Game Theory (STAT 155) Homework #1

All the problems are worth 4 points each and will be graded on a 0/1/2/3/4 scale.

1. (m-subtraction game): Given positive integers $a_1 < a_2 < \ldots < a_m$ define the $m-$subtraction game where a position consists of a pile of chips, and a legal move is to remove from the pile a_i chips for some $i \in \{1, 2, \ldots, m\}$. The player who cannot move loses. Show that the labelling of the set of states $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ in to P or N is periodic after some time i.e. there exists n_0 and k such that for all $n \geq n_0$ the labels of n and $n + k$ are the same.

   Solution: Suppose that the allowed subtractions $a_1 < \cdots < a_m$ are fixed and known. Since $a_1 > 1$, the number of chips in the pile must decrease by at least one every turn. In other words, the game started with n chips in the pile will terminate in at most n turns. This means the game is progressively bounded with $B(n) \leq n$. Since this is a progressively bounded impartial combinatorial game, we know by the theorem from lecture, that its states can all be classified into N or P. (This is just a sanity check to make sure that the question is well-defined.)

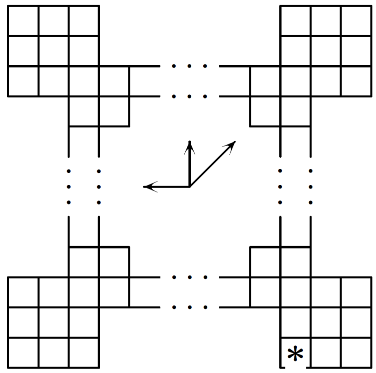
   For each state $n \in \mathbb{Z}_{\geq 0}$, let $c(n)$ be either N or P depending on whether n is an N state or P state in this subtraction game. In other words, we can think of c as the "classification function" $c : \mathbb{Z}_{\geq 0} \to \{N, P\}$. Observe that the only legal moves from state n are those which result in a state in the set $\{n - a_m, n - a_{m-1}, \ldots n - a_2, n - a_1\}$. So, we have that $c(n)$ depends only on the sequence $A(n) = (c(n - a_m), c(n - a_{m-1}), \ldots c(n - a_2), c(n - a_1))$, which we can regard as a string of N's and P's with m total characters. (More specifically, $c(n) = N$ if this string consists only of P's, and $c(n) = P$ if this string has at least one N.) A slightly weaker statement is then also true: that $c(n)$ depends only on the sequence $S(n) = (c(n - a_m), c(n - a_m + 1), \ldots c(n - 2), c(n - 1))$, which we can regard as a string of N's and P's with a_m total characters.

   The number of strings of N's and P's with a_m total characters is 2^{a_m}, which is a finite (but potentially very large) positive integer. There are infinitely many n in $\mathbb{Z}_{\geq 0}$ but only finitely many possible values of $S(n)$, so it follows by the pigeonhole principle that there must exist a pair $n_0, n_1 \in \mathbb{Z}_{\geq 0}$ with $n_0 < n_1$ such that $S(n_0) = S(n_1)$. Then define $k = n_1 - n_0$. We claim that, for $n \geq n_0 - 1$, we have $c(n) = c(n + k)$, which is the desired result.

   To show this, we use induction on $i \in \mathbb{Z}_{\geq 0}$ to prove the statement that we have both $S(n_0 + i) = S(n_1 + i)$ and $c(n_0 + i) = c(n_1 + i)$; since $n_1 = n_0 + k$, this will prove the desired claim about the periodicity of $c(\cdot)$. First consider the base case of $i = 0$. By construction, we have $S(n_0) = S(n_1)$. Moreover, we know that $c(n_0)$ depends only on a substring of $S(n_0)$ and that $c(n_1)$ depends only on a substring of $S(n_1)$, so $S(n_0) = S(n_1)$ also implies $c(n_0) = c(n_1)$. Hence, the base case is true.

