Game Theory Homework #4, 02/17/2021

All the problems are worth 4 points each and will be graded on a 0/1/2/3/4 scale. Due on Wednesday 02/24/2021 before 11:59 pm to be uploaded via Gradescope.

1. Two smart students form a study group in some math class where homework is handed in jointly by each group. In the last homework of the semester, each of the two students can choose to either work ('W') or party ('P'). If at least one of them solves the homework that week (chooses 'W'), then they will both receive 10 points. But solving the homework incurs a substantial effort, worth -7 points for a student doing it alone, and an effort worth -2 points for each student, if both students work together. Partying involves no effort, and if both students party, they both receive 0 points. Assume that the students do not communicate prior to deciding whether they will work or party. Write this situation as a matrix game and determine all Nash equilibria.

   Solution. The matrix of the game is as follows:

	W	P
W	$(8,8)$	$(3,10)$
P	$(10,3)$	$(0,0)$

   In order to find all Nash equilibria (x^*,y^*), we need to consider four possible cases: (1) x^* and y^* are both pure, (2) x^* and y^* are both not pure, (3) x^* is pure and y^* is not pure, and (4) y^* is pure and x^* is not pure. We enumerate these cases below:

   (1) The pair (W,W) is not a Nash equilibrium, since Player I can increase her payoff from 8 to 10 by switching her strategy. The pair (W,P) is a Nash equilibrium, since the payoff to Player I decreases from 3 to 0 if she switches strategies and since the payoff to Player II decreases from 10 to 8 if he switches strategies. Symmetrically, (P,W) is also a Nash equilibrium. Finally, (P,P) is not a Nash equilibrium, since Player I can increase her payoff from 0 to 3 by switching her strategy. Thus, the pure Nash equilibria are (W,P) and (P,W).

   (2) Write $x^*=(x_1^*,1-x_1^*)$ and $y^*=(y_1^*,1-y_1^*)$, and note that this case entails $0<x_1^*,y_1^*<1$. Since both strategies are fully mixed, both strategies are fully equalizing. That is, we must have $8x_1^*+3(1-x_1^*)=10x_1^*$, which is solved by $x_1^*=\frac{3}{5}$. Likewise, $8y_1^*+3(1-y_1^*)=10y_1^*$ is solved by $y_1^*=\frac{3}{5}$. Thus, $x^*=(\frac{3}{5},\frac{2}{5})^{\mathrm{T}}$ and $y^*=(\frac{3}{5},\frac{2}{5})^{\mathrm{T}}$ is the only Nash equilibrium where both strategies are not pure.

   (3) If $x^*=(1,0)^{\mathrm{T}}$, then $y^*=(y_1^*,1-y_1^*)$ must satisfy the equalization $8=10$, but this is clearly impossible. Similarly, if $x^*=(0,1)^{\mathrm{T}}$, then $y^*=(y_1^*,1-y_1^*)$ must satisfy the equalization $3=0$, which is also impossible. Therefore, there are no Nash equilibria where Player I's strategy is pure and Player II's strategy is not pure.

   (4) The same proof as above, after interchanging the roles of the players, shows that there are no Nash equilibria where Player I's strategy is not pure and Player II's strategy is pure.

   Therefore, all of the Nash equilibria in this game are (W,P), (P,W), and $((\frac{3}{5},\frac{2}{5})^{\mathrm{T}},(\frac{3}{5},\frac{2}{5})^{\mathrm{T}}$. □

2. Consider the two player general sum game with payoff matrix

$$\begin{pmatrix} (1,1) & (2,0) \\ (2,0) & (-1,5) \end{pmatrix}.$$

   Find safety strategies (for both players), and all mixed and pure Nash Equilibria, and the respective expected payoffs for both players in each case.

   Solution. The payoff matrix for Player I is

$$A=\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

   We know that a safety strategy x_* for Player I is any strategy satisfying $\min_{y\in\Delta_2} x_*^{\mathrm{T}}Ay=\max_{x\in\Delta_2}\min_{y\in\Delta_2} x^{\mathrm{T}}Ay$. As we have seen before, it suffices to replace the minimum over $y\in\Delta_2$ with a minimum over $y\in\{(1,0)^{\mathrm{T}},(0,1)^{\mathrm{T}}\}$. Hence for $x=(x_1,1-x_1)^{\mathrm{T}}$ we have

$$\min_{y\in\Delta_2} x^{\mathrm{T}}Ay=\min\{x_1+2(1-x_1),2x_1-(1-x_1)\}.$$

   A straightforward calculation shows that this is solved at $x_1=\frac{3}{4}$, so Player I's safety strategy is $x_*=(\frac{3}{4},\frac{1}{4})^{\mathrm{T}}$. Her guaranteed expected payoff under this strategy is $1\cdot\frac{3}{4}+2\cdot\frac{1}{4}=\frac{5}{4}$.

