1. Homework 3 Solutions

Exercise 1.5.3

(a) Proof by induction:

(i) Induction Base: If A and B are countable then so is $A \cup B$. First, we can expand $A \cup B$ as $A \cup (B \setminus A)$ and note that $B \setminus A \subseteq B$ thus it is countable or finite by Theorem 1.5.7. If $B \setminus A \subseteq B$ is countable, then we can prove the base statement similar to how we prove $\mathbb{Z} \sim \mathbb{N}$ i.e. we can order $A \cup B$ as $\{a_1, b_1, a_2, b_2, a_3, b_3, ...\}$ where $a_i \in A$ and $b_i \in B \setminus A$. More precisely we can define a 1-1 and onto map $f : \mathbb{N} \to A \cup B$ as follows:

$$f(n) = \begin{cases} f_1((n+1)/2) & \text{if } n \text{ is odd} \\ f_2(n/2) & \text{if } n \text{ is even} \end{cases}$$

where f_1 and f_2 are 1-1 and onto maps from $\mathbb{N}$ to A and $B \setminus A$, respectively. Note that if either A or $B \setminus A$ is finite, we continue by listing only A after $B \setminus A$ is exhausted. More precisely; let m be the number of elements in $B \setminus A$, we define a 1-1 and onto map $f : \mathbb{N} \to A \cup B$ as follows;

$$f(n) = \begin{cases} b_n & \text{if } n \leq m \text{ is odd} \\ f_1(n-m) & \text{otherwise,} \end{cases}$$

where f_1 is 1-1 and onto map from $\mathbb{N}$ to A and $B \setminus A = \{b_1, b_2, ..., b_n\}$. In either case bijectivity (another word for one-to-one and onto) of f follows from the fact that f_1 and f_2 are bijective (1-1 and onto).

(ii) Induction Hypothesis: We claim that if $(A_1 \cup A_2 \cup .... \cup A_n)$ is countable for countable $A_i's$, so is $(A_1 \cup A_2 \cup .... \cup A_{n+1})$. Let $A = (A_1 \cup A_2 \cup .... \cup A_n)$ and $B = A_{n_1}$, then the statement follows from the induction base.

(b) As we have seen in Section 1.1, induction does not extend to ∞-case. Again, the reason is that ∞ is not a natural number whereas induction provides us with a proof for why a hypothesis is true for any *natural number n*.

(c) **Remark:** The idea, as we saw in the class, is to deconstruct A_i's, form new finite sets by using the "diagonal argument" and then enumerate these new finite sets. The sets A_i's are not necessarily disjoint but that is fine since if multiply counting these repeating elements in the union is giving us a countable set, excluding their multiplicity, we should still obtain an at most countable set. Since A_i's are infinite, the union will in fact be countable (as opposed to at most countable). Here we will solidify this hand-wavy argument and write a proof by first forming disjoint sets B_i's and then using the "diagonal argument" on the sets B_i's.

Similar to part (a), we first construct disjoints sets to avoid technicalities and to make sure that the function we construct will be 1-1. Let $B_1 = A_1$, and $B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i)$. That is, B_n's are disjoint and countable or finite sets. We want to bijectively map a row R_n of the rearranged $\mathbb{N}$ onto B_n but the subtlety is that B_n might be finite. Moreover there might be only finitely many countable B_n's. Note that, it is easy to show countable or finite union of finite sets is countable or finite. Let D denote the union of finite B_n's. Now we divide the proof into two cases:

(i) There are only finitely many countable B_n's and the rest are finite. Then the result follows from part (a).

(ii) There are countable many countable B_n's and the rest of B_n's are finite. To overcome this, we re-index countable B_n's and have $\bigcup_i^\infty A_i = \bigcup_i^\infty B_i \bigcup D$. Now, if we show that the union of 'leftover' countably many countable, pairwise disjoint B_i's is countable, then the result follows from again part(a).

