FUNCTIONS * RELATIONS
OPERATIONS * OPERATIONS
RELATIONS * FUNCTIONS

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November 26, 2023

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1 Preliminaries

1.1 The language of functions

1.1.1 Mathematical structures

Modern Mathematics is concerned with *mathematical structures*. A "mathematical structure" consists of one or more sets equipped with data of certain type.

This informal initial definition already covers practically all fundamental types of structures that a mathematician encounters on a daily basis.

1.1.2 The concept of a function

An example of a mathematical structure is provided by the familiar concept of a function. A function of n variables consists of

- a list of n sets
$$X_1, \dots, X_n \tag{1}$$
- a set Y
- an assignment
$$x_1, \dots, x_n \longmapsto y \tag{2}$$
that assigns a *single* element y of set Y to *every* list $x_1, \dots, x_n$ such that
$$x_1 \in X_1, \dots, x_n \in X_n . \tag{3}$$

1.1.3 The domain of a function

The list of sets, (1), is called the *domain* of the function. We shall also call it the *source-list* and will refer to n as the *length* of that list.

1.1.4 The antidomain of a function

The set Y is called the *antidomain* of the function. We shall also refer to it as the *target*.

1.1.5 The argument-list and the value of a function

We shall refer to $x_1, \dots, x_n$ satisfying Condition (3) as the *argument-list*. The single element $y \in Y$ that is assigned to it is then called the *value* of the function on that particular argument-list.

If the *name* of the function is, say, f, its value on the list $x_1, \dots, x_n$ is denoted
$$f(x_1, \dots, x_n) \tag{4}$$

1.1.6 The arrow representation of a function

The symbolic representation of a function

$$f : X_1, \dots, X_n \longrightarrow Y \tag{5}$$

at a glance supplies the following information: *the function's name*, often represented by a symbol, its domain, and its target. In (5) the name of the function is 'f', the domain is the list of sets $X_1, \dots, X_n$, and the target is the set denoted Y.

It is often more convenient to place the name of a function above the arrow representating the function

$$X_1, \dots, X_n \xrightarrow{\quad f \quad} Y \ .$$

1.1.7 Equality of functions

Two functions are declared to be *equal* if

- *their domains are equal,*
- *their targets are equal,*
- *and their assignments are equal.*

In particular, a function

$$V_1, \dots, V_m \xrightarrow{\quad f \quad} W$$

can be equal to a function

$$X_1, \dots, X_n \xrightarrow{\quad g \quad} Y$$

only when

$$m = n, \qquad V_1 = X_1, \ \dots, \ V_m = X_m \ , \qquad \text{and} \qquad W = Y.$$

1.1.8 Functions of zero variables

When $n = 0$, the domain of a function is the empty list of sets. The arrow representation of such a function would be thus

$$\xrightarrow{\quad f \quad} Y \tag{6}$$

There is only one argument list in this case, namely the empty list. The function assigns to it a single element $y \in Y$. In particular,

$$f \quad \longleftrightarrow \quad \text{the value of } f \text{ on the empty argument-list}$$

defines a canonical identification between functions (6) and elements of the target-set Y.

1.1.9 Functions constant in the i-th variable

If the value (4) does not depend on x_i, we say that f is *constant in i-th variable.*

1.1.10

We shall denote the set of all functions (5) by

$$\mathrm{Funct}(X_1, \dots, X_n; Y) \tag{7}$$

or

$$Y^{X_1, \dots, X_n}. \tag{8}$$

1.1.11 Lists with omitted entries

Since lists with certain entries having been omitted are frequently encountered in Mathemtics, we have the notation to denote such lists. For example,

$$x_1, \dots, \hat{x}_i, \dots, x_n \tag{9}$$

stands for the list of length $n - 1$

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$$

while

$$x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n \tag{10}$$

stands for the list of length $n - 2$

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n,$$

and so on.

1.1.12 Freezing a variable in a function of n-variables

For any $1 \leq i \leq n$ and any $a \in X_i$, assignment

$$x_1, \dots, \hat{x}_i, \dots, x_n \longmapsto f(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n)$$

defines a function of $n - 1$ variables

$$X_1, \dots, \hat{X}_i, \dots, X_n \longrightarrow Y. \tag{11}$$

We shall denote function (11) by $\mathrm{ev}^i_a f$.

1.1.13 The associated evaluation functions of one variable

Assignment

$$x_i \mapsto \mathrm{ev}^i_{x_i} f$$

defines a function of a single variable

$$X_i \longrightarrow \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n; Y) \tag{12}$$

We shall denote function (12) by $\mathrm{ev}^i f$ and call it the i-th *evaluation function* associated with a function f.

1.1.14 Adjunction correspondence

Assignment

$$f \longmapsto \mathrm{ev}^i f$$

defines a canonical bijection

$$\mathrm{Funct}(X_1, \dots, X_n\,;\, Y) \longleftrightarrow \mathrm{Funct}(X_i, \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,;\, Y)) \tag{13}$$

whose inverse is given by sending a function

$$\phi \in \mathrm{Funct}(X_i, \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,;\, Y))$$

to the function

$$X_1, \dots, X_n \longrightarrow Y, \qquad x_1, \dots, x_n \longmapsto \big(\phi(x_i)\big)(x_1, \dots, \hat{x}_i, \dots, x_n)\,.$$

Correspondence (13) is a manifestation of what is perhaps the single most important phenomenon in Modern Mathematics known as a *pair of adjoint functors*. This is not your first encounter with this phenomenon—you encountered it in some fundamental theorems of basic Mathematical curriculum, but it is the first time that you are expressly told about it.

1.1.15

In order to describe the conceptual mechanics behind the concept of *adjoint* functors, one needs to introduce a proper language, the language of *morphisms* and *categories*, cf. Chapters 3 and 4.

1.1.16 Adjunction correspondence in exponential notation

Canonical identification (13) in exponential notation (8) acquires particularly suggestive form

$$Y^{X_1, \dots, X_n} \longleftrightarrow \left(Y^{X_1, \dots, \hat{X}_i, \dots, X_n}\right)^{X_i}\,. \tag{14}$$

1.1.17 Surjective functions

A function (5) is said to be *surjective* if

$$\text{for every } y \in Y \text{ there exists an argument-list } x_1, \dots, x_n \text{ such that } f(x_1, \dots, x_n) = y\,. \tag{15}$$

You are likely to be familiar with an informal expression "a function f is onto" instead of being surjective. I encourage you to use the term surjective.

1.1.18 Injective functions

A function (5) is said to be *injective* if it has the property

$$\text{if } f(x_1, \dots, x_n) = f(x'_1, \dots, x'_n), \text{ for two argument-lists, then the two argument-lists are equal.} \tag{16}$$

You are likely to be familiar with an informal expression "a function f is one-to-one" instead of of being injective.

1.1.19 Bijective functions

A function is said to be *bijective* if it is both surjective and injective. This terminology is used primarily for functions of a single variable.

1.2 Composition of functions

1.2.1 Postcomposition

Given a function (5) and a function $g : Y \to Y'$, their composition yields the function

$$g \circ f : X_1, \dots, X_n \longrightarrow Y', \qquad x_1, \dots, x_n \longmapsto g(f(x_1, \dots, x_n)). \tag{17}$$

1.2.2

Postcomposition with a function g is itself a function between the function sets

$$g_\bullet \, : \, \mathrm{Funct}(X_1, \dots, X_n; Y) \longrightarrow \mathrm{Funct}(X_1, \dots, X_n; Y'), \qquad f \longmapsto g \circ f. \tag{18}$$

1.2.3 Precomposition

Given a function (5) and a function-list $h_1, \dots, h_n$,

$$X'_1, \dots, X'_m \xrightarrow{h_1} X_1 \quad , \quad \dots \quad , \quad X'_1, \dots, X'_m \xrightarrow{h_n} X_n \tag{19}$$

their composition yields the function

$$f \circ (h_1, \dots, h_n) \, : \, X'_1, \dots, X'_m \longrightarrow Y, \qquad x'_1, \dots, x'_m \longmapsto f(\, h_1(x'_1, \dots, x'_m), \dots, h_n(x'_1, \dots, x'_m) \,). \tag{20}$$

1.2.4

Precomposition with a function-list $h_1, \dots, h_n$ is itself a function between the function sets

$$(h_1, \dots, h_n)^\bullet \, : \, \mathrm{Funct}(X_1, \dots, X_n \, ; \, Y) \longrightarrow \mathrm{Funct}(X'_1, \dots, X'_m \, ; \, Y), \qquad f \longmapsto f \circ (h_1, \dots, h_n). \tag{21}$$

1.2.5 Invertible functions of a single variable

Composition of functions of a single variable produces a function of a single variable. We say that $f : \mathrm{X} \to Y$ is a *left-invertible* function, if there exists a function $g : Y \to \mathrm{X}$ such that

$$g \circ f = id_{\mathrm{X}} \, . \tag{22}$$

We say that $f : \mathrm{X} \to Y$ is a *right-invertible* function, if there exists a function $g : Y \to \mathrm{X}$ such that

$$f \circ g = id_Y \, . \tag{23}$$

Exercise 1 *Show that, if g is a left inverse of f and h is a right inverse of f, then $g = h$.*

1.2.6

We denote that unique left- and right-inverse by f^{-1}.

Exercise 2 *Show that a left-invertible function f is injective and a right-invertible function is surjective.*

In particular, an invertible function is bijective.

Exercise 3 *Show that a bijective function is invertible.*

Lemma 1.1 *Suppose that $f : X \to Y$ is injective. Then there is a natural correspondence between left inverses of f and functions $h : Y \setminus f_* X \longrightarrow X$.*

Proof. The target of a function f is the union of disjoint sets

$$Y' := f_* X \qquad \text{and} \qquad Y'' := Y \setminus f_* X .$$

Exercise 4 *Show that $g : Y \to X$ is a left inverse of f if and only if the restriction of g to Y' is the function*

$$y \longmapsto \textit{ the unique } x \in X \textit{ such that } f(x) = y .$$

Thus, the set of left inverses of f is in bijective correspondence with the set of functions $Y'' \to X$,

$$\text{Left Inverses}(f) \longleftrightarrow \text{Funct}(Y'', X) , \qquad g \longmapsto g_{|Y''} .$$

$\square$

Since the function set $\text{Funct}(Y'', X)$ is not empty as long as either X is not empty or Y'' is empty, we obtain the following two corollaries.

Corollary 1.2 *A function $f : X \to Y$ with $X \neq \emptyset$ is left-invertible if and only if f is injective. A function $f : \emptyset \to Y$ is left invertible if and only $Y = \emptyset$, i.e., if and only if f is bijective.*

Corollary 1.3 *A function $f : X \to Y$ with $X \neq \emptyset$ is bijective if and only if it has a unique left-inverse. That unique left-inverse is also a right-inverse.*

1.2.7 Finite sets

We say that a set is *finite* if every left-invertible function $f : X \to X$ is invertible.

1.2.8 Infinite sets

Accordingly, we say that a set X is *infinite*, if it admits a left-invertible function $f : X \to X$ that is not right-invertible.

1.2.9 Axiom of Infinity

The so called *Axiom of Infinity* of Set Theory asserts existence of an infinite set.

Existence of an infinite set cannot be proven using the remaining axioms of Set Theory. In fact, the remaining axioms of Set Theory are consistent with the assertion that every set is finite.

We shall prove later that Axiom of Infinity is equivalent to existence of the *semiring* $(\mathbf{N}, 0, 1, +, \cdot)$ of natural numbers.

1.3 The language of relations

1.3.1 Statements

A *statement* is a well-formed sentence that is either true or false. Any human language whose vocabulary is extended by adding various, previously defined, mathematical terms, is acceptable.

1.3.2 A relation is a function whose values are statements

A *relation* on sets $X_1, \dots, X_n$ is a function of n variables

$$\rho : X_1, \dots, X_n \longrightarrow \text{Statements}, \qquad x_1, \dots, x_n \longmapsto \rho(x_1, \dots, x_n). \tag{24}$$

We say in this case that ρ is an *n-ary* relation. We also say that the relation is *between* elements of sets $X_1, \dots, X_n$.

1.3.3

Statement $\rho(x_1, \dots, x_n)$, i.e., the value of ρ on the argument list $x_1, \dots, x_n$, needs not refer to some or even to anyone of the element variables x_i.

1.3.4 Total relations

Statement $\rho(x_1, \dots, x_n)$ may hold for every list of arguments. Such a relation is sometimes referred to as *a total* relation.

1.3.5 Void relations

Statement $\rho(x_1, \dots, x_n)$ may fail for every list of arguments. Such a relation is sometimes referred to as *a void* relation.

1.3.6 Nullary, unary, binary, ternary, ... relations

For small values of n, instead of speaking about 0-ary, 1-ary, 2-ary, 3-ary, ..., relations, we speak of *nullary, unary, binary, ternary, ...,* relations.

1.3.7 $\{$nullary relations$\}$ $\longleftrightarrow$ $\{$statements$\}$

According to Section 1.1.8, there is a canonical identification between *nullary relations* and *statements*.

1.3.8

Since a nullary relation reduces to a single statement, and since every statement either holds or fails, a nullary relation is either total or void.

1.3.9 Relations *on* a set

When all sets X_i in the domain coincide with a set X, we speak of an n-ary relation *on* X.

1.4 Operations on sets

1.4.1

An n*-ary operation* on a set Y is a function

$$\mu : X_1, \dots, X_n \longrightarrow Y \tag{25}$$

where all the sets $X_1, \dots, X_n$ are equal to Y.

1.4.2 $\{$nullary operations on $Y\} \longleftrightarrow Y$

To declare a nullary operation on a set Y is equivalent to supplying a single element of Y. For this reason, nullary operations on Y are thought of as "distinguished" elements of Y. In particular, there is a canonical bijection between the set of nullary operations on Y and the set Y itself.

1.4.3 Induced operations

Given a list of n functions of m variables,

$$f_1, \dots, f_n \in \mathrm{Funct}(X_1, \dots, X_m\,; Y), \tag{26}$$

let us assign to the argument list

$$x_1, \dots, x_m$$

the list of values

$$f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)$$

and then apply the operation μ. Composite assignment

$$x_1, \dots, x_m \longmapsto f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m) \longmapsto \mu\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big)$$

defines a function $X_1, \dots, X_m \longrightarrow Y$. We shall denote this function by $\mu_\bullet(f_1, \dots, f_n)$.

Assignment

$$f_1, \dots, f_n \longmapsto \mu_\bullet(f_1, \dots, f_n) \tag{27}$$

defines then an n-ary operation $\mu_\bullet$ on the set of functions $\mathrm{Funct}(X_1, \dots, X_m\,; Y)$. We refer to it as the operation *induced by* μ.

1.4.4

Operations on sets of Y-valued functions induced by operations defined on Y have been playing an essential role in Mathematics since the time when the foundations of Differential and Integral Calculus had been laid down nearly 400 years ago.

1.5 Canonical operations on $\mathscr{P}X$

1.5.1 Canonical operations

A general set X has no distinguished elements, hence it is not equipped with any distinguished nullary operation. Similarly, there are no distinguished binary, ternary, etc., operations on a general set. The identity function

$$\mathrm{id}_X : X \longrightarrow X, \qquad x \longmapsto x, \tag{28}$$

is the only distinguished unary operation.

Certain sets, however, are *naturally* equipped with various operations. We refer to such operations as *canonical*. An example of prime importance is provided by the set of all subsets, $\mathscr{P}X$, of an arbitrary set X. A shorter designation for $\mathscr{P}X$ is the *power-set of* X.

1.5.2 Canonical nullary operations on $\mathscr{P}X$

The power-set of a general nonempty set has exactly two distinguished elements: the empty subset $\emptyset$ and X. In other words, $\mathscr{P}X$ is equipped with exactly two canonical nullary operations.

1.5.3 The complement of a subset

The power-set of a general set has a canonical unary operation

$$\complement : \mathscr{P}X \longrightarrow \mathscr{P}X, \qquad A \longmapsto \complement A := \{x \in X \mid x \notin A\}, \tag{29}$$

that sends a subset $A \subseteq X$ to its *complement*. We shall usually denote the complement of a subset $A \subseteq X$ by A^c and use symbol $\complement$ to denote the complement operation.

1.5.4 Involutions on a set

Note that $\complement^2 := \complement \circ \complement$ is the identity operation. A unary operation $\mu : X \to X$ with this property is called an *involution* (on a set X). The identity operation id_X is a *trivial* involution.

1.5.5 Canonical unary operations on $\mathscr{P}X$

The power-set $\mathscr{P}X$ of a nonempty set is equipped with exactly two unary operations, both of them involutions on $\mathscr{P}X$: the identity operation $\mathrm{id}_{\mathscr{P}X}$ and the complement operation $\complement$.

1.5.6 Canonical binary operations on $\mathscr{P}X$

Union of two sets,

$$A, B \longmapsto A \cup B,$$

intersection of two sets,

$$A, B \longmapsto A \cap B,$$

difference of two sets,

$$A, B \longmapsto A \setminus B,$$

are examples of canonical binary operations on the power-set.

1.5.7

Any one of the above three operations can be expressed in terms of any of the remaining two and of the complement operation. For example, the union and the intersection operations are linked to each other by the following pair of identities

$$A \cap B = \complement(\complement A \cup \complement B) \qquad \text{and} \qquad A \cup B = \complement(\complement A \cap \complement B) \qquad (A, B \subseteq X) \tag{30}$$

called *de Morgan laws.*

Note also the following identities

$$A \cup \complement A = X, \qquad A \cap \complement A = \emptyset \qquad \text{and} \qquad A \setminus B = A \cap \complement B = \complement(\complement A \cup B)^c \qquad (A, B \subseteq X).$$

Exercise 5 *Find the identity expressing* $\cap$ *in terms of* $\setminus$ *and* $\complement$*, and prove it.*

1.6 Operations on Statements

1.6.1 Basic binary operations on sentences

The following table contains the list of basic binary operations on sentences (symbols P and Q stand for arbitrary sentences).

Read:	Symbolic notation	Name
P *and* Q	$P \wedge Q$	Conjunction
P *or* Q	$P \vee Q$	Alternative
if P, *then* Q	$P \Rightarrow Q$	Implication
P *if and only if* Q	$P \Leftrightarrow Q$	Equivalence

1.6.2 Negation

The negated sentence P will be symbolically denoted $\neg P$. In many languages, negating a sentence is performed according to rules that depend on the syntactical structure of that sentence. For this reason, it is difficult or impossible to provide one single reading of the negated sentence $\neg P$. We can circumvent this difficulty by saying, instead, "the negation of P" or "P negated", when we need to refer to $\neg P$.

1.6.3 Validity of the corresponding statements

Assuming that P and Q are statements,

- $P \wedge Q$ holds precisely when P and Q hold;
- $P \vee Q$ holds precisely when P or Q holds;
- $P \Rightarrow Q$ fails if P holds and Q fails, otherwise it holds;
- $P \Leftrightarrow Q$ holds precisely when P and Q both hold or both fail;
- $\neg P$ holds precisely when P fails.

In particular, Conjunction, Alternative, Implication, Equivalence, define binary operations on the set of Statements, while Negation defines a unary operation.

1.6.4 Operations on Statements = Relations on Statements

On the set of statements the concepts of an n-ary operation and of an n-ary relation coincide.

1.6.5 Operations on relations

Any operation on Statements induces the corresponding operations on the sets of relations, $\mathrm{Rel}(X_1, \dots, X_n)$, between elements of sets $X_1, \dots, X_n$.

1.6.6

Thus, given relations $\rho, \sigma \in \mathrm{Rel}(X_1, \dots, X_n)$, we can form the relations $\neg\rho$, $\rho \vee \sigma$, $\rho \wedge \sigma$, $\rho \Rightarrow \sigma$ and $\rho \Leftrightarrow \sigma$. They assign to an argument list $x_1, \dots, x_n$ the statements

$$\neg\rho(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \vee \sigma(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \wedge \sigma(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \Rightarrow \sigma(x_1, \dots, x_n)$$

and, respectively,

$$\rho(x_1, \dots, x_n) \Leftrightarrow \sigma(x_1, \dots, x_n).$$

1.7 Quantification

1.7.1 Universal quantification

Given a relation ρ between elements of sets $X_1, \dots, X_n$, and a subset $A_i \subseteq X_i$, by assigning to a list $x_1, \dots, \hat{x}_i, \dots, x_n$ the statement

$$\textit{for all } x_i \in X_i,\ \rho(x_1, \dots, x_n) \tag{31}$$

we obtain an $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$. Instead of "for all", we can also say "for every" with the same meaning.

Symbolically, statement (31) is represented

$$\forall_{x_i \in A_i}\, \rho(x_1, \dots, x_n)\,. \tag{32}$$

1.7.2 Universal quantification over a subset

The above construction defines what is called *universal quantification over a subset.* By assigning to a relation $\rho \in \mathrm{Rel}(X_1, \dots, X_n)$ the resulting relation $\forall_{x_i \in A_i} \rho$, we obtain a function

$$\forall_{x_i \in A_i} : \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}\left(X_1, \dots, \hat{X}_i, \dots, X_n\right), \qquad \rho \longmapsto \forall_{x_i \in A_i} \rho\,. \tag{33}$$

1.7.3

Statement (32) is naturally the value of a canonically defined function

$$\forall^i : \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}\left(X_1, \dots, \mathscr{P}X_i, \dots, X_n\right), \qquad x_1, \dots, A_i, \dots, x_n \longmapsto \forall_{x_i \in A_i} \rho(x_1, \dots, x_n)\,., \tag{34}$$

that preserves the number of arguments of ρ and replaces set X_i, the i-th entry in the domain-list, by its power-set $\mathscr{P}X_i$. Superscript i indicates that we are quantifying the relation with respect to the i-th variable.

1.7.4 " Statement S is a special case of statement T "

Suppose $\rho : X \longrightarrow$ Statements is a (unary) relation on a set X. Consider the statements obtained by universal quantification of relation ρ over subsets $A \subseteq X$ and $B \subseteq X$

$$S := \text{“ } \forall_{x \in A}\, \rho(x) \text{ ”} \qquad \text{and} \qquad T := \text{“ } \forall_{x \in B}\, \rho(x) \text{ ”}. \tag{35}$$

Note that, if $A \subseteq B$, then

$$S \Longrightarrow T. \tag{36}$$

If so, we shall say that *statement S is a special case of statement T*.

In general, given two statements S and T, we shall say that S is a special case of T if there exist

a unary relation ρ on a certain set X and subsets $A \subseteq B \subseteq X$

such that S and T have the form as in (35).

1.7.5 " Statement S trivially implies statement T "

Note that in order to establish implication (36), one deoes not need to know anything about a set X, a relation ρ on X, or subsets A and B. One only needs to know that both statements are obtained by *universal* quantification of the *same* certain unary relation over two subsets $A \subseteq B$ of X.

This is one of those situations when mathematicians are likely to say that a statement S *trivially* implies a statement T.

1.7.6 Existential quantification

Assigning to a list $x_1, \dots, \hat{x}_i, \dots, x_n$ the statement

$$\textit{there exists } x_i \in X_i \textit{ such that } \rho(x_1, \dots, x_n) \tag{37}$$

defines another an $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$.

Symbolically, statement (37) is represented

$$\exists_{x_i \in X_i}\, \rho(x_1, \dots, x_n).$$

Exercise 6 *Formulate the definitions of functions $\exists_{x_i \in A_i}$ and $\exists^i$ in analogy with the definitions given in Sections 1.7.2 and 1.7.3.*

1.7.7

Suppose $\rho : X \longrightarrow$ Statements is a (unary) relation on a set X. Consider the statements obtained by existential quantification of relation ρ over subsets $A \subseteq X$ and $B \subseteq X$

$$S := \text{“ } \exists_{x \in A}\, \rho(x) \text{ ”} \qquad \text{and} \qquad T := \text{“ } \exists_{x \in B}\, \rho(x) \text{ ”}. \tag{38}$$

Note that, if $A \subseteq B$, then

$$T \Longrightarrow S. \tag{39}$$

Also in this case we say that statement T *trivially implies* statement S.

1.7.8 The direct image function f_*

Operations of quantification are frequently iterated. An example of this is present in the definition of the *direct image* function associated with an arbitrary function (5).

$$f_* : \mathscr{P}X_1, \dots, \mathscr{P}X_n \longrightarrow \mathscr{P}Y \tag{40}$$

where

$$A_1, \dots, A_n \ \longmapsto\ f_*(A_1, \dots, A_n) := \left\{ y \in Y \mid \exists_{x_1 \in X_1} \dots \exists_{x_n \in X_n} f(x_1, \dots, x_n) = y \right\} . \tag{41}$$

Changing the order of iterated quantifiers *of the same type* produces relations that are equipotent. This allows to use compressed notation like

$$\exists_{x_1 \in X_1, \dots, x_n \in X_n}$$

in place of

$$\exists_{x_1 \in X_1} \cdots \exists_{x_n \in X_n}$$

in the definition of f_*,

$$A_1, \dots, A_n \ \longmapsto\ f_*(A_1, \dots, A_n) := \left\{ y \in Y \mid \exists_{x_1 \in X_1, \dots, x_n \in X_n} f(x_1, \dots, x_n) = y \right\} . \tag{42}$$

1.7.9 Caveat

Changing the order in which a universal and an existential quantifier are applied has usually a drammatic effect, however. Thus, given $i \neq j$,

$$\forall_{x_i \in X_i} \exists_{x_j \in X_j} \rho(x_1, \dots, x_n) \tag{43}$$

denotes the statement:

$$\textit{for all } x_i \in X_i\,,\ \textit{there exists } x_j \in X_j \ \textit{ such that } \rho(x_1, \dots, x_n) \tag{44}$$

while

$$\exists_{x_j \in X_j} \forall_{x_i \in X_i} \rho(x_1, \dots, x_n) \tag{45}$$

denotes the nonequivalent statement:

$$\textit{there exists } x_j \in X_j \ \textit{ such that, for all } x_i \in X_i\,,\ \rho(x_1, \dots, x_n)\ . \tag{46}$$

In particular, relations (43) and (45) are almost never equipotent.

1.7.10

Relations with several levels of quantifications are frequently encountered in important definitions and constructions. Processing with understanding such relations can pose a serious challenge to a beginner and is one of the reasons why Mathematics is considered to be difficult.

For example, the statement

$$\forall_{\varepsilon \in \mathbf{R}^+} \exists_{i \in \mathbf{N}} \forall_{j \in \mathbf{N}} \ (i \leq j \ \Rightarrow\ |x_j - a| < \varepsilon) \tag{47}$$

describes the fact that a sequence of real numbers (x_n) converges to a point a of the real line. Here, $\mathbf{R}^+$ denotes the set of positive real numbers and $\mathbf{N}$ denotes the set of natural numbers. The statement is about sequences (x_n) of real numbers and points a of the real line. It defines a binary relation between elements of these two sets. The relation is the result of applying one-after-another universal and existential quantification to the statement that has the form of implication

$$i \leq j \;\Rightarrow\; |x_j - a| < \varepsilon \,. \tag{48}$$

Here x_j denotes the j-th term of the sequence (x_n). Statement (48) is a statement about natural numbers i and j, a sequence of real numbers (x_n), a point of the real line a, and a positive real number ε. As such, it is a 5-ary relation. Application of three consecutive quantifications yields the binary relation defined in (47).

What you see here is a typical example of statements encountered in Mathematical Analysis.

Exercise 7 *Let* $\rho : X_1, X_2 \longrightarrow \text{Statements}$ *be a binary relation. Consider the statements*

$$S := \text{“ } \exists_{x_1 \in A_1} \forall_{x_2 \in A_2} \rho(x_1, x_2) \text{ ”} \qquad \textit{and} \qquad T := \text{“ } \exists_{x_1 \in B_1} \forall_{x_2 \in B_2} \rho(x_1, x_2) \text{ ”}$$

where A_1 *and* B_1 *are subsets of* X_1 *while* A_2 *and* B_1 *are subsets of* X_2*. Under what condition on* A_1*,* A_2*,* B_1 *and* B_2*, statement* S *implies statement* T*?*

1.8 Binary relations *on* a set: a vocabulary of terms

1.8.1

Binary relations on a set X call for a special attention in view of the central role they play in every area of Mathematics.

1.8.2 Infix notation

In view of the fact that binary relations have been used by mathematicians long before the concept of a general relation was formulated and are still the most frequently encountered type of relation, special notation has been employed when binary relations are mentioned. The symbolic expression

$$x_1 \rho x_2$$

has the meaning:

Statement $\rho(x_1, x_2)$ holds.

1.8.3 Tilde notation

More likely, however, you will see expressions like

$$x_1 \sim x_2 \,, \tag{49}$$

since symbol $\sim$ and its variants have been adopted as a generic symbol denoting a binary relation. The meaning of (49) is:

the binary relation in question, denoted $\sim$*, holds for elements* $x_1 \in X_1$ *and* $x_2 \in X_2$*.*

The difference between the *functional* notation and the *tilde* notation, when talking about binary relations, is similar to the difference between *direct speech* and *indirect speech*: compare the statements

$$3 < 5$$

and

$$\textit{inequality } 3 < 5 \textit{ holds}.$$

1.8.4 Various types of binary relations on a set

A binary relation ρ on a set X is said to be:

reflexive if

$$\forall_{x \in X}\, \rho(x, x) \tag{50}$$

symmetric if

$$\forall_{x,y \in X} \left(\rho(x, y) \Rightarrow \rho(y, x) \right) \tag{51}$$

antisymmetric if

$$\forall_{x,y \in X} \left(\rho(x, y) \Rightarrow \neg\rho(y, x) \right) \tag{52}$$

weakly antisymmetric if

$$\forall_{x,y \in X} \left(\rho(x, y) \wedge \neg\rho(y, x) \;\Rightarrow\; x = y \right) \tag{53}$$

transitive if

$$\forall_{x,y,z \in X} \left(\rho(x, y) \wedge \rho(y, z) \;\Rightarrow\; \rho(x, z) \right) \tag{54}$$

1.8.5

Of all the properties that a binary relation ρ on a set X may have, by far the most important is its *transitivity*.

1.8.6 Preorder relations

A transitive and reflexive relation is called a *preorder* or a *quasiorder*.

1.8.7 Equivalence relations

A symmetric preorder is called an *equivalence relation*. The set

$$[x]_\rho := \{y \in X \mid \rho(x, y)\} \tag{55}$$

of elements ρ*-related* to x is then called the *equivalence class of an element* $x \in X$. Since ρ is symmetric, $[x]_\rho = [y]_\rho$ precisely when $\rho(x, y)$.

Exercise 8 *Show that, for any* $x, y \in X$,

$$[x]_\rho \cap [y]_\rho \neq \emptyset,$$

if and only if,

$$[x]_\rho = [y]_\rho .$$

1.8.8 The set of equivalence classes of an equivalence relation

The set of equivalence classes of an equivalence relation ρ

$$\left\{C \subseteq X \mid \exists_{x \in X}\, C = [x]_\rho\right\} \tag{56}$$

makes a frequent appearance in every area of modern Mathematics. It has been one of the Mathematics most important constructs.

1.8.9

Set (56) is an example of a *family of subsets of X* and it appears, for example, in the construction of a *quotient of a set X by a binary relation on X*. For this reason, set (56) is often denoted $X_{/\rho}$ and the surjective function

$$[\ ]_\rho : X \longrightarrow X_{/\rho}, \qquad x \longmapsto [x]_\rho, \tag{57}$$

is called the *canonical quotient map*. We revisit this construction in Chapter 5 devoted to the concept of a *quotient structure*.

1.8.10 A remark about terminology: a *map*, a *mapping*

The term *map* is very frequently employed today as an alternative term for *function*. This use became established among Mathematical Analysts who preferred to reserve the term 'function' for real- or complex-valued functions. The word *map* is an abbreviated form of the word *mapping*, which is a calque from German word *Abbildung*, introduced early in the 20th Century by topologists, writing in German, to denote functions between spaces of real or complex-valued functions, and between more general spaces.

1.8.11 The equivalence relation canonically associated with a preorder

Suppose that ρ is a preorder relation on a set X.

Exercise 9 *Show that the conjunction of ρ and its opposite relation ρ^{op}*

$$\rho \wedge \rho^{\mathrm{op}} : X, X \longrightarrow \text{Statements}, \qquad x, y \longmapsto \text{“}\ \rho(x,y) \wedge \rho(y,x)\ \text{”}, \tag{58}$$

is an equivalence relation on X.

We shall refer to a pair of elements satisfying $\rho \wedge \rho^{\mathrm{op}}$ as *ρ-equivalent*.

1.8.12 Order relations

A weakly antisymmetric preorder is called an *order relation*. A preorder is an order relation precisely when (58) is the *weakest equivalence relation* on X, i.e., when $\rho \wedge \rho^{\mathrm{op}}$ is equipotent with the equality relation $=$.

1.8.13 Sharp order relations

An antisymmetric transitive relation is called a *sharp-order relation*.

1.8.14 Preordered sets

A set X equipped with a preorder relation will be called a *preordered set*. We shall use the generic symbol $\precsim$ to denote the preorder relation. When using the term 'preordered set', remember that it is not a set, it is a *mathematical structure*: a *set equipped with a binary relation* $(X, \precsim)$.

1.8.15 Ordered sets

When $\precsim$ is weakly-antisymmetric, i.e., when $\precsim$ is an order relation, we shall be using generic symbol $\preccurlyeq$ to denote it and we shall refer to a set X equipped with an order relation, $(X, \preccurlyeq)$, as an *ordered set*.

1.8.16 Comments about terminology and notation

To emphasize that elements of an ordered set are not necessarily *comparable*, the adverb "partially" is often placed in front of "ordered". Those who insisted on using the term "partially ordered set" soon began to abbreviate it in typed texts as "p. o. sets." When the abbreviation dots got lost, a monstrous term "poset" was born. *Do not use that term.*

1.8.17 Linearly ordered sets

Ordered sets whose elements are *comparable*, i.e., satisfy the condition

$$\forall_{x,y \in X} \; x \preccurlyeq y \lor y \preccurlyeq x\,, \tag{59}$$

are called *linearly*, or *totally*, ordered.

Naturally defined linear orders are scarce, unlike (partial) orders.

1.8.18 Well-ordered sets

Even scarcer are *well-ordering* relations, i.e., order relations for which every *nonempty* subset $A \subseteq X$ has the smallest element. A prime example of a well-ordered set is the set of natural numbers $\mathbf{N}$, cf. Section 3.3.11.

1.8.19 $|A| = |B|$

We say that subsets A and B of a set X *have the same cardinality* if there exists a bijection $f : A \to B$. One expresses this by writing

$$|A| = |B|\,. \tag{60}$$

Assignment

$$A, B \;\longmapsto\; \text{“ } |A| = |B| \text{ ”} \tag{61}$$

defines a binary relation on the power-set of X.

Exercise 10 *Show that* (61) *is an equivalence relation.*

1.8.20 Caveat

Note that we do *not* define the *cardinality* of a set X. We only define a binary relation *between* subsets of $\mathscr{P}X$.

1.8.21 $|A| \leq |B|$

Let us define the binary relation on $\mathscr{P}X$

$$A, B \longmapsto \text{“ } |A| \leq |B| \text{ ”} \tag{62}$$

by replacing the word 'bijection' in the definition of (60) by 'injective'. In other words, we mean by $|A| \leq |B|$ that there exists an *injective* function $A \rightarrowtail B$.

Exercise 11 *Show that* (62) *is a preorder relation on* $\mathscr{P}X$.

It is a nontrivial fact, established early in development of Set Theory, that existence of injective functions $A \rightarrowtail B$ and $B \rightarrowtail A$ implies existence of a bijection. We state it here without proof.

Theorem 1.4 (Cantor,Bernstein,Schröder) *For any sets* A *and* B,

$$|A| \leq |B| \wedge |B| \leq |A| \iff |A| = |B|. \tag{63}$$

□

1.8.22 $|A| = \mathfrak{c}$

We write

$$|A| = \mathfrak{c} \tag{64}$$

and say that a set A has the *cardinality of continuum* if

$$|A| = |\mathbf{R}|. \tag{65}$$

The following exercise is an application of Theorem 1.4.

Exercise 12 *Let* $A \subseteq \mathbf{R}$ *be a subset of the real line that contains an interval* (a, b). *Show that* $|A| = \mathfrak{c}$.

1.8.23 'Continuum Hypothesis'

The statement

$$\forall_{A \subseteq \mathbf{R}} \ |A| < \infty \vee |A| = \aleph_0 \vee |A| = \mathfrak{c}. \tag{66}$$

is known as *Continuum Hypothesis*. It was conjectured to be a theorem of Set Theory. All attempts to prove it were futile. Kurt Gödel proved that (66) was consistent with Axioms of Set Theory. In the early 1960-ties, Paul Cohen proved that its negation was also consistent with Axioms of Set Theory. Theorems of Gödel and Cohen together mean that assertion (66) cannot be proved or disproved. Such assertions are known as being *undecidable*.

1.8.24 Various approaches to the concept of the 'size' of a set

It is natural to *define* $|A|$ as the equivalence class of A with respect to the same-cardinality relation on $\mathscr{P}X$.

1.8.25 $|A| < \infty$ or $|A| = \infty$

When A is an infinite set, it is common to write

$$|A| = \infty. \tag{67}$$

Symbol ∞ here has no independent meaning. One should consider whole Expression (67) as saying that A *is an infinite set*. Accordingly,

$$|A| < \infty. \tag{68}$$

expresses the fact that set A is finite.

1.8.26 $|A| = n$

For a finite set, the expression

$$|A| = n. \tag{69}$$

means that there exists a bijection between A and the interval

$$\mathbf{n} := \{0, \dots, n-1\}$$

of the set of natural numbers $\mathbf{N}$. Element $n \in \mathbf{N}$ is then called the *number of elements of* A. Since $\mathbf{N}$ is equipped with a canonical linear order, a bijection $\mathbf{n} \leftrightarrow A$ is the same as linearly-ordering set A,

$$a_1 < \dots < a_n.$$

where $n \in \mathbf{N}$ is a certain natural number that This corresponds to 'counting' the elements of A.

1.8.27 $|A| = \aleph_0$

We say that a set A is *countably infinite* or, an *infinite countable* set, if there exists a bijection between A and the set of natural numbers $\mathbf{N}$. In this case it is common to write

$$|A| = \aleph_0 \tag{70}$$

and to say that A *has the cardinality aleph zero*. Such sets are the departure point for developing *theory of cardinal numbers* within the scope of theory of well-ordered sets. These advanced topics are covered in a course on Set Theory.

1.8.28 A canonical ordered-set structure on the power-set $\mathscr{P}X$ of a set X

The set of all subsets of a given set X is guaranteed to exist by one of the Axioms of Set Theory. Informally referred to as the *power-set of* X, it is denoted $\mathscr{P}X$. Inclusion of subsets,

$$\subseteq : \mathscr{P}X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad A, B \longmapsto \text{“}A \subseteq B\text{”}. \tag{71}$$

is a *canonical* order relation on $\mathscr{P}X$. Note that the inclusion relation is defined in terms of the *membership relation*

$$\in : X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad x, A \longmapsto \text{“ } x \in y \text{ ”}, \tag{72}$$

by

$$A \subseteq B := \text{“ } \forall_{x \in A}\, x \in B \text{ ”}. \tag{73}$$

Both $(\mathscr{P}X, \subseteq)$ and the *opposite* ordered set, $(\mathscr{P}X, \supseteq)$, play a central role in Mathematics.

1.9 Induced relations

1.9.1

Given a list of n functions of m variables (26) and an n-ary relation $\rho \in \mathrm{Rel}_n Y$, the composite $\rho_\bullet(f_1, \dots, f_n)$ is an m-ary relation

$$\rho_\bullet(f_1, \dots, f_n) : X_1, \dots, X_m \longrightarrow \text{Statements}, \qquad x_1, \dots, x_m \longmapsto \rho\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big). \quad (74)$$

Universal quantification over $x_1 \in X_1, \dots, x_m \in X_m$ transforms (74) into a statement. If we assign thas statement to function-list $f_1, \dots, f_n$,

$$f_1, \dots, f_n \longmapsto \forall_{x_1 \in X_1, \dots, x_m \in X_m}\, \rho\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big), \tag{75}$$

we obtain an n-ary relation on the set of functions $\mathrm{Funct}(X_1, \dots, X_m; Y)$. Here

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n}$$

is an abbreviation for

$$\forall_{x_1 \in X_1} \cdots \forall_{x_n \in X_n}.$$

We shall denote relation (75) by $\hat{\rho}$ and will refer to it as the *induced relation* or as the relation *induced by* ρ.

Exercise 13 *Show that the relation induced by a preorder is a preorder.*

1.9.2 Induced relations on $\mathrm{Rel}(X_1, \dots, X_n)$

Recalling that, on the set of statements, there is no difference between relations and operations, we observe that any binary operation on the set of statements induces a binary relation on the set of relations $\mathrm{Rel}(X_1, \dots, X_n)$.

1.9.3 The equipotence relation on $\mathrm{Rel}(X_1, \dots, X_n)$

We shall denote by $\Longleftrightarrow$ the relation induced by Equivalence $\Leftrightarrow$, an *operation* on the set of statemnets. We shall refer to this induced relation as the *equipotence relation*. Following an old habit, mathematicians refer to equipotent relations as *equivalent*. This is one of many uses of the term *equivalent* by mathematicians. One should remember that an *equivalence relation* is a generic term for a binary relation on any set that is *reflexive,symmetric, and transitive*, cf. Section 1.8.7.

1.9.4 Caveat

One must be careful to distinguish, $\rho \Leftrightarrow \sigma$, the relation obtained by applying the induced binary *operation*

$$\mathrm{Rel}(X_1, \dots, X_n), \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}(X_1, \dots, X_n), \qquad \rho, \sigma \longmapsto \rho \Leftrightarrow \sigma, \tag{76}$$

to the pair ρ, σ, from $\rho \Longleftrightarrow \sigma$. The latter is a single *statement*, namely

$$\rho \Longleftrightarrow \sigma \quad := \quad \text{“} \forall_{x_1 \in X_1, \dots, x_n \in X_n}\, \rho(x_1, \dots, x_n) \Leftrightarrow \sigma(x_1, \dots, x_n) \text{”}.$$

The former, $\rho \Leftrightarrow \sigma$, is a function

$$X_1, \dots, X_n \longrightarrow \text{Statements}\,, \qquad x_1, \dots, x_n \;\longmapsto\; \text{“}\; \rho(x_1, \dots, x_n) \Leftrightarrow \sigma(x_1, \dots, x_n) \;\text{”}\,. \tag{77}$$

This distinction disappears when the domain-list $X_1, \dots, X_n$ is empty, i.e., when ρ and σ are *statements*.

1.9.5 Equipotence classes of statements

By definition, there are just two equipotence classes of statements

$$\top := \{S \in \text{Statements} \mid S \text{ holds}\} \qquad \text{and} \qquad \bot := \{S \in \text{Statements} \mid S \text{ does not hold}\}\,. \tag{78}$$

There is a canonical bijective correspondence

$$\left\{ \begin{array}{c} \text{Equipotence classes of relations} \\ \rho : X_1, \dots, X_n \longrightarrow \text{Statements} \end{array} \right\} \longleftrightarrow \text{Funct}\,(X_1, \dots, X_n\,;\, \{\top, \bot\})\,. \tag{79}$$

1.9.6 The implication relation on $\mathbf{Rel}(X_1, \dots, X_n)$

We shall denote by $\Longrightarrow$ the relation on $\text{Rel}(X_1, \dots, X_n)$ that is induced by Conditional $\Rightarrow$, an *operation* on the set of statements. We shall refer to this induced relation as the *implication relation*.

1.9.7 Caveat

A warning similar to the one issued in 1.9.4 is due: do not confuse $\rho \Rightarrow \sigma$ with $\rho \Longrightarrow \sigma$.

1.9.8 The canonical preorder on $\mathbf{Rel}(X_1, \dots, X_n)$

Since Conditional $\Rightarrow$ is a *preorder* relation on the set of statements, the induced relation $\Longrightarrow$ is a preorder relation. It is not only a canonical preorder on the set of relations, it is, in fact, a vital part of any reasoning process. Any form of rigorous reasoning employs the implication relation.

1.9.9 Terminology: *implies, is weaker than, is stronger than*

Given two relations $\rho, \sigma \in \text{Rel}(X_1, \dots, X_n)$ such that

$$\rho \Longrightarrow \sigma\,, \tag{80}$$

we shall say that ρ *implies* σ or that ρ is *weaker* than σ. In that case we shall σ is *stronger* than ρ.

The terms “weaker” and “stronger” is not an ideal terminology: ρ is both weaker and stronger than σ precisely when ρ and $_s$ are *equipotent*, not equal.

1.9.10

The implication preorder induces a canonical order relation on the set of equipotence classes of relations and canonical correspondence (79) *identifies* that set with the set of $\{\top,\bot\}$-valued functions equipped with the order relation induced by the order relation on $\{\top,\bot\}$ such that $\top$ is *greater* than $\bot$.

Lemma 1.5 *Any transitive relation on the set of statements that is stronger than* $\Leftrightarrow$ *is equipotent to*

$$\Leftrightarrow,\quad \Rightarrow,\quad \Leftarrow, \tag{81}$$

or is a total relation.

Proof. Transitivity of ρ means that

$$\forall_{P,Q,R\in\text{Statements}}\quad \rho(P,Q)\wedge\rho(Q,R)\Rightarrow\rho(P,R)\,.$$

It follows that a transitive relation stronger than $\Leftrightarrow$ has the properties

$$\forall_{P,P',Q\in\text{Statements}}\quad P\Leftrightarrow P'\wedge\rho(P,Q)\Rightarrow\rho(P',Q)$$

and

$$\forall_{P,Q,Q'\in\text{Statements}}\quad \rho(P,Q)\wedge Q\Leftrightarrow Q'\Rightarrow\rho(P,Q')$$

which, in view of *symmetry* of relation $\Leftrightarrow$, imply the stronger properties

$$\forall_{P,P',Q\in\text{Statements}}\quad P\Leftrightarrow P'\Rightarrow\big(\rho(P,Q)\Leftrightarrow\rho(P',Q)\big) \tag{82}$$

and

$$\forall_{P,Q,Q'\in\text{Statements}}\quad Q\Leftrightarrow Q'\Rightarrow\big(\rho(P,Q)\Leftrightarrow\rho(P,Q')\big)\,. \tag{83}$$

It follows that ρ defines a binary operation $\cdot_\rho\in\mathrm{Op}_2\{\top,\bot\}$,

$$T_1\cdot_\rho T_2:=\begin{cases}\top & \text{if } \rho(P_1,P_2) \text{ for any } P_1\in T_1 \text{ and } P_2\in T_2\\ \bot & \text{otherwise}\end{cases}$$

where $T_1,T_2\in\{\top,\bot\}$.

Since ρ is stronger than $\Leftrightarrow$, one has $T_1\cdot_\rho T_2=\top$ whenever $T_1=T_2$. This leaves four possibilities

$\top\cdot\bot=\bot\cdot\top=\bot$ In this case ρ is equipotent to $\Leftrightarrow$.

$\top\cdot\bot=\bot\cdot\top=\top$ In this case ρ is a total relation ,

$\top\cdot\bot=\bot\wedge\bot\cdot\top=\top$ In this case ρ is equipotent to $\Rightarrow$.

$\top\cdot\bot=\top\wedge\bot\cdot\top=\bot$ In this case ρ is equipotent to $\Leftarrow$.

$\square$

1.10 Functions of n variables viewed as $(n+1)$-ary relations

1.10.1

Given sets $X_1, \dots, X_n$ and Y, and a function of n variables

$$f : X_1, \dots, X_n \longrightarrow Y, \tag{84}$$

we can associate with it an $(n+1)$-ary relation

$$\rho_f : X_1, \dots, X_n, Y \longrightarrow \text{Statements}, \qquad x_1, \dots, x_n, y \longmapsto \text{“ } f(x_1, \dots, x_n) = y \text{ ”}. \tag{85}$$

Functions corresponding to reflexive relations

1.10.2

The $(n+1)$-ary relation ρ_f has the following property:

$$\textit{for every list of elements } x_1 \in X_1, \dots, x_n \in X_n, \textit{ there exists a unique } y \in Y, \textit{ such that } \rho(x_1, \dots, x_n, y). \tag{86}$$

1.10.3

Given any $(n+1)$-ary relation satisfying property (86), we can define a function (84) where $f(x_1, \dots, x_n)$ is defined to be that unique element $y \in Y$ such that

$$\rho(x_1, \dots, x_n, y).$$

Let us denote this function f_ρ.

