Line Integrals

$$\int_C \vec{F} \cdot d\vec{r}$$

Easy to do directly?

yes ↙ Do directly

↓ No

Is $\vec{F}$ conservative?

yes ↙ FTLI

↓ No

Dimension

2D ↙ Green's Theorem

↓ 3D Stokes' Theorem

Directly: Calculate $\int_C \vec{F} \cdot d\vec{r}$, $\vec{F} = \langle x^2, 3y, 3xz \rangle$

C is the straight line segment from $(1,0,1)$ to $(0,2,-1)$.

①

$x(t) = (1-t) \cdot 1 + t \cdot 0 = 1 - t$

$y(t) = (1-t) \cdot 0 + t \cdot 2 = 2t$

$z(t) = (1-t) \cdot 1 + t(-1) = 1 - 2t.$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad 0 \le t \le 1.$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle (1-t)^2, 3(2t), 3(1-t)(1-2t) \rangle \cdot \langle -1, 2, -2 \rangle dt$$

$$= \int_0^1 \left(-(1-t)^2 + 12t - 6(1-t)(1-2t) \right) dt$$

$$= \int_0^1 (-13t^2 + 32t - 7)\, dt$$

$$= -\frac{13}{3} t^3 + 16t^2 - 7t \Big|_0^1$$

$$= \frac{14}{3}.$$

$\int_C \vec{F}\cdot d\vec{r}$, $\vec{F} = \langle xy^2, x^2y \rangle$, C is the arc of $y = \sin x$ from $(0,0)$ to $(3\pi, 0)$

$\text{curl}\,\vec{F} = 0$.

$\vec{F} = \langle M, N, 0 \rangle$ and check that

$\text{curl}\,\vec{F}\cdot\vec{k} = 0$ (for 2D).

$$\Leftrightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(xy^2) = 2xy - 2xy = 0$$

Find f : $\vec{F} = \nabla f$.

$$\langle xy^2, x^2y \rangle = \langle f_x, f_y \rangle$$

$f_x = xy^2$ $\qquad f = \int xy^2\,dx$

$$= \frac{1}{2}x^2y^2 + g(y)$$

$f_y = x^2y \Rightarrow f = \int x^2y\,dy$

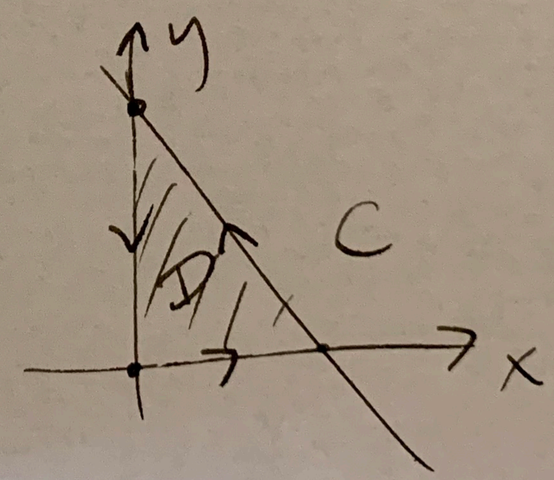
$$= \frac{1}{2}x^2y^2 + h(x)$$

$$f = \frac{1}{2}x^2y^2$$

FTLI: $\int_C \vec{F}\cdot d\vec{r} = f(3\pi, 0) - f(0,0)$

$$= \frac{1}{2}(3\pi)^2\cdot 0^2 - \frac{1}{2}0^2\cdot 0^2 = 0$$

Green's Theorem: $\int_C \vec{F}\cdot d\vec{r}$, $\vec{F} = \langle x^3, 2xy \rangle$ and C is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,6)$ ccw.



Conservative?

$$\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2) = 2y. \quad \text{No}$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_D 2y \, dx\,dy$$

$$= \int_0^2 \int_0^{6-3x} 2y \, dy\,dx$$

$$= 24.$$

Stokes': Find $\int_C \vec{F}\cdot d\vec{r}$, $\vec{F} = \langle \sin x, \cos y, xz \rangle$ and C is the curve parameterized by $\vec{r}(t) = \langle \cos t, \sin t, \cos t \sin t \rangle$ with $0 \le t \le 2\pi$.

