Game Theory (STAT 155) Worksheet #4, 02/15/2021

1. Solve the games with the following matrices.

(a)

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution. Let's blindly use equalization to find a candidate for the optimal strategies. The only x that solves

$$x_1 = -x_1 + 2x_2 = -x_1 + x_2 + 3x_3, \tag{1}$$

is $x^* = (\frac{3}{7}, \frac{3}{7}, \frac{1}{7})^T$, and the only y that solves

$$y_1 - y_2 - y_3 = 2y_2 + y_3 = 3y_3 \tag{2}$$

is $y_* = (\frac{5}{7}, \frac{1}{7}, \frac{1}{7})^T$. But now the question remains: is this pair of solutions actually optimal? Under this strategy, Player I can guarantee herself a payoff of at least $\frac{3}{7}$, and Player II can guarantee himself a loss of at most $\frac{3}{7}$, so the strategies are optimal. The value of the game is $\frac{3}{7}$.

□

(b)

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 3/2 & 1 & 1 \\ 1 & 1 & 4/3 & 1 \\ 1 & 1 & 1 & 5/4 \end{pmatrix}$$

Solution. Since the matrix is symmetric, it follows that there will be a pair of optimal strategies where Players I and II will play the same strategy. As before, let's try equalization to find what these are. Some calculation shows that we can take $x_* = y_* = (\frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5})^T$. Then Player I guarantees herself a payoff of at least $\frac{11}{10}$, and Player II guarantees himself a loss of at most $\frac{11}{10}$. Therefore, the value of the game is $\frac{11}{10}$. □

(c)

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Solution. Note that the last column is dominated by the average of the first and the third columns. Then the last row is dominated by the average of the first two rows. So we can instead look for optimal strategies in the reduced game

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0. \\ 0 & 0 & 4 \end{pmatrix}$$

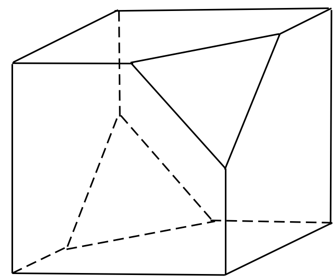
Now we observe that this game is symmetric, so there must be a pair of optimal strategies which agree for Players I and II. Using equalization gives us $x^*_{\text{reduced}} = y^*_{\text{reduced}} = (\frac{6}{13}, \frac{4}{13}, \frac{3}{13})^T$, so this implies that $x_* = y_* = (\frac{6}{13}, \frac{4}{13}, \frac{3}{13}, 0)^T$ is optimal. Also, the value of the game is $\frac{12}{13}$. □

2. Consider a cube in $\mathbb{R}^3$. Cut two opposite vertices and get the following set:

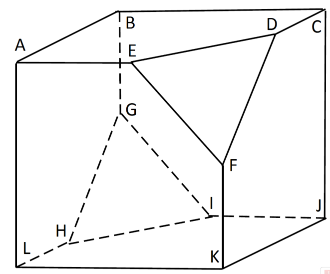
$$\{(x, y, z) \in \mathbb{R}^3 : -1 \le x, y, z \le 1,\ |x + y + z| \le 2\}.$$

(see the picture).

This is a polyhedron. Consider the following game: the first player picks a vertex of the polyhedron, the second player picks one of the faces. If the face contains the vertex, the first player gets $1 from the second player, else she pays $1 to the second player. Find the value of the game and some pair of safety strategies.



Solution. Name the vertices of the cube as in the following picture:



Let's introduce a symmetry that allows us to reduce this game to a much smaller game. Note that Player I has only two types of moves, to play one of the vertices $\{E, G, D, I, F, H\}$ or to play one of the vertices in $\{A, B, C, J, K, L\}$. Likewise, Player II has only two types of moves, to play a pentagonal face or to play a triangular face. So, we may hope to find an optimal strategy where Player I's moves are represented by the sets $\{E, G, D, I, F, H\}$ and $\{A, B, C, J, K, L\}$, where under either move she chooses one of the six elements of that set uniformly at random, and where Player II's moves are represented by the words "Pentagon" and "Triangle", where Pentagon means he chooses one of the six pentagons uniformly at random, and Triangle means he chooses one of the two triangles uniformly at random.