Exercise 14 *Show that $f_\rho = f_\sigma$ if and only if ρ and σ are equipotent.*

1.11 Composing relations

1.11.1

Suppose that two relations are given,

$$\text{an } (m+1)\text{-ary relation between elements of sets } X_0, \dots, X_m \,,$$

denoted σ, and

$$\text{an } (n+1)\text{-ary relation between elements of sets } X_m, \dots, X_{m+n+1} \,,$$

denoted ρ. Assigning to a list $x_1, \dots, \hat{x}_m, \dots, x_{m+n+1}$ the statement

$$\textit{there exists } x_m \in X_m \textit{ such that } \sigma(x_0, \dots, x_m) \textit{ and } \rho(x_m, \dots, x_{m+n+1}) \tag{87}$$

defines an $(m+n+1)$-ary relation between elements of sets

$$X_1, \dots, \hat{X}_m, \dots, X_{m+n+1} \,.$$

Symbolically, statement (87) is represented

$$\exists_{x_m \in X_m} \left(\sigma(x_0, \dots, x_m) \wedge \rho(x_m, \dots, x_{m+n+1})\right) .$$

1.11.2

We call the relation defined above, the *composite of ρ and σ* and denote it $\rho \circ \sigma$.

1.12 Cartesian product $X_1 \times \cdots \times X_n$

1.12.1

Given a list of sets $X_1, \ldots, X_n$, let us form its Cartesian product

$$X_1 \times \cdots \times X_n . \tag{88}$$

By definition, its elements are ordered n-tuples $(x_1, \ldots, x_n)$ of elements $x_1 \in X_1, \ldots, x_n \in X_n$.

1.12.2 The concept of an ordered n-tuple

What is an ordered n-tuple? There is not much difference between lists of length n and ordered n-tuples. When we speak of an ordered n-tuple, we always think of it being a *single* entity, while when we speak of a list of length n, we think of n separate entities.

1.12.3

To illustrate this further, the assignment

$$x, y \mapsto x + y \qquad (x, y \in \mathbf{N})$$

defines a function of 2 variables on the set of natural numbers $\mathbf{N}$, while the assignment

$$(x, y) \mapsto x + y \qquad (x, y \in \mathbf{N})$$

defines a function of a single variable on the Cartesian square $\mathbf{N} \times \mathbf{N}$ of $\mathbf{N}$. The targets of both functions are the same, namely the set of natural numbers.

1.12.4 The equality principle

The principal property built into the concept of an ordered n-tuple is the following equality principle

$$(x_1, \ldots, x_m) = (y_1, \ldots, y_n)$$

if and only if $m = n$ and $x_i = y_i$ for all $1 \leq i \leq n$.

1.12.5 The standard set-theoretic model of an ordered pair

The actual model of an ordered n-tuple is of little importance. It is possible to prove existence of such a model using only basic set theoretic concepts. For example, the axiom of Set Theory called Axiom of a Pair states that, for any x and y, the set $\{x, y\}$, whose elements are x and y, exists. Thus, $\{x\} = \{x, x\}$ and $\{x, y\}$ exist and therefore also the following set

$$\big\{\{x\}, \{x, y\}\big\} \tag{89}$$

exists. This set is a model of an *ordered pair*, i.e., of an ordered a 2-tuple.

Exercise 15 *Show that*

$$\big\{\{x\},\{x,y\}\big\} = \big\{\{x'\},\{x',y'\}\big\}$$

if and only if $x = x'$ *and* $y = y'$.

1.12.6

If $x \in X$ and $y \in Y$, then (89) is a *family* of subsets of $X \cup Y$, i.e., it is a subset of the power-set of $X \cup Y$,

$$\big\{\{x\},\{x,y\}\big\} \subseteq \mathscr{P}(X \cup Y) \, .$$

Accordingly, the Cartesian product $X \times Y$ is realized as the appropriate subset of the power-set of the power-set of $X \cup Y$,

$$X \times Y := \big\{ \, P \in \mathscr{P}(\mathscr{P}(X \cup Y)) \mid \exists_{x\in X} \exists_{y\in Y} P = \{\{x\},\{x,y\}\} \, \big\} \, ,$$

which demonstrates its existence.

1.12.7

Having a model of on ordered pair, the ordered pair

$$\big((x,y),z\big)$$

becomes a model of an ordered triple and the Cartesian product

$$(X \times Y) \times Z$$

becomes a model of $X \times Y \times Z$. By induction on n, one can construct a model of an ordered n-tuple

$$(x_1, \dots, x_n)$$

and of

$$X_1 \times \cdots \times X_n \, ,$$

There are other, more convenient models.

1.12.8 An ordered n-tuple as a function

A convenient model of an ordered n-tuple $(x_1, \dots, x_n)$ is provided by a function

$$\xi : \{1, \dots, n\} \longrightarrow X_1 \cup \cdots \cup X_n \tag{90}$$

whose value at i is, for every $1 \le i \le n$, an element of X_i.

In this model, the Cartesian product $X_1 \times \cdots \times X_n$ is represented as a subset of the set of all functions (90).

1.12.9 Universal functions of n-variables

We shall say that a function

$$\tau : X_1, \dots, X_n \longrightarrow T \tag{91}$$

is a *universal* function with the domain-list $X_1, \dots, X_n$, if *every* function (5) can be produced from τ by postcomposing τ with a *unique* function $\tilde{f} : T \to Y$,

$$f = \tilde{f} \circ \tau.$$

In that case, the bijective correspondence

$$\mathrm{Funct}(X_1, \dots, X_n; Y) \longleftrightarrow \mathrm{Funct}(T, Y), \qquad f \longleftrightarrow \tilde{f}, \tag{92}$$

identifies the set of Y-valued functions of n-variables, with the domain-list $X_1, \dots, X_n$, with the set of functions of a single variable $T \to Y$.

1.12.10 The canonical function of n-variables $X_1, \dots, X_n \longrightarrow X_1 \times \dots \times X_n$

For every list of sets $X_1, \dots, X_n$, there exists a canonical function of n-variables with that list as its domain. It assigns to an argument list $x_1, \dots, x_n$ the corresponding ordered n-tuple,

$$X_1, \dots, X_n \longrightarrow X_1 \times \dots \times X_n, \qquad x_1, \dots, x_n \longmapsto (x_1, \dots, x_n), \tag{93}$$

1.12.11

The canonical function has the universal property defined in Section 1.12.9. Indeed,

$$f \quad \longmapsto \quad \left(\tilde{f} : X_1 \times \dots \times X_n \to Y, \quad (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)\right)$$

is a bijective correspondence and f is produced by postcomposing function (93) with $\tilde{f}$.

1.12.12 The case of functions of zero variables

When $n = 0$, Cartesian product of the empty list of sets consists of functions from the *empty* set of natural numbers to the union of the empty family of sets. The latter, as we already know, is the empty set. In other words, Cartesian product of the empty list of sets is the set of functions

$$\emptyset^{\emptyset} = \mathrm{Funct}(\emptyset, \emptyset) = \{\mathrm{id}_{\emptyset}\}, \tag{94}$$

and that set has a unique element, namely the identity function associated with the empty set. Exponential notation $\emptyset^{\emptyset}$, cf. (8) is particularly apt in this case. We observe that foundations of Set Theory themeselves are telling us that 0^0 is well defined and equals to 1.

1.12.13 Canonical identification $\mathrm{Op}_0(Y) \longleftrightarrow \mathrm{Funct}(\emptyset^{\emptyset}, Y)$

In particular, nullary operations on a set Y, i.e., Y-valued functions of zero of variables, are canonically identified with functions $\emptyset^{\emptyset} \to Y$.

1.12.14

Every statement containing references to functions of n-variables can be now replaced by an equivalent statement containing references exclusively to functions of a single variable.

This explains why the use of the concept of a function of n-variables has practically disappeared from modern mathematical language. This is also the reason why Cartesian product is today present everywhere where normally one would be mentioning functions of n-variables: Cartesian product

$$X_1 \times \cdots \times X_n$$

is the *target* of the universal function of n-variables (93).

1.12.15 Canonical projections $(\pi_i)_{i \in \{1,\dots,n\}}$

The Cartesian product is more than just a set, it is a *mathematical structure*, like a relation or a function. One should consider the Cartesian product to consist of a set $X_1 \times \cdots \times X_n$ equipped with a list of functions

$$\pi_1, \dots, \pi_n , \tag{95}$$

called the *canonical projections*, where π_i is defined as

$$\pi_i : X_1 \times \cdots \times X_n \longrightarrow X_i, \qquad (x_1, \dots, x_n) \mapsto x_i . \tag{96}$$

Having just set $X_1 \times \cdots \times X_n$ alone would not suffice to recover the list of sets $X_1, \dots, X_n$. For example, $X_1 \times \cdots \times X_n$ is the empty set whenever at least one set X_i is empty.

1.12.16 *Naturality* of Cartesian product

Cartesian product assigns to a list of sets $X_1, \dots, X_n$ a single set $X_1 \times \cdots \times X_n$ equipped with the list of functions $\pi_1, \dots, \pi_n$. A function-list

$$X_1 \xrightarrow{f_1} X'_1 \,, \ \dots \ , \ X_n \xrightarrow{f_n} X'_n \,, \tag{97}$$

induces a function between the corresponding Cartesian-product sets

$$f_1 \times \cdots \times f_n : X_1 \times \cdots \times X_n \longrightarrow X'_1 \times \cdots \times X'_n \,, \qquad (x_1, \dots, x_n) \longmapsto (f_1(x_1), \dots, f_n(x_n)) . \tag{98}$$

Moreover, the assignment

$$f_1, \dots, f_n \longmapsto f_1 \times \cdots \times f_n$$

commutes with the operations of function composion.

Exercise 16 *Given a function-list*

$$X'_1 \xrightarrow{f'_1} X''_1 \,, \ \dots \ , \ X'_n \xrightarrow{f'_n} X''_n \,,$$

show that

$$(f'_1 \times \cdots \times f'_n) \circ (f_1 \times \cdots \times f_n) \ = \ (f'_1 \circ f_1, \dots, f'_n \circ f_n) .$$

Mathematicians refer to such behavior as *naturality* of the assignment

$$X_1, \dots, X_n \longmapsto X_1 \times \cdots \times X_n .$$

1.12.17 The graph of a relation

Given a relation ρ between elements of sets $X_1, \ldots, X_n$, the following subset of the Cartesian product,

$$\Gamma_\rho := \{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n \mid \rho(x_1, \ldots, x_n)\} \tag{99}$$

is guaranteed to exist by the axioms of Set Theory. This is the set of those ordered n-tuples for which statement $\rho(x_1, \ldots, x_n)$ holds. One calls it the *graph* of ρ.

Exercise 17 *Let ρ and σ be two relations between elements of sets $X_1, \ldots, X_n$. Show that ρ is weaker than σ if and only if*

$$\Gamma_\rho \subseteq \Gamma_\sigma .$$

1.12.18

In particular, relations ρ and σ are equipotent if and only if their graphs are equal

$$\Gamma_\rho = \Gamma_\sigma .$$

1.12.19 Correspondences

The graph of a relation provides another example of a mathematical structure. It involves the list of the following data:

- a list of sets $X_1, \ldots, X_n$,
- a subset $C \subseteq X_1 \times \cdots \times X_n$.

Having just the set C alone would not suffice to recover the list of sets $X_1, \ldots, X_n$.

A structure of this kind begs for a name. I propose to call it a *correspondence between elements of sets* $X_1, \ldots, X_n$ or, an *n-correspondence*, in short.

1.12.20

When all sets X_i are one and the same set X, we shall speak of n-correspondences *on* X.

1.12.21 1-correspondences

In particular, 1-correspondences on X are the same as *subsets* of X.

1.12.22

In practice, we still be denoting a correspondence by the symbol denoting the subset C of $X_1 \times \cdots \times X_n$.

1.12.23 Caveat

In fact, a common practice among mathematicians is to call precisely this structure a *relation*. This approach to the concept of a relation, while being much less intuitive than the 'statements-valued function' approach, it allows one to place theory of relations entirely within the realm of Set Theory. For example, relations with a given domain (1) form a well defined set.

1.12.24

The main advantage of such a restrictive notion of a relation is that it frees a mathematician from any concerns about what is and what is not a *statement* while still being sufficient for studying the whole of Mathematics.

Indeed, given a correspondence C between elements of sets $X_1, \dots, X_n$, let $\rho_C(x_1, \dots, x_n)$ be the statement

$$(x_1, \dots, x_n) \in C\ .$$

This defines a relation between elements of sets $X_1, \dots, X_n$.

Exercise 18 *Show that any relation ρ is equipotent to the relation ρ_{Γ_ρ}.*

Exercise 19 *Show that, for any correspondence C, one has $C = \Gamma_{\rho_C}$.*

1.12.25

We shall express the operations on relations, introduced in Sections 1.6.5–1.7, in terms of their graph correspondences. For this we need to introduce some notation.

Exercise 20 *Given a relation ρ, show that*

$$\Gamma_{\neg\rho} = \complement\Gamma_\rho\ . \tag{100}$$

Exercise 21 *Given relations ρ and σ with the same domain, show that*

$$\Gamma_{\rho\vee\sigma} = \Gamma_\rho \cup \Gamma_\sigma \qquad \textit{and} \qquad \Gamma_{\rho\wedge\sigma} = \Gamma_\rho \cap \Gamma_\sigma\ . \tag{101}$$

1.12.26

The above two exercises demonstrate that the operations of negation, alternative and conjunction of relations translate into the operations of taking the complement, the union, and the intersection, of correspondences.

Exercise 22 *Given relations ρ and σ with the same domain, show that*

$$\Gamma_{\rho\Rightarrow\sigma} = \complement\Gamma_\rho \cup \Gamma_\sigma\ . \tag{102}$$

1.12.27 The function-list canonically associated with an n-correspondence

By post-composing the canonical inclusion $\iota : C \hookrightarrow X_1 \times \cdots \times X_n$ with the list of canonical projections $\pi_1, \dots, \pi_n$, we obtain a list of functions

$$\begin{array}{c} C \\ {\scriptstyle\partial_1}\big\downarrow \cdots \big\downarrow{\scriptstyle\partial_n} \\ X_1, \dots, X_n \end{array} \tag{103}$$

that is canonically associated with the correspondence. Here $\partial_i := \pi_i \circ \iota$, $1 \leq i \leq n$.

1.12.28 Oriented graphs

When $n = 2$ and X_1 and X_2 are the same set X, a list (103) is called an *oriented graph*. Elements of X are referred to, in this case, as *vertices* and elements of C are referred as *oriented edges*, or *arrows*, of the graph.

1.12.29 2-Correspondences as oriented graphs

In particular, 2-correspondences on a set X can be viewed as oriented graphs with vertices being elements of X, such that no two oriented edges have the same source and the same target.

1.13 The language of diagrams

1.13.1

Situations involving several functions are frequently expressed and analyzed in the language of oriented graphs, represented visually as diagrams drawn on a blackboard, or on a page. Arrows in a diagram represent functions. Vertices represent their domains and targets. *Oriented paths* in such graphs represent composable lists of functions.

1.13.2 Commutative diagrams

When composition of two paths with the same origin and the same terminus produces the same result, we call such a diagram *commutative*. Most common examples of commutative diagrams have a form of a *commutative sqaure*,

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & S \arrow[l, "\beta"'] \arrow[d, "\delta"] \\ T & Y \arrow[l, "\gamma"] \end{tikzcd} . \tag{104}$$

Commutativity of square diagram (104) expresses the equality

$$\alpha \circ \beta = \gamma \circ \delta .$$

1.13.3

Commutativity of a diagram is often signaled by placing a symbol $\circlearrowleft$, or its cousins: $\circlearrowright$, $\circlearrowleft$, or $\circlearrowright$, between two composable paths of arrows originating and terminating in a common vertex.

1.13.4

Diagrams are employed not only to illustrate situations that can be discussed without introducing diagrams. It has been long observed that employing diagrams can greatly clarify and enhance analysis of complex scenarios. We shall illustrate it here by considering one example. Later in these notes you will see many more appearances of commutative diagrams.

1.13.5 An example

Consider a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \tag{105}$$

We do not know whether it is possible to complete diagram (105) to a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} , \tag{106}$$

we observe, however, that diagram (105) defines in a canonical manner a binary relation between elements of Y_1 and Y_2,

$$\rho : Y_1, Y_2 \longrightarrow \text{Statements}, \qquad y_1, y_2 \longmapsto \text{“ } \exists_{x_1 \in X_1}\, y_1 = \chi_1(x_1) \wedge y_2 = (\chi_2 \circ \gamma)(x_1) \text{ ”}. \tag{107}$$

1.13.6

It is clear that

$$\forall_{y_1 \in Y_1} \exists_{y_2 \in Y_2} \rho(y_1, y_2)$$

if and only if function χ_1 is surjective.

1.13.7

Let $y_2, y_2' \in Y_2$ be two elements in relation with a given element $y_1 \in Y_1$. Then, there are elements $x_1, x_1' \in X_1$ such that

$$y_1 = \chi_1(x_1) = \chi_1(x_1'), \qquad y_2 = (\chi_2 \circ \gamma)(x_1) \qquad \text{and} \qquad y_2' = (\chi_2 \circ \gamma)(x_1').$$

By combining this with commutativity of diagram (105) we obtain a chain of equalities

$$\varphi_2(y_2) = (\varphi_2 \circ \chi_2 \circ \gamma)(x_1) = (\alpha \circ \varphi_1)(\chi_1(x_1)) = (\alpha \circ \varphi_1)(\chi_1(x_1')) = (\varphi_2 \circ \chi_2 \circ \gamma)(x_1') = \varphi_2(y_2').$$

If φ_2 is injective, then $y_2 = y_2'$ and relation (107) defines a function

$$\beta : Y_1 \longrightarrow Y_2, \qquad y_1 \longmapsto \text{the unique } y_2 \in Y_2 \text{ such that } \rho(y_1, y_2).$$

1.13.8 Diagram chasing

The method we used to construct relation (107) and then to prove that under suitable hypotheses (107) defines a function, is referred to as *diagram chasing*.

1.13.9

Let us represent surjective functions by two-headed arrows $\twoheadrightarrow$ and injective functions by tailed arrows $\rightarrowtail$. We established the following fact.

Lemma 1.6 *Every commutative diagram*

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, phantom, "\circlearrowleft"] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, tail, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \tag{108}$$

admits a completion to a commutative diagram (106)*. Moreover, function β that makes diagram* (106) *commutative is unique.* □

Exercise 23 *Prove uniqueness of β.*

1.13.10

Consider now an arbitrary commutative diagram

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] & X_1 \arrow[l, "\chi_1"'] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \tag{109}$$

If arrow χ_2 admits a right inverse $\xi : Y_2 \to X_2$, then

$$\gamma := \xi \circ \beta \circ \chi_1$$

obviously makes the diagram

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd}$$

commute.

1.13.11

We can sum our discussion up in the following lemma.

Lemma 1.7 *Consider a diagram*

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] & X_1 \arrow[l, two heads, "\chi_1"'] \\ Z_2 & Y_2 \arrow[l, tail, "\varphi_2"] & X_2 \arrow[l, two heads, "\chi_2"] \end{tikzcd} \tag{110}$$

The following three properties of diagram (110) *are equivalent:*

(a) *it admits a completion to a commutative diagram* (105);

(b) *it admits a completion to a commutative diagram* (109).

(c) *it admits a completion to a commutative diagram*

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dl, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \qquad (111)$$

Implications (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) rely on Axiom of Choice, cf. Section 1.15.15, or one has to add the hypothesis to the effect that χ_2 has a right inverse (such functions are said to be *split surjections*).

1.13.12 $\sim$-commutative diagrams

A $\sim$-*commutative diagram* is a slight yet a very significant generalization of a commutative diagram, cf. 1.13.2. Whole areas of advanced modern Mathematics and Mathematical Physics are devoted to studying phenomena expressed in the language of $\sim$-commutative diagrams.

Commutativity of a diagram means that two composable paths of arrows (representating functions between sets), that have a common source and a common target, are equal. If that common target, call it T, is equipped with a binary relation $\sim$, then *equality* may be replaced by the condition that the corresponding composite functions satisfy the relation induced by $\sim$ on the set of T-valued functions.

Since $\sim$ is not necessarily symmetric, one needs to indicate which composite function appears as the *left* argument and which appears as the *right* argument of the relation in question.

This can be represented in a diagram by placing a small arrow (ideally, a bent arrow) near the common target of two composable paths of arrows, as is shown in the following simple example. A square-shaped diagram

$$\begin{tikzcd} X \arrow[d, "\upsilon"'] & S \arrow[l, "\varphi"'] \arrow[d, "\psi"] \arrow[dl, phantom, "\curvearrowleft\sim"] \\ T & Y \arrow[l, "\chi"] \end{tikzcd}$$

expresses the statement

$$\forall_{s \in S}\, \chi(\psi(s)) \sim \upsilon(\varphi(s))\,,$$

i.e., the composite arrow $\chi \circ \psi$ is in relation, induced by $\sim$, with the composite arrow $\upsilon \circ \varphi$.

When the binary relation on the target is clear from the context, the label ($\sim$ here) may be omitted.

1.14 Power-set functions induced by a function $f : X \to Y$

1.14.1 The *image-of-a-subset* and the *preimage-of-a-subset* functions f_* and f^*

Given a function $f : X \longrightarrow Y$, there are two associated functions between the power-sets

$$\mathscr{P}X \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathscr{P}Y \ , \tag{112}$$

where the associated *image* function is defined by

$$f_*(A) := \left\{ y \in Y \mid \exists_{x \in A}\ f(x) = y \right\} \qquad (A \subseteq X) \tag{113}$$

and the associated *preimage* function is defined by

$$f^*(B) := \left\{ x \in X \mid \exists_{y \in B}\ f(x) = y \right\} \qquad (B \subseteq Y). \tag{114}$$

1.14.2

Function (113) is a single-variable case of the direct-image function f_* introduced in Section 1.7.8 and associated with an arbitrary function f of n variables.

1.14.3 A comment about notation

What I here denote by $f_*(A)$ and $f^*(B)$ is usually denoted $f(A)$ and $f^{-1}(B)$. This is all right as long as there is no need to consider the assignments

$$A \mapsto f(A) \qquad \text{and} \qquad B \mapsto f^{-1}(B)$$

as functions between the corresponding power-sets. When such a need arises, one needs an appropriate notation to denote the image and the preimage functions associated with f. This is why I adopted the *lower-* and the *upper-star* notation that is universally used in Modern Mathematics to denote all sorts of functions that are naturally associated with a given function.

1.14.4

This has yet another advantage: it often allows us to skip parentheses around the arguments of functions f_* and f^* in the interest of keeping notation as simple as possible, without affecting the intended meaning. Thus, we shall, generally, write f_*A and f^*B instead of $f_*(A)$ and $f^*(B)$.

1.14.5

I will say later why in some cases we mark the associated function by placing $*$ as a *subscript* while in other cases—as a *superscript*.

1.14.6 The *fiber* of a function $f : X \to Y$ at $y \in Y$

The preimage f^*B of a singleton subset $B = \{y\}$ is referred to as the *fiber of* f *at* y. It is usually denoted $f^{-1}y$ or $f^{-1}(y)$. We shall denote it $f^*\{y\}$.

1.14.7 Caveat

One must be careful not to confuse notation $f^{-1}(y)$, when it is used to denote the *fiber* of f at y, with notation $f^{-1}(y)$ used to denote the *value* of the *inverse* function. The inverse function, denoted f^{-1}, is defined only when f is invertible. In that case, the fiber of f at $y \in Y$ is given by

$$f^*\{y\} = \{f^{-1}(y)\}.$$

1.14.8 The characteristic function of a subset

Given a subset $A \subset X$, its *characteristic function* is defined by

$$\chi_A : X \to \mathbf{F}_2, \qquad \chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}, \tag{115}$$

where $\mathbf{F}_2 = \{0, 1\}$ denotes the 2-element field.

Assignment

$$A \longmapsto \chi_A$$

yields a canonical identification

$$\chi : \mathscr{P}X \longleftrightarrow \mathrm{Funct}(X, \mathbf{F}_2). \tag{116}$$

Exercise 24 *Prove that, given a function $f : X \to Y$ and a subset $B \subset Y$, one has*

$$f^\bullet \chi_B = \chi_{f^*B}. \tag{117}$$

In other words, the preimage function $f^* : \mathscr{P}Y \to \mathscr{P}X$ can be viewed also as the precomposition function

$$f^\bullet : \mathrm{Funct}(Y, \mathbf{F}_2) \longrightarrow \mathrm{Funct}(X, \mathbf{F}_2).$$

1.14.9

Identity (117) can be also expressed by saying that the following square diagram of functions

$$\begin{tikzcd} \mathscr{P}X \arrow[r, "\chi"] & \mathrm{Funct}(X, \mathbf{F}_2) \\ \mathscr{P}Y \arrow[u, "f^*"] \arrow[r, "\chi"'] & \mathrm{Funct}(Y, \mathbf{F}_2) \arrow[u, "f^\bullet"'] \end{tikzcd}$$

commutes.

1.14.10

Note how close the definitions of the image and of the preimage are to each other: they are both defined by *existential* quantification of the *same* binary relation

$$X, Y \longrightarrow \text{Statements}, \qquad x, y \longmapsto \rho(x, y) \ := \ \text{“ } f(x) = y \text{ ”} \tag{118}$$

over the corresponding subsets $A \subseteq X$ and $B \subseteq Y$, respectively. We often refer to f_* as the *direct image map* and to f^* as the *inverse image map.*

1.14.11 Comments about the usual "definitions" of the image and the preimage functions.

The image function is usually "defined" by

$$f_*A := \{f(x) \mid x \in A\}\,. \tag{119}$$

This should be considered only as an *informal definition* since it violates the requirement that brace notation we use to define a subset of Y *must* be of the form

$$\{y \in Y \mid \rho(y)\}$$

where ρ is a unary relation on Y. Additionally, note the equality of sets

$$f^*B = \{x \in X \mid f(x) \in B\}\,. \tag{120}$$

The right-hand-side of (120) is how the inverse image is usually defined. Such a definition, however, obfuscates the fact that f^* is a "twin sister of f_*".

1.14.12 The *conjugate image* function $f_!$

These two concepts or, if you wish, constructions, naturally associated with every function $f : X \longrightarrow Y$, are omnipresent. One encounters them nearly in every mathematical argument involving functions between sets. What remains a very little known fact is that f^* has yet another "sibling,"

$$f_! : \mathscr{P}(X) \longrightarrow \mathscr{P}(Y), \qquad A \longmapsto \complement f_*(\complement A)\,, \tag{121}$$

that I propose to call the *conjugate image* function.

The name, "conjugate image" stems from the fact that $f_!$ is the *conjugate* of f_* by the *complement operation*,

$$f_! = \complement \circ f_* \circ \complement. \tag{122}$$

Caveat: the *inner* complement operation is applied to a subset of X whereas the *outer* complement operation is applied to a subset of Y. When fully expanded the value of $f_!$ on a subset A of X equals

$$f_!A = Y \setminus f_*(X \setminus A).$$

Exercise 25 *Show that*

$$f_!A = \{y \in Y \mid \forall_{x \in X}\, f(x){=}y \Rightarrow x{\in}A\}\,. \tag{123}$$

Exercise 26 *Let $A \subseteq X$ and $B \subseteq Y$. Show that*

$$A \subseteq f^*B \quad \textit{if and only if} \quad f_*A \subseteq B\,. \tag{124}$$

1.14.13

Exercise 26 expresses the fact that f_*, f^* form what in the language of ordered sets is known as a *Galois connection,* cf. Section 6.15.

Exercise 27 *Show that*

$$f^* \circ f_* = \mathrm{id}_{\mathscr{P}X} \quad \textit{if and only if} \quad f \ \textit{is injective}. \tag{125}$$

Exercise 28 *Show that*

$$f_* \circ f^* = \mathrm{id}_{\mathscr{P}Y} \quad \textit{if and only if} \quad f \ \textit{is surjective}. \tag{126}$$

Exercise 29 *Show that*

$$f^*(\complement B) = \complement(f^* B). \tag{127}$$

1.14.14

Identities (122) and (127) can be expressed by a pair of commutative square diagrams

$$\begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{P}X \arrow[d, "f_!"] \\ \mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] & \mathscr{P}Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{P}X \\ \mathscr{P}Y \arrow[u, "f^*"] \arrow[r, leftrightarrow, "\complement"'] & \mathscr{P}Y \arrow[u, "f^*"'] \end{tikzcd} \tag{128}$$

that can be combined into a single diagram

$$\begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[d, bend right, "f_*"'] \arrow[dr, phantom, "\circlearrowleft\circlearrowright"] & \mathscr{P}X \arrow[d, bend left, "f_!"] \\ \mathscr{P}Y \arrow[u, "f^*"'] \arrow[r, leftrightarrow, "\complement"'] & \mathscr{P}Y \arrow[u, "f^*"] \end{tikzcd} \tag{129}$$

in which both squares commute.

1.14.15

I used two different circle-arrows to make you aware that in the left diagram in (128), the composite arrows have their source at one of the *top* vertices and their target in the diagonally opposite *bottom* vertex. In the right diagram in (128) the roles are reversed: the composite arrows have their source at one of the *bottom* vertices and their target in the diagonally opposite *top* vertex.

Normally, I will be marking commutativity of any (portion of a) diagram by using the circle-arrow symbol that I consider the most appropriate.

Exercise 30 *Show that*

$$f^* B \subseteq A \quad \textit{if and only if} \quad B \subseteq f_! A\,. \tag{130}$$

1.14.16

Exercise 30 expresses the fact that $f^*, f_!$ form what in the language of ordered sets is known as a *Galois connection.*

Exercise 31 *Given an n-ary relation ρ between elements of sets $X_1, \dots, X_n$, let ρ_i be the $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$ defined in Section 1.7.6. Show that*

$$\Gamma_{\rho_i} = (\pi_{\hat{i}})_* \Gamma_\rho \tag{131}$$

where

$$\pi_{\hat{i}} : X_1 \times \cdots \times X_n \longrightarrow X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n \tag{132}$$

removes from an ordered n-tuple its i-th component,

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, \hat{x}_i, \dots, x_n).$$

Exercise 32 *Let $g : Y \to Z$ be a function. Show that*

$$(g \circ f)_* = g_* \circ f_*, \qquad (g \circ f)^* = f^* \circ g^* \qquad \textit{and} \qquad (g \circ f)_! = g_! \circ f_! . \tag{133}$$

Exercise 33 *Show that all three functions*

$$(id_X)_*, \qquad (id_X)^* \qquad \textit{and} \qquad (id_X)_!, \tag{134}$$

are equal to the identity function $\mathrm{id}_{\mathscr{P}X}$ of power-set $\mathscr{P}X$.

An immediate consequence of identities (133) and (134) is that, for every invertible function f, one has

$$(f^{-1})_* = (f_*)^{-1}. \tag{135}$$

Exercise 34 *Show that, for an invertible function f, one has*

$$f^* = (f^{-1})_* .$$

Exercise 35 *Let ρ^i be the $(n-1)$-ary relation defined in Section 1.7.1. Show that*

$$\Gamma_{\rho^i} = (\pi_{\hat{i}})_! \Gamma_\rho . \tag{136}$$

1.14.17 Pull-back of a relation

Given a function-list (97), we refer to the associated precomposition functions

$$(f_1, \dots, f_n)^\bullet : \mathrm{Rel}\left(X_1', \dots, X_n'\right) \longrightarrow \mathrm{Rel}(X_1, \dots, X_n), \qquad \rho' \longmapsto (f_1, \dots, f_n)^\bullet \rho' . \tag{137}$$

as the *pull-back* functions.

Exercise 36 *Show that*

$$\Gamma_{(f_1, \dots, f_n)^\bullet \rho} = (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'} . \tag{138}$$

Exercise 37 *Let $f : X \to X'$ be a function and ρ' be a binary relation on X'. Which properties of ρ', from the list given in Section 1.8.4, are inherited by $(f, f)^\bullet \rho'$?*

1.14.18 Push-forward of a relation

We define the *push-forward* functions

$$(f_1, \dots, f_n)_\sharp : \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}\left(X'_1, \dots, X'_n\right), \qquad \rho \longmapsto (f_1, \dots, f_n)_\sharp \rho, \tag{139}$$

where $(f_1, \dots, f_n)_\sharp \rho$ is the relation

$$x'_1, \dots, x'_n \longmapsto \text{“ } \exists_{x_1 \in X_1, \dots, x_n \in X_n}\ \rho(x_1, \dots, x_n) \wedge f_1(x_1) = x'_1 \wedge \dots \wedge f_n(x_n) = x'_n \text{ ”}. \tag{140}$$

Exercise 38 *Show that*

$$\Gamma_{(f_1, \dots, f_n)_\sharp \rho} = (f_1 \times \dots \times f_n)_* \Gamma_\rho . \tag{141}$$

1.14.19

The analog of Identity (141),

$$\Gamma_{(f_1, \dots, f_n)_\natural \rho} = (f_1 \times \dots \times f_n)_! \Gamma_\rho , \tag{142}$$

exists also for the *conjugate* direct image function and *conjugate push-forward* functions

$$(f_1, \dots, f_n)_\natural : \mathrm{Rel}(X_1, \dots, X_n) \longrightarrow \mathrm{Rel}\left(X'_1, \dots, X'_n\right), \qquad \rho \longmapsto (f_1, \dots, f_n)_\natural \rho, \tag{143}$$

where $(f_1, \dots, f_n)_\natural \rho$ is the relation

$$x'_1, \dots, x'_n \longmapsto \text{“ } \forall_{x_1 \in X_1, \dots, x_n \in X_n}\ \left(f_1(x_1) = x'_1 \wedge \dots \wedge f_n(x_n) = x'_n \right) \Rightarrow \rho(x_1, \dots, x_n) \text{ ”}. \tag{144}$$

Exercise 39 *Prove Identity* (142).

1.15 Families of sets

1.15.1

A *family of sets* is, by definition, a set whose elements are themselves sets. In a restrictive approach to Set Theory every set is requiered to be of this form. It is possible to develop all of Mathematics within such a restrictive framework.

1.15.2 Notation

A general practice is to denote *elements* of sets by lower case Latin alphabet letters:

$$a, b, c, d, e, f, g, h, i, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, x, z,$$

and to denote *sets* by capital letters:

$$A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, X, Z.$$

1.15.3 Boldface notation

Certain particularly common sets are denoted by upright boldface letters in order to make them stand out wherever they appear. Thus, $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, became standard notation for the sets of natural numbers, of integers, of rational numbers, of real numbers and, respectively, of complex numbers.

A bad habit that infected publishing practice like a noxious virus and that **should not be followed**, is to use in printed texts in place of those boldface letters their "blackboard" equivalents: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

1.15.4 Families of sets

A set whose elements are sets is often referred to as a *family of sets*. We shall denote families of sets by capital calligraphic letters:

$$\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D},\mathscr{E},\mathscr{F},\mathscr{G},\mathscr{H},\mathscr{I},\mathscr{J},\mathscr{K},\mathscr{L},\mathscr{M},\mathscr{O},\mathscr{P},\mathscr{Q},\mathscr{R},\mathscr{S},\mathscr{T},\mathscr{U},\mathscr{V},\mathscr{W},\mathscr{X},\mathscr{Y},\mathscr{Z}.$$

1.15.5 The union of a family of subsets of a set

Given a family of subsets $\mathscr{A}$ of a set X, the *union of* $\mathscr{A}$ is the set

$$\bigcup \mathscr{A} \;:=\; \{x \in X \mid \exists_{A\in\mathscr{A}}\, x \in A\}\,. \tag{145}$$

The existence of such a set is guaranteed by the axioms of Set Theory. It is the *smallest* subset of X *containing* each member set $A \in \mathscr{A}$. An alternative notation:

$$\bigcup_{x\in\mathscr{A}} A\,. \tag{146}$$

1.15.6 The intersection of a family of subsets of a set

The set

$$\bigcap \mathscr{A} \;:=\; \{x \in X \mid \forall_{A\in\mathscr{A}}\, x \in A\} \tag{147}$$

is called the *intersection* of (family) $\mathscr{A}$. It is the *greatest* subset of X *contained in* each member set $A \in \mathscr{A}$. An alternative notation

$$\bigcap_{x\in X} A\,. \tag{148}$$

1.15.7

Union and intersection define two canonical functions

$$\mathscr{P}X \underset{\bigcap}{\overset{\bigcup}{\leftleftarrows}} \mathscr{P}\mathscr{P}X\,. \tag{149}$$

Exercise 40 *Let* $\mathscr{A} \subseteq \mathscr{B}$ *(we say, in this case, that* $\mathscr{A}$ *is a* subfamily *of* $\mathscr{B}$*. Show that*

$$\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B} \qquad \textit{and} \qquad \bigcap \mathscr{A} \supseteq \bigcap \mathscr{B}. \tag{150}$$

1.15.8 Union and intersection of the *empty* family of subsets

If $\mathscr{A} = \{A\}$ consists of a single set A, then

$$\bigcup \mathscr{A} = A = \bigcap \mathscr{A}.$$

Since the empty family $\emptyset$ of subsets of X is contained in every family of subsets, in particular in the singleton family $\{\emptyset\}$, the union of the empty family is contained in set $\emptyset$,

$$\bigcup \emptyset \subseteq \bigcup \{\emptyset\} = \emptyset,$$

hence it is the empty set.

Since the empty family $\emptyset$ of subsets of X is contained in the singleton family $\{X\}$, the intersection of the empty family of subsets of X contains set X,

$$\bigcap \emptyset \supseteq \bigcap \{X\} = X,$$

hence it equals X.

1.15.9

The above argument demonstrates that the union of the empty family of subsets of X is the empty set independently of what set X is.

On the other hand, the intersection of the empty family of subsets of X equals X, hence it *does* depend on X.

1.15.10 Selectors of a family

A function $\xi : \mathscr{X} \longrightarrow \bigcup \mathscr{X}$ satisfying the property

$$\forall_{X \in \mathscr{X}} \ \xi(X) \in X \tag{151}$$

is called a *selector* or a *choice function* of family $\mathscr{X}$.

1.15.11 A comment about the use of the quantifier notation

Mathematicians, unless they are logicians or axiomatic-set-theorists, prefer to limit the use of the quantifier symbols in their formulae to those rare occasions when their use clarifies, not obfuscates, the meaning. The reason is partly a reflection of their habits, partly is related to the physiology of human brain perception of abstract symbolic expressions. The defining property of a selector (151) can be also written as:

$$\xi(X) \in X \quad \textit{for every} \quad X \in \mathscr{X}. \tag{152}$$

or, more tersely,

$$\xi(X) \in X \qquad (X \in \mathscr{X}). \tag{153}$$

Each expression (151)–(153) carries exactly the same meaning and can be read in the same way. From now on you will be frequently exposed to notation (153) that eliminates the need to use quantifier symbols in phrases involving only universal quantifiers.

1.15.12 Axiom of Choice

For obvious reasons, no selector exists if family $\mathscr{X}$ contains the empty set $\emptyset$. It is not obvious, however, that a selector exists *for every* family of nonempty sets. *Axiom of Choice* states just that. That statment was proven to be independent of other axioms of Set Theory. Some mathematicians do not accept it automatically while all mathematicians are, generally, cautious when they are forced to use it. Much of Mathematics can be developed without assuming its validity.

1.15.13 The product of a family of sets

The set of all selectors of family $\mathscr{X}$ forms the set

$$\prod \mathscr{X}\,, \qquad \text{alternately denoted} \qquad \prod_{X \in \mathscr{X}} X\,, \tag{154}$$

which is called the *product* of (family) $\mathscr{X}$.

1.15.14

Axiom of Choice says:

$$\textit{The product of a family of nonempty sets is nonempty.} \tag{155}$$

1.15.15 An equivalent form of Axiom of Choice

$$\textit{Every surjective function } f : \mathrm{X} \to Y \textit{ is right-invertible.} \tag{156}$$

1.15.16 Independence of Axiom of Choice

It was established long ago that Axiom of Choice is consistent with the remaining axioms of Set Theory. This means that if there are contradictory statements in Mathematics provable with the aid of Axiom of Choice, then there are contradictory statements provable without Axiom of Choice.

It took much longer to resolve the open question whether Axiom of Choice is, or is not, a consequence of the remaining axioms of Set Theory. This was finally resolved by a brilliant mathematician, Paul Cohen, whose demonstrated strength was in Harmonic and Functional Analysis, not in Set Theory or Mathematical Logic. He proved that Axiom of Choice is *not* a consequence of axioms of Set Theory. Statements in Mathematics that are consistent but not provable are said to be *independent* of axioms of Set Theory.

1.16 Canonical functions between the sets-of-families

1.16.1

As we saw in Sections 1.14.1 and 1.14.12, every function $f : X \to Y$ induces three functions between the corresponding power-sets

$$\begin{tikzcd} \mathscr{P}Y \arrow[d, "f^*"] \\ \mathscr{P}X \arrow[u, bend left=50, "f_*"] \arrow[u, bend right=50, "f_!"'] \end{tikzcd} \quad . \tag{157}$$

Families of subsets of X are elements of the power-set-of-the-power-set $\mathscr{P}\mathscr{P}X$ and similary for families of subsets of Y. In particular, each of the three functions in diagram (157) induces three functions between the corresponding sets of families of subsets:

$$\begin{array}{ccc} (f_*)_* & (f_*)^* & (f_*)_! \\ (f^*)_* & (f^*)^* & (f^*)_! \\ (f_!)_* & (f_!)^* & (f_!)_! \,. \end{array} \tag{158}$$

One can omit parentheses provided one carefully observes the spacing that distinguishes between, e.g., $f_*{}^*$ and $f^*{}_*$.

$$\begin{array}{ccc} f_{**} & f_*{}^* & f_{*!} \\ f^*{}_* & f^{**} & f^*{}_! \\ f_{!*} & f_!{}^* & f_{!!} \end{array} \quad . \tag{159}$$

Exercise 41 *Find all functions in diagram (159) that are functions from $\mathscr{P}\mathscr{P}X$ to $\mathscr{P}\mathscr{P}Y$.*

1.16.2

Of these nine canonical functions between sets of families of subsets, four play an important role in Topology, Measure Theory, Mathematical Analysis, where families of subsets are essential objects of study.

1.16.3

Let $\mathscr{A} \subset \mathscr{P}X$ be a family of subsets of X, let $\mathscr{B} \subset \mathscr{P}Y$ be a family of subsets of Y.

Exercise 42 *Show that*

$$f_*\left(\bigcup \mathscr{A}\right) = \bigcup f_{**}\mathscr{A} \qquad \textit{and} \qquad f^*\left(\bigcup \mathscr{B}\right) = \bigcup f^*{}_*\mathscr{B} \tag{160}$$

and express each identity in the form of a commutative diagram.

Exercise 43 *Show that*

$$f^*\left(\bigcap \mathscr{B}\right) = \bigcap f^*{}_*\mathscr{B} \qquad \textit{and} \qquad f_!\left(\bigcap \mathscr{A}\right) = \bigcap f_{!*}\mathscr{A} \tag{161}$$

and express each identity in the form of a commutative diagram.[1]

Exercise 44 *Show that*

$$f_*\left(\bigcap \mathscr{A}\right) \subseteq \bigcap f_{**}\mathscr{A} \qquad \textit{and} \qquad f_!\left(\bigcup \mathscr{A}\right) \supseteq \bigcup f_{!*}\mathscr{A}\,. \tag{162}$$

In general, $\subseteq$ cannot be replaced by $=$ in (162).

1.17 Indexed families of sets

1.17.1

An indexed family of sets $(X_i)_{i\in I}$ is, by definition, a function from a certain set I to the power-set of a certain set U,

$$I \longrightarrow \mathscr{P}(U)\,, \qquad i \mapsto X_i\,.$$

The standard notation for the value at $i \in I$ is X_i. The set I is referred to as the *indexing set*.

1.17.2 The union and the intersection of an indexed family

Let us denote by $\mathscr{X}$ the *image* of this function in $\mathscr{P}(U)$. It is a family of sets. The union and the intersection of $\mathscr{X}$ are called, respectively, the *union* and the *intersection* of $(X_i)_{i\in I}$, and denoted

$$\bigcup_{i\in I} X_i \qquad \text{and} \qquad \bigcap_{i\in I} X_i\,.$$

Explicitly,

$$\bigcup_{i\in I} X_i \;:=\; \{x \mid \exists_{i\in I}\; x \in X_i\} \tag{163}$$

and

$$\bigcap_{i\in I} X_i \;:=\; \{x \mid \forall_{i\in I}\; x \in X_i\}\,. \tag{164}$$

1.17.3

When the indexing set I is empty, the comments made about the union and the intersection of an empty family of subsets apply, cf. 1.15.9.

[1] A hint for both exercises: recall that $\bigcup$ and $\bigcap$ define certain canonical functions, cf. (149).

1.17.4 Selectors of an indexed family

Functions

$$I \longrightarrow \bigcup_{i\in I} X_i\,, \qquad i \mapsto x_i\,, \tag{165}$$

satisfying

$$x_i \in X_i \qquad (i \in I)\,,$$

could be called *selectors* of indexed family $(X_i)_{i\in I}$. They are more frequently called I*-tuples* because in the case

$$I = \{1, \dots, n\}\,,$$

they correspond to ordered n-tuples of elements of $\bigcup_{i\in I} X_i$.

1.17.5 "Tuple" notation

Standard notation for an I-tuple is $(x_i)_{i\in I}$. The subscript $i \in I$ is usually omitted when the indexind set is understood from the context.

1.17.6 The product of an indexed family of sets

Predictably, the set of all I-tuples of $(X_i)_{i\in I}$ is called the *product* of $(X_i)_{i\in I}$ and is denoted

$$\prod_{i\in I} X_i\,. \tag{166}$$

1.17.7

For $I = \{1, 2\}$, the product is naturally identified with the Cartesian product

$$X_1 \times X_2\,,$$

and, for $I = \{1, \dots, n\}$, it provides the most convenient model of the Cartesian product

$$X_1 \times \cdots \times X_n\,.$$

1.17.8 Canonical projections (π_J)

Restricting a function (165) to a subset $J \subseteq I$ defines a function

$$\pi_J : \prod_{i\in I} X_i \longrightarrow \prod_{i\in J} X_i\,, \tag{167}$$

called the *canonical projection* (associated with a subset J of the indexing set. We have encountered these functions in Section 1.12.15 where $I = \{1, \dots n\}$ and $J = \{i\}$.

1.17.9 Notation

In the interest of keeping notation simple, when, e.g., $J = \{2, 5, 7\}$, we write

$$\pi_{2,5,7} \quad \text{instead} \quad \pi_{\{2,5,7\}}$$

or, even, as

$$\pi_{257}$$

when it it is clear from the context that the elements of J are natural numbers less than 10.

A general rule is to separate the items in a list of subscripts or superscripts by commas when notation is, otherwise, ambiguous, and to omit the commas when no ambiguity arises.

1.17.10 Composition of correspondences

Given correspondences

$$C \subseteq X_0 \times \cdots \times X_{m+1} \qquad \text{and} \qquad D \subseteq X_{m+1} \times \cdots \times X_{m+n+1} ,$$

their preimages under the canonical projections

$$\pi^*_{0,\dots,m+1}C \qquad \text{and} \qquad \pi^*_{m+1,\dots,m+n+1}D$$

are correspondences between elements of sets

$$X_0, \dots , X_{m+n+1} .$$

In particular, we can form their intersection

$$\pi^*_{0,\dots,m+1}C \,\cap\, \pi^*_{m+1,\dots,m+n+1}D$$

and project it into $X_0 \times \cdots \times \hat{X}_{m+1} \times \cdots \times X_{m+n+1}$,

$$(\pi_{\widehat{m+1}})_*(\pi^*_{0,\dots,m+1}C \,\cap\, \pi^*_{m+1,\dots,m+n+1}D) , \tag{168}$$

where

$$\pi_{\widehat{m+1}} \;=\; \pi_{0,\dots,\widehat{m+1},\dots,m+n+1} .$$

We shall denote (168) by $C \circ D$.