   Similarly, the payoff matrix for Player II is

$$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Thus, a strategy $y = (y_1, 1-y_1)^{\mathrm{T}}$ is a safety strategy for Player II iff it is a maximizer of the function

$$\min_{x \in \Delta_2} x^{\mathrm{T}} A y = \min\{y_1, 5(1-y_1)\}.$$

We easily see that the unique maximizer is $y_1 = \frac{5}{6}$, hence $y_* = (\frac{5}{6}, \frac{1}{6})^{\mathrm{T}}$ is a safety strategy for Player II. His guaranteed expected payoff under this strategy is $1 \cdot \frac{5}{6} + 5 \cdot \frac{1}{6} = \frac{5}{6}$.

Next, we search for the Nash equilibria. As in the previous problem there are four cases, which we enumerate in the same order.

(1) By checking all four cells, we see that there is no pure Nash equilibrium.

(2) If both strategies are not pure, then they are fully equalizing. Setting up the equalization equations leads us to the same equations from the first part of the problem, so we see that the only fully mixed Nash equilibrium is given by $((\frac{5}{6}, \frac{1}{6})^{\mathrm{T}}, (\frac{3}{4}, \frac{1}{4})^{\mathrm{T}})$. The same calculation also shows that the expected payoffs to the players under this pair of strategies is $(\frac{5}{4}, \frac{5}{6})$.

(3,4) As in the previous problem, this game admits no Nash equilibrium under which one players strategy is pure and the other's is not pure.

$\square$

3. Two cheetahs and three antelopes: Two cheetahs each chase one of three antelopes. If they catch the same one, they have to share. The antelopes are Large, Small, and Tiny, and their values to the cheetahs are ℓ, s and t. Write the 3×3 matrix for this game. Assume that $t < s < \ell < 2s$. Find a condition for a symmetric fully mixed equilibrium to exist and find the latter as well as all the pure equilibria in that case.

Solution. First let's write down the payoff matrix for this game. Write L, S, and T for the strategies of hunting the large, small, and tiny antelope, respectively. Then the game is just:

	L	S	T
L	$(\frac{l}{2}\frac{l}{2})$	(l, s)	(l, t)
S	(s, l)	$(\frac{s}{2}, \frac{s}{2})$	(s, t)
T	(t, l)	(t, s)	$(\frac{t}{2}, \frac{t}{2})$

In order to find all the pure Nash equilibria, let's underscore the maximal payoffs to Player I in the columns and the maximal payoffs to Player II the in rows. (Note that, since $l < 2s$, we have $s > \frac{l}{2}$.)

	L	S	T
L	$(\frac{l}{2}\frac{l}{2})$	$(\underline{l}, \underline{s})$	$(\underline{l}, t)$
S	$(\underline{s}, \underline{l})$	$(\frac{s}{2}, \frac{s}{2})$	(s, t)
T	$(t, \underline{l})$	(t, s)	$(\frac{t}{2}, \frac{t}{2})$

We see that there are two pure Nash equilibria: (S, L) and (L, S).

Next, we look for a symmetric fully mixed equilibrium. For a strategy $x = (x_1, x_2, x_3)^{\mathrm{T}} \in \Delta_3$ the pair (x, x) is a fully mixed symmetric equilibrium iff x is a fully equalizing strategy. This means that x must satisfy the equations:

$$1 = x_1 + x_2 + x_3,$$
$$x_1, x_2, x_3 > 0,$$
$$\frac{l}{2} x_1 + l x_2 + l x_3 = s x_1 + \frac{s}{2} x_2 + s x_3 = t x_1 + t x_2 + \frac{t}{2} x_3$$

Using the first equation we can rewrite the last line as

$$l\left(1 - \frac{x_1}{2}\right) = s\left(1 - \frac{x_2}{2}\right) = t\left(1 - \frac{x_3}{2}\right)$$

