TWISTED * SETS

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Contents

1 The category of twisted objects

1.1 Category $\mathcal{C}^{(1)}$

1.1.1 Twisted objects

Let $c \in \mathcal{C}_0$ be an object of a category $\mathcal{C}$ and

$$\tau \in \mathrm{End}_{\mathcal{C}}\, c \tag{1}$$

be its endomorphism. We shall refer to (c, τ) as a *twisted object* (in $\mathcal{C}$). Given a twisted object (c, τ) we shall refer to c as *the underlying object of* $\mathcal{C}$ and to endomorphism (1) as *the twist*.

1.1.2 Intertwining morphisms

Given two *twisted objects* (c, τ) and (c', τ'), we say that $\alpha \in \mathrm{Hom}_{\mathcal{C}}(c, c')$ is an *intertwining morphism* if $\alpha \circ \tau = \tau' \circ \alpha$, i.e., if the following diagram commutes

$$\begin{tikzcd}
c \arrow[r, "\alpha"] & c' \\
c \arrow[u, "\tau"] \arrow[r, "\alpha"] \arrow[ur, phantom, "\circlearrowright"] & c' \arrow[u, "\tau'"']
\end{tikzcd}$$

1.1.3 Morphisms between twisted objects

By definition, morphisms between twisted objects

$$(c, \tau) \longrightarrow (c', \tau') \tag{2}$$

are the intertwining morphisms.

1.1.4 Notation

We shall denote the category of twisted objects in $\mathcal{C}$ by $\mathcal{C}^{(1)}$.

2 The category of twisted sets

2.1 Category $\mathbf{Set}^{(1)}$

2.1.1 Twisted sets coincide with unary algebraic structures

The category of *twisted sets* is the same as the category of *unary algebraic structures*. We shall denote it $\mathbf{Set}^{(1)}$.

2.1.2 Homomorphisms

For this reason we shall refer to morphisms

$$f : (X, \tau) \to (X, \tau') \tag{3}$$

as *homomorphisms*.

2.1.3 Notation

In order to keep notation simple, we will omit parentheses around the argument of τ when the argument is denoted by a single symbol.

2.1.4 Three canonical twisted-set structures on the power-set of a twisted set

The twist operation induces three canonical operations on $\mathscr{P}X$, hence we have three twisted set structures on the power-set of a twisetd set,

$$(\mathscr{P}X, \tau_*)\,, \qquad (\mathscr{P}X, \tau^*) \qquad \text{and} \qquad (\mathscr{P}X, \tau_!)\,. \tag{4}$$

Exercise 1 *Show that a homomorphism* (3) *induces a homomorphism*

$$f_* : (\mathscr{P}X, \tau_*) \longrightarrow (\mathscr{P}X', \tau'_*)\,. \tag{5}$$

Exercise 2 *Show that a homomorphism* (3) *induces a homomorphism*

$$f^* : (\mathscr{P}X', \tau'^*) \longrightarrow (\mathscr{P}X, \tau^*)\,. \tag{6}$$

Exercise 3 *Show that a homomorphism* (3) *induces a homomorphism*

$$f_! : (\mathscr{P}X, \tau_!) \longrightarrow (\mathscr{P}X', \tau'_!)\,. \tag{7}$$

2.2 Induced twisted set structures

Consider the set of functions $\mathrm{Funct}(W, X)$ from a set W to a set X.

2.2.1 The target-induced structure

Post-composition with a unary operation $\tau \in \mathrm{Op}_1 X$ is a unary operation on $\mathrm{Funct}(W, X)$,

$$f \longmapsto \tau_{\bullet} f = \tau \circ f \qquad (f \in \mathrm{Funct}(W, X))\,. \tag{8}$$

2.2.2 The source-induced structure

Pre-composition with a unary operation $\sigma \in \mathrm{Op}_1 W$ is a unary operation on $\mathrm{Funct}(W, X)$,

$$f \longmapsto \sigma^{\bullet} f = f \circ \sigma \qquad (f \in \mathrm{Funct}(W, X)) \,. \tag{9}$$

Exercise 4 *Show that the post-composition operation $\tau_{\bullet}$ commutes with the pre-composition operation $\sigma^{\bullet}$.*

2.2.3 Bitwisted sets

We shall call a set (Y, σ, τ) equipped with two commuting with each other unary operations a *bitwisted set.*

2.2.4 Characterization of homomorphisms

Homomorphisms of twisted sets $(W, \sigma) \longrightarrow (X, \tau)$ afford an elegant characterization in terms of the bitwisted structure of the set of functions $(\mathrm{Funct}(W, X), \tau_*, \sigma^*)$.

Exercise 5 *Show that*

$$f \in \mathrm{Hom}\big((W, \sigma), (X, \tau)\big) \;\Leftrightarrow\; \tau_{\bullet} f = \sigma^{\bullet} f \,. \tag{10}$$

2.2.5 The evaluation-at-w homomorphism

Recall the definition of the evaluation-at-w function

$$\mathrm{ev}_w : \mathrm{Funct}(W, X) \longrightarrow X \,, \qquad f \longmapsto \mathrm{ev}_w(f) = f(w) \,. \tag{11}$$

Exercise 6 *Show that* ev_w *is a homomorphism*

$$\big(\mathrm{Funct}(W, X), \tau_{\bullet}\big) \longrightarrow (X, \tau) \,.$$

2.3 The twisted set canonically associated with an (almost) well-ordered set $(X, \preccurlyeq)$

2.3.1 *Almost* well-ordered sets

Let $(X, \preccurlyeq)$ be an ordered set in which every nonempty, *bounded below* subset has the smallest element. We shall say in this case that $(X, \preccurlyeq)$ is *almost well-ordered.*

2.3.2 Well-ordered sets

An almost well-ordered set is *well-ordered* precisely when X is bounded below which is equivalent to $(X, \preccurlyeq)$ having the smallest element.

2.3.3 The associated successor operation

We define the associated *successor operation* on X as follows

$$x^+ = \begin{cases} x & \text{when } x \text{ is a maximal element of } (X, \preccurlyeq) \\ \min\{y \in X \mid x \prec y\} & \text{otherwise} \end{cases} \tag{12}$$

2.4 Invariant subsets

2.4.1 The definition

We say that a subset $E \subseteq X$ is *invariant under* τ or, τ*-invariant*, if it is closed under operation τ,

$$\forall_{x \in E}\; \tau(x) \in E\,. \tag{13}$$

2.4.2

In the language of Algebra, an invariant subset of a twised set is the same as a *substructure* of a unary structure.

2.4.3

A subset A of X is invariant if and only if the diagram

$$\begin{tikzcd}
A \arrow[r, hook] & X \\
A \arrow[r, hook] & X \arrow[u, "\tau"']
\end{tikzcd}$$

admits a completion to a commutative diagram

$$\begin{tikzcd}
A \arrow[r, hook] & X \\
A \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[u, "\tau"']
\end{tikzcd}$$

2.4.4 The *induced* operation on an invariant subset

The operation represented by the blue arrow is unique. We refer to it as the operation *induced by* τ *on* A.

Indeed, if τ' and τ'' are two such operations and ι denotes the canonical inlusion of X' into X, then

$$\iota \circ \tau' = \tau \circ \iota = \iota \circ \tau''.$$

Since ι is *injective*, it is a monomorphism in the category of sets. Hence, $\tau' = \tau''$.

Generally, the induced operation on an invariant subset $A \subseteq X$ is denoted τ, the same way as the operation on X.

2.4.5 Equivalent definitions in terms of the adjoint pair τ_*, τ^* of operations on $\mathscr{P}X$

Property (13) is equivalent to either one of the following two conditions

$$\tau_* E \subseteq E \tag{14}$$

and

$$E \subseteq \tau^* E\,. \tag{15}$$

2.4.6 The complete lattice $(\mathscr{P}_{\mathrm{inv}}(X,\tau),\subseteq)$ of invariant subsets

Let us denote the set of all invariant subsets of (X,τ) by $\mathscr{P}_{\mathrm{inv}}(X,\tau)$.

Exercise 7 *Show that intersection*

$$\bigcap \mathscr{E}$$

and union

$$\bigcup \mathscr{E}$$

of any family of invariant subsets $\mathscr{E} \subseteq \mathscr{P}X$ is invariant.

It follows that $(\mathscr{P}_{\mathrm{inv}}(X,\tau),\subseteq)$ is a complete sublattice of $(\mathscr{P}X,\subseteq)$.

2.4.7 Galois connection f_*, f^*

Let f be a homomorphism, cf. (3), and $A \subseteq X$ and $A' \subseteq X'$ be invariant subsets.

Exercise 8 *Show that f_*A is τ'-invariant and f^*A' is τ-invariant.*

It follows that the Galois connection f_*, f^* between the corresponding power-sets induces a Galois connection between the lattices of invariant subsets

$$\begin{array}{c} (\mathscr{P}_{\mathrm{inv}}(X',\tau'),\subseteq) \\ {\scriptstyle f_*}\big\uparrow\ \big\downarrow{\scriptstyle f^*} \\ (\mathscr{P}_{\mathrm{inv}}(X,\tau),\subseteq) \end{array} \qquad . \tag{16}$$

2.4.8 The invariant subset *generated* by a subset

Given a subset $A \subseteq X$, we denote by $\langle A\rangle$ the smallest invariant subset of X that contains A. By Exercise 7 such a subset exists and coincides with

$$\bigcap\{E \subseteq X \mid A \subseteq X \text{ and } E \text{ is invariant}\}\,. \tag{17}$$

Exercise 9 *Show that*

$$\forall_{A,B\subseteq X}\ A\subseteq B \Rightarrow \langle A\rangle\subseteq\langle B\rangle\,. \tag{18}$$

Exercise 10 *Show that*

$$\forall_{A\subseteq X}\ \langle\langle A\rangle\rangle = \langle A\rangle\,. \tag{19}$$

3 The orbital structure of a twisted set

3.1 Orbits

3.1.1 The *orbit* of an element

Given an element $x \in X$ in a twisted set, we shall denote by $\langle x \rangle$ the smallest invariant subset containing x. This set coincides with $\langle \{x\} \rangle$. We shall refer to it as *the orbit of* x.

Exercise 11 *Show that $E \subseteq X$ is an invariant subset if and only if*

$$\forall_{x \in X}\; E \ni x \Leftrightarrow E \supseteq \langle x \rangle\,. \tag{20}$$

3.1.2 The set of orbits X_τ

Each orbit being an invariant subset means that the set of orbits, denoted X_τ, is a subset of $\mathscr{P}_{\mathrm{inv}}(X, \tau)$.

Exercise 12 *Show that $E \subseteq X$ is invariant if and only if*

$$E = \bigcup_{x \in E} \langle x \rangle\,.$$

3.1.3

It follows that invariant subsets coincide with unions of families of orbits. In other words, ordered set $(X_\tau, \subseteq)$ is sup-dense in $(\mathscr{P}_{\mathrm{inv}}(X, \tau), \subseteq)$ and $(\mathscr{P}_{\mathrm{inv}}(X, \tau), \subseteq)$ is the sup-closure of $(X_\tau, \subseteq)$.

3.1.4 The relation between the orbits of x and τx

Exercise 13 *Show that, for every $x \in X$, the subset*

$$\{x\} \cup \langle \tau x \rangle \tag{21}$$

is invariant.