1.17.11

Explicitly, $C \circ D$ consists of $(m + n + 1)$-tuples

$$(x_0, \dots , \hat{x}_{m+1}, \dots , x_{m+n+1})$$

for which there exists $x_{m+1} \in X_{m+1}$ such that

$$(x_0, \dots , x_{m+1}) \in C \qquad \text{and} \qquad (x_{m+1}, \dots , x_{m+n+1}) \in D .$$

1.17.12

It follows that for $C = \Gamma_\rho$ and $D = \Gamma_\sigma$, one has

$$\Gamma_{\rho\circ\sigma} = \Gamma_\rho \circ \Gamma_\sigma . \tag{169}$$

2 The language of mathematical structures

2.1 Mathematical structures

2.1.1 The concept of a mathematical structure

A list of sets

$$X_1, \dots, X_n$$

equipped with some 'data' is what a mathematical structure is. As such, a mathematical structure can be thought of as an ordered pair

$$(X_1, \dots, X_n ; \text{'data'})$$

2.1.2

This sinple concept became a focal point of modern Mathematics because it allows to view many apparently distant phenomena as manifestations of the same general laws.

2.1.3

Functions, operations, relations, are obvious examples of mathematical structures.

2.1.4 Structures of functional type

Sets X equipped with a family $\mathscr{O} \subset \mathrm{Funct}(X, \mathbf{R})$ of real-valued functions on X,

$$(X, \mathscr{O}),$$

are a backbone of Analysis. Think, for example, of a subset X of Euclidean space $\mathbf{R}^n$ and $\mathscr{O}$ being the set of all infinitely differentiable functions on X.

2.1.5 Structures of topological type

Sets X equipped with a family $\mathscr{A} \subset \mathscr{P}X$ of subsets

$$(X, \mathscr{A})$$

are the central objects in Topology, Geometry, Measure Theory, Combinatorics.

2.1.6 Example: topological spaces

A set X equipped with a family of subsets $\mathscr{T} \subset \mathscr{P}X$ closed under formation of *finite* intersections and arbitrary unions is called a *topological space*. Members of $\mathscr{T}$ are referred to as *open subsets*.

2.1.7 Example: measurable spaces

A set X equipped with a family of subsets $\mathscr{M} \subset \mathscr{P}X$ closed under formation of *countable* intersections and under the complement operation $\complement$, cf. Section 1.5.3, is called a *measurable space*. Members of $\mathscr{M}$ are referred to as *measurable subsets*.

2.2 Algebraic structures

2.2.1 $(X, (\mu_i)_{i\in I})$

A set X equipped with an indexed family $(\mu_i)_{i\in I}$ of operations on X is called an *algebraic structure*. Groups, rings, fields, vector spaces, etc., are all examples of algebraic structures.

2.2.2 The *signature* of an algebraic structure

The function

$$\nu : I \longrightarrow \mathbf{N}, \qquad i \longmapsto \nu(i) := \text{the arity of operation } \mu_i \tag{170}$$

is called the *signature* of algebraic structure $\big(X, (\mu_i)_{i\in I}\big)$.

2.2.3 The associated algebraic structure on the power-set

The power-set of X, equipped with the family of direct-image operation $(\mu_i)_*$, cf. Section 1.7.8, forms an algebraic structure

$$\big(\mathscr{P}X, ((\mu_i)_*)_{i\in I}\big)$$

of the same signature.

2.2.4

When the family of operations is finite, we prefer to employ the *list-of-operations* notation

$$(X\,;\, li\mu n)\,.$$

2.2.5 Example: a binary structure

A *binary structure* consists of a set X equipped with a single binary operation on X,

$$(X; \mu_2)\,.$$

Here the list has length 1. Here, I chose the subscript $_2$ to signal that the arity of that single operation is 2.

2.2.6 Multiplicative notation: xy

Traditionally, the generic term for a binary operation has been *multiplication*, and the value $\mu_2(x, y)$ is written as xy or, by using *infix* notation, as

$$x \cdot y\,, \quad x * y\,, \quad \text{et caetera.}$$

2.2.7 Multiplicative notation: AB, aB, Ab

Similarly, the result of the direct-image operation

$$(\mu_2)_*(A, B)$$

applied to a pair of subsets $A, B \subseteq X$, is denoted AB or, when using infix notation, as

$$A \cdot B\,, \quad A * B\,, \quad \text{et caetera.}$$

2.2.8 Cosets of a subset

We skip braces when one of the sets is a singleton set. Thus, sets $\{a\}B$ are generally denoted

$$aB \qquad (a \in A) \tag{171}$$

and sets $A\{b\}$ are denoted

$$Ab \qquad (b \in A)\,. \tag{172}$$

Sets (172) form a family of *right cosets of A* while sets (171) form a family of *left cosets of B*.

2.2.9 Coset ternary relations

Consider the following two relations

$$\rho_r : X, \mathscr{P}X, X \longrightarrow \text{Statements}\,, \qquad x, A, y \;\longmapsto\; \text{“ } Ax \ni y \text{ ”}\,, \tag{173}$$

and

$$\rho_l : X, \mathscr{P}X, X \longrightarrow \text{Statements}\,, \qquad x, A, y \;\longmapsto\; \text{“ } xA \ni y \text{ ”}\,, \tag{174}$$

2.2.10 A-divisor relations

By freezing the subset variable we obtain the corresponding two *A-divisor* binary relations on X,

$${}_A| \;: X, X \longrightarrow \text{Statements}\,, \qquad x, y \;\longmapsto\; x\,{}_A|\,y := \text{“ } Ax \ni y \text{ ”}\,, \tag{175}$$

and

$$|_A : X, X \longrightarrow \text{Statements}\,, \qquad x, y \;\longmapsto\; x\,|_A\,y := \text{“ } xA \ni y \text{ ”}\,. \tag{176}$$

We can read $x\,{}_A|\,y$ as

x is a right A-divisor of y

which means that

$$\exists_{a \in A}\; ax = y\,.$$

Similarly, we can read $x\,{}_A|\,y$ as

x is a left A-divisor of y

which means that

$$\exists_{a \in A}\; xa = y\,.$$

2.2.11 The *opposite* binary structure

By flipping the arguments in a binary operation we obtain another binary operation on X

$$x, y \;\longmapsto\; \mu_2^{\mathrm{op}}(x, y) := \mu_2(y, x) \qquad (x, y \in X)\,. \tag{177}$$

The binary structure $\left(X, \mu_2^{\mathrm{op}}\right)$ is referred to as the *opposite of* $\left(X, \mu_2^{\mathrm{op}}\right)$ and is often denoted $X, \mu_2)^{\mathrm{op}}$.

When using generic multiplicative notation it is highly advisable to mark elements of X considered as elements of the opposite binary structure, with the $^{\mathrm{op}}$ tag. Then Definition (177) of the opposite operation becomes

$$x^{\mathrm{op}} y^{\mathrm{op}} := (yx)^{\mathrm{op}} \qquad (x, y \in X)\,. \tag{178}$$

2.2.12 Left- and Right Cancellation Properties

If

$$\forall_{x,y,z\in X}\ xy{=}xz \Longrightarrow y{=}z \tag{179}$$

we say that $(X,\cdot)$ satisfies *Left Cancellation Property* or is *left-cancellative*.

If

$$\forall_{x,y,z\in X}\ xz{=}yz \Longrightarrow x{=}y \tag{180}$$

we say that $(X,\cdot)$ satisfies *Right Cancellation Property* or is *right-cancellative*.

2.2.13 Left- and right-identity elements

An element $e\in X$ is said to be a *left identity* if the following identity is satisfied

$$\forall_{x\in X}\ ex = x \tag{181}$$

and is said to be a *right identity* if the identity

$$\forall_{x\in X}\ xe = x \tag{182}$$

holds.

Note that e^{op} is a right identity in the opposite structure precisely when e is a left identity, and vice-versa.

Exercise 45 *Let e be a left identity in $(X,\cdot)$. Show that $\{e\}$ is a left identity in $(\mathscr{P}X,\cdot_*)$.*

2.2.14

A binary structure may admit none, one, or many left- or right-identity elements. For example, for the operation

$$X,X \longrightarrow X, \qquad x_1,x_2 \longmapsto x_1,$$

that discards the second entry from the argument list, *every* element is a right identity, and none is a left identity as long as X is not a singleton set.

2.2.15 Unital binary structures

If a binary structure admits at least one left identity, say $e\in X$, and at least one right identity, say $e'\in X$, then they coincide in view of the double equality

$$e = ee' = e'.$$

In this case we refer to that unique double-sided identity as *the identity element* of a binary structure. If we consider the identity element e as a distinguished element of X, i.e., as a *nullary* operation, μ_0, then the algebraic structure $(X;\mu_0,\mu_2)$ is referred to as a *unital binary structure*.

2.2.16

The defining pair of Identities (181) and (181) is equivalently described as commutativity of the left and, respectively, right triangles in the diagram

$$
\begin{tikzcd}
 & X \arrow[dl, Rrightarrow, "\mu_0"'] \arrow[dd, equal] \arrow[dr, Rrightarrow, "\mu_0"] & \\
X,X \arrow[dr, "\mu_2"'] \arrow[r, phantom, "\circlearrowleft"] & {} \arrow[r, phantom, "\circlearrowright"] & X,X \arrow[dl, "\mu_2"] \\
 & X &
\end{tikzcd}
\quad . \tag{183}
$$

2.2.17 Left- and right inverses of an element

If elements $x, y \in X$ satisfy equality

$$xy = e\,,$$

where $e \in X$ is the identity, then x is said to be a *left inverse of y* and y is said to be a *right inverse* of x. In this case we also say that x is a right-invertible, while y is a *left-invertible* element.

Note that in the opposite structure x^{op} is a right inverse while y^{op} is a left inverse.

2.2.18 Pointed sets

A set equipped just with a nullary operation (X, μ_0) is frequently encountered in Topology where it would be called a *pointed set*, and the preferred notation would be (X, x_0).

2.2.19 Idempotents

A *square* of any element $x \in X$ in a binary structure is defined to be

$$x^2 := x \cdot x\,.$$

Elements $e \in X$ such that $e^2 = e$ are called *idempotents*.

2.2.20

An attempt to define x^3 requires performing two multiplications and produces two outcomes

$$x \cdot x^2 \qquad \text{and} \qquad x^2 \cdot x\,.$$

2.2.21

An attempt to define x^4 requires performing three multiplications and produces five outcomes

$$
\begin{tikzcd}
& x^2\cdot x^2 \arrow[dl, dash] \arrow[dr, dash] & \\
x\cdot(x\cdot x^2) \arrow[d, dash] & & (x^2\cdot x)\cdot x \arrow[d, dash] \\
x\cdot(x^2\cdot x)) \arrow[rr, dash] & & (x\cdot x^2)\cdot x
\end{tikzcd}
\tag{184}
$$

2.2.22 Power associative binary structures

A binary structure is said to be *power associative* if, for every positive integer n, the product of n copies of an arbitrary element, calculated by applying $n-1$ times multiplication, produces one and the same result regardless of how we group the arguments.

At this point it is necessary to mention that construction of the sequences of powers of an element in a binary structure is accomplished by *Recursive Definition.* Existence of recursively defined sequences is one of the key features of of the set of natural numbers $\mathbf{N}$, more precisely, of its being *well-ordered,* cf. Section 3.3.11 below.

2.2.23 Semigroups

A single most important property of a binary operation on a set is known as *associativity* and is expressed in the form of the identity

$$\forall_{x,y,z\in X}\,(xy)z = x(yz) \tag{185}$$

or, alternatively, as commutativity of the diagram

$$
\begin{tikzcd}
& X,X,X \arrow[dl, Rrightarrow, "\mu_2"'] \arrow[dr, Rrightarrow, "\mu_2"] & \\
X,X \arrow[rr, phantom, "\circlearrowleft"] \arrow[dr, "\mu_2"'] & & X,X \arrow[dl, "\mu_2"] \\
& X &
\end{tikzcd}
\quad .
\tag{186}
$$

An associative binary structure is called a *semigroup.*

Exercise 46 *Show that* $(\mathscr{P}X, \cdot_*)$ *is a semigroup if* $(X, \cdot)$ *is a semigroup.*

2.2.24

Associativity of a binary operation, of course, implies its power-associativity, but no vice-versa.

In multiplicative notation, powers of an element x are written x^n. In additive notation x^n becomes nx. Here $n \in \mathbf{N}\setminus\{0\}$.

2.2.25 Commutative binary structures

A second important property of a binary operation on a set is known as *commutativity* and is expressed in the form of the identity

$$\forall_{x,y\in X}\, xy = yx\,. \tag{187}$$

A binary structure $(X,\cdot)$ is commuttaive precisely when

$$\mu_2 = \mu_2^{\mathrm{op}}\,,$$

i.e., when $(X,\cdot)$ coincides with the opposite structure

$$(X,\cdot) = (X,\cdot)^{\mathrm{op}}\,.$$

Exercise 47 *Show that $(\mathscr{P}X,\cdot_*)$ is a commuttaive binary structure if $(X,\cdot)$ is commuttaive.*

2.2.26 Terminology: *abelian* groups

We encounter commutative semigroups, monoids, semirings, rings, etc. Commutative *groups*, however, are traditionally called *abelian groups*. This tradition predates introduction of general algebraic structures.

2.2.27 Unital semigroups, i.e., monoids

Unital semigroups, i.e., unital binary structures $(X\,;\mu_0,\mu_2)$ with an associative multiplication are called *monoids*. They appear in nearly every aspect of Modern Mathematics.

Exercise 48 *Show that $\big(\mathscr{P}X,\{\mu_0\},(\mu_2)_*\big)$ is a monoid if (X,μ_0,μ_2) is a monoid.*

2.2.28 Left- and right-invertible elements in a monoid

If

$$xx' = e \qquad \text{and} \qquad yy' = e\,,$$

then the short calculation

$$(xy)(y'x') = x((yy')x') = x(ex') = xx' = e$$

demonstrates that the subset of X formed by all right-invertible elements is closed under multiplication. Let us denote it X^{inv}. Similarly, the subset of X formed by all left-invertible elements is closed under multiplication. Let us denote it ${}^{\mathrm{inv}}X$.

2.2.29 Invertible elements

If

$$xx' = e = x''x\,,$$

then the short calculation that makes use of associativity of multiplication,

$$x' = ex' = (x''x)x' = x''(xx') = x''e = x''\,,$$

demonstrates that, in a monoid, any element that admits a left and and a right inverse, has precisely a single left and a single right inverse, and they necessary coincide. That unique double inverse of an element x is usually denoted x^{-1} and is referred to as *the inverse of* x.

The subset of invertible elements of a monoid is sometimes denoted X^*, at other times it may be denoted $G(X)$. We established above that

$$X^* = {}^{\mathrm{inv}}X \cap X^{\mathrm{inv}}.$$

2.2.30

Assignment

$$X \longrightarrow X, \qquad x \longmapsto x^{-1},$$

defines a unary operation on X^* and the set of invertible elements X^* equipped with the identity element, the inverse-element operation, and multiplication, is known as the *group of invertible elements of* $(X; \mu_0, \mu_2)$.

2.2.31 Groups

A *group* is an algebraic structure

$$(X; \mu_0, \mu_1, \mu_2)$$

such that $(X; \mu_0, \mu_2)$ is a monoid and μ_1 is a unary operation that sends an arbitrary element $x \in X$ to its inverse element, x^{-1}.

2.2.32

The defining pair of identities $x^{-1}x = e = xx^{-1}$ is equivalently described as commutativity of the left and, respectively, of the right triangle in the diagram

$$\begin{tikzcd}
& X \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_1"] \arrow[dd, "e_X"] & \\
X,X \arrow[dr, "\mu_2"'] \arrow[r, phantom, "\circlearrowleft"] & {} \arrow[r, phantom, "\circlearrowright"] & X,X \arrow[dl, "\mu_2"] \\
& X &
\end{tikzcd} \tag{188}$$

where e_X denotes the *constant* function

$$X \longrightarrow X, \qquad x \longmapsto e \qquad (x \in X).$$

Note that $e_X : X \to X$ is the composite function

$$X \longrightarrow \emptyset^{\emptyset} \xrightarrow{\tilde{e}} X$$

where $X \longrightarrow \emptyset^{\emptyset}$ is the unique function from X to the singleton set $\emptyset^{\emptyset}$ and $\tilde{e}$ is the function of a single variable canonically corresponding to the function of zero variables $e : \longrightarrow X$, cf. Section 1.12.13.

2.2.33 Caveat

The algebraic structure $(\mathscr{P}X\,;\,\{\mu_0\},(\mu_1)_*,(\mu_2)_*)$ associated with a group $(X\,;\,\mu_0,\mu_1,\mu_2)$ is *not* a group:

$$\{\mu_0\} \subseteq A\cdot A^{-1} \quad \text{but} \quad \{\mu_0\} \neq A\cdot A^{-1}$$

if A has at least two elements.

2.2.34 The canonical monoid structure on $\mathrm{Op}_1(X)$

Composition $\circ$ is a canonical binary operation on the set of all unary operations $\mathrm{Op}_1(X)$ on an arbitrary set X. The identity operation id_X is a distinguished element of $\mathrm{Op}_1(X)$. Composition of functions is associative and id_X is an identity element for the operation of composition.

Thus, $(\mathrm{Op}_1(X),\mathrm{id}_X,\circ)$ is a monoid and $\mathrm{Op}_1(X)$ provides an example of a set that is equipped with a canonical structure of a monoid.

2.2.35 Fixed points of a unary operation

Given an operation $\tau\in\mathrm{Op}_1(X)$ and an element of $x\in X$, we say that x is a *fixed point of* τ if

$$\tau(x)=x\,.$$

The set of fixed points of τ is often denoted

$$X^\tau\,.$$

2.2.36 A *retraction* of a set onto its subset

An idempotent τ in monoid $(\mathrm{Op}_1(X),\mathrm{id}_X,\circ)$ is called a *retraction*. If $Y=\tau_*X$ is the image of τ, then we often say that τ is a *retraction of a set X onto its subset Y*.

A unary operation τ is a retarction if and only if its image is contained in its set of fixed points,

$$\tau_*X\subseteq X^\tau\,.$$

2.2.37 The permutation group of a set

Invertible unary operations on a set X are, traditionally, called *permutations* (of elements of X). They form the group of permutations, denoted

$$\Sigma_X\,,\quad S_X\,,\quad \mathrm{Per}\,X\,,\quad \text{or}\quad \mathrm{Sym}\,X\,.$$

It is one of the most important groups in Mathemamtics.

2.2.38 Actions of sets on other sets

A set X, equipped with a family of unary operations $(\lambda_a)_{a\in A}$ indexed by a set A, is referred to as a set equipped with an *action of set A*. A short designation for this structure is an *A-set*.

An action of a set A on a set X is the same as a function

$$\lambda: A\longrightarrow \mathrm{Op}_1(X)\,. \tag{189}$$

We shall use, in general, notation (X,λ) to denote A-sets where λ is a function (189).

2.2.39 Standard multiplicative notation

The value of operation λ_a on an element $x \in X$ is frequently denoted ax.

2.2.40 Example: the *left* and the *right regular* actions of a semigroup

Given an element $a \in X$ of a a binary structure $(X, \cdot)$, left and, respectively, right multiplication by a define two actions of X on set X,

$$L_a : X \longrightarrow X, \qquad x \longmapsto L_a(x) := ax, \tag{190}$$

and

$$R_a : X \longrightarrow X, \qquad x \longmapsto R_a(x) := xa. \tag{191}$$

Multiplication in $(X, \cdot)$ is associative if and only if unary operations L_a and R_b commute with each other,

$$\forall_{a,b \in X}\ L_a R_b = R_b L_a. \tag{192}$$

2.2.41 Example: the *adjoint* action of the group of invertible elements of a monoid

Given an invertible element $g \in X^*$ of a monoid $(X, e, \cdot)$, the formula

$$\mathrm{ad}_g : X \longrightarrow X, \qquad x \longmapsto \mathrm{ad}_g(x) := gxg^{-1}, \tag{193}$$

defines an action of X^* on set X.

2.2.42 The *conjugacy* class of an element

For any element $x \in X$ and an invertible element $g \in X^*$, the element $\mathrm{ad}_g(x)$ is called the *conjugate of x by g* and is frequently denoted ${}^g x$.

The set of all conjugates of an element $x \in X$,

$$\left\{ y \in X \mid \exists_{g \in X^*}\ y = {}^g x \right\}, \tag{194}$$

is called the *conjugacy class of x*.

Exercise 49 *Show that, for any $g, h \in X^*$, one has*

$$\mathrm{ad}_{gh} = \mathrm{ad}_g \circ \mathrm{ad}_h, \qquad \mathrm{ad}_e = \mathrm{id}_X \qquad \textit{and} \qquad \mathrm{ad}_{g^{-1}} = \left(\mathrm{ad}_g\right)^{-1}. \tag{195}$$

2.2.43 *Normal* subsets

A subset $A \subseteq X$ is said to be *normal* if, for every invertible element $g \in X^*$,

$${}^g A = A,$$

i.e., A is a fixed point of operations

$$(\mathrm{ad}_g)_* : \mathscr{P}X \longrightarrow \mathscr{P}X, \qquad A \longmapsto {}^g A := gAg^{-1}, \qquad (g \in X^*). \tag{196}$$

Exercise 50 *Show that A is a normal subset if and only if it is closed under operations* (196).

Solution. Note that

$$ {}^{g}A \subseteq A \;\Leftrightarrow\; A = {}^{g^{-1}}({}^{g}A) \subseteq {}^{g^{-1}}A\,. $$

Since the inverse-element operation is bijective, we have

$$ \left(\forall_{g\in G}\ {}^{g}A \subseteq A\right) \;\Leftrightarrow\; \left(\forall_{g\in G}\ A \subseteq {}^{g}A\right)\,. $$

□

Exercise 51 *Let $A \subseteq X$ be a subset of a monoid $(X, e, \cdot)$. Show that, for every invertible element $g \in X^*$,*

$$ \forall_{a,b,g\in G}\ (ag)\,|_{A}\,(bg) \;\Leftrightarrow\; a\,|_{{}^{g}A}\,b\,. $$

Exercise 52 *Let $A \subseteq G$ be a subset of a group G and $g \in G$. Show that ${}^{g}A$ is a subgroup if and only if A is a subgroup.*

2.2.44

Semigroups, monoids, groups, are encountered everywhere where mathematical considerations are involved.

2.2.45

Algebraic structures involving two binary operations lead to algebraic structures known as semirings and rings. They will be introduced and discussed later.

2.3 Relational structures

2.3.1

Sets X equipped with an indexed family $(\rho_i)_{i\in I}$ of relations on X are called *relational structures.* Such structures are encountered in all areas of Mathematics and especially so in Mathematical Logic and in Incidence Geometry.

2.3.2 Binary relational structures

Particularly important are *binary relational structures,* i.e., sets equipped with a single binary relation. We discuss them in Chapter 6 devoted to binary relations.

2.3.3 Example : (pre)ordered sets

(Pre)ordered sets, introduced in Sections 1.8.14–1.8.15, are examples of binary relational structures.

2.4 Substructures

2.4.1

For every type of a mathematical structure there is usally naturally defined notion of a *substructure.*

2.4.2 Subsets

For sets equipped with no "data", a substructure is the same as a subset.

2.4.3 Algebraic substructures

For sets X equipped with a family of operations $\left(\mu_i\right)_{i\in I}$, a substructure consists of a subset, say $Y \subsetneq X$, *closed* under each operation μ_i, i.e., such that the following diagram

$$\begin{tikzcd} X,\dots,X \arrow[r, "\mu_i"] & X \\ Y,\dots,Y \arrow[u, hook] & Y \arrow[u, hook] \end{tikzcd} \tag{197}$$

admits completion to the commutative diagram

$$\begin{tikzcd} X,\dots,X \arrow[r, "\mu_i"] & X \\ Y,\dots,Y \arrow[u, hook] \arrow[r, dashed, "\bar{\mu}_i"'] \arrow[ur, phantom, "\circlearrowright"] & Y \arrow[u, hook] \end{tikzcd} . \tag{198}$$

Note that function $\bar{\mu}_i$ is unique when it exists, and its values coincide with the corresponding values of μ_i,

$$\forall_{x_1,\dots,x_n\in X}\ \bar{\mu}_i(x_1,\dots,x_n) = \mu_i(x_1,\dots,x_n)\,.$$

We refer to μ_i as the operation *induced by* μ_i on Y.

2.4.4

Note that $\bar{\mu}_i$ is *not* the *restriction* of μ_i to Y. The restriction of a function has a smaller domain and the *same* target. In particular, the restriction of μ_i produces a function

$$Y_1,\dots,Y_n \longrightarrow X\,,$$

not an operation on a subset Y.

2.4.5

If a subset Y is closed under *every* operation μ_i, then $\left(Y,\left(\bar{\mu}_i\right)_{i\in I}\right)$ is called a *substructure* of a structure $\left(X,\left(\mu_i\right)_{i\in I}\right)$.

This is how we define *subgroups, submonoids, subsemigroups, subrings, vector subspaces,* etc.

2.4.6 The ordered set of substructures Substr $\left(X,(\mu_i)_{i\in I}\right)$

Let us denote by Substr $\left(X,(\mu_i)_{i\in I}\right)$ the set of substructures of $\left(X,(\mu_i)_{i\in I}\right)$, i..e, the set of subsets of X that are closed under every operation μ_i. It is ordered by inclusion.

Exercise 53 *Let* $\mathcal{Y} \subsetneq \mathrm{Substr}\left(X,(\mu_i)_{i\in I}\right)$ *be a family of substructures of* $\left(X,(\mu_i)_{i\in I}\right)$. *Show that the intersection of memebers of* $\mathcal{Y}$ *is a substructure.*

2.4.7

The union of a family of substructures is not a substructure, in general. For example, the union of two vector subspaces $V' \cup V''$ of a vector space V is closed under addition of vectors if and only if either $V' \subseteq V''$ or $V'' \subseteq V'$.

Exercise 54 *Suppose that $V' \cup V''$ is closed under addition of vectors. Show that either $V' \subseteq V''$ or $V'' \subseteq V'$.*

Solution. Suppose that $V'' \not\subseteq V'$. Let $v' \in V'$ and $v'' \in V'' \setminus V'$, and assume that $v' + v'' \in V' \cup V''$. If $v' + v'' \in V'$, then

$$v'' = (v' + v'') - v' \in V' .$$

Since $v'' \notin V'$, we deduce that

$$v' + v'' \in (V' \cup V'') \setminus V' \subseteq V'' .$$

It follows that

$$v' = (v' + v'') - v'' \in V'' ,$$

i.e., $V' \subseteq V''$. □

2.4.8 Locally filtered families of subsets

We shall say that a family of subsets $\mathscr{Y} \subseteq \mathscr{P}X$ is *locally filtered* if, for every *finite subset*

$$F \subseteq \bigcup \mathscr{Y} ,$$

there exists a member $Y \in \mathscr{Y}$, such that

$$F \subseteq Y .$$

Exercise 55 *Let $\mathscr{Y} \subsetneq \operatorname{Substr}\left(X, (\mu_i)_{i\in I}\right)$ be a locally filtered family of substructures of $\left(X, (\mu_i)_{i\in I}\right)$. Show that the union of members of $\mathscr{Y}$ is a substructure.*

2.4.9 The substructure $\langle A \rangle$ generated by a subset $A \subseteq X$

Given a subset $A \subseteq X$, the intersection of the family

$$\mathscr{Y}_A := \left\{ Y \in \operatorname{Substr}\left(X, (\mu_i)_{i\in I}\right) \mid Y \supseteq A \right\}$$

is the smallest substructure containing subset A. We shall denote it $\langle A \rangle$ and call it the substructure *generated by A*.

Exercise 56 *Show that*

$$\forall_{A,B\subseteq X}\ A\subseteq B \Longrightarrow \langle A\rangle\subseteq\langle B\rangle . \tag{199}$$

Exercise 57 *Show that*

$$\forall_{A\subseteq X}\ \langle A\rangle = \langle\langle A\rangle\rangle . \tag{200}$$

2.4.10 Invariant subsets

Subsets $Y \subseteq X$ closed under a *unary* operation $\tau : X \to X$ are frequently encountered outside of Algebra, for example in Theory of Group Actions, Theory of Dynamical Systems, Topology, Operator Theory. Such sets are said to be *invariant* or, more precisely, τ-invariant. In the language of diagrams invariance of a subset Y is expressed by saying that

$$\begin{tikzcd} X \arrow[r, "\tau"] & X \\ Y \arrow[u, hook] & Y \arrow[u, hook] \end{tikzcd} \tag{201}$$

admits completion to the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\tau"] & X \\ Y \arrow[u, hook] \arrow[r, dashed, "\tilde{\tau}"'] \arrow[ur, phantom, "\circlearrowright"] & Y \arrow[u, hook] \end{tikzcd} . \tag{202}$$

Invariance of a subset $Y \subseteq X$ is expressed in terms of the *direct* image function by the condition

$$\tau_* Y \subseteq Y \tag{203}$$

or, in terms of the *inverse* image function, by the equivalent condition

$$Y \subseteq \tau^* Y . \tag{204}$$

2.4.11 Coinvariant subsets

If we reverse the relation in Condition (204),

$$Y \supseteq \tau^* Y , \tag{205}$$

then we obtain a *dual* condition that is equivalent to

$$\tau_! Y \supseteq Y . \tag{206}$$

We shall say in such case that Y is a *coinvariant* subset or, more precisely, a τ*-coinvariant* subset.

2.5 Subgroups

2.5.1

A subset $H \subseteq G$ of a group G that is closed under the idenity operation,

$$H \ni e ,$$

under the inverse-element operation,

$$H^{-1} \subseteq H$$

and under multiplication,

$$H \cdot H \subseteq H ,$$

is called a *subgroup of* G. Each of these properties of a subset $H \subseteq G$ admits an equivalent characterization in terms of the corresponding divisor relations, cf. Section 2.2.10.

Exercise 58 *Show that H-divisor relation $|_H$ is reflexive if and only if $H \ni e$.*

Exercise 59 *Show that*

$$(|_H)^{\mathrm{op}} \;\Leftrightarrow\; |_{H^{-1}} \;.$$

Exercise 60 *Show that relation $|_H$ is symmetric if and only if $H = H^{-1}$.*

Exercise 61 *Show that*

$$|_H \circ |_H \Leftrightarrow\; |_{H \cdot H} \;.$$

Exercise 62 *Show that relation $|_H$ is transitive if and only if $H \cdot H \subseteq H$.*

2.5.2

It follows that relation $|_H$ is an equivalence relation precisely when H is a subgroup of G, and the same for the other H-divisor relation ${}_H|$.

2.5.3

Note that

$$\forall_{x,y \in G}\; x\,{}_H|\,y \;\Leftrightarrow\; x^{-1}\,|_{H^{-1}}\,y^{-1}\,. \tag{207}$$

Subset H is closed under each of the group operations if and only if the set of inverses H^{-1} is closed under the same operation.

Moreover, a binary relation ρ on G is reflexive, symmetric, or transitive, precisely when the relation

$$x, y \;\longmapsto\; \rho\left(x^{-1}, y^{-1}\right) \qquad (x, y \in G)$$

is reflexive, symmetric or, respectively, transitive. It follows that in Exercises 58–62 we could replace the left H-divisor relation $|_H$ by the right H-divisor relation ${}_H|$.

2.5.4

The equivalence class of an element $x \in G$ for relation $|_H$ coincides with the left coset xH. Right multiplication by x, defines a function

$$R_x : H \longrightarrow Hx$$

whose inverse is right multiplication by x^{-1}.

Similarly, left multiplication by x, defines a function

$$L_x : H \longrightarrow xH$$

whose inverse is left multiplication by x^{-1}.

In particular, right and left cosets of H have the same cardinality as H.

2.5.5 Terminology: the *order* of a group G

In Group Theory the number of elements of a *finite* group G is denoted $|G|$ and called the *order of* G. Accordingly, the groups whose underlying sets are infinite are referred to as groups of *infinite order*.

2.5.6 Terminology: the *order* of an element $g \in G$

The *order of an element* $g \in G$ is defined as the order of the subgroup $\langle g \rangle \subseteq G$ generated by element g and is denoted $|g|$. Groups generated by a single element are said to be *cyclic*.

2.5.7 The *index* of a subgroup $H \subseteq G$

The cardinality of the set of right cosets ${}_{H\backslash}G$ coincides with the cardinality of the set of left cosets $G_{/H}$. When those sets are finite, the number of right cosets of H in G, which coincides with the number of left cosets, is called the *index of a subgroup* H *in* G and is denoted $|G:H|$.

2.5.8

When the sets of cosets are infinite, we say that H *is a subgroup of infinite index*.

2.5.9

Infinite groups may have finite subgroups and may also have infinite subgroups of finite index.

2.5.10

Since G is the union of disjoint right cosets and each coset of H has the same cardinality as H, the number of elements in a finite group G is the number of elements in H multiplied by the number of cosets. This simple counting argument was discovered more than 200 years ago and bears the name of franco-italian mathematician Lagrange.

Theorem 2.1 (Lagrange) *For any subgroup H of a finite group G, one has the*

$$|G| = \sum_{C \in G/H} |C| = \sum_{C \in G/H} |H| = |G:H| \cdot |H|. \tag{208}$$

□

Corollary 2.2 *The order of any subgroup H of a finite group G divides the order of G.*

□

Corollary 2.3 *The order of any element g of a finite group G divides the order of G.*

□

Corollary 2.4 *A group whose order is prime p has no nontrivial proper subgroups. Any element $g \neq e$ in such a group has order p, hence*

$$\langle g \rangle = G.$$

In particular, such a group is cyclic.

$\square$

Exercise 63 *Show that a finite subgroup H in an infinite group G has infinite index.*

Exercise 64 *Show that a subgroup H of finite index in an infinite group G is infinite.*

3 Morphisms

3.1 Interactions between mathematical structures

3.1.1

If mathematical structures are *objects* of mathematical theories, studying a given structure is nearly always executed by observing how that structure *interacts* with other structures of the same type. Binary interactions between structures are expressed in the language of *morphisms*.

3.1.2 The concept of a *concrete* morphism

A *morphism*

$$f : (X, \mathrm{data}) \longrightarrow (X', \mathrm{data}') \tag{209}$$

is most commonly understood to be a function between the *underlying sets*

$$f : X \longrightarrow X'$$

that *respects* the corresponding data. We refer to such morphisms as being *concrete morphisms*.

3.1.3

It is assumed that the data must be of the same type. The term 'respects' can be replaced by : 'is compatible with'. The meaning of this term is nearly always natural for each type of data. We shall illustrate this for some types of mathematical structures mentioned above.

The crucial expectation when introducing a suitable concept of a morphism between sets equipped with data is that the composite $g \circ f$ of two composable morphisms

$$(X, \mathrm{data}) \xrightarrow{f} (X', \mathrm{data}') \xrightarrow{g} (X'', \mathrm{data}'')$$

is again a morphism.

Additionally, it is expected that the identity operation id_X is an endomorphism of

$$(X, \mathrm{data}) \tag{210}$$

irrespective of what type of data we may consider.

3.1.4 Terminology : an *endo*morphism

When $(X, \mathrm{data}) = (X', \mathrm{data}')$ a morphism (209) is referred to as an *endomorphism of* (X, data)..

3.1.5 The monoid of endomorphisms $\mathrm{End}(X, \mathrm{data})$

In agreement with the requirements spelled out in Section 3.1.3, the set of endomorphisms of (210) is equipped with a canonical monoid structure

$$\big(\mathrm{End}(X, \mathrm{data}), \mathrm{id}_X, \circ\big)\,. \tag{211}$$

This is the most important source of monoids in Mathematics and its applications.

3.1.6 Terminology: an *iso*morphism

When there exists a morphism

$$g : (X', \mathrm{data}') \longrightarrow (X, \mathrm{data}) \tag{212}$$

such that

$$f \circ g = id_{X'} \qquad \text{and} \qquad g \circ f = \mathrm{id}_X , \tag{213}$$

we say that f is an *isomorphism between* $(X, data)$ and (X', data'). A pair of morphisms satisfying pair of equalities (213) is said to be *inverse to each other*.

3.1.7 Terminology: an *auto*morphism

An endomorphism of (X, data) that is an isomorphism is said to be an *automorphism of* $(X, data)$.

3.1.8 The group of automorphisms Aut(X, data)

Automorphisms form a subset of the monoid $\mathrm{End}(X, \mathrm{data})$ that is closed under composition and formation of inverses. It is therefore naturally equipped with a structure of a group. This is the most important source of groups in Mathematics and its applications.

3.1.9 The arrow notation

Morphisms are represented graphically as arrows. Every arrow has its source and its target, each being a structure of the same type. They are referred to as the *source* and the *target* of a morphism.

3.2 Morphisms between algebraic structures

3.2.1 Homomorphisms

Suppose that a set X is equipped with an n-ary operation μ and a set X' is equipped with an n-ary operation μ'. We say that a function $f : X \to X'$ is *compatible* with the operations if

$$\forall_{x_1,\dots,x_n \in X}\; f\big(\mu(x_1, \dots , x_n)\big) = \mu\big(f(x_1), \dots , f(x_n)\big) . \tag{214}$$

Algebraists refer to such functions as *homomorphisms*.

3.2.2

The definition of a morphism between sets equipped with an n-ary operation can be also expressed as commutativity of the following square diagram

$$\begin{tikzcd} X', \dots , X' \arrow[r, "\mu'"] & X' \\ X, \dots , X \arrow[u, "f \,\cdots\, f"] \arrow[r, "\mu"] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[u, "f"'] \end{tikzcd} . \tag{215}$$

3.2.3

The above definition can be easily extended to general algebraic structures. A morphism

$$(X, (\mu_i)_{i\in I}) \longrightarrow (X', (\mu_i')_{i\in I})$$

is a function $f : X \to X'$ such that it is a homomorphism

$$(X, \mu_i) \longrightarrow (X', \mu_i')$$

for *each* $i \in I$. Notice that μ_i and μ_i' must have the same 'arity' for every $i \in I$.

The concept of a homomorphism provides the most natural definition of a morphism between algebraic structures.

3.2.4 Example: morphisms between pointed sets

A morphism from a pointed set (X, x_0) to a pointed set (X', x_0') is, by definition, a function $f : X \to X'$ such that

$$f(x_0) = x_0'. \tag{216}$$

3.2.5 Example: morphisms between A-sets

A morphism from an A-set (X, λ) to an A-set (X', λ'), cf. Section 2.2.38, is, by definition, a function $f : X \to X'$ such that

$$\forall_{a\in A}\ f \circ \lambda_a = \lambda_a' \circ f \tag{217}$$

or, equivalently, in multiplicative notation,

$$\forall_{a\in A, x\in X}\ f(ax) = af(x). \tag{218}$$

3.2.6

Condition (217) can be expressed as commutativity of square diagrams

$$\begin{tikzcd} X' \arrow[r, "\lambda_a'"] & X' \\ X \arrow[u, "f"] \arrow[r, "\lambda_a"'] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[u, "f"'] \end{tikzcd} \tag{219}$$

for all $a \in A$.

3.2.7 *Anti*homomorphisms between binary structures

Homomorphisms between binary structures

$$(A, \cdot)^{\mathrm{op}} \longrightarrow (A', \cdot') \qquad \text{or} \qquad (A, \cdot) \longrightarrow (A', \cdot')^{\mathrm{op}}$$

are the same as *antihomomorphisms* $(A, \cdot) \longrightarrow (A', \cdot')$, i.e., functions $f : A \to A'$ that satisfy the condition

$$\forall_{a,b\in A}\ f(ab) = f(b)f(a). \tag{220}$$

3.2.8 Actions of binary structures $(A, \cdot)$ on sets

The set of unary operations $\mathrm{Op}_1(X)$ of any set X is canonically equipped with a structure of a monoid. When a set A, equipped with a binary operation $\cdot$, is acting on a set X, it is usually assumed that the action function λ in (189) is a homomorphism of binary algebraic structures, i.e., that

$$\forall_{a,b\in A}\ \lambda_{a\cdot b} = \lambda_a \circ \lambda_b\,. \tag{221}$$

Condition (221) is equivalently expressed as the identity that closely resembles Associativity

$$\forall_{a,b\in A}\,\forall_{x\in X}\ (a\cdot b)x = a(bx)\,. \tag{222}$$

3.2.9

If the same generic multiplicative notation is used for the binary operation in A and for the action of A on X, then the requirement that λ be a homomorphism takes the form of the identity

$$\forall_{a,b\in A}\,\forall_{x\in X}\ (ab)x = a(bx) \tag{223}$$

that is indistinguishable from Associativity. And for a good reason: Associativity of a binary algebraic structure $(A, \cdot)$ expresses the fact that the structure acts on set A by left-multiplication.

Exercise 65 *Show that a binary algebraic structure $(A, \cdot)$ is associative if and only if the left-multiplication function, cf.* (190)*,*

$$L : A \longrightarrow \mathrm{Op}_1(A)\,, \qquad a \longmapsto L_a\,, \tag{224}$$

is a homomorphism of binary algebraic structures.

3.2.10 Right actions

What we described above is also known as a *left* action of a binary structure $(A, \cdot)$. A *right action* is an *antihomomorphism*

$$\varrho : A \longrightarrow \mathrm{Op}_1(X)\,. \tag{225}$$

Exercise 66 *Show that a binary algebraic structure $(A, \cdot)$ is associative if and only if the right-multiplication function, cf.* (191)*,*

$$R : A \longrightarrow \mathrm{Op}_1(A)\,, \qquad a \longmapsto R_a\,, \tag{226}$$

is an antihomomorphism of binary algebraic structures.

3.2.11

Generic multiplicative notation for right actions places an element $a \in A$ that acts on $x \in X$ on the *right*

$$\varrho_a(x) = xa\,. \tag{227}$$

This is where the terms *left* and *right* action come from.

The property of ϱ being an antihomomorphism then again has the form of the familiar associativity condition

$$\forall_{a,b\in A}\,\forall_{x\in X}\ x(ab) = (xa)b\,. \tag{228}$$

3.2.12

The left and the right regular actions of a semigroup on itself, introduced in Section 2.2.40 are particularly important in Group Theory and in Theory of Group Actions.

3.3 Semirings

3.3.1 Sets equipped with two binary operations

Suppose a set X is equipped with two binary operations, denoted $*$ and $\cdot$, respectively.

3.3.2 Left Distributivity Property

If the operations of left multiplication by a,

$$L_a \in \mathrm{Op}_1 X \qquad (a \in X),$$

cf. (190), act on X as *endomorphisms of binary structure* $(X, *)$, i.e., if

$$L_a \in \mathrm{End}(X, *) \qquad (a \in X),$$

we say that operation $\cdot$ *left-distributes over operation* $*$. Left Distributivity of $\cdot$ over $*$ is equivalent to the following identity

$$\forall_{a,x,y \in X} \quad a \cdot (x * y) = a \cdot x * a \cdot y . \tag{229}$$

3.3.3 Right Distributivity Property

If the operations of right multiplication by a,

$$R_a \in \mathrm{Op}_1 X \qquad (a \in X),$$

cf. (191), act on X as *endomorphisms of binary structure* $(X, *)$, i.e., if

$$R_a \in \mathrm{End}(X, *) \qquad (a \in X),$$

we say that operation $\cdot$ *right-distributes over operation* $*$. Right Distributivity of $\cdot$ over $*$ is equivalent to the following identity

$$\forall_{a,x,y \in X} \quad (x * y) \cdot a = x \cdot a * y \cdot a . \tag{230}$$

3.3.4 Commutative semigroups

The binary operation in a commutative semigroup is often referred to as *addition* and $+$ is the generic symbol for such an operation.

3.3.5 Semirings

Suppose a commuttive semigroup $(S, +)$ is equipped with a secondary operation $\cdot$, referred to as *multiplication*, that is both left and right distributive over addition. We call

$$(S, +, \cdot)$$

a *semiring*. We say that a semiring is *associative*, *commutative*, *unital*, if the multiplicative binary structure $(S, \cdot)$ is associative, commutative or, respectly, unital.

3.3.6 0 and 1 in a semiring

The identity element of the additive semigroup $(S,+)$ is referred to as the *zero* element, if it exists, and is denoted 0.

The identity element of the multiplicative binary structure $(S,\cdot)$ is denoted 1, when it exists, and is simply referred to as the *identity element* or the *unit element* (of the semiring).

3.3.7

In general, $s\cdot 0$ may not equal 0. This is so, however, if the additive semigroup $(S,+)$ is *cancellative*, cf. Section (2.2.12).

Exercise 67 *Show that in a semiring-with-zero*

$$\forall_{s\in S}\ 0\cdot s = 0 = s\cdot 0 \tag{231}$$

if addition is cancellative.

3.3.8 Rings

When the additive semigroup of a semiring is a group, we say that a semiring is a ring.

3.3.9 The ordered unital semiring-with-zero of natural numbers $(\mathbf{N}, 0, 1, +, \cdot, \leq)$

A principal example of a semiring is provided by the set of natural numbers equipped with the standard addition and multiplication operations. Its existence is equivalent to existence of an infinite set. We prove that and we establish some of its key features by studying *twisted sets*, i.e., unary algebraic structures (X,μ_1). We do this in separate sets of notes.

3.3.10

One such feature is that the additive semigroup $(\mathbf{N},+)$ and the multiplicative semigroup $(\mathbf{N}\setminus\{0\},\cdot)$ are *cancellative*, cf. Section 2.2.12.

3.3.11

Another feature is presence of the order relation that can be expressed entirely in terms of the operation of addition

$$\forall_{m,n\in\mathbf{N}}\ (\,m\leq n\ \Leftrightarrow\ \exists_{l\in\mathbf{N}}\ l+m=n\,)\,. \tag{232}$$

and that has the following properties:

(i) natural number l in (232) is unique and is denoted $n-m$;

(ii) $\forall_{n\in\mathbf{N}}\ 0\leq n<1 \Rightarrow 0=n$;

(iii) $\forall_{m,n\in\mathbf{N}}\ m<n \Leftrightarrow m+1\leq n$;

(iv) $(\mathbf{N},\leq)$ is a well-ordered set, cf. Section 1.8.18;

(v) $(\mathbf{N},+,\leq)$ and $(\mathbf{N},\cdot,\leq)$ are *ordered semigroups*, i.e.,

$$\forall_{m,n,m',n'\in\mathbf{N}} \quad m\leq m' \wedge n\leq n' \implies m+n \leq m'+n' \wedge mn \leq m'n' \,.$$

The following lemma is frequently used.

Lemma 3.1 (Euclid) *For every $m \in \mathbf{N}$ and $n \in \mathbf{N}\setminus\{0\}$, there exist unique $q, r \in \mathbf{N}$ such that*

$$m = qn + r \qquad \textit{and} \qquad 0 \leq r < n \,. \tag{233}$$

Proof. Consider the set

$$E := \{l \in \mathbf{N} \mid m < ln\} \,.$$

Since $1 \leq n$, one has $m \cdot 1 \leq mn$; hence $mn \in E$ and E is not empty. Let k be its smallest element. Since every element of E is greater than 0, there exists $q := k - 1 \in \mathbf{N}$ and

$$qn \leq m < qn + n \,.$$

Equivalently, $r := m - qn$ satisfies the double inequality

$$0 \leq r < n \,.$$

If q and r satisfy Equality (233), then $q + 1$ is the smallest element of set E; hence, representation (233) of m is unique. □

3.4 Morphisms between n-ary relations

3.4.1

A general approach to binary interactions between n-ary relations with the same domain-list consists of using a binary relation $\sim$ on $\mathrm{Rel}(X_1, \dots, X_n)$ to verify, for given $\rho, \rho' \in \mathrm{Rel}(X_1, \dots, X_n)$, whether $\rho \sim \rho'$ or not.

