1.) $g(x,y) = x$

$g_x = 1$, $g_y = 0 \Rightarrow$ no max and min values on the interior of the region given by $2x^2+6xy+9y^2 \leq 9$.

Check for max and min on the boundary:

Constraint: $2x^2+6xy+9y^2=9$

$$h(x,y) = 2x^2+6xy+9y^2 = 9$$

$\nabla g = \lambda \nabla h$ $\quad g_x = \lambda h_x \quad g_y = \lambda h_y$

$$1 = \lambda(4x+6y) \qquad 0 = \lambda(6x+18y)$$

$$0 = \lambda(x+3y)$$

$$\Rightarrow x = -3y$$

$$1 = \lambda(-6y)$$

$$\lambda = -\frac{1}{6y} \Rightarrow x = \frac{1}{2\lambda}$$

$$h(x,y) = 0 \Rightarrow 2\left(\frac{1}{2\lambda}\right)^2 + 6\left(\frac{1}{2\lambda}\right)\left(-\frac{1}{6\lambda}\right) + 9\left(\frac{-1}{6\lambda}\right)^2 = 9$$

$$\frac{9}{36\lambda^2} = 9 \Rightarrow \lambda = \pm\frac{1}{6}$$

$\Rightarrow (3,-1), (-3,1)$

max at $g(3,-1) = 3$, min at $g(-3,1) = -3$

2) a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim\limits_{(x,y)\to(0,0)} \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}$

$= \lim\limits_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{x-y}$

$= \lim\limits_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0$

b.) $\lim\limits_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$

Consider the path $y=mx$.

$\lim\limits_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+m^2x^2} = \lim\limits_{(x,mx)\to(0,0)} \frac{3mx^2}{(1+m^2)x^2}$

$= \frac{3m}{1+m^2}$.

Different values of m correspond to different paths which will give different limits, so the limit does not exist.