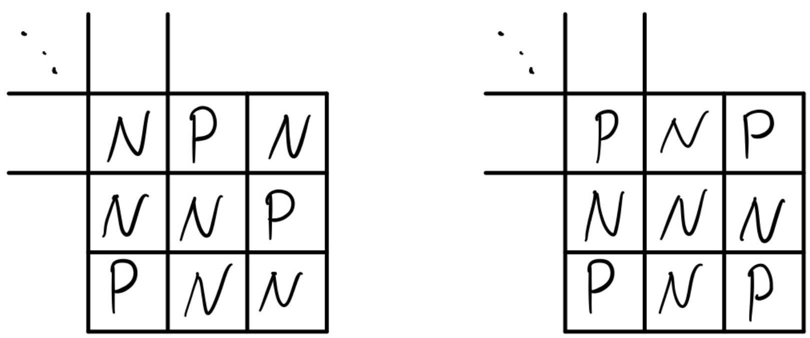
   For the inductive step, suppose that we have $S(n_0 + i) = S(n_1 + i)$ and $c(n_0 + i) = c(n_1 + i)$, and let us show that this implies $S(n_0 + i + 1) = S(n_1 + i + 1)$ and $c(n_0 + i + 1) = c(n_1 + i + 1)$. Note that $S(n + i + 1)$ is obtained by starting with $S(n + i)$, removing the first element, and appending $c(n + i + 1)$ to the end. The inductive hypothesis guarantees $S(n_0 + i) = S(n_1 + i)$ and $c(n_0 + i) = c(n_1 + i)$, so this impies $S(n_0 + i + 1) = S(n_1 + i + 1)$. Moreover, we know that $c(n_0 + i + 1)$ depends only on a substring of $S(n_0 + i + 1)$ and that $c(n_1 + i + 1)$ depends only on a substring of $S(n_1 + i + 1)$, so $S(n_0 + i + 1) = S(n_1 + i + 1)$ also implies $c(n_0 + i + 1) = c(n_1 + i + 1)$. This completes the induction.

2. Two players play a game on the board, which is shown on the figure. Initially there is a stone in the cell marked by $*$. A player can move the stone left, up, or up-right. A player who is unable to move loses. Which player has the winning strategy?



Solution: First, let's show that the game is progressively bounded. By taking any legal move, the coordinates of the stone in $\mathbb{R}^2$ change by adding one of the vectors $(-1, 0), (0, 1)$ or $(1, 1)$ to its current coordinates. In particular, the distance of the stone along the direction $(-0.6, 0.8)$ always increases by at least 0.2. Since the board is bounded in this direction, the game must terminate in a finite number of steps from any starting position. In particular, all the positions can be classified as either N or P.

We claim now that the second player has a winning strategy. To see this, consider the state B connecting the bottom-right 3×3 grid with the rest of the board. (You can guess that this is a good state to examine, since it's a "bottleneck state": any path from the starting state to a terminal state must pass through B.) Now we consider two cases: If B is in N, then we can fill out the 3×3 grid and determine that $*$ is in P. Otherwise B is in P, and we can fill out the 3×3 grid and determine that $*$ is in P. In either case, $*$ is in P, so the second player has a winning strategy.



(Note that we did not really have to consider anything outside of the 3×3 grid in order to analyze this game. In some sense, that means this game is "decided very early", assuming that both players are completely rational. In particular, this means that it is not important what is the size of the main board, whether the board is hollow, etc.)

3. Take two progressively bounded impartial combinatorial games G_a, G_b with state spaces X_a, X_b and set of legal moves M_a, M_b, respectively. Consider the sum $G = G_a + G_b$ of the two games with state space

$$X = X_a \times X_b = \{(x_a, x_b) : x_a \in X_a, x_b \in X_b\},$$

where legal moves consist of picking any coordinate and playing a legal move of that coordinate. Now call a state (x_a, x_b) as a (P, P) state if x_a and x_b are P states in the games G_a and G_b respectively. Similarly define $(N, P), (P, N), (N, N)$ respectively.
(0) Show G is progressively bounded.
(1) Show a (P, P) state is a P state in G.
(2) Show an (N, P) or a (P, N) state is an N state in G.
(3) Are (N, N) states always N states in G? Prove if yes, or provide a counterexample otherwise.

