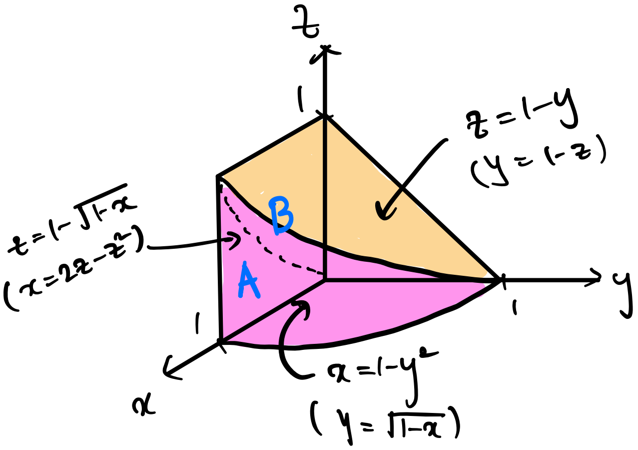
MATH 53 PRACTICE EXAM 3

Please note that this practice test is designed to give you examples of the types of questions that will appear on the actual exam. Your performance on these sample questions should not be taken as an absolute prediction of your performance on the actual exam. When you take the actual exam, the questions you see will cover some of the same content included in the practice exam, as well as some content not tested on the practice exam.

(1) Rewrite the following integral as an equivalent iterated integral in the five other orders:

$$\int_0^1 \int_0^{1-y^2} \int_0^{1-y} dz\,dx\,dy.$$



$$\begin{cases} 0 \le y \le 1 \\ 0 \le x \le 1-y^2 \\ 0 \le z \le 1-y \end{cases}$$

The projection of intersection curve onto xz plane is $1-z = y = \sqrt{1-x} \iff z = 1-\sqrt{1-x}$
$\iff x = 1-(1-z)^2 = 2z - z^2$

① $dx\,dy\,dz$: $\begin{cases} 0 \le z \le 1 \\ 0 \le y \le 1-z \\ 0 \le x \le 1-y^2 \end{cases} \Rightarrow \int_0^1 \int_0^{1-z} \int_0^{1-y^2} dx\,dy\,dz$

② $dx\,dz\,dy$: $\begin{cases} 0 \le y \le 1 \\ 0 \le z \le 1-y \\ 0 \le x \le 1-y^2 \end{cases} \Rightarrow \int_0^1 \int_0^{1-y} \int_0^{1-y^2} dx\,dz\,dy$

③ $dy\,dx\,dz$: $\begin{cases} 0 \le z \le 1 \\ 0 \le x \le 1-y^2 \le 1 \\ 0 \le y \le \min\{\sqrt{1-x}, 1-z\} \end{cases}$

$$\Rightarrow \int_0^1 \int_0^1 \int_0^{\min\{\sqrt{1-x},\,1-z\}} dy\,dx\,dz = \underbrace{\int_0^1 \int_0^{2z-z^2}}_{B} \int_0^{1-z} dy\,dx\,dz + \underbrace{\int_0^1 \int_{2z-z^2}^1}_{A} \int_0^{\sqrt{1-x}} dy\,dx\,dz$$

④ $dy\,dz\,dx$: $\begin{cases} 0 \le x \le 1 \\ 0 \le z \le 1-y \le 1 \\ 0 \le y \le \min\{\sqrt{1-x}, 1-z\} \end{cases}$

$$\Rightarrow \int_0^1 \int_0^1 \int_0^{\min\{\sqrt{1-x},\,1-z\}} dy\,dz\,dx = \underbrace{\int_0^1 \int_0^{1-\sqrt{1-x}}}_{A} \int_0^{\sqrt{1-x}} dy\,dz\,dx + \underbrace{\int_0^1 \int_{1-\sqrt{1-x}}^1}_{B} \int_0^{1-z} dy\,dz\,dx$$

⑤ $dz\,dy\,dx$: $\begin{cases} 0 \le x \le 1 \\ 0 \le y \le \sqrt{1-x} \\ 0 \le z \le 1-y \end{cases} \Rightarrow \int_0^1 \int_0^{\sqrt{1-x}} \int_0^{1-y} dz\,dy\,dx$

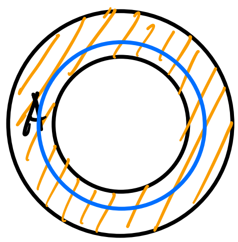
(2) True or False. Give a short explanation for each. You may cite any definitions or theorems that have been covered so far.

