Prof. Ming Gu, 861 Evans, tel: 2-3145
Email: mgu@math.berkeley.edu

Math128A: Sample Final Exam

This is a closed book exam, with the exception of a one-page cheat sheet on one side only. You are allowed to cite any results, up to Section 6.6 but excluding those in the exercises, from the textbook. Results from anywhere else will need to be justified. Completely correct answers given without justification will receive little credit. Partial solutions will get partial credit.

Problem	Maximum Score	Your Score
1	12.5	
2	12.5	
3	12.5	
4	12.5	
5	12.5	
6	12.5	
7	12.5	
8	12.5	
Total	100	

Your Name and SID: ______________________

Your Section: ______________________

1. Let $h > 0$. Develop a finite difference method to approximate the derivative $f'(x_0)$ using function values $f(x_0 - h), f(x_0 + 2\,h)$.

   (a) What is the order of your method?

   (b) Turn your method into a second order method with extrapolation.

2. Consider the iteration

$$x_{k+1} = -\,(\alpha - 1)\; x_k + \frac{1}{2} x_k^2, \quad k = 0, 1, \cdots,$$

where $0 < \alpha \leq 1$ is given.

   (a) Show that the iteration converges to 0 for $0 < x_0 < 2\,\alpha$.
   (**hint:** show that $0 < x_{k+1} < x_k$ for all k)

   (b) Assume that the iteration converges to 0 for an initial point x_0. What is the order of convergence for $0 < \alpha < 1$? What is the order of convergence for $\alpha = 1$?

3. Determine the constants c and x_0 so that the following quadrature

$$\int_{-\infty}^{\infty} e^{-|x|}\, f(x)\, dx \approx c\, f(x_0)$$

is exact for $f(x) = 1,\, x$.

4. By performing Gaussian elimination with partial pivoting, find the permutation matrix P, the lower triangular matrix L with 1's on its diagonal, and the upper triangular matrix U so that

$$P\,A = L\,U, \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

5. Given an initial value ODE

$$y' = f(t, y), \quad \text{for} \quad a \le t \le b, \;\; y(a) = \alpha. \quad (1)$$

Consider the multi-step method for solving (1)

$$w_{k+1} = -4w_k + 5w_{k-1} + h\left(\alpha\, f(t_k, w_k) + \beta\, f(t_{k-1}, w_{k-1})\right), \quad k = 1, 2, \cdots, N-1 \quad (2)$$

with starting values w_0, w_1.

(a) Choose the constants α and β so the method (2) is of 2nd order.

6. Compute $\mathbf{det}(A)$ with Gaussian Elimination, where $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 4 & \alpha \end{pmatrix}$.

   For which values of α is $\mathbf{det}(A) = 0$?

7. Determine the exact conditions on the coefficients a, b, c, d, e under which the following function is a cubic spline (i.e., $f \in C^2(-\infty, \infty)$):

$$f(x) = \begin{cases} a(x-1)^2 + b\,x^3, & \text{if} \quad x \in (-\infty, 0], \\ c(x-1)^2, & \text{if} \quad x \in [0, 2], \\ d(x-2)^3 + e(x-1)^2, & \text{if} \quad x \in [2, \infty). \end{cases}$$

8. Does the function $f(t,y) = t^2\,y^3$ satisfy the Lipschitz condition on the domain
$$\mathcal{D} \stackrel{def}{=} \{(t,y)\ |\ 0 \le t \le 1,\ -\infty < y < \infty\}?$$