3.4.2

Given two n-ary relations whose domain-lists are arbitrary and not necessarily equal

$$\rho : X_1, \dots, X_n \longrightarrow \text{Statements} \qquad \text{and} \qquad \rho' : X_1', \dots, X_n' \longrightarrow \text{Statements}\,,$$

we may use a function-list (97) to *pull-back* ρ' to the domain-list of ρ and then *declare* that function-list a *$\sim$-morphism from ρ to ρ'* if

$$\rho \sim (f_1, \dots, f_n)^{\bullet}\rho' \,. \tag{234}$$

Note that the identity-list

$$\mathrm{id}_{X_1}, \dots, \mathrm{id}_{X_n}$$

is a $\sim$-morphism precisely when $\rho \sim \rho'$.

3.4.3

This approach requires that every domain-list $X_1, \dots, X_n$ has been equipped with a binary relation $\sim$.

There is a canonical way to equip sets $\mathrm{Rel}(X_1, \dots, X_n)$ with binary relations that are induced by a single binary relation on the common target of all relations, the set of statements.

3.4.4 Definition of a $\sim$-morphism

Given a binary relation $\sim$ on the set of statements, we *declare* a function-list $f_1, \dots, f_n$ to be a $\sim$-*morphism from ρ to ρ'* if condition (234) holds for the relation induced by $\sim$ on $\mathrm{Rel}(X_1, \dots, X_n)$.

3.4.5

Note that we use the same symbol $\sim$ to denote the original relation on the set of statements, and the induced relation on $\mathrm{Rel}(X_1, \dots, X_n)$. This is a common practice and rarely leads to confusion if used with care. The actual meaning is usually clear from the context.

This practice is analogous to using the same symbol $+$ for addition of real numbers, as well as the *induced* operation of addition of real-valued functions.

3.4.6 $\Rightarrow$-morphisms, $\Leftarrow$-morphisms, $\Leftrightarrow$-morphisms

An essential feature of the language of morphisms is the expectation that morphisms can be *composed* and that the composition law is associative. This requirement narrows the choice of the binary relations $\sim$ on the set of statements to transitive relations.

Recall that any transitive relation on the set of statements that is stronger than the *equipotence* relation $\Leftrightarrow$, is necessarily equipotent to $\Leftrightarrow$, $\Rightarrow$, $\Leftarrow$, or is a total relation, cf. Lemma 1.5. The first three are, in practice, the only choices for $\sim$ that lead to a nontrivial notion of a morphism between relations.

Note that a function-list (97) is a $\Leftrightarrow$-morphism if and only if it is at once a $\Rightarrow$-morphism and a $\Leftarrow$-morphism.

3.4.7

Composition of two $\sim$-morphisms, where $\sim$ is one of those three relations $\Rightarrow$, $\Leftarrow$, or $\Leftrightarrow$, and both morphisms have the same type, is again a $\sim$-morphism and of the same type.

3.4.8

The definition of a $\sim$-morphism for each of those three choices of $\sim$ is expressed by means of the corresponding diagram

$$(\Rightarrow\text{-}\mathbf{morphism}) \qquad \begin{tikzcd} X'_1, \dots, X'_n \arrow[dr, "\rho'"] \arrow[d, phantom, "\Uparrow"] & \\ X_1, \dots, X_n \arrow[u, "f_1 \ \cdots \ f_n"] \arrow[r, "\rho"'] & \text{Statements} \end{tikzcd} \tag{235}$$

$$\begin{tikzcd} X'_1, \dots, X'_n \arrow[dr, "\rho'"] & \\ {} \arrow[r, phantom, "\Downarrow"] & \text{Statements} \\ X_1, \dots, X_n \arrow[uu, "f_1 \cdots f_n"] \arrow[ur, "\rho"'] & \end{tikzcd} \qquad (\Leftarrow\text{-}\textbf{morphism}) \tag{236}$$

$$\begin{tikzcd} X'_1, \dots, X'_n \arrow[dr, "\rho'"] & \\ {} \arrow[r, phantom, "\Updownarrow"] & \text{Statements} \\ X_1, \dots, X_n \arrow[uu, "f_1 \cdots f_n"] \arrow[ur, "\rho"'] & \end{tikzcd} \qquad (\Leftrightarrow\text{-}\textbf{morphism}) \tag{237}$$

Exercise 68 *Show that $f_1, \dots, f_n$ is a $\sim$-morphism from ρ to ρ' if and only if*

$$\Gamma_\rho \subseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'}, \qquad \Gamma_\rho \supseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'} \qquad \textit{or, respectively,} \qquad \Gamma_\rho = (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'}, \tag{238}$$

depending on whether $\sim$ is $\Rightarrow$, $\Leftarrow$, or $\Leftrightarrow$.

3.4.9 Characterization of $\Rightarrow$-morphisms

Recall that

$$(f_1 \times \cdots \times f_n)_* \Gamma_\rho \subseteq \Gamma_{\rho'} \;\Leftrightarrow\; \Gamma_\rho \subseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'}. \tag{239}$$

It follows from (238)–(239) that the following conditions are equivalent.

(a) $f_1, \dots, f_n$ is a $\Rightarrow$-morphism from ρ to ρ'.

(b) ρ is *weaker* than the pull-back of ρ' by $f_1, \dots, f_n$, cf. Section 1.14.17.

(c) ρ' is *stronger* than the push-forward of ρ by $f_1, \dots, f_n$, cf. Section 1.14.18.

3.4.10 Characterization of $\Leftarrow$-morphisms

There is a similar characterization of $\Leftarrow$-morphims. It is based on the middle part of (238) and on the equivalence

$$(f_1 \times \cdots \times f_n)_! \Gamma_\rho \supseteq \Gamma_{\rho'} \;\Leftrightarrow\; \Gamma_\rho \supseteq (f_1 \times \cdots \times f_n)^* \Gamma_{\rho'}. \tag{240}$$

The following conditions are equivalent.

(a) $f_1, \dots, f_n$ is a $\Leftarrow$-morphism from ρ to ρ'.

(b) ρ is *stronger* than the pull-back of ρ' by $f_1, \dots, f_n$.

(c) ρ' is *weaker* than the conjugate push-forward of ρ by $f_1, \dots, f_n$, cf. Section 1.14.19.

3.4.11 Terminology

In practice, $\Rightarrow$-morphisms are usually referred simply as *morphisms* (between relations), $\Leftrightarrow$-morphisms are frequently referred to as *strict morphisms*, while $\Leftarrow$-morphisms rarely make their appearance in actual arguments built by mathematicians.

3.4.12

Functions $\forall_{x_i \in A_i}$ and $\forall^i$, cf. (33) and (34), defined by *universal* quantification, are $\Rightarrow$-morphisms of binary relational structures if we equip the sets of relations with the *implication* relation $\Longrightarrow$.

This fact, known since at least the times of Aristotle, is hardly ever mentioned, yet it is constantly used when reasoning is based on rules of Logic.

Exercise 69 *Is $\forall_{x_i \in A_i}$ or $\forall^i$ a $\Leftrightarrow$-morphism?*

Exercise 70 *Is $\exists_{x_i \in A_i}$ or $\exists^i$ a $\sim$-morphism for $\sim$ being $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$?*

3.4.13 Morphisms between relational structures

When

$$X_1 = \cdots = X_n = X\,, \qquad X'_1 = \cdots = X'_n = X' \qquad \text{and} \qquad f_1 = \cdots = f_n = f\,,$$

we shall be denoting the pulled-back relation $(f, \dots, f)^{\bullet}\rho'$ by $f^{\bullet}\rho'$.

We say that $f : X \to X'$ is a $\sim$*-morphism from a relational structure* (X, ρ) *to a relational structure* (X', ρ') if

$$\rho \sim f^{\bullet}\rho'\,.$$

3.5 The ordered $*$-monoid of 2-correspondences $(\mathscr{P}(X \times X); \Delta_X, \tau_*, \circ; \subseteq)$

3.5.1

If $C \subseteq X \times X$ and $D \subseteq X \times X$, then their composition $C \circ D$, defined in Section 1.17.10,

$$C \circ D = \{(x, y) \in X \times X \mid \exists_{z \in X}\ (x, z) \in C \wedge (z, y) \in D\}\,, \tag{241}$$

is contained in $X \times X$. In particular, the set of 2-correspondences on a set X, equipped with the composition operation, is a semigroup.

3.5.2 (Pre)ordered binary algebraic structures

Let $(B, \cdot, \precsim)$ be a set equipped with a binary algebraic operation and a preorder relation $\precsim$. We say that $(B, \cdot, \precsim)$ is a preordered binary algebraic structure if

$$\forall_{a, a', b, b' \in B}\ \ a \precsim a' \wedge b \precsim b' \Rightarrow ab \precsim a'b'\,. \tag{242}$$

Exercise 71 *Show that, if $C \subseteq C'$ and $D \subseteq D'$, then*

$$C \circ D \subseteq C' \circ D'\,, \tag{243}$$

i.e., $(\mathscr{P}(X \times X), \circ, \subseteq)$ is an ordered semigroup.

3.5.3 The diagonal subsets $\Delta_n(X) \subset X^n$

The subset

$$\Delta_n(X) := \{(x_1, \dots, x_n) \in X \times \cdots \times X \,|\, x_1 = \cdots = x_n\} \tag{244}$$

is referred to as the *n-diagonal.*

3.5.4 The diagonal function $\Delta : X \longrightarrow X \times X$ and its image Δ_X

The 2-diagonal set is usually denoted Δ_X. It coincides with the image of the *diagonal function*

$$\Delta : X \longrightarrow X \times X\,, \qquad x \longmapsto (x, x)\,. \tag{245}$$

Exercise 72 *Show that,*

$$\forall_{C \subseteq X \times X}\ \Delta_X \circ C = C = C \circ \Delta_X\,, \tag{246}$$

i.e., Δ_X is an identity element for $\circ$. In particular, $(\mathscr{P}(X{\times}X), \Delta_X, \circ, \subseteq)$ is an ordered monoid.

3.5.5 The graph homomorphism $\Gamma : (\mathrm{Op}_1 X, \mathrm{id}_X, \circ) \longrightarrow (\mathscr{P}(X{\times}X), \Delta_X, \circ)$

The graph-of-a-function correspondence

$$f \longmapsto \Gamma_f := \{(x, y) \in X{\times}X \mid f(x) = y\}$$

is an injective homomorphism of monoids

$$(\mathrm{Op}_1 X, \mathrm{id}_X, \circ) \longrightarrow (\mathscr{P}(X{\times}X), \Delta_X, \circ)\,.$$

It identifies the monoid of unary operations on X with a submonoid of 2-correspondences on X.

3.5.6 Antiinvolutions

Let $(B, \cdot)$ be a binary algebraic structure. An operation $\alpha : B \longrightarrow B$ is said to be an *antiinvolution* if it satisfies the identities

$$\alpha \circ \alpha = \mathrm{id}_B \qquad \text{and} \qquad \forall_{a,b \in B}\ \alpha(ab) = \alpha(b)\alpha(a)\,. \tag{247}$$

3.5.7 $*$-binary structures

A binary structure equipped with an antiinvolution, $(B, \alpha, \cdot)$, is called a $*$-binary structure.

Exercise 73 *Let $e \in B$ be a left identity element for $(B, \cdot)$. Show that $\alpha(e)$ is a right identity for $(B, \cdot)$.*

Solution. For any $a \in B$, one has

$$a\alpha(e) = (\alpha(\alpha(a))\alpha(e) = \alpha(e\alpha(a)) = \alpha(\alpha(a)) = a\,.$$

□

Exercise 74 *Let $e \in B$ be a right identity element for $(B, \cdot)$. Show that $\alpha(e)$ is a left identity for $(B, \cdot)$.*

In Section 2.2.15 we observed that, if a binary structure $(B, \cdot)$ admits both a left and a right identity, then they are equal. It follows that if a $*$-binary structure contains a one-sided identity element e, then this element is necessarily a two-sided identity and e is fixed by the antiinvolution

$$\alpha(e) = e\,. \tag{248}$$

3.5.8 The flip operation on $X{\times}X$

Let us denote by τ the operation on set $X{\times}X$ that transposes the factors in $X{\times}X$,

$$\tau(x_1, x_2) := (x_2, x_1)\,. \tag{249}$$

Exercise 75 *Show that τ_* is an antiinvolution on the monoid of 2-correspondences $(\mathscr{P}(X{\times}X), \Delta_X, \circ)$, i.e.,*

$$\tau_*(C{\circ}D) = \tau_* D \circ \tau_* C\,. \tag{250}$$

In particular, $(\mathscr{P}(X{\times}X); \Delta_X, \tau_*, \circ\,; \subseteq)$ is an ordered $*$-monoid.

3.5.9

In Mathematics and, especially, in Mathematical Physics, $*$-structures play an important role. We encounter $*$-semigroups, $*$-monoids, $*$-groups, and $*$-rings, i..e, rings equipped with an antiinvolution for addition and multiplication.

3.5.10

The ring of $n{\times}n$-matrices equipped with matrix transposition is an example of a $*$-ring that you are familiar with. Another example is provided by the field of complex numbers $\mathbf{C}$ equipped with complex conjugation.

Theory of $*$-rings of linear operators has been one of the most active areas of Mathematics during the last 80 years. One of its multiple applications to Mathematical Physics has been Constructive Quantum Field Theory.

3.5.11 The preordered $*$-structure of binary relations $(\mathbf{Rel}_2 X; =, (\ )^{\mathrm{op}}, \circ\,; \Rightarrow)$

The set of binary relations on a set X is canonically equipped with a preordered $*$-structure. Implication $\Rightarrow$ is the preorder, composition of relations is the (nonassociative) binary operation and the opposite-relation operation is the antiinvolution.

3.5.12 The graph homomorphism $\Gamma : (\mathbf{Rel}_2 X; =, (\ )^{\mathrm{op}}, \circ\,; \Rightarrow) \to (\mathscr{P}(X{\times}X); \Delta_X, \tau_*, \circ\,; \subseteq)$

Exercise 76 *Show that the graph of the equality relation $=$ on X equals Δ_X.*

Exercise 77 *Show that, for any binary relation ρ on X, one has*

$$\Gamma_{\rho^{\mathrm{op}}} = \tau_* \Gamma_\rho\,. \tag{251}$$

Exercise 78 *Show that, for any binary relations ρ and σ on X, one has*

$$\Gamma_{\rho\circ\sigma} = \Gamma_\rho \circ \Gamma_\sigma\,. \tag{252}$$

By combining Exercises 17, 76, 77, and 78, we conclude that the graph function

$$\Gamma : \mathrm{Rel}_2 X \longrightarrow \mathscr{P}(X{\times}X)$$

is a surjective homomorphism of preordered $*$-structures.

3.5.13 Graph characterizations of various types of binary relations

Each of the important properties of a binary relation ρ on a set X admits a characterization in terms of the graph Γ_ρ of ρ.

3.5.14 Subidempotent correspondences

Let us denote by $\mathscr{P}^{\text{subid}}(X{\times}X)$ the set

$$\{C \subseteq X{\times}X \| C \circ C \subseteq C\} \tag{253}$$

of *subidempotent* correspondences.

Exercise 79 *Show that ρ is transitive if and only if*

$$\Gamma_\rho \circ \Gamma_\rho \subseteq \Gamma_\rho , \tag{254}$$

i.e., Γ_ρ is a subidempotent in ordered semigroup $(\mathscr{P}(X{\times}X), \circ, \subseteq)$.

Exercise 80 *Let $\mathscr{C} \subseteq \mathscr{P}^{\text{subid}}(X{\times}X)$. Show that*

$$\bigcap \mathscr{C} \in \mathscr{P}^{\text{subid}}(X{\times}X) ,$$

i.e., the family of subidempotent correspondences $\mathscr{P}^{\text{subid}}(X{\times}X)$ is closed under intersection of arbitrary subfamilies.

3.5.15

Given $C \subseteq X{\times}X$, the family

$$\mathscr{C}_C^{\text{subid}} := \left\{D \in \mathscr{P}^{\text{subid}}(X{\times}X) \,|\, D \supseteq C\right\} \tag{255}$$

contains $X{\times}X$, hence is not nempty. According to Exercise 80,

$$C^{\text{subid}} := \bigcap \mathscr{C}_C^{\text{subid}} \tag{256}$$

is the smallest subidempotent correspondence containing C.

3.5.16 A weakest transitive relation stronger than ρ

Suppose that $\rho \Rightarrow \sigma$ and σ is a transitive relation. This is equivalent to

$$\Gamma_\rho \subseteq \Gamma_\sigma \qquad \text{and} \qquad \Gamma_\sigma \in \mathscr{P}^{\text{subid}}(X{\times}X) .$$

Then

$$\Gamma_\rho \subseteq \left(\Gamma_\rho\right)^{\text{subid}} \subseteq \Gamma_\sigma ,$$

which is equivalent to

$$\rho \Rightarrow \rho' \Rightarrow \sigma$$

where ρ' is any relation with graph $\left(\Gamma_\rho\right)$.

In particular, we established that, for any relation $\rho \in \mathrm{Rel}_2 X$, there exists a *weakest transitive relation stronger than* ρ. Its graph is the smallest subidempotent correspondence containing Γ_ρ.

3.5.17 A weakest reflexive relation stronger than ρ

Exercise 81 *Show that ρ is reflexive if and only if*

$$\Gamma_\rho \supseteq \Delta_X . \tag{257}$$

Exercise 82 *Show that, for a reflexive relation ρ, one has*

$$\Gamma_\rho \subseteq \Gamma_\rho \circ \Gamma_\rho . \tag{258}$$

Exercise 83 *Let $\rho \in \mathrm{Rel}_2 X$. Show that*

$$\text{if } \rho \Rightarrow \sigma \text{ and } \sigma \text{ is reflexive, then } \rho \Rightarrow \rho\vee= \ \Rightarrow \sigma . \tag{259}$$

Here $\rho\vee=$ denotes the alternative of ρ and the equality relation on X.

In other words, for any relation $\rho \in \mathrm{Rel}_2 X$, relation $\rho\vee=$ is a *weakest reflexive relation stronger than* ρ.

Exercise 84 *Show that ρ is a preorder if and only if*

$$\Gamma_\rho \circ \Gamma_\rho = \Gamma_\rho , \tag{260}$$

i.e., Γ_ρ is an idempotent in semigroup $(\mathscr{P}(X\times X), \circ)$.

3.5.18 A weakest preorder stronger than ρ

Since the intersection of any family of correspondences containing Δ_X contains Δ_X, the intersection of any family of idempotents, i.e., subidempotents containing Δ_X, is an idempotent.

Thus,

$$\bigcap\{D \subseteq X\times X \mid D\circ D = D \text{ and } D \supseteq C\} \tag{261}$$

is the smallest idempotent correspondence containing C.

Exercise 85 *Let $\rho \in \mathrm{Rel}_2 X$. Show that there exists a preorder ρ' satisfying the following universal property:*

$$\text{if } \rho \Rightarrow \sigma \text{ and } \sigma \text{ is a preorder, then } \rho \Rightarrow \rho' \Rightarrow \sigma . \tag{262}$$

In other words, for any relation $\rho \in \mathrm{Rel}_2 X$, there exists a *weakest preorder stronger than* ρ.

3.5.19 A weakest symmetric relation stronger than ρ

Exercise 86 *Show that ρ is symmetric if and only if its graph Γ_ρ is τ-invariant, i.e.,*

$$\Gamma_\rho \subseteq \tau_* \Gamma_\rho . \tag{263}$$

Exercise 87 *Show that* (263) *implies (and therefore is equivalent to) the stronger condition*

$$\Gamma_\rho = \tau_* \Gamma_\rho . \tag{264}$$

In other words, a relation ρ is symmetric if and only if its graph is a fixed point of τ_*.

Exercise 88 *Let* $\rho \in \mathrm{Rel}_2 X$.

$$\text{if } \rho \Rightarrow \sigma \text{ and } \sigma \text{ is symmetric, then } \rho \Rightarrow \rho \vee \rho^{\mathrm{op}} \Rightarrow \sigma. \tag{265}$$

In other words, for any relation $\rho \in \mathrm{Rel}_2 X$, relation $\rho \vee \rho^{\mathrm{op}}$ is a *weakest symmetric relation stronger than* ρ.

Exercise 89 *Show that* ρ *is an equivalence relation if and only if* Γ_ρ *is a* τ*-invariant idempotent in* $(\mathscr{P}(X \times X), \Delta_X, \circ)$.

3.5.20 A weakest equivalence relation stronger than ρ

Intersection of any family of τ-invariant subsets of $\mathscr{P}(X \times X)$ is τ-invariant. Thus,

$$\bigcap \{D \subseteq X \times X \mid D \circ D = D,\ D \subseteq \tau_* D \text{ and } D \supseteq C\} \tag{266}$$

is the smallest τ-invariant idempotent correspondence containing C.

Exercise 90 *Let* $\rho \in \mathrm{Rel}_2 X$. *Show that there exists an equivalence relation* ρ' *satisfying the following universal property:*

$$\text{if } \rho \Rightarrow \sigma \text{ and } \sigma \text{ is an equivalence relation, then } \rho \Rightarrow \rho' \Rightarrow \sigma. \tag{267}$$

In other words, for any relation $\rho \in \mathrm{Rel}_2 X$, there exists a *weakest equivalence relation stronger than* ρ.

3.5.21

Exercise 91 *Show that* ρ *is antisymmetric if and only if*

$$\Gamma_\rho \cap \tau_* \Gamma_\rho = \emptyset. \tag{268}$$

Exercise 92 *Show that* ρ *is weakly antisymmetric if and only if*

$$\Gamma_\rho \cap \tau_* \Gamma_\rho \subseteq \Delta_X. \tag{269}$$

Exercise 93 *Show that* ρ *is an order relation if and only if* Γ_ρ *is an idempotent in* $(\mathscr{P}(X \times X), \Delta_X, \circ)$ *and*

$$\Gamma_\rho \cap \tau_* \Gamma_\rho = \Delta_X.$$

3.6 Morphisms between structures of functional type

3.6.1

Suppose that a set X is equipped with a family of functions

$$\mathscr{O} \subset \mathrm{Funct}(X, \mathbf{R})$$

and a set X' is equipped with a family of functions

$$\mathscr{O}' \subset \mathrm{Funct}(X', \mathbf{R}).$$

We say that a function $f : X \to X'$ is a morphism if, for every $\phi' \in \mathscr{O}'$, the composite function $f^\bullet \phi' = \phi' \circ f$ belongs to $\mathscr{O}$,

$$\forall_{\phi' \in \mathscr{O}'}\ f^\bullet \phi' \in \mathscr{O}. \tag{270}$$

3.6.2

An equivalent form of condition (270) is

$$(f^{\bullet})_*\mathcal{O}' \subset \mathcal{O}\,. \tag{271}$$

This, in turn, can be expressed in the language of diagrams: a function $f : X \to X'$ is a morphism if the diagram

$$\begin{tikzcd} \mathcal{O} \arrow[d, hook] & \mathcal{O}' \arrow[d, hook] \\ \mathrm{Funct}(X, \mathbf{R}) & \mathrm{Funct}(X', \mathbf{R}) \arrow[l, "f^{\bullet}"'] \end{tikzcd}$$

admits a completion to a commutative square diagram

$$\begin{tikzcd} \mathcal{O} \arrow[d, hook] & \mathcal{O}' \arrow[l] \arrow[d, hook] \\ \mathrm{Funct}(X, \mathbf{R}) & \mathrm{Funct}(X', \mathbf{R}) \arrow[l, "f^{\bullet}"'] \end{tikzcd}$$

3.7 Morphisms between structures of topological type

3.7.1

Suppose that a set X is equipped with a family of subsets $\mathscr{A} \subset \mathscr{P}X$ and a set X' is equipped with a family of subsets $\mathscr{A}' \subset \mathscr{P}X'$. We say that a function $f : X \to X'$ is a morphism if the preimage under f of every member of family $\mathscr{A}'$ is a member of $\mathscr{A}$,

$$\forall_{A' \in \mathscr{A}'}\, f^{\bullet}A' \in \mathscr{A}\,. \tag{272}$$

3.7.2

An equivalent form of condition (272) is

$$(f^{\bullet})_*\mathscr{A}' \subset \mathscr{A}\,. \tag{273}$$

Notice the similarity to condition (271).

3.7.3

Condition (273) can be expressed by saying that the diagram

$$\begin{tikzcd} \mathscr{A} \arrow[d, hook] & \mathscr{A}' \arrow[d, hook] \\ \mathscr{P}Y & \mathscr{P}Y' \arrow[l, "f^{*}"'] \end{tikzcd}$$

admits a completion to a commutative square diagram

$$\begin{tikzcd} \mathscr{A} \arrow[d, hook] & \mathscr{A}' \arrow[l] \arrow[d, hook] \\ \mathscr{P}Y & \mathscr{P}Y' \arrow[l, "f^{*}"'] \end{tikzcd}$$

3.7.4 Continuous functions

When $\mathscr{A}$ and $\mathscr{A}'$ have the meaning of being the families of *open subsets* in a topological spaces, i.e., when $(X, \mathscr{A})$ and $(X', \mathscr{A}')$ are topological spaces, cf. Section 2.1.6, we obtain the definition of a morphism between topological spaces. This is precisely how a continuous function is defined.

3.7.5 Measurable functions

When $\mathscr{A}$ and $\mathscr{A}'$ have the meaning of being the families of *measurable subsets* in a measurable spaces, i.e., when $(X, \mathscr{A})$ and $(X', \mathscr{A}')$ are measurable spaces, cf. Section 2.1.7, we obtain the definition of a morphism between measurable spaces. This is precisely how a measurable function is defined.

3.7.6

Another condition that can be interpreted as saying that f respects distinguished families of subsets reads

$$\forall_{A \in \mathscr{A}} \, f_* A \in \mathscr{A}' \tag{274}$$

or, equivalently,

$$(f_*)_* \mathscr{A} \subset \mathscr{A}' \,. \tag{275}$$

Either condition can serve as a definition of a morphism between structures of topological type. It is however the former, (272), that plays a fundamental role in Topology and Measure Theory, not the latter, (274).

4 The language of categories

4.1 The concept of a category

4.1.1

Whatever definition of a morphism between mathematical structures one adopts, it always has the following features

- any morphism α has a *source* and a *target* that are mathematical structures of the same type
- if the source $s(\alpha)$ of a morphism α coincides with the target $t(\beta)$ of a morphism β, then their composition $\alpha \circ \beta$ is defined and

$$t(\alpha \circ \beta) = t(\alpha) \qquad \text{and} \qquad s(\alpha \circ \beta) = s(\beta)$$

- composition of morphisms is associative, i.e.,

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

  for any composable list α, β, γ of morphisms

4.1.2

The above observations led to the introduction of the concept of a *category*. In a nutshell, a *category* $\mathcal{C}$ consists of two classes, a class $\mathcal{C}_0$ of *objects* and a class $\mathcal{C}_1$ of *morphisms*, equipped with an associative operation of composition of morphisms.

4.1.3

Various classes of mathematical structures equipped with appropriate classes of morphisms form natural categories. Studying the category of groups is what Group Theory does. Studying the category of rings is what Ring Theory does. Algebraic geometers study the category of *algebraic varieties* and the bigger category of *algebraic schemes*. Topologists study the category of *topological spaces*, and so on.

4.1.4

Every mathematical theory can be expressed in a categorical language. This usually provides an added degree of clarity to a theory and yields insights that are otherwise lost.

4.2 Basic vocabulary

4.2.1 Epimorphisms

A morphism α is said to be a *epimorphism* if, for any diagram

$$\bullet \underset{\psi}{\overset{\phi}{\leftleftarrows}} \bullet \xleftarrow{\alpha} \bullet\,, \tag{276}$$

equality $\phi \circ \alpha = \psi \circ \alpha$ implies $\phi = \psi$.

Exercise 94 *Show that the composite of two epimorphisms is an epimorphism.*

Exercise 95 *Show that if $\alpha \circ \beta$ is an epimorphism, then α is an epimorphism.*

Exercise 96 *Show that a function f is an epimorphism in the category of sets if and only if f is surjctive.*

4.2.2 Monomorphisms

A morphism α is said to be a *monomorphism* if, for any diagram

$$\bullet \xleftarrow{\alpha} \bullet \overset{\phi}{\underset{\psi}{\leftleftarrows}} \bullet\ , \tag{277}$$

quality $\alpha \circ \phi = \alpha \circ \psi$ implies $\phi = \psi$.

Exercise 97 *Show that the composite of two monomorphisms is a monomorphism.*

Exercise 98 *Show that if $\alpha \circ \beta$ is a monomorphism, then β is an monomorphism.*

Exercise 99 *Show that a function f is a monomorphism in the category of sets if and only if f is injective.*

4.2.3 Initial objects

An object i is said to be *initial* if, for every object c, there exists a unique morphism $i \to c$.

4.2.4 Terminal objects

An object t is said to be *termial* if, for every object c, there exists a unique morphism $c \to t$.

4.3 Endomorphisms

4.3.1

Morphisms whose source and target coincide with an object c are referred as *endomorphisms* of object c.

4.3.2 The identity endomorphism

Let ι be an endomorpism of an object c such that

$$\forall_\alpha\ s(\alpha){=}c \Longrightarrow \alpha{\circ}\iota = \alpha$$

and

$$\forall_\beta\ t(\beta){=}c \Longrightarrow \iota{\circ}\beta = \beta\,.$$

We shall call it an *identity endomorphism*.

Exercise 100 *Suppose that ι and ι' are identity endomorphisms of an object c. Show that $\iota = \iota'$.*

It follows that, if c admits an identity endomorphism, then it is unique. We denote this unique identity endomorphism id_c or 1_c.

4.3.3 Unital categories

A category is said to be *unital* if every object admits an identity endomorphism.

4.3.4 A right inverse of a morphism

We say that β is a *right inverse* of a morphism α if

$$s(\alpha) = t(\beta)\,, \qquad t(\alpha) = s(\beta) \qquad \text{and} \qquad \alpha \circ \beta = \mathrm{id}_{t(\alpha)}\ .$$

Existence of the identity endomorphism $\mathrm{id}_{t(\alpha)}$ is a necessary condition for α to be right-invertible.

4.3.5 Split epimorphisms

Exercise 101 *Show that a right-invertible morphism is an epimorphism.*

On this account, we call a right-invertible morphism a *split epimorphism*. A *splitting* of a epimorphism is, by definition, any of its right inverses.

Exercise 102 *Show that the composite of two split epimorphisms is a split epimorphism.*

4.3.6 A left inverse of a morphism

We say that β is a *left inverse* of a morphism α if

$$s(\alpha) = t(\beta)\,, \qquad t(\alpha) = s(\beta) \qquad \text{and} \qquad \beta \circ \alpha = \mathrm{id}_{s(\alpha)}\ .$$

Existence of the identity endomorphism $\mathrm{id}_{s(\alpha)}$ is a necessary condition for α to be left invertible.

4.3.7 Split monomorphisms

Exercise 103 *Show that a left-invertible morphism is a monomorphism.*

On this account, we call a left-invertible morphism a *split monomorphism*. A *splitting* of a monomorphism is, by definition, any of its left inverses.

4.3.8 The inverse of a morphism

Exercise 104 *Let β be a right inverse of α and β' be a left inverse of α. Show that $\beta = \beta'$.*

Solution. In view of associativity of composition of morphisms, one has

$$\beta = \mathrm{id}_{s(\alpha)} \circ \beta = (\beta' \circ \alpha) \circ \beta = \beta' \circ (\alpha \circ \beta) = \beta' \circ \mathrm{id}_{t(\alpha)} = \beta'\ .$$

□

It follows that existence of a (two-sided) inverse of α is equivalent to existence of a right and of a left inverse. Moreover, a two-sided inverse is unique when it exists. We denote it α^{-1}.

4.3.9 Isomorphisms

An invertible morphism, i.e., a morphism that admits an inverse, is called an *isomorphism*. Objects c and d are said to be *isomorphic* if there exists an isomorphism $c \to d$. Symbolically, this is expressed by $c \simeq d$.

4.3.10

According to Exercise 104 an isomorphism is morphism that is, at once, a split epimorphism and a split monomorphism.

4.3.11 Arrow notation

We signal that a morphism $\alpha : c \to d$ is a monomorphism, an epimorphism, or an isomorphism, by employing the following arrow notation

monomorphism	$\alpha : c \rightarrowtail d$	(278)
epimorphism	$\alpha : c \twoheadrightarrow d$	(279)
isomorphism	$\alpha : c \xrightarrow{\sim} d\,.$	(280)

4.3.12 The semigroup of endomorphisms

Equipped with composition as its binary operation, the set of endomorphisms of an object c of any category becomes a semigroup, denoted

$$\mathrm{End}_{\mathcal{C}}\, c\,. \tag{281}$$

The semigroups of endomorphisms of various mathematical structures play a fundamental role in nearly every area of Mathematics and Mathematical Pysics.

4.3.13 The monoid of endomorphisms

If object c admits an identity endomorphism, then

$$(\mathrm{End}_{\mathcal{C}}\, c, \mathrm{id}_c, \circ)$$

is a monoid. For example, the monoid of unary operations $\mathrm{Op}_1(X)$ on a set X is precisely the monoid of endomorphsisms of X viewed as an object of the category of sets.

4.3.14 The group of autorphisms

An invertible endomorphism of c is called an *automorphism*. The set $\mathrm{Aut}_{\mathcal{C}}\, c$ of automorphisms of c contains id_c and is closed under the operations of composition and passing to the inverse element. It coincides with the group of invertible elements in the monoid $\mathrm{End}_{\mathcal{C}}\, c$ of endomorphisms of c.

4.3.15 An action of a set A on an object of a category

If A is a set and c is an object of a category $\mathscr{C}$, we have a ready definition of an *action of A on c* if we notice that $\mathrm{Op}_1(X)$ in (189) coincides with the monoid of endomorphisms of X in the category of sets. Thus, an action of a set A on an object c is defined to be a function

$$L : A \longrightarrow \mathrm{End}_{\mathscr{C}}\, c\,. \tag{282}$$

4.3.16 An action of a binary structure $(A, \cdot)$ on an object of a category

We say that a binary structure $(A, \cdot)$ acts on an object c if the function in (282) is a homomorphism of binary structures.

4.3.17 An action of a monoid $(A, e, \cdot)$ on an object of a *unital* category

We say that a monoid $(A, \cdot)$ acts on an object c of a unital category if the function in (282) is a homomorphism of monoids.

4.3.18 Representation Theory of Groups

Classical Representation Theory studies group actions on the objects of the category of vector spaces over a field k. Such actions are referred to as *k-linear representations* of a given group. The cases $k = \mathbf{R}$ and $k = \mathbf{C}$ produce Real and, respectively, Complex Representation Theory.

4.3.19 Category of k-linear representations of a group

Given a group G, its k-linear representations form naturally objects of a category, and determination of the structure of that category is a central topic of Representation Theory.

4.3.20

Representation Theory has been, beginning from its roots in Linear Algebra in the latter part of 19th Century, an essential area of Mathematics, that had enormous impact on the development of Mathematical Physics in 20th Century. The shear wealth of the methods it employs and applications it produces is a reason why learning Representation Theory is simultaneously obligatory and takes several years of very intensive study.

5 Quotient structures

5.1 The concept of a quotient set

5.1.1 The quotient set $X_{/\eta}$

We begin by revisiting a few terms briefly discussed when the concept of an equivalence relation was first introduced. Given an equivalence relation η, we denote by

$$X_{/\eta} := \left([\ ]_\eta\right)_* X \tag{283}$$

the image of the function

$$[\ ]_\eta : X \longrightarrow \mathscr{P}X, \qquad x \longmapsto [x]_\eta . \tag{284}$$

We refer to (283) as the *quotient of set X by equivalence relation η*, and we call the function induced by (284)

$$[\ ]_\eta : X \longrightarrow X_{/\eta}, \qquad x \longmapsto [x]_\eta , \tag{285}$$

the (canonical) *quotient map*. By definition, (285) is surjective.

5.1.2

If θ is an equivalence relation stronger than η, then

$$\forall_{x \in X}\ [x]_\eta \subseteq [x]_\theta$$

and $[x]_\theta$ is the union of η-equivalence classes $[y]_\eta$ of elements that are θ-equivalent to x. In particular, assignment

$$x \longmapsto [x]_\theta \qquad (x \in X)$$

factorizes

$$x \longmapsto [x]_\eta \longmapsto [x]_\theta$$

and, if we denote the corresponding function $X_{/\eta} \longrightarrow X_{/\theta}$ by $[\ ]_{\eta\theta}$, then we obtain the commutative triangle diagram

$$\begin{tikzcd} & X \arrow[dl, "{[\ ]_\eta}"'] \arrow[dr, "{[\ ]_\theta}"] & \\ X_{/\eta} \arrow[rr, two heads, "{[\ ]_{\eta\theta}}"'] & & X_{/\theta} \end{tikzcd} . \tag{286}$$

5.1.3 The fiber equivalence relation $=_f$

Let $f : X \to X'$ be an arbitrary function. We shall denote the pull-back to X of the equality relation on X' by $=_f$,

$$x =_f y \quad := \quad \text{“}\ f(x) = f(y)\ \text{”}, \tag{287}$$

and we shall call $=_f$ the *fiber equivalence relation* associated with function f.

5.1.4 The canonical factorization of a function $f : X \to X'$

Let $\iota_f : f_* X \hookrightarrow X'$ be the inclusion function.

Lemma 5.1 *Every function $f : X \to X'$ is represented as the composite*

$$f \;=\; \iota_f \circ \tilde{f} \circ [\;]_{=_f} \tag{288}$$

for a unique bijection $\tilde{f} : X_{/=_f} \longrightarrow f_* X$.

The assertion of Lemma 5.1 is illustrated by the following commutative square diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "{[\;]_{=_f}}"'] & X' \\ X_{/=_f} \arrow[r, "\sim", "\tilde{f}"'] & f_* X \arrow[u, hook, "\iota_f"'] \end{tikzcd} \tag{289}$$

Proof. Suppose that

$$\iota_f \circ g \circ [\;]_{=_f} = \iota_f \circ h \circ [\;]_{=_f} . \tag{290}$$

In view of ι_f being a monomorphism, cf. Exercise 99, Equality (290) implies that

$$g \circ [\;]_{=_f} = h \circ [\;]_{=_f} . \tag{291}$$

In view of $[\;]_{=_f}$ being an epimorphism, cf. Exercise 96, Equality (290) implies that $g = h$. This proves uniqueness of $\tilde{f}$.

For an element $E \in X_{/=_f}$, the restriction of f to E is a constant function. Denote by $\tilde{f}(E)$ the single value of $f_{|E}$. This defines a function $\tilde{f}(E) : X_{/=_f} \longrightarrow f_* X$ such that $f = \iota_f \circ \tilde{f} \circ [\;]_{=_f}$.

Suppose that $\tilde{f}(E) = \tilde{f}(F)$ for some $E, F \in X_{/=_f}$. Then $f(x) = f(y)$ for arbitrary $x \in E$ and $y \in F$. It follows that $x \in F$ and $y \in E$, i.e., $E = F$. In other words, $\tilde{f}$ is injective. Since $\tilde{f}$ has the same image as f, function $\tilde{f}$ is surjective. ☐

5.1.5 The quotient of X by an arbitrary binary relation

Let $\rho \in \mathrm{Rel}_2 X$ be an arbitrary binary relation on a set X and η_ρ be a weakest equivalence relation stronger than ρ, cf. Section 3.5.20. We shall denote by $[x]_\rho$ the equivalence class $[x]_{\eta_\rho}$ and we shall denote by $X_{/\rho}$ the quotient set $X_{/\eta_\rho}$.

5.1.6 The universal property of the canonical quotient map

Lemma 5.2 *Let $f : X \to X'$ be a function satisfying the following condition*

$$\forall_{x,y \in X}\, f(x) = f(y) . \tag{292}$$

Then there exists a unique function $\bar{f} : X_{/\rho} \longrightarrow X'$, such that

$$f = \bar{f} \circ [\;]_\rho . \tag{293}$$

Exercise 105 *Prove uniqueness of $\bar{f}$.*

Proof. Here we prove existence of $\bar{f}$. Condition (292) means that relation ρ is weaker than relation $=_f$. In particular, $=_f$ is stronger than a weakest equivalence relation η_ρ stronger than ρ. Hence, according to (286), canonical quotient map $[\ ]_{=_f}$ factorizes

$$[\ ]_{=_f} = [\ ]_{\eta_\rho,=_f} \circ [\ ]_{\eta_\rho}.$$

Note that, by definition, $[\ ]_\rho = [\ ]_{\eta_\rho}$. By combining this with factorization (288), we obtain equality (293) if we put

$$\bar{f} := \iota_f \circ \bar{\bar{f}} \circ [\ ]_{\eta_\rho,=_f}.$$

□

5.1.7 Morphisms $(X,\rho) \longrightarrow (X',=)$

Notice that functions $f : X \to X'$ satisfying Condition (292) are nothing but morphisms of binary relational structures

$$(X,\rho) \longrightarrow (X',=). \tag{294}$$

Lemma 5.2 thus establishes a canonical bijection between morphisms (294) and arbitrary functions

$$X_{/\rho} \longrightarrow X'.$$

5.1.8 Morphisms $(X',=) \longrightarrow (X,\rho)$

Let

$$X^{\setminus\rho} := \{x \in X \mid \rho(x,x)\}. \tag{295}$$

Exercise 106 *Show that a function $g : X' \to X$ is a morphism*

$$(X',=) \longrightarrow (X,\rho) \tag{296}$$

if and only if

$$g_* X \subseteq X^{\setminus\rho}.$$

5.1.9

Accordingly, morphisms (296) factorize

$$g = \iota_g \circ \underline{g}$$

where $\underline{g}$ is the function $X' \to X^{\setminus\rho}$ induced by g.

Correspondence $g \leftrightarrow \underline{g}$ defines a canonical bijection between the set of morphisms (296) and the set of arbitrary functions

$$X' \longrightarrow X^{\setminus\rho}.$$

5.1.10 Duality between the concepts of a subset and of a quotient set

The axioms of Set Theory provide just one mechanism to produce subsets of a given set X: for a given *unary* relation $\sigma \in \mathrm{Rel}_1 X$ form its graph,

$$\Gamma_\sigma = \{x \in X \mid \sigma(x)\}\,. \tag{297}$$

Note that the graph of σ coincides with $X^{\setminus\rho}$,

$$\Gamma_\sigma = X^{\setminus\rho}\,,$$

for the binary relation

$$\rho(x,y) \quad = \quad \text{“ } x{=}y \wedge \sigma(x) \text{ ”}\,.$$

Note that the graph of ρ is the direct image of Γ_σ under the diagonal function, cf. (245),

$$\Gamma_\rho = \Delta_* \Gamma_\sigma\,.$$

The dual nature of the universal properties of the canonical inclusion map $X^{\setminus\rho} \hookrightarrow X$ and of the canonical quotient map $X \twoheadrightarrow X_{/\rho}$ reveals the dual nature of the concepts of a subset and of a quotient set.

5.2 Congruence relations

5.2.1 Congruence equivalence relations

Let (X,μ) be an n-ary algebraic structure. We say that an equivalence relation η on X is a μ-*congruence equivalence* if

$$\forall_{\substack{x_1,\dots,x_n\\ y_1,\dots,y_n}} \quad \eta(x_1,y_1) \wedge \dots \wedge \eta(x_n,y_n) \implies \eta\left(\mu(x_1,\dots,x_n),\mu(y_1,\dots,y_n)\right)\,. \tag{298}$$

In other words, η is a congruence equivalence if replacing entries in the argument list

$$x_1,\dots,x_n$$

by η-equivalent entries

$$y_1,\dots,y_n$$

produces η-equivalent results when we apply operation μ to both argument lists.

5.2.2 The quotient structure $(X_{/\eta},\bar\mu)$

An equivalence relation η is a μ-congruence equivalence if and only if the diagram

$$\begin{array}{ccc} X_{/\eta},\dots,X_{/\eta} & & X_{/\eta} \\ {[\;]_\eta}\big\uparrow\cdots\big\uparrow{[\;]_\eta} & & \big\uparrow{[\;]_\eta} \\ X,\dots,X & \xrightarrow{\mu} & X \end{array} \tag{299}$$

can be completed to a commutative diagram

$$\begin{tikzcd} X_{/\eta}, \ldots, X_{/\eta} \arrow[r, "\bar{\mu}"] & X_{/\eta} \\ X, \ldots, X \arrow[u, twoheadrightarrow, "{[\ ]_\eta}"] \arrow[u, phantom, "\cdots"] \arrow[r, "\mu"'] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[u, twoheadrightarrow, "{[\ ]_\eta}"'] \end{tikzcd} . \tag{300}$$

We refer to $(X_{/\eta}, \bar{\mu})$ as the *quotient* of (X, μ) by a congruence relation η.

5.2.3 The operation $\bar{\mu}$ on $X_{/\eta}$ induced by μ

The value

$$\bar{(}E_1, \ldots, E_n) \qquad (E_1, \ldots, E_n \in X_{/\eta})$$

is the equivalence class

$$[\mu(x_1, \ldots, x_n)]_\eta \tag{301}$$

where $x_1 \in E_1, \ldots, x_n \in E_n$. In view of η being a μ-congruence, class (301) does not depend on the choice of representatives of classes $E_1, \ldots, E_n$.

5.2.4 $\left(X_{/\eta}, (\bar{\mu}_i)_{i\in I}\right)$

Let $(X, (\mu_i)_{i\in I})$ be a general algebraic structure. We say that η is a *congruence equivalence* if η is a μ_i-congruence equivalence for every $i \in I$. We refer to $\left(X_{/\eta}, (\bar{\mu}_i)_{i\in I}\right)$ as the *quotient* of $\left(X, (\mu_i)_{i\in I}\right)$ by congruence equivalence η.

5.2.5 General congruence relations

We say that an arbitrary binary relation $\rho \in \mathrm{Rel}_2 X$ is a μ*-congruence* if a weakest equivalence relation η_ρ stronger than ρ, is a μ-congruence equivalence.

The corresponding quotient structures $\left(X_{/\rho}, \bar{\mu}\right)$ and $\left(X_{/\rho}, (\bar{\mu}_i)_{i\in I}\right)$ are formed with respect to congruence equivalence η_ρ.

5.2.6 A criterion for a binary relation to be a μ-congruence

Here is a simple criterion of ρ being a μ-congruence relation.

> *Let $x_1, \ldots, x_n$ be any argument list. Let us replace any entry x_j by an element y_i such that*
>
> $$\rho(x_j, y_j)\,. \tag{302}$$
>
> *Then*
>
> $$\eta_\rho\left(\mu(x_1, \ldots, x_n)\,,\ \mu(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n)\right).$$

5.3 Group congruence equivalences

5.3.1

Let η be a congruence equivalence on a group $(G, e, (\ )^{-1}, \cdot)$

Exercise 107 *Show that $[e]_\eta$ is a normal subgroup of G.*

Solution. Below we shall use abbreviated notation $[x] = [x]_\eta$. Since η is a congruence equivalence for the inverse-element operation, one has

$$\forall_{x \in [e]}\ x^{-1} \in [e^{-1}] = [e]\,,$$

i.e., $[e]$ is closed under the inverse-element operation.

Since η is a congruence equivalence for the operation of multiplication, one has

$$\forall_{x,y \in [e]}\ xy \in [e \cdot e] = [e]\,,$$

i.e., $[e]$ is closed under multiplication, and

$$\forall_{x \in [e]}\ \forall_{g \in G}\ gxg^{-1} \in \left[geg^{-1}\right] = [e]\,,$$

i.e., $[e]$ is closed under conjugation by an arbitrary element $g \in G$. □

Exercise 108 *Show that K is a normal subgroup of G if and only if*

$$\forall_{x,y \in G}\ xH \cap Hy \neq \emptyset \Rightarrow xH = Hy\,.$$

Exercise 109 *Show that K is a normal subgroup of G if and only if the right K-divisor relation $|_K$, cf. Section 2.2.10, is a congruence relation on a group G.*

Exercise 110 *Show that K is a normal subgroup of G if and only if the right and left K-divisor relations are equipotent*

$$|_K \ \Leftrightarrow\ {}_K| \,. \tag{303}$$

Exercise 111 *Show that η is a congruence equivalence on a group G if and only if η is equipotent to the K-divisor relation for a normal subgroup K of G.*

5.3.2 The quotient group G/K

When K is a normal subgroup of a group G, we signal this by employing notation

$$K \lhd G\,. \tag{304}$$

The quotient group $G_{/|_K}$ is denoted G/K.

