Math 53 Practice Final

1) a.) Boundary of S_1: C_1 the unit circle $x^2+y^2=1$ counterclockwise

Boundary of S_2: C_2 the unit circle $x^2+y^2=1$ clockwise.

C_1 & C_2 are the same curve with opposite orientations, so

$\oint_{C_1} \vec{F}\cdot d\vec{r} = -\oint_{C_2} \vec{F}\cdot d\vec{r}$. By Stokes' Theorem, we have

$$\iint_{S_1} (\nabla\times\vec{F})\cdot d\vec{S} = \oint_{C_1} \vec{F}\cdot d\vec{r} = -\oint_{C_2} \vec{F}\cdot d\vec{r} = -\iint_{S_2} (\nabla\times\vec{F})\cdot d\vec{S}$$

b.) No relation.

c.) $$\iint_{S} (\nabla\times\vec{F})\cdot d\vec{S} = \iint_{S_1} (\nabla\times\vec{F})\cdot d\vec{S} + \iint_{S_2} (\nabla\times\vec{F})\cdot d\vec{S} = 0$$

2) $\vec{r}_\theta \times \vec{r}_\alpha = a(b+a\cos\alpha)\langle \cos\alpha\cos\theta, \cos\alpha\sin\theta, \sin\alpha\rangle$

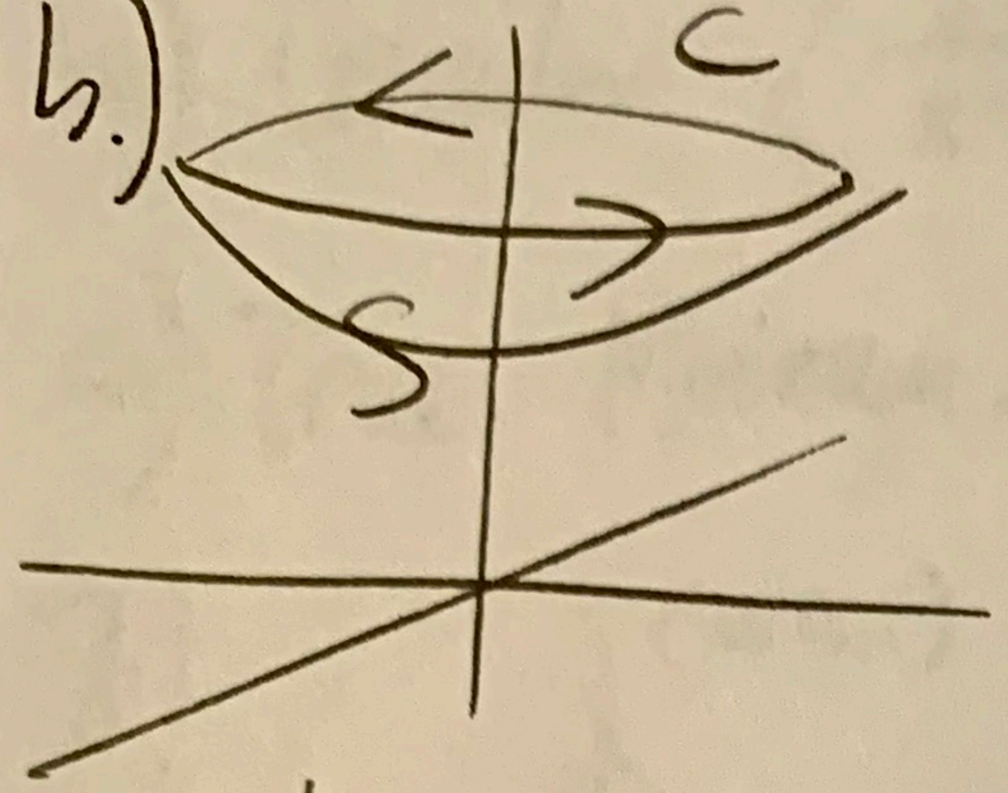
$$dS = \|\vec{r}_\theta\times\vec{r}_\alpha\| = a(b+a\cos\alpha)\|\langle\cos\alpha\cos\theta, \cos\alpha\sin\theta, \sin\alpha\rangle\|$$
$$= a(b+a\cos\alpha)\sqrt{(\cos\alpha\cos\theta)^2+(\cos\alpha\sin\theta)^2+(\sin\alpha)^2}$$
$$= a(b+a\cos\alpha)$$

