# Math 135: Introduction to Set Theory
# Autumn 2000
# Final Examination

20 December 2000
5:00pm - 8:00pm

This exam counts as forty percent of your course grade. There are nine questions.

**1.** Complete the following definitions. If your definition involves a term introduced in this course, you must define that term as well.

a. The set $X$ is a *subset* of the set $Y$ if and only if _______.

b. If $a$ and $b$ are sets then the set $c$ is the *ordered pair* of $a$ and $b$ if and only if _______.

c. The set $R$ is a *relation* between the sets $A$ and $B$ if and only if _______.

d. The set $x$ is *transitive* if and only if _______.

e. The set $\alpha$ is an *ordinal* if and only if _______.

**2.** State the following axioms. Your statement should be presented in the formal language of set theory, though you may explain your formal answer using mathematical English.

- Union Axiom
- Power Set Axiom
- Replacement Axiom

**3.** Suppose that $A$ and $B$ are two sets and $f : \mathcal{P}A \to \mathcal{P}B$ is a function satisfying

$$X \subseteq Y \subseteq A \Rightarrow f(X) \subseteq f(Y)$$

Define $L := \bigcap\{X \subseteq A \mid f(X) \subseteq X\}$. Show that $f(L) = L$.

**4.** Let $A \subseteq \omega$ be a set of natural numbers satisfying $\bigcup A = A$. Show that either $A = \varnothing$ or $A = \omega$.

**5.** Show that if $r \in \mathbb{R}$ is positive (ie $r > 0$), then there is some $s \in \mathbb{R}$ with $s \cdot s = r$.

**6.** For $X \subseteq \mathbb{R}$, let $\mathcal{C}(X, \mathbb{R})$ be the set of continuous functions from $X$ to $\mathbb{R}$. Calculate $\|\mathcal{C}(\mathbb{R}, \mathbb{R})\|$. (Hint: You may use any basic fact about continuity that you know from calculus. Show that if $E : \mathbb{Q} \hookrightarrow \mathbb{R}$ is the standard embedding of the rationals into the real numbers and $r : \mathcal{C}(\mathbb{R}, \mathbb{R}) \to \mathcal{C}(E(\mathbb{Q}), \mathbb{R})$ is the restriction map, $f \mapsto f \restriction_{E(\mathbb{Q})}$, then $r$ is injective.)

**7. Without using any other form of the axiom of choice** show directly that Zorn's Lemma implies that every surjective function has a right inverse.

**8.** Show that for any set $a$ one has $\operatorname{rank} \mathcal{P}a = (\operatorname{rank} a)^+$.

**9.** Compute $(\omega + 1)^4$.