5.3.3 The kernel of a group homomorphism

Exercise 112 *Show that the fiber equivalence $=_f$ of a group homomorphism $f : G \longrightarrow G'$ is a group congruence.*

The equivalence class $[e]_{=_f}$ of the identity element in G is, accordingly, a normal subgroup. It is called the *kernel of f* and is denoted $\operatorname{Ker} f$.

5.3.4 The image of a group homomorphism

The direct image of G under a group homomorphism f is referred to as the *image of* f and is denoted $\mathrm{Im}\, f$.

5.3.5 The canonical factorization of a group homomorphism $f : G \to G'$

For group homomorphisms, Lemma 5.1 acquires the following form.

Lemma 5.3 *Every group homomorphism $f : G \to G'$ is represented as the composite*

$$f = \iota_f \circ \tilde{f} \circ [\;]_{\equiv_f} \tag{305}$$

for a unique isomorphism $\tilde{f} : G/\mathrm{Ker}\, f \xrightarrow{\sim} \mathrm{Im}\, f$.

The assertion of Lemma 5.3 is illustrated by the following commutative square diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, two heads, "{[\;]_{\equiv_f}}"'] & G' \\ G/\mathrm{Ker}\, f \arrow[r, "\sim", "\tilde{f}"'] & \mathrm{Im}\, f \arrow[u, hook, "\iota_f"'] \end{tikzcd} \tag{306}$$

5.4 Ring congruence equivalences

5.4.1

Let η be a congruence equivalence on a ring $(R, 0, -, +, \cdot)$ Here

$$x \longmapsto -x \qquad (x \in R)$$

denotes the additive inverse operation.

In view of Exercise 107, the equivalence class of zero, $[0]_\eta$, is a subgroup of the additive group $(R, 0, -, +)$.

5.4.2 Ideals

We say that a subgroup J of the additive group of a ring is a *right*, respectly, a *right ideal*, if

$$\forall_{r \in R}\; Jr \subseteq J, \qquad \text{respectly,} \qquad rJ \subseteq J.$$

When both conditions hold, we say that J is a *twosided ideal*.

Exercise 113 *Show that $[0]_\eta$ is a twosided ideal in R.*

5.4.3

A congruence equivalence η on a ring $(R, 0, -, +, \cdot)$ is, in particular, a congruence equivalence on its additive group. Hence, η is equipotent to the relation

$$\sim_J : R, R \longrightarrow \text{Statements}, \qquad r, s \longmapsto r \sim_J s := \text{“ } r - s \in J \text{ ”}, \tag{307}$$

where $J = [0]_\eta$.

Exercise 114 *Show that $\sim_J$ is a congruence equivalence on a ring $(R, 0, -, +, \cdot)$ if and only if J is a twosided ideal in R.*

5.4.4 The quotient ring R/J

When J is a twosided ideal the quotient ring $R_{/\sim_J}$ is denoted R/J.

5.4.5 The kernel of a ring homomorphism

Exercise 115 *Show that the fiber equivalence $=_f$ of a ring homomorphism $f : G \longrightarrow G'$ is a ring congruence.*

The equivalence class $[e]_{=_f}$ of the identity element in R is, accordingly, a twosided ideal. It is called the *kernel of f* and is denoted $\operatorname{Ker} f$.

5.4.6 The canonical factorization of a ring homomorphism $f : R \to R'$

For ring homomorphisms, Lemma 5.1 acquires the following form.

Lemma 5.4 *Every ring homomorphism $f : G \to R'$ is represented as the composite*

$$f = \iota_f \circ \tilde{f} \circ [\ ]_{=_f} \tag{308}$$

for a unique isomorphism $\tilde{f} : R/\operatorname{Ker} f \xrightarrow{\sim} \operatorname{Im} f$.

The assertion of Lemma 5.3 is illustrated by the following commutative square diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, twoheadrightarrow, "{[\ ]_{=_f}}"'] & R' \\ R/\operatorname{Ker} f \arrow[r, "\sim", "\tilde{f}"'] & \operatorname{Im} f \arrow[u, hook, "\iota_f"'] \end{tikzcd} \tag{309}$$

5.4.7 The lattice of ideals $\Lambda(R)$

In what follows by an “ideal” we shall mean a *twosided* ideal. Denote by $\Lambda(R)$ the subset of $\mathscr{P}R$ consisting of all ideals in a ring R.

Exercise 116 *Show that intersection*

$$\bigcap \mathcal{J}$$

of any family of ideals $\mathcal{J} \subseteq \Lambda(R)$ is an ideal.

5.4.8 The sum of a family of ideals $\sum \mathcal{J}$

Given a family $\mathcal{J} \subseteq \Lambda(R)$ of ideals, let us denote by $\sum \mathcal{J}$ be the *additive* span of the union of $\mathcal{J}$,

$$\bigcup \mathcal{J}.$$

Elements of $\sum \mathcal{J}$ are represented, not necessarily uniquely, as finite sums

$$\sum_{j \in \mathcal{J}'} x_j \qquad (\mathcal{J}' \text{ an arbitrray finite subfamily of } \mathcal{J}).$$

Exercise 117 *Show that $\sum \mathcal{J}$ is an ideal.*

By construction, $\sum \mathcal{J}$ is the smallest ideal that contains every member of $\mathcal{J}$.

5.4.9

A general construction that we have encountered several times before, that employs intersection of the family of ideals containing every member of family $\mathcal{J}$

$$\{I \in \Lambda(R) \mid \forall_{J \in \mathcal{J}}\; I \supseteq J\},$$

guarantees existence of the smallest ideal that contains all members of $\mathcal{J}$.

We just encountered the situation when this smallest *upper bound* of $\mathcal{J}$ in the ordered set $\Lambda(R)$ admits a simple explicit description.

5.4.10 Funtorial properties of $\Lambda(R)$

Let $f : R \longrightarrow R'$ be a ring homomorphism.

Exercise 118 *Show that for any ideal J' in R', the preimage $f^* J'$ is an ideal in R.*

Exercise 119 *Show that if f is surjective, then the direct image $f_* J$ of any ideal J in R is an ideal in R'.*

Exercise 120 *Show that if f is surjective, then f_* induces a bijection*

$$\{J \in \Lambda(R) \mid J \supset \operatorname{Ker} f\} \longleftrightarrow \Lambda(R'). \tag{310}$$

5.5 The K-group construction

5.5.1 Stable equivalence of formal differences

Let $(A, +)$ be a commutative semigroup. Consider the Cartesian square $A \times A$ of A equipped with the relation

$$\rho\big((a,b),(a',b')\big) \;:=\; \text{“}\; a + b' = a' + b \;\text{”} \tag{311}$$

and the componentwise addition operation

$$(a,b) + (a',b') := (a + a', b + b'). \tag{312}$$

We think of pairs (a,b) as *formal* differences $a - b$. Relation ρ describes formal differences that would represent the same element of A if such an element existed.

Exercise 121 *Let $a, b, c, d, a', b' \in A$. Show that*

$$\rho\left((a,b),(a',b')\right) \Rightarrow \rho\left((a,b)+(c,d)\,,\,(a',b')+(c,d)\right).$$

It follows that relation ρ is a $+$-congruence. It is reflexive and symmetric. It is *not* transitive, however, unless operation $+$ is *cancellative*, cf. Section (2.2.12).

5.5.2 Stabilization

Consider a relation slightly stronger than ρ,

$$\hat{\rho}\left((a,b),(a',b')\right) := \text{“}\ \exists_{c\in A}\ a+b'+c = a'+b+c\ \text{”}. \tag{313}$$

Relation $\hat{\rho}$ is obtained by replacing the equality relation $=$ in the definition of ρ by the stronger *stable equality* relation

$$x =^{\mathrm{st}} y := \text{“}\ \exists_{z\in A}\ x+z = y+z\ \text{”}. \tag{314}$$

Mathematicians refer to the operation on $\mathrm{Rel}_2 A$,

$$\rho \longmapsto \hat{\rho},$$

as *stabilization.*

Exercise 122 *Show that $\hat{\rho}$ is transitive.*

In particular, $\hat{\rho}$ is an equivalence relation stronger than ρ. If η is any equivalence relation stronger than ρ, then

$$\rho\circ\rho \Rightarrow \eta\circ\eta \Rightarrow \eta. \tag{315}$$

Exercise 123 *Show that $\hat{\rho} \Rightarrow \rho\circ\rho$.*

By combining this with (315), we deduce that $\rho\circ\rho$ and $\hat{\rho}$ are equipotent. In particular, $\rho\circ\rho$ is an equivalence relation even when ρ is not, and both $\rho\circ\rho$ and $\hat{\rho}$ are weakest equivalence relations stronger than ρ.

5.5.3 The K-group of a commutative semigroup

Denote by $K(A,+)$ the quotient set $(A\times A)_{/\rho}$ and let $(K(A,+),+)$ be the corresponding quotient semigroup. To simplify notation, we shall denote the class $[(a,b)]_\rho$ by $[a-b]$. We shall asume below that $A \neq \emptyset$.

Exercise 124 *Show that, for any $c \in A$, class $[c-c]$ is an identity element in $K(A,+)$.*

Exercise 125 *Show that, for any $a, b \in A$, class $[a-b]$ has an inverse in $K(A,+)$.*

It follows that $K(A,+)$ is, in fact, an abelian group.

5.5.4 Canonical homomorphism $\partial : (A,+) \longrightarrow (K(A,+),+)$

Let $c \in A$ be an arbitrary element of A. Let

$$\partial : A \longrightarrow K(A,+)\,, \qquad a \;\longmapsto\; [(a+c)-c] \tag{316}$$

Exercise 126 *Show that*

$$\forall_{a,c,d}\; [(a+c)-c] \;=\; [(a+d)-d]\,.$$

In particular, ∂ does not depend on the choice of an element $c \in A$, hence is canonically associated with semigroup $(A,+)$.

Exercise 127 *Show that* $\partial : (A,+) \longrightarrow (K(A,+),+)$ *is a homomorphism.*

5.5.5 Universal property of $\partial : (A,+) \longrightarrow (K(A,+),+)$

By construction, one has the following identity in $K(A,+)$,

$$[a-b] + \partial(b) = [a-b] + [(b+c)-c] = [(a+b+c)-(b+c)] = \partial(a)\,. \tag{317}$$

It follows that any element of group $(A,+)$ is the sum of an element of $\partial_* A$ and of the inverse of an element of $\partial_* A$. In other words, the image of homomorphism ∂ generates $K(A,+)$ and therefore any group homomorphism $\phi : K(A,+) \longrightarrow G$ is uniquely determined by the composite $\phi \circ \partial$.

Lemma 5.5 *Given a group* $(G, e, (\ )^{-1}, \cdot\,)$, *precomposition with* ∂ *defines*

$$\partial^\bullet : \phi \;\longmapsto\; f := \varphi \circ \partial \tag{318}$$

defines a bijection between the set of group homomorphisms

$$\phi : K(A,+) \longrightarrow G$$

and the set of semigroup homomorphisms

$$f : (A,+) \longrightarrow (G,\cdot)\,.$$

Proof. Injectivity of (318) is a consequence of group $K(A,+)$ being generated by the image of ∂. Let

$$\varphi : A{\times}A \longrightarrow G\,, \qquad (a,b) \;\longmapsto\; f(a)f(b)^{-1} \tag{319}$$

Exercise 128 *Show that* $\varphi : (A{\times}A,+) \longrightarrow (G,\cdot\,)$ *is a semigroup homomorphism.*

Exercise 129 *Show that*

$$\varphi(a,b) = \varphi(a',b')$$

whenever

$$\rho\left((a,b),(a',b')\right)\,.$$

It follows that φ induces a homomorphism from the quotient semigroup $(K(A,+)$ to semigroup $(G,\cdot\,)$.

Exercise 130 *Suppose that* $(G, e, (\ )^{-1}, \cdot)$ *and* $(G', e', (\ )^{-1}, \cdot)$ *are groups. Show that, for any semigroup homomorphism*

$$f : (G', \cdot) \longrightarrow (G, \cdot),$$

one has

$$f(e') = e \qquad \text{and} \qquad \forall_{x\in G'}\, f(x^{-1}) = f(x)^{-1}.$$

In other words, a semigroup homomorphism between groups is automatically a group homomorphism.

The proof of Lemma 5.5 is complete.

5.5.6

Suppose that a secondary binary operation $\cdot$ on A is given. Let us define the following operation on $A{\times}A$,

$$(a, b) \cdot (c, d) \ := \ (ac + bd, ad + bc). \tag{320}$$

Exercise 131 *Show that operation* (320) *is associative if operation* $\cdot$ *on* A *is associative.*

Exercise 132 *Show that operation* (320) *left-distributes over addition on* $A{\times}A$ *if operation* $\cdot$ *on* A *left-distributes over addition on* A.

Exercise 133 *Show that* $(1, 0)$ *is the identity element for operation* (320) *if* 1 *is an identity element for* $\cdot$ *on* A *and* $(A, +)$ *is a cancellative semigroup, cf. Section* 2.2.12.

Exercise 134 *Show that*

$$\forall_{a,b,c,d,a',b'\in A}\ \rho\big((a, b), (a', b')\big) \ \Rightarrow\ \rho\big(\,(a, b)\cdot(c, d)\,,\ (a', b')\cdot(c, d)\,\big)$$

if operation $\cdot$ *on* A *right-distributes over operation* $+$ *on* A.

5.5.7

It follows that $K(A, +, \cdot)$ for a semiring is automatically equipped with the structure of a ring.

5.5.8

If $(A, 1, +, \cdot)$ is a unital semiring, then $[(1 + c) - c]$ is a multiplicative identity in $K(A, +, \cdot)$.

5.5.9 The ring of integers $(\mathbf{Z}, 0, 1, +, \cdot)$

The ring of integers $(\mathbf{Z}, 0, 1, +, \cdot)$ is defined as

$$K(\mathbf{N}, 0, 1, +, \cdot).$$

We identify elements $[n - 0] \in \mathbf{Z}$ with $n \in \mathbf{N}$ and we denote elements $[0 - n] \in \mathbf{Z}$ by $-n$.

Euclid's Lemma 3.1 extends to the set of integers.

Lemma 5.6 *For every $m \in \mathbf{Z}$ and $n \in \mathbf{N}\setminus\{0\}$, there exist unique $q \in \mathbf{Z}$ and $r \in \mathbf{N}$ such that*

$$m = qn + r \qquad \textit{and} \qquad 0 \leq r < n\,. \tag{321}$$

We shall still refer to Lemma 5.6 as Euclid's Lemma even though neither Euclid, nor other mathematicians of Classical Period considered *negative* numbers.

Proof. Apply Lemma 3.1 to $-m$ if $m < 0$,

$$-m = qn + r\,.$$

If $r = 0$, then

$$m = (-q)n$$

is the desired representation. If $0 < r < n$, then

$$m = (-1 - q)n + (n - r)$$

is such a representation.

If

$$m = q'n + r' \qquad \text{and} \qquad 0 \leq r' < n$$

is another representation and $q \leq q'$, then

$$0 \leq (q' - q)n = r - r' \leq r < n$$

and, therefore,

$$0 \leq q' - q < 1\,,$$

in view of the multiplicative semigroup $(\mathbf{N}\setminus\{0\},\cdot)$ being cancellative, cf. Section 3.3.10. It follows that $q' - q = 0$ and, therefore also $r - r' = 0$. □

5.5.10 Subgroups of the additive group of integers

As a corollary of Lemma 5.6 we obtain a complete description of additive subgroups $A \subseteq \mathbf{Z}$. It contains If A is not the trivial subgroup $\{0\}$, then A contains $a \neq 0$ and then either a or $-a$ belong to $\mathbf{N}\setminus\{0\}$. Thus, $A \cap (\mathbf{N}\setminus\{0\})$ is a nonempty subset of $\mathbf{N}$ and, therefore, it has the smallest element, say n.

Let $m \in A$. In representation (321) both m and qn belong to A, hence $r \in A$. Since $0 \leq r < n$ and n is the smallest positive element of A, it follows that $r = 0$, i.e., $A = (n)$.

Corollary 5.7 *Every subgroup A of the additive group of integers has the form*

$$(n) := \mathbf{Z}n = \left\{m \in \mathbf{Z} \mid \exists_{q \in \mathbf{Z}}\, m = qn\right\} \tag{322}$$

for a unique $n \in \mathbf{N}$.

□

Note that every subgroup A of $\mathbf{Z}$ is canonically isomorphic to $\mathbf{Z}$,

$$\mathbf{Z} \xrightarrow[\sim]{\times n} (n)\ . \tag{323}$$

5.5.11 The greatest common divisor of a subset $E \subseteq \mathbf{Z}$

Given a set of integers $E \subseteq \mathbf{Z}$, let (E) denote the additive subgroup of $\mathbf{Z}$ generated by E. The natural number $n \in \mathbf{N}$ such that

$$(E) = (n)$$

is called the *greatest common divisor* of E and is denoted $\gcd E$.

5.5.12 The least common multiple of a subset $E \subseteq \mathbf{Z}$

The least common multiple of a set of integers $E \subseteq \mathbf{Z}$ is the natural number $n \in \mathbf{N}$ such that

$$\bigcap_{m \in E} (m) = (n) .$$

We denote it $\operatorname{lcm} E$.

5.5.13 The lattice of ideals of the ring of integers $(\mathbf{Z}, 0, 1, +, \cdot)$

Note that every additive subgroup of $\mathbf{Z}$ is closed under multiplication by integers, hence is an ideal in the ring of integers. As an ordered set $(\Lambda(\mathbf{Z}), \subseteq)$ is isomorphic to the set of natural numbers $(\mathbf{N}, |^{\mathrm{op}})$ equipped with the *opposite* divisor relation:

$$\forall_{m,n \in \mathbf{N}} \ (m) \subseteq (n) \Leftrightarrow n | m . \tag{324}$$

5.6 (Central) *localization* of a ring

5.6.1 Central multiplicative subsets in a ring

Let S be a multiplicatively closed subset of a ring $(R, 0, -, +, \cdot)$. We call it *central* if S is contained in the *center* of R,

$$Z(R) := \{z \in R \mid \forall_{r \in R} \ rz = zr\} \tag{325}$$

5.6.2 S-fractions

We shall represent elemenets of $R \times S$ as *formal* fractions

$$\frac{r}{s} \qquad (r \in R, s \in S) .$$

5.6.3 Calculus of S-fractions

Multiplication of fractions is defined to be the product operation on $R \times S$,

$$\frac{r}{s} \cdot \frac{r'}{s'} := \frac{rr'}{ss'} . \tag{326}$$

Addition of fractions is defined by the familiar formula

$$\frac{r}{s} + \frac{r'}{s'} := \frac{rs' + sr'}{ss'} . \tag{327}$$

5.6.4 The algebraic structure $\mathrm{Frac}_S(R)$ of S-fractions

We shall denote by $\mathrm{Frac}_S(R)$ the algebraic structure

$$(R{\times}S, +, \cdot\,)\,. \tag{328}$$

Associativity of multiplication of fractions is an immediate consequence of associativity of multiplication in ring R.

5.6.5

Addition of fractions is not associative, in general. Neither is multiplication right- or left-distributive over addition.

5.6.6

If R has a multiplicative identity element 1, then the fraction

$$\frac{1}{1}$$

is a multiplicative identity element in $\mathrm{Frac}_S(R)$, the fraction

$$\frac{0}{1}$$

is an additive identity element, and

$$\frac{r}{1} + \frac{-r}{1} = \frac{0}{1}\,,$$

i.e., fractions with denominator 1 are additively invertible. On the other hand, fractions with denominator $\neq 1$ are not additively invertible.

5.6.7

Consider the multiplicative version of relation (311)

$$\rho\left(\frac{r}{s}, \frac{r'}{s'}\right) \;:=\; \text{“}\; rs' = sr' \;\text{”}\,. \tag{329}$$

Exercise 135 *Let $r_1, r_1', r_2 \in R$ and $s_1, s_1', s_2 \in S$. Show that*

$$\rho\left(\frac{r_1}{s_1}, \frac{r_1'}{s_1'}\right) \;\Rightarrow\; \rho\left(\frac{r_1}{s_1}\cdot\frac{r_2}{s_2}, \frac{r_1'}{s_1'}\cdot\frac{r_2}{s_2}\right).$$

Exercise 136 *Let $r_1, r_1', r_2 \in R$ and $s_1, s_1', s_2 \in S$. Show that*

$$\rho\left(\frac{r_1}{s_1}, \frac{r_1'}{s_1'}\right) \;\Rightarrow\; \rho\left(\frac{r_1}{s_1}+\frac{r_2}{s_2}, \frac{r_1'}{s_1'}+\frac{r_2}{s_2}\right).$$

It follows that relation ρ is a congruence for both multiplication and addition of S-fractions. Relation ρ is reflexive and symmetric. It is *not* transitive, however, if the operations of multiplication by elements of S are not injective.

5.6.8 Multiplicative stabilization

Consider a relation slightly stronger than ρ,

$$\rho^S\left(\frac{r}{s}, \frac{r'}{s'}\right) := \text{“ } \exists_{t\in S}\ rs't = sr't \text{ ”}. \tag{330}$$

Relation ρ^S is obtained by replacing the equality relation $=$ in the definition of ρ by the stronger *multiplicatively stable equality* relation

$$x =^S y := \text{“ } \exists_{t\in S}\ xs = ys \text{ ”}. \tag{331}$$

Exercise 137 *Show that ρ^S is transitive.*

In particular, ρ^S is an equivalence relation stronger than ρ.

Exercise 138 *Show that $\rho^S \Rightarrow \rho \circ \rho$.*

By combining this with (315), we deduce that $\rho \circ \rho$ and ρ^S are equipotent. In particular, $\rho \circ \rho$ is an equivalence relation even when ρ is not, and both $\rho \circ \rho$ and ρ^S are weakest equivalence relations stronger than ρ.

5.6.9 $R[S^{-1}]$

Let $R[S^{-1}]$ be the quotient by ρ of the algebraic structure $\mathrm{Frac}_S(R)$ of S-fractions.

Exercise 139 *Show that*

$$\left[\frac{0}{s}\right]$$

is an additive identity element of $R[S^{-1}]$ and

$$\left[\frac{s}{s}\right]$$

is a multiplicative identity..

Exercise 140 *Show that addition in $R[S^{-1}]$ is associative.*

5.6.10

Since

$$\frac{r}{s} + \frac{-r}{s} = \frac{0}{s},$$

every element of $R[S^{-1}]$ is additively invertible and the additive structure of $R[S^{-1}]$ is an abelian group.

Exercise 141 *Show that multiplication distributes over addition in $R[S^{-1}]$.*

In other words, $R[S^{-1}]$ is a unital ring. It is often called, not entirely correctly, the *ring of S-fractions*. We shall refer to it as the *S-localization* of R.

Exercise 142 *Describe $R[S^{-1}]$ when $S \ni 0$.*

5.6.11 Canonical homomorphism $\lambda : R \longrightarrow R[S^{-1}]$

Let $s \in S$ be an arbitrary element of S. Let

$$\lambda : R \longrightarrow R[S^{-1}], \qquad r \longmapsto \left[\frac{rs}{s}\right]. \tag{332}$$

Exercise 143 *Show that*

$$\forall_{r\in R} \forall_{s,t\in S} \left[\frac{rs}{s}\right] = \left[\frac{rt}{t}\right].$$

In particular, λ does not depend on the choice of an element $s \in S$, hence is canonically associated with ring R and multiplicative subsemigroup S of $(R, \cdot)$.

Exercise 144 *Show that $\lambda : R \longrightarrow R[S^{-1}]$ is a ring homomorphism.*

5.6.12 Universal property of $\lambda : R \longrightarrow R[S^{-1}]$

We say that a homomorphism $f : R \longrightarrow R'$ into a unital ring R' *inverts* elements of S if

$$\forall_{s\in S} \ f(s) \text{ is invertible in } R'.$$

Lemma 5.8 *Given a unital ring R', precomposition with λ defines*

$$\lambda^{\bullet} : \phi \longmapsto f := \varphi \circ \lambda \tag{333}$$

defines a bijection between the set of ring homomorphisms

$$\phi : R[S^{-1}] \longrightarrow R'$$

and the set of ring homomorphisms

$$f : R \longrightarrow R'$$

that invert elements of S.

5.6.13 Zero divisors

We say that a nonzero element $r \in R$ is a *right zero divisior* if there exists a nonzero element $q \in R$ such that

$$qr = 0.$$

In this case q is said to be a *left zero divisior*.

5.6.14 Domains

A ring without zero divisors is called by ring theorists a *domain*. Commutative rings with identity and without zero divisiors ae also called *integral domains*.

5.7 Commutative Algebra

5.7.1

Theory of commutative unital rings developed into a separate area of Mathematics situated between Algebra and Geometry. It is called "Commutative Algebra" and has been one of the vital fields of research over the last century.

5.7.2 Notation: $\mathfrak{a}, \mathfrak{b}, \ldots$

Let A denote a commutative unital ring. In Commutative Algebra ideals of A are traditionally denoted by lower case Fraktur letters $\mathfrak{a}, \mathfrak{b}, \ldots$.

5.7.3 Notation: $(a_1, \ldots, a_n)$

The smallest ideal in A containing a subset E is denoted (E). We write

$$(a_1, \ldots, a_n) \tag{334}$$

when $E = \{a_1, \ldots, a_n\}$.

Exercise 145 *Show that*

$$a \in (a_1, \ldots, a_n) \quad \textit{if and only if} \quad \exists_{c_1,\ldots,c_n \in A} \quad a = c_1 a_1 + \cdots + c_n a_n\,. \tag{335}$$

5.7.4 Principal ideals

Ideals generated by a single element are called *principal*.

5.7.5 Principal ideal domains

Integral domains with all ideals being principal are called *principal ideal domains*. The ring of integers $\mathbf{Z}$ is an example, cf. Section 5.5.13.

5.7.6 Prime ideals

A proper ideal $\mathfrak{p} \subsetneq A$ is said to be *prime* if its complement $\complement\mathfrak{p}$ is closed under multiplication.

Exercise 146 *Show that $\mathfrak{p}$ is prime if and only if $A/\mathfrak{p}$ is an integral domain.*

Solution. Note that $\mathfrak{p} = [0]$ is the zero element in the quotient ring $A/\mathfrak{p}$ and

$$[a] = [0] = \mathfrak{p} \iff ab \in \mathfrak{p}\,.$$

Accordingly, $\mathfrak{p}$ being a prime ideal is equivalent to

$$\forall_{a,b \in A} \quad [ab]{=}\mathfrak{p} \iff [a]{=}\mathfrak{p} \vee [b]{=}\mathfrak{p}\,.$$

□

5.7.7 Localization of A at a prime ideal $\mathfrak{p}$

The ring $A[(\complement\mathfrak{p})^{-1}]$ is called the *localization of A at a prime ideal* $\mathfrak{p}$ and is denoted $A_{\mathfrak{p}}$.

5.7.8

In particular, a commutative unital ring A is an integral domain precisely when the zero ideal (0) is prime.

5.7.9 The *field of fractions* of an integral domain

If A is an integral domain, then the localization of A at the zero ideal $A_{(0)}$ is a field. This field is called, not entirely correctly, the *field of fractions*, or the *field of quotients*, of integral domain A.

5.7.10 The prime spectrum $\operatorname{Spec} A$ of a commutative unital ring A

The set of *proper* prime ideals of A is called the *prime spectrum of A* and is denoted

$$\operatorname{Spec} A .$$

5.7.11 Functorial properties of prime ideals

Let $f : A \to A'$ be a homomorphism of commutative unital rings and $\mathfrak{p}'$ be a prime ideal in A'.

Exercise 147 *Show that $f^*\mathfrak{p}'$ is a prime ideal in A.*

Assignment

$$A \longmapsto \operatorname{Spec} A \tag{336}$$

is *functorial*. This means that every homomorphism of commutative unital rings $f : A \to A'$ induces a canonical function

$$\operatorname{Spec} A \xleftarrow{\ f^*\ } \operatorname{Spec} A' \quad . \tag{337}$$

Since assignment $f \longmapsto f^*$ reverses the source and the target, we say that the double assignment

$$A \longmapsto \operatorname{Spec} A , \qquad f \longmapsto f^* , \tag{338}$$

is a ***contravariant*** *functor* from the category of commutative unital rings to the category of ordered sets.

5.7.12 *Zariski closed* subsets $V(E) \subseteq \operatorname{Spec} A$

Let us consider the function

$$V : \mathscr{P}A \longrightarrow \mathscr{P}(\operatorname{Spec} A) , \qquad E \longmapsto V(E) := \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq E\} , \tag{339}$$

that assigns to a subset $E \subseteq A$, the set of prime ideals containg E. It is a morphism of ordered sets

$$V : (\mathscr{P}A, \subseteq) \longrightarrow \big(\mathscr{P}(\operatorname{Spec} A), \supseteq\big) . \tag{340}$$

Exercise 148 *Show that, for any family of subsets $\mathscr{E} \subseteq \mathscr{P}A$, one has*

$$V\big(\bigcup \mathscr{E}\big) = \bigcap_{E \in \mathscr{E}} V(E) = \bigcap V_*\mathscr{E} . \tag{341}$$

5.7.13

In the language of ordered sets Identity (341) reads: function (340) is a sup *-exact morphism* of ordered sets, cf. Section 6.11.3.

5.7.14

When $E = \{a\}$ is a singleton set, we denote $V(E)$ by $V(a)$. Note that

$$V(0) = \operatorname{Spec} A \qquad \text{and} \qquad V(1) = \emptyset. \tag{342}$$

Exercise 149 *Show that, for any $a, a' \in A$, one has*

$$V(a) \cup V(a') = V(aa'). \tag{343}$$

Lemma 5.9 *For any subsets $E, E' \subsetneq A$, one has*

$$V(E) \cup V(E') = V(EE') \tag{344}$$

where $EE' := \{b \in A \mid \exists_{a \in E} \exists_{a' \in E'}\ b = aa'\}$.

Proof. Straightforward calculation employing distributivity of $\bigcap$ over $\cup$ yields

$$\begin{aligned}
V(E) \cup V(E') &= V\left(\bigcup_{a \in E} \{a\}\right) \cup V\left(\bigcup_{a' \in E'} \{a'\}\right) \\
&= \left(\bigcap_{a \in E} V(a)\right) \cup \left(\bigcap_{a' \in E'} V(a')\right) \\
&= \bigcap_{\substack{a \in E \\ a' \in E'}} V(a) \cup V(a') \\
&= \bigcap_{\substack{a \in E \\ a' \in E'}} V(aa') \\
&= V(EE').
\end{aligned}$$

In the third equality we used Exercise 149. □

5.7.15 Zariski topology

Denote by $\mathcal{Z}$ the image of morphism V, cf. (339) or (340),

$$\mathcal{Z} := V_* \mathcal{P} A \subseteq \mathcal{P}(\operatorname{Spec} A). \tag{345}$$

In view of Exercise 148, family $\mathcal{Z}$ is closed under arbitrary intersections. In view of Lemma 5.9 and the second formula in (342), family $\mathcal{Z}$ is closed under finite unions.

Such a family of subsets has the meaning of the family of all *closed subsets* in a *topological space*, cf. Section 7.1.18. This topological structure on the prime spectrum of a commutative unital ring was introduced by Alexander Grothendieck who named it *Zariski topology*.

Exercise 150 *Let $f : A \to A'$ be a homomorphism of unital rings. Show that, for any subset $E \subseteq A$, one has*

$$f^{**}V(E) = V(f_*E)\,. \tag{346}$$

Identity (346) expresses commutativity of the square diagram

$$\begin{tikzcd} \mathscr{P}(\operatorname{Spec} A) \arrow[r, "f^{**}"] & \mathscr{P}(\operatorname{Spec} A') \\ \mathscr{P}A \arrow[u, "V"] \arrow[r, "f_*"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathscr{P}A' \arrow[u, "V"'] \end{tikzcd} \tag{347}$$

Solution. One has

$$\begin{aligned} f^{**}V(E) &= \{\mathfrak{p}' \in \operatorname{Spec} A' \mid f^*\mathfrak{p}' \in V(E)\} \\ &= \{\mathfrak{p}' \in \operatorname{Spec} A' \mid f^*\mathfrak{p}' \supseteq E\} \\ &= \{\mathfrak{p}' \in \operatorname{Spec} A' \mid \mathfrak{p}' \supseteq f_*E\} \\ &= V(f_*E)\,. \end{aligned}$$

In the second equality we used the fact that f_*, f^* form a *Galois connection*, cf. Exercise 26. $\square$

It follows that the preimage of any Zariski closed subset of $\operatorname{Spec} A$ by f^* is a Zariski closed subset of $\operatorname{Spec} A'$. In Topology this property of a function is known as *continuity*.

Double assignment (338) thus becomes a functor from the category of commutative unital rings to the category of topological spaces.

5.7.16 Spec $\mathbf{Z}$

Every prime ideal in the ring of integers $\mathbf{Z}$ is the principal ideal (n) generated by 0 or by a prime natural number $p \in \mathbf{N}$.

A proper subset $Z \subsetneq \operatorname{Spec}\mathbf{Z}$ is Zariski closed if and only if it is a finite subset not containing (0). On the other hand, $\operatorname{Spec}\mathbf{Z}$ is the smallest Zariski closed subset containing (0).

6 Binary relations

6.1 Preliminaries

6.1.1 Canonical identification $\mathbf{Rel}(X,Y) \longleftrightarrow \mathbf{Rel}(Y,X)$

Given a binary relation
$$\rho : X, Y \longrightarrow \text{Statements}, \tag{348}$$
the *opposite* relation is defined by flipping the two arguments
$$\rho^{\mathrm{op}} : Y, X \longrightarrow \text{Statements}, \qquad y, x \longmapsto \rho(x,y). \tag{349}$$
Note that $(\rho^{\mathrm{op}})^{\mathrm{op}} = \rho$. In particular, assignment
$$\rho \longmapsto \rho^{\mathrm{op}}$$
defines a canonical identification of the sets of binary relations
$$\mathrm{Rel}(X,Y) \longleftrightarrow \mathrm{Rel}(Y,X).$$

6.1.2 A canonical involution on $\mathbf{Rel}_2(X)$

When $X = Y$, operation $(\ )^{\mathrm{op}}$ is a (canonical) involution on the set of binary relations on X.

6.2 Morphisms between binary relations

6.2.1 The definition expressed in terms of a $\Longrightarrow$-commutative diagram

The concept of a morphism between n-ary relations was subjected to a thorough analysis in Section 3.4. That analysis led us to focus on three types of morphisms: $\Rightarrow$-morphisms, $\Leftarrow$-morphisms, and $\Leftrightarrow$-morphisms.

For $n = 2$, in all three cases a morphism from a binary relation
$$\rho : X, Y \longrightarrow \text{Statements}$$
to a binary relation
$$\rho' : X', Y' \longrightarrow \text{Statements}$$
consists of a pair of functions $f : X \to X'$ and $g : Y \to Y'$ such that
$$\rho \Longrightarrow (f,g)^{\bullet}\rho', \qquad \rho \Longleftarrow (f,g)^{\bullet}\rho', \qquad \text{and, respectively,} \qquad \rho \Longleftrightarrow (f,g)^{\bullet}\rho'.$$
This triple definition of a morphism can be expresed by means of the corresponding diagrams

$$(\Rightarrow\text{-}\mathbf{morphism}) \qquad \begin{tikzcd} X', Y' \arrow[dr, "\rho'"] & \\ & \text{Statements} \\ X, Y \arrow[uu, "f", shift left] \arrow[uu, "g"', shift right] \arrow[ur, "\rho"'] \end{tikzcd} \tag{350}$$

$$
(\Leftarrow\text{-}\textbf{morphism}) \qquad
\begin{tikzcd}
X', Y' \arrow[dr, "\rho'"] & \\
\arrow[r, phantom, "\Downarrow"] & \text{Statements} \\
X, Y \arrow[uu, shift right, "f"] \arrow[uu, shift left, "g"'] \arrow[ur, "\rho"'] &
\end{tikzcd}
\tag{351}
$$

$$
(\Leftrightarrow\text{-}\textbf{morphism}) \qquad
\begin{tikzcd}
X', Y' \arrow[dr, "\rho'"] & \\
\arrow[r, phantom, "\Updownarrow"] & \text{Statements} \\
X, Y \arrow[uu, shift right, "f"] \arrow[uu, shift left, "g"'] \arrow[ur, "\rho"'] &
\end{tikzcd}
\tag{352}
$$

6.2.2

Below we shall follow the common practice, explained in Section 3.4.11, of refering to $\Rightarrow$-morphisms simply as *morphisms*, and to $\Leftrightarrow$-morphims as *strict morphisms*.

6.2.3 Morphisms between binary relational structures

When $X = Y$, $X' = Y'$ and $f = g$, we obtain the corresponding definitions of a $\sim$-morphism, were $\sim$ is $\Rightarrow$, $\Leftarrow$, or $\Leftrightarrow$, from a binary relational structure (X, ρ) to a binary relational structure (X, ρ'), cf. Section 3.4.13.

6.3 Membership relation $\in$ is a *universal relation*

6.3.1

Membership relation

$$
\in_S \,:\, S, \mathscr{P}S \longrightarrow \text{Statements}, \qquad s, E \longmapsto \text{“ } s \in E \text{ ”}, \tag{353}
$$

associated with an arbitrary set S, and its opposite relation $\ni_S$, are the *only* relations whose existence is guaranteed by axioms of Set Theory. In other words, $\in_S$ is the only primitive relation of all of Mathematics. Remarkably, it is also a *universal* binary relation,

Theorem 6.1 (a) *For every binary relation ρ, there exists a set S and a $\Leftrightarrow$-morphism $\rho \to \in_S$,*

$$
\begin{tikzcd}
X, Y \arrow[r, "\rho"] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dr, phantom, "\Updownarrow"] & \text{Statements} \\
S, \mathscr{P}S \arrow[ur, "\in"'] &
\end{tikzcd}
, \tag{354}
$$

i.e., ρ is equipotent to a pull-back of membership relation on some set S.

(b) *There exists a $\Leftrightarrow$-morphism* (354) *with $S = X$ and $f = \mathrm{id}_X$, and such a morphism is unique. In particular, there exists a canonical bijective correspondence*

$$\left\{ \begin{array}{c} \textit{Equipotence classes of binary relations} \\ \rho : X, Y \longrightarrow \text{Statements} \end{array} \right\} \longleftrightarrow \mathrm{Funct}(Y, \mathscr{P}X) \tag{355}$$

which sends a function $g : Y \to \mathscr{P}X$ to the equipotence class of the pulled-back relation $(\mathrm{id}_X, g)^* \in$.

6.3.2

Part (a) of Theorem 6.1 is a consequence of Part (b). The pulled-back relation $(\mathrm{id}_X, g)^* \in$ has the form

$$x, y \longmapsto \text{“ } x \in g(y) \text{ ”}. \tag{356}$$

If $h : Y \to \mathscr{P}X$ is another function and $g \neq h$, then there exists $a \in X$ such that sets $g(a)$ and $h(a)$ differ,

$$g(a) \neq h(a).$$

It follows that there exists $x \in X$ such that

$$\text{either} \quad x \in g(a) \text{ and } x \notin h(a) \quad \text{or} \quad x \notin g(a) \text{ and } x \in h(a).$$

The same expressed symbolically,

$$\exists_{x \in X} \left(x \in g(a) \wedge x \notin h(a) \right) \vee \left(x \notin g(a) \wedge x \in h(a) \right).$$

In particular, the corresponding pulled-back relations are not equipotent.

6.3.3 Two *set-of-relatives* functions

Given a binary relation (348), there are two evaluation functions canonically associated with it

$$\mathrm{ev}^1\rho : X \longrightarrow \mathrm{Rel}_1(Y) \qquad \text{and} \qquad \mathrm{ev}^2\rho : Y \longrightarrow \mathrm{Rel}_1(X)$$

cf. (12). By composing them with the graph functions

$$\mathrm{Rel}_1(Y) \xrightarrow{\ \Gamma\ } \mathscr{P}Y \qquad \text{and} \qquad \mathrm{Rel}_1(X) \xrightarrow{\ \Gamma\ } \mathscr{P}X\,,$$

we obtain a pair of functions

$$X \longrightarrow \mathscr{P}Y, \qquad x \longmapsto [x\rangle_\rho := \{y \in Y \mid \rho(x, y)\}, \tag{357}$$

and, respectively,

$$Y \longrightarrow \mathscr{P}X, \qquad y \longmapsto {}_\rho\langle y] := \{x \in X \mid \rho(x, y)\}. \tag{358}$$

When the context allows that, we shall simplify notation by omitting the subscript denoting the relation. We shall refer to $[x\rangle$ as the set of *right relatives* of $x \in X$, and to $\langle y]$ as the set of *left relatives* of $y \in Y$.

Accordingly, we shall refer to (357) as the *right-relatives* function, and to (358) as the *left-relatives* function.

6.3.4

The pair of assignments

$$
\begin{tikzcd}
 & \rho \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
{[\;\rangle_{\rho}} & & {{}_{\rho}\langle\;]}
\end{tikzcd}
\tag{359}
$$

defines the pair of functions

$$
\begin{tikzcd}
 & \mathrm{Rel}(X,Y) \arrow[dl] \arrow[dr] & \\
\mathrm{Funct}(X,\mathscr{P}Y) & & \mathrm{Funct}(Y,\mathscr{P}X)
\end{tikzcd}
\tag{360}
$$

Exercise 151 *Show that*

$$
{}_{\rho^{\mathrm{op}}}\langle\;] = [\;\rangle_{\rho} \qquad \textit{and} \qquad [\;\rangle_{\rho^{\mathrm{op}}} = {}_{\rho}\langle\;].
$$

6.3.5

The pulled-back relation $\left(\mathrm{id}_X, {}_{\rho}\langle\;]\right)^{*}\in$,

$$
x, y \longmapsto \text{“ } x \in {}_{\rho}\langle y] \text{ ”},
$$

is, by the definition of ${}_{\rho}\langle y]$, equipotent to ρ. This completes a proof of Theorem 6.1.

6.3.6 The $\ni\rho\in$-diagram

Our proof exhibits a $\Leftrightarrow$-commutative diagram canonically associated with every binary relation ρ,

$$
\begin{tikzcd}
\mathscr{P}Y, Y \arrow[drr, "\ni"] & & \\
X\,,\,Y \arrow[u, "{[\;\rangle}"] \arrow[u, equal] \arrow[d, equal] \arrow[d, "{\langle\;]}"'] \arrow[rr, "\rho"] \arrow[urr, phantom, "\Updownarrow"] \arrow[drr, phantom, "\Updownarrow"'] & & \text{Statements} \\
X, \mathscr{P}X \arrow[urr, "\in"'] & &
\end{tikzcd}
\tag{361}
$$

whose deeper meaning will be revealed when we formulate the concept of a *Galois connection*, cf. Section 6.15.

6.3.7 An indispensible equipment for a modern mathematician: "algebraic glasses"

Let me pause to make a remark before resuming our excursion into the land of binary relations. Even the *nearest surroundings* of the concept of a binary relation reveal *omnipresence* of the central mechanism of Modern Mathematics, that of a *pair of adjoint functors*, cf. Section 1.1.15. That mechanism will accompany us all the way to the end of our inquiry, and will manifest itself both in *where* we get to and *how* we get there. This is one of the many miracles of Mathematics, perhaps its most important. In order to be able to see it one must, however, be wearing "algebraic glasses."

6.3.8 Terminology: initial and terminal elements

When $[x\rangle = Y$, we say that an element $x \in X$ is *initial* (for a given binary relation ρ). When $\langle y] = X$, we say that an element $y \in Y$ is *terminal.*

When ρ is an order relation on a set X, an initial element is unique when it exists. In Theory of Ordered Sets that unique initial element is said to be the *smallest element* of (X,ρ).

Similarly, a terminal element in an ordered set is unique when it exists. That unique element is said to be the *greatest element* of (X,ρ).

6.3.9 The preorders on X and Y canonically associated with $\rho \in \mathbf{Rel}(X,Y)$

We shall denote by $\gtrsim$ the relation $\subseteq$ pulled-back to X from $\mathscr{P}Y$ by $[\ \rangle$,

$$x \gtrsim x' \quad \text{if} \quad [x\rangle \subseteq [x'\rangle \qquad (x, x' \in X)\,. \tag{362}$$

We shall denote by $\lesssim$ the relation $\subseteq$ pulled-back to Y from $\mathscr{P}X$ by $\langle\ ]$,

$$y \lesssim y' \quad \text{if} \quad \langle y] \subseteq \langle y'] \qquad (y, y' \in Y)\,. \tag{363}$$

6.3.10

By construction, $\gtrsim$ is the *strongest* binary relation on X such that $[\ \rangle : (X,\lesssim) \to (\mathscr{P}Y,\subseteq)$ is a morphism. Similarly, $\lesssim$ is the *strongest* binary relation on Y such that $\langle\ ] : (Y,\lesssim) \to (\mathscr{P}X,\subseteq)$ is a morphism.

Exercise 152 *Consider the relation ρ_f associated with a function $f : X \to Y$ between arbitrary sets, cf.* (85). *Describe the relations $\gtrsim$ and $\lesssim$.*

6.3.11

Suppose $X = Y$. In that case all three relations, ρ, $\lesssim$ and $\gtrsim$, are members of the same set $\mathrm{Rel}_2(X)$ of binary relations on X, which is preordered by the $\Longrightarrow$ relation. This leads to the following natural questions that I am stating as exercises.

Exercise 153 *Characterize binary relations $\rho \in \mathrm{Rel}_2(X)$ such that $\rho \Longrightarrow \lesssim$.*[2]

Exercise 154 *Characterize binary relations $\rho \in \mathrm{Rel}_2(X)$ such that $\lesssim \Longrightarrow \rho$.*

Exercise 155 *State the analogs of the above two exercises for $\gtrsim$ instead of $\lesssim$.*

[2] *Characterize* means: 1° Find a property of ρ, that can be stated directly in terms of ρ and is as simple as possible, that holds precisely when $\rho \Longrightarrow \lesssim$; 2° then prove that.

6.4 Functions $\mathscr{P}X \leftrightarrows \mathscr{P}Y$ canonically associated with a binary relation.

6.4.1 $R : \mathscr{P}X \longrightarrow \mathscr{P}Y$

Given a subset $A \subseteq X$ we obtain a family $([x\rangle)_{x\in A}$ of subsets of Y whose intersection,

$$RA := \bigcap_{x\in A}[x\rangle = \{y \in Y \mid \forall_{x\in A}\, \rho(x,y)\}, \tag{364}$$

consists of those elements of Y that are right relatives of *every* element of A.

Exercise 156 *Given a family $\mathscr{A} \subseteq \mathscr{P}X$ of subsets of X, show that*

$$\bigcap R_*\mathscr{A} = R\left(\bigcup \mathscr{A}\right). \tag{365}$$

6.4.2 The set of upper bounds of a subset of an ordered set

In Theory of Ordered Sets, RA is called the *set of upper bounds of a subset* $A \subseteq X$. A subset $A \subseteq X$ is said to be *bounded above* when RA is not empty.

6.4.3 Right-directed binary relations

We say that ρ is *right-directed* if $RA \neq \emptyset$ for every *finite* subset $A \subseteq X$.

6.4.4 Directed sets

An ordered set whose order relation is right-directed is called a *directed set*. Directed sets are indispensable in all questions related to *convergence* in Topology and in Algebra.

6.4.5 Supremum of a subset

If $RA = [\alpha\rangle$, for some element $\alpha \in X$, we shall say that α is a *supremum* of a subset $A \subseteq X$. This term is yet another adoption from Theory of Ordered Sets. A supremum of A is unique when ρ is an order relation on X. In that case we denote it

$$\sup A. \tag{366}$$

For a general relation ρ, any two suprema[3] are $\gtrsim$-equivalent, cf. Section 6.3.9 and Exercise 9. One can still use notation (366) understanding, however, that the element of X denoted $\sup A$ is defined uniquely only up to a $\gtrsim$-equivalence.