Exercise 14 *Show that*

$$\langle x \rangle \subseteq x \cup \langle \tau x \rangle\,. \tag{22}$$

Exercise 15 *Show that, for every $x \in X$, one has*

$$\langle x \rangle \supseteq \langle \tau x \rangle\,. \tag{23}$$

Lemma 3.1 *One has*

$$\langle x \rangle = \{x\} \cup \langle \tau x \rangle \tag{24}$$

and

$$\tau_* \langle x \rangle = \langle \tau x \rangle\,. \tag{25}$$

Proof. Decomposition (24) follows by combining Exercise 14 with Exercise 15.

By applying τ_* to both sides of (22), we obtain

$$\tau_*\langle x\rangle \subseteq \{\tau x\} \cup \tau_*\langle \tau x\rangle \subseteq \langle \tau x\rangle\,.$$

In view of the fact that $\tau_*\langle x\rangle$ is an invariant subset and $\tau_*\langle x\rangle \ni \tau x$, we obtain with help of Exercise 11 that

$$\tau_*\langle x\rangle \supseteq \langle \tau x\rangle\,.$$

□

Corollary 3.2 *One has*

$$\forall_{x,y\in X}\ \langle x\rangle \supseteq \langle y\rangle \implies \langle \tau x\rangle \supseteq \langle \tau y\rangle\,. \tag{26}$$

Proof. In view of Equality (25), one has

$$\langle \tau x\rangle = \tau_*\langle x\rangle \supseteq \tau_*\langle y\rangle = \langle \tau y\rangle\,.$$

□

3.1.5 The orbit epimorphism $\langle\ \rangle : (X,\tau) \twoheadrightarrow (X_\tau, \tau_*)$

Let

$$[x] = \big\{y \in X \mid \langle x\rangle = \langle y\rangle\big\}\,. \tag{27}$$

Obviously,

$$x \in [x] \subseteq \langle x\rangle\,. \tag{28}$$

If $y \in [x]$, then, in view of Corollary 3.2,

$$\langle \tau x\rangle = \langle \tau y\rangle\,,$$

i.e., $[\tau x] = [\tau y]$. In particular, the function that assigns to each element $x \in X$ its orbit,

$$\langle\ \rangle : (X,\tau) \longrightarrow (X_\tau, \tau_*)\,, \qquad x \longmapsto \langle x\rangle\,, \tag{29}$$

is a homomorphism. It is, of course, surjective, by the definition of X_τ. Surjective homomorphisms are called *epimomorphisms.* The fibers of function (29) are subsets $[x]$ defined in (27).

3.1.6 Orbits fixed by τ_*

Recall that an element $x \in X$ of a unary operation τ is said to be a *fixed point* if

$$\tau x = x\,. \tag{30}$$

Lemma 3.3 *If there exists $y \neq x$ such that $\langle x\rangle = \langle y\rangle$, then orbit $\langle x\rangle$ is fixed by τ_*,*

$$\tau_*\langle x\rangle = \langle \tau x\rangle = \langle x\rangle\,. \tag{31}$$

Proof. If $\langle x\rangle = \langle y\rangle$ and $y \neq x$, then

$$\tau_*\langle x\rangle = \langle \tau x\rangle \supseteq \langle x\rangle \setminus \{x\} \ni y$$

according to Decomposition (24) combined with Equality (25). Hence

$$\langle x\rangle \supseteq \langle \tau x\rangle \supseteq \langle y\rangle = \langle x\rangle .$$

□

Lemma 3.4 *For any element $x \in X$, one has*

$$\langle x\rangle = \langle \tau x\rangle \iff \forall_{y\in\langle x\rangle}\ \langle x\rangle{=}\langle y\rangle . \tag{32}$$

In other words, an orbit $\langle x\rangle \in X_\tau$ is a fixed point of τ_* if and only if $[x] = \langle x\rangle$.

Proof. The right-hand-side statement in (32) is equivalent to equality

$$[x] = \langle x\rangle . \tag{33}$$

Since $x \in [x]$ and $\langle x\rangle$ is, by definition, the smallest invariant subset containg x, it suffices to show that equality $\langle x\rangle = \langle \tau x\rangle$ implies invariance of $[x]$. Indeed, for every $y \in [x]$, one has

$$\langle x\rangle = \langle y\rangle \supseteq \langle \tau y\rangle = \langle \tau x\rangle = \langle x\rangle .$$

□

Corollary 3.5 *For any $x, y \in X$, one has*

$$\langle x\rangle \supsetneq \langle y\rangle \iff \langle x\rangle{\neq}\langle \tau x\rangle \wedge \langle \tau x\rangle{\supseteq}\langle y\rangle . \tag{34}$$

Proof. If $\langle x\rangle \supsetneq \langle y\rangle$, then $x \notin \langle y\rangle$. Otherwise $\langle x\rangle \subseteq \langle y\rangle$ and that contradicts the hypothesis. Then $\langle x\rangle \neq \langle \tau x\rangle$, by Lemma 3.4, and $\langle y\rangle$ is contained in $\langle x\rangle\setminus\{x\}$ which is, in view of Decomposition (24), contained in $\langle \tau x\rangle$.

If $\langle x\rangle{\neq}\langle \tau x\rangle$ and $\langle \tau x\rangle{\supseteq}\langle y\rangle$, then

$$\langle x\rangle \supsetneq \langle \tau x\rangle \supseteq \langle y\rangle .$$

□

4 The orbital preorder of a twisted set

4.1 Preliminaries on preordered sets

4.1.1 Notation

Below $(P, \precsim)$ denotes a preordered set, i.e., a set equipped with reflexive and transitive relation denoted $\precsim$.

4.1.2 The associated equivalence relation

We shall say that two elements p and q of a preordered set are *equivalent* if

$$p \precsim q \qquad \text{and} \qquad q \precsim p\,. \tag{35}$$

We shall denote this by $p \simeq q$.

By definition, relation $\simeq$ is reflexive, symmetric and transitive, i.e., it is an equivalence relation on P.

4.1.3 Intervals $[p, q]$

Consider the family of right-relatives sets

$$[p\rangle := \{r \in P \mid p \precsim r\} \qquad (p \in P) \tag{36}$$

and the family of left-relatives sets

$$\langle q] := \{r \in P \mid r \precsim q\} \qquad (q \in P)\,. \tag{37}$$

We define the family of right interval sets $[p, q]$ as intersections of the corresponding right- and left-relatives sets,

$$[p, q] := [p\rangle \cap \langle q]\,. \tag{38}$$

Note that $[p, p]$ is the set of equivalence classes of the associated equivalence relation $\simeq$. We shall denote it $[p]$.

4.1.4

A preorder relation is an order relation if and only if the associated equivalence relation $\simeq$ coincides with the idenity relation $=$.

4.1.5 The associated sharp-order relation

We shall say that p is *less than* q if

$$p \precsim q \qquad \text{and} \qquad p \not\simeq q\,. \tag{39}$$

We shall denote this by $p \prec q$.

By definition, relation $\prec$ is antisymmetric and transitive, i.e., it is a sharp-order relation on P.

4.1.6 Intervals $[p, q)$

We shall also encounter intervals

$$[p, q) \coloneqq \{r \in P \mid p \precsim r \prec q\}\,. \tag{40}$$

One obviously has the decomposition of $[p, q]$,

$$[p, q] = [p, q) \cup [q]\,, \tag{41}$$

into the union of disjoint subsets.

4.1.7 Linearly-preordered sets

We say that a preordered set $(P, \precsim)$ is *linearly-preordered* if

$$\forall_{p,q \in P}\ p{\prec}q \vee q \precsim p\,. \tag{42}$$

4.1.8 Minimal elements

Recall that an element $m \in P$ of a preordered set $(P, \precsim)$ is *minimal* if

$$\forall_{p \in P}\ m \precsim p\,. \tag{43}$$

We shall denote the set of minimal elements of a subset $A \subseteq P$ by $\min A$.

4.1.9

In an ordered set, the set of minimal elements of A has no more than one element. In theory of ordered sets, $\min A$ serves as the standard notation for the unique minimal element when it exists.

4.1.10 Well-preordered sets

We say that a preordered set $(P, \precsim)$ is *well-preordered* if

$$\forall_{A \subseteq P}\ A{\neq}\emptyset \implies \min A \neq \emptyset \tag{44}$$

4.1.11 Almost well-preordered sets

If for every nonempty and bounded below subset $A \subseteq P$, the set of minimal elements $\min A$ is nonempty, we shall say that $(P, \precsim)$ is *almost well-preordered.*

4.1.12

A well-preordered set is the same as an almost well-preordered set that is bounded below.

4.2 Preliminaries on induced relations

4.2.1 The relation on X induced by a function $f : X \longrightarrow (S, \sim)$

Recall that a function $f : X \longrightarrow S$ whose target is S is equipped with a binary relation $\sim$, induces a binary relation on X:

$$\sim^{\mathrm{ind}} : X, X \longrightarrow \text{Statements}, \qquad x, y \longmapsto \text{“ } f(x) \sim f(y) \text{ ”}. \tag{45}$$

4.2.2

The induced relation is *reflexive*, respectively, *transitive*, if $\sim$ is reflexive, respectively, transitive.

Exercise 16 *Suppose that $\sim$ is weakly antisymmetric. Show that $\sim^{\mathrm{ind}}$ is weakly antisymmetric if and only if f is injective.*

4.2.3 The relation induced by a preorder is a preorder

It follows that the relation induced by a preorder relation is a preorder relation but is not an order relation even if $\sim$ is, unless f is injective.

4.3 The orbital preorder

4.3.1 The preorder on a twisted set induced by the orbit epimorphism

We define the *orbital preorder* of a twisted set to be the relation induced by the orbit epimorphism (29) and the *reverse* containment relation on the set od orbits X_τ,

$$\precsim : X, X \longrightarrow \text{Statements}, \qquad x, y \longmapsto \text{“ } \langle x \rangle \supseteq \langle y \rangle \text{ ”}. \tag{46}$$

In view of Exercise 11, one has

$$\forall_{x,y \in X} \; x \precsim y \Leftrightarrow \langle x \rangle \ni y \,. \tag{47}$$

4.3.2 The right-relatives sets of the orbital preorder relation coincide with orbits

Given an element $x \in X$, let

$$[x\rangle := \{ y \in X \mid x \precsim y \} \tag{48}$$

denote the set of right-relatives of $x \in X$ for the orbital preorder relation.

Exercise 17 *Show that*

$$[x\rangle = \langle x \rangle \,. \tag{49}$$

4.3.3 Properties of the orbital preorder

Below we collect in the form of exercises several properties of the orbit preorder that are, essentially, the corresponding properties, of orbits, restated in terms of $\precsim$.

Exercise 18 *Show that*

$$\forall_{x\in X}\ x \precsim \tau x\,. \tag{50}$$

Exercise 19 *Show that*

$$\forall_{x,y\in X}\ x\precsim y \Longrightarrow \tau x\precsim\tau y\,. \tag{51}$$

Exercise 20 *Suppose that $x \simeq y$ while $x \neq y$. Show that $x \simeq \tau x$.*

Solution. Since $\langle x\rangle \ni y$ and $x \neq y$, Decomposition (24) implies that $\langle \tau x\rangle \ni y$, hence

$$\langle\tau x\rangle \supseteq \langle y\rangle \supseteq [y] = [x] \ni x\,,$$

i.e., $\tau x \precsim x$. □

Exercise 21 *Show that $x \simeq \tau x$ if and only if x is a maximal element of $(X, \precsim)$, i.e., it satisfies the following property*

$$\forall_{y\in X}\ x\precsim y \Longrightarrow x \simeq y\,. \tag{52}$$

4.3.4

Exercises 20 and 21 have the following corollaries.