(a) The region $A = \{(x, y) : |x^2 + y^2 - 2| < 1\}$ is a simply connected region in the xy-plane.

(b) If $\mathbf{F}(x, y, z)$ is a vector field whose divergence is zero everywhere, and $f(x, y, z)$ is a scalar function, then $\operatorname{div}(f\mathbf{F})$ is zero.

(c) Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous first partial derivatives in a open connected region R, and let C be a piecewise smooth curve in R. If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in R, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in R.

(a) We have $-1 < x^2+y^2-2 < 1 \iff 1 < x^2+y^2 < 3$, so A is an annulus between two circles:



A is NOT simply connected since it has a hole. More precisely, the blue circle ($x^2+y^2=2$) can't be contracted to a point without leaving A.

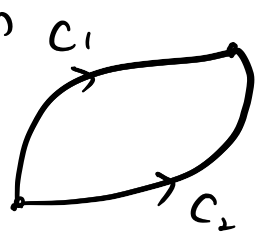
$\boxed{\text{False}}$

(b) Let $F(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$. Then $\operatorname{div}(F) = 0 \Rightarrow \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$.

Then
$$\operatorname{div}(fF) = \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$
$$= \frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} + \frac{\partial f}{\partial y}Q + f\frac{\partial Q}{\partial y} + \frac{\partial f}{\partial z}R + f\frac{\partial R}{\partial z}$$
$$= (\nabla f)\cdot F + f\cdot \operatorname{div}(F) = (\nabla f)\cdot F \quad (\operatorname{div}(F) = 0)$$

This is not zero in general. For example, let $F(x,y,z) = (y,0,0)$ and $f(x,y,z) = x$, then $\operatorname{div}(F) = 0$ but $\operatorname{div}(fF) = y \neq 0$. $\boxed{\text{False}}$

(c)



Let C_1, C_2 be curves with same initial & terminal points. Then $C_1 + (-C_2)$ forms a closed path, so $\int_{C_1+(-C_2)} F\cdot dr = 0$ by assumption. This implies

$$0 = \int_{C_1} F\cdot dr + \int_{-C_2} F\cdot dr \iff \int_{C_1} F\cdot dr = -\int_{-C_2} F\cdot dr = \int_{C_2} F\cdot dr.$$

Hence $\int_C F\cdot dr$ is independent of path. $\boxed{\text{True}}$

(3) Find the mass of the wire in the shape of a helix

$$\mathbf{r}(t) = \frac{1}{\sqrt{2}}(\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}), \quad 0 \le t \le 6\pi$$

where the density of the wire is $\rho(x, y, z) = 1 + z$.

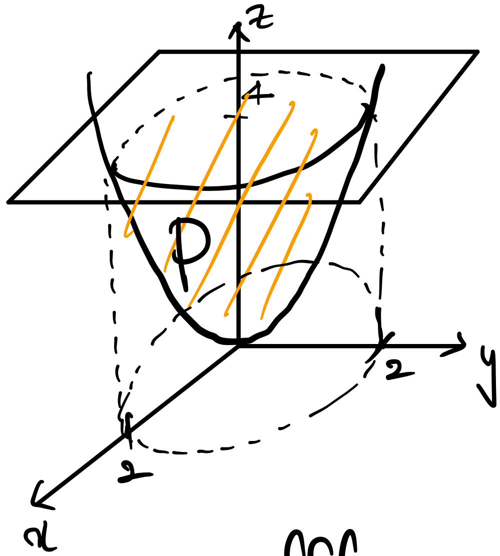
Let C be the helix. Then $r'(t) = \left(-\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\right)$

$$\text{mass} = \int_C \rho \, ds = \int_0^{6\pi} \rho(r(t)) \, |r'(t)| \, dt$$

$$= \int_0^{6\pi} \left(1 + \frac{t}{\sqrt{2}}\right) \cdot \sqrt{\left(-\frac{1}{\sqrt{2}}\sin t\right)^2 + \left(\frac{1}{\sqrt{2}}\cos t\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \, dt$$

$$= \int_0^{6\pi} \left(1 + \frac{t}{\sqrt{2}}\right) dt = \left[t + \frac{t^2}{2\sqrt{2}}\right]_0^{6\pi} = 6\pi + \frac{36\pi^2}{2\sqrt{2}} = \boxed{6\pi + 9\sqrt{2}\pi^2}$$

(4) Consider the solid P bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Suppose the density at each point is proportional to the distance between the point and the z-axis. Find the moment of inertia about the axis of symmetry.



The density is $\rho(x,y,z) = k\sqrt{x^2+y^2}$ for some constant k, and the axis of symmetry of P is z-axis.

