End}_{\mathscr{C}}\, c\,. \tag{290}$$

4.3.16 An action of a binary structure $(A, \cdot)$ on an object of a category

We say that a binary structure $(A, \cdot)$ acts on an object c if the function in (290) is a homomorphism of binary structures.

4.3.17 An action of a monoid $(A, e, \cdot)$ on an object of a *unital* category

We say that a monoid $(A, \cdot)$ acts on an object c of a unital category if the function in (290) is a homomorphism of monoids.

4.3.18 Representation Theory of Groups

Classical Representation Theory studies group actions on the objects of the category of vector spaces over a field k. Such actions are referred to as *k-linear representations* of a given group. The cases $k = \mathbf{R}$ and $k = \mathbf{C}$ produce Real and, respectively, Complex Representation Theory.

4.3.19 Category of k-linear representations of a group

Given a group G, its k-linear representations form naturally objects of a category, and determination of the structure of that category is a central topic of Representation Theory.

4.3.20

Representation Theory has been, beginning from its roots in Linear Algebra in the latter part of 19th Century, an essential area of Mathematics, that had enormous impact on the development of Mathematical Physics in 20th Century. The shear wealth of the methods it employs and applications it produces is a reason why learning Representation Theory is simultaneously obligatory and takes several years of very intensive study.