Corollary 4.1 *One has $[x] = \{x\}$ precisely in one of the following two mutually exclusive cases*

(a) $x \prec \tau x$;

(b) $x = \tau x$, *i.e., x is a fixed point of τ.*

□

Corollary 4.2 *The following two conditions are equivalent.*

(a) *x is a maximal element of $(X, \precsim)$ and $[x]$ is a singleton set*;

(b) $x = \tau x$, *i.e., x is a fixed point of τ.*

□

Corollary 4.3 *One has $[x] \supsetneq \{x\}$ if and only if x is a maximal element and $x \neq \tau x$. In this case $[x] = \langle x\rangle$.* □

Exercise 22 *Show that the following statements are equivalent*:

(a) $x \prec y$;

(b) *x is not a maximal element and $\tau x \precsim y$.*

4.3.5 Orbits are linearly-preordered

Lemma 4.4 *Every orbit $\langle x\rangle$ is linearly-preordered, cf. Section 4.1.7.*

Proof. Consider the subset of $\langle x\rangle$,

$$E := \{y \in \langle x\rangle \mid \forall_{z\in\langle x\rangle}\ y{\prec}z \vee z{\precsim}y\}\,. \tag{53}$$

If $x \precsim z$, then either $x \prec z$ or $x \simeq z$. If $x \simeq z$, then, in particular, $z \precsim x$. This demonstrates that $x \in E$.

If $z \precsim y$, then

$$z \precsim y \precsim \tau y$$

by Exercise 18. If $z \not\precsim y$ and $y \in E$, then

$$y \prec z$$

which implies, in view of Exercise 22, that

$$\tau y \precsim z\,.$$

This holds for arbitrary $z \in \langle x\rangle$, hence $\tau y \in E$.

We demonstrated that $x \in E$ and E is an invariant subset of $\langle x\rangle$. Hence $E = \langle x\rangle$. □

Corollary 4.5 *Let y and z be two different elements of $\langle x\rangle$. If $[y]$ or $[z]$ are singleton sets, then*

$$y \prec z \qquad \text{or} \qquad z \prec y\,. \tag{54}$$

□

Corollary 4.6 *For any $x, y \in X$, one has*

$$[x, \tau y] = [x, y] \cup [\tau y] \tag{55}$$

where $[x, y]$ denotes the interval set introduced in Section 4.1.3.

Proof. If $z \in [x, \tau y]$ and $z \notin [x, y]$, then

$$y \prec z\,,$$

in view of Lemma 4.4. Then $\tau y \precsim z$, according to Exercise 22, and therefore $y \simeq z$. □

Corollary 4.7 *For any $x \in X$ and any $y, z \in \langle x\rangle$ such that*

$$y{\neq}z \qquad \text{and} \qquad \tau y{\simeq}\tau z\,, \tag{56}$$

one has:

(a) *$y \simeq \tau y$ or $z \simeq \tau z$;*

(b) *either y or z is a maximal element of X ;*

(c) $\langle x\rangle = [x,y]$ *or* $\langle x\rangle = [x,z]$.

Proof. By Corollary 4.6, one has

$$y \prec z \qquad \text{or} \qquad z \preceq y\,. \tag{57}$$

In view of Decomposition (24),

$$y \prec z \Longrightarrow \tau y \preceq z \qquad \text{and} \qquad y \neq z \wedge z \preceq y \Longrightarrow \tau z \precsim y\,.$$

By combining this with Alternative (57) and with the hypothesis $\tau y \simeq \tau z$, we obtain Part (a).

Part (b) follows from Part (a) by combining Corollary 4.7 with Exercise 21.

Part (c) is an immediate corollary of Part (b). □

4.3.6 Connectedness

Let us say that elements x and y are *connected* if

$$\langle x\rangle \cap \langle y\rangle \neq \emptyset \tag{58}$$

or, equivalently, if there exists z such that

$$x \precsim v \qquad \text{and} \qquad y \precsim v\,. \tag{59}$$

Let us denote this by $x \curlywedge y$.

Exercise 23 *Show that*

$$\forall_{x,y\in X}\; x \curlywedge y \Longrightarrow \tau x \curlywedge \tau y\,.$$

Solution. If x, y, v satisfy (59), then

$$\tau x \precsim \tau v \qquad \text{and} \qquad \tau y \precsim \tau v$$

in view of Exercise 19. □

Lemma 4.8 *Connectedness is an equivalence relation.*

Proof. Reflexivity and Symmetry hold by definition. Suppose (59) and

$$y \precsim w \qquad \text{and} \qquad z \precsim w$$

hold. In view of $\langle y\rangle$ being linearly-preordered, either $v \precsim w$ or $w \precsim v$. If the former, then

$$x \precsim w \qquad \text{and} \qquad z \precsim w\,.$$

If the latter, then

$$x \precsim v \qquad \text{and} \qquad z \precsim v\,.$$

□

4.3.7 Connected components

We refer to the equivalence classes of the connectedness relation as the *connected components* of (X, τ).

Exercise 24 *Show that connected components are invariant subsets.*

Solution. Since $x \precsim \tau x$, one has $x \curlywedge \tau x$. □

4.3.8 $\pi_0(X, \tau)$

The family of connected components of (X, τ) is a subset of $\mathscr{P}X$ that is denoted $\pi_0(X, \tau)$. Different connected components are disjoint since they are equivalence classes of an equivalence relation. It follows that a twisted set (X, τ) decomposes into the union of its connected components

$$X = \bigcup_{X' \in \pi_0(X,\tau)} X' \tag{60}$$

and each component is an invariant subset of (X, τ).

4.3.9 Connected twisted sets

We say that a twisted set (X, τ) is *connected* if any two elements are connected. This is equivalent to X forming a single connected component.

4.3.10 Principal twisted sets

Connected twisted sets may have a very complex structure. Their simplest building blocks are orbits. We shall say that a twisted set (X, τ) is *principal* if there exists $o \in X$ such that $X = \langle o \rangle$. We shall refer to that element as an *origin* of X.

4.3.11

We know that an element $x \in X$ is maximal if and only if $\langle x \rangle = [x]$. In this case $\langle x \rangle = \langle y \rangle$ for any $y \in \langle x \rangle$, hence *every* element of $\langle x \rangle$ is an origin of $\langle x \rangle$.

4.3.12 Uniqueness of an origin of a principal twisted set

An origin is unique precisely when either $x \prec \tau x$, cf. Corollary 4.2, or $[x] = \{x\}$, and the latter occurs whens x is a fixed point of τ, i.e., when $x = \tau x$.

4.3.13 Terminology : *the* origin versus *an* origin

In view of the fact that an origin, in general, is not unique, from now one we will be distinguishing between two uses of that term. When we say say *an origin* of a principal twisted set $(\mathcal{O}, \sigma)$, it means we are talking about an *arbitrary* element of $o \in \mathcal{O}$ such that

$$\langle o \rangle = \mathcal{O} \,. \tag{61}$$

We shall be saying *the* origin of $(\mathcal{O}, \sigma)$ when that element $o \in \mathcal{O}$ is assumed to *have been* selected.

This important distinction reflects two uses of the term *principal twisted set*: as a generic term for

$$\text{a twisted set } (\mathcal{O}, \sigma) \text{ with an element satisfying } (61) \tag{62}$$

or as a term for

$$\text{a twisted set } (\mathcal{O}, \sigma) \text{ with a } \textbf{distinguished} \text{ element satisfying } (61)\ . \tag{63}$$

In the last case we may be also using notation $(\mathcal{O}, o, \sigma)$ that transforms a principal twisted set into a nullary-unary algebraic structure.

4.3.14 Orbits are well-preordered

Theorem 4.9 *Every orbit is well-preordered.*

Proof. Consider the subset of $\langle x \rangle$,

$$E := \{y \in \langle x \rangle \mid \forall_{A \subseteq \langle x \rangle}\ A \cap [x, y] \neq \emptyset \Rightarrow \min A \neq \emptyset\}\,. \tag{64}$$

Let A be a subset of $\langle x \rangle$.

Since $[x]$ is the set of minimal elements of $\langle x \rangle$, $A \cap [x] \neq \emptyset$ implies that $A \cap [x] = \min A$.

Exercise 25 *Show that, for every $A \subseteq \langle x \rangle$, one has*

$$A \cap [x] \neq \emptyset \ \Rightarrow\ A \cap [x] = \min A\,. \tag{65}$$

Suppose that $A \cap [x, \tau y] \neq \emptyset$. If

$$A \cap [x, y] \neq \emptyset \qquad \text{and} \qquad y \in E\,,$$

then $\min A \neq \emptyset$. If $A \cap [x, y] = \emptyset$, then

$$\forall_{z \in A}\ y \prec z$$

and, by Exercise 22, one has

$$\forall_{z \in A}\ \tau y \precsim z\,,$$

i.e., $A \subseteq \langle \tau y \rangle$. Then, by Exercise 25, one has

$$\emptyset \neq A \cap [x, \tau y] = A \cap [\tau y] = \min A\,.$$

□

Corollary 4.10 *Preordered set $(X, \precsim)$ is almost well-ordered.*

Proof. A subset $A \subseteq X$ is bounded below precisely when it is contained in $[a\rangle$ for some $a \in X$. According to Exercise 17, $[a\rangle = \langle a \rangle$, hence A is a bounded below subset of $(X, \precsim)$ if and only if it is a subset of a well-ordered subset $\langle a \rangle$, for some $a \in X$. □

Exercise 26 *Show that, for any $x, y \in X$, either $\langle x \rangle \cap \langle y \rangle = \emptyset$ or there exists $v \in X$, such that*

$$\langle x \rangle \cap \langle y \rangle = \langle v \rangle\,. \tag{66}$$

4.4 Classification of orbits

4.4.1

We establish some preliminary facts about the structure of an arbitrary orbit.

Lemma 4.11 *For any $x, y \in X$, and any $z \in [x, y)$, one has*

$$\forall_{x,y \in X} \forall_{v \in [x,y)} \; v \prec \tau v, \tag{67}$$

τ is injective on $v \in [x, y)$ and $x \notin \tau_* \langle x \rangle$.

Proof. Interval $[x, y)$ is nonempty precisely when $x \prec y$. No $v \in [x, y)$ is a maximal element of $(X, \precsim)$, hence, by Exercise 21, one has

$$v \neq \tau v$$

which, in view of $v \precsim \tau v$, means that $v \prec \tau v$.

Let $v, w \in [x, y)$ and $v \neq w$. Since $v \prec \tau v$ and $w \prec \tau w$, equivalence classes $[v]$ and $[w]$ are singleton sets. By Corollary 4.5, either $v \prec w$ or $w \prec v$. If the former, then

$$v \prec \tau v \precsim w \prec \tau w$$

which implies that $\tau v \neq \tau w$. In particular, $\tau v \neq \tau w$. If the latter, then

$$w \prec \tau w \precsim v \prec \tau v$$

which implies that $\tau v \neq \tau w$ as well.

Interval $[x, y)$ is nonempty if and only if $[x, y) \ni x$. This means that $x \prec \tau x$ and $x \notin \tau_* \langle x \rangle$.