3) $f(x,y) = \dfrac{-4x}{x^2+y^2+1}$

Some level sets: $2 = \dfrac{-4x}{x^2+y^2+1}$

$\iff (x+1)^2 + y^2 = 0$

$-2 = \dfrac{-4x}{x^2+y^2+1}$

$\iff (x-1)^2 + y^2 = 0$

$1 = \dfrac{-4x}{x^2+y^2+1}$

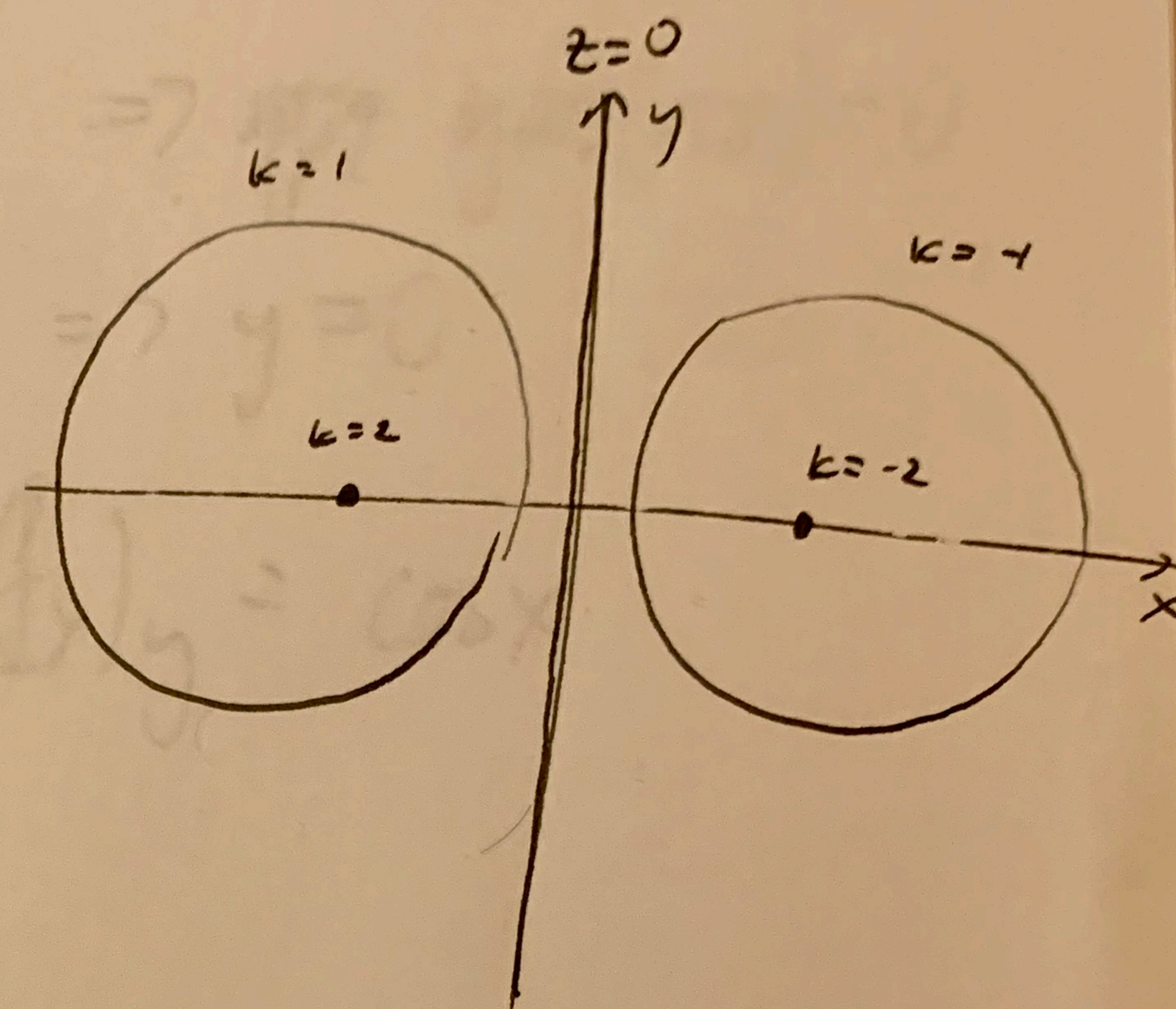
$\iff (x+2)^2 + y^2 = 3$

$-1 = \dfrac{-4x}{x^2+y^2+1}$

$\iff (x-2)^2 + y^2 = 3$

$0 = \dfrac{-4x}{x^2+y^2+1}$

$\iff x = 0$



$(-1,0)$ is an abs. max

$(1,0)$ is an abs. min

$f_x = \dfrac{4(x^2-y^2-1)}{(x^2+y^2+1)^2}$, $f_y = \dfrac{8xy}{(x^2+y^2+1)^2}$

$f_x = 0,\ f_y = 0 \Rightarrow y = 0,\ x = \pm 1$

$$f_{xx} = \frac{(x^2+y^2+1)^2(8x) - 4(x^2-y^2-1)\,2(x^2+y^2+1)^2(2x)}{(x^2+y^2+1)^4}$$

$$f_{yy} = \frac{(x^2+y^2+1)^2(8x) - 8xy(2(x^2+y^2+1)(2y))}{(x^2+y^2+1)^4}$$

$$f_{yx} = (f_y)_x = \frac{(x^2+y^2+1)^2\,8y - 8xy(2(x^2+y^2+1)(2x))}{(x^2+y^2+1)^4}$$

4.) $f(x,y) = y\sin x$

$f_x = y\cos x \quad = 0$

$f_y = \sin x \quad = 0 \qquad x = n\pi \Rightarrow y\cos(n\pi) = 0$

$\Rightarrow y = 0$

$f_{xx} = -y\sin x$

$f_{yy} = 0 \qquad f_{xy} = (f_x)_y = \cos x$

$D(x,y) = -\cos^2 x$

$D(n\pi, 0) = -\cos^2(n\pi) < 0$

$(n\pi, 0)$ are all saddle ~~point~~ points.

5.) a) Let $F(x,y,z) = x^3 + z^2 + z\cos y$

$F_x = 3x^2$, $F_y = -z\sin y$, $F_z = 2z + \cos y$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2}{2z+\cos y}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0,0)} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{z\sin y}{2z+\cos y}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,0,0)} = 0$$

b) $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= 2(x+y+z)\left(1 - \sin(r+s) + \cos(r+s)\right)$$

$$\left.\frac{\partial w}{\partial r}\right|_{r=1,\,s=-1} = 2(2+1+0)(1-0+1)$$

$$= 2 \cdot 3 \cdot 2$$

6.) $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)\cdot\frac{\partial v}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 w}{\partial u^2}\cdot\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial w}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial u^2}\cdot\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial w}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial w}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}$$

$\frac{\partial v}{\partial y} = x$, $\frac{\partial^2 v}{\partial y^2} = 0$, $\frac{\partial u}{\partial x} = x$, $\frac{\partial^2 u}{\partial x^2} = 1$

$\frac{\partial v}{\partial x} = y$, $\frac{\partial^2 v}{\partial x^2} = 0$, $\frac{\partial u}{\partial y} = -y$, $\frac{\partial^2 u}{\partial y^2} = -1$

$$W_{xx} = W_{uu}\cdot x^2 + W_u + W_{vv}\cdot y^2 + 0$$

$$W_{yy} = W_{uu}\cdot y^2 + (-1)W_u + W_{vv}\cdot x^2 + 0$$

$$W_{xx} + W_{yy} = (x^2+y^2)W_{uu} + (x^2+y^2)W_{vv}$$

$$= (x^2+y^2)(W_{uu}+W_{vv})$$

$$= (x^2+y^2)(f_{uu}+f_{vv}) = (x^2+y^2)\cdot 0 = 0.$$

7.) $f(x,y) = x^2\cos(\pi y)$ $\quad f(2,1) = 4\cos(-\pi) = -4$

$f_x = 2x\cos(\pi y)$ $\quad f_x(2,-1) = 4\cos(-\pi) = -4$

$f_y = -\pi x^2 \sin(\pi y)$ $\quad f_y(2,-1) = -\pi \cdot 4 \cdot \sin(-\pi) = 0$

$z + 4 = -4(x-2)$

Trace : $x = 2$ $\quad f(2,y) = 4\cos(\pi y)$

Distance $= \displaystyle\int_{-1}^{1} \sqrt{1 + 16\pi^2 \sin^2(\pi y)}\, dy$

(This is enough, integral is too hard to do).