Hence the moment of inertia about the axis of symmetry is

$$I_0 = \iiint_P (x^2+y^2)\,\rho(x,y,z)\,dV$$

$$= \iiint_P (x^2+y^2)\cdot k\sqrt{x^2+y^2}\,dV$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot kr\cdot r\,dz\,dr\,d\theta \quad \text{(use cylinderical coordinate)}$$

$$= \int_0^{2\pi}\int_0^2 kr^4(4-r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 k(4r^4-r^6)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[k\left(\frac{4}{5}r^5-\frac{1}{7}r^7\right)\right]_0^2 d\theta = 2\pi k\cdot\frac{256}{35} = \boxed{\frac{512}{35}k\pi}$$

Note that the bound "2" for the integral over r is obtained by the radius of the disk that is the projection of P to xy-plane.

Also, the bounds "z^2" and "4" comes from paraboloid $z = x^2+y^2 = r^2$ and plane $z=4$.

(5) Let R be the region bounded by the lines $x-2y=0$, $x-2y=-4$, $x+y=4$, and $x+y=1$. Evaluate $\int_R\int 3xy\, dA$ using the change of coordinates $x=\frac{1}{3}(2u+v)$ and $y=\frac{1}{3}(u-v)$.

First, the boundaries of R transforms in u, v as

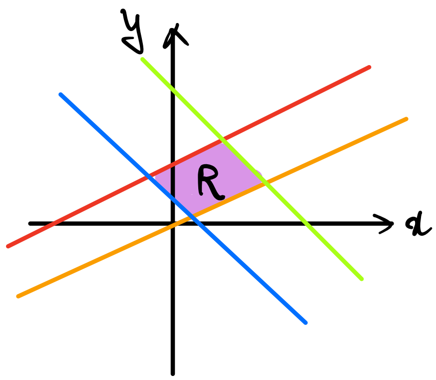
$x-2y=0 \longrightarrow \frac{1}{3}(2u+v)-2\cdot\frac{1}{3}(u-v)=0$, $v=0$

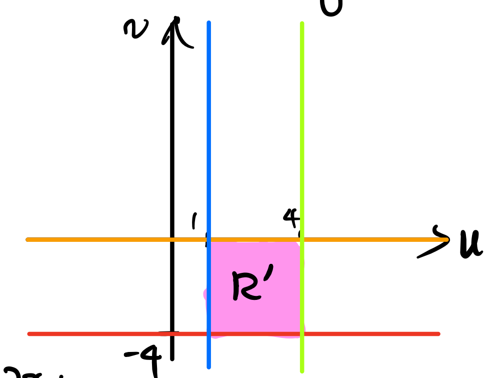
$x-2y=-4 \longrightarrow \frac{1}{3}(2u+v)-2\cdot\frac{1}{3}(u-v)=-4$, $v=-4$.

$x+y=4 \longrightarrow \frac{1}{3}(2u+v)+\frac{1}{3}(u-v)=4$, $u=4$

$x+y=1 \longrightarrow \frac{1}{3}(2u+v)+\frac{1}{3}(u-v)=1$, $u=1$.

Hence the transformed region is a reetangle.





The Jacobian is $J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{vmatrix} = -\frac{1}{3}$.

Hence,

$$\iint_R 3xy\,dx\,dy = \iint_{R'} 3\cdot\frac{1}{3}(2u+v)\cdot\frac{1}{3}(u-v)\cdot\left|-\frac{1}{3}\right| du\,dv$$

$$= \int_{-4}^{0}\int_{1}^{4}\frac{1}{9}\cdot(2u^2-uv-v^2)\,du\,dv$$

$\frac{2}{3}63=42$

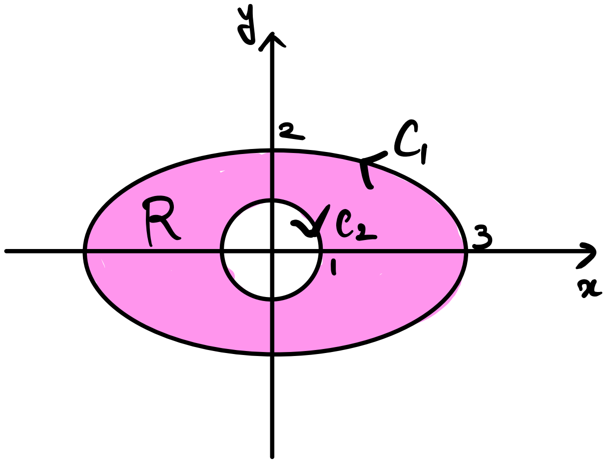
$$= \frac{1}{9}\int_{-4}^{0}\left[\frac{2}{3}u^3-\frac{1}{2}u^2v-uv^2\right]_1^4 dv$$

$$= \frac{1}{9}\int_{-4}^{0}\left(42-\frac{15}{2}v-3v^2\right)dv = \boxed{\frac{164}{9}}$$