$\square$

Lemma 4.12 *Suppose that $x \prec y$ and y is a maximal element. Then, there exists a unique $v \in X$ such that*

$$x \precsim v \prec \tau v \simeq y. \tag{68}$$

Proof. Under the hypothesis, one has

$$\langle x \rangle = [x, y) \cup [y]$$

where $[x, y) \neq \emptyset$. Since $x \in [x, y) \subsetneq \langle x \rangle$, interval $[x, y)$ is not an invariant subset, hence

$$\tau_*[x, y) \cap [y] \neq \emptyset.$$

Suppose there are $v, w \in [x, y)$ such that

$$v \neq w \qquad \text{and} \qquad \tau v, \tau w \in [y].$$

Since

$$\tau v \simeq y \simeq \tau w,$$

Lemma 4.7 asserts that either v or w are maximal which means that $\langle x \rangle \subseteq [x, y)$ contradicting the fact that $[x, y) \subsetneq \langle x \rangle$. This proves that $\tau_*[x, y) \cap [y]$ is a singleton set $\{z\}$ and that there exists a unique $v \in [x, y)$ such that $\tau v = z$. $\square$

4.4.2 Three orbital types

There are three types of orbits.

(i) $\langle x\rangle = [x]$.

(ii) $\langle x\rangle = [x, y]$ where $x \prec y$.

(iii) No $y \in \langle x\rangle$ is a maximal element of $(X, \precsim)$.

4.4.3 Type (i) orbits : $\langle x\rangle = [x]$

Type (i) orbits coincide, according to Lemma 3.4, with fixed points of the twisted set of orbits (X_τ, τ_*). They are the orbits of maximal elements of (X, τ).

Some examples:

$$
\begin{tikzcd} x \arrow[mapsto, loop right] \end{tikzcd} , \quad
\begin{tikzcd} x \arrow[r, mapsto, bend right] & \tau x \arrow[l, mapsto, bend right] \end{tikzcd} , \quad
\begin{tikzcd} & \bullet \arrow[dl, mapsto] \\ x \arrow[dr, mapsto] & \\ & \tau x \arrow[uu, mapsto] \end{tikzcd} , \quad
\begin{tikzcd} & \bullet \arrow[dl, mapsto] & \\ x \arrow[dr, mapsto] & & \bullet \arrow[ul, mapsto] \\ & \tau x \arrow[ur, mapsto] & \end{tikzcd} , \quad
\begin{tikzcd} & \bullet \arrow[dl, mapsto] & \bullet \arrow[l, mapsto] \\ x \arrow[dr, mapsto] & & \\ & \tau x \arrow[r, mapsto] & \bullet \arrow[uu, mapsto] \end{tikzcd} , \quad \ldots \tag{69}
$$

4.4.4 Type (ii) orbits : $\langle x\rangle = [x, y]$ where $x \prec \tau x$

Type (ii) orbits are the orbits of nonmaximal elements that contain a maximal element.

Some examples:

$$
\begin{tikzcd} x \arrow[r, mapsto] & y \arrow[mapsto, loop right] \end{tikzcd} , \quad
\begin{tikzcd} x \arrow[r, mapsto] & \tau x \arrow[r, mapsto, bend right] & y \arrow[l, mapsto, bend right] \end{tikzcd} , \quad
\begin{tikzcd} & & \bullet \arrow[dl, mapsto] \\ x \arrow[r, mapsto] & \tau x \arrow[dr, mapsto] & \\ & & y \arrow[uu, mapsto] \end{tikzcd} , \quad
\begin{tikzcd} & & \bullet \arrow[dl, mapsto] & \\ x \arrow[r, mapsto] & \tau x \arrow[dr, mapsto] & & \bullet \arrow[ul, mapsto] \\ & & y \arrow[ur, mapsto] & \end{tikzcd} , \quad \ldots \tag{70}
$$

and

$$
\begin{tikzcd} x \arrow[r, mapsto] & \tau x \arrow[r, mapsto] & y \arrow[mapsto, loop right] \end{tikzcd} , \quad
\begin{tikzcd} x \arrow[r, mapsto] & \tau x \arrow[r, mapsto] & y \arrow[r, mapsto, bend right] & \tau y \arrow[l, mapsto, bend right] \end{tikzcd} , \quad
\begin{tikzcd} & & & \bullet \arrow[dl, mapsto] \\ x \arrow[r, mapsto] & \tau x \arrow[r, mapsto] & y \arrow[dr, mapsto] & \\ & & & \tau y \arrow[uu, mapsto] \end{tikzcd} , \quad \ldots \tag{71}
$$

and so on, with the initial interval terminating in an orbit of type (i).

One could think of an orbit of type (i) as the *degenerate* case of an orbit of type (ii).

4.4.5 Type (iii) orbits : orbits without maximal elements

Type (iii) orbits are all the remaining orbits, i.e., orbits without maximal elements. One could think of an orbit of type (iii) as the *limit* case of an orbit of type (ii), when the initial interval never reaches the terminal orbit of type (i).

$$
x \longmapsto \tau x \longmapsto \tau\tau x \longmapsto \tau\tau\tau x \longmapsto \ldots \tag{72}
$$

4.4.6 Behavior of τ on an orbit of type (i)

We begin from the following simple observation.

Exercise 27 *Operation τ is surjective on an invariant subset $A \subseteq X$ if and only if $A \in \mathscr{P}X$ is a fixed point of τ.*

In particular, τ is surjective on orbits of type (i) and is *not* surjective on orbits of other types, since orbits of type (i) are precisely the fixed points of τ_* on the set of orbits X_τ.

4.4.7 Behavior of τ on an orbit of type (ii)

On orbits of type (ii) operation τ is not surjective. It is not injective either.

Lemma 4.13 *If $x \prec y$ for some maximal element y, then*

(a) *there exists $v \in X$ such that $x \precsim v \prec \tau v$ and τv is a maximal element; in particular $\langle x \rangle = [x, \tau v]$;*

(b) *there exists $w \neq v$ such that $\tau w = \tau v$; such w is necessarily equivaelent to $\langle \tau v \rangle$.*

Proof. Lemma 4.12 yields Part (a). Since $\langle \tau x \rangle$ is an orbit of type (i), operation τ_* is surjective on it, hence there exists $w \simeq \tau v$ such that $\tau w = \tau v$. Since $v \prec \tau v$, one has $v \neq w$. In view of Part (b) of Corollary 4.7, any element $w \in \langle x \rangle$ such that $w \neq v$ and $\tau w = \tau v$ is maximal, hence is equivalent to τv. □

By Lemma 4.13, an orbit of type (ii) that, by definition, has the form $[x, y]$, has *two* distinguished elements, the *origin*, x, and the element v such that $v \prec \tau v$ and τv is a maximal element of (X, τ). In orbit drawings (70)–(71), $\tau v = y$. One needs to remember, however, that

$$\forall_{y \in X}\ \tau v \precsim y \Rightarrow \langle x \rangle = [x, \tau v] = [x, y]\,. \tag{73}$$

4.4.8 Behavior of τ on an orbit of type (iii)

We shall say that $\langle x \rangle$ is an *infinite orbit* if

$$\forall_{y \in \langle x \rangle}\ y \prec \tau y\,. \tag{74}$$

4.4.9 Characterization of orbits of type (iii)

Lemma 4.14 *For any orbit $\langle x \rangle \in X_\tau$, the following conditions are equivalent.*

(a) *$\langle x \rangle$ is an infinite orbit.*

(b) *$\langle x \rangle$ is an orbit of type* (iii).

(c) *$\tau : \langle x \rangle \longrightarrow \langle x \rangle$ is injective but not surjective.*

In particular, an orbit of type (iii) is an *infinite set* (recall that infinite sets are defined as sets admitting a left-invertible but not right-invertible unary operation).

Proof. Implication (a)⇒(c) is a corollary of Lemma 4.11.

(b)⇒(a) By Exercise 21, for every $y \in \langle x \rangle$, one has $y \neq \tau y$. Since $y \precsim \tau y$, this means that $y \prec \tau y$.

(c)⇒(b) Injectivity of τ precludes type (ii), lack of surjectivity precludes type (i). □

5 Homomorphisms

5.1 Preliminaries

5.1.1 A homomorphic image of an invariant subset

Let

$$f : (W, \sigma) \longrightarrow (X, \tau) \tag{75}$$

be a homomorphism of twisted sets.

Exercise 28 *Show that the image f_*E of an invariant subset $E \subseteq W$ is an invariant subset of X.*

Solution. One has

$$\tau_*(f_*E) = (\tau_* \circ f_*)(E) = (\tau \circ f)_*E = (f \circ \sigma)_*(E) = (f_* \circ \sigma_*)(E) = f_*(\sigma_*E) \subseteq f_*(E) .$$

□

5.1.2

If we represent a twisted set (W, σ) as the union of its connected components

$$W = \bigcup_{V \in \pi_0(W,\sigma)} V \tag{76}$$

and recall that connected components are disjoint, then a homomorphism

$$f : (W, \sigma) \longrightarrow (X, \tau)$$

is completely described by the family of its restrictions to connected components

$$\left(f_{|V}\right)_{V \in \pi_0(W,\sigma)}$$

and the assignment

$$f \longmapsto \left(f_{|V}\right)_{V \in \pi_0(W,\sigma)}$$

defines a canonical bijective correspondence

$$\mathrm{Hom}\big((W,\sigma),(X,\tau)\big) \longleftrightarrow \prod_{V \in \pi_0(W,\sigma)} \mathrm{Hom}\big((V,\sigma),(X,\tau)\big) . \tag{77}$$

5.2 A *quotient set* of X : the dual concept to the concept of a *subset* of X

5.2.1

This subchapter reviews a material as basic as it is indispensable, a material that belongs to every 'Primer of basic concepts of Mathematics'. I did not include it in the Notes on *Functions, Operations, Relations* only for one reason: in order to grasp the essence of the concept of a quotient set one must be already familiar with the concepts of a category and of an *initial object*.

5.2.2 A quotient function $p : X \twoheadrightarrow Q$

Given an equivalence relation $\sim$ on a set X, *a quotient function* $p : X \twoheadrightarrow Q$ is defined to be a function that satisfies the following *universal property*,

$$\begin{gathered}\textit{for every function } f : X \to X' \textit{ such that} \\ \forall_{x,y\in X}\ x{\sim}y \Rightarrow f(x){=}f(y) \qquad (79) \\ \textit{there exists a unique function } \bar{f} : Q \to X' \textit{ such that} \\ f = \bar{f} \circ \pi\,.\end{gathered} \tag{79}$$

5.2.3

By definition, a quotient function is an *initial object* in the category whose objects are functions from a fixed set X to any set, and satisfying condition (79). A function $\varphi : X' \to X''$ such that the diagram

$$\begin{tikzcd} X' \arrow[rr, "\varphi"] & & X'' \\ & X \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd} \tag{80}$$

commutes, is a morphism from f to g.

5.2.4

Since any two initial objects in a unital category are isomorphic by a unique isomorphism, the choice of an actual model for a quotient function is not important.

5.2.5 The *fiber equivalence* relation $=_f$ of a function $f : X \longrightarrow X'$

Given a function $f : X \longrightarrow X'$ between arbitrary sets, the *fiber equivalence relation* of f is defined as follows:

$$x =_f y := \text{“ } f(x) = f(y) \text{ ”}. \tag{81}$$

A surjective function $p : X \twoheadrightarrow Q$ is a quotient function for an equivalence relation $\sim$ precisely when $\sim$ is equipotent with the *fiber equivalence relation* $=_p$ of p,

5.2.6 *Coimage* of a function $f : X \longrightarrow X'$

A quotient of X by the fiber equivalence relation of f is called a *coimage* of f.