$$\text{Area}(D) = \int_0^{2\pi}\int_0^{2\pi} a(b+a\cos\alpha)\,d\theta\,d\alpha = 2\pi a\int_0^{2\pi}(b+a\cos\alpha)\,d\alpha$$
$$= (2\pi a)(2\pi b).$$

3.) a.) $\text{curl}\,\vec{F} = \langle bx^2e^z, -2bxye^z + axye^z, -axe^z \rangle$

(want) $= \langle x^2e^z, 0, -2xe^z \rangle \Rightarrow b = 1, a = 2$

b.)



$$\iint_S \vec{G}\cdot d\vec{S} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

C is the circle of radius $\sqrt{3}$ in the plane $z=2$, counterclockwise. $\vec{r}(t) = \langle \sqrt{3}\cos t, \sqrt{3}\sin t, 2\rangle$, $0 \le t \le 2\pi$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi} \langle 2(\sqrt{3}\cos t)(\sqrt{3}\sin t)e^2, 0, 3\cos^2 t(\sqrt{3}\sin t)e^2\rangle\cdot\langle -\sqrt{3}\sin t, \sqrt{3}\cos t, 0\rangle dt$$

$$= \int_0^{2\pi} -6\sqrt{3}\sin^2 t\cos t\, e^2\,dt = -2\sqrt{3}e^2\sin^3 t\Big|_0^{2\pi} = 0$$

4.) $|\vec{r}\,'(t)| = \sqrt{1^2 + (-\sin t)^2 + (\cos t)^2} = \sqrt{2}$

$$\text{Mass} = \int_0^{2\pi}(t^2 + \cos^2 t + \sin^2 t)\,|\vec{r}\,'(t)|\,dt = \sqrt{2}\int_0^{2\pi}(t^2+1)\,dt$$

$$= \sqrt{2}\left(\frac{t^3}{3} + t\right)\Big|_0^{2\pi} = \sqrt{2}\left(\frac{8}{3}\pi^3 + 2\pi\right)$$

5.) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1 \Leftrightarrow u^2+v^2+w^2 \le 1.$

$dx\,dy\,dz = (abc)\,du\,dv\,dw$. $\text{Vol}(E) = \iiint_E dx\,dy\,dz = \iiint_B abc\,du\,dv\,dw$

$= abc\iiint_B du\,dv\,dw = abc\,\text{Vol}(\text{Ball of radius } 1) = \frac{4}{3}\pi abc.$