Conservative?

③

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & \cos y & xz \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(\cos y), -\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial z}(\sin x), \frac{\partial}{\partial x}(\cos y) - \frac{\partial}{\partial y}(\sin x) \right\rangle$$

$$= \langle 0, -z, 0 \rangle = \langle 0, 0, 0 \rangle. \quad \text{No}$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot d\vec{S}$$

Parameterize S:

$z = xy$ is where C lies.

$\vec{r}(x,y) = \langle x, y, xy \rangle$

$\vec{r}_x = \langle 1, 0, y \rangle$

$\vec{r}_y = \langle 0, 1, x \rangle$

$$\vec{n} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \langle -y, -x, 1 \rangle$$

≥ 0, upward normal

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \langle 0, -xy, 0 \rangle \cdot \langle -y, -x, 1 \rangle \, dx\,dy$$

$$= \iint_D x^2 y \, dx\,dy = \int_0^{2\pi} \int_0^1 (r^2 \cos^2\theta)(r \sin\theta) \, r \, dr \, d\theta$$

$$= \left(\int_0^1 r^4 \, dr \right) \left(\int_0^{2\pi} \cos^2\theta \sin\theta \, d\theta \right)$$

$$= \left(\frac{1}{5} \right) \left(-\frac{\cos^3\theta}{3} \right) \Big|_0^{2\pi} = 0$$

Surface Integrals

$$\iint_S \vec{F} \cdot d\vec{S}$$

Do you need to find $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$?

yes ↙ ↓ No

Stokes'

Easy to evaluate directly?

Yes ↙ ↓ No

Do directly

Divergence Theorem

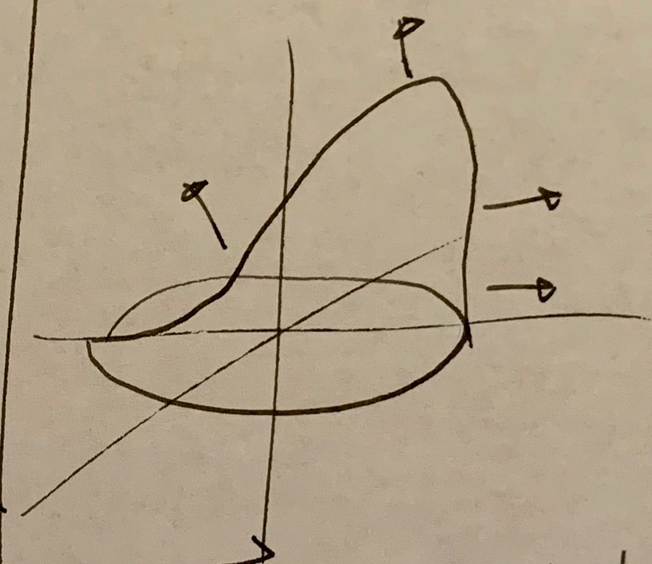
Find $\iint_S \vec{F} \cdot d\vec{S}$, where

$\vec{F} = \langle 2x, 2z-2x, 2x-2z \rangle$ and

$S = \{(x,y,z) \mid z = e^y \underline{(1-x^2-y^2)}(1-y^5)\cos x,$

and $\underline{x^2+y^2 \le 1}\}$,

with an upward pointing normal.



