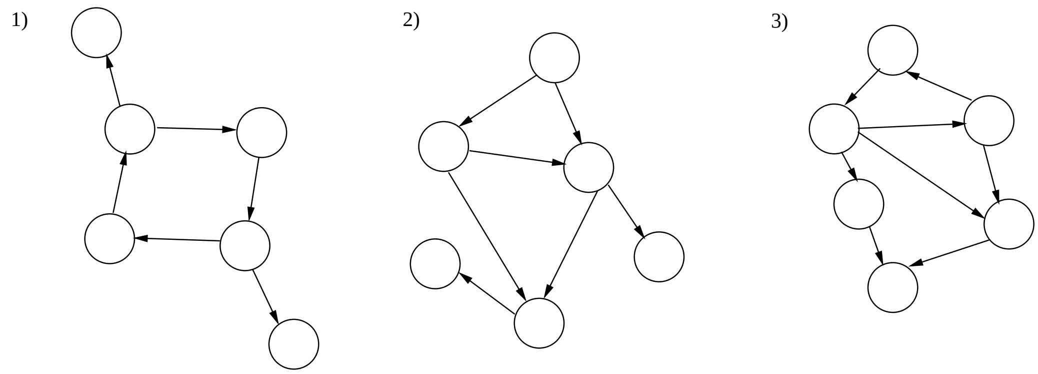
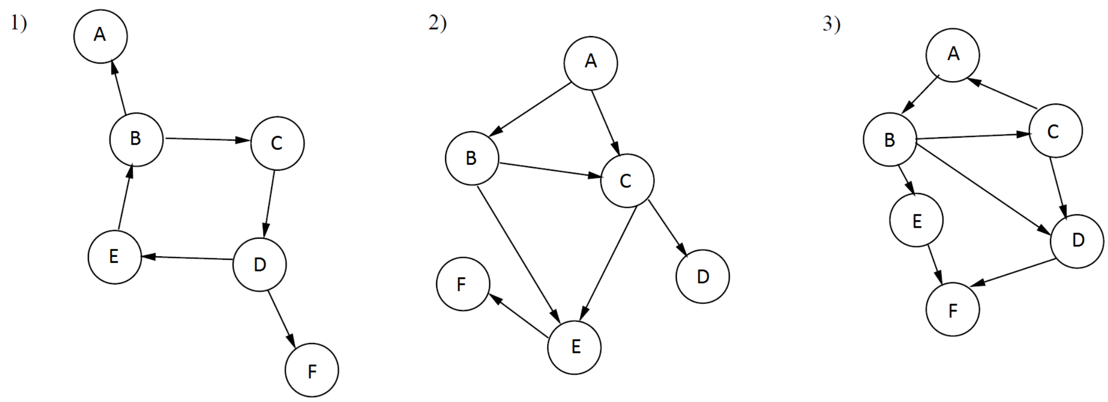
Game Theory (STAT 155) Worksheet #1, 01/25/2021

1. Two people play a game on a directed graph. There is a stone in one of the vertices of the graph. Players move the stone across directed edges in turns. A player who is unable to move loses. For each of the graphs on the figure find the sets **N** and **P**. For which of the graphs the game is progressively bounded?



Solution

For convenience, let's label the vertices of the graph as in the picture below:



1) Let's work backwards from the terminal states to see what we can say about each of the states in this game. First, observe that, if the stone starts in state A, then the starting player has no legal moves. Thus, the previous player wins, hence $A \in P$. The same logic shows that $F \in P$. Now look at state B: there is at least one (in fact, exactly one) outgoing arrow from B into a state in P (namely, A), and this implies $B \in N$. The same logic shows $D \in N$. Finally, look at stat C: all of the outgoing arrows (actually, there is only one outgoing arrow) point to states in N, hence $C \in P$. The same logic shows $E \in P$. Therefore, we have shown $P = \{A, C, E, F\}$ and $N = \{B, D\}$.

Now let's show that this game is not progressively bounded. Imagine that the stone starts in state B. If the first player moves the stone to state C, then the second player moves the stone to D, then the first player moves the stone to E, then the second player moves the stone to B, then we have executed four moves but ended up in the same position we started in. This sequence of moves can repeat infinitely many times. So, it is possible that, starting in state B, the game continues forever without reaching a terminal position. (In other words, this game is not progressively bounded since the directed graph detailing its states and moves contains a directed cycle.)