(6) Let C_1 be the boundary of the the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, oriented in the counterclockwise direction, and let C_2 be the boundary of the circle $x^2+y^2=1$, oriented in the clockwise direction. Let R be the region inside the ellipse and outside the circle. Use Green's Theorem to evaluate the line integral

$$\int_C 2xydx + (x^2+2x)dy,$$

where $C = C_1 + C_2$ is the boundary of R.



Note that both C_1 and C_2 are positively oriented as boundaries of R. By Green's theorem,

$$\int_C 2xy\,dx + (x^2+2x)\,dy = \iint_R \frac{\partial}{\partial x}(x^2+2x) - \frac{\partial}{\partial y}(2xy)\,dA$$

$$= \iint_R 2x+2-2x\,dA = 2\iint_R dA$$

$$= 2 \times \text{area}(R)$$

The area of ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ can be computed as follows: using parametrization $(x,y) = (3\cos t, 2\sin t)$, $0 \le t \le 2\pi$ and the formula $\text{area(ellipse)} = \int_{C_1} x\,dy$ (follows from Green's theorem), we have

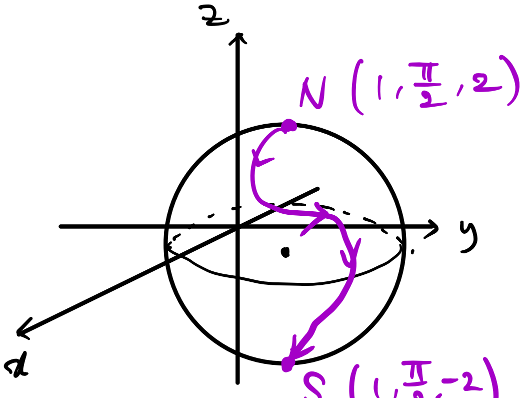
$$\text{area(ellipse)} = \int_0^{2\pi} 3\cos t \cdot 2\cos t\,dt = 6\cdot\int_0^{2\pi}\cos^2 t\,dt$$

$$= 6\cdot\int_0^{2\pi}\frac{1}{2}(1+\cos 2t)\,dt = 6\cdot\left[\frac{t}{2}+\frac{1}{4}\sin 2t\right]_0^{2\pi} = 6\pi.$$

Hence answer is $2\,\text{area}(R) = 2\left(6\pi - \pi\cdot 1^2\right) = \boxed{10\pi}$

<u>Rmk</u> The area of ellipse also can be computed using change of variable $x=3u$, $y=2v$.

(7) Find the work done by $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ on an object moving along a curve from the north pole to the south pole on the surface of a sphere of radius 2 centered at $(1, \frac{\pi}{2}, 0)$. Assume that north is in the positive z direction.



The coordinates of north/south poles are $(1, \frac{\pi}{2}, 2)$ and $(1, \frac{\pi}{2}, -2)$.

We'll first show that F is conservative.

Let's assume $F = \nabla f$ and try to find f. We have

$$\begin{cases} f_x = e^x \cos y + yz & \cdots ① \\ f_y = xz - e^x \sin y & \cdots ② \\ f_z = xy + z & \cdots ③ \end{cases}$$

By integrating ① over x, we have $f(x,y,z) = e^x \cos y + xyz + g(y,z)$ for some $g(y,z)$ that is a function only in y and z. Then $f_y = -e^x \sin y + xy + g_y(y,z)$, and compairing with ② gives $g_y = 0$, i.e. $g(y,z) = h(z)$ for some h. (doesn't depend on y).

Now $f_z = xy + h'(z) = xy + z$ by ③, so we can choose $h(z) = \frac{1}{2}z^2$ and $f(x,y,z) = e^x \cos y + xyz + \frac{1}{2}z^2$. One can check that $\nabla f = F$.

By fundamental theorem of line integral, we have

$$\int_C F \cdot dr = f(S) - f(N) = \left(1 \cdot \frac{\pi}{2} \cdot (-2) + \frac{1}{2}(-2)^2\right) - \left(1 \cdot \frac{\pi}{2} \cdot 2 - \frac{1}{2} \cdot 2^2\right) = \boxed{-2\pi}$$