Game Theory (STAT 155) Worksheet #6 03/01/2021

1. Find some pure Nash equilibrium in the following congestion game:

   - $k = 3$ players,
   - $m = 3$ facilities $\{1 \dots m\}$,
   - For player i , there is a set S_i of strategies that are subsets of facilities, $s \subseteq \{1 \dots m\}$, $S_1 = S_2 = S_3 = \{\{1,2\},\{3,2\},\{1,3\}\}$.
   - For facility j , there is a cost vector $c_j \in \mathbb{R}^k$, where $c_j(n)$ is the cost of facility j when it is used by n players.

   $$c_1 = (1,2,3), \quad c_2 = (3,2,1), \quad c_3 = (2,2,2).$$

   Solution. We know that this is a congestion game, so we can get a pure Nash equilibrium by iteratively increasing the utility of each player. In principle we can start this procedure from anywhere, but our life may be made slightly easier if we choose a starting configuration which seesm like it may be close to stable. Indeed, it seems like we should make sure that the maximum number of players use facility 2, since this becomes more efficient as more players use it. A good second choice is facility 3, since it does not worsen as more players use it. Keeping this in mind, let's start with the configuration of strategies $\{2,3\},\{2,3\},\{2,3\}$. At this point, the cost to every player is 3 units.

   Now we consider changing the strategy of one individual player. Since all players are currently playing the same strategy, we can alter that of Player I. If she changes her strategy from $\{2,3\}$ to $\{1,2\}$, then her cost decreases to 2 units. If she changes her strategy from $\{2,3\}$ to $\{1,3\}$, then her cost remains 3 units.

   Now we are at the configuration of strategies $\{1,2\},\{2,3\},\{2,3\}$. Let's consider altering the strategy of Player II. If he changes his strategy to $\{1,2\}$, his cost remains 2 units. If he changes his strategy to $\{1,3\}$, his cost increases to 4 units. (Recall that, even if his strategy change incurs a change in cost for the other players, he does not care. All of his utility is encoded in the costs included in his own payoff matrix.) So, Player II has no incentive to change strategies. Similarly, Player II has no incentive to change strategies.

   In particular, this shows that the configuration of strategies $\{1,2\},\{2,3\},\{2,3\}$ is a (pure) Nash equilibrium. $\square$

2. Show that the following games are not potential:

   $$\begin{pmatrix} (5,3) & (1,5) \\ (0,4) & (3,3) \end{pmatrix}, \quad \begin{pmatrix} (2,1) & (3,1) & (2,3) \\ (2,1) & (4,2) & (2,2) \\ (3,2) & (4,0) & (1,3) \end{pmatrix}$$

   Solution. Every potential game must have a pure Nash equilibrium, so in the first game we can simple note that there is no pure Nash equilibrium. This follows by checking the four cells of the matrix and noting that, in each case, some player has an incentive to deviate.

   For the second game, let us make a general observation. Suppose that a two-player game is a potential game with potential function ϕ. Then for any cycle of strategies $(s_1,s_2) \to (s_1',s_2) \to (s_1',s_2') \to (s_1,s_2') \to (s_1,s_2)$, we have

   $$\begin{aligned}
   &\Big(u_1(s_1',s_2) - u_1(s_1,s_2)\Big) + \Big(u_2(s_1',s_2') - u_2(s_1',s_2)\Big) + \Big(u_1(s_1',s_2) - u_1(s_1',s_2')\Big) + \Big(u_2(s_1,s_2) - u_2(s_1',s_2)\Big) \\
   &= \Big(\phi(s_1',s_2) - \phi(s_1,s_2)\Big) + \Big(\phi(s_1',s_2') - \phi(s_1',s_2)\Big) + \Big(\phi(s_1',s_2) - \phi(s_1',s_2')\Big) + \Big(\phi(s_1,s_2) - \phi(s_1',s_2)\Big) \\
   &= \Big(\phi(s_1,s_2) - \phi(s_1,s_2)\Big) + \Big(\phi(s_1',s_2) - \phi(s_1',s_2)\Big) + \Big(\phi(s_1',s_2') - \phi(s_1',s_2')\Big) + \Big(\phi(s_1,s_2') - \phi(s_1,s_2')\Big) \\
   &= 0 + 0 + 0 + 0 \\
   &= 0.
   \end{aligned} \tag{1}$$

   In other words, the change in utility as we wind around any cycle must be zero. So, to check that a game is not a potential game, it suffices to find a cycle for which the sum of the change in utility as we wind around the cycle is non-zero.

