1. Homework 2 Solutions

Exercise 1.3.2

(a) Consider B=$\{0\}$. $\inf B = 0 = \sup B$.
(b) Not possible, finite sets must contain their infimum and supremum.
(c) Let $B = \{x : x \in \mathbb{Q} \text{ and } 0 < x \leq 1\}$.

Exercise 1.3.6

(a) Let $s = \sup A$ and $t = \sup B$. Then for any $a \in A$ and any $b \in B$ we know that $a \leq s$ and $b \leq t$. Then adding these two gives us that $a + b \leq s + t$ for any a, b. Since every element of $A + B$ is given by a pair of a, b, $s + t$ is an upper bound.
(b) Now let u be an arbitrary upper bound of $A + B$, and temporarily fix $a \in A$. Then for any $b \in B$, $a + b \leq u$ by assumption on u, in particular this means $b \leq u - a$ for any $b \in B$. This means $u - a$ is an upper bound of B. The least upper bound t is less than or equal to $u - a$ as desired. Note that, this holds for an arbitrary a.
(c) Let $u = \sup(A + B)$, then by part (a) we know that $s + t$ is an upper bound. So we only need to show that it is the least i.e $s + t \leq u$. Now to do so: By part (b), we have $a \leq u - t$ for any $a \in A$, then $u - t$ is an upper bound of A meaning that $s \leq u - t$ i.e $s + t \leq u$ which is what was to be shown.
(d) Since $s + t$ is an upper bound of $A + B$ (part (a)), by Lemma 1.3.8, showing that $s + t - \epsilon$ is not an upper bound for any $\epsilon > 0$ proves that $s + t$ is the least upper bound. Again by Lemma 1.3.8 (in the opposite direction), for $\epsilon/2$ there exist $a \in A$ and $b \in B$ such that $s - \epsilon/2 < a$ and $t - \epsilon/2 < b$. Thus, $s - \epsilon/2 + t - \epsilon/2 < a + b$ that is $s + t - \epsilon < a + b$. This shows $s + t - \epsilon$ is not an upper bound of $A + B$ for any $\epsilon > 0$.

Exercise 1.3.11

(a) True. Since $\sup B$ is an upper bound of B, it is greater than or equal to any element of B and thus A which makes it an upper bound of A. But $\sup A$ is the least upper bound.
(b) True. Let $(\inf B + \sup A)/2 = c$. Then $c > \sup A$ meaning that $c > a\ \forall a \in A$. And $c < \inf B$ meaning that $c < b\ \forall b \in B$.
(c) False. Consider $A = (-1, 0)$ and $B = (0, 1)$, $c = 0$. Then A has supremum 0 while B has infimum 0, so $\sup A \not< \inf B$.

Exercise 1.4.8

(a) Let $A = (-1, 1) \cap \mathbb{Q}$ and let $B = (-1, 1) \cap \mathbb{I}$ (set of irrationals). Then $A \cap B = \emptyset$ and $\sup A = \sup B = 1$ is contained neither in A nor in B.
(b) Let $J_n = (-\frac{1}{n}, \frac{1}{n})$ then $\bigcap_1^\infty J_n = \{0\}$.
(c) Consider the nested, unbounded, closed sets $L_n = [n, \infty)$, then $\bigcap_1^\infty L_n = \emptyset$.
(d) Not possible. Considers the sets $J_n = \bigcap_{i=1}^n I_i$. It goes without saying that J_n's are nonempty and nested. J_n's are also bounded and closed, because every J_n is the intersection of finitely many bounded and closed sets. Also, since $\bigcap_{n=1}^\infty J_n = \bigcap_{n=1}^\infty \left(\bigcap_{i=1}^n I_n\right) = \bigcap_{n=1}^\infty I_n$, by the Nested Interval Property, $\bigcap_{n=1}^\infty I_n \neq \emptyset$.