Name: ____
SID: ____

MATH 135: SET THEORY
FINAL EXAMINATION

There are **eight** questions. The first question has **seven** parts. Write your answers in the space provided. If you need more space, you may continue your answers on the blank pages at the end of the exam. Unless explicitly stated to the contrary, your proofs may use the Axiom of Choice. **Please submit all of the pages, including the extra blank pages even if you have not written on them.**

For all of the parts of Question 1, you may use only the language of set theory having $\in$ as its sole non-logical primitive. However, if you have defined some term in another part of the problem, you may then use that defined term.

1a. (5 pts) Give a precise formal definition of the relation

$$z = \langle x, y \rangle \ .$$

Name or initials: ______________________________

1b.(5 pts) Give a precise formal definition of

$$f : A \to B \; .$$

Name or initials: ______________________________

1c. (5 pts) Express the Empty Set Axiom.

Name or initials: ____

1d. (5 pts) Express the Axiom of Choice (in our official formulation).

Name or initials: ____________________

1e. (5 pts) Give a precise formal definition of the condition

x is a `transitive set` .

Name or initials: ______________________________

1f. (5 pts) Give a precise formal definition of the condition that

R is a `well-ordering` of X .

Name or initials: ______________________________

1g. (5 pts) Give a precise formal definition of the condition

$$\kappa \text{ is a } \texttt{cardinal} \, .$$

Name or initials: ____

2. (15 pts) **Prove:** Let R be any relation and let $X = \operatorname{fld}(R)$ be the field of R. Define $f : \omega \to \mathcal{P}(X \times X)$ by recursion via $f(0) = R$ and $f(n^+) := f(n) \cup f(n)^{-1} \cup f(n) \circ f(n)$. Let $E := \bigcup \operatorname{ran} f$. **Prove:** that for any equivalence relation F on X with $R \subseteq F$, we have $E \subseteq F$.

Name or initials: ______________________________

3. (15 pts) Let X be any set and let $R \subseteq X \times X$ be a transitive, irreflexive relation on X. **Prove:** there is a linear ordering $S \subseteq X \times X$ with $R \subseteq S$.

Name or initials: ______________________________

4. (15 pts) **Prove:** that $\daleth$, the class of all cardinals, is not a set. That is, there is no set $\daleth$ such that $\forall t[t \in \daleth \leftrightarrow t$ is a cardinal $]$.

Name or initials: ______________________________

5. (15 pts) **Prove:** that

$$\aleph_\omega < \aleph_\omega^{\aleph_0}$$

Hint: Let $g : \aleph_\omega \to {}^\omega\aleph_\omega$ be any function. Show that for every $n \in \omega$,

$$\aleph_\omega \neq \{\alpha \in \aleph_\omega \ : \ (\exists \beta \in \aleph_n)\ \alpha = g(\beta)(n)\} \ .$$

Use this result to construct some $f \in {}^\omega\aleph_\omega \setminus \operatorname{ran}(g)$.

Name or initials: ____

6. (15 pts) **Prove:** that ordinal addition is associative. That is, for all ordinals α, β, and γ, we have

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

Name or initials: ______________________________

7. (15 pts) Let $X \subseteq \mathbb{R}$ be a subset of the real numbers which is well-ordered with respect to the ordering induced from the usual ordering on $\mathbb{R}$. **Prove:** $\operatorname{card}(X) \leq \aleph_0$.

Name or initials: ______________________________

8. (15 pts) Let X be any set, E an equivalence relation on X, and κ any cardinal. We suppose that for each $x \in X$, $\mathrm{card}([x]_E) \leq \kappa$. **Prove:** $\mathrm{card}(X) \leq \mathrm{card}(X/E) \cdot \kappa$.

Name or initials: ____

For which problem should this work be credited?

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