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MATH 135: SET THEORY
FINAL EXAMINATION

For all of the parts of Question 1, you may use only the language of set theory having $\in$ as its sole non-logical primitive. However, if you have defined some term in another part of the problem, you may then use that defined term.

1a. (5 pts)

$$z = \langle x, y\rangle \iff \forall t(t \in z \leftrightarrow (\forall s[s \in t \leftrightarrow t = x] \text{ or } \forall s[s \in t \leftrightarrow (s = x \text{ or } s = y)]))$$

1b.(5 pts)

$$\begin{aligned} f : A \to B \quad \iff \quad & \forall t[t \in f \to \exists a \exists b(a \in A \;\&\; b \in B \;\&\; t = \langle a, b\rangle)] \\ & \& \; \forall a[a \in A \to \exists b\langle a, b\rangle \in f] \\ & \& \; \forall x \forall y \forall z([\langle x, y\rangle \in f \;\&\; \langle x, z\rangle \in f] \to y = z) \end{aligned}$$

1c. (5 pts) Express the Empty Set Axiom.

$$\exists z \forall t(\neg t \in z)$$

1d. (5 pts) Express the Axiom of Choice (in our official formulation).

$$\forall R \exists f \exists A \exists B(f : A \to B \;\&\; \forall t(t \in f \to t \in R) \;\&\; \forall a[\exists b\langle a, b\rangle \in R \to \exists c\langle a, c\rangle \in f])$$

1e. (5 pts)

$$x \text{ is a } \texttt{transitive set} \iff \forall y \forall z([z \in y \;\&\; y \in x] \to z \in x)$$

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1f. (5 pts)

$$
\begin{aligned}
R \text{ is a } \texttt{well-ordering} \text{ of } X \iff\ & \forall t[t \in R \to \exists x \exists y(x \in X \ \&\ y \in X \ \&\ t = \langle x, y\rangle)] \\
& \&\ \forall x(\neg\langle x, x\rangle) \\
& \&\ \forall x \forall y \forall z[(\langle x, y\rangle \in R \ \&\ \langle y, z\rangle \in R) \to \langle x, z\rangle \in R] \\
& \&\ \forall x \forall y[(x \in X \ \&\ y \in X) \to (\langle x, y\rangle \in R \text{ or } x = y \text{ or } \langle y, x\rangle \in R)] \\
& \&\ \forall Y([\forall t(t \in Y \to t \in Y) \ \&\ \exists t(t \in Y)] \\
& \qquad \to \exists s[s \in Y \ \&\ \forall u(u \in Y \to s = u \text{ or } \langle s, u\rangle \in R)])
\end{aligned}
$$

1g.(5 pts)

$$
\begin{aligned}
\kappa \text{ is a } \texttt{cardinal} \iff\ & \forall R[\forall t(t \in R \leftrightarrow \exists x \exists y[t = \langle x, y\rangle \ \&\ x \in \kappa \ \&\ y \in \kappa]) \to R \text{ is a } \texttt{well-ordering} \text{ of } \kappa \\
& \&\ \kappa \text{ is a } \texttt{transitive set} \\
& \&\ \forall \alpha[\alpha \in \kappa \to \forall f(f : \alpha \to \kappa \to \exists \beta(\beta \in \kappa \ \&\ \forall \gamma \langle \gamma, \beta\rangle \notin f)]
\end{aligned}
$$

2. (15 pts) **Prove:** Let R be any relation and let let $X = \text{fld}(R)$ be the field of R. Define $f : \omega \to \mathcal{P}(X \times X)$ by recursion via $f(0) = R$ and $f(n^+) := f(n) \cup f(n)^{-1} \cup f(n) \circ f(n)$. Let $E := \bigcup \text{ran } f$. **Prove:** that E is an equivalence relation on X and that for any equivalence relation F on X with $R \subseteq F$, we have $E \subseteq F$.

Proof: First we check that E is an equivalence relation. If $x \in X$, then there is some $y \in X$ so that $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$. If $\langle x, y\rangle \in R = f(0)$. Either way, $\langle x, x\rangle \in R \circ R = f(0) \circ f(0) \subseteq f(1) \subseteq E$. Thus, E is reflexive on X. If $\langle x, y\rangle \in E$, then for some n, we have $\langle x, y\rangle \in f(n)$, which implies that $\langle y, x\rangle \in f(m^+) \subseteq E$. Finally, if $\langle x, y\rangle \in E$ and $\langle y, z\rangle \in E$, then for some n and m we have $\langle x, y\rangle \in f(m)$ and $\langle y, z\rangle \in f(n)$. Note that whenever $k \leq \ell$, $f(k) \subseteq f(\ell)$ (one sees this by induction — the

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1c. (5 pts) Express the Empty Set Axiom.

Name or initials: ____

1d. (5 pts) Express the Axiom of Choice (in our official formulation).

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1e. (5 pts) Give a precise formal definition of the condition

$x$is a `transitive set` .

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1f. (5 pts) Give a precise formal definition of the condition that

R is a `well-ordering` of X .

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1g.(5 pts) Give a precise formal definition of the condition

$$\kappa \text{ is a } \texttt{cardinal} \text{ .}$$

Name or initials: ____

2. (15 pts) **Prove:** Let R be any relation and let let $X = \mathrm{fld}(R)$ be the field of R. Define $f : \omega \to \mathcal{P}(X \times X)$ by recursion via $f(0) = R$ and $f(n^+) := f(n) \cup f(n)^{-1} \cup f(n) \circ f(n)$. Let $E := \bigcup \operatorname{ran} f$. **Prove:** that E is an equivalence relation on X and that for any equivalence relation F on X with $R \subseteq F$, we have $E \subseteq F$.

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3. (15 pts) Let X be any set and let $R \subseteq X \times X$ be a transitive, irreflexive relation on X. **Prove:** there is a linear ordering $S \subseteq X \times X$ with $R \subseteq S$.

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4. (15 pts) **Prove:** that ת, the class of all cardinals, is not a set. That is, there is no set ת such that $\forall t[t \in ת \leftrightarrow t$ is a cardinal].

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5. (15 pts) **Prove:** that

$$\aleph_\omega < \aleph_\omega^{\aleph_0}$$

Hint: Let $g : \aleph_\omega \to {}^\omega\aleph_\omega$ be any function. Show that for every $n \in \omega$, $\aleph_\omega \operatorname{ran} g \restriction \aleph_n \neq \emptyset$. Use this result to construct some $f \in {}^\omega\aleph_\omega \smallsetminus \operatorname{ran}(g)$.

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6. (15 pts) **Prove:** that ordinal addition is associative. That is, for all ordinals α, β, and γ, we have

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

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7. (15 pts) Let $X \subseteq \mathbb{R}$ be a subset of the real numbers is well-ordered with respect to the ordering induced from the usual ordering on $\mathbb{R}$. **Prove:** $\mathrm{card}(X) \leq \aleph_0$.

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8. (15 pts) Let X be any set, E an equivalence relation on X, and κ any cardinal. We suppose that for each $x \in X$, $\mathrm{card}([x]_E) \leq \kappa$. **Prove:** $\mathrm{card}(X) \leq \mathrm{card}(X/E) \cdot \kappa$.

Name or initials: ______________________________

For which problem should this work be credited?

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