   To do this, consider the "bottom-left square" of the matrix for the game. This is just the cycle of strategies $(2,1) \to (3,1) \to (3,2) \to (2,2) \to (2,1)$, or, visually:

   $$\begin{pmatrix} (2,1) & (3,1) & (2,3) \\ \underline{(2,1)} & \underline{(4,2)} & (2,2) \\ \underline{(3,2)} & \underline{(4,0)} & (1,3) \end{pmatrix}$$

As we move from $(2,1) \to (3,1)$, Player I's payoff changes from 2 to 3, so increases by 1. As we move from $(3,1) \to (3,2)$, Player II's payoff changes from 2 to 0, so decreases by 2. As we move from $(3,2) \to (2,2)$, Player I's payoff remains at 4, so does not change. And, as we move from $(2,2) \to (2,1)$, Player II's payoff changes from 2 to 1, so decreases by 1. The total change in utility is $1 - 2 + 0 - 1 = -2$ which is non-zero, so this is not a potential game. □

3. Consider the following general-sum game of n players. For each i from 1 to n the i-th player picks a natural number k_i from 1 to 1000. Denote the set of natural divisors of k as $D(k)$. The utility functions are the following:

$$u_i(k_i, k_{-i}) = \sum_{d \in D(k_i)} \#\{j : k_j \neq 0 \bmod d\}.$$

Show that this game has a pure Nash equilibrium.

Solution. Let's show that this game is a congestion game:

- Facilities are all possible divisors.
- Each player picks a number, which can be seen as picking the subset of divisors. (So, the set of possible subsets that are legal to pick is exactly the set of sets of numbers in $\{1, \dots 1000\}$ which correspond to the divisor set of a number in $\{1, \dots 1000\}$.)
- We can view utility (which we aim to maximize) as the opposite of cost (which we aim to minimize). From the expressions of the utility functions, we see that, if a player chooses a set of divisors including d, then this player incurs the cost $-\#\{j : k_j \neq 0 \bmod d\}$. This value is minus one times the number of players who don't use this divisor. Thus, for example, Player I pays for each facility that she uses, and the payment for each facility depends only on the number of players who use this facility. Likewise for all other players.

This is exactly the definition of a congestion game. In particular, this is a potential game, so it has a pure Nash equilibrium.

(During the discussion section, the question arose of whether we can use the "greedy algorithm" to actually find a pure Nash equilibrium for this game. This is surely possible, but it may take many steps to terminate, so I would not try to attempt it by hand. Instead I wrote a script to execute the greedy algorithm (nothing fancy, just checking the updates by brute force), and it found some pure Nash equilibria for some values of n to be:

n	pure Nash equilibirum
5	$(840, 960, 840, 792, 900)$
6	$(840, 720, 840, 900, 924, 936)$
7	$(840, 990, 840, 900, 924, 936, 960)$
8	$(840, 960, 840, 720, 924, 900, 936, 990)$
9	$(840, 960, 840, 720, 840, 990, 900, 924, 936)$
10	$(840, 840, 864, 720, 840, 900, 924, 936, 990, 960)$

It runs relatively fast and we can even check that for $n = 25$, a pure Nash equilibrium is

$$(840, 840, 840, 960, 840, 900, 840, 720, 840, 720, 840, 900, 720, 924, 936, 960, 990, 924, 900, 936, 840, 990, 864, 924, 936).$$

There seems to be some sort of structure here, but it appears pretty hard to untangle exactly what's going on.) □