5.2.7 The standard construction of a quotient set $X_{/\sim}$

A concrete model for the target of a quotient function is provided by the set of equivalence classes of $\sim$. We denote this family of subsets of X by $X_{/\sim}$. By definition, it is a subset of $\mathscr{P}X$.

5.2.8 The canonical quotient function $\pi : X \twoheadrightarrow X_{/\sim}$

The function that assigns to an element $x \in X$ the corresponding equivalence class

$$\pi : x \longmapsto \bar{x} \qquad (x \in X) \tag{82}$$

is called *the canonical quotient function*. Note that by design, the canonical quotient function is surjective, hence any quotient function is surjective.

Proposition 5.1 (The canonical image-coimage factorization) *Every function $f : X \longrightarrow X'$ admits a canonical factorization*

$$f = i_f \circ \bar{f} \circ p_f \tag{83}$$

where

$$p_f : X \twoheadrightarrow X_{/=_f}, \qquad x \longmapsto \bar{x}, \tag{84}$$

is the canonical coimage-of-f surjection, where

$$i_f : f_* X \hookrightarrow X', \qquad x' \longmapsto x', \tag{85}$$

is the canonical image-of-f inclusion, and

$$\bar{f} : X_{/\sim_f} \xrightarrow{\simeq} f_* X \tag{86}$$

is a bijection.

Proof. The diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "p_f"'] & X' \\ X_{/=_f} & f_* X \arrow[u, hook, "i_f"'] \end{tikzcd}$$

admits a unique completion to the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "p_f"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \\ X_{/=_f} \arrow[r, "\bar{f}"'] & f_* X \arrow[u, hook, "i_f"'] \end{tikzcd} .$$

Uniqueness of arrow $\bar{f}$ follows from the following simple general observation.

Exercise 29 *Let $g : X \twoheadrightarrow Y$ be a surjective, and $g' : Y' \hookrightarrow X'$ be an injective, function. Show that functions $h, k : Y \longrightarrow Y'$ are equal if and only if*

$$g' \circ h \circ g = g' \circ k \circ g .$$

Existence of $\bar{f}$ follows from the universal properties of the canonical coimage-of-f surjection, cf. (84), and of the canonical image-of-f inclusion, cf. (85).

If we apply first the universal property of i_f, we obtain a diagonal function $X \twoheadrightarrow f_*X$ such that the right top triangle in Diagram (87),

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "p_f"'] \arrow[dr] & X' \\ X_{/=_f} \arrow[r, "h"'] & f_*X \arrow[u, hook, "i_f"'] \end{tikzcd} , \tag{87}$$

commutes. Next, we apply the universal property of p_f to obtain the blue arrow $h : Y \longrightarrow Y'$ in the same diagram.

If we proceed in reverse order, the universal property of p_f yields a diagonal function $X_{/=_f} \longrightarrow X'$ that the left top triangle in the following diagram,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "p_f"'] & X' \\ X_{/=_f} \arrow[r, "k"'] \arrow[ur] & f_*X \arrow[u, hook, "i_f"'] \end{tikzcd} , . \tag{88}$$

commutes. Next, we apply the universal property of i_f to obtain the blue arrow $h : Y \longrightarrow Y'$ in the same diagram.

Equality $h = k$ follows from Exercise 29. □

5.3 Quotients of a twisted set

5.3.1 Congruences on a twisted set

An equivalence relation $\sim$ on X is said to be a *congruence relation* for a unary operation τ if it satisfies the condition

$$\forall_{x,y \in X}\; x \sim y \Rightarrow \tau x \sim \tau y . \tag{89}$$

5.3.2

An example of a congruence relation is the *connectedness* relation introduced in Section 4.3.6, cf. Exercise 23.

5.3.3 The induced operation on the quotient by a congruence relation

Lemma 5.2 *Given an equivalence relation $\sim$ and a unary operation τ on X, the diagram*

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, two heads, "p"'] & X \arrow[d, two heads, "p"] \\ Q & Q \end{tikzcd}$$

where p is a quotient-by-$\sim$ function, can be completed to a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, two heads, "p"] \\ Q \arrow[r, "\bar{\tau}"'] & Q \end{tikzcd} \tag{90}$$

if and only if relation $\sim$ is a congruence. A function $\bar{\tau}$ that makes Diagram 90 commute is unique.

Proof. Uniqueness of $\bar{\tau}$ is an immediate consequence of a quotient function p being surjective.

If $x \sim y$ and $\sim$ is a congruence for operation τ, then $\tau x \sim \tau y$, hence $q(\tau x) \sim q(\tau y)$. Thus, the composite function $q \circ \tau$ satisfies Condition (79). By the universal property of function p, there exists a unique function $\bar{\tau} : Q \to Q$ such that $\bar{\tau} \circ q = q \circ \tau$.

Vice-versa, suppose that there is an operation $\bar{\tau}$ making Diagram (90) commute. This happens precisely when

$$\forall_{x,y \in X}\ x {\sim_p} y \Rightarrow \tau x {\sim_p} \tau y\,. \tag{91}$$

Since $\sim_p$ is equipotent with $\sim$, Condition (91) holds precisely when the following condition holds

$$\forall_{x,y \in X}\ x {\sim} y \Rightarrow \tau x {\sim} \tau y\,,$$

i.e., when $\sim$ is a congruence for τ. $\square$

5.3.4 The quotient twisted set $(Q, \bar{\tau})$

The twisted set $(Q, \bar{\tau})$ is referred to as a *quotient of (X, τ) by congruence $\sim$*.

Exercise 30 *Show that the set*

$$p_* \circ \tau_* \circ p^*\{q\} = \{r \in Q \mid \exists_{x \in X}\ p(x) = q \wedge p(\tau x) = r\}$$

is, for every $q \in Q$, a singleton set if and only if relation $\sim$ is a congruence for τ.

Solution. Since subsets $p^*\{q\}$ coincide with equivalence classes of the fiber equivalence of p, set $p_* \circ \tau_* \circ p^*\{q\}$ is a singleton set precisely when the image under τ of any fiber equivalence class of p is contained in a single equivalence class of that fiber equivalence. In othere words, if Condition (91) is satisfied.

As was pointed out above, this is equivalent to τ being a congruence for operation τ.

5.3.5

The value of $\bar{\tau}$ on an element $q \in Q$ is the unique element of the singleton set $p_* \circ \tau_* \circ p^*\{q\}$.

Exercise 31 *Show that p is a homomorphism $(X, \tau) \longrightarrow (Q, \bar{\tau})$ when $\sim$ is a congruence.*

Exercise 32 *Let $f : (X, \tau) \longrightarrow (X', \tau')$ be a homomorphism. Show that the induced bijection $\bar{f}$ present in the coimage-image Factorization of f, cf. Proposition 5.1 is an isomorphism.*

5.4 Homomorphisms from a principal twisted set

5.4.1 A homomorphic image of a principal twisted set

Consider an orbit $\langle w\rangle$ of an element $w \in W$ in a twisted set (W, σ).

Exercise 33 *Show that the image $f_*\langle w\rangle$ is the orbit of $f(w)$ in (X, τ).*

Solution. The proof is very similar to the proof of Identity (25).
Since $f(w) \in f_*\langle w\rangle$ and $f_*\langle w\rangle$ is an invariant subset of (X, τ), by Exercise 28, we have

$$\langle f(w)\rangle \subseteq f_*\langle w\rangle .$$

Consider the subset of (w),

$$E := \{v \in \langle w\rangle \mid f(v) \in \langle f(w)\rangle\} .$$

By definition $w \in E$. If $v \in E$, then

$$f(\sigma v) = \tau(f(v)) \in \langle f(w)\rangle$$

since $\langle f(w)\rangle$ is an invariant subset of (X, τ). It follows that E is an invariant subset of $\langle w\rangle$ that contains w, hence $E = \langle w\rangle$ and therefore

$$f_*\langle w\rangle \subseteq \langle f(w)\rangle .$$

□

5.4.2 Hom$((\mathcal{O}, o, \sigma), (\mathcal{O}', o', \sigma'))$

Consider two principal twisted sets $(\mathcal{O}, \sigma)$ and $(\mathcal{O}', \sigma')$ with origins $o \in \mathcal{O}$ and $o' \in \mathcal{O}'$, respectively.

Lemma 5.3 (a) *Any homomorphism $f : (\mathcal{O}, o, \sigma) \longrightarrow (\mathcal{O}', o', \sigma')$ is an epimorphism and $(\mathcal{O}', o', \sigma')$ is isomorphic to a quotient of principal twisted set $(\mathcal{O}, o, \sigma)$.*

(b) *Any monomorphism $f : (\mathcal{O}, o, \sigma) \longrightarrow (\mathcal{O}', o', \sigma')$ is an isomorphism.*

(c) *Any two homomorphisms $(\mathcal{O}, o, \sigma) \longrightarrow (\mathcal{O}', o', \sigma')$ are equal.*

Proof. By Exercise 33, the image of a homomorphism $f : (\mathcal{O}, o, \sigma) \longrightarrow (\mathcal{O}', o', \sigma')$ is the orbit of $o' = f(o) = \mathcal{O}'$. Thus, $(\mathcal{O}', o', \sigma')$ is isomorphic to a quotient of $(\mathcal{O}, o, \sigma)$ by the fiber equivalence $=_f$ associated with homomorphism f.

Part (b) is an immediate corollary of Part (i).

Given two homomorphisms f and g, let

$$E := \{u \in \mathcal{O} \mid f(u) = g(u)\} .$$

One has

$$f(o) = o' = g(o) ,$$

hence $o \in E$. If $u \in E$, then

$$f(\sigma u) = \sigma'(f(u)) = \sigma'(g(u)) = g(\sigma u) ,$$

i.e., E is an invariant subset of $\mathcal{O} = \langle o\rangle$ that contains o. Hence $E = \mathcal{O}$. □

5.4.3 The structure of a homomorphism $f:(\mathcal{O},o,\sigma)\longrightarrow(X,x,\tau)$

Consider the image-coimage factorization of an arbitrary homomorphism $f:(\mathcal{O},o,\sigma)\longrightarrow(X,x,\tau)$,

$$\begin{tikzcd}
(\mathcal{O},\sigma) \arrow[r, "f"] \arrow[d, two heads, "p_f"'] \arrow[dr, phantom, "\circlearrowleft"] & (X,\tau) \\
\left(\mathcal{O}_{/=_f},\bar{\sigma}\right) \arrow[r, "\bar{f}"'] & (f_*\mathcal{O},\tau) \arrow[u, hook, "i_f"']
\end{tikzcd} \quad . \tag{92}$$

By Exercise 33, one has

$$f_*\mathcal{O} = \langle f(o)\rangle = \langle x\rangle .$$

In particular, canonical inclusion i_f depends only on x, not on f. It follows that f exists if and only if the orbit of x is a quotient of $(\mathcal{O},\sigma)$ and, by Lemma 5.3, such a homomorphism $(\mathcal{O},o,\sigma)\longrightarrow(X,x,\tau)$ is unique when it exists. We established the following fact.

