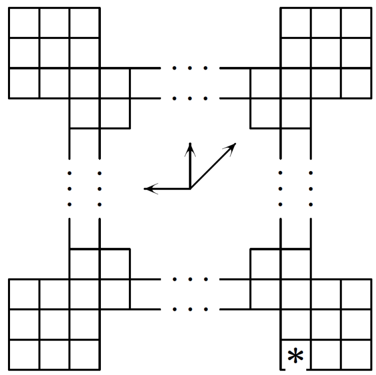
Game Theory Homework #1, 01/24/2021

All the problems are worth 4 points each and will be graded on a 0/1/2/3/4 scale. Due on Wednesday 02/03/2021 before 11:59 pm to be uploaded via bcourses.

1. (m-subtraction game): Given positive integers $a_1 < a_2 < \ldots < a_m$ define the $m-$subtraction game where a position consists of a pile of chips, and a legal move is to remove from the pile a_i chips for some $i \in \{1, 2, \ldots, m\}$. The player who cannot move loses. Show that the labelling of the set of states $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ in to P or N is periodic after some time i.e. there exists n_0 and k such that for all $n \geq n_0$ the labels of n and $n+k$ are the same.

2. Two players play a game on the board, which is shown on the figure. Initially there is a stone in the cell marked by $*$. A player can move the stone left, up, or up-right. A player who is unable to move loses. Is this progressively bounded? if yes, which player has the winning strategy? (Hint: Consider a cell through which the stone must pass through and then consider the two cases depending on whether the configuration given by the stone in that cell is P or N.)



3. Take two progressively bounded impartial combinatorial games G_a, G_b with state spaces X_a, X_b and set of legal moves M_a, M_b, respectively. Consider the sum $G = G_a + G_b$ of the two games with state space

$$X = X_a \times X_b = \{(x_a, x_b) : x_a \in X_a, x_b \in X_b\},$$

where legal moves consist of picking any coordinate and playing a legal move of that coordinate. Now call a state (x_a, x_b) as a (P, P) state if x_a and x_b are P states in the games G_a and G_b respectively. Similarly define $(N, P), (P, N), (N, N)$ respectively.
(0) Show G is progressively bounded.
(1) Show a (P, P) state is a P state in G.
(2) Show an (N, P) or a (P, N) state is an N state in G.
(3) Are (N, N) states always N states in G? Prove if yes, or provide a counterexample otherwise.

4. Consider the following game: The game starts with two piles of chips say with n_1 and n_2 chips denoted by (n_1, n_2). The players alternate moves, and each move consists of throwing away one of the piles and then dividing the contents of the other pile into two piles (each of which has at least one chip). For instance, if the piles had 15 and 9 chips in them, a legal move would be to throw away the pile with 15 chips and split the other pile of 9 chips into piles of 4 and 5. The game ends when no legal moves can be made, which happens when there is 1 chip in each pile. As usual, the first player who cannot make a legal move loses.

   (1) Show that the game is progressively bounded.
   (2) Classify the starting positions into N or P states (Hint: Work out the cases for small n_1, n_2, by hand and identify a pattern).

5. In class we showed the following theorem:

   Theorem 0.1. *In a progressively bounded impartial combinatorial game under normal play, $X = N \cup P$. That is, from any initial position, one of the players has a winning strategy. Moreover,*
   • *P: Every move leads to N.*
   • *N: Some move leads to P (hence cannot contain terminal positions).*

   Let now P' and N' be disjoint subsets of X (set of positions) such that $X = N' \cup P'$ and:
   • P': Every move leads to N'.
   • N': Some move leads to P'.

   Show that $N' = N$ and $P' = P$.