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MATH 135: SET THEORY
SOLUTIONS TO FINAL EXAM

1a. State precisely the Power Set Axiom in the language $\mathcal{L}(\in)$ having only the binary relation symbol $\in$.

$$(\forall x)(\exists y)(\forall t)[t \in y \leftrightarrow (\forall u)(u \in t \to u \in x)]$$

1b. Give a formal definition of the expression X `is a transitive set` in the language of set theory $\mathcal{L}(\in)$ having only the binary relation symbol $\in$.

$$(\forall t)(\forall u)[(u \in t \ \&\ t \in X) \to u \in X]$$

1c. Give a formal definition of the expression α `is an ordinal` (or as we wrote this in class $\alpha \in \mathbb{ON}$) in the language $\mathcal{L}(\in)$ having only the binary relation symbol $\in$.

α `is transitive` $\&\ (\forall x)(\forall y)[(x \in \alpha \ \&\ y \in \alpha) \to (x \in y \vee x = y \vee y \in x)] \ \&\ \&\ (\forall x)[x \in \alpha \to x$ `is transit`

1d. Give a formal definition of the expression $Y = \operatorname{ran}(R)$ in the language of set theory $\mathcal{L}(\in)$ having only the binary relation symbol $\in$.

We define first:

$$t = \{x, y\} :\Longleftrightarrow (\forall u)[u \in t \leftrightarrow (u = x \vee u = y)]$$

We then define

$$t = \langle x, y\rangle :\Longleftarrow t = \{\{x\}, \{x, y\}\}$$

Finally,

$$Y = \operatorname{ran}(R) :\Longleftrightarrow (\forall t)[t \in Y \leftrightarrow (\exists x)(\langle x, t\rangle \in R)]$$

1e. State precisely the Replacement Axiom Scheme in the language of set theory $\mathcal{L}(\in)$ having only the binary relation symbol $\in$.

Let $\varphi = \varphi(x, y, t_1, \ldots, t_n)$ be a formula of set theory having free variables amongst x, y, t_1, $\ldots$, t_n and for which the variables A and B do not appear.

This instance of Replacement states:

$$(\forall t_1) \cdots (\forall t_n)(\forall A)[(\forall x)(\forall y)(\forall z)(\phi(x, y, t_1, \ldots, t_n) \ \&\ \phi(x, z, t_1, \ldots, t_n) \to y = z) \to (\exists B)(\forall y)(y \in B \leftrightarrow (\exists x)$$

1f. State precisely the Empty Set Axiom in the language of set theory $\mathcal{L}(\in)$ having only the binary relation symbol $\in$.

$$(\exists x)(\forall y)(\neg y \in x)$$

2. Let X be any set and $R \subseteq X \times X$ any relation on X. By the recursion theorem on ω there is a unique function $f : \omega \to \mathcal{P}(X \times X)$ satisfying $f(0) = I_X \cup R \cup R^{-1}$ and $f(n^+) = f(n) \circ f(n)$ for all $n \in \omega$. Let $E := \bigcup \text{ran}(f)$. **Show** that for any equivalence relation E' on X with $R \subseteq E'$, we have $E \subseteq E'$.

Proof. We argue by induction that for every $n \in \omega$ we have $f(n) \subseteq R'$. Consider the case of $n = 0$. By hypothesis, $R \subseteq E'$. Since E' is an equivalence relation on X and is thus reflexive on X, $I_X \subseteq E'$. Finally, since E' is symmetric and $R \subseteq E'$, we have $R^{-1} \subseteq E'$. Thus, $f(0) = R \cup I_X \cup R^{-1} \subseteq E'$. Suppose now that we know $f(n) \subseteq E'$. Suppose that $t \in f(n^+)$. Since $f(n^+) = f(n) \circ f(n)$, there are x, y and z for which $t = \langle x, z \rangle$, $\langle x, y \rangle \in f(n)$ and $\langle y, z \rangle \in f(n)$. Since $f(n) \subseteq E'$, we have $\langle x, y \rangle \in E'$ and $\langle y, z \rangle \in E'$. As E' is a transitive relation, $t = \langle x, z \rangle \in E'$. Thus, $f(n^+)$. From we conclude that $E = \bigcup \text{ran}(f) \subseteq E'$, as required. $\square$

3. **Show** that for every set X there is some set K so that for every $x \in X$ one has $x \prec K$.

Proof. Let $K := \mathcal{P}(\bigcup X)$. Consider any $x \in X$. Then $x \subseteq \bigcup X$, so that $x \preceq \bigcup X$ and by Cantor's theorem, $\bigcup X \prec \mathcal{P}(\bigcup X) = K$. Thus, $x \prec K$. $\square$

4. **Show** that if X is a nonempty finite set, then X has at least one maximal element (with respect to $\subseteq$).

Proof. We argue by induction on $\text{card}(X)$ with the case of $\text{card}(X) = 0$ being trivial. Suppose now that $\text{card}(X) = n^+$ are we already know the result for n. Fix a bijection $f : n^+ \to X$ and set $X' := f[[n]]$ and $x := f(n)$. If $n = 0$, then x is the maximal element. Otherwise, $X' \neq \varnothing$ so by induction there is some $y \in X'$ maximal in X'. Observe that because f is a bijection, $x \neq y$. So there are two cases to consider: $y \subset x$ or $y \not\subset x$. In the former case, x is maximal in X for if there were some $z \in X$ with $x \subset z$, then necessarily $z \in X'$ and $y \subset z$ contradicting maximality of y in X'. In the latter case, y is maximal in X for we know that $y \not\subset x$ and any other z would have to come form X' and we know that y is maximal in X'. $\square$

5. **Show** that there is an **infinite** set N and an **onto** function $f : \omega \to N$ for which $f(0) = \varnothing$ and for every $n \in \omega$ one has $f(n^+) = \{f(n)\}$.

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6. Prove that for sets A, B and C, one has ${}^{C}(A \times B) \approx {}^{C}A \times {}^{C}B$. (Yes, we talked about this in class. I want to see your detailed proof.)

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7. Recall that for a set X, the product is the set

$$\prod X := \{f \in \mathcal{P}(X \times \bigcup X) \ : \ (\forall x)[x \in X \to f(x) \in x\} \ .$$

Show that if X is a set of nonempty disjoint sets, then $\bigcup X \prec \prod X$. (Note: There are two parts to this. You need to establish that $\bigcup X \preceq \prod X$ and that $\bigcup X \not\approx \prod X$.)

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8. Show that for every set X there is an inductive set I with $X \subseteq I$.

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