6.4.6 $\mathscr{P}X \longleftarrow \mathscr{P}Y : L$

Given a subset $B \subseteq Y$ we obtain a family $(\langle y])_{y\in B}$ of subsets of X whose intersection,

$$LB := \bigcap_{y\in B}\langle y] = \{x \in X \mid \forall_{y\in B}\, \rho(x,y)\}, \tag{367}$$

consists of those elements of X that are left relatives of *every* element of B.

[3]The plural of the neuter noun *supremum* is *suprema*.

6.4.7 The set of lower bounds of a subset of an ordered set

In Theory of Ordered Sets, LB is called the *set of lower bounds of a subset* $B \subseteq Y$. A subset $B \subseteq Y$ is said to be *bounded below* when LB is not empty.

6.4.8 Left-directed binary relations

We say that ρ is *left-directed* if $LB \neq \emptyset$ for every *finite* subset $B \subseteq Y$.

6.4.9 Infimum of a subset

If $LB = \langle \beta]$, for some element $\beta \in Y$, we shall say that β is an *infimum* of a subset $B \subseteq Y$. That element is unique when ρ is an order relation on Y. In that case we denote it

$$\inf B\,. \tag{368}$$

For a general relation ρ, any two infima are $\precsim$-equivalent and the use of notation (368) is subject to the same caveat as in the case of $\sup A$.

6.4.10 $R\emptyset = Y$ and $L\emptyset = X$

Note that $R\emptyset = Y$ and $L\emptyset = X$. In particular, $\emptyset \subseteq X$ is bounded above precisely when X is not empty. Suprema of $\emptyset \subseteq X$ are precisely initial elements of X, cf. Section 6.3.8.

Similarly, $\emptyset \subseteq Y$ is bounded below precisely when Y is not empty. Infima of $\emptyset \subseteq Y$ are precisely terminal elements of Y.

Exercise 157 (a) *Show that*

$$RX = \{y \in Y \mid y \text{ is a terminal element of } Y\} \tag{369}$$

and

$$LY = \{x \in X \mid x \text{ is an initial element of } X\}\,. \tag{370}$$

(b) *Show that*

$$\xi \in X \text{ is a supremum of } X \quad \Leftrightarrow \quad \xi \text{ is a smallest element of preordered set } (X, \succsim) \tag{371}$$

and

$$\upsilon \in Y \text{ is an infimum of } Y \quad \Leftrightarrow \quad \upsilon \text{ is a smallest element of preordered set } (Y, \precsim)\,. \tag{372}$$

6.4.11 Example: $(\mathscr{P}X, \subseteq)$

Given a subset $\mathscr{A} \subseteq \mathscr{P}X$, i.e., a family of subsets of a set X, the union of $\mathscr{A}$ is the smallest subset of X that contains every member of $\mathscr{A}$ and the intersection of $\mathscr{A}$ is the greatest subset of X that is contained in every member of $\mathscr{A}$. Thus, the supremum and infimum exist for every subset of the ordered set $(\mathscr{P}X, \subseteq)$ and they coincide with the union and, respectively, the intersection of family $\mathscr{A}$,

$$\sup \mathscr{A} = \bigcup \mathscr{A} \qquad \text{and} \qquad \inf \mathscr{A} = \bigcap \mathscr{A}\,. \tag{373}$$

6.4.12

Identity (365) and a similar identity for L can be both expressed in the form of a pair of commutative diagrams

$$\begin{tikzcd} \mathscr{P}X \arrow[d, "\cap"'] & \mathscr{P}Y \arrow[l, "L"'] \arrow[d, "\cup"] \arrow[dl, phantom, "\circlearrowleft"] \\ \mathscr{P}X & \mathscr{P}Y \arrow[l, "L"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{P}X \arrow[r, "R"] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{P}Y \\ \mathscr{P}X \arrow[u, "\cup"] \arrow[r, "R"'] & \mathscr{P}Y \arrow[u, "\cap"'] \end{tikzcd} . \tag{374}$$

Exercise 158 *Consider the* membership *relation*

$$\in \, : X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad x, A \longmapsto \text{“} x \in A \text{”}. \tag{375}$$

Determine RA and $L\mathscr{A}$ for $A \subseteq X$ and $\mathscr{A} \subseteq \mathscr{P}X$.

Exercise 159 *Consider the set-containment relation* (71)*. Determine $R\mathscr{A}$ and $L\mathscr{B}$ for $\mathscr{A} \subseteq \mathscr{P}X$ and $\mathscr{B} \subseteq \mathscr{P}X$.*

Exercise 160 *Consider the relation ρ_f associated with a function $f : X \to Y$ between arbitrary sets, cf.* (85)*. Determine $R\mathscr{A}$ and $L\mathscr{B}$ for $\mathscr{A} \subseteq \mathscr{P}X$ and $\mathscr{B} \subseteq \mathscr{P}X$.*

Exercise 161 *Show that R is a morphism of ordered sets $(\mathscr{P}X, \subseteq) \to (\mathscr{P}Y, \supseteq)$ and L is a morphism of ordered sets $(\mathscr{P}Y, \subseteq) \to (\mathscr{P}X, \supseteq)$.*

6.4.13 The preorders on $\mathscr{P}X$ and $\mathscr{P}Y$ canonically associated with $\rho \in \mathbf{Rel}(X, Y)$

Let us denote by $\overset{\mathrm{R}}{\subseteq}$ the relation $\supseteq$ pulled-back from $\mathscr{P}Y$ to $\mathscr{P}X$ by function R,

$$A \overset{\mathrm{R}}{\subseteq} A' \quad \text{if} \quad RA \supseteq RA' \qquad (A, A' \subseteq X), \tag{376}$$

and let us denote by $\overset{\mathrm{L}}{\subseteq}$ the relation $\supseteq$ pulled-back from $\mathscr{P}X$ to $\mathscr{P}Y$ by function L,

$$B \overset{\mathrm{L}}{\subseteq} B' \quad \text{if} \quad LB \supseteq LB' \qquad (B, B' \subseteq Y). \tag{377}$$

In view of the universal property of the pulled-back relation, cf. Section 3.4.13, relation $\overset{\mathrm{R}}{\subseteq}$ is a strongest relation on $\mathscr{P}X$ for which R is a morphism into $(\mathscr{P}Y, \supseteq)$. Similarly, $\overset{\mathrm{L}}{\subseteq}$ is a strongest relation on $\mathscr{P}Y$ for which L is a morphism into $(\mathscr{P}X, \supseteq)$. By combining this remark with Exercise 161, we deduce that both $\overset{\mathrm{R}}{\subseteq}$ and $\overset{\mathrm{L}}{\subseteq}$ are stronger than the containment relation,

$$\subseteq \implies \overset{\mathrm{R}}{\subseteq} \qquad \text{on } \mathscr{P}X \tag{378}$$

and

$$\subseteq \implies \overset{\mathrm{L}}{\subseteq} \qquad \text{on } \mathscr{P}Y. \tag{379}$$

In the following exercises, A denotes an arbitrary subset of X and B denotes an arbitrary subset of Y.

Exercise 162 *Show that*

$$A \subseteq LB \quad \textit{if and only if} \quad RA \supseteq B. \tag{380}$$

Exercise 163 *Show that*

$$A \subseteq LRA \qquad \textit{and} \qquad RLB \supseteq B. \tag{381}$$

Exercise 164 *Show that*

$$LRLB = LB \qquad \textit{and} \qquad RA = RLRA. \tag{382}$$

6.4.14

In other words, the pair of functions (L, R) satisfies the following identities

$$LRL = L \qquad \text{and} \qquad RLR = R\,, \tag{383}$$

and the unary operation LR on X, as well as the unary operation RL on Y, are *idempotent*, i.e.,

$$(LR)^2 = LR \qquad \text{and} \qquad (RL)^2 = RL\,. \tag{384}$$

Exercise 165 *Show that*

$$RL[x\rangle = [x\rangle\,. \tag{385}$$

Exercise 166 *Show that*

$$\rho(x,y) \Leftrightarrow L[x\rangle \subseteq \langle y] \Leftrightarrow [x\rangle \supseteq R\langle y]\,. \tag{386}$$

Solution. One has the following sequence of equivalences and implications

$$\rho(x,y) \Leftrightarrow [x\rangle \ni y \Leftrightarrow [x\rangle \supseteq \{y\} \Rightarrow L[x\rangle \subseteq \langle y] \Rightarrow RL[x\rangle \supseteq R\langle y] \overset{(385)}{\Rightarrow} [x\rangle \supseteq R\langle y] \Rightarrow [x\rangle \ni y \Leftrightarrow \rho(x,y)\,.$$

$\square$

6.4.15 Ordered sets $\mathscr{L}(\rho)$ and $\mathscr{R}(\rho)$

Consider the images of power-sets $\mathscr{P}Y$ and $\mathscr{P}X$ under L and, respectively, R,

$$\mathscr{L}(\rho) := L_*\mathscr{P}Y \qquad \text{and} \qquad \mathscr{R}(\rho) := R_*\mathscr{P}X\,. \tag{387}$$

Exercise 167 *Show that, for any subsets B_1 and B_2 of Y, one has*

$$LB_1 \subseteq LB_2 \qquad \textit{if and only if} \qquad LB_1 \overset{\mathrm{R}}{\subseteq} LB_2 \tag{388}$$

and, for any subsets A_1 and A_2 of X, one has

$$RA_1 \subseteq RA_2 \qquad \textit{if and only if} \qquad RA_1 \overset{\mathrm{L}}{\subseteq} RA_2\,. \tag{389}$$

It follows that on $\mathscr{L}(\rho) \subseteq \mathscr{P}X$ the preorder relation $\overset{\mathrm{R}}{\subseteq}$ is equipotent with the order relation $\subseteq$, and similarly for the preorder relation $\overset{\mathrm{L}}{\subseteq}$ on $\mathscr{R}(\rho) \subseteq \mathscr{P}Y$. In particular, both preorders are order relations on those two subsets of the corresponding power-sets.

6.4.16 A canonical isomorphism $(\mathscr{L}(\rho), \subseteq) \simeq (\mathscr{R}(\rho), \supseteq)$

Restriction of $R : \mathscr{P}X \to \mathscr{P}Y$ to $\mathscr{L}(\rho)$ induces a morphism of ordered sets

$$(\mathscr{L}(\rho), \subseteq) \longrightarrow (\mathscr{R}(\rho), \supseteq)\,.$$

Similarly, restriction of $L : \mathscr{P}Y \to \mathscr{P}X$ to $\mathscr{R}(\rho)$ induces a morphism

$$(\mathscr{R}(\rho), \supseteq) \longrightarrow (\mathscr{L}(\rho), \subseteq)\,.$$

In view of identities (382), those two morphisms are inverse to each other.

6.4.17 Case: ρ a symmetric relation

For a symmetric relation on a set X, one has $R = L$ and Identities (383) become a single identity

$$R^3 = R\,. \tag{390}$$

Moreover, $\mathscr{R}(\rho) = \mathscr{L}(\rho)$ and R induces a canonical *antiinvolution* of $(\mathscr{L}(\rho), \subseteq)$, i.e., a morphism

$$(\mathscr{L}(\rho), \subseteq) \xrightarrow{R} (\mathscr{L}(\rho), \subseteq)^{\mathrm{op}} \qquad \text{such that} \qquad R \circ R = \mathrm{id} \tag{391}$$

where $(\mathscr{L}(\rho), \subseteq)^{\mathrm{op}}$ coincides with $(\mathscr{L}(\rho), \supseteq)$.

6.4.18 $(\mathscr{P}X, \subseteq)$ represented as $(\mathscr{L}(\rho), \subseteq)$

When ρ is the *nonequality* relation $\neq$ on a set X, we obtain

$$(\mathscr{P}X, \subseteq) = (\mathscr{L}(\neq), \subseteq) \qquad \text{and} \qquad R = \complement\,. \tag{392}$$

Exercise 168 *Prove* (392).

6.5 A close look at the definition of a morphism between binary relations

6.5.1

For the *equality* relation $=$, the class of $=$-comutative diagrams coincides with the class of commutative diagrams.

6.5.2 $2 \times 2 \times 2$ characterizations of a morphism

Let

$$\begin{tikzcd} X', Y' \\ X, Y \arrow[u, shift left, "g"'] \arrow[u, shift right, "f"] \end{tikzcd} \tag{393}$$

be a pair of functions. The definition of a morphism between binary relations can be expressed in the language of $\supseteq$- and $\subseteq$-commutative diagrams and the canonical functions associated to the relations involved.

Lemma 6.2 *The following statements are equivalent:*

(mo) *Pair of functions* (393) *is a morphism.*

(r_*) *The square*

$$\begin{tikzcd} X' \arrow[r, "{[\ )}"] & \mathscr{P}Y' \\ X \arrow[u, "f"] \arrow[r, "{[\ )}"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g_*"'] \end{tikzcd} \tag{394}$$

is $\supseteq$*-commutative.*

(R_*) *The square*

$$\begin{tikzcd} \mathscr{P}X' \arrow[r, "R"] & \mathscr{P}Y' \\ \mathscr{P}X \arrow[u, "f_*"] \arrow[r, "R"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g_*"'] \end{tikzcd} \tag{395}$$

is $\supseteq$*-commutative.*

(r^*) *The square*

$$\begin{tikzcd} X' \arrow[r, "{[\ \rangle}"] & \mathscr{P}Y' \arrow[d, "g^*"] \\ X \arrow[u, "f"] \arrow[r, "{[\ \rangle}"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \end{tikzcd} \tag{396}$$

is $\supseteq$*-commutative.*

(R^*) *The square*

$$\begin{tikzcd} \mathscr{P}X' \arrow[r, "R"] & \mathscr{P}Y' \arrow[d, "g^*"] \\ \mathscr{P}X \arrow[u, "f_*"] \arrow[r, "R"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \end{tikzcd} \tag{397}$$

is $\supseteq$*-commutative.*

(l_*) *The square*

$$\begin{tikzcd} \mathscr{P}X' & Y' \arrow[l, "{\langle\ ]}"'] \\ \mathscr{P}X \arrow[u, "f_*"] \arrow[ur, phantom, "\subseteq"] & Y \arrow[l, "{\langle\ ]}"] \arrow[u, "g"'] \end{tikzcd} \tag{398}$$

is $\subseteq$*-commutative.*

(L_*) *The square*

$$\begin{tikzcd} \mathscr{P}X' & \mathscr{P}Y' \arrow[l, "L"'] \\ \mathscr{P}X \arrow[u, "f_*"] \arrow[ur, phantom, "\subseteq"] & \mathscr{P}Y \arrow[l, "L"] \arrow[u, "g_*"'] \end{tikzcd} \tag{399}$$

is $\subseteq$*-commutative.*

(l^*) *The square*

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & Y' \arrow[l, "{\langle\ ]}"'] \\ \mathscr{P}X \arrow[ur, phantom, "\subseteq"] & Y \arrow[l, "{\langle\ ]}"] \arrow[u, "g"'] \end{tikzcd} \tag{400}$$

is $\subseteq$*-commutative.*

(L*) *The square*

$$
\begin{tikzcd}
\mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{P}Y' \arrow[l, "L"'] \\
\mathscr{P}X \arrow[ur, phantom, "\subseteq"] & \mathscr{P}Y \arrow[l, "L"] \arrow[u, "g_*"']
\end{tikzcd}
\tag{401}
$$

is $\subseteq$*-commutative.*

Proof of (mo) $\Leftrightarrow$ (r$_*$). For any $x \in X$ and $y' \in Y'$, one has the following equivalences of statements:

$$\exists_{y \in Y}\, \rho(x,y) \wedge g(y) = y' \;\Leftrightarrow\; g_*[x\rangle \ni y'$$

and

$$\rho'\left(f(x), y'\right) \;\Leftrightarrow\; [f(x)\rangle \ni y' \,.$$

Hence, the conditional statement

$$\begin{array}{c} \rho(x,y) \wedge g(y) = y' \\ \Downarrow \\ \rho'\left(f(x), y'\right) \end{array}$$

is equivalent to the conditional statement

$$\begin{array}{c} g_*[x\rangle \ni y' \\ \Downarrow \\ [f(x)\rangle \ni y' \end{array} \,,$$

i.e.,

$$\begin{array}{c} \exists_{y \in Y}\, \rho(x,y) \wedge g(y) = y' \\ \Downarrow \\ \rho'\left(f(x), y'\right) \end{array} \quad \Leftrightarrow \quad \begin{array}{c} g_*[x\rangle \ni y' \\ \Downarrow \\ [f(x)\rangle \ni y' \end{array} \,. \tag{402}$$

By applying, first $\forall_{y' \in Y'}$, then $\forall_{x \in X}$, to both sides of equivalence (402), we deduce equivalence of Conditions (mo) and (r$_*$). □

Exercise 169 *Prove equivalence of Conditions* (r$_*$) *and* (R$_*$).

Exercise 170 *Prove equivalence of Conditions* (r$_*$) *and* (r*).

6.5.3 2×2 characterizations of a $\Leftrightarrow$-morphism

Lemma 6.3 *The following statements are equivalent:*

(fa) *Pair of functions* (393) *is a* $\Leftrightarrow$*-morphism, cf. Section 3.4.6.*

($r^*_{\circlearrowright}$) *The square*

$$\begin{tikzcd} X' \arrow[r, "{[\;\rangle}"] & \mathscr{P}Y' \arrow[d, "g^*"] \\ X \arrow[u, "f"] \arrow[r, "{[\;\rangle}"'] \arrow[ur, phantom, "\circlearrowright"] & \mathscr{P}Y \end{tikzcd} \tag{403}$$

is commutative.

($R^*_{\circlearrowright}$) *The square*

$$\begin{tikzcd} \mathscr{P}X' \arrow[r, "R"] & \mathscr{P}Y' \arrow[d, "g^*"] \\ \mathscr{P}X \arrow[u, "f_*"] \arrow[r, "R"'] \arrow[ur, phantom, "\circlearrowright"] & \mathscr{P}Y \end{tikzcd} \tag{404}$$

is commutative.

($l^*_{\circlearrowleft}$) *The square*

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & Y' \arrow[l, "{\langle\;]}"'] \\ \mathscr{P}X \arrow[ur, phantom, "\circlearrowleft"] & Y \arrow[l, "{\langle\;]}"] \arrow[u, "g"'] \end{tikzcd} \tag{405}$$

is commutative.

($L^*_{\circlearrowleft}$) *The square*

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{P}Y' \arrow[l, "L"'] \\ \mathscr{P}X \arrow[ur, phantom, "\circlearrowleft"] & \mathscr{P}Y \arrow[l, "L"] \arrow[u, "g_*"'] \end{tikzcd} \tag{406}$$

is commutative.

Exercise 171 *Prove equivalence of Conditions* (fa) *and* ($r^*_{\circlearrowright}$) *by using the proof of equivalence of Conditions* (mo) *and* (r_*) *as a model.*

6.5.4 Right-saturated morphisms

We shall say that a pair (393) is a *right-saturated morphism* if square (394) commutes.

6.5.5 Left-saturated morphisms

We shall say that a pair (393) is a *left-saturated morphism* if square (398) commutes.

6.5.6 Saturated subsets

In the special case when $X \subseteq X'$ and $Y \subseteq Y'$ and f, g is the pair of canonical inclusion functions, the inclusion morphism is right-saturated precisely when every right relative in Y' of any element $x \in X$, belongs to Y.

Similarly, the inclusion morphism is left-saturated precisely when every left relative in X' of any element $y \in Y$, belongs to X.

6.5.7 Up-sets and down-sets

In Theory of Ordered Sets right-saturated subsets of an ordered set are often referred to as *up-sets* whereas left-saturated subsets are referred to as *down-sets*.

6.6 Filters

6.7

Left-directed upsets are called *filters*. Filters in $(\mathscr{P}X, \subseteq)$ play a fundamental role in Topology, Algebra and Model Theory. In that case they are often referred to as filters *on* a set X. A standard definition of a filter $\mathscr{F}$ of subsets usually requires a filter to be *proper* which is equivalent to the requirement that $\emptyset \notin \mathscr{F}$.

We shall denote the set of proper filters on a set X by $\operatorname{Filt} X$.

6.7.1 Principal filters $\mathscr{P}_A$

A *principal filter* on a set X is the filter generated by a single subset $A \subseteq X$. It is the family of those subsets of X that contain A as their subset. In other words, $\mathscr{P}_A$ coincides with the set of right relatives $[A\rangle$ of relation $\subseteq$ on $\mathscr{P}X$.

6.7.2 Ultrafilters

Maximal proper filters are traditionally called *ultrafilters*. Wherever the language of filters of subsets is used, ultrafilters in $(\mathscr{P}X, \subseteq)$ take a central position.

6.7.3 Principal ultrafilters

Exercise 172 *Show that a principal filter $\mathscr{P}_A$ is an ultrafilter if and only if A is a singleton set $A = \{a\}$.*

In this case it is common to denote $\mathscr{P}_{\{a\}}$ by $\mathscr{P}_a$ or $\mathscr{P}_a X$.

6.7.4 Finite-Intersection Property

Exercise 173 *Prove that $\mathscr{A} \subseteq \mathscr{P}X$ is contained in a proper filter if and only if it satisfies the following condition*

$$\forall_{\mathit{finite}\ \mathscr{A}' \subseteq \mathscr{A}} \quad \bigcap \mathscr{A}' \neq \emptyset . \tag{407}$$

Condition (407) is known as the *Finite-Intersection Property*. We shall denote by $\mathrm{FIP}(X)$ the set of all families of subsets of X that satisfy this condition.

6.8 Right- and left-complete binary relations

6.8.1

We shall say that a binary relation ρ is *right-complete* when every subset $A \subseteq X$ has a supremum, cf. Section 6.4.5, i.e.,

$$[\ \rangle_* X = \mathscr{R}(\rho) . \tag{408}$$

We shall say that a binary relation ρ is *left-complete* when every subset $B \subseteq Y$ has an infimum, cf. Section 6.4.9, i.e.,

$$\langle\ ]_* Y = \mathscr{L}(\rho)\,. \tag{409}$$

6.8.2 Bicomplete binary relations

We shall say that a binary relation ρ is *bicomplete* if it is both right- and left-complete. A central fact of our theory is comprised by the following proposition.

Proposition 6.4 *For every binary relation ρ, ordered set $(\mathscr{L}(\rho), \subseteq)$ is bicomplete.*

Proof. Given a family $\mathscr{B}$ of subsets of Y, intersection of family $L_*\mathscr{B} = \{LB \mid B \in \mathscr{B}\}$ is the greatest subset of X contained in each member of family $L_*\mathscr{B}$ and, according to the left commutative square in (374), is a member of $\mathscr{L}(\rho)$:

$$\bigcap L_*\mathscr{B} = L\left(\bigcup \mathscr{B}\right)\,.$$

Every subset of $\mathscr{E} \subseteq \mathscr{L}(\rho)$ is a family of subsets of X of the form $L_*\mathscr{B}$ for some $\mathscr{B} \subseteq \mathscr{P}Y$. Indeed,

$$\mathscr{E} = L_*(R_*\mathscr{E})\,.$$

It follows that every subset of $\mathscr{L}(\rho)$ has infimum in $(\mathscr{L}(\rho), \subseteq)$. A similar argument demonstrates that every subset of $\mathscr{R}(\rho)$ has infimum in $(\mathscr{R}(\rho), \subseteq)$ and, henceforth, has supremum in $(\mathscr{R}(\rho), \supseteq)$.

Since $(\mathscr{L}(\rho), \subseteq)$ is isomorphic to $(\mathscr{R}(\rho), \supseteq)$, cf. Section 6.4.16, every subset of $\mathscr{L}(\rho)$ has supremum in $(\mathscr{L}(\rho), \subseteq)$. □

Note how elegant is the above argument.

Exercise 174 *Given $\mathscr{E} \subseteq \mathscr{P}X$, find a formula for* $\sup \mathscr{E}$ *in* $(\mathscr{L}(\rho), \subseteq)$.

Exercise 175 *Determine $[\ \rangle_* X$, $\langle\ ]_* Y$, $\mathscr{R}(\rho_f)$, and $\mathscr{L}(\rho_f)$, for the relation associated with a function $f : X \to Y$ between arbitrary sets, cf.* (85).

6.8.3 Restricted notions of completeness

Given a family of subsets $\mathscr{A} \subseteq \mathscr{P}X$, we could say that a binary relation ρ is (right) *$\mathscr{A}$-complete* if

$$[\ \rangle_* X \supseteq R_*\mathscr{A} \tag{410}$$

and, similarly, given a family of subsets $\mathscr{B} \subseteq \mathscr{P}Y$, we could say that ρ is (left) *$\mathscr{B}$-complete* if

$$\langle\ ]_* Y \supseteq L_*\mathscr{B}\,. \tag{411}$$

6.8.4 Completeness restricted to nonempty *bounded* subsets

Following the practice of Theory of Ordered Sets, let us say that a subset $A \subseteq X$ is *right-bounded* if $RA \neq \emptyset$, and a subset $B \subseteq Y$ is *left-bounded* if $LA \neq \emptyset$. Let us denote the family of all nonempty right-bounded subsets of X by

$$\mathscr{P}_{\triangleright} X := \{A \subseteq X \mid A \neq \emptyset \wedge RA \neq \emptyset\} \tag{412}$$

and the family of all nonempty right-bounded subsets of X by

$$\mathscr{P}_{\triangleleft} Y := \{B \subseteq Y \mid B \neq \emptyset \wedge LB \neq \emptyset\} . \tag{413}$$

Exercise 176 *Show that the functions L and R induce mutually inverse functions*

$$R_* \mathscr{P}_{\triangleright} X \underset{R}{\overset{L}{\rightleftarrows}} L_* \mathscr{P}_{\triangleleft} Y \quad . \tag{414}$$

Lemma 6.5 *If a binary relation ρ is right $\mathscr{P}_{\triangleright} X$-complete and*

$$L_* \{[x\rangle \mid x \in X \wedge [x\rangle \neq \emptyset\} \ \subseteq \ \langle\ ]_* Y , \tag{415}$$

then ρ is left $\mathscr{P}_{\triangleright} Y$-complete.

Proof. If $B \in \mathscr{P}_{\triangleleft} Y$ and ρ is right $\mathscr{P}_{\triangleright} X$-complete, then

$$LB = RLB = L[\alpha\rangle$$

where $\alpha \in X$ is a supremum of RLB and $[\alpha\rangle \supseteq B \neq \emptyset$. Condition (415) implies that $LB = L[\alpha\rangle = \langle\beta]$ for a certain $\beta \in Y$. ☐

Condition (415) is obviously satisfied when ρ is a preorder relation on a set X. In that case, one has

$$\forall_{x \in X} \ \langle x] = L[x\rangle \wedge R\langle x] = [x\rangle .$$

Corollary 6.6 *A preorder relation on a set X is right $\mathscr{P}_{\triangleright} X$-complete if and only if it is left $\mathscr{P}_{\triangleleft} X$-complete.* ☐

6.8.5 Terminology: *complete* (pre)ordered sets

Such sets, in Theory of Ordered Sets, are said to be *complete*.

6.8.6 Finite completeness

The case of the family of all *finite* subsets is of particular importance. We shall refer to the corresponding binary relations as being *finitely* right-, or left-complete.

6.9 Semilattices

6.9.1 $\vee$-semilattices

An ordered set $(X, \preccurlyeq)$ is finitely right-complete precisely when it has the *smallest* element (frequently denoted 0), that coincides with supremum of $\emptyset \subseteq X$, and any two-element subset has a supremum which in this case becomes a commutative binary operation on X,

$$a, b \longmapsto a \vee b := \sup\{a, b\} \qquad (a, b \in X), \tag{416}$$

referred to as the *join* operation. Join is, by definition, commutative and every element is an *idempotent*,

$$\forall_{a \in X} \; a \vee a = a. \tag{417}$$

Exercise 177 *Show that join is associative and that*

$$\forall_{a \in X} \; 0 \vee a = a,$$

i.e., 0 is an identity element for binary operation $\vee$.

6.9.2 The associated monoid $(X, 0, \vee)$

In Theory of Ordered Sets, finitely right-complete ordered sets are called $\vee$- or *join semilattices*. The associated commutative monoid $(X, 0, \vee)$ satisfies Identity (417). In Algebra, any comutative monoid $(A, e, \cdot)$ such that every element is idempotent, is called a *semilattice*.

6.9.3

Connection between $\vee$-semilattices and semilattices in the sense of Algebra goes either way.

Exercise 178 *Given a semilattice $(A, e, \cdot)$, define the binary relation on X,*

$$\rho : A, A \longrightarrow \text{Statements}, \qquad a, b \longmapsto \text{“ } ab = b \text{ ”}. \tag{418}$$

Show that ρ is an order relation, e is the smallest element, and

$$\forall_{a,b \in A} \; \sup\{a, b\} = ab.$$

6.9.4

The monoid canonically associated to a $\vee$-semilattice, and the $\vee$-semilattice canonically associated to a semilattice in the sense of Algebra are constructions that are *natural*, i.e., they extend in a unique manner to a pair of *functors* between the corresponding categories.

For algebraic structures, homomorphisms provide a natural definition of a *morphism*, cf. Section 3.2.1. This does not correspond, however, to the standard definition of a morphism between ordered sets. Homomorphisms between semilattices correspond to, what we shall call below, *finitely* sup*-exact* morphisms.

6.9.5 $\wedge$-semilattices

An ordered set $(X, \preccurlyeq)$ is finitely left-complete precisely when it has the *greatest* element (frequently denoted 1), that coincides with infimum of $\emptyset \subseteq X$, and any two-element subset has a infimum which in this case becomes a commutative binary operation on X,

$$a, b \longmapsto a \wedge b := \inf\{a, b\} \qquad (a, b \in X), \tag{419}$$

referred to as the *meet* operation. Like join, meet is, by definition, commutative and every element is an *idempotent*,

$$\forall_{a \in X}\ a \wedge a = a .$$

If $(X, \preccurlyeq)$ is a meet semilattice, then $(X, \preccurlyeq)^{\mathrm{op}}$ is a join semilattice, and vice-versa. In particular, meet is an associative operation and 1 serves as its identity element.

In Theory of Ordered Sets, finitely left-complete ordered sets are called $\wedge$- or *meet semilattices*.

6.9.6 Terminology: lattices

In Theory of Ordered Sets, finitely bicomplete ordered sets are called *lattices*.

6.10 Complete lattices

6.10.1

In Theory of Ordered Sets, bicomplete ordered sets are called *complete lattices*.

In Section 6.4.11 we saw that power-set $(\mathscr{P}X, \subseteq)$ is a complete lattice. Above we established that $(\mathscr{L}(\rho), \subseteq)$ and $(\mathscr{R}(\rho), \subseteq)$ are complete lattices for any binary relation ρ.

6.10.2 The complete lattice of algebraic substructures

Let $\big(X, (\mu_i)_{i \in I}\big)$, be an algebraic structure, cf. Section 2.2.1. A subset $A \subseteq X$ that is *closed* under every operation μ_i, $i \in I$, inherits those operations from X and is an algebraic structure of the same type. We call such "inherited" structure a *substructure* of $\big(X, (\mu_i)_{i \in I}\big)$.

The family of all subsets of X that are closed under operations μ_i forms an ordered subset of power-set $(\mathscr{P}X, \subseteq)$. We shall denote it $\mathrm{Substr}\big(X, (\mu_i)_{i \in I}\big)$.

It follows immediately from the definition that intersection of any family $\mathscr{A} \subseteq \mathscr{P}X$ of subsets that are closed under operations μ_i is itself closed under those operations. Thus, every family of substructures has infimum.

6.10.3 An algebraic substructure generated by a subset

Given any subset $E \subseteq X$, the intersection of the family of subsets that are closed under the operations and containg E, is the *smallest* subset of X with that property. We call this the *substructure generated by a subset* $E \subseteq X$ and often denote it

$$\langle E \rangle .$$

Note that the above mentioned family is not empty: E is contained in X and X is closed under the operations by definition.

6.10.4 Notation : $\langle a_1, \dots, a_n \rangle$

When

$$\{a_1, \dots, a_n\}$$

it is a common practice to denote $\langle E \rangle$ by

$$\langle a_1, \dots, a_n \rangle . \tag{420}$$

6.10.5

Given a family $\mathscr{A} \subseteq \mathrm{Substr}(X, (\mu_i)_{i \in I})$, any substructure that contains every member of $\mathscr{A}$ will contain also their union,

$$\bigcup \mathscr{A} . \tag{421}$$

The union of even two subsets closed under an algebraic operation is closed under that operation only in exceptional circumstances (consider for example, the union of two lines passing through the origin in the 2-dimensional vector space). The substructure generated by subset (421) of X is, obviously, the smallest substructure containing every member of family $\mathscr{A}$.

In conclusion, for every $\mathscr{A} \subseteq \mathrm{Substr}(X, (\mu_i)_{i \in I})$, supremum and infimum exist and are given by

$$\sup \mathscr{A} = \left\langle \bigcup \mathscr{A} \right\rangle \qquad \text{and} \qquad \inf \mathscr{A} = \bigcap \mathscr{A} . \tag{422}$$

Thus, the ordered set $\left(\mathrm{Substr}(X, (\mu_i)_{i \in I}), \subseteq\right)$ is a complete lattice, irrespective of what family of algebraic operations we consider.

6.11 Right- and left-exact functions

6.11.1

Given binary relations ρ and ρ' as in Section 6.2.1, and an arbitrary function

$$f : X \to X' ,$$

we shall say that f is *right-exact* if

$$f_* : (\mathscr{P}X, \overset{\mathrm{R}}{\subseteq}) \longrightarrow (\mathscr{P}X', \overset{\mathrm{R}}{\subseteq}) \tag{423}$$

is a morphism of preordered sets, cf. Section 6.4.13. Similarly, given an arbitrary function

$$g : Y \to Y' ,$$

we shall say that g is *left-exact* if

$$g_* : (\mathscr{P}Y, \overset{\mathrm{L}}{\subseteq}) \longrightarrow (\mathscr{P}Y', \overset{\mathrm{L}}{\subseteq}) \tag{424}$$

is a morphism of preordered sets.

6.11.2 Exact functions between binary relational structures

When $X = Y$, $X' = Y'$ and a function f is both right- and left-exact, we shall say that f is *exact*.

6.11.3 $\sup$- and $\inf$-exact functions

We shall say that f is $\sup$*-exact* if, for every subset $A \subseteq X$ that has a supremum, say $\alpha \in X$, its image under f has $f(\alpha)$ as a supremum. Similarly, we shall say that g is $\inf$*-exact* if, for every subset $B \subseteq Y$ that has an infimum, say $\beta \in Y$, its image under g has $g(\beta)$ as a supremum.

Exercise 179 *Show that*

$$\forall_{x_1,x_2 \in X} \ \sup\{x_1, x_2\} = x_1 \Leftrightarrow x_1 \succsim x_2 \tag{425}$$

and

$$\forall_{y_1,y_2 \in Y} \ \inf\{y_1, y_2\} = y_1 \Leftrightarrow y_1 \precsim y_2 \,. \tag{426}$$

Exercise 180 *Show that a* $\sup$*-exact function* f *is a morphism of preorded sets*

$$f : (X, \succsim) \longrightarrow (X', \succsim)\,. \tag{427}$$

Show that an $\inf$*-exact function* g *is a morphism of preorded sets*

$$g : (Y, \precsim) \longrightarrow (Y', \precsim)\,. \tag{428}$$

Proposition 6.7 *A right-exact function* $f : X \to X'$ *is* $\sup$*-exact. If* ρ *is right-complete and* f *is* $\sup$*-exact, then* f *is right-exact.*

Similarly, a left-exact function $g : Y \to Y'$ *is* $\inf$*-exact. If* ρ *is left-complete and* g *is* $\inf$*-exact, then* g *is left-exact.*

Proof. Given subsets $A_1 \overset{\mathrm{R}}{\subseteq} A_2$ of X, with suprema α_1 and α_2, respectively, we observe that

$$[\alpha_1\rangle = RA_1 \supseteq RA_2 = [\alpha_2\rangle\,.$$

If f is $\sup$-exact, then

$$Rf_*A_1 = [f(\alpha_1)\rangle \qquad \text{and} \qquad [f(\alpha_2)\rangle = Rf_*A_2\,.$$

Since f is a morphism (427), cf. Exercise 180 one has

$$[f(\alpha_1)\rangle \supseteq [f(\alpha_2)\rangle\,.$$

It follows that $f_*A_1 \overset{\mathrm{R}}{\subseteq} f_*A_2$.

If ρ is right-complete, then the above argument applies to any pair A_1, A_2 of subsets of X.

The other case is proved similarly, or is reduced to the one already proven by passing to the opposite relations ρ^{op} and $(\rho')^{\mathrm{op}}$. □

6.11.4 Example: *supremum* as a right-exact function

Let $E \subseteq S$ be a subset of an ordered set $(S,\preccurlyeq)$ such that every subset $A \subseteq E$ has supremum in $(S,\preccurlyeq)$. Recall that in an ordered set a supremum element is unique when it exists. Consider the corresponding morphism of ordered sets

$$\mathbf{sup} : (\mathscr{P}E,\subseteq) \longrightarrow (S,\preccurlyeq)\,, \qquad A \longmapsto \sup A\,. \tag{429}$$

Lemma 6.8 *Supremum morphism* (429) *is right-exact.*

Proof. Recall that, for an order relation ρ on a set S, the associated preorder $\succsim$ coincides with ρ^{op}. Any function $f : S \to S'$ between sets equipped with a binary relation is a morphism $(S,\rho) \to (S',\rho')$ if and only if it is a morphism for the opposite relations, $(S,\rho^{\mathrm{op}}) \to (S',(\rho')^{\mathrm{op}})$. In particular, for functions between ordered sets, we can replace condition stating that f in (427) is a morphism by the equivalent condition that $f : (S,\rho) \to (S',\rho')$ is a morphism.

We first prove that morphism (429) is sup-exact, then recall that $(\mathscr{P}E,\subseteq)$ is a complete lattice and, finally, invoke Proposition 6.7.

Let $\mathscr{A} \subseteq \mathscr{P}E$ be a family of subsets of E. The following calculation[4]

$$R\,\mathbf{sup}_*\mathscr{A} = R\left(\bigcup_{A\in\mathscr{A}}\{\sup A\}\right) = \bigcap_{A\in\mathscr{A}} R\{\sup A\} = \bigcap_{A\in\mathscr{A}} [\sup A\rangle = \bigcap_{A\in\mathscr{A}} RA = R\left(\bigcup\mathscr{A}\right) = \left[\sup\left(\bigcup\mathscr{A}\right)\right\rangle \tag{430}$$

demonstrates that $\mathbf{sup}_*\mathscr{A}$ has supremum in $(S,\preccurlyeq)$ and that

$$\sup \mathbf{sup}_*\mathscr{A} = \mathbf{sup}\left(\bigcup\mathscr{A}\right)\,. \tag{431}$$

Thus, morphism (429) is sup-exact. □

6.11.5 A basis of a complete lattice

When supremum morphism (429) is surjective, we say that subset $E \subseteq S$ is sup*-dense* in $(S,\preccurlyeq)$. Morphism (429) is an isomorphism precisely when every element $s \in S$ is the supremum of a unique subset of E. In this case we shall refer to E as a *basis* of complete lattice $(S,\preccurlyeq)$.

6.11.6 Description of right-exact morphisms *from* a complete lattice with a basis

If

$$f : (S,\preccurlyeq) \longrightarrow (S',\preccurlyeq') \tag{432}$$

is a right-exact morphism and φ is its restriction to E, then

$$f(s) = \sup \varphi_* A \qquad \text{where } A \subseteq E \text{ is the unique subset such that } s = \sup A$$

which shows that the restriction function

$$\mathrm{Hom}^{\mathrm{rex}}_{\mathsf{OrdSet}}\left((S,\preccurlyeq),(S',\preccurlyeq')\right) \longrightarrow \mathrm{Funct}(E,S')\,, \qquad f \longmapsto f_{|E}\,, \tag{433}$$

[4]For added clarity, when sup stands for the morphism between ordered sets, (429), I make it stand out in the above calculation by using the boldface font.

from the set of right-exact morphisms $(S,\preccurlyeq) \to (S',\preccurlyeq')$ to the set of arbitrary functions $E \to S'$, is injective.

Any function $\varphi : E \to S'$ canonically induces a right-exact morphism

$$\tilde{\varphi} : (\mathscr{P}E, \subseteq) \longrightarrow (S',\preccurlyeq'), \qquad \tilde{\varphi} := \sup \circ \varphi_*,$$

provided

$$\textit{every member of family } \varphi_*\mathscr{P}E \textit{ has supremum in } (S',\preccurlyeq'). \tag{434}$$

Then, by precomposing $\tilde{\varphi}$ with the inverse isomorphism

$$\mathbf{sup}^{-1} : (S,\preccurlyeq) \longrightarrow (\mathscr{P}E, \subseteq)$$

we obtain a right-exact morphism $f_\varphi : (S,\preccurlyeq) \longrightarrow (S',\preccurlyeq')$ whose restriction to E coincides with φ. It follows that (433) defines a bijective correspondence between right-exact morphisms (432) and functions $\varphi : E \to S'$ that satisfy Condition (434).

In particular, we establish the following lemma.

Lemma 6.9 *If $E \subseteq S$ is a basis of a complete lattice $(S,\preccurlyeq)$, then restriction to E defines a canonical bijection between the set of right-exact morphisms from $(S,\preccurlyeq)$ to any right-complete ordered set $(S',\preccurlyeq')$, and the set of arbitrary functions $E \to S'$.*

$\square$

6.11.7 Right-exact morphisms $(\mathscr{P}X, \subseteq) \longrightarrow (\mathscr{P}Y, \supseteq)$

As a corollary we obtain the following important result.

Theorem 6.10 *Every right exact morphism*

$$f : (\mathscr{P}X, \subseteq) \longrightarrow (\mathscr{P}Y, \supseteq) \tag{435}$$

between the power-sets of arbitrary sets X and Y has the form of the canonical R-function associated with some binary relation $\rho : X, Y \longrightarrow \mathrm{Statements}$, cf. Section 6.4.1.

More precisely, correspondence

$$R \longleftarrow\!\shortmid\, \rho$$

defines a canonical bijection

$$\mathrm{Hom}^{\mathrm{rex}}_{\mathrm{OrdSet}}\left((\mathscr{P}X, \subseteq), (\mathscr{P}Y, \supseteq)\right) \longleftrightarrow \left\{ \begin{array}{c} \textit{Equipotence classes of binary relations} \\ \rho : X, Y \longrightarrow \mathrm{Statements} \end{array} \right\}. \tag{436}$$

Proof. Every function

$$\varphi : X \longrightarrow \mathscr{P}Y \tag{437}$$

has the form of the right-relatives function

$$[\ \rangle : X \longrightarrow \mathscr{P}Y$$

for the binary relation

$$\rho_\varphi : X, Y \longrightarrow \text{Statements}, \qquad x, y \longmapsto \text{“ } \varphi(x) \ni y \text{ ”}. \tag{438}$$

According to the proof of Lemma 6.9, for any function (437),

$$f := \sup_{(\mathscr{P}Y, \supseteq)} \circ \varphi$$

is a right-exact morphism (435). Explicitly, for any $A \subseteq X$, one has

$$f(A) = \sup_{(\mathscr{P}Y, \supseteq)} \varphi_* A = \bigcap_{x \in A} \varphi_*(x) = \bigcap_{x \in A} [x\rangle = RA$$

where R is the canonical R-function associated with relation ρ_φ.

Moreover, according to Lemma 6.9, any right-exact morphism $f : (\mathscr{P}X, \subseteq) \longrightarrow (\mathscr{P}Y, \supseteq)$ has this form for a unique function (437).

Finally, equipotence classes of binary relations $\rho : X, Y \longrightarrow$ Statements are in bijective correspondence with the associated right-relatives functions $[\;\rangle : X \to \mathscr{P}Y$. □

6.11.8 Complete description of right- and left-exact morphisms between power-sets

By pre- or post-composing the canonical R-functions of binary relations $\rho : X, Y \longrightarrow$ Statements,

$$\complement \circ R, \qquad R \circ \complement, \qquad \complement \circ R \circ \complement,$$

with the complement antiinvolution $\complement$ that is both right- and left-exact, we obtain also complete description of right-exact morphisms between power-sets ordered by any combination of $\subseteq$ and $\supseteq$ relations. As a consequence, we also obtain a complete decription of left-exact morphisms between power-sets.

We can make the corresponding descriptions even more explicit.

Corollary 6.11 *Every right exact morphism*

$$f : (\mathscr{P}X, \subseteq) \longrightarrow (\mathscr{P}Y, \subseteq) \tag{439}$$

between the power-sets of arbitrary sets X and Y has the form

$$\rho_* : A \longmapsto \bigcup_{x \in A} [x\rangle \tag{440}$$

for some binary relation $\rho : X, Y \longrightarrow$ Statements, and that relation is unique up to equipotence.

Exercise 181 *Show that*

$$\rho_* = \complement \circ R_\neg \tag{441}$$

where $R_\neg$ is the R-function associated with the negated relation $\neg\rho$.

Corollary 6.12 *Every left exact morphism*

$$g : (\mathscr{P}Y, \subseteq) \longrightarrow (\mathscr{P}X, \subseteq) \tag{442}$$

between the power-sets of arbitrary sets X and Y has the form

$$_*\rho : B \longmapsto \bigcup_{y \in B} \langle y] \tag{443}$$

for some binary relation $\rho : X, Y \longrightarrow$ Statements, and that relation is unique up to equipotence.

6.11.9 Finitely exact functions

By analogy with our discussion of restricted notions of completeness of a binary relation, we can also consider similarly restricted notions of exactness of functions. Below we shall focus on the case of *finitely* right-, left-, sup-, and inf-, exact functions.

We begin from a pair of simple observations.

Exercise 182 *Show that*

$$\forall_{x_1,x_2\in X}\ x_1 \succsim x_2 \Leftrightarrow \sup\{x_1,x_2\} = x_1\,. \tag{444}$$

Lemma 6.13 *A finitely* sup*-exact function* $f : X \to X'$ *is a morphism* $(X, \succsim) \to (X', \succsim')$.

Proof. If f is finitely sup-exact, then

$$x_1 \succsim x_2 \;\Leftrightarrow\; \sup\{x_1, x_2\} = x_1 \;\Rightarrow\; i \sup f_*\{x_1, x_2\} = x_1 = \sup\{f(x_1), f(x_2)\} = x_1\,.$$

□

It follows that all four types of finitely exact functions are necessarily morphisms $(X, \succsim) \to (X', \succsim')$ or, respectively, $(Y, \precsim) \to (Y', \precsim')$.

6.11.10 Finitely biexact functions between power-sets

Note that a morphism $f : (\mathscr{P}X, \subseteq) \to (\mathscr{P}Y, \subseteq)$ is finitely right-exact if and only if

$$f(\emptyset) = \emptyset \qquad \text{and} \qquad \forall_{A_1,A_2\subseteq X}\ f(A_1 \cup A_2) = f(A_1) \cup f(A_2)\,. \tag{445}$$

It is finitely left-exact if and only if

$$f(X) = Y \qquad \text{and} \qquad \forall_{A_1,A_2\subseteq X}\ f(A_1 \cap A_2) = f(A_1) \cap f(A_2)\,. \tag{446}$$

For any subsets E and F of any set S the pair of conditions

$$E \cap F = \emptyset \qquad \text{and} \qquad E \cup F = S \tag{447}$$

is equivalent to the sigle condition

$$\complement E = F\,.$$

Lemma 6.14 *A finitely biexact morphism* $f : (\mathscr{P}X, \subseteq) \to (\mathscr{P}Y, \subseteq)$ *is a homomorphism of algebraic unary structures*

$$f : (\mathscr{P}X, \complement) \longrightarrow (\mathscr{P}Y, \complement)\,,$$

i.e., $\complement \circ f = f \circ \complement$.

Exercise 183 *Prove Lemma 6.14.*

6.11.11 Complete description of exact morphisms between power-sets

We are ready to prove that every biexact function from $(\mathscr{P}X, \subseteq)$ to $(\mathscr{P}Y, \subseteq)$ coincides with the preimage morphism ϕ^* of a certain function $\phi : Y \to X$.