Corollary 5.4 *The evaluation-at-o homomorphism, cf. Section 2.2.5,*

$$\mathrm{ev}_o:\big(\mathrm{Hom}\big((\mathcal{O},\sigma),(X,\tau)\big),\tau_*\big)\longrightarrow(X,\tau), \qquad f\longmapsto f(o), \tag{93}$$

is a monomorphism of twisted sets that identifies the twisted set of homomorphisms from $(\mathcal{O},\sigma)$ to (X,τ) with a twisted subset of X.

The image of the evaluation-at-o function consists of those $x\in X$ whose orbits are isomorphic to the quotient of $(\mathcal{O},\sigma)$ by a certain congruence equivalence relation.

□

5.4.4

The above corollary reduces the question of classifying arbitrary homomorphisms from a principal twisted set $(\mathcal{O},\sigma)$ to two separate tasks:

(a) the classification of congruence equivalence relations on $(\mathcal{O},\sigma)$;

(b) the determination of orbital types of elements of (X,τ).

5.5 The monoid of endomorphisms of a principal twisted set $(\mathcal{O},\sigma)$

5.5.1

By applying Lemma 5.4 to the special case $(X,\tau)=(\mathcal{O},\sigma)$, we obtain a complete description of endomorphisms of $(\mathcal{O},\sigma)$.

Corollary 5.5 *Evaluation-at-the-origin defines a canonical isomorphism of twisted sets*

$$\big(\mathrm{End}(\mathcal{O},\sigma),\sigma_*\big)\simeq(\mathcal{O},\sigma), \qquad f\longleftrightarrow f(o). \tag{94}$$

□

5.5.2 The family of endomorphisms $(\sigma_{ou})_{u\in\mathcal{O}}$ of a principal twisted set $(\mathcal{O},\sigma)$

We shall denote by σ_{ou} the unique endomorphism of $(\mathcal{O},\sigma)$ that sends the origin of the twisted set to element $u\in\mathcal{O}$. This σ_{ou} is a unique homomorphism $(\mathcal{O},o,\sigma)\longrightarrow(\mathcal{O},u,\sigma)$.

5.5.3 $\sigma_{ou}\circ\sigma_{ov}=\sigma_{ov}\circ\sigma_{ou}$

Consider the subset of $\mathcal{O}$

$$E\coloneqq\{u\in\mathcal{O}\mid\forall_{v\in\mathcal{O}}\ \sigma_{\mathrm{O}u}\circ\sigma_{\mathrm{O}v}=\sigma_{\mathrm{O}v}\circ\sigma_{\mathrm{O}u}\}.$$

Since $\sigma_{oo}=\mathrm{id}_{\mathcal{O}}$, the origin of $\mathcal{O}$ belongs to E. If $u\in E$, then, for every $v\in\mathcal{O}$, straightforward calculation

$$\begin{aligned}\sigma_{o,\sigma u}\circ\sigma_{ov}&=\sigma_{\bullet}\sigma_{ou}\circ\sigma_{ov}\\&=(\sigma\circ\sigma_{ou})\circ\sigma_{ov}\\&=\sigma\circ(\sigma_{ou}\circ\sigma_{ov})\\&=\sigma\circ(\sigma_{ov}\circ\sigma_{ou})\\&=(\sigma\circ\sigma_{ov})\circ\sigma_{ou}\\&=(\sigma_{ov}\circ\sigma)\circ\sigma_{ou}\\&=\sigma_{ov}\circ(\sigma\circ\sigma_{ou})\\&=\sigma_{ov}\circ\sigma_{o,\sigma u}\end{aligned}$$

demonstrates that $\sigma u\in E$. Thus, E is an invariant subset of $\langle o\rangle$ that contains o, hence $E=\langle o\rangle=\mathcal{O}$.

Exercise 34 *Copy the above calculation and, for each equality, provide a correct reason, e.g., including a reference to a previously proven fact.*

We established commutativity of the monoid of endomorphisms of $(\mathcal{O},\sigma)$.

Lemma 5.6 *The monoid of endomorphisms of a principal twisted set is commutative.*

□

5.5.4 The canonical commutative monoid structure on a principal twisted set $(\mathcal{O},o,\sigma)$

By translating the monoid structure from $\mathrm{End}(\mathcal{O},\sigma)$ to E by means of the canonical bijective correspondence (94), we obtain a canonical commutative monoid structure $(\mathcal{O},o,+)$:.

$$\mathrm{id}_{\mathcal{O}}\longleftrightarrow o\qquad\text{and}\qquad\sigma_{ou}\circ\sigma_{ov}\longleftrightarrow u+v. \tag{95}$$

5.5.5 The meaning of $u + v$

By definition, $u + v$ is the element of $\mathcal{O}$ obtained by applying σ_{ov} to o and then applying σ_{ou} :

$$o \xmapsto{\sigma_{ov}} v \xmapsto{\sigma_{ou}} u + v \quad .$$

Commutativity of operation $+$,

$$u + v = v + u \qquad (u, v \in \mathcal{O})$$

is a translation of the identity

$$\sigma_{ou} \circ \sigma_{ov} = \sigma_{ov} \circ \sigma_{ou} \qquad (u, v \in \mathcal{O})\,.$$

5.5.6

In view of Exercise 33, one has

$$(\sigma_{ou})_* \mathcal{O} = \langle u \rangle \qquad (u \in \mathcal{O})$$

which demonstrates that σ_{ou} is an epimorphism if and only if u is an origin of $\mathcal{O}$.

If the origin $o \in \mathcal{O}$ is not maximal, then

$$o \prec \sigma o \precsim u$$

for every element $u \neq o$, hence $\mathrm{id}_{\mathcal{O}}$ is the only epimorphism. This establishes the 'if' part of the following lemma.

Lemma 5.7 *The origin $o \in \mathcal{O}$ is not a maximal element if and only if $\mathrm{id}_{\mathcal{O}}$ is the only surjective endomorphism.*

5.5.7

The 'only if' part will be proved simultaneously with describing what happens when the origin is a maximal element.

Lemma 5.8 *The origin $o \in \mathcal{O}$ is a maximal element if and only if every endomorphism of $(\mathcal{O}, \sigma)$ is an automorphism.*

Proof. If the origin is a maximal element, then $\mathcal{O} = [o]$ and any element of $\mathcal{O}$ is an origin. Given two elements $u, v \in \mathcal{O}$, endomorphism σ_{uv} is the unique endomorphism of $(\mathcal{O}, \sigma)$ that sends u to v and, similarly, σ_{vu} is the unique endomorphism that sends v to u. The composite endomorphisms $\sigma_{uv} \circ \sigma_{vu}$ and $\sigma_{vu} \circ \sigma_{uv}$ are, respectively, the unique endomorphisms that send u to u and, respectively, v to v. Since $\mathrm{id}_{\mathcal{O}}$ has these properties, one has

$$\sigma_{uv} \circ \sigma_{vu} = \mathrm{id}_{\mathcal{O}} = \sigma_{vu} \circ \sigma_{uv}\,. \tag{96}$$

The 'only if' part of Lemma 5.7 follows from the 'if' part of Lemma 5.8 and vice-versa, the 'only if' part of Lemma 5.8 follows from the 'if' part of Lemma 5.7. This completes the proof of both lemmata.

5.5.8

Note how elegant is the above criss-cross argument that proves simultaneously two lemmata by using in the proof of one part of one lemma, the other part of the other lemma, and vice-versa.

5.5.9

The following corollary of Lemmata 5.7 and 5.8 completes our investigation, begun in Sections 4.4.7 and 4.4.8, of the behavior of the twist operation on orbits of arbitrary elements in twisted sets.

Corollary 5.9 *The origin $o \in \mathcal{O}$ is a maximal element if and only if σ is bijective.*

Proof. Indeed, σ is an endomorphism of $(\mathcal{O}, \sigma)$ and every endomorphism is bijective according to Lemma 5.8.

Vice-versa, let o be an origin of $\mathcal{O}$. If o is not maximal, then $o \prec \sigma o$ and therefore $\sigma = \sigma_{o,\sigma o}$ is not surjective. ☐

5.5.10 G-torsors

If a group G acts on a set X so that, for every $x, y \in X$, there exists a *unique* element of $g \in G$ that sends x to y, then we say that X is a G*-torsor*.

5.5.11 The canonical pairing $X, X \longrightarrow G$ associated with a G-torsor

The function

$$X, X \longrightarrow G, \qquad x, y \longmapsto \text{the unique } g \in G \text{ such that } gx = y\,, \tag{97}$$

is canonically associated with a G-torsor X. Distinguishing any element $o \in X$ identifies X with G,

$$X \longrightarrow G, \qquad x \longmapsto \text{the unique } g \in G \text{ such that } go = x\,.$$

In the following proposition we reinterpret several facts that we established above.

Proposition 5.10 (a) *When $(\mathcal{O}, \sigma)$ has a unique origin, its group of automorphisms* $\mathrm{Aut}(\mathcal{O}, \sigma)$ *is the trivial group* $\{\mathrm{id}_{\mathcal{O}}\}$ *and $\mathcal{O}$ is canonically identified with its own endomorphism monoid* $\mathrm{End}(\mathcal{O}, \sigma)$.

(b) *When $(\mathcal{O}, \sigma)$ has more than one origin, its group of automorphisms* $G = \mathrm{Aut}(\mathcal{O}, \sigma)$ *coincides with* $\mathrm{End}(\mathcal{O}, \sigma)$ *and $\mathcal{O}$ is a G-torsor. The associated pairing has the form*

$$\mathcal{O}, \mathcal{O} \longrightarrow G, \qquad u, v \longmapsto \sigma_{uv}\,. \tag{98}$$

In particular, every element $u \in \mathcal{O}$ defines a bijection

$$\mathcal{O} \longleftrightarrow G, \qquad v \longmapsto \sigma_{uv}\,. \tag{99}$$

☐

5.6 The twisted set of endomorphisms of a commutative semigroup $(C,\cdot)$

5.6.1 The twisted set of unary operations $(\mathrm{Op}_1 B,\tau)$ of a binary structure $(B,\cdot)$

The set of unary operations on a binary algebraic structure $(B,\cdot)$ is naturally equipped with a twist

$$\tau : \mathrm{Op}_1 B \longrightarrow \mathrm{Op}_1 B\,, \qquad f \longmapsto \tau f\,, \tag{100}$$

where

$$\tau f : B \longrightarrow B\,, \qquad b \longmapsto bf(b)\,. \tag{101}$$

5.6.2

Consider the subset of $(\mathrm{Op}_1 B,\tau)$ formed by endomorphisms of $(B,\cdot)$,

$$\mathrm{End}(B,\cdot) \subset \mathrm{Op}_1 B\,. \tag{102}$$

Exercise 35 *Show that* $\mathrm{End}(B,\cdot)$ *is an invariant subset of twisted set* (100) *if multiplication* $\cdot$ *is associative and commutative.*

5.6.3 The evaluation-at-g homomorphism $(\mathrm{End}_{\mathrm{Sgr}} C,\tau) \longrightarrow (C,\lambda_g)$

To avoid necessity of using multiply nested pairs of parentheses, we shall denote the monoid of endomorphisms of $(C,\cdot)$ by $\mathrm{End}_{\mathrm{Sgr}} C$.

Let g be an element of a commutative semigroup $(C,\cdot)$.

