MATH 53 PRACTICE FINAL

Please note that this practice test is designed to give you examples of the types of questions that will appear on the actual exam. Your performance on these sample questions should not be taken as an absolute prediction of your performance on the actual exam. When you take the actual exam, the questions you see will cover some of the same content included in the practice exam, as well as some content not tested on the practice exam.

(1) (a) Suppose that the surface S_1 is the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and S_2 is the lower hemisphere $x^2 + y^2 + z^2 = 1, z \leq 0$, oriented by the outward unit normal vector $\mathbf{n}$ on $x^2 + y^2 + z^2 = 1$. If $\mathbf{F}$ is a smooth vector field, what is the relationship between the integrals $\int\int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ and $\int\int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$?

(b) Does the relation above hold for the integrals $\int\int_{S_1} \mathbf{G} \cdot d\mathbf{S}$ and $\int\int_{S_2} \mathbf{G} \cdot d\mathbf{S}$ for general vector fields $\mathbf{G}$?

(c) Let $S = S_1 \cup S_2$. Find $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$

(2) Suppose a doughnut D has a parameterization in the from $r(\theta, \alpha)$, $0 \leq \theta, \alpha \leq 2\pi$, such that the normal vector to the surface of D is

$$a(b + a\cos\alpha)\langle \cos\alpha\cos\theta, \cos\alpha\sin\theta, \sin\alpha\rangle$$

where $0 < a < b$ are fixed constants. Find the surface area of D.

(3) (a) Find constants a and b such that $\mathbf{G} = \operatorname{curl} \mathbf{F}$, where $\mathbf{G} = \langle x^2e^z, 0, -2xe^z \rangle$ and $\mathbf{F} = axye^z, 0, bx^2ye^z\rangle$.

(b) Find $\int\int_S \mathbf{G} \cdot d\mathbf{S}$, where S is the portion of the surface $-x^2 - y^2 + z^2 = 1$ with $1 \leq z \leq 2$ where $\mathbf{G} = \langle x^2e^z, 0, -2xe^z \rangle$.

(4) Find the mass of a wire in the shape of the helix $x = t, y = \cos t, z = \sin t$, with $0 \leq t \leq 2\pi$, if the mass density function at any point is equal to the square of the distance from the origin.

(5) Find the volume of the ellipsoid $(\frac{x}{a})^2+(\frac{y}{b})^2+(\frac{z}{c})^2 \leq 1$ using transformations $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$. Find the flux of the vector field $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$ flowing through the surface of this ellipsoid.

(6) True or false. Provide explanations.

(a) There exists a vector field $\mathbf{G}$ such that $\mathbf{F} = \text{curl } \mathbf{G}$, where $\mathbf{F} = \langle xz, xyz, -y^2 \rangle$.

(b) If $\mathbf{F} = \langle M, N \rangle$ and $M_y = N_x$ in an open region R, then $\mathbf{F}$ is conservative.

(c) If f is a smooth scalar function and ∇f is never zero, then the minimum and maximum of f on a closed and bounded domain must occur on the boundary.

(7) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{x^2}, x + y^2 + z, y + \sin z \rangle$ and C is the triangle connecting the points (2, 0, 0), (0, 1, 0), (0, 0, 1), in the counterclockwise direction.

(8) Find the absolute extrema of the function $f(x, y) = \sin xy$ on the closed region given by $0 \leq x \leq \pi$ and $0 \leq y \leq 1$.

(9) A butterfly net has a circular rim of radius 5. A bug catcher fixes the net in such a way that the rim is in the xz plane with the center at the origin. We can observe that the net entirely billows out in the positive y direction and the velocity of air is given by the vector field $\mathbf{F} = \langle x^4 + 2y^2, 3 - y^2, 2yz - 4x^3 z \rangle$. Find the flux of this vector field through the net, oriented in the positive y direction.

(10) Suppose the variables x, y, z are constrained by some equation $g(x, y, z) = 3$. Suppose at the origin, $g = 3$ and $\nabla g = \langle 2, -1, -1 \rangle$. Find $\partial z/\partial x$ at the origin. Hint: the equation $g = 3$ implicitly defines z as a function of x and y.

(11) Heron's formula gives the area of a triangle with sides x, y, z with $A = \sqrt{s(s-x)(s-y)(s-z)}$, where $s = \frac{p}{2}$ and p is the perimeter of the triangle. Assume that $s \neq x, y, z$ and find values for x, y, z that maximize the triangle's area.

(12) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $F = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle$ and $\mathbf{r}(t) = \langle t^2 + 1, t^2 - 1, t^2 - 2t \rangle$ and $0 \leq t \leq 2$.