Theorem 6.15 *If $f : (\mathscr{P}X, \subseteq) \to (\mathscr{P}Y, \subseteq)$ is right-exact and finitely left-exact then it is left-exact.*
Any such function coincides with the preimage morphism

$$f = \phi^* \tag{448}$$

for a unique function

$$\phi : Y \longrightarrow X .$$

Proof. The first assertion of the theorem is an immediate corollary of Lemma 6.14. By combining sup-exactness of f with its finite inf-exactness we obtain the following chain of equalities

$$Y = f(X) = f\left(\bigcup_{x \in X} \{x\}\right) = \bigcup_{x \in X} f\big(\{x\}\big) .$$

Finite inf-exactness implies that sets $f\big(\{x\}\big)$ are disjoint for different $x \in X$,

$$\forall_{x_1, x_2 \in X} \; x_1 \neq x_2 \Rightarrow f\big(\{x_1\}\big) \cap f\big(\{x_2\}\big) = \emptyset .$$

Thus, f coincides with the *fiber* function

$$X \longrightarrow \mathscr{P}Y , \qquad x \longmapsto \phi^*\{x\} ,$$

of the function

$$\phi : Y \longrightarrow X , \qquad y \longmapsto \text{the unique } x \in X \text{ such that } f\big(\{x\}\big) \ni y .$$

□

6.12 The canonical tower of morphisms

6.12.1

Every binary relation defines a canonical morphism to the binary relation

$${}_L\!\subseteq \,:\, \mathscr{P}Y, \mathscr{P}X \longrightarrow \text{Statements} , \qquad B, A \longmapsto \text{“ } LB \subseteq A \text{ ”} . \tag{449}$$

Exercise 184 *Show that the diagram*

$$\begin{tikzcd}
\mathscr{P}Y, \mathscr{P}X \arrow[dr, "{}_L\subseteq"] & \\
\arrow[r, phantom, "\Uparrow"] & \text{Statements} , \\
X, Y \arrow[uu, shift left, "{[\,\rangle}"] \arrow[uu, shift right, "{\langle\,]}"'] \arrow[ur, "\rho"'] &
\end{tikzcd} \tag{450}$$

is $\Longrightarrow$-commutative, i.e., $[\,\rangle, \langle\,] : (X, Y, \rho) \to (\mathscr{P}Y, \mathscr{P}X, {}_L\!\subseteq)$ is a morphism.

6.12.2

Diagram (450) admits completion to the diagram

$$\begin{tikzcd}
\mathscr{P}Y, \mathscr{P}X \arrow[ddr, "{}_L\subseteq"] & \\
\mathscr{R}(\rho), \mathscr{L}(\rho) \arrow[u, hook] \arrow[dr, dotted] & \\
[\,\rangle_* X, \langle\,]_* Y \arrow[u, hook] \arrow[r, dotted] \arrow[dr, phantom, "\Uparrow"] & \text{Statements} \\
X, Y \arrow[u, two heads, "{[\,\rangle}", "{\langle\,]}"'] \arrow[ur, "\rho"'] &
\end{tikzcd} \; . \tag{451}$$

The dotted arrows are the corresponding restrictions of relation ${}_L\subseteq$. In particular, the middle and the top triangles in (451) strictly commute.

We shall refer to diagram (451) as the *canonical tower associated with a binary relation* ρ.

6.12.3

Consider the bottom triangle of the canonical tower:

$$\begin{tikzcd}
[\,\rangle_* X, \langle\,]_* Y \arrow[dr, "{}_L\subseteq"] \arrow[d, phantom, ""] & \\
 \arrow[r, phantom, "\Uparrow"] & \text{Statements} \\
X, Y \arrow[uu, "{[\,\rangle}", "{\langle\,]}"'] \arrow[ur, "\rho"'] &
\end{tikzcd} \; . \tag{452}$$

Lemma 6.16 *Let us denote by R' and L' the corresponding R- and, respectively, L-functions of the relation*

$${}_L\subseteq \,:\, [\,\rangle_* Y, \langle\,]_* X \longrightarrow \text{Statements}\,. \tag{453}$$

One has the following commutative diagrams

$$\begin{tikzcd}
\mathscr{P}([\,\rangle_* Y) \arrow[r, "R'"] & \mathscr{P}(\langle\,]_* X) \\
\mathscr{P}X \arrow[u, "{[\,\rangle_*}"] \arrow[r, "R"'] \arrow[ur, phantom, "\circlearrowright"] & \mathscr{P}Y \arrow[u, "{\langle\,]_*}"']
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathscr{P}([\,\rangle_* Y) & \mathscr{P}(\langle\,]_* X) \arrow[l, "L'"'] \\
\mathscr{P}X \arrow[u, "{[\,\rangle_*}"] \arrow[ur, phantom, "\circlearrowleft"] & \mathscr{P}Y \arrow[u, "{\langle\,]_*}"'] \arrow[l, "L"]
\end{tikzcd} \; . \tag{454}$$

Proof. In view of the definition of relation (453), one has, for $A \subseteq X$,

$$R'\big([\,\rangle_* A\big) = \{\langle y] \in \langle\,]_* Y \mid \forall_{x\in A}\, L[x\rangle \subseteq \langle y]\}\,.$$

In view of (386), one has

$$RA = \{y \in Y \mid L[x\rangle \subseteq \langle y]\}\,.$$

Hence $\langle\,]_* RA = R'[\,\rangle_* A$. The commutativity of the second square in (454) has a similar prooof.

$\square$

6.12.4 Biexact morphisms

We shall say that a morphism between binary relations $f, g : (X, Y, \rho) \to (X', Y', \rho')$ is *biexact* if f is right-exact and g is left-exact.

Corollary 6.17 *Canonical morphism* (452) *is biexact.*

Proof. Given subsets $A_1, A_2 \subseteq X$, one has the following sequence of equivalences and implications

$$A_1 \overset{\mathrm{R}}{\subseteq} A_2 \Leftrightarrow RA_1 \supseteq RA_2 \Rightarrow R'([\ \rangle_* A_1) = \langle\ ]_* RA_1 \supseteq \langle\ ]_* RA_2 = R'([\ \rangle_* A_2) \Leftrightarrow [\ \rangle_* A_1 \overset{\mathrm{R}'}{\subseteq} [\ \rangle_* A_2 \,.$$

Right-exactness of $[\ \rangle$ follows. Left-exactness of $\langle\ ]$ is a corollary of the second commutative square in (454). □

6.12.5 Right- and left-dense functions

Given binary relations ρ and ρ' as in Section 6.2.1, and an arbitrary function $f : X \to X'$, we shall say that f is *right-dense* if

$$R \circ L = R \circ f_* \circ f^* \circ L \,. \tag{455}$$

Similarly, given an arbitrary function $g : Y \to Y'$, we shall say that g is *left-dense* if

$$L \circ R = L \circ g_* \circ g^* \circ R \,. \tag{456}$$

Lemma 6.18 *Let $f, g : (X, Y, \rho) \to (X', Y', \rho')$ be a $\Leftrightarrow$-morphism between binary relations. If g is left-dense, then f is right-exact and vice-versa: if f is right-dense, then g is left-exact.*

Proof. Let $A_1, A_2 \subseteq X$. Lemma 6.3, in view of f, g being a $\Leftrightarrow$-morphism, yields the sequence of equivalences and implications

$$A_1 \overset{\mathrm{R}}{\subseteq} A_2 \Leftrightarrow RA_1 \supseteq RA_2 \Leftrightarrow g^* R f_* A_1 \supseteq g^* R f_* A_2 \Rightarrow g_* g^* R f_* A_1 \supseteq g_* g^* R f_* A_2 \Rightarrow L g_* g^* R f_* A_1 \subseteq L g_* g^* R f_* A_2 \,. \tag{457}$$

Left-density of g yields the sequence of equivalences

$$L g_* g^* R f_* A_1 \subseteq L g_* g^* R f_* A_2 \Leftrightarrow LR f_* A_1 \subseteq LR f_* A_2 \Leftrightarrow R f_* A_1 \supseteq R f_* A_2 \Leftrightarrow f_* A_1 \overset{\mathrm{R}}{\subseteq} f_* A_2 \,. \tag{458}$$

By combining (457) and (458), we deduce that f is right-exact. Left-exactness of g follows in a similar manner from the double hypothesis that f, g is a $\Leftrightarrow$-morphism and f is right-dense. □

6.12.6 Bidense morphisms

We shall say that a morphism between binary relations $f, g : (X, Y, \rho) \to (X', Y', \rho')$ is *bidense* if f is right-dense and g is left-dense.

Corollary 6.19 *A bidense $\Leftrightarrow$-morphism between binary relations is biexact.*

□

6.12.7

Consider the middle triangle of the canonical tower :

$$
\begin{tikzcd}
\mathscr{R}(\rho), \mathscr{L}(\rho) \arrow[dr, "{}_L\subseteq"] & \\
 & \text{Statements} \\
{[\;\rangle_* X, \langle\;]_* Y} \arrow[uu, hook] \arrow[uu, phantom, shift left=4, "\circlearrowright"] \arrow[ur, "{}_L\subseteq"'] &
\end{tikzcd}
\tag{459}
$$

6.12.8 Calculations in $(\mathscr{R}(\rho), \mathscr{L}(\rho), {}_L\subseteq)$

Let us denote by $[\;\rangle''$ and $\langle\;]''$ the right-relatives function and, respectively, the left-relatives function of the relation

$$
{}_L\subseteq \,:\, \mathscr{R}(\rho), \mathscr{L}(\rho) \longrightarrow \text{Statements}\,, \tag{460}
$$

and let us denote by R'' and by L'' the associated R- and L-functions between the power-sets of $\mathscr{R}(\rho)$ and $\mathscr{L}(\rho)$.

Lemma 6.20 *One has the following commutative diagrams*

$$
\begin{tikzcd}
\mathscr{P}\mathscr{L} \arrow[r, "L''"] & \mathscr{P}\mathscr{R} \\
\mathscr{R} \arrow[u, "{[\;\rangle''}"] \arrow[r, "L"'] \arrow[ur, phantom, "\circlearrowright"] & \mathscr{L} \arrow[u, "{\langle\;]''}"']
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathscr{P}\mathscr{L} & \mathscr{P}\mathscr{R} \arrow[l, "R''"'] \\
\mathscr{R} \arrow[u, "{[\;\rangle''}"] \arrow[ur, phantom, "\circlearrowleft"] & \mathscr{L} \arrow[u, "{\langle\;]''}"'] \arrow[l, "R"]
\end{tikzcd}
\tag{461}
$$

where $\mathscr{R} = \mathscr{R}(\rho)$ *and* $\mathscr{L} = \mathscr{L}(\rho)$.

Proof. Given families $\mathscr{E} \subseteq \mathscr{L}$ and $\mathscr{F} \subseteq \mathscr{R}$, one has

$$
L''\mathscr{E} = \{F \in \mathscr{R} \mid \forall_{E\in\mathscr{E}}\, LF \subseteq E\} = \{F \in \mathscr{R} \mid LF \subseteq \bigcap \mathscr{E}\} = \left\langle \bigcap \mathscr{E} \right]'' \tag{462}
$$

and

$$
\begin{aligned}
R''\mathscr{F} &= \{E \in \mathscr{E} \mid \forall_{F\in\mathscr{F}}\, LF \subseteq E\} = \{E \in \mathscr{E} \mid \forall_{F\in\mathscr{F}}\, F \supseteq RE\} \\
&= \{E \in \mathscr{E} \mid \bigcap \mathscr{F} \supseteq RE\} = \{E \in \mathscr{E} \mid L \bigcap \mathscr{F} \subseteq E\} = \left[\bigcap \mathscr{F} \right\rangle''\,.
\end{aligned}
\tag{463}
$$

This shows that relation $(\mathscr{R}(\rho), \mathscr{L}(\rho), {}_L\subseteq)$ is complete which is not surprising in view of the fact that it is canonically isomorphic to the complete lattices $(\mathscr{R}(\rho), \mathscr{R}(\rho), \supseteq)$ and $(\mathscr{L}(\rho), \mathscr{L}(\rho), \subseteq)$.

Since

$$
\bigcap \{E \in \mathscr{L} \mid LF \subseteq E\} = LF\,,
$$

we obtain the identity

$$
L''[F\rangle'' = \left\langle \bigcap [F\rangle'' \right]'' = \langle LF]'' \tag{464}
$$

that is equivalent to the commutativity of the left square in (461).

Exercise 185 *Prove commutativity of the right square is in* (461).

Lemma 6.21 *Canonical inclusion morphism* (459) *is bidense.*

Proof. Let $g : \langle\ ]_* Y \hookrightarrow \mathscr{L}$ be the canonical inclusion. Given a family $\mathscr{E} \subseteq \mathscr{L}$, one has

$$g_* g^* \mathscr{E} = \mathscr{E} \cap \langle\ ]_* Y\,.$$

Given $F \in \mathscr{R}$, one has

$$[F\rangle'' \cap \langle\ ]_* Y = \{\langle y] \in \mathscr{L} \mid LF \subseteq \langle y]\}\,,$$

hence

$$\bigcap \big([F\rangle'' \cap \langle\ ]_* Y\big) = \bigcap \{\langle y] \in \mathscr{L} \mid LF \subseteq \langle y]\} = LF\,.$$

Combined with identities (462) and (464), this yields the identity

$$L''\big([F\rangle'' \cap \langle\ ]_* Y\big) = \langle LF] = L''[F\rangle''$$

and left-density of $g : \langle\ ]_* Y \hookrightarrow \mathscr{L}$ follows.

Exercise 186 *Prove right-density of the canonical inclusion* $f : [\ \rangle_* X \hookrightarrow \mathscr{R}$.

By combining Lemma 6.21 with Lemma 6.18, we obtain the following corollary.

Corollary 6.22 *Canonical inclusion* (459) *is a bidense* $\Leftrightarrow$ *-morphism, hence it is biexact.*

□

6.13 Completion of a binary relation

6.13.1

The bottom and the middle morphisms in the canonical tower (465),

$$\begin{tikzcd}
\mathscr{R}(\rho), \mathscr{L}(\rho) \arrow[drr, "L^{\subseteq}"] & & \\
{[\ \rangle_* X, \langle\ ]_* Y} \arrow[u, hook] \arrow[rr, "L^{\subseteq}"] & & \text{Statements} \\
X, Y \arrow[u, twoheadrightarrow, "{[\ \rangle\ \ \langle\ ]}"] \arrow[urr, "\rho"'] & &
\end{tikzcd}\,, \tag{465}$$

are bidense and biexact. Composition preserves left- and right-density as well as left- and right-exactness, hence also their composite is both bidense and biexact.

6.13.2

The source of the composite morphism is the original binary relation ρ, the target is a complete relation that is canonically isomorphic to complete lattices $(\mathscr{R}(\rho), \mathscr{R}(\rho), \supseteq)$ and $(\mathscr{L}(\rho), \mathscr{L}(\rho), \subseteq)$.

6.13.3

In Theory of Ordered Sets those two lattices provide standard models for the biexact embedding of an arbitrary ordered set onto a bidense subset of a complete lattice.

6.13.4 The extended real line $(\bar{\mathbf{R}}, \leq)$

When applied to the standard linear order on the set of rational numbers $(\mathbf{Q}, \leq)$, we obtain a model for the *extended real line* $(\bar{\mathbf{R}}, \leq)$ where $\bar{\mathbf{R}}$ is the union of three disjoint sets

$$\bar{\mathbf{R}} = \{-\infty\} \cup \mathbf{R} \cup \{\infty\}. \tag{466}$$

If one uses complete lattice $(\mathscr{L}(\mathbf{Q}, \leq), \subseteq)$ as the model for the completion, then

(i) Real numbers $r \in \mathbf{R}$ correspond to the sets of lower bounds of nonempty bounded below subsets of $(\mathbf{Q}, \leq)$. In other words, $(\mathbf{R}, \leq)$ coincides with the ordered subset $(L_* \mathscr{P}_{\triangleright} \mathbf{Q}, \subseteq)$ of $(\mathscr{L}(\mathbf{Q}, \leq), \subseteq)$, cf. Section 6.8.4.

(ii) Extended real number $-\infty$ corresponds to $\emptyset$ viewed as the set of lower bounds of *not* bounded below subsets of $\mathbf{Q}$ (such subsets of $\mathbf{Q}$ are automatically not empty).

(iii) Extended real number ∞ corresponds to $\mathbf{Q}$ viewed as the set of lower bounds of the *empty* subset of $(\mathbf{Q}, \leq)$.

Exercise 187 *Make a similar description of* $\mathbf{R}$, $-\infty$, *and* ∞, *if one uses complete lattice* $(\mathscr{R}(\mathbf{Q}, \leq), \subseteq)$ *instead.*

You can take either of the above two descriptions as *the* definition of $-\infty$, $\mathbf{R}$ and ∞. It is wiser, however, to define *an* extended real line as *any* completion of ordered set $(\mathbf{Q}, \leq)$, and then to define $-\infty$ and ∞ as the smallest and, respectively, the greatest elements, and to define points of *a* real line to be the remaining elements.

6.14 Bimorphisms between binary relations

6.14.1

A binary relation provides an example of a mathematical structure where apart from the obvious definition of a morphism, there is yet another natural if less obvious definition.

6.14.2

We shall say that a pair of functions

$$f : X \to X' \qquad \text{and} \qquad Y \longleftarrow Y' : g \tag{467}$$

is a *$\sim$-bimorphism from ρ to ρ'* if

$$(f, \mathrm{id}_Y)^{\bullet} \rho' \sim (\mathrm{id}_X, g)^{\bullet} \rho \tag{468}$$

where $\sim$ is one of the three relations $\Rightarrow$, $\Leftarrow$, or $\Leftrightarrow$.

This triple definition of a bimorphism can be expresed by means of the corresponding diagrams

$$
(\Rightarrow\textbf{-bimorphism}) \qquad
\begin{tikzcd}
X', Y' \arrow[drr, "\rho'"] & & \\
X, Y' \arrow[u, "f"] \arrow[d, "g"'] & \arrow[r, phantom, "\Downarrow"] & \text{Statements} \\
X, Y \arrow[urr, "\rho"'] & &
\end{tikzcd}
\tag{469}
$$

$$
(\Leftarrow\textbf{-bimorphism}) \qquad
\begin{tikzcd}
X', Y' \arrow[drr, "\rho'"] & & \\
X, Y' \arrow[u, "f"] \arrow[d, "g"'] & \arrow[r, phantom, "\Uparrow"] & \text{Statements} \\
X, Y \arrow[urr, "\rho"'] & &
\end{tikzcd}
\tag{470}
$$

$$
(\Leftrightarrow\textbf{-bimorphism}) \qquad
\begin{tikzcd}
X', Y' \arrow[drr, "\rho'"] & & \\
X, Y' \arrow[u, "f"] \arrow[d, "g"'] & \arrow[r, phantom, "\Updownarrow"] & \text{Statements} \\
X, Y \arrow[urr, "\rho"'] & &
\end{tikzcd}
\tag{471}
$$

Exercise 188 *Suppose* f, g *is a* $\sim$*-bimorphism from* ρ *to* ρ' *and* f', g' *is a* $\sim$*-bimorphism from* ρ' *to* ρ''*. Show that* $f' \circ f, g \circ g'$ *is a* $\sim$*-bimorphism from* ρ *to* ρ''*.*

Lemma 6.23 *The following conditions are equivalent:*

(a) *A pair of functions* (467) *is a* $\Rightarrow$*-bimorphism.*

(b) $\forall_{y' \in Y'}\ f^*\langle y'] \subseteq \langle g(y')]$.

(c) $\forall_{x \in X}\ [f(x)\rangle \subseteq g^*[x\rangle$.

6.14.3

Condition (b) of Lemma 6.23 says that the diagram

$$
\begin{tikzcd}
\mathscr{P}X' & Y' \arrow[l, "\langle\ \,]"'] \arrow[d, "g"] \\
\mathscr{P}X \arrow[from=u, "f^*"'] \arrow[ur, phantom, "\subseteq"] & Y \arrow[l, "\langle\ \,]"]
\end{tikzcd}
\tag{472}
$$

is $\subseteq$-commutative, while Condition (c) says that the diagram

$$
\begin{tikzcd}
X' \arrow[r, "{[\ \rangle}"] & \mathscr{P}Y' \\
X \arrow[u, "f"] \arrow[r, "{[\ \rangle}"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g^*"']
\end{tikzcd}
\tag{473}
$$

is $\supseteq$-commutative.

Proof of Lemma 6.23. For every $x \in X$ and $y' \in Y'$, one has the following two sequences of equivalent statements

$$[f(x)\rangle \ni y' \iff \rho'(f(x), y') \iff f(x) \in \langle y'] \iff x \in f^*\langle y']$$

and

$$g^*[x\rangle \ni y' \iff [x\rangle \ni g(y') \iff \rho(x, g(y')) \iff x \in \langle g(y')] \,.$$

If any of the statements of the top sequence implies any of the statements in the bottom sequence, then any other statement of the top sequence implies any statement of the bottom one.

The first statement in the formulation of Lemma 6.23 says that

$$\forall_{x \in X} \left(\forall_{y' \in Y'}\ \rho'(f(x), y')\right) \Rightarrow \rho(x, g(y'))$$

which is equivalent to the statement

$$\forall_{y' \in Y'} \left(\forall_{x \in X}\ \rho'(f(x), y')\right) \Rightarrow \rho(x, g(y')) \,.$$

Condition (b) of Lemma 6.23 is equivalent to the statement

$$\forall_{x \in X} \left(\forall_{y' \in Y'}\ [f(x)\rangle \ni y'\right) \Rightarrow g^*[x\rangle \ni y' \,.$$

Finally, Condition (c) of Lemma 6.23 is equivalent to the statement

$$\forall_{y' \in Y'} \left(\forall_{x \in X}\ x \in f^*\langle y']\right) \Rightarrow x \in \langle g(y')] \,.$$

Thus, each of the conditions in Lemma 6.23 implies the remaining two. ☐

6.14.4

Note that the proof uses the fact that interchanging the order in which universal quantification is applied produces equivalent statements. The same holds for existential quantification of relations. This is an analogue of Fubini's Theorem stating that (under a mild integrability hypothesis) interchanging the order of integration in evaluation of an iterated double integral produces the same result.

Beware that interchanging the order in which universal and existential quantification are applied produces statements that are rarely equivalent.

Exercise 189 *Show that if f, g is a $\sim$-bimorphism from ρ to ρ', then g, f is a $\sim$-bimorphism from $(\rho')^{\mathrm{op}}$ to ρ^{op}.*

Later we shall see that this means that the category of binary relations and bimorphisms is equipped with a canonical $*$*-category* structure, i.e., it is a category with a canonical *anti-involution*. The exact meaning of these terms will be explained later. Here it is sufficient to say that $*$*-structures* play a fundamental role in Mathematics and, especially, in Mathematical Physics.

Corollary 6.24 *The following conditions are equivalent:*

(a) *A pair of functions* (467) *is a* $\Rightarrow$*-bimorphism.*

(b') $\forall_{B'\subseteq Y'}\ f^*LB' \subseteq L(g_*B')$.

(c') $\forall_{A\subseteq X}\ R(f_*A) \subseteq g^*RA$.

Proof. Condition (b) of Lemma 6.23 implies that

$$\forall_{y'\in B'}\ f^*\langle y'] \subseteq \langle g(y')]$$

which, in turn, implies that

$$\bigcap_{y'\in B'} f^*\langle y'] \subseteq \bigcap_{y'\in B'} \langle g(y')]\,. \tag{474}$$

The left-hand-side of (474) equals, in view of identity (161),

$$f^*\Bigl(\bigcap_{y'\in B'}\langle y']\Bigr)$$

while the right-hand-side equals

$$\bigcap_{x\in g_*B'}\langle x] = L(g_*B')\,.$$

This demonstrates that statement (b) implies statement (b').

Exercise 190 *Write down two proofs of the equivalence*

$$\text{statement (c)} \iff \text{statement (c')}\,,$$

an explicit proof and a proof that deduces this from the already proven equivalence of statements (b) and (b').

The reverse implications (b') $\Longrightarrow$ (b) and (c') $\Longrightarrow$ (c) hold because

$$g_*\{y'\} = \{g(y')\},\quad f_*\{x\} = \{f(x)\},\quad L\{y\} = \langle y],\quad \text{and}\quad R\{x\} = [x\rangle\,,$$

hence statement (b) is a *special case* of statement (b') and statement (c) is a *special case* of statement (c'), cf. Section 1.7.4. $\square$

6.14.5

Condition (b′) of Corollary 6.24 says that the diagram

$$\begin{tikzcd}
\mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{P}Y' \arrow[l, "L"'] \arrow[d, "g_*"] \\
\mathscr{P}X \arrow[ur, phantom, "\subseteq"] & \mathscr{P}Y \arrow[l, "L"]
\end{tikzcd} \tag{475}$$

is ⊆-commutative, while Condition (c′) says that the diagram

$$\begin{tikzcd}
\mathscr{P}X' \arrow[r, "R"] & \mathscr{P}Y' \\
\mathscr{P}X \arrow[u, "f_*"] \arrow[r, "R"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g^*"']
\end{tikzcd} \tag{476}$$

is ⊇-commutative.

Exercise 191 *State for ⇐-bimorphisms the analog of Lemma 6.23 and draw the analogs of diagrams* (472) *and* (473).

Exercise 192 *State for ⇐-bimorphisms the analog of Corollary 6.24 and draw the analogs of diagrams* (475) *and* (476).

6.14.6

We can compose bimorphisms of either type but the result is going to be a ⇒-bimorphism or a ⇐-bimorphism, if the bimorphisms we compose are both ⇒- or ⇐-bimorphisms.

6.15 Galois connections

6.15.1

The following characterization of ⇔-bimorphisms is an immediate corollary of Lemma 6.23.

Corollary 6.25 *The following conditions are equivalent:*

(a) *A pair of functions* (467) *is a* ⇔ *-bimorphism.*

(b) $\forall_{y' \in Y'}\ f^*\langle y'] = \langle g(y')]\,.$

(c) $\forall_{x \in X}\ [f(x)\rangle = g^*[x\rangle\,.$

6.15.2

Condition (b) in Corollary 6.25 says that the square diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & Y' \arrow[l, "\langle\ ]"'] \arrow[d, "g"] \arrow[dl, phantom, "\circlearrowleft"] \\ \mathscr{P}X & Y, \arrow[l, "\langle\ ]"] \end{tikzcd} \tag{477}$$

is commutative, while Condition (c) says that the square diagram

$$\begin{tikzcd} X' \arrow[r, "\lbrack\ \rangle"] & \mathscr{P}Y' \\ X \arrow[u, "f"] \arrow[r, "\lbrack\ \rangle"'] \arrow[ur, phantom, "\circlearrowright"] & \mathscr{P}Y \arrow[u, "g^*"'] \end{tikzcd} \tag{478}$$

is commutative.

Exercise 193 *State the analogue of Corollary 6.24 for $\Leftrightarrow$-bimorphisms and represent two out of three conditions as commutativity of appropriate diagrams.*

Exercise 194 *Show that, if f, g is a $\Leftrightarrow$-bimorphism, then f is a morphism $(X, \succsim) \longrightarrow (X', \succsim)$ of preordered sets and, likewise, g is a morphism of preordered sets $(Y', \precsim) \longrightarrow (Y, \precsim)$. Here $\succsim$ and $\precsim$ are the corresponding canonical preorders associated with the binary relations ρ and ρ', cf. Section 6.3.9.*

6.15.3 Galois connections

A *Galois connection* is a traditional term for a $\Leftrightarrow$-bimorphism between ordered sets, i.e., when $X = Y$, $X' = Y'$, while ρ and ρ' are order relations. From now on, we shall adopt this term to arbitrary $\Leftrightarrow$-bimorphisms between arbitrary binary relations.

Exercise 195 *Let $(X, \preccurlyeq)$ be an ordered set and*

$$\mathscr{P}_{\sup}X := \{A \subseteq X \mid \sup A \text{ exists}\}$$

be the set of those subsets of X that have supremum. Show that the pair of functions

$$\begin{tikzcd} (X, \preccurlyeq) \arrow[d, shift left, "\langle\ ]"] \\ (\mathscr{P}_{\sup}X, \subseteq) \arrow[u, shift left, "\sup"] \end{tikzcd} \tag{479}$$

forms a Galois connection.

Exercise 196 *State the analogous assertion for the infimum function.*[5]

[5] Hint: if you do this mindlessly, you are likely to state it incorrectly and you will receive no credit.

6.15.4 Equipotence classes of binary relations represented as Galois connections

The proof of Theorem 6.1 provides a complete description of all Galois connections from the membership relation on a set X that we will denote $\in_X$, to the opposite membership relation on a set Y that we will denote $\ni_Y$.

Theorem 6.26 *Every Galois connection from $\in_X$ to $\ni_Y$ has the form*

$$[\,\rangle_\rho\,, {}_\rho\langle\,]$$

for some binary relation $\rho : X, Y \to \text{Statements}$. In particular, there is a canonical bijective correspondence

$$\left\{ \begin{array}{c} \textit{Galois connections} \\ \textit{from } \in_X \textit{ to } \ni_Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Equipotence classes of binary relations} \\ \rho : X, Y \longrightarrow \text{Statements} \end{array} \right\}. \tag{480}$$

6.16 Left and right adjoints: existence and exactness properties

6.16.1 Terminology

When f, g forms a $\Leftrightarrow$-connection, we say that f is a *left adjoint* of g and that g is a *right adjoint* of f.

6.16.2 Uniqueness up to an equivalence

If f_1 and f_2 are left adjoints of g, then

$$\forall_{x \in X}\, [f_1(x)\rangle = g^*[x\rangle = [f_2(x)\rangle\,,$$

i.e., f_1 is $\succsim$-equivalent to f_2.

Vice-versa, if f_2 is a left adjoint of g and f_1 is $\succsim$-equivalent to f_2, then also f_1 is a left adjoint of g.

Similarly for right adjoints of any function $f : X \to X'$: any two right adjoints $Y' \to Y$ of f are $\precsim$-equivalent and any function $Y' \to Y$ $\precsim$-equivalent to a right adjoint of f is a right adjoint of f.

6.16.3 Existence of left and right adjoints

It remains to characterize functions $f : X \to X'$ that admit a right adjoint, and functions $g : Y' \to Y$ that admit a left adjoint.

For a given function $f : X \to X'$ and binary relations ρ and ρ', let us consider the diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] & \langle\,]_* Y' \arrow[l, hook'] & Y' \arrow[l, two heads, "\langle\,]"'] \\ \mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\,]_* Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\,]"'] \end{tikzcd} \;. \tag{481}$$

The canonical inlusions are marked by hooked arrows, the canonical surjections—by two arrowheads.

Lemma 1.7 guarantees that the following three conditions are equivalent:

(i) One has

$$f^*_*\langle\,]_*Y' \subseteq \langle\,]_*Y\,,$$

i.e., diagram (481) can be completed to a commutative diagram

$$\begin{tikzcd}
\mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] & \langle\,]_*Y' \arrow[l, hook'] \arrow[d, "f^*"'] & Y' \arrow[l, two heads, "\langle\,]"'] \\
\mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\,]_*Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\,]"']
\end{tikzcd}\quad. \tag{482}$$

(ii) Diagram (481) can be completed to a commutative diagram

$$\begin{tikzcd}
\mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] & \langle\,]_*Y' \arrow[l, hook'] \arrow[d, "f^*"'] & Y' \arrow[l, two heads, "\langle\,]"'] \arrow[d, "g"] \\
\mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\,]_*Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\,]"']
\end{tikzcd} \tag{483}$$

for some function $g : Y' \to Y$.

(iii) Diagram (481) can be completed to a commutative diagram

$$\begin{tikzcd}
\mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] & \langle\,]_*Y' \arrow[l, hook'] & Y' \arrow[l, two heads, "\langle\,]"'] \arrow[d, "g"] \\
\mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\,]_*Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\,]"']
\end{tikzcd} \tag{484}$$

for some function $g : Y' \to Y$.

6.16.4

Condition (i) means that f is a morphism of structures of "topological type",

$$(X, \mathscr{A}) \longrightarrow (X', \mathscr{A}')\,,$$

where $\mathscr{A} = \langle\,]_*Y$ and $\mathscr{A}' = \langle\,]_*Y'$.

6.16.5

Condition (iii) means that g is a right adjoint of f.

6.16.6

Given a subset $B' \subseteq Y'$, one has

$$f^*LB' = f^*\Big(\bigcap\{\langle y] \mid y \in B'\}\Big) = \bigcap_{y' \in B'} f^*[y'\rangle \tag{485}$$

and the right-hand-side of (485) belongs to $\mathscr{L}(\rho)$ if Condition (i) holds. In particular, each of the previous three conditions is equivalent to the condition:

(iv) Diagram (481) can be completed to a commutative diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[r, hook'] \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[d, "f^*"'] \arrow[r, hookleftarrow] & \langle\ ]_* Y' \arrow[d, "f^*"'] & Y' \arrow[l, two heads, "\langle\ ]"'] \arrow[d, "g"] \\ \mathscr{P}X \arrow[r, hook'] & \mathscr{L}(\rho) \arrow[r, hookleftarrow] & \langle\ ]_* Y & Y \arrow[l, two heads, "\langle\ ]"'] \end{tikzcd} \tag{486}$$

for some function $g : Y' \to Y$.

This concludes characterization of functions $f : X \to X'$ that admit a right adjoint. There is an analogous characterization of functions $g : Y' \to Y$ that admit a left adjoint.

Corollary 6.27 *If f, g is a $\Leftrightarrow$-connection, then f is right-exact and g is left-exact.*

In other words, if f admits a right adjoint, then f is right-exact. Similarly, if g admits a left adjoint, then g is left-exact.

Proof. Let A_1 and A_2 be subsets of X such that $RA_1 \supseteq RA_2$. Then,

$$Rf_* A_1 = g^* RA_1 \supseteq g^* RA_2 = Rf_* A_2 .$$

Similarly, if B_1 and B_2 are subsets of Y such that $LB_1 \supseteq LB_2$, then

$$Lg_* B_1 = f^* LB_1 \supseteq f^* LB_2 = Lg_* B_2 .$$

□

Note that Corollary 6.27 is a stronger version of the assertion of Exercise 194.

Corollary 6.28 *When $X = Y$ and $X' = Y'$, and $f : X \to X'$ admits both a right adjoint and a left adjoint, then f is exact, cf. Section 6.11.2.* □

Note that the right and the left adjoint of f, are, generally, not equal to each other.

6.16.7 Example: f^* and its two adjoints f_* and $f_!$

A fundamentally important example of this happening is provided by the direct image, inverse image, and conjugate image, associated with a function $f : X \to Y$, and operating between power-sets $(\mathscr{P}X, \subseteq)$ and $(\mathscr{P}Y, \subseteq)$, cf. (157). Of this trio, f_* is right-exact but, generally, not left-exact, $f_!$ is left-exact but, generally, not right-exact, whereas f^* is both left- and right-exact. And indeed,

$$f^* \text{ has both a left and a right adjoints, namely } f_* \text{ and } f_! . \tag{487}$$

According to Theorem 6.15 the preimage morphisms are the *only* exact functions from $(\mathscr{P}Y, \subseteq)$ to $(\mathscr{P}X, \subseteq)$.

6.16.8

A particularly interesting phenomenon, known in Category Theory as *equivalence of categories*, occurs when a function f admits a left adjoint g that is simultaneously a right adjoint of f. This concept has been a cornerstone of Mathematics for the last sixty years.

6.16.9 Example: Relations versus correspondences

Let X' be the set of n-ary relations $\mathrm{Rel}(S_1, \ldots, S_n)$ between elements of sets $S_1, \ldots, S_n$, preordered by implication $\Longrightarrow$. Let X be the power-set of the Cartesian product $S_1 \times \cdots \times S_n$, ordered by inclusion $\subseteq$. The function $X \to X'$ introduced in Section 1.12.24, that assigns in a canonical manner to a correspondence $C \subseteq S_1 \times \cdots \times S_n$, a relation $\rho_C \in \mathrm{Rel}(S_1, \ldots, S_n)$, is both a left and a right adjoint of the graph function $\Gamma : X' \to X$, cf. Section 1.12.17.

7 Topology

7.1 Binary relations between elements and subsets of a given set

7.1.1

The only primitive relation of Mathematics, membership relation, cf. Section 6.3.1, tells us which elements of a set X belong to which subsets $A \subseteq X$ and nothing else. It is an important example of a binary relation between elements and subsets of a given set, for this reason we shall denote an arbitrary relation of this kind by a Greek minuscule letter *epsilon*,

$$\varepsilon : X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad x, A \;\longmapsto\; \text{“ } x\varepsilon A \text{ ”}. \tag{488}$$

We shall refer to relations (488) as *ε-relations*.

7.1.2 The *conjugate* ε-relation

The set $\mathrm{Rel}(X, \mathscr{P}X$ of ε-relations on X is equipped with distinguished elements $\in$ and $\notin$, and with two unary operations $\mathscr{P}X \longrightarrow \mathscr{P}X$,

$$\neg_{\bullet} : \varepsilon \longmapsto \neg \circ \varepsilon \qquad \text{and} \qquad (\mathrm{id}_X, \complement)^{\bullet} : \varepsilon \longmapsto \varepsilon \circ (\mathrm{id}_X, \complement)\,, \tag{489}$$

and their composition $\neg_{\bullet} \circ \complement^{\bullet}$ is the operation

$$\varepsilon \longmapsto \varepsilon^c := \neg \circ \varepsilon \circ \,. \tag{490}$$

We shall refer to (490) as the *conjugation* operation and to ε^c as the *conjugate* of ε.

7.1.3

Conjugation is an involution of $\mathrm{Rel}(X, \mathscr{P}X)$ *up to equipotence*, i.e., the its square is equipotent to the identity operation on $\mathrm{Rel}(X, \mathscr{P}X)$,

$$\forall_{\varepsilon \in \mathrm{Rel}(X,\mathscr{P}X)}\ \varepsilon^{cc} \Longleftrightarrow \varepsilon\,.$$

7.1.4 Unary operations on the power-set

The left-relatives function of an ε-relation is a unary operation $\langle\ ] : \mathscr{P}X \to \mathscr{P}X$ and the surjective function

$$\mathrm{Rel}(X, \mathscr{P}X) \longrightarrow \mathrm{Op}_1 \mathscr{P}X\,, \qquad \varepsilon \;\longmapsto\; {}_{\varepsilon}\langle\ ] \tag{491}$$

induces a canonical bijective correspondence

$$\left\{ \begin{array}{c} \text{equipotence classes of relations} \\ \varepsilon : X, \mathscr{P}X \longrightarrow \text{Statements} \end{array} \right\} \longleftrightarrow \mathrm{Op}_1 \mathscr{P}X\,. \tag{492}$$

7.1.5

The set $\mathrm{Op}_1\mathscr{P}X$ of unary operations on $\mathscr{P}X$ is equipped with distinguished elements $\mathrm{id}_{\mathscr{P}X}$ and $\complement$, and with two unary operations $\mathrm{Op}_1\mathscr{P}X \longrightarrow \mathrm{Op}_1\mathscr{P}X$,

$$\complement_\bullet : \lambda \longmapsto \complement \circ \lambda \qquad \text{and} \qquad \complement^\bullet : \lambda \longmapsto \lambda \circ \complement\,. \tag{493}$$

Exercise 197 *Show that function* (491) *is a homomorphism*

$$\left(\mathrm{Rel}(X, \mathscr{P}X)\,; \in, \notin, \neg_\bullet, (\mathrm{id}_X, \complement)^\bullet\right) \longrightarrow \left(\mathrm{Op}_1\mathscr{P}X\,; \mathrm{id}_{\mathscr{P}X}, \complement, \complement_\bullet, \complement^\bullet\right)\,. \tag{494}$$

In particular, one has

$$_{\varepsilon^c}\langle\ ] = \complement \circ \langle\ ] \circ \complement\,. \tag{495}$$

7.1.6 Properties of the $_\varepsilon\langle\ ]$-operation transferred to relation ε

As an operation on a set equipped with an order relation, operation $_\varepsilon\langle\ ]$ may have numerous properties. It may be an endomorphism of the ordered set $(\mathscr{P}X, \subseteq)$, it may be idempotent, it may be left- or right-exact, it may be finitely left- or right-exact, etc. In each case we shall transfer the corresponding property to the relation itself and say, for example, that (488) is *monotonic, idempotent, left-* or *right-exact, finitely left-* or *right-exact,* etc.

Exercise 198 *Show that ε is monotonic if and only if ε^c is monotonic.*

7.1.7 ε-invariant and ε-coinvariant families

Recall, that given a unary operation on any set S, there are two distinguished families of subsets $E \subseteq S$, *invariant* and *coinvariant* ones, cf. Sections 2.4.10 and 2.4.11. As usual, when $S = \mathscr{P}X$, we shall refer to such subsets as *families of subsets.*

We shall say that a family $\mathscr{E} \subseteq \mathscr{P}X$ of subsets of X is

($\mathcal{P}\mathcal{P}X^\varepsilon$) *$\varepsilon$-invariant* if $\mathscr{E}$ is invariant under operation $_\varepsilon\langle\ ]$, and we shall denote the set of such families by $\mathscr{P}\mathscr{P}X^\varepsilon$;

($\mathcal{P}\mathcal{P}X_\varepsilon$) *$\varepsilon$-coinvariant* if $\mathscr{E}$ is coinvariant under operation $_\varepsilon\langle\ ]$, and we shall denote the set of such families by $\mathscr{P}\mathscr{P}X_\varepsilon$.

Exercise 199 *Show that $\mathscr{E}$ is ε-invariant if and only if the family of the complements $\complement_*\mathscr{E}$ is ε^c-invariant.*

7.1.8

7.1.9 X-indexed families $(\mathscr{E}_x)_{x \in X}$ of families of subsets of X.

The right-relatives function of a relation ε is a family of families of subsets of X and the assignment

$$\varepsilon \longmapsto [\ \rangle_\varepsilon$$

induces a canonical bijective correspondence

$$\left\{ \begin{array}{c} \text{equipotence classes of relations} \\ \varepsilon : X, \mathscr{P}X \longrightarrow \text{Statements} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} X\text{-indexed families } (\mathscr{E}_x)_{x \in X} \\ \text{of families of subsets of } X \end{array} \right\}. \qquad (496)$$

The equipotence class of membership relation $\in$ corresponds to the family of *principal ultrafilters*

$$X \longrightarrow \mathscr{P}\mathscr{P}X, \qquad x \longmapsto \mathscr{P}_x X, \qquad (497)$$

cf. Section 6.7.3.

7.1.10 Invariant and coinvariant ε-relations

We shall say that relation (488) is

(a) *invariant* if each family $[x\rangle$ is ε-invariant;

(b) *coinvariant* if each family $[x\rangle$ is ε-coinvariant.

Lemma 7.1 (a) *A relation ε is invariant if and only if $\langle\,] \circ \langle\,] \subseteq \langle\,]$.*

(b) *A relation ε is coinvariant if and only if $\langle\,] \circ \langle\,] \supseteq \langle\,]$.*

Note. The target of $\langle\,]$ is the power-set ordered by relation $\subseteq$. Accordingly, sets of $\mathscr{P}X$-valued functions are ordered by the induced relation, denoted by the same symbol.

Proof. $\Leftrightarrow$-connection $\langle\,]_*, \langle\,]^*$ yields, for any $x \in X$, equivalence of statements

$$\langle\,]_*[x\rangle \subseteq [x\rangle \quad \Leftrightarrow \quad [x\rangle \subseteq \langle\,]^*[x\rangle. \qquad (498)$$

One has

$$[x\rangle = \{A \subseteq X \mid x \in \langle A]\} \qquad \text{and} \qquad \langle\,]^*[x\rangle = \{A \subseteq X \mid \langle A] \in [x\rangle\} = \{A \subseteq X \mid x \in \langle\langle A]]\}.$$

The right-hand-side of (498) is, therefore, equivalent to

$$\forall_{A \subseteq X}\; x \in \langle A] \Rightarrow x \in \langle\langle A]]. \qquad (499)$$

By quantifying over $x \in X$ each statement in (498) and (499), we obtain equivalence of invariance of relation ε with the statement

$$\forall_{A \subseteq X}\; \langle A] \subseteq \langle\langle A].$$

The latter means that $\langle\,] \subseteq \langle\,] \circ \langle\,]$.

Exercise 200 *Prove Part* (b).

Hint. Utilize Galois connection $\langle\,]^*, \langle\,]_!$.

Exercise 201 *Show that*

$$_\varepsilon\langle\,] \circ {}_\varepsilon\langle\,] \subseteq {}_\varepsilon\langle\,] \qquad \textit{if and only if} \qquad {}_{\varepsilon^c}\langle\,] \circ {}_{\varepsilon^c}\langle\,] \supseteq {}_{\varepsilon^c}\langle\,].$$

7.1.11 Idempotent relations

Recall that relation (488) is said to be *idempotent* if the associated operation $\langle\ ]$ is idempotent, i.e.,

$$\langle\ ] \circ \langle\ ] = \langle\ ]\,.$$

Corollary 7.2 *A relation ε is idempotent in either of the following two cases*

(a) *ε is invariant and $\varepsilon \Longrightarrow \in$.*

(b) *ε is coinvariant and $\in \Longrightarrow \varepsilon$.*

Proof. A relation ε being weaker than membership relation $\in$ is equivalent to

$$_{\varepsilon}\langle\ ] \subseteq {}_{\in}\langle\ ]\,. \tag{500}$$

Noting that $_{\in}\langle\ ] = \mathrm{id}_{\mathscr{P}X}$, by composing both sides of (500) with $_{\varepsilon}\langle\ ]$, we obtain

$$_{\varepsilon}\langle\ ] \circ {}_{\varepsilon}\langle\ ] \subseteq {}_{\varepsilon}\langle\ ]\,.$$

The reverse containment is, in view of Lemma 7.1, equivalent to ε being invariant.

Exercise 202 *Prove Part* (b).

□

Exercise 203 *Show that*

$$\varepsilon \Longrightarrow \in \qquad \text{if and only if} \qquad \in \Longrightarrow \varepsilon^{c}\,.$$

Corollary 7.3 (a) *If a relation ε is idempotent, monotonic and weaker than $\in$, then the family*

$$\mathscr{E} := \langle\ ]_{*}\mathscr{P}X \tag{501}$$

is a sup*-closed subset of $(\mathscr{P}X, \subseteq)$ and*

$$\forall_{A\subseteq X}\ \langle A] = \max\{E \in \mathscr{E} \mid E \subseteq A\}\,. \tag{502}$$

(b) *If ε is idempotent, monotonic and stronger than $\in$, then family* (501) *is an* inf*-closed subset of $(\mathscr{P}X, \subseteq)$ and*

$$\forall_{A\subseteq X}\ \langle A] = \min\{E \in \mathscr{E} \mid E \supseteq A\}\,. \tag{503}$$

Proof. Let $\mathscr{A} \subseteq \mathscr{P}X$ be a family of subsets of X. One has

$$\forall_{A\in\mathscr{A}}\ \langle A] \subseteq \bigcup_{B\in\mathscr{A}} \langle B]\,.$$

This implies, if ε is monotonic, that

$$\forall_{A\in\mathscr{A}}\ \langle\langle A]] \subseteq \left\langle \bigcup_{B\in\mathscr{A}} \langle B] \right]. \tag{504}$$

If ε is idempotent, then (504) becomes

$$\forall_{A\in\mathscr{A}}\ \langle A] \subseteq \left\langle \bigcup_{B\in\mathscr{A}} B \right]$$

which is equivalent to

$$\bigcup_{A\in\mathscr{A}} \langle A] \subseteq \left\langle \bigcup_{B\in\mathscr{A}} B \right].$$

If ε is weaker than $\in$, then

$$\left\langle \bigcup_{B\in\mathscr{A}} B \right] \subseteq \bigcup_{A\in\mathscr{A}} \langle A].$$

This proves that the supremum of family $\langle\ ]_*\mathscr{A}$ in $(\mathscr{P}X, \subseteq)$ is a member of $\mathscr{E}$.

Moreover, given an arbitrary subset $A \subseteq X$ and a member $E \in \mathscr{E}$, contained in A, one has

$$E = \langle\langle E]] \subseteq \langle A] \subseteq A\,,$$

and this means that the family

$$\{E \in \mathscr{E} \mid E \subseteq A\}$$

has the greatest element, namely $\langle A]$.