Exercise 36 *Show that*

$$\mathrm{ev}_g : \mathrm{End}_{\mathrm{Sgr}} C \longrightarrow C\,, \qquad \alpha \longmapsto \alpha(g)\,, \tag{103}$$

is a homomorphism $(\mathrm{End}_{\mathrm{Sgr}} C,\tau) \longrightarrow (C,\lambda_g)$ *of twisted sets.*

Solution. Verification of this statement is by straightforward calculation

$$\mathrm{ev}_g(\tau\alpha) = (\tau\alpha)(g) = g\cdot\alpha(g) = g\cdot\mathrm{ev}_g(\alpha) = \lambda_g(\mathrm{ev}_g\alpha)\,.$$

$\square$

Lemma 5.11 *If* (C,λ_g) *is a principal twisted set with origin* $g \in C$, *then* $(\mathrm{End}_{\mathrm{Sgr}} C,\tau)$ *is a principal twisted set with origin* id_C, *and the evaluation-at-*g *homomorphism* (103) *is an isomorphism of principal twisted sets.*

Proof. The image of the evaluation homomorphism, (103), is an invariant subset of (C,λ_g) and it contains

$$g = \mathrm{ev}_g(\mathrm{id}_C)\,,$$

hence it coincides with the orbit of g,

$$\langle g\rangle = C\,.$$

This proves surjectivity of homomorphism (103).

Suppose $\alpha,\beta \in \mathrm{End}_{\mathrm{Sgr}} C$ and $\mathrm{ev}_g \alpha = \mathrm{ev}_g \beta$. Consider the set

$$E := \{c \in C \mid \alpha(c) = \beta(c)\}.$$

If $c \in E$, then

$$\alpha\left(\lambda_g c\right) = \alpha(gc) = \alpha(g) \cdot \alpha(c) = \beta(g) \cdot \beta(c) = \beta(gc) = \beta\left(\lambda_g c\right),$$

i.e., $\lambda_g c \in E$. Hence E is an invariant subset of principal twisted C that contains its origin g. It follows that $E = C$ and homomorphism (103) is injective. ☐

5.6.4 The commutative semiring structure on C when (C,λ_g) is a principal twisted set

The product $\alpha \cdot \beta$ of two endomorphisms of any commutative semigroup is an endomorphism and the operation of composition $\circ$ distributes on the left and on the right over the operation of multiplication $\cdot$.

In particular, $(\mathrm{End}_{\mathrm{Sgr}} C, \mathrm{id}_C, \cdot, \circ)$ is a semiring with identity for every commutative semigroup $(C, \circ)$.

Corollary 5.12 *If (C,λ_g) is a principal twisted set with origin $g \in C$, then*

$$(\mathrm{End}_{\mathrm{Sgr}} C, \mathrm{id}_C, \cdot, \circ) \tag{104}$$

is a commutative semiring with identity.

Proof. Let us consider the following subset of $\mathrm{End}\, C := \mathrm{End}_{\mathrm{Sgr}} C$,

$$E := \{\alpha \in \mathrm{End}\, C \mid \forall_{\beta \in \mathrm{End}\, C}\ \alpha \circ \beta = \beta \circ \alpha\}.$$

Since the identity endomorphism id_C commutes with every unary operation on set C, one has $\mathrm{id}_C \in E$.

If $\alpha \in E$, then, for every $c \in C$, straightforward calculation

$$\begin{aligned}
(\tau\alpha \circ \beta)(c) &= (\tau\alpha)\big(\beta(c)\big)\\
&= \beta(c) \cdot (\alpha \circ \beta)(c)\\
&= \beta(c) \cdot (\beta \circ \alpha)(c)\\
&= \beta(c) \cdot \beta(\alpha(c))\\
&= \beta\big(c \cdot \alpha(c)\big)\\
&= \beta\big(\tau\alpha(c)\big)\\
&= (\beta \circ \tau\alpha)(c)
\end{aligned}$$

demonstrates that $\tau\alpha \in E$. Hence, E is an invariant subset of $\langle \mathrm{id}_C \rangle$ that contains id_C and therefore coincides with $\langle \mathrm{id}_C \rangle$ and the latter equals $\mathrm{End}_{\mathrm{Sgr}} C$ according to Lemma 5.11, ☐

5.6.5 The twisted set of endomorphisms of a commutative monoid $(C, e, \cdot)$

Given a monoid $(C, e, \cdot)$, let us consider the subset of the set of endomorphisms of a semigroup $(C, \cdot)$,

$$\mathrm{End}_{\mathrm{Mon}} C := \mathrm{End}(C, e, \cdot) \subset \mathrm{End}(C, \cdot)\,, \tag{105}$$

formed by endomorphisms that preserve the identity element $e \in C$.

Exercise 37 *Show that* $\mathrm{End}_{\mathrm{Mon}} C$ *is an invariant subset of twisted set* $\left(\mathrm{End}_{\mathrm{Sgr}} C, \tau\right)$.

Solution. If $\alpha(e) = e$, then $\tau\alpha(e) = e \cdot \alpha(e) = e \cdot e = e$. □

5.6.6

Let $\varepsilon : C \longrightarrow C$, be the constant function

$$\varepsilon(c) = e \qquad (c \in C)\,. \tag{106}$$

We have the following variant of Lemma 5.11

Lemma 5.13 *If* $\left(C, \lambda_g\right)$ *is a principal twisted set with origin* $e \in C$, *then* $\left(\mathrm{End}_{\mathrm{Mon}} C, \tau\right)$ *is a principal twisted set with origin* ε, *and the evaluation-at-*g *homomorphism*

$$\mathrm{ev}_g : \mathrm{End}_{\mathrm{Mon}} C \longrightarrow C\,, \qquad \alpha \longmapsto \alpha(g)\,, \tag{107}$$

is an isomorphism of principal twisted sets.

Proof. The proof is nearly the same as in the semigroup case, cf. the proof of Lemma 5.11. The only difference is that we now verify that the subset of $\mathrm{End}\, C := \mathrm{End}_{\mathrm{Mon}} C$,

$$E := \{\alpha \in \mathrm{End}\, C \mid \forall_{\beta \in \mathrm{End}\, C}\ \alpha \circ \beta = \beta \circ \alpha\}\,,$$

contains constant function ε. Indeed, for every β and $c \in C$, straightforward calculation

$$(\varepsilon \circ \beta)(c) = \varepsilon\big(\beta(c)\big) = e = \beta(e) = \beta\big(\varepsilon(c)\big) = (\beta \circ \varepsilon)(c)\,,$$

demonstrates that

$$\varepsilon \circ \beta = \varepsilon = \beta \circ \varepsilon\,. \tag{108}$$

□

5.6.7

Being a subsemiring of commutative semiring (5.12), the semiring with zero,

$$\left(\mathrm{End}_{\mathrm{Sgr}} C, \varepsilon, \mathrm{id}_C, \cdot, \circ\right), \tag{109}$$

is commutative.

5.6.8 The meaning of the canonical semiring operations on $\mathrm{End}_{\mathrm{Mon}} C$

In this semiring operation $\cdot$ plays the role of 'addition', operation $\circ$ plays the role of 'multiplication', constant function ε plays the role of 'zero', and the identity function id_C plays the role of 'one'.

5.6.9 Terminology: a semiring-with-zero

Identity (108) means that 'multiplication by zero' sends every element of the semiring to 'zero'. When a semiring that contains an identity element for addition has this property, we say that the semiring is a *semiring-with-zero*.

5.7 The canonical commutative semiring-with-zero structure on $\mathcal{O}$ when $(\mathcal{O}, o, \sigma)$ is a principal twisted set

5.7.1

Let

$$C = \mathrm{End}(\mathcal{O}, \sigma)$$

be the monoid of endomorphisms of a principal twisted set with origin $o \in \mathcal{O}$. Here,

$$e = \mathrm{id}_{\mathcal{O}} \qquad \text{and} \qquad f \cdot g = f \circ g\,.$$

5.7.2 Caveat: *internal* versus *external* composition

Since monoid C is itself the monoid of endomorphisms of a certain object in a certain category, in the monoid of endomorphisms of $(\mathrm{End}(\mathcal{O}, \tau), \mathrm{id}_{\mathcal{O}}, \circ)$ we have now two types of composition:

internal composition $\alpha \cdot \beta : f \longmapsto \alpha(f) \circ \beta(f)$ (composition of the *values* of α and β at $f \in \mathrm{End}(\mathcal{O}, \tau)$, performed in $\mathrm{End}(\mathcal{O}, \tau)$)

external composition $\alpha \circ \beta$ (composition of α and β, performed in $\mathrm{End}_{\mathrm{Mon}}(\mathrm{End}(\mathcal{O}, \tau))$.

5.7.3 The meaning of the canonical semiring operations on $\mathrm{End}_{\mathrm{Mon}}(\mathrm{End}(\mathcal{O}, \tau))$

addition : internal composition $\alpha \cdot \beta$;

multiplication : external composition $\alpha \circ \beta$;

zero : the constant function ε on $\mathrm{End}(\mathcal{O}, \tau)$ that sends every f to $\mathrm{id}_{\mathcal{O}}$;

one : the identity operation $\mathrm{id}_{\mathrm{End}(\mathcal{O}, \tau)}$ on set $\mathrm{End}(\mathcal{O}, \tau)$.

5.7.4

Composition of two evaluation bijections

$$\mathrm{End}_{\mathrm{Mon}}(\mathrm{End}(\mathcal{O},\sigma),\mathrm{id}_{\mathcal{O}},\circ) \xrightarrow{\mathrm{ev}_\sigma} \mathrm{End}(\mathcal{O},\sigma) \xrightarrow{\mathrm{ev}_o} \mathcal{O} \quad , \tag{110}$$

where

$$\alpha \longmapsto \alpha(\sigma) \longmapsto \big(\alpha(\sigma)\big)(o)\,, \tag{111}$$

equips $\mathcal{O}$ with a canonical structure of a commutative semiring-with-zero.

5.7.5 The family of endomorphisms $(\alpha_{ou})_{u\in\mathcal{O}}$ of monoid $\mathrm{End}(\mathcal{O},\sigma)$

We shall denote by α_{ou} the unique endomorphism of monoid

$$\big(\mathrm{End}(\mathcal{O},\sigma),\mathrm{id}_{\mathcal{O}},\circ\big)$$

that sends σ to $\sigma_{ou}\in\mathcal{O}$. This α_{ou} is a unique homomorphism of principal twisted sets

$$\big(\mathrm{End}(\mathcal{O},\sigma),\sigma,\tau\big) \longrightarrow \big(\mathrm{End}(\mathcal{O},\sigma),\sigma_{ou},\sigma_{\bullet}\,\big)\,.$$

In terms of α_{ou}, the composite assignment $\mathrm{ev}_o\circ\mathrm{ev}_\sigma$, cf. (111), has the following form

$$\alpha_{ou} \longmapsto \sigma_{ou} \longmapsto u \qquad (u\in\mathcal{O})\,. \tag{112}$$

Exercise 38 *Verify the following identity by a straightforward calculation using Definition* (112)

$$\alpha_{o,u+v} = \alpha_{ou}\cdot\alpha_{ov} \qquad (u,v\in\mathcal{O})\,. \tag{113}$$

Exercise 39 *Verify the identity*

$$\alpha_{o,\sigma u} = \tau\alpha_{ou} \qquad (u\in\mathcal{O})\,. \tag{114}$$

5.7.6 The meaning of α_{ov} as an endomorphism of $\big(\mathrm{End}(\mathcal{O},\sigma),\mathrm{id}_{\mathcal{O}},\circ\big)$

Identity (114) means that

$$\alpha_{o,\sigma u}(f) = f\circ\alpha_{ou}(f) \qquad \big(f\in\mathrm{End}(\mathcal{O},\sigma)\,;u\in\mathcal{O}\big)\,.$$

In particular,

$$\alpha_{oo}(f)=\mathrm{id}_{\mathcal{O}}\,, \qquad \alpha_{o,\sigma o}(f)=f\circ\mathrm{id}_{\mathcal{O}}=f\,, \qquad \alpha_{o,\sigma\sigma o}(f)=f\circ f\,, \qquad \alpha_{o,\sigma\sigma\sigma o}(f)=f\circ f\circ f\,, \quad \dots$$

where $\circ$ denotes composition of endomorphisms of principal twisted set $(\mathcal{O},\sigma)$.

