MATH 110, Spring 2021, midterm test solutions.

1. (10pp.) Let v_1, v_2, $\ldots$, v_n be a basis of a vector space V. Determine, with proof, the dimension of $\operatorname{span}(v_1+v_2, v_1+v_2+v_3, v_1+v_2+v_3+v_4, \ldots, v_1+\cdots+v_n)$.

Solution: Suppose a linear combination of the given vectors is zero:

$$a_1(v_1+v_2)+a_2(v_1+v_2+v_3)+\cdots+a_{n-1}(v_1+\cdots+v_n)=0.$$

This can be rewritten as

$$(a_1+a_2+\cdots+a_{n-1})v_1+(a_1+a_2+\cdots+a_{n-1})v_2+(a_2+a_3+\cdots+a_{n-1})v_3+\cdots+a_{n-1}v_n=0.$$

Since v_1, v_2, $\ldots$, v_n are linearly independent as a basis in V, the coefficient of each v_j must be zero. That means that

$$\begin{aligned} a_{n-1} &= 0\\ a_{n-2}+a_{n-1} &= 0\\ a_{n-3}+a_{n-2}+a_{n-1} &= 0\\ \dots\dots\dots\dots &= 0\\ a_1+a_2+a_3+\cdots+a_{n-1} &= 0. \end{aligned}$$

Substituting the first equality $a_{n-1}=0$ into the second, we obtain $a_{n-2}=0$, which implies $a_{n-3}=0$, etc. until we obtain $a_1=0$, This proves the given vectors are linearly independent. Hence the dimension of their span is equal to the number of the vectors, i.e., $n-1$.

Answer: $n-1$.

2. (10pp.) Let $V = \mathbb{R}^4$, let $W_1 = \{(x_1, x_2, x_3, x_4) : x_2 + x_4 = 0,\ x_j \in \mathbb{R} \text{ for all } j\}$, and let $W_2 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 - x_4 = 0,\ x_j \in \mathbb{R} \text{ for all } j\}$.

(a) Prove that W_1 and W_2 are subspaces of V.

Proof: The sets W_j, $j = 1, 2$, are contained in V because the vectors in either W_j are real and have length 4. Next, if (x_1, x_2, x_3, x_4) and $(y_1, y_2, y_3, y_4) \in W_1$ and $a,\ b \in \mathbb{R}$, then $(ax_2 + by_2) + (ax_4 + by_4) = a(x_2 + x_4) + b(y_2 + y_4) = 0$, hence $a(x_1, x_2, x_3, x_4) + b(y_1, y_2, y_3, y_4) \in W_1$. Likewise, if (x_1, x_2, x_3, x_4) and $(y_1, y_2, y_3, y_4) \in W_1$ and $a,\ b \in \mathbb{R}$, then $(ax_1 + by_1) + (ax_2 + by_2) + (ax_3 + by_3) - (ax_4 + by_4) = a(x_1 + x_2 + x_3 - x_4) + b(y_1 + y_2 + y_3 - y_4) = 0$, hence $a(x_1, x_2, x_3, x_4) + b(y_1, y_2, y_3, y_4) \in W_1$. So, both W_1 and W_2 are closed under addition and scalar multipliction, and are therefore subspaces of V.

(b) Is the sum $W_1 + W_2$ direct? Explain why or why not.

Solution: The sum $W_1 + W_2$ is not direct because, say, the nonzero vector $(1, 0, -1, 0)$ belongs to both W_1 and W_2, so the intersection $W_1 \cap W_2$ is nonzero.

(c) Determine $\dim(W_1 + W_2)$.

Solution: We will show that $W_1 + W_2 = V$, and so $\dim(W_1 + W_2) = 4$. Indeed, any vector $(x_1, x_2, x_3, x_4) \in V$ can be written as a sum of a vector in W_1 and a vector in W_2, e.g., as

$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3 - x_2 - x_4, -x_2) + (0, 0, x_2 + x_4, x_4 + x_2).$$

3. (10pp.) Let V be the vector space of all real-valued polynomials in x and y of total degree at most 2, i.e. $V = \operatorname{span}\{1, x, y, x^2, xy, y^2\}$. The list $(1, x, y, x^2, xy, y^2)$ is a basis of V. You do **NOT** need to prove it. Consider the linear operator (do **NOT** check linearity)

$$T \in \mathcal{L}(V) \; : \; (Tf)(x,y) = \frac{\partial}{\partial x} f(x,y) + \frac{\partial}{\partial y} f(x,y).$$

(a) Find the matrix representation of T in this basis used for the domain and the codomain.

Solution: We calculate $T(1) = 0$, $T(x) = 1$, $T(y) = 1$, $T(x^2) = 2x$, $T(xy) = x + y$, $T(y^2) = 2y$, so the matrix representation of T with respect to the basis $(1, x, y, x^2, xy, y^2)$ used on both sides is

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) What are $\dim \operatorname{null} T$ and $\dim \operatorname{range} T$? Justify your answers.

Solution: By 3.117, $\dim \operatorname{range} T$ equals the column rank of $\mathcal{M}(T)$ (or its row rank by 3.118). Notice that the even-numbered column are linearly independent because they each have a nonzero component at different slots. They also span the other columns since the first column is zero, the third column is a copy of the second, and the fifth is half the sum of the fourth and the sixth. Therefore $\dim \operatorname{range} T = 3$. Now, by the Fundamental Theorem of Linear Maps, $\dim \operatorname{null} T = dim V - \dim \operatorname{range} T = 6 - 3 = 3$.

Answers: 3 and 3.

4. (10pp.) Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x + 2y + 3z, x - y - z)$ and the linear functional $\varphi : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x - 10y$. (**NO** need to prove they are linear.)

(a) Write down the domain and co-domain of the linear functional $T'(\varphi)$.

Solution: $T'(\varphi) = \varphi \circ T \in (\mathbb{R}^3)'$, so the domain of $T'(\varphi)$ is $\mathbb{R}^3$ and the codomain is $\mathbb{R}$.

(b) Write down the action of $T'(\varphi)$. (E.g., if your functional were from $\mathbb{R}^4$ to $\mathbb{R}$ and added up all coordinates, your formula would be $(x_1, x_2, x_3, x_4) \mapsto x_1 + x_2 + x_3 + x_4$.)

Solution: $T'(\varphi)(x_1, x_2, x_3) = \varphi(T(x_1, x_2, x_3)) = \varphi(x_1 + 2x_2 + 3x_3, x_1 - x_2 - x_3) = x_1 + 2x_2 + 3x_3 - 10(x_1 - x_2 - x_3) = -9x_1 + 12x_2 + 13x_3$.

Answer: $T'(\varphi) : (x_1, x_2, x_3) \mapsto -9x_1 + 12x_2 + 13x_3$,

(c) Determine the dimension of $\operatorname{null} T'$.

Solution: First observe that $\operatorname{range} T = \mathbb{R}^2$ because $(1, 0) = T(1/3, 1/3, 0)$ and $(0, 1) = T(2/3, -1/3, 0)$. Therefore, $(\operatorname{range} T)^0 = \{0\}$. So, by 3.107 (a), $\dim \operatorname{null} T' = \dim(\operatorname{ran} T)^0 = 0$.

Answer: 0.