Recall that, in class, we showed that a game being progressively bounded is sufficient for its state space to be able to be partitioned into N and P. This game is an example that the condition is not necessary. That is, this game is not progressively bounded, but nonetheless we can classify all states into N and P.

In section, there was also another interesting point made. If the game starts in state C, then the second player has a winning strategy since $C \in P$. However, something even stronger is true: the first player only has one possible move at each step. (In some sense, this means that the first player does not really get to make any choices.) In other games that we'll see (including those in the next parts of this problem), the losing player may have many possible moves at some steps, but none of them can give a winning strategy.

2) Again let's work backwards from the terminal positions in order to find which states are in N and P. The states D and F have no outgoing arrows, so, if the stone starts there, then the current player has no legal moves and loses. Therefore, $D, F \in P$. Then, we see that both E and C have an outgoing arrow into a state in P, hence $E, C \in N$. Next consider state B. It has two outgoing arrows, both of which point to states in N, hence $B \in P$. Lastly, we note that A has two outgoing arrows, one of which leads to a state in P and one of which leads to a state in N. Since A has at least one outgoing arrow to a state in P, we have $A \in N$. Therefore, we have shown $P = \{B, D, F\}$ and $N = \{A, C, E\}$.

Next let's show that this game is progressively bounded. Write $X = \{A, B, C, D, E, F\}$. We claim that there is a function $b : X \to \mathbb{N}$ such that the game started in state $x \in X$ will terminate in at most $b(x)$ turns. Of course, $b(D) = b(F) = 0$ works. Now note that, from state E, you can only go to state F, so the number of moves is bounded above by $b(E) = 1 + b(F) = 1$. Similarly, from C we can only move to E or to D, so we can put $b(C) = 1 + \max(b(D), b(E)) = 2$. Next we have $b(B) = 1 + \max(b(E), b(C)) = 3$, and finally $b(A) = 1 + \max(b(B), b(C)) = 4$. We have constructed a function b as claimed, so the game is progressively bounded.

3) As always, we work backwards to determine the classification of states in this game. It is clear that F is a terminal position, so $F \in P$. Then both D and E have arrows going into a site in P, so $D, E \in N$. Now what can we say about A, B, and C? In order to classify B into P or N, we need to know whether it has some outgoing arrow into P. We know that $D, E \in N$, so the classification of B is determined by whether C is in P or N. Likewise, C points to $D \in N$, so the classification of C depends on whether A is in P or N. And A points only to B, so the classification of A depends on whether B is in P or N. Inuitively, this suggests that the states $\{A, B, C\}$ cannot be classified into either N or P.

To prove this, let's show that both players have a strategy for which, if the game starts in any of the states $\{A, B, C\}$, neither player will lose. (Here, not losing is *not* the same as winning!) Both players use the following strategy:

- If at my turn the stone is in $\{A, B, C\}$, then I move it according to the cycle rules $A \to B \to C \to A$.
- If at my turn the stone is in $\{D, E\}$, then I move it to F.
- If at my turn the stone is in F, then I lose. (This move will never have to be executed, as we see below.)

Let's imagine being one of the players, and play according to the strategy above. If the initial position was in the cycle, there are two cases to consider:

- The stone never left the cycle. Then, I did not lose.
- The stone leaves the cycle at some point. Let k be the index of the first turn at which the stone leaves the cycle. By minimality, this means that the stone is in the cycle at the $(k-1)$th move. Since I never move the stone out of the cycle, this means that $(k-1)$th move was not mine. So the $(k-1)$th move was the other players'. This means that turn $k+1$ is my turn at the stone is in either D or E. This means I move the stone to F. Not only do I not lose, but actually I win.

In both cases we didn't lose, so each player has a strategy which allows her to never lose. Thus, no one can have a winning strategy, and positions A, B, C are neither in N nor in P.

Since there are states that cannot be classified into either N or P, this implies (by the main theorem from lecture) that this game is not progressively bounded. Alternatively, one can observe directly that there is a sequence of moves (both players going through the cycle forever) under which the game never terminates.