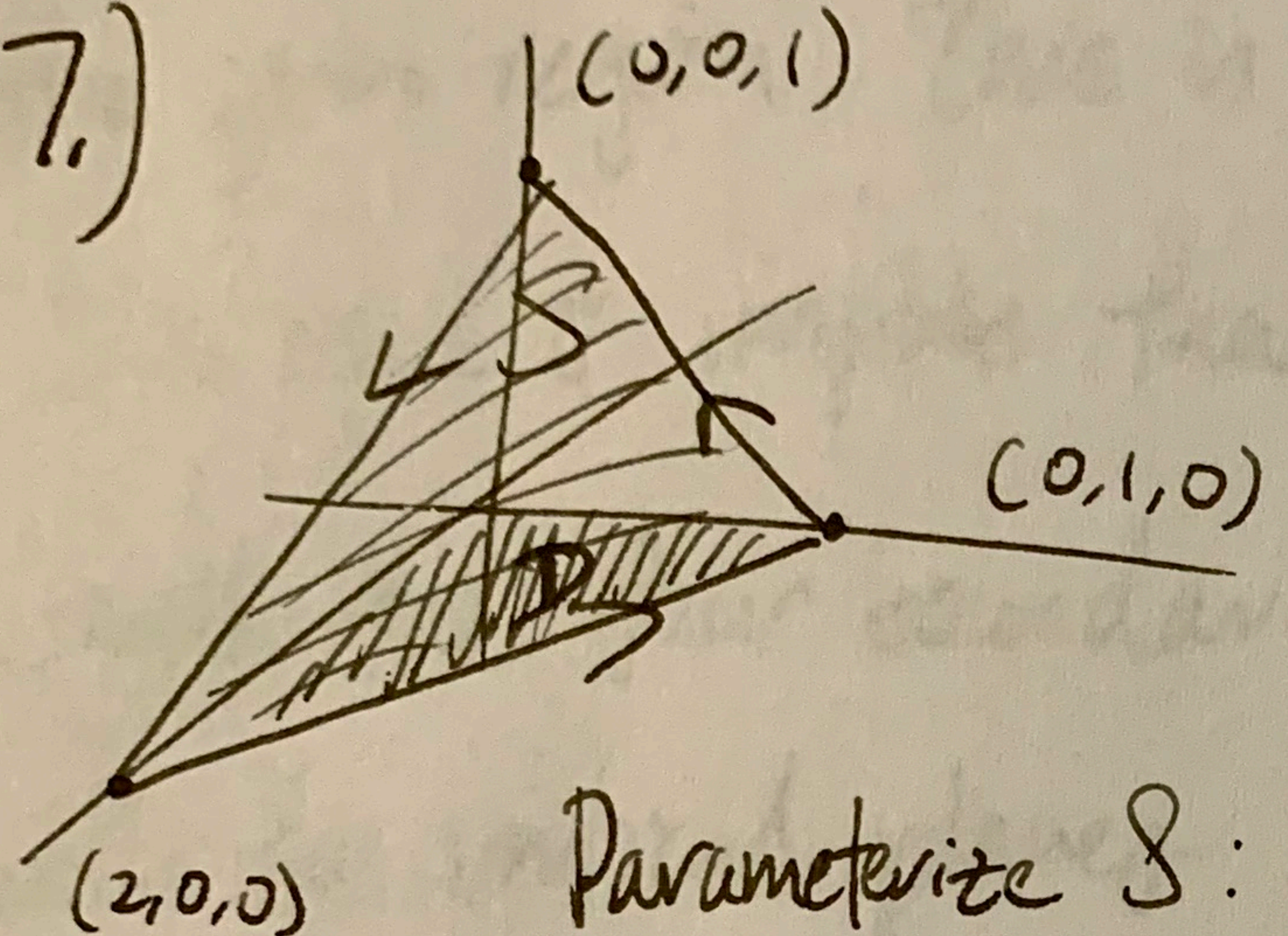
$$\text{Flux} = \iint_{\partial E}\vec{F}\cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\,dV = \iiint_E 6\,dV = 6\iiint_E dV = 8\pi abc$$

6.) a.) $\operatorname{div}\vec{F} = \operatorname{div}\operatorname{curl}\vec{G} = 0$, but $\operatorname{div}\vec{F} = z + xz \neq 0$, so false.

b.) Consider $\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$. False

c.) True. Mininum and maximum can't occur on the inside since $\nabla f \neq 0$.

7.)