Exercise 204 *Prove Part* (b).

□

Exercise 205 *Show that*

$$_{\varepsilon^c}\langle\ ]_*\mathscr{P}X \;=\; \complement_*\left({}_{\varepsilon}\langle\ ]_*\mathscr{P}X\right)\,. \tag{505}$$

In other words, the family $\langle\ ]_*\mathscr{P}X$ for the *conjugate* of ε consists of the *complements* of members of the family $\langle\ ]_*\mathscr{P}X$ for ε.

7.1.12 Interior relations

A relation (488) is said to be an *interior* relation if it is

(a) weaker than $\in$, i.e., $\varepsilon \Longrightarrow \in$,

(b) *invariant*,

(c) finitely left-axact.

7.1.13 Adherence relations

A relation (488) is said to be an *adherence* relation if it is

(a) stronger than $\in$, i.e., $\in \Longrightarrow \varepsilon$,

(b) *coinvariant*,

(c) finitely right-axact.

Exercise 206 *Show that ε is an interior relation if and only if ε^c is an adherenece relation.*

7.1.14

Both for interior and for adherence relations, operation ${}_\varepsilon\langle\ ]$ is a *retraction* of $\mathscr{P}X$ onto $\mathscr{E}$, cf. (501).

7.1.15

Finite left-exactness, as well as finite right-exactness, of a function between ordered sets—either one of these conditions implies that the function is a morphism of ordered sets. Then, according to Corollary 7.3, family $\mathscr{E}$ is sup-closed if ε is an interior relation, and it is inf-closed if ε is an adherence relation.

7.1.16 *Open* subsets

If operation $\langle\ ]$ is finitely left-exact, then $\mathscr{E}$ is finitely inf-closed and we conclude that, for every interior relation on X, family $\mathscr{E}$ is

$$\text{sup-}\textit{closed, and finitely } \text{inf-}\textit{closed.} \tag{506}$$

In this case, we shall refer to members of family $\mathscr{E}$ as *open subsets*.

7.1.17 Topologies on a set X

A family of subsets of a set X satisfying double Condition (506) is called *a topology* on X. We shall signal that a family is a topology by employing generic notation $\mathscr{T}$.

7.1.18 *Closed* subsets

If $\langle\ ]$ is finitely right-exact, then $\mathscr{E}$ is finitely sup-closed and we conclude that, for every adherence relation on X, family $\mathscr{E}$ is

$$\textit{finitely } \text{sup-}\textit{closed, and } \text{inf-}\textit{closed.} \tag{507}$$

In this case, we shall refer to members of family $\mathscr{E}$ as *closed subsets*. We shall also signal that a family satisfies double Condition (507) by adopting generic notation $\mathscr{Z}$ for such families.

7.1.19 Closed versus open subsets

Since the complement operation $\complement$ is an isomorphism of $(\mathscr{P}X, \subseteq)$ with $(\mathscr{P}X, \supseteq)$, we note that

$$\forall_{\mathscr{E} \subseteq \mathscr{P}X} \quad \begin{array}{c}\text{family } \mathscr{E} \\ \text{is a topology}\end{array} \quad \Leftrightarrow \quad \begin{array}{c}\text{family of the complements } \complement_* \mathscr{E} \\ \text{satisfies Condition (507).}\end{array} \tag{508}$$

7.2 *Topologizing* a set

7.2.1 Six equivalent approaches to topologizing a set

The chain of assignments

$$\varepsilon \longmapsto {}_\varepsilon\langle\ ] \longmapsto \mathscr{E} := {}_\varepsilon\langle\ ]_* \mathscr{P}X \tag{509}$$

induces the commutative diagram of canonical identifications

$$
\begin{tikzcd}
\left\{\begin{matrix}\text{equipotence classes}\\ \text{of interior relations}\end{matrix}\right\} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "(\ )^c"] & \left\{\begin{matrix}\text{finitely left-exact}\\ \text{operations } \lambda \in \mathrm{Op}_1\mathscr{P}X\\ \text{such that } \lambda^2 = \lambda \subseteq \mathrm{id}_{\mathscr{P}X}\end{matrix}\right\} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\mathrm{ad}_{\mathtt{C}}"] & \left\{\begin{matrix}\text{topologies}\\ \text{on } X\end{matrix}\right\} \arrow[d, leftrightarrow, "\mathtt{C}_*"] \\
\left\{\begin{matrix}\text{equipotence classes}\\ \text{of adherence relations}\end{matrix}\right\} \arrow[r, leftrightarrow] & \left\{\begin{matrix}\text{finitely right-exact}\\ \text{operations } \lambda \in \mathrm{Op}_1\mathscr{P}X\\ \text{such that } \mathrm{id}_{\mathscr{P}X} \subseteq \lambda = \lambda^2\end{matrix}\right\} \arrow[r, leftrightarrow] & \left\{\begin{matrix}\text{finitely sup-closed,}\\ \text{and inf-closed families}\\ \text{of subsets of } X\end{matrix}\right\},
\end{tikzcd}
\tag{510}
$$

where $\mathrm{ad}_{\mathtt{C}}$ is the adjoint action of the involution $\mathtt{C}$, cf. Section 2.2.41. Commutativity of this diagram is a consequence of Exercise 197, formula (495) and Exercise 205.

Elements of any of the six sets appearing in Diagram (510) are the "data" that transform an arbitrary set X into a mathematical structure of a certain type.

To *topologize* a set X is to provide a list of six elements, one in each of the above six sets, that correspond to each other under those canonical correspondences. This is what we should accept as the definition of a topological structure on a set.

7.2.2 Terminology

When a topological structure is fixed, the corresponding elements in those six sets, traversing Diagram (510) clockwise, have the following *interpretation*,

(a) the relation $x, A \;\longmapsto\;$ " *x is an interior point of a subset A* " ;

(b) the operation $A \;\longmapsto\;$ " the *interior* of A " ;

(c) the topology ;

(d) the family of all closed subsets ;

(e) the operation $A \;\longmapsto\;$ " the *closure* of A " ;

(f) the relation $x, A \;\longmapsto\;$ " *x is an adherenece point of a subset A* " .

7.2.3 Topological spaces: the standard definition

Any one of the above six types of mathematical structures can be chosen to serve as the definition of a *topological space*. The definition that has long been accepted as *standard* reads:

$$\textit{a set equipped with a topology } (X, \mathscr{T})\,.$$

7.2.4 Standard notation

The interior of a subset A of a topological space is usually denoted $\mathring{A}$ or $\mathrm{Int}\,A$. The closure of a subset A is denoted either $\bar{A}$ or $\mathrm{Cl}\,A$.

7.2.5 An alternative approach to topologizing a set

Departing from an ε-relation on a set X, we focused our attention exclusively on the associated operation on the power-set of X. There is an equivalent approach that focuses on the X-indexed family of families of subsets of X,

$$\big([x\rangle\big)_{x\in X}, \tag{511}$$

cf. Section (7.1.9).

Exercise 207 *Show that, for every ε-relation and every $x \in X$, one has*

$$[x\rangle_{(\mathrm{id}_X,\complement)^\bullet\varepsilon} = \complement_*[x\rangle_\varepsilon \tag{512}$$

where $\complement_*$ denotes the direct image operation on $\mathscr{P}\mathscr{P}X$ induced by the complement operation on $\mathscr{P}X$.

Exercise 208 *Show that, for every ε-relation and every $x \in X$, one has*

$$[x\rangle_{\neg\bullet\varepsilon} = \complement[x\rangle_\varepsilon \tag{513}$$

where $\complement$ denotes the complement operation on $\mathscr{P}\mathscr{P}X$.

7.2.6

As a corollary of equalities (512) and (513), we obtain the following lemma.

Lemma 7.4 *One has*

$$[\ \rangle_{(\mathrm{id}_X,\complement)^\bullet\varepsilon} = \complement_*\circ[\ \rangle_\varepsilon, \qquad [\ \rangle_{\neg\bullet\varepsilon} = \complement\circ[\ \rangle_\varepsilon \qquad \textit{and} \qquad [\ \rangle_{\varepsilon^c} = \complement\circ\complement_*\circ[\ \rangle_\varepsilon. \tag{514}$$

7.2.7

Note that operations $\complement$ and $\complement_*$ commute

$$\complement\circ\complement_* = \complement_*\circ\complement.$$

7.2.8

The canonical Galois connection, $\langle\ ], [\ \rangle$, cf. Section 6.3.6, translates properties of operation $\langle\ ]$ on $\mathscr{P}X$ into the corresponding properties of the family of families of subsets of X,

Lemma 7.5 (a) *A relation ε is monotonic if and only if each $[x\rangle$ is a right-saturated subset of $(\mathscr{P}X, \subseteq)$, cf. Section 6.5.6.*

(b) *If ε is finitely left exact, then each $[x\rangle$ is a finitely* inf*-closed subset of $(\mathscr{P}X, \subseteq)$.*

(c) *If ε is monotonic and each $[x\rangle$ is a finitely* inf*-closed subset of $(\mathscr{P}X, \subseteq)$ then ε is finitely left exact.*

(d) *A relation ε is finitely left exact if and only if each $[x\rangle$ is a filter of subsets of X, cf. Section 6.6.*

(e) *A relation ε is weaker than $\in$ if and only if each $[x\rangle$ is contained in the principal ultrafilter $\mathscr{P}_x X$, cf. Section 6.7.3.*

Proof of (a). Given $x \in X$ and $A, B \subseteq X$, one has the statement

$$x \in \langle A] \Rightarrow x \in \langle B] \quad \Leftrightarrow \quad [x\rangle \ni A \Rightarrow [x\rangle \ni B\,.$$

It, in turn, implies the statement

$$(A \subseteq B) \Rightarrow \big(x \in \langle A] \Rightarrow x \in \langle B]\big) \quad \Leftrightarrow \quad (A \subseteq B) \Rightarrow \big([x\rangle \ni A \Rightarrow [x\rangle \ni B\big)\,.$$

By applying $\forall_{x \in X} \forall_{A,B \subseteq X}$ to each side of the above statement equivalence, and by recalling that changing the order in which universal quantification is applied produces an equivalent statement, we obtain the equivalence of statements

$$\forall_{A,B \subseteq X}\ A \subseteq B \Rightarrow \langle A] \subseteq \langle B] \quad \Leftrightarrow \quad \forall_{x \in X}\ \big(\forall_{A,B \subseteq X}\ A \subseteq B \Rightarrow \big([x\rangle \ni A \Rightarrow [x\rangle \ni B\big)\big)\,.$$

The left-hand-side is monotonicity of ε, the right-hand-side is right-saturation of each $[x\rangle$ in $(\mathscr{P}X, \subseteq)$.

7.2.9 A comment about proofs

The above argument is constructed to reveal the structure of a formal proof, i.e., a proof acceptable from the standpoint of Mathematical Logic. This structure is always *implicit* in any *logically correct* proof of any statement even though the actual phrasing of the argument usually is more colloquial. The degree of colloquiality of the argument my vary but not the degree of how *rigorous* the argument is.

I suggest that you write down your own, more colloquially phrased, proof of Part (a). You will likely discover that it is natural to split a colloquial proof of equivalence of two statements into separate proofs of *necessity* and *sufficiency* of one of the statements for validity of the other one.

Exercise 209 *Prove Part* (b).

Exercise 210 *Prove Part* (c).

Exercise 211 *Prove Part* (d).

Exercise 212 *Prove Part* (e).

Lemma 7.6 *Assignment*

$$\varepsilon \longmapsto \big([x\rangle_\varepsilon\big)_{x \in X} \tag{515}$$

induces a canonical identification

$$\left\{ \begin{array}{c} \text{equipotence classes} \\ \text{of interior relations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{families of } \langle\ ]\text{-invariant filters } (\mathscr{F}_x)_{x \in X} \\ \text{such that } \forall_{x \in X}\ \mathscr{F}_x \subseteq \mathscr{P}_x X \end{array} \right\}. \tag{516}$$

Exercise 213 *Prove Lemma 7.6.*

7.2.10 Terminology and notation: *points, neighborhoods* and *the neighborhoods filter* of a point

When a topological structure is fixed, elements of the topologized set are referred to as *points*, members of filter $[x\rangle$ are referred to as *neighborhoods of* x, and the family of subsets $[x\rangle$ is referred to as *the neighborhoods filter* of point x, and it will be subsequently denoted $\mathcal{N}_x$.

In this case, family $[x\rangle_{\varepsilon^c}$ consists of all subsets of X whose closure contains point x,

$$[x\rangle_{\varepsilon^c} = \left\{ A \subseteq X \mid x \in \bar{A} \right\}$$

as follows from the third equality in (514).

Exercise 214 *Describe the neighborhood filters $\mathcal{N}_{(0)}$ and $\mathcal{N}_{(p)}$ of points in* Spec $\mathbf{Z}$, *cf. Section 5.7.16.*

7.2.11 *Pas de sept de la Topologie*

The above seven approaches to topologizing a set should never be lost from sight when investigating topological spaces. Arguments are frequently built by graciously moving from one approach to another one, and so on.

7.3 Morphisms

7.3.1 Concrete morphisms between *ε-spaces*

There are a priori six different candidates for the role of a *concrete* morphism between *ε-spaces*

$$(X, \varepsilon) \longrightarrow (X', \varepsilon'), \tag{517}$$

i.e., between sets equipped with an ε-relation.

Four of these involve covariant functions between the corresponding power-sets, f_* and $f_!$, and are morphisms of two types from ε to ε',

$$\begin{tikzcd} X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\ \arrow[r, phantom, "\Uparrow"] & \text{Statements} \\ X, \mathscr{P}X \arrow[uu, shift right, "f"] \arrow[uu, shift left, "f_*"'] \arrow[ur, "\varepsilon"'] & \end{tikzcd} , \qquad \begin{tikzcd} X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\ \arrow[r, phantom, "\Downarrow"] & \text{Statements} \\ X, \mathscr{P}X \arrow[uu, shift right, "f"] \arrow[uu, shift left, "f_*"'] \arrow[ur, "\varepsilon"'] & \end{tikzcd} \tag{518}$$

and

$$\begin{tikzcd} X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\ \arrow[r, phantom, "\Uparrow"] & \text{Statements} \\ X, \mathscr{P}X \arrow[uu, shift right, "f"] \arrow[uu, shift left, "f_!"'] \arrow[ur, "\varepsilon"'] & \end{tikzcd} , \qquad \begin{tikzcd} X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\ \arrow[r, phantom, "\Downarrow"] & \text{Statements} \\ X, \mathscr{P}X \arrow[uu, shift right, "f"] \arrow[uu, shift left, "f_!"'] \arrow[ur, "\varepsilon"'] & \end{tikzcd} . \tag{519}$$

Two of these involve contravariant function f^* and are bimorphisms of two types from ε to ε',

$$
\begin{tikzcd}
X', \mathscr{P}X' \arrow[drr, "\varepsilon'"] & & \\
X, \mathscr{P}X' \arrow[u, "f"] \arrow[u, equal, shift left=2] \arrow[d, equal] \arrow[d, shift left=2, "f^*"] & \Uparrow & \text{Statements} \\
X, \mathscr{P}X \arrow[urr, "\varepsilon"'] & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X', \mathscr{P}X' \arrow[drr, "\varepsilon'"] & & \\
X, \mathscr{P}X' \arrow[u, "f"] \arrow[u, equal, shift left=2] \arrow[d, equal] \arrow[d, shift left=2, "f^*"] & \Downarrow & \text{Statements} \\
X, \mathscr{P}X \arrow[urr, "\varepsilon"'] & &
\end{tikzcd}
\,. \tag{520}
$$

7.3.2

We shall refer to these six types of concrete morphisms associated with a function f between the underlying sets as being *morphisms between ε-spaces* of type

$$
{}_*\!\!\Rightarrow\,, \quad {}_*\!\!\Leftarrow\,, \quad {}_!\!\Rightarrow\,, \quad {}_!\!\Leftarrow\,, \quad {}^*\!\!\Rightarrow\,, \quad \text{or} \quad {}^*\!\!\Leftarrow\,.
$$

7.3.3 A relation between ${}_*\!\!\Rightarrow$- and ${}_!\!\Leftarrow$-morphisms

There exists a close relation between ${}_*\!\!\Rightarrow$- and ${}_!\!\Leftarrow$-morphisms.

Lemma 7.7 *For any function $f : X \longrightarrow X'$ and any pair of ε-relations on X and X', one has*

$$
\begin{tikzcd}
X', \mathscr{P}X' \arrow[drr, "\varepsilon'"] & & \\
& \Uparrow & \text{Statements} \\
X, \mathscr{P}X \arrow[uu, "f"] \arrow[uu, shift right=2, "f_*"'] \arrow[urr, "\varepsilon"'] & &
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
X', \mathscr{P}X' \arrow[drr, "\varepsilon'^c"] & & \\
& \Downarrow & \text{Statements} \\
X, \mathscr{P}X \arrow[uu, "f"] \arrow[uu, shift right=2, "f_!"'] \arrow[urr, "\varepsilon^c"'] & &
\end{tikzcd}
\,. \tag{521}
$$

In other words,

$$
f \text{ is a } {}_*\!\!\Rightarrow\text{-morphism } (X, \varepsilon) \to (X, \varepsilon') \quad \Longleftrightarrow \quad f \text{ is a } {}_!\!\Leftarrow\text{-morphism } (X, \varepsilon^c) \to (X, \varepsilon'^c)\,. \tag{522}
$$

Proof. To keep notation simple, let us adopt the following notation

$$
\langle\,] := {}_{\varepsilon}\langle\,]\,, \qquad \langle\,]' := {}_{\varepsilon'}\langle\,]\,, \qquad \langle\,]^c := {}_{\varepsilon^c}\langle\,]\,, \qquad \text{and} \qquad \langle\,]'^c := {}_{\varepsilon'^c}\langle\,]\,. \tag{523}
$$

Equivalence in the middle of diagram (521) is the composite of a series of equivalences and identifications

$$
\begin{tikzcd}
\forall_{x\in X, A\subseteq X}\ x\varepsilon A \Rightarrow f(x)\varepsilon' f_* A \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \text{the left diagram in (521)} \arrow[d, Leftrightarrow] \\
\forall_{x\in X, A\subseteq X}\ x\in \langle A] \Rightarrow x\in f^*\langle f_* A]' \arrow[d, Leftrightarrow] & \text{the right diagram in (521)} \arrow[d, Leftrightarrow] \\
\forall_{A\subseteq X}\ \langle A] \subseteq f^*\langle f_* A]' \arrow[d, Leftrightarrow, "\complement"] & \forall_{x\in X, B\subseteq X}\ x\varepsilon^c B \Leftarrow f(x)\varepsilon'^c f_! B \arrow[d, Leftrightarrow] \\
\forall_{A\subseteq X}\ \complement\langle \complement\complement A] \supseteq \complement f^*\langle \complement\complement f_* \complement\complement A]' \arrow[d, Leftrightarrow, "\complement A = B"', "(122)"] & \forall_{x\in X, B\subseteq X}\ x\in \langle B]^c \Leftarrow x\in f^*\langle f_! B]'^c \arrow[d, Leftrightarrow] \\
\forall_{B\subseteq X}\ \complement\langle \complement B] \supseteq f^*\complement\langle \complement f_! B]' \arrow[r, equal, "(495)"] & \forall_{B\subseteq X}\ \langle B]^c \supseteq f^*\langle f_! B]'^c
\end{tikzcd}
$$

□

7.3.4 A relation between $^*\!\Rightarrow$- and $^*\!\Leftarrow$-morphisms

There exists a close relation between $^*\!\Rightarrow$- and $^*\!\Leftarrow$-morphisms.

Lemma 7.8 *For any function $f : X \longrightarrow X'$ and any pair of ε-relations on X and X', one has*

$$
\begin{tikzcd}
X', \mathscr{P}X' \arrow[drr, "\varepsilon'"] & & & & X', \mathscr{P}X' \arrow[drr, "\varepsilon'^c"] & & \\
X, \mathscr{P}X' \arrow[u, "f"] \arrow[u, equal, shift left=2] \arrow[d, equal] \arrow[d, shift left=2, "f^*"] \arrow[rr, phantom, "\Uparrow"] & & \text{Statements} & \Longleftrightarrow & X, \mathscr{P}X' \arrow[u, "f"] \arrow[u, equal, shift left=2] \arrow[d, equal] \arrow[d, shift left=2, "f^*"] \arrow[rr, phantom, "\Downarrow"] & & \text{Statements} \\
X, \mathscr{P}X \arrow[urr, "\varepsilon"'] & & & & X, \mathscr{P}X \arrow[urr, "\varepsilon^c"'] & &
\end{tikzcd}
\quad . \tag{524}
$$

In other words,

$$
f \text{ is a } {}^*\!\Rightarrow\text{-morphism } (X,\varepsilon) \to (X,\varepsilon') \quad \Longleftrightarrow \quad f \text{ is a } {}^*\!\Leftarrow\text{-morphism } (X,\varepsilon^c) \to (X,\varepsilon'^c)\ . \tag{525}
$$

Proof. Equivalence in the middle of diagram (524) is the composite of a series of equivalences and identifications

$$\begin{tikzcd}
\forall_{x\in X, A'\subseteq X'}\ x\varepsilon f^*A' \Rightarrow f(x)\varepsilon' A' \arrow[r, Longleftrightarrow] \arrow[d, Updownarrow] & \text{the left diagram in (524)} \arrow[d, Updownarrow] \\
\forall_{x\in X, A'\subseteq X'}\ x \in \langle f^*A'] \Rightarrow f(x) \in \langle A']' \arrow[d, Updownarrow] & \text{the right diagram in (524)} \arrow[d, Updownarrow] \\
\forall_{A'\subseteq X'}\ \langle f^*A'] \subseteq f^*\langle A']' \arrow[d, Updownarrow, "\complement"] & \forall_{x\in X, B'\subseteq X'}\ x\varepsilon^c f^*B' \Leftarrow f(x)\varepsilon'^c B' \arrow[d, Updownarrow] \\
\forall_{A'\subseteq X'}\ \complement\langle f^*\complement\complement A'] \supseteq \complement f^*\langle \complement\complement A']' \arrow[d, Updownarrow, "\complement A = B"', "(127)"] & \forall_{x\in X, B'\subseteq X'}\ x \in \langle f^*B']^c \Leftarrow f(x) \in \langle B']'^c \arrow[d, Updownarrow] \\
\forall_{B'\subseteq X'}\ \complement\langle \complement f^*B'] \supseteq f^*\complement\langle \complement B']' \arrow[r, equal, "(495)"] & \forall_{B'\subseteq X'}\ \langle f^*B']^c \supseteq f^*\langle B']'^c
\end{tikzcd}$$

□

7.3.5 Relations between $_*\!\Rightarrow$-morphisms and $^*\!\Rightarrow$-morphisms

There exist close relations between $_*\!\Rightarrow$-morphisms and $^*\!\Rightarrow$-morphisms as long as either the source or the target ε-relation is *monotonic*.

Lemma 7.9 *If ε' is monotonic, then, for any function $f : X \longrightarrow X'$ and any ε, one has*

$$\begin{tikzcd}
X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\
& \text{Statements} \\
X, \mathscr{P}X \arrow[uu, "f"] \arrow[uu, shift right=3ex, "f_*"'] \arrow[ur, "\varepsilon"'] \arrow[u, phantom, "\Uparrow"]
\end{tikzcd}
\quad\Longrightarrow\quad
\begin{tikzcd}
X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] \arrow[d, equal] & \\
X, \mathscr{P}X' \arrow[u, shift left, "f"] \arrow[d, equal] \arrow[r, phantom, "\Uparrow"] & \text{Statements} \\
X, \mathscr{P}X \arrow[u, shift right, "f^*"', leftarrow] \arrow[ur, "\varepsilon"'] &
\end{tikzcd} \tag{526}$$

In other words, if ε' is monotonic, then

$$f \text{ is a } {}_*\!\Rightarrow\text{-morphism } (X,\varepsilon) \to (X,\varepsilon') \quad\Longrightarrow\quad f \text{ is a } {}^*\!\Rightarrow\text{-morphism } (X,\varepsilon) \to (X,\varepsilon')\ . \tag{527}$$

Proof. Implication in the middle of diagram (526) is the composite of a series of equivalences and of a implications

$$\begin{tikzcd}
\text{the left diagram in (526)} \arrow[rr, Longrightarrow] \arrow[d, Updownarrow] & & \text{the right diagram in (526)} \arrow[d, Updownarrow] \\
\forall_{A\subseteq X}\ \langle A] \subseteq f^*\langle f_*A]' \arrow[dr, Rightarrow, "f_*f^*\subseteq \mathrm{id}_{X'}"'] & & \forall_{A'\subseteq X'}\ \langle f^*A'] \subseteq f^*\langle A']' \\
& \forall_{A'\subseteq X'}\ \langle f^*A'] \subseteq f^*\langle f_*f^*A']' \arrow[ur, Rightarrow, "\varepsilon' \text{ monotonic}"'] &
\end{tikzcd}$$

Lemma 7.10 *If ε is monotonic, then, for any function $f : X \longrightarrow X'$ and any ε', one has*

$$
\begin{tikzcd}
X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\
 & \text{Statements} \\
X, \mathscr{P}X \arrow[uu, shift left=2, "f"] \arrow[uu, shift right=2, "f_*"'] \arrow[ur, "\varepsilon"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
\quad \Longleftarrow \quad
\begin{tikzcd}
X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] \arrow[d, equal] & \\
X, \mathscr{P}X' \arrow[u, bend left, "f"] \arrow[d, equal] \arrow[r, phantom, "\Uparrow"] & \text{Statements} \\
X, \mathscr{P}X \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
\tag{528}
$$

In other words, if ε is monotonic, then

$$
f \text{ is a } {}_*{\Rightarrow}\text{-morphism } (X, \varepsilon) \to (X, \varepsilon') \quad \Longleftarrow \quad f \text{ is a } {}^*{\Rightarrow}\text{-morphism } (X, \varepsilon) \to (X, \varepsilon') \,. \tag{529}
$$

Proof. Implication in the middle of diagram (528) is the composite of a series of equivalences and of a implications

$$
\begin{tikzcd}
\text{the left diagram in (528)} \arrow[d, Leftrightarrow] & & \text{the right diagram in (528)} \arrow[ll, Rightarrow] \arrow[d, Leftrightarrow] \\
\forall_{A \subseteq X} \ \langle A] \subseteq f^* \langle f_* A]' & & \forall_{A' \subseteq X'} \ \langle f^* A'] \subseteq f^* \langle A']' \arrow[dl, Rightarrow, "\varepsilon \text{ monotonic}"'] \\
 & \forall_{A \subseteq X} \ \langle f^* f_* A] \subseteq f^* \langle f_* A]' \arrow[ul, Rightarrow, "\mathrm{id}_X \subseteq f^* f_*"] &
\end{tikzcd}
$$

□

Corollary 7.11 *If both ε and ε' are monotonic, then*

$$
f \text{ is a } {}_*{\Rightarrow}\text{-morphism } (X, \varepsilon) \to (X, \varepsilon') \quad \Longleftrightarrow \quad f \text{ is a } {}^*{\Rightarrow}\text{-morphism } (X, \varepsilon) \to (X, \varepsilon') \,. \tag{530}
$$

□

7.3.6 Relations between ${}_!{\Leftarrow}$-morphisms and ${}^*{\Leftarrow}$-morphisms

In view of the fact that ε' is monotonic precisely when ε is monotonic (cf. Exercise 198), Lemmata 7.9–7.10 and Corollary 7.11 have their analogs also for ${}_!{\Leftarrow}$-morphisms and ${}^*{\Leftarrow}$-morphisms.

Lemma 7.12 *If ε' is monotonic, then, for any function $f : X \longrightarrow X'$ and any ε, one has*

$$
\begin{tikzcd}
X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\
 & \text{Statements} \\
X, \mathscr{P}X \arrow[uu, shift left=2, "f"] \arrow[uu, shift right=2, "f_!"'] \arrow[ur, "\varepsilon"'] \arrow[u, phantom, "\Downarrow"] &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
X', \mathscr{P}X' \arrow[dr, "\varepsilon'"] \arrow[d, equal] & \\
X, \mathscr{P}X' \arrow[u, bend left, "f"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow"] & \text{Statements} \\
X, \mathscr{P}X \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
\tag{531}
$$

In other words, if ε' is monotonic, then

$$f \text{ is a } {}_!{\Leftarrow}\text{-morphism } (X,\varepsilon) \to (X,\varepsilon') \quad \Longrightarrow \quad f \text{ is a } {}^*{\Leftarrow}\text{-morphism } (X,\varepsilon) \to (X,\varepsilon') \,. \tag{532}$$

Exercise 215 *Prove Lemma 7.12 by closely following the proof of Lemma 7.9.*

Exercise 216 *Prove Lemma 7.13 by closely following the proof of Lemma 7.10.*

Lemma 7.13 *If ε is monotonic, then, for any function $f : X \longrightarrow X'$ and any ε', one has*

$$\begin{tikzcd} X',\mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\ & \text{Statements} \\ X,\mathscr{P}X \arrow[uu, shift right, "f"] \arrow[uu, shift left, "f_!"'] \arrow[ur, "\varepsilon"'] & \end{tikzcd} \quad \Longleftarrow \quad \begin{tikzcd} X',\mathscr{P}X' \arrow[dr, "\varepsilon'"] & \\ X,\mathscr{P}X' \arrow[u, shift right, "f"] \arrow[u, shift left, equal] \arrow[d, shift right, equal] \arrow[d, shift left, "f^*"'] & \text{Statements} \\ X,\mathscr{P}X \arrow[ur, "\varepsilon"'] & \end{tikzcd} \tag{533}$$

In other words, if ε is monotonic, then

$$f \text{ is a } {}_!{\Leftarrow}\text{-morphism } (X,\varepsilon) \to (X,\varepsilon') \quad \Longleftarrow \quad f \text{ is a } {}^*{\Leftarrow}\text{-morphism } (X,\varepsilon) \to (X,\varepsilon') \,. \tag{534}$$

Corollary 7.14 *If both ε and ε' are monotonic, then*

$$f \text{ is a } {}_!{\Leftarrow}\text{-morphism } (X,\varepsilon) \to (X,\varepsilon') \quad \Longleftrightarrow \quad f \text{ is a } {}^*{\Leftarrow}\text{-morphism } (X,\varepsilon) \to (X,\varepsilon') \,. \tag{535}$$

□

Corollary 7.15 *Let $f : X \to X'$ be a function and let ε and ε' be monotonic. The following conditions are equivalent.*

(a) *f is a ${}_*{\Rightarrow}$-morphism $(X,\varepsilon) \to (X,\varepsilon')$.*

(b) *f is a ${}^*{\Rightarrow}$-morphism $(X,\varepsilon) \to (X,\varepsilon')$.*

(c) *f is a ${}_!{\Leftarrow}$-morphism $(X,\varepsilon^c) \to (X,\varepsilon'^c)$.*

(d) *f is a ${}^*{\Leftarrow}$-morphism $(X,\varepsilon^c) \to (X,\varepsilon'^c)$.*

□

7.3.7 Morphisms between monotonic ε-spaces

Corollary 7.15 suggests what should serve as a definition of a morphism between monotonic ε-spaces (517). We declare a function $f : X \to X'$ to be a *morphism* if it satisfies any one of the four equivalent conditions of Lemma 7.15.

7.3.8 *Continuous* functions as morphisms between adherence ε-spaces

For adherence spaces, the four equivalent conditions of Lemma 7.15, translated into the language of the *closure* and the *interior* operations, read as follows.

(a′) $f_* \circ \mathrm{Cl} \subseteq \mathrm{Cl} \circ f_*$.

(b′) $\mathrm{Cl} \circ f^* \subseteq f^* \circ \mathrm{Cl}$.

(c′) $f_! \circ \mathrm{Int} \supseteq \mathrm{Int} \circ f_!$.

(d′) $\mathrm{Int} \circ f^* \supseteq f^* \circ \mathrm{Int}$.

Condition (a′) is, essentially, the classical definition of a *continuous* function. It says that

$$\begin{gathered}\text{if } x \in X \text{ is an adherence point of a subset } A \subseteq X,\\ \text{then } f(x) \text{ is an adherence point of the image } f_*A \subseteq X'.\end{gathered} \tag{536}$$

7.3.9 *Adherence* is the *primary* structure in classical approach to Topology

Note that here, in contrast to what Diagram (510) may suggest, ${}_{\varepsilon}\langle\ ]$ is the closure operation, while ${}_{\varepsilon^c}\langle\ ]$ is the interior operation. In other words, the classical approach to Continuity is based on *adherence* as the primary structure of a topologized set, with the interior structure appearing as the *conjugate* of that primary structure.

7.4 Compactology

7.4.1 A subset covered by a family of subsets

Given a subset $E \subseteq X$ and a family of subsets $\mathscr{C} \subseteq \mathscr{P}X$, we say that E *is covered by* $\mathscr{C}$ if

$$E \subseteq \bigcup \mathscr{C}\,. \tag{537}$$

We shall say in this case that $\mathscr{C}$ *is a cover of* E. *To be covered* is an ε-relation on $\mathscr{P}X$. We shall denote it $E \Subset \mathscr{C}$[6]

7.4.2 $\mathscr{A}$-compact subsets

Given a family $\mathscr{A}$ of subsets of a set X, we shall say that a subset $K \subseteq X$ is $\mathscr{A}$*-compact* if, for every cover of K by a family $\mathscr{A}' \subseteq \mathscr{A}$, there exists a *finite* subfamily $\mathscr{A}'' \subseteq \mathscr{A}'$ that covers K.

7.4.3 Principal filter $\mathscr{P}_{\mathscr{A}}$ on $\mathscr{P}X$

If $\mathscr{A} \subseteq \mathscr{P}X$ is a family of subsets of X, then $\mathscr{P}_{\mathscr{A}}$ denotes the family of all families of subsets that contain $\mathscr{A}$ as their subfamily.

[6]Be aware that symbol $A \Subset B$ may be used also for the relation *subset A is contained in the interior of subset B*.

7.4.4 Principal filter $\mathscr{P}_{\mathscr{P}_A}$ on $\mathscr{P}X$

When we apply this to $\mathscr{A} = \mathscr{P}_A$, we obtain the family of subsets of X, then $\mathscr{P}_{\mathscr{P}_a}$ denotes the family of all families of subsets that contain $\mathscr{P}_A$ as their subfamily.

Exercise 217 *Show that* $\mathscr{P}_{\mathscr{P}_A} = \mathfrak{E}(A)$ *where*

$$\mathfrak{E}(A) := \{\mathscr{E} \subseteq \mathscr{P}X \mid A \in \mathscr{E}\}. \tag{538}$$

7.4.5 $\mathscr{A}$-prime filters

Let $\mathscr{A}'' \subseteq \mathscr{P}X$ be a *finite* family of subsets.

Exercise 218 *Show that*

$$\operatorname{Filt} X \cap \bigcap \mathfrak{E}_* \mathscr{A}'' = \operatorname{Filt} X \cap \mathfrak{E}\left(\bigcap \mathscr{A}''\right) \tag{539}$$

and

$$\operatorname{Filt} X \cap \bigcup \mathfrak{E}_* \mathscr{A}'' \subseteq \operatorname{Filt} X \cap \mathfrak{E}\left(\bigcup \mathscr{A}''\right). \tag{540}$$

We say that a filter $\mathscr{F}$ is *$\mathscr{A}$-prime* if, for every finite nonempty family of nonempty subsets $\mathscr{A}'' \subseteq \mathscr{A}$, one has

$$\bigcup \mathscr{A}'' \in \mathscr{F} \implies \exists_{A \in \mathscr{A}''}\, A \in \mathscr{F}. \tag{541}$$

We shall denote the set of $\mathscr{A}$-prime filters on a set X by $\operatorname{Filt}^{\mathscr{A}\text{-prime}} X$.

In the case of $\mathscr{A} = \mathscr{P}X$, we talk of *prime* filters.

7.4.6 Functorial properties of prime filters

Suppose that $\mathscr{A} \subseteq \mathscr{P}X$ and $\mathscr{B} \subseteq \mathscr{P}Y$. Let $f : (X, \mathscr{A}) \longrightarrow (Y, \mathscr{B})$ be a morphism, cf. (273), i.e.,

$$\mathscr{A} \supseteq f^*{}_* \mathscr{B} \tag{542}$$

or, equivalently,

$$f^{**} \mathscr{A} \supseteq \mathscr{B}. \tag{543}$$

Exercise 219 *Show that, if $\mathscr{F}$ is an $\mathscr{A}$-prime filter, then $f^{**}\mathscr{F}$ is a $\mathscr{B}$-prime filter.*

7.4.7 $\mathscr{A}$-generated filters

We say that a filter $\mathscr{F}$ is *$\mathscr{A}$-generated* if $\mathscr{F} \cap \mathscr{A}$ generates $\mathscr{F}$, i.e., if

$$\forall_{F \in \mathscr{F}} \exists_{A \in \mathscr{F} \cap \mathscr{A}}\, A \subseteq F. \tag{544}$$

Lemma 7.16 *Every maximal $\complement_* \mathscr{A}$-generated filter is $\mathscr{A}$-prime.*

Here $\complement_* \mathscr{A}$ is the family of complements of members of $\mathscr{A}$.

Proof. Let $\mathscr{A}'' \subseteq \mathscr{A}$ be a finite nonempty family of nonempty subsets of X such that

$$\bigcup \mathscr{A}'' \in \mathscr{F}.$$

We shall prove, by induction on the cardinality of $\mathscr{A}''$, that there exists $A \in \mathscr{A}''$ such that $A \in \mathscr{F}$.

The assertion is obviously satisfied when $\mathscr{A}''$ consists of a single member subset. Assume that $\mathscr{A}''$ has at least two member subsets and let $B \in \mathscr{A}''$ be any member of $\mathscr{A}''$. One has

$$B \notin \mathscr{F} \quad \Leftrightarrow \quad \forall_{F \in \mathscr{F}}\, F \cap \complement B \neq \emptyset. \tag{545}$$

By taking into account that $\complement B \in \complement_* \mathscr{A}$ and $\mathscr{F}$ is $\complement_* \mathscr{A}$-generated, we deduce that $B \notin \mathscr{F}$ if and only if the family

$$\{F \cap \complement B \mid F \in \mathscr{F} \cap \complement_* \mathscr{A}\} \tag{546}$$

satisfies Finite-Interscetion Property, cf. Section 6.7.4. The smallest filter that contains this family is $\complement_* \mathscr{A}$-generated and contains $\mathscr{F}$. Since $\mathscr{F}$ is a maximal $\complement_* \mathscr{A}$-generated filter, the two filters coincide. In particular,

$$\complement B \in \mathscr{F}.$$

Thus, if $B \notin \mathscr{F}$, then

$$\mathscr{F} \ni \complement B \cap \bigcup \mathscr{A}'' = \complement B \cap \left(B \cup \bigcup \mathscr{A}'' \setminus \{B\}\right) = \complement B \cap \bigcup \left(\mathscr{A}'' \setminus \{B\}\right)$$

and, since

$$\complement B \cap \bigcup \left(\mathscr{A}'' \setminus \{B\}\right) \subseteq \bigcup \mathscr{A}'' \setminus \{B\},$$

also

$$\mathscr{F} \ni \bigcup \mathscr{A}'' \setminus \{B\}.$$

Family $\mathscr{A}'' \setminus \{B\}$ has fewer number of members than $\mathscr{A}''$ and therefore, by inductive hypothesis, there exists $A \in \mathscr{A}'' \setminus \{B\}$ such that $A \in \mathscr{F}$. □

7.4.8

In the special case $\mathscr{A} = \mathscr{P}X$, we obtain the following sharper result.

Corollary 7.17 *A filter is prime if and only if it is an ultrafilter.*

Proof. For obvious reasons every filter is $\mathscr{P}X$-generated and $\complement_* \mathscr{P}X = \mathscr{P}X$. It suffices to show that every prime filter is a maximal filter, i.e., an ultrafilter.

Exercise 220 *Show that $\mathscr{F}$ is an ultrafilter if*

$$\forall_{E \in \mathscr{P}X} \left((\forall_{F \in \mathscr{F}}\, E \cap F \neq \emptyset) \;\Leftrightarrow\; E \in \mathscr{F} \right). \tag{547}$$

Exercise 221 *Let $\mathscr{F}$ be an arbitrary filter and $E \subseteq X$. Suppose that*

$$\forall_{F \in \mathscr{F}}\, E \cap F \neq \emptyset,$$

and, therefore, family $\{E\} \cup \mathscr{F}$ satisfies Finite-Intersection Property. Show that

$$\complement E \notin \mathscr{F}.$$

Taking into account that

$$E \cup \complement E = X \in \mathcal{F}\,, \qquad \complement E \notin \mathcal{F}\,, \qquad \text{and} \qquad \mathcal{F} \text{ is prime,}$$

one concludes that $E \in \mathcal{F}$. $\square$

Exercise 222 *Show that*

$$\forall_{x \in X} \forall_{A \subseteq X} \left(\mathcal{P}_{\{x\}} \in \mathfrak{E}(A) \iff x \in A \right). \tag{548}$$

Lemma 7.18 *Let $\mathcal{A}'' \subseteq \mathcal{P}X$ be a* ***finite*** *family of subsets. Then, $\mathcal{A}''$ covers X if and only if family $\mathfrak{E}_*\mathcal{A}''$ covers $\mathrm{Filt}^{\mathcal{A}\text{-prime}}X$. Symbolically,*

$$X \Subset \mathcal{A}'' \iff \mathrm{Filt}^{\mathcal{A}\text{-prime}} X \Subset \mathfrak{E}_*\mathcal{A}''\,.$$

$\square$

Exercise 223 *Prove that*

$$\begin{array}{c} \textit{a family } \mathcal{A}' \subseteq \mathcal{P}X \textit{ does not admit any} \\ \textit{finite subfamily } \mathcal{A}'' \textit{ that covers } X \end{array} \iff \complement_*\mathcal{A}' \in \mathrm{FIP}(X)\,. \tag{549}$$

Corollary 7.19 *One has*

$$\forall_{\mathcal{B} \in \mathrm{FIP}(X)} \left(\complement_*\mathcal{A}' \subseteq \mathcal{B} \implies \forall_{A \in \mathcal{A}'}\ \mathcal{B} \notin \mathfrak{E}(A) \right) \tag{550}$$

and, therefore,

$$\mathrm{FIP}(X) \cap \mathcal{P}_{\complement_*\mathcal{A}'} \cap \bigcup \mathfrak{E}_*\mathcal{A}' = \emptyset\,. \tag{551}$$

It follows that, if no finite subfamily $\mathcal{A}''$ covers X, then any maximal $\complement_*\mathcal{A}$-generated filter that contains $\complement_*\mathcal{A}'$ does not belong to any $\mathfrak{E}(A)$, $(A \in \mathcal{A}')$. Since each such filter is $\mathcal{A}$-prime, family $\mathfrak{E}_*\mathcal{A}$ does *not* cover $\mathrm{Filt}^{\mathcal{A}\text{-prime}}X$.

By combining this with Lemma 7.18, we obtain the following result.

Proposition 7.20 *Let $\mathcal{A} \subseteq \mathcal{P}X$ be an arbitrary family of subsets of X. One has*

$$\begin{array}{c} \textit{a family } \mathcal{A}' \subseteq \mathcal{P}X \textit{ admits a finite} \\ \textit{subfamily } \mathcal{A}'' \textit{ that covers } X \end{array} \iff \mathrm{Filt}^{\mathcal{A}\text{-prime}} X \Subset \mathfrak{E}_*\mathcal{A}'\,. \tag{552}$$

$\square$

Corollary 7.21 *For any family $\mathcal{A} \subseteq \mathcal{P}X$, the set of $\mathcal{A}$-prime filters is $\mathfrak{E}_*\mathcal{A}$-compact.*

$\square$

7.5 Compactification of a topological space ($\mathcal{A} = \mathcal{T}$ and $\complement_*\mathcal{A} = \mathcal{Z}$)

7.5.1 A canonical embedding $X \hookrightarrow \mathrm{Filt}\, X$

Assignment

$$x \longmapsto \mathcal{P}_{\{x\}} \tag{553}$$

defines a canonical embedding of X into the set of filters on X, whose image is contained in the $\mathfrak{E}_*\mathcal{A}$-compact set $\mathrm{Filt}^{\mathcal{A}\text{-prime}}X$.

7.5.2

If $\mathscr{A}$ is finitely inf-closed, i.e., closed under formation of finite intersections, then, according to Identity (539), also family $\mathfrak{E}_*\mathscr{A}$ is finitely inf-closed and assignment

$$A \longmapsto \mathfrak{E}(A) \qquad (A \in \mathscr{A}) \tag{554}$$

is a homomorphism of the monoids

$$(\mathscr{A}, X, \cap) \longrightarrow (\mathfrak{E}_*\mathscr{A}, \operatorname{Filt} X, \cap) . \tag{555}$$

7.5.3

Any family $\mathscr{B}$ of subsets of any set X that is closed under finite intersections is a base of a topology $\mathscr{T}$. That topology is the sup-closure $\mathscr{B}^{\cup}$ of $\mathscr{B}$.

7.5.4 $\mathscr{B}$-compactness $\Leftrightarrow$ $\mathscr{T}$-compactness

Exercise 224 *If $\mathscr{B}$ is a* base *of a topology $\mathscr{T}$, then $\mathscr{B}$-compactness and $\mathscr{T}$-compactness are equivalent.*

7.5.5

Intersection of open subset $\mathfrak{E}(U)$ of $\operatorname{Filt} X$ with the image of Canonical Embedding (553),

$$\left\{\mathscr{P}_{\{x\}} \mid \mathscr{P}_{\{x\}} \in \mathfrak{E}(U)\right\} = \left\{\mathscr{P}_{\{x\}} \mid U \in \mathscr{P}_{\{x\}}\right\} = \left\{\mathscr{P}_{\{x\}} \mid x \in U\right\} , \tag{556}$$

coincides with the image of open subset $U \subseteq X$ proving at once that Canonical Embedding identifies a topological space $(X, \mathscr{T})$ with a dense subspace of $(\operatorname{Filt} X, (\mathfrak{E}_*\mathscr{T})^{\cup})$.

7.5.6 A distinguished fundamental system of open neighborhoods of $\mathscr{F} \in \operatorname{Filt} X$

The family

$$\{\mathfrak{E}(U) \mid \mathscr{F} \in \mathfrak{E}(U)\} = \{\mathfrak{E}(U) \mid U \in \mathscr{F}\} \tag{557}$$

is a fundamental system of open neighborhoods of a point $\mathscr{F} \in \operatorname{Filt} X$.

7.5.7 A distinguished fundamental system of neighborhoods of $\mathscr{P}_{\{x\}} \in \operatorname{Filt} X$

By (557), principal filter $\mathscr{P}_{\{x\}}$ has as its fundamental of neighbourhoods,

$$\{\mathfrak{E}(U) \mid U \in \mathscr{P}_{\{x\}}\} = \{\mathfrak{E}(U) \mid x \in U\} = \mathfrak{E}_*(\mathscr{N}_x \cap \mathscr{T}) , \tag{558}$$

the direct image under $\mathfrak{E}$ of the fundamental system of all open neighborhoods of x in $(X, \mathscr{T})$.

We established the following fact.

Theorem 7.22 (Compactification Theorem) *Canonical Embedding* (553) *identifies an arbitrary topological space $(X, \mathscr{T})$ with a dense subspace of* $(\operatorname{Filt} X, (\mathfrak{E}_*\mathscr{T})^{\cup})$. *That subspace is contained in the compact subset of maximal $\mathscr{T}$-prime filters*

$$\operatorname{Filt}^{\mathscr{T}\text{-prime}} X \subseteq \operatorname{Filt} X .$$

□

7.5.8 A canonical embedding $X \hookrightarrow \mathbf{Filt}^{\mathscr{T}\text{-prime}} X$

The intermediate embedding of $(X, \mathscr{T})$ onto a dense subset of compact set $\mathrm{Filt}^{\mathscr{T}\text{-prime}} X$, equipped with the topology induced from $(\mathrm{Filt}\, X, (\mathfrak{E}_* \mathscr{T})^{\cup})$, is referred to as a *compactification* of $(X, \mathscr{T})$.