2. Find the set of P-positions for the subtraction games with subtraction sets

   (a) $S = \{1, 3, 5\}$;

   Solution

   Let's work very carefully and classify the first few positions. We have $0 \in P$ by the definition of the game. This implies that $1, 3, 5 \in N$, since they all have at least one outgoing arrow into a 0, which is in P. What about 2? The only outgoing arrow from 2 is to 1, which is in N, so $2 \in P$. Similarly, $4 \in P$, since its outgoing arrows point to 1 and 3 which are both in N. Continuing in this way, we get the following table:

0	1	2	3	4	5	6
P	N	P	N	P	N	P

   At this point we might start to guess that $P = 2\mathbb{N}$ and $N = \mathbb{N} \setminus 2\mathbb{N}$. In section we saw how to prove this using induction, but there's also a more general approach that is useful for the other subtraction games. Let's go over both of these proofs.

   Lemma In the subtraction game with $S = \{1, 3, 5\}$, we have $P = 2\mathbb{N}$ and $N = \mathbb{N} \setminus 2\mathbb{N}$.

   Proof 1. Let's prove by induction that $\{k \in \mathbb{N} : k \text{ even}, k \le m\} \subseteq P$ and $\{k \in \mathbb{N} : k \text{ odd}, k \le m\} \subseteq N$. If we are successful, then this implies

$$2\mathbb{N} = \bigcup_{m=1}^{\infty} \{k \in \mathbb{N} : k \text{ even}, k \leq m\} \subseteq P, \text{ and } \mathbb{N} \setminus 2\mathbb{N} = \bigcup_{m=1}^{\infty} \{k \in \mathbb{N} : k \text{ odd}, k \leq m\} \subseteq N$$

which is the desired claim. So, let $S(m)$ denote the statement "$\{k \in \mathbb{N} : k \text{ even }, k \leq m\} \subseteq P$ and $\{k \in \mathbb{N} : k \text{ odd }, k \leq m\} \subseteq N$". We need to check that $S(0)$ is true, and that $S(m)$ being true implies $S(m+1)$ is true.

Checking that $S(0)$ is true is easy, since $\{k \in \mathbb{N} : k \text{ even}, k \leq 0\} = \{0\}$ and $\{k \in \mathbb{N} : k \text{ odd}, k \leq 0\}$ is empty. Recalling that $0 \in P$ for any subtraction game finishes this step.

Now we suppose that $S(m)$ is true, and we use it to show that $S(m+1)$ is true. Suppose that k is even and $k \leq m+1$. Then the outgoing arrows from state k are $k-1, k-3$, and $k-5$. Observe that all of these states are odd and less than or equal to m. So, by the inductive hypothesis, they are in N. Since all outgoing arrows from k are in N, we have shown $k \in P$. This gives $\{k \in \mathbb{N} : k \text{ even}, k \leq m+1\} \subseteq P$, which is half of the statement we need. For the other half, suppose that k is odd and $k \leq m+1$. Then the outgoing arrows from state k are $k-1, k-3$, and $k-5$, which are all even and less than or equal to m. By the inductive hypothesis, they are in P. Since k has at least one outgoing arrow to a state in P (in fact, in this case, all of its arrows are going to states in P), it follows that $k \in N$. Hence we have proven $\{k \in \mathbb{N} : k \text{ odd}, k \leq m+1\} \subseteq N$, which is the other half we needed. This completes the induction. □

Proof 2. Consider a state k. Since the only possible moves from k are to $k-1, k-3$, and $k-5$, the classification of k into N or P can only depend on the classifications of values in the list $(k-5, k-4, k-3, k-2, k-1)$. Write $A_k = \{N, P\}^5$ for this classification. Now observe that $A_6 = A_4 = (P, N, P, N)$. Therefore, a state $k \geq 6$ is in P if and only if $k-2$ is in P. This finishes the proof. □

(b) $S = \{2, 3\}$;

Solution

Analogously to part (a) we compute the classifications of some small numbers:

0	1	2	3	4	5	6	7
P	P	N	N	N	P	P	N

Again, the last 3 elements coincide with the first 3, so there is a period of length 5, and again it starts from the beginning. The answer:

$$n \in P \iff n = 1 \pmod 5 \text{ or } n = 0 \pmod 5.$$

(c) $S = \{2, 7, 8\}$;

Solution

Analogously to the previous parts, compute the first 23 values:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
P	P	N	N	P	P	N	N	N	N	P	N	N	N	P	P	N	N	N	P	P	N	N

In this case the period of length 5 starts from the position of 10 chips. There is a preperiod of length 10 in this case. The answer is

$$P = \{0, 1, 4, 5\} \bigcup \{5k : k \geq 2\} \bigcup \{5k + 4 : k \geq 2\}$$

(d) Who wins each of these games if play starts at 100 chips, the first player or the second?