In other words, α_{ou} has the meaning of raising f to the 'u-th power'. We could introduce notation f^u for the value of α_{ou} on f, where u is an *arbitrary* element of principal twisted set $(\mathcal{O},\sigma)$. When there is a unique origin in $(\mathcal{O},\sigma)$, this is a perfectly suitable notation.

5.7.7 The commutative unital ring $\mathbf{Z}_{(\mathcal{O},\sigma)}$

When there is more than one origin, then, as we know, cf. Corollary 4.3, *every* element $o \in \mathcal{O}$ is an origin and such a definition of the u-th power depends on the choice of $o \in \mathcal{O}$.

On the other hand, we have a *canonical* isomorphism of principal twisted sets

$$\mathrm{End}_{\mathrm{Mon}}(\mathrm{End}(\mathcal{O},\sigma),\mathrm{id}_{\mathcal{O}},\circ) \xrightarrow{\mathrm{ev}_\sigma} \mathrm{End}(\mathcal{O},\sigma) \tag{115}$$

that is also an isomorphism of monoids

$$\mathrm{End}_{\mathrm{Mon}}(\mathrm{End}(\mathcal{O},\sigma),\varepsilon,\cdot) \xrightarrow{\mathrm{ev}_\sigma} (\mathrm{End}(\mathcal{O},\sigma),\mathrm{id}_{\mathcal{O}},\circ) \tag{116}$$

Since in this case $\mathrm{End}(\mathcal{O},\sigma)$ is an abelian group, $\mathrm{End}_{\mathrm{Mon}}(\mathrm{End}(\mathcal{O},\sigma),\mathrm{id}_{\mathcal{O}},\circ)$ is a *commutative unital ring*. We shall denote by $\mathbf{Z}_{(\mathcal{O},\sigma)}$ the corresponding canonical unital ring structure on the monoid

$$(\mathrm{End}(\mathcal{O},\sigma),\mathrm{id}_{\mathcal{O}},\circ).$$

5.7.8 Addition in $\mathbf{Z}_{(\mathcal{O},\sigma)}$

The operation of 'addition' in $\mathbf{Z}_{(\mathcal{O},\sigma)}$ is composition of endomorphisms of $(\mathcal{O},\sigma)$

$$f+g := f\circ g. \tag{117}$$

5.7.9 Multiplication in $\mathbf{Z}_{(\mathcal{O},\sigma)}$

The operation of 'multiplication' in $\mathrm{End}(\mathcal{O},\sigma)$, is calculated as follows. Let α and β be the unique endomorphisms of monoid $\mathrm{End}(\mathcal{O},\sigma)$ such that

$$\alpha(\sigma)=f \qquad \text{and} \qquad \beta(\sigma)=g. \tag{118}$$

Then

$$fg := (\alpha\circ\beta)(\sigma) = \alpha(\beta(\sigma)) = \alpha(g). \tag{119}$$

5.7.10

The above description of addition and multiplication of endomorphisms of $(\mathcal{O},\sigma)$ is valid irrespective of whether $\mathrm{End}(\mathcal{O},\sigma)$ is a group or only a monoid.

5.8 Recursively defined functions

5.8.1

In this chapter we assume that $(\mathcal{O},\sigma)$ is of type (iii), in the terminology of Section 4.4.2, i.e., that

$$\forall_{u\in\mathcal{O}}\ u \prec \sigma u.$$

5.8.2 Recursive data

By recursive data we mean a pair

$$x, (\tau_u)_{u\in\mathcal{O}} \tag{120}$$

where $x \in X$ and $(\tau_u)_{u\in\mathcal{O}}$ is a family of operations $\tau_u \in \mathrm{Op}_1 X$ indexed by elements of $\mathcal{O}$.

5.8.3

Consider the sets

$$F = \left\{ f \in \mathrm{Funct}(\mathcal{O}, X) \mid f(o){=}x \wedge \left(\forall_{u\in\mathcal{O}}\ f(\sigma u) = \tau_u(f(u)) \right) \right\} \tag{121}$$

and, for every $v \in \mathcal{O}$,

$$F_v = \left\{ f \in \mathrm{Funct}([o,v], X) \mid f(o){=}x \wedge \left(\forall_{u\in[o,v)}\ f(\sigma u) = \tau_u(f(u)) \right) \right\}. \tag{122}$$

5.8.4 The restriction functions

One has obvious restriction functions

$$\mathrm{res}_v : F \longrightarrow F_v \qquad (v \in \mathcal{O}) \tag{123}$$

and

$$\mathrm{res}_{wv} : F_w \longrightarrow F_v \qquad (v \precsim w). \tag{124}$$

5.8.5 The one-step extension functions

One has also the one-step extension functions

$$\mathrm{ext}_v : F_v \longrightarrow F_{\sigma v}, \qquad \mathrm{ext}_v f(u) = \begin{cases} f(u) & \text{for } u \in [o,v] \\ \tau_v(f(v)) & \text{for } u = \sigma v \end{cases}. \tag{125}$$

5.8.6

An obvious identity

$$\mathrm{res}_{\sigma v, v} \circ \mathrm{ext}_v = \mathrm{id}_{F_v} \qquad (v \in \mathcal{O}) \tag{126}$$

demonstrates that the one-step restriction functions are surjective.

5.8.7

Moreover, $f \in F_{\sigma v}$ precisely when its restriction to $[o,v]$ belongs to F_v and $f(\sigma v) = \tau_v(f(v))$. In other words, when

$$f = \mathrm{ext}_v(\mathrm{res}_{\sigma v, v} f).$$

This demonstrates that ext_v and $\mathrm{res}_{\sigma v, v}$ are inverses one of another.

Since F_o is a singleton set consisting of the function

$$[o] = \{o\} \longrightarrow X, \qquad o \longmapsto x,$$

a standard by now argument demonstrates that the set

$$E := \{v \in \mathcal{O} \mid F_v \text{ is a singleton set}\}$$

is an invariant subset of $\mathcal{O}$ that contains the origin, hence $E = \mathcal{O}$.

5.8.8

A function $f : \mathcal{O} \longrightarrow X$ belongs to F precisely when

$$\forall_{v \in \mathcal{O}} \ \mathrm{res}_v f \in F_v . \tag{127}$$

Define the function $f : \mathcal{O} \longrightarrow X$ as follows:

$$f(v) := f_v(v) \qquad (v \in \mathcal{O}) \tag{128}$$

where f_v is the unique element of F_v.

Exercise 40 *Show that* $\mathrm{res}_v f = f_v$.

Solution. Consider the set

$$E := \{v \in \mathcal{O} \mid \mathrm{res}_v f = f_v\} .$$

By definition, $\mathrm{res}_o f(o) = f_o(o) = x$. Thus $o \in E$. If $v \in E$, then

$$\mathrm{res}_{\sigma v} f(u) = f_v(u) \qquad (u \in [o, v])$$

while

$$\mathrm{res}_{\sigma v} f(\sigma v) = f_{\sigma v}(\sigma v)$$

by the definition of f. Thus E is an invariant subset of $\mathcal{O}$ that contains o, hence $E = \mathcal{O}$.

5.8.9

We established a fundamentally important fact.

Theorem 5.14 (Recursion Theorem) *For any recursive data* (120) *indexed by elements of a principal twisted set of type* (iii), *there exists a unique function* $f : \mathcal{O} \longrightarrow X$ *that satisfies Recursion Identities used to define set* (121). □

Corollary 5.15 *For every principal twisted set of type (iii) and every element* $x \in X$ *of a twisted set* (X, τ), *there exists a unique homomorphism*

$$(\mathcal{O}, o, \sigma) \longrightarrow (X, x, \tau) ,$$

i.e., $(\mathcal{O}, o, \sigma)$ *is an initial object in the category of nullary-unary algebraic structures* $\mathbf{Set}^{(01)}$.

5.9 Free twisted sets

5.9.1

For any twisted set (W, σ) and any subset $A \subseteq W$, Let us consider the restriction-to-A functions

$$\mathrm{res}_A : \mathrm{Hom}((W,\sigma),(X,\tau)) \longrightarrow \mathrm{Funct}(A,X)\,, \qquad f \longmapsto f_{|A}\,. \tag{129}$$

We say that

(a) *A generates* (W,σ) if, for every twisted set (X,τ), the restriction function is injective.

(b) A is an *independent* subset of (W,σ) if, for every twisted set (X,τ), (129) is surjective.

(c) *A freely generates* (W,σ) if, for every twisted set (X,τ), (129) is bijective.

5.9.2

The following corollaries are immediate consequences of Lemma 5.15.

Corollary 5.16 *A twisted set is freely generated by a single element if and only if it is a principal twisted set of type* (iii).

□

Corollary 5.17 *Any principal twisted set is a quotient of a principal twisted set of type* (iii).

□

5.9.3 Existence of a principal twisted set of type (iii)

Every principal twisted set of type (iii) is infinite, since σ is an injective but not a surjective unary operation on it.

Let W be an infinite set and σ be an injective unary operation on W.

The nonempty subset $W \setminus \sigma_* W$ coincides with the set of minimal elements in $(W, \precsim)$ and it freely generates a subset of W that we will denote W_{free}. The orbit $\langle o \rangle$ of any element $o \in W \setminus \sigma_* W$ is, of course, a principal twisted set.

On the remainder $W_{\mathrm{inv}} \coloneqq W \setminus W_{\mathrm{free}}$ the twist operation σ is invertible, since *every* element belongs to the image of σ while σ is injective. The remainder is the union of disjoint orbits of type (i) and of connected components being nested unions of orbits of type (iii).

5.9.4 Natural numbers $(\mathbf{N}, 0, \sigma)$

Taking into account that a principal twisted set of type (iii) is unique up to a unique isomorphism, so a choice of a particular model is of little importance, we adopt notational and terminological conventions that are in constant use in Mathematics.

We denote such a set by $\mathbf{N}$, its origin by 0, and refer to 0 as 'zero'. We refer to elements of $\mathbf{N}$ as *natural numbers*, and we denote $\sigma 0$ by '1', $\sigma 1$ by '2', $\sigma 2$ by '3', and so on.

5.9.5 The well-ordered set $(\mathbf{N}, \leq)$

The associated preorder $\precsim$ on a principal twisted set of type (iii) is an order. We shall denote it $\leq$. The orbit epimorphism, cf. Section 3.1.5, is an isomorphism of $(\mathbf{N}, \leq)$ with the ordered set of orbits $(\mathbf{N}_{\sigma}, \supseteq)$.

5.9.6 The first infinite ordinal ω

In theory of ordered sets this well-ordered set is denoted ω, and is called *the first infinite ordinal*,

5.9.7 The first infinite cardinal $\aleph_0$

In theory of cardinality, in Set Theory, it is denoted $\aleph_0$ and is called *the first infinite cardinal*.