MATH 110, Spring 2021, midterm test.

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Show all relevant work in logical sequence and indicate all answers clearly. Cross out all work you do not wish considered. Books and notes are allowed. Electronic devices are not allowed during the test.

1. (10pp.) Let v_1, v_2, …, v_n be a basis of a vector space V. Determine, with proof, the dimension of $\operatorname{span}(v_1+v_2, v_1+v_2+v_3, v_1+v_2+v_3+v_4, \ldots, v_1+\cdots+v_n)$.

2. (10pp.) Let $V = \mathbb{R}^4$, let $W_1 = \{(x_1, x_2, x_3, x_4) : x_2 + x_4 = 0,\ x_j \in \mathbb{R} \text{ for all } j\}$, and let $W_2 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 - x_4 = 0,\ x_j \in \mathbb{R} \text{ for all } j\}$.

(a) Prove that W_1 and W_2 are subspaces of V.

(b) Is the sum $W_1 + W_2$ direct? Explain why or why not.

(c) Determine $\dim(W_1 + W_2)$.

3. (10pp.) Let V be the vector space of all real-valued polynomials in x and y of total degree at most 2, i.e. $V = \operatorname{span}\{1, x, y, x^2, xy, y^2\}$. The list $(1, x, y, x^2, xy, y^2)$ is a basis of V. You do **NOT** need to prove it. Consider the linear operator (do **NOT** check linearity)

$$T \in \mathcal{L}(V) \ : \ (Tf)(x, y) = \frac{\partial}{\partial x} f(x, y) + \frac{\partial}{\partial y} f(x, y).$$

(a) Find the matrix representation of T in this basis used for the domain and the codomain.

(b) What are $\dim \operatorname{null} T$ and $\dim \operatorname{range} T$? Justify your answers.

4. (10pp.) Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x + 2y + 3z, x - y - z)$ and the linear functional $\varphi : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x - 10y$. (**NO** need to prove they are linear.)

(a) Write down the domain and co-domain of the linear functional $T'(\varphi)$.

(b) Write down the action of $T'(\varphi)$. (E.g., if your functional were from $\mathbb{R}^4$ to $\mathbb{R}$ and added up all coordinates, your formula would be $(x_1, x_2, x_3, x_4) \mapsto x_1 + x_2 + x_3 + x_4$.)

(c) Determine the dimension of null T'.

Justify your answers.