Solution

In all the cases 100 is a P-position, so the 2nd wins.

3. Consider the following game: the initial position is number 4. Two players make moves in turns: at each turn the player increases the current number by one of its divisors, but not by the number itself (e.g. one can increase 12 by 1, 2, 3, 4 or 6, but not by 12). The player who makes the number larger than 1000 loses. Who wins in this game, the first or the second player?

Solution

First, note that the game is impartial and progressively bounded, so all the positions in either N or P. Now we employ the technique of strategy-stealing:

(a) If $6 \in P$ then $4 \in N$, since there is a move $4 \to 6$.

(b) If $6 \in N$ then $5 \in P$, since the only move from 5 goes to 6. But then we see that $4 \in N$, since there is a move from 4 to 5.

In either cases $4 \in N$, so the first player wins.

4. This is not a problem to solve, but rather an example of a *partisan* combinatorial game; we'll study these more in detail in class this week. A partisan game is a combinatorial game in which the players either have different win conditions, or in which there is some step at which the players have different sets of possible moves. (So, partisan combinatorial games are just combinatorial games that are not impartial.)

(Hex) Hex is played on a rhombus-shaped board tiled with hexagons (see the picture below). Each player is assigned a color, either blue or yellow, and two opposing sides of the board. The players take turns coloring in empty hexagons. The goal for each player is to link their two sides of the board with a chain of hexagons in their own color. Thus, the terminal positions of Hex are the full or partial colorings of the board that have a chain crossing.

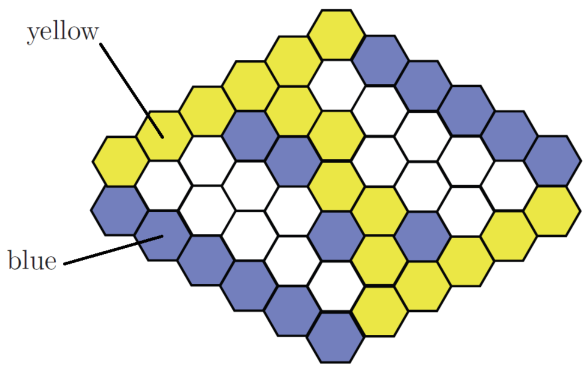


FIGURE 1.9. A completed game of Hex with a yellow chain crossing.

There are two important observations to make about Hex.

- Any fully-colored board has a either blue chain crossing or a yellow chain crossing, but not both. Thus the game cannot end in a tie. (This point is not immediately obvious, and requires a little bit of topology to prove. For now, just take it as a fact.)
- Having extra tiles can only help you. In other words, if the game ends early and the yellow player wins with tiles $\{x_1, \ldots x_n\}$, then any configuration in which the yellow player has tiles $\{x_1, \ldots x_n, x_{n+1}\}$ is also a winning position.

It turns out that these observations tell us that the first player has some winning strategy. We will now show that such a strategy exists, but we will not be able to describe it explicitly. (In fact, for large Hex boards, the winning strategy is unknown!)

First note that this game is progressively bounded. It is not an impartial game, so our usual classification theorem does not apply. (The notions of N and P don't make much sense in the partisan setting.) However, there is a similar theorem (we will see this in lecture later) which guarantees that, if the game cannot end in a tie, then at least one of the players has a winning strategy. The first observation above gives that the game cannot end in a tie, so the theorem applies. Now assume for the sake of contradiction that the second player is the one with the winning strategy.

Now suppose that I am the first player. Our assumption is that, no matter what move I make, the second player will be able to win. So, I put my tile in a randomly-chosen spot on the board. Now the second player goes in a strategic position. I *mirror* their move by flipping the board across its horizonal axis and playing the analogous position. (If this strategy requires me to play in a spot where I placed my initial random move, then I play another random move instead.) Proceeding in this way, my board is always the mirror image of my opponents board, plus one piece.

By assumption, the second player wins after some finite number of turns. But my position is the mirror image of theirs, plus an extra tile, so, by the second observation, I also win! Since the first observation guarantees that there cannot be crossings of both colors, this is a contradiction. Therefore, the second player does not have a winning strategy, hence the first player has a winning strategy.

Note also that this strategy stealing argument relied heavily on the symmetry of the Hex board. In variants of Hex played on boards of different shapes, this argument mught not apply!