Find $\vec{G}$ such that $\vec{F} = \nabla \times \vec{G}$ and

set $\vec{G} = \langle G_1, G_2, 0 \rangle$

If $\vec{F} = \nabla \times G$, then

$$-\frac{\partial G_2}{\partial z} = 2x$$

$$\frac{\partial G_1}{\partial z} = 2z - 2x$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \underline{2x - 2z}$$

$$-\frac{\partial G_2}{\partial z} = 2x \Rightarrow G_2 = -2xz + M(x,y)$$

$$\frac{\partial G_1}{\partial z} = 2z - 2x \Rightarrow G_1 = z^2 - 2xz + N(x,y)$$

$$\frac{\partial}{\partial x}\left(-2xz + M(x,y)\right) - \frac{\partial}{\partial y}\left(z^2 - 2xz + N(x,y)\right)$$

$$= -2z + \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 2x - 2z$$

$$\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 2x$$

Choose anything that works: $N = 0$

⑥

$$\frac{\partial M}{\partial x} = 2x \Rightarrow M = x^2$$

$$G = \langle z^2 - 2xz,\ x^2 - 2xz,\ 0 \rangle$$

Parameterize boundary of S:

$$\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle \qquad 0 \le t \le 2\pi$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\nabla \times \vec{G}) \cdot d\vec{S}$$

$$= \oint_C \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \vec{G}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^{2\pi} (\cos^2 t)(\cos t)\, dt = \int_0^{2\pi} (1 - \sin^2 t) \cos t\, dt$$

$$= \sin t - \frac{1}{3}\sin^3 t \Big|_0^{2\pi} = 0.$$

Directly: $\iint_S \vec{F}\cdot d\vec{S}$, $\vec{F} = \langle x, y, z^2 \rangle$ and S is the helicoid parameterized by $\vec{r}(u,v) = \langle u\cos v, u\sin v, v \rangle$ with $0 \le u \le 2$ and $0 \le v \le 4\pi$.

$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= \langle \sin v, -\cos v, u\cos^2 v + u\sin^2 v \rangle$$

$$= \langle \sin v, -\cos v, \underset{\substack{\uparrow \\ \ge 0}}{u} \rangle \checkmark$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{n}\,du\,dv$$

$$= \iint_D \langle u\cos v, u\sin v, v^2 \rangle \cdot \langle \sin v, -\cos v, u \rangle\,du\,dv$$

$$= \int_0^{4\pi}\int_0^2 (\cancel{u\cos v\sin v} - \cancel{u\sin v\cos v} + uv^2)\,du\,dv$$

$$= \left(\int_0^2 u\,du\right)\left(\int_0^{4\pi} v^2\,dv\right)$$

$$= \frac{128\pi^3}{3}$$

Divergence Theorem:

$\iint_S \vec{F}\cdot d\vec{S}$, $\vec{F} = \langle xy^2, x^2z, e^y \rangle$, S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outwards, without the top surface.

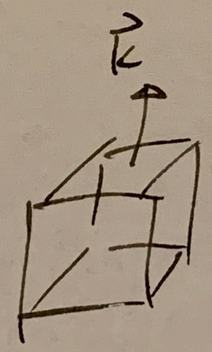
Let T be the top surface. $S \cup T$ is closed enclosing a solid region E.

$$\iint_{S\cup T} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dx\,dy\,dz$$

$$= \iiint_E \left(\frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(x^2 z) + \frac{\partial}{\partial z}(e^y)\right) dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} y^2\, dx\,dy\,dz = 2\cdot 2\cdot \frac{y^3}{3}\Big|_{-1}^{1}$$

$$= \frac{8}{3}$$



Find $\iint_T \vec{F}\cdot d\vec{S}$:

T: $\vec{r}(x,y) = \langle x, y, 1\rangle$

$\vec{r}_x = \vec{i}$, $\vec{r}_y = \vec{j}$ $\Rightarrow$ $\vec{n} = \vec{k}$

$$\iint_T \vec{F}\, d\vec{S} = \iint_D \langle xy, x^2, e^y\rangle \cdot \langle 0,0,1\rangle\, dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} e^y\, dx\,dy = 2\int_{-1}^{1} e^y\, dy$$

$$= 2(e - e^{-1})$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S\cup T} \vec{F}\cdot d\vec{S} - \iint_T \vec{F}\cdot d\vec{S}$$

$$= \frac{8}{3} - 2(e - e^{-1}).$$