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$\operatorname{curl}\vec{F} = \langle 0,0,1\rangle$$

Parameterize S: $\vec{r}(x,y) = \langle x, y, 1-\frac{x}{2}-y\rangle$

$$\left.\begin{aligned}\vec{r}_x &= \langle 1, 0, -\tfrac{1}{2}\rangle \\ \vec{r}_y &= \langle 0, 1, -1\rangle\end{aligned}\right\} \quad \vec{r}_x \times \vec{r}_y = \langle \tfrac{1}{2}, 1, 1\rangle$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \langle 0,0,1\rangle \cdot \langle \tfrac{1}{2}, 1, 1\rangle \, dx\,dy$$

$$= \iint_D dx\,dy = \int_0^2 \int_0^{1-\frac{x}{2}} dy\,dx = 1$$

8.) $f_x = y\cos xy$ and $f_y = x\cos xy$. Each point (x,y) satisfying $xy = \frac{\pi}{2}$ is a critical point. These points yield the value $f(x,y) = \sin\frac{\pi}{2} = 1$, which is an absolute maximum since sine is bounded above by 1. The only other critical point is $(0,0)$ in the given region. This is an absolute minimum since

$0 \le xy \le \pi$ implies that $0 \le \sin xy \le 1$.

Check the four boundaries of the region by taking traces with the vertical planes $x=0$, $x=\pi$, $y=0$, and $y=1$ to find the other extrema. $\sin xy = 0$ for all points on the x-axis, all points on the y-axis, and at $(\pi, 1)$. These will give an absolute minimum.

9.) Use divergence theorem. Let D be the disk of radius 5 centered at the origin in the xz-plane, oriented towards the negative y-axis. Let E be the interior of $S \cup D$. $\operatorname{div}\vec{F} = 0$, so $\iint_{S\cup D} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = 0$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S\cup D} \vec{F}\cdot d\vec{S} - \iint_D \vec{F}\cdot d\vec{S} = -\iint_D \vec{F}\cdot d\vec{S} = -\iint_D \langle x^4, 3, -4x^3 z\rangle \cdot \langle 0,-1,0\rangle\, dA$$

$$= -\iint_D -3\, dA = 3\iint_D dA = 3\pi 5^2 = 75\pi$$

10.) $g=3$, $g_x=2$, $g_y=-1$, $g_z=-1$.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial z}} = -\frac{2}{-1} = 2.$$

11.) Square A : we want to maximize $f(x,y,z) = s(s-x)(s-y)(s-z)$.

$g(x,y,z) = x+y+z-p$ (since $x+y+z=p$)

$$\begin{aligned} f_x &= \lambda g_x & & s(-1)(s-y)(s-z) = \lambda \\ f_y &= \lambda g_y & \Rightarrow\ & s(s-x)(-1)(s-z) = \lambda \\ f_z &= \lambda g_z & & s(s-x)(s-y)(-1) = \lambda \end{aligned}$$

①, ②

①: $\cancel{-s}(s-y)\cancel{(s-z)} = \cancel{-s}(s-x)\cancel{(s-z)} \Rightarrow s-y=s-x \Rightarrow x=y$

②: $\cancel{-s}\cancel{(s-x)}(s-z) = \cancel{-s}\cancel{(s-x)}(s-y) \Rightarrow s-z=s-y \Rightarrow z=y$

$\Rightarrow x=y=z$. An equilateral triangle

12.) $\vec{F}$ is conservative since curl $\vec{F}=0$. $\vec{F}=\nabla f$

$\Rightarrow \langle yze^{xy}, e^{xz}, xye^{xz} \rangle = \langle f_x, f_y, f_z \rangle$

$$\left.\begin{aligned} f_x &= yze^{xy} \Rightarrow f = ye^{xz} \\ f_y &= e^{xy} \Rightarrow f = ye^{xz} \\ f_z &= xye^{xz} \Rightarrow f = ye^{xz} \end{aligned}\right\} \Rightarrow f = ye^{xz}$$

Then $\int_C \vec{F}\cdot d\vec{r} = f(5,3,0) - f(1,-1,0) = 3e^{5\cdot 0} - (-1)e^{1\cdot 0} = 3+1$
$= 4$