In order to show that $\bigcup_i^\infty B_i$: We know that there exist bijections $f_i : R_i \to B_i$, define

$f : \mathbb{N} \to \bigcup_i^\infty B_i$ as follows:

$$f(n) = \begin{cases} f_1(n) & \text{if } n \in C_1 \\ f_2(n) & \text{if } n \in C_2 \\ \vdots & \vdots \end{cases}$$

f is 1-1 since R_i's are disjoint, and it is onto because for $b \in \bigcup_i^\infty B_i \Rightarrow b \in B_i$ for some i and $\exists n \in R_i$ such that $f(n) = b$ because f_i is onto. Hence, f is 1-1 and onto and $\bigcup_i^\infty B_i$ is countable as desired

Exercise 2.2.1

Definition 1.1. A sequence (x_n) verconges to x if there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Claim: The vercongence of (x_n) to x is equivalent to boundedness of x_n. That is, a sequence (x_n) verconges to x **iff** x_n is bounded.
Proof: ($\Rightarrow$) Observe that if for all $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$, then in particular for $N = 1$ we get $|x_n - x| < \epsilon$ for all $n \in \mathbb{N}$. Hence, using the lower side of the triangle inequality we obtain,

$$|x_n| - |x| < \epsilon$$

and thus,

$$|x_n| < \epsilon + |x| \quad \forall n.$$

($\Leftarrow$) Assume (x_n) bounded. That is there exists $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. Then for *any* $x \in \mathbb{R}$ and any $n \in \mathbb{N}$ we have

$$|x_n - x| \leq |x_n| + |x| < M + |x|.$$

Hence take $\epsilon = M + |x|$ in the definition to see that (x_n) verconges to x. Note that x was arbitrary here and that any bounded sequence (x_n) will verconge to any $x \in \mathbb{R}$.

Example: Next, we know that the sequence given by $(-1)^n$ is divergent (proven in class), moreover $(|(-1)^n - 0|) < 3$, $(|(-1)^n - 1|) < 3$ for any $n \in \mathbb{N}$. In other words, it verconges to both 0 and 1.

Exercise 2.2.2 Due to the fact that these notes are written as solutions, we skip the 'sketch' work in the following proofs, although in general, I recommend using it for practical reasons.

(a) For a given $\epsilon > 0$, let N be a natural number greater than $\frac{1}{\epsilon}$, then for all $n \geq N$ we have:

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{5(2n+1) - 2(5n+4)}{5(5n+4)}\right| = \left|\frac{-3}{5(5n+4)}\right| = \frac{3}{5(5n+4)} < \frac{3}{25n} < \frac{1}{n} < \epsilon,$$

therefore $\left(\frac{2n+1}{5n+4}\right) \to \frac{2}{5}$.

(b) For a given $\epsilon > 0$, let N be a natural number greater than $\frac{2}{\epsilon}$, then for all $n \geq N$ we have:

$$\left|\frac{2n^2}{n^3+3} - 0\right| < \left|\frac{2n^2}{n^3}\right| < \frac{2}{n} < \epsilon.$$

therefore $\left(\frac{2n^2}{n^3+3}\right) \to 0$.

(c) For a given $\epsilon > 0$, let N be a natural number greater than $\frac{1}{\epsilon^3}$, then for all $n \geq N$ we have:

$$\text{Note that } n > \frac{1}{\epsilon^3} \implies \frac{1}{n} < \epsilon^3 \implies \frac{1}{\sqrt[3]{n}} < \epsilon, \text{ thus } \left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| < \frac{1}{\sqrt[3]{n}} < \epsilon.$$

therefore $\left(\frac{\sin(n^2)}{\sqrt[3]{n}}\right) \to 0$.

Exercise 2.2.4

(a) $a_n = (-1)^n$

(b) Not possible. Let (a_n) be a convergent sequence with an infinite number of 1s and let $\lim a_n = a$, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$. Since there are infinitely many $a_n = 1$, there exists an $a_n = 1$ where $n \geq N$, thus $|1 - a| < \epsilon$. Note that this holds for any $\epsilon > 0$, therefore a has to be equal to 1.

(c) Consider the sequence where every natural number n is followed by n many 1s, i.e, $(a_n) = (1, 1, 2, 1, 1, 3, 1, 1, 1, 4, 1, 1, 1, 1, 5, ...)$.