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Math128A: Final Exam

This is a closed book exam, with the exception of a one-page cheat sheet on one side only. You are allowed to cite any results, up to Section 6.6 but excluding those in the exercises, from the textbook. Results from anywhere else will need to be justified. Completely correct answers given without justification will receive little credit. Partial solutions will get partial credit.

Problem	Maximum Score	Your Score
1	12.5	
2	12.5	
3	12.5	
4	12.5	
5	12.5	
6	12.5	
7	12.5	
8	12.5	
Total	100	

By signing below, you certify that you completed this exam all by yourself. Exams without this signature will not be graded.

Your Name and SID: _____

Your Signature: _____

1. Let $h > 0$. Develop an $O(h^3)$ four point finite difference method to approximate the derivative $f'(x_0)$ using function values $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$ and $f(x_0 + 2\,h)$.

2. Consider the iteration

$$x_{k+1} = 1 - \mathbf{cos}\left(\alpha\, x_k^2\right), \quad k = 0, 1, \cdots,$$

where $0 < \alpha < 1$ is given.

(a) Show that the iteration converges to 0 for any $x_0 \in \mathcal{R}$.

(b) What is the order of convergence?

3. Determine the constants c and x_0 so that the following quadrature

$$\int_{-1}^{1} |x| \; f(x)\, dx \approx c\, f(x_0)$$

is exact for $f(x) = 1,\, x$.

4. Let

$$A = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

For what values of α is the matrix A invertible?

5. Given an initial value ODE

$$y' = f(t, y), \quad \text{for} \quad a \leq t \leq b, \;\; y(a) = \alpha. \quad (1)$$

By integrating y' on the interval $[t_{i-1},\, t_{i+1}]$, we obtain the following equation

$$y(t_{i+1}) - y(t_{i-1}) = \int_{t_{i-1}}^{t_{i+1}} f(t, y(t))\, dt \qquad (2)$$

Derive a two-step implicit method based on equation (2). What is the order of your method?

6. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{R}^{2\times 2}$.

   - Under what conditions is A symmetric?
   - Under what conditions is A symmetric positive definite?
   - Under what conditions is A strictly diagonally dominant?

7. Find the quadratic polynomial $P(x)$ so that

$$P(0) = 2, \quad P(1) = 1, \quad P(2) = 0.$$

What is $P(-1)$?

8. Does the function $f(t,y) = \frac{y}{1+y}$ satisfy the Lipschitz condition on the domain
$$\mathcal{D} \stackrel{def}{=} \{(t,y) \mid 0 \leq t \leq 1,\ -\infty < y < \infty\}?$$

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