Now write $A = l(1 - \frac{x_1}{2})$ for this common value. Rearranging the last displayed equation, we get

$$x_1 = 2\left(1 - \frac{A}{l}\right), \quad x_2 = 2\left(1 - \frac{A}{s}\right), \quad x_3 = 2\left(1 - \frac{A}{t}\right),$$

and plugging this into the first equation, we can find A:

$$1 = x_1 + x_2 + x_3 = 2\left(3 - A\left(\frac{1}{l} + \frac{1}{s} + \frac{1}{t}\right)\right) \implies A = \frac{5/2}{\frac{1}{l} + \frac{1}{s} + \frac{1}{t}}.$$

By construction, $(x_1, x_2, x_3)^{\mathrm{T}}$ is a symmetric Nash equilibirum provided that x_1, x_2, and x_3 are all strictly positive. But the ordering $t < s < l$ implies $x_3 < x_2 < x_1$, so it suffices find a condition on l, s, and t such that $x_3 > 0$. Of course $x_3 > 0$ if and only if $A < t$. So, using the calculations above, this is equivalent to

$$\frac{5/2}{\frac{1}{l} + \frac{1}{s} + \frac{1}{t}} < t \qquad \Leftrightarrow \qquad \frac{2}{l} + \frac{2}{s} > \frac{3}{t}.$$

In summary, the condition $2/l + 2/s > 3/t$ implies that there exists a symmetric fully mixed Nash equilibrium, namely $x = (2(1 - A/l), 2(1 - A/s), 2(2 - A/t))^{\mathrm{T}}$. □

4. Volunteering dilemma: There are n players in a game show. Each player is put in a separate room. If some of the players volunteer to help the others, then each volunteer will receive 1000 and each of the remaining players will receive 1500. If no player volunteers, then they all get zero. Show that for this game the set of symmetric (mixed) Nash equilibria contains exactly one element. Let p_n denote the probability that player 1 volunteers in this equilibrium. Find p_2 and show that

$$\lim_{n\to\infty} np_n = \log(3).$$

Solution. First, let's show that there are no pure symmetric Nash equilibria. Indeed, if all players volunteer, then they all get payoff 1000, and any player can deviate to increase her payoff to 1500. On the other hand, if all players don't volunteer, they all get payoff 0, and any player can deviate and increase her payoff to 1000. Thus, the set of all the symmetric Nash equilibria consists of just fully mixed Nash equilibria. Let p_n be the probability that Player I (and symmetrically, all players) volunteers, and note that we have $0 < p_n < 1$.

In order for this to be a Nash equilibrium, Player I must be indifferent about her two possible moves. In order to find her payoff under either move, let's calculate the following:

- If Player I does not volunteer, then she gets a payoff of 1500 on the event that at least one of the $n-1$ remaining players volunteers, and a payoff of 0 on the event that all $n-1$ remaining players do not volunteer. Of course, these are just certain outcomes of a binomial random variable, since there are $n-1$ players each with an independent choice of probability p_n. In particular, the event that at least one of the $n-1$ remaining players volunteer has probability $1-(1-p_n)^{n-1}$, and the event that all $n-1$ remaining players do not volunteer is $(1-p_n)^{n-1}$. Thus, Player I's expected payoff when she does not volunteer is $1500(1-(1-p_n)^{n-1})$.
- If Player I volunteers, then she gets a payoff of 1000, no matter what any other player does.

Thus, the number p_n corresponds to a symmetric fully mixed Nash equilibrium iff $1000 = 1500 - 1500(1-p_n)^{n-1}$, which is solved by

$$p_n = 1 - 3^{-\frac{1}{n-1}}.$$

This shows that there is exactly one symmetric mixed Nash equilibrium.

In particular, we get $p_2 = 2/3$. Moreover, we can compute the limit statement by using Taylor's approximation of the exponential, $e^x = 1 + x + o(x)$ when $x \to 0$:

$$np_n = n\left(1 - 3^{-\frac{1}{n-1}}\right) = n\left(1 - e^{-\frac{\log(3)}{n-1}}\right) = n\left(1 - \left(1 - \frac{\log(3)}{n-1} + o(n^{-1})\right)\right) = \frac{n}{n-1}\log 3 + o(1).$$

Of course, this implies $np_n \to \log(3)$ as claimed. □