(So far, we are just justifying a heuristic to help us find what may be an optimal strategy. Later we will later check that this strategy is optimal. However, we can also prove right away that there must exist an optimal strategy of this form. To do this, consider the map that sends a point $x = (x_1, x_2, x_3)$ to $(-x_2, -x_3, -x_1)$. This map preserves the overall shape of the polyhedron, but permutes its vertices and faces. The orbit of any vertex is described by the cycles

$$E \to G \to D \to I \to F \to H \to E \text{ and}$$
$$A \to B \to C \to J \to K \to L \to A,$$

while the orbit of any face is desribed by the cycles

$$EDF \to GIH \to DFE \text{ and}$$
$$ABCDE \to BCJIG \to CJKFD \to JKLHI \to KLAEF \to LABGH \to ABCDE.$$

Now we see that any optimal strategy "pushes forward" to another optimal strategy by applying this map. Taking the average over all iterates leads to a strategy which is optimal and which assigns uniform probability to each element of every orbit.)

Now we find the 2×2 payoff matrix for this reduced game.

- Suppose Player I plays $\{E, G, D, I, F, H\}$ and Player II chooses Pentagons. Since each vertex in this set is contained in exactly two of the six pentagon faces, Player I's has a payoff of $+1$ on an event with probability $\frac{2}{6} = \frac{1}{3}$. Similarly, she has a payoff of -1 on an event of probability $\frac{4}{6} = \frac{2}{3}$. Therefore, the expected payoff of this outcome is $-\frac{1}{3}$.
- Suppose Player I plays $\{E, G, D, I, F, H\}$ and Player II chooses Triangles. Since each vertex in this set is contained in exactly one of two triangular faces, we see that she has a payoff of $+1$ on an event of probability $\frac{1}{2}$ and a payoff of -1 on an event of probability $\frac{1}{2}$. So her expected payoff is 0.
- If Player I plays $\{A, B, C, J, K, L\}$ and Player II plays Pentagons, then similar reasoning shows that her expected payoff is 0.
- Suppose that Player I plays $\{A, B, C, J, K, L\}$ and Player II plays Triangles. Note that none of these vertices are contained in either triangle, so Player I has a deterministc payoff of -1.

Therefore, the payoff matrix for this reduced game is

	Pentagons	Triangles
$\{E, G, D, I, F, H\}$	$-\frac{1}{3}$	0
$\{A, B, C, J, K, L\}$	0	-1

Now a direct computation shows that a pair of optimal strategies for the reduced game is the symmetric pair $(\frac{3}{4}, \frac{1}{4})^T$. Moreover, the value of the game is $-\frac{1}{4}$. Finally, we interpret this as a strategy in the original game.

- Player I's optimal strategy is to play $\{E, G, D, I, F, H\}$ with probability $\frac{3}{4}/6 = \frac{1}{8}$ each, and to play $\{A, B, C, J, K, L\}$ with probability $\frac{1}{4}/6 = \frac{1}{24}$ each.
- Player II's optimal strategy is to play pentagonal faces with probability $\frac{3}{4}/6 = \frac{1}{8}$ each, and to play triangular faces with probability $\frac{1}{4}/2 = \frac{1}{8}$ each. (So, overall, Player II's strategy is just to pick a face uniformly at random.)

$\square$

3. Player II chooses a number $j \in \{1, 2, \ldots, n\}$ and I tries to guess what it is. If she guesses correctly, she wins 1. If she guesses too high, she loses 1. If she guesses too low, there is no payoff. Set up the matrix and solve.

Solution. The matrix has ones on the main diagonal, minus ones above the main diagonal, and zeros below.

Let's try to find fully equalizing strategies in this game:

$$\begin{aligned}
1 &= x_1 + x_2 + \cdots + x_n, \\
x_1 &= x_2 - x_1 \implies x_2 = 2x_1 \\
&= x_3 - x_2 - x_1 \implies x_3 = 4x_1 \\
&= x_4 - x_3 - x_2 - x_1 \implies x_4 = 8x_1 \\
&\ldots \\
&= x_n - x_{n-1} - \cdots - x_1 \implies x_n = 2^{n-1}x_1.
\end{aligned}$$

The solution is $x^* = \frac{1}{2^n - 1}(1, 2, 4, \ldots, 2^{n-1})^T$.

Analogously, one can find the fully equalizing strategy of the second player $y^* = \frac{1}{2^n-1}(2^{n-1}, 2^{n-2}, \ldots, 2, 1)^T$.

The value of the game is $\frac{1}{2^n-1}$.

$\square$