Solution: (0) Suppose that $B_a(\cdot)$ and $B_b(\cdot)$ are functions describing upper bounds on the number of turns that either game can have. Since, in G, a turn consists of choosing a game to play in and then playing in that game, we see that the game started in position (x_a, x_b) will end in at most $B_a(x_a) + B_b(x_b)$ turns. Therefore, the game G is progressively bounded with $B(x_a, x_b) \leq B_a(x_a) + B_b(x_b)$.

(1, 2) It is most convenient (and, in some sense, unavoidable) to prove statements (1) and (2) simultaneously. To do this, define the following sets for each $n \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned}
(P, P)_n &= \{(x_a, x_b) \in X : x_a \in P_a, x_b \in P_b, B(x_a, x_b) \leq n\}, \\
(P, N)_n &= \{(x_a, x_b) \in X : x_a \in P_a, x_b \in N_b, B(x_a, x_b) \leq n\}, \text{ and} \\
(N, P)_n &= \{(x_a, x_b) \in X : x_a \in N_a, x_b \in P_b, B(x_a, x_b) \leq n\}.
\end{aligned}$$

(We could also define $(N, N)_n$ in an analogous way, but we won't need to do that for this problem.)

We will prove by induction on $n \in \mathbb{Z}_{\geq 0}$ that we have $(P, P)_n \subseteq P$, $(P, N)_n \subseteq N$, and $(N, P)_n \subseteq N$. For the base case, consider $n = 0$: The set $(P, P)_0$ consists of all pairs of terminal states, so $(P, P)_0 \subseteq P$ holds by the definition of normal play. Also, the sets $(P, N)_0$ and $(N, P)_0$ are empty, since in either case there is some coordinate with a position in N, and for any positions in N there is always at least one legal move to make (namely, the legal move which goes to some state in P). Thus the base case is true.

For the inductive step, suppose that we have $(P, P)_n \subseteq P$, $(P, N)_n \subseteq N$, and $(N, P)_n \subseteq N$. We now make three arguments to show that we have $(P, P)_{n+1} \subseteq P$, $(P, N)_{n+1} \subseteq N$, and $(N, P)_{n+1} \subseteq N$:

- Consider any state $(x_a, x_b) \in (P, P)_{n+1}$. One type of legal move from this state is to move to (x_a, y_b) for some $y_b \in X_b$ such that $x_b \to y_b$ is a legal move in G_b. Since $x_a \in P_b$ in G_b, we have $y_b \in N_b$ in G_b. Note also that $B(x_a, y_b) \leq B(x_a, x_b) - 1 \leq n$. Therefore, we have $(x_a, y_b) \in (P, N)_n$, so the inductive hypothesis gives $(x_a, y_b) \in N$. The other type of possible legal move from (x_a, x_b) is to move to (y_a, x_b) for some $y_a \in X_a$ such that $x_a \to y_a$ is a legal move in G_a, and the same argument shows that $(y_a, x_b) \in (N, P)_n \subseteq N$, again by the inductive hypothesis. We have shown that every move from (x_a, x_b) leads to a state in N, hence $(x_a, x_b) \in P$. In other words, we have proven $(P, P)_{n+1} \subseteq P$.
- Consider any state $(x_a, x_b) \in (P, N)_{n+1}$. Since $x_b \in N_b$ in G_b, we know that there exists a legal move $x_b \to y_b$ in G_b such that $y_b \in P_b$. Therefore, a legal move in G is to move (x_a, x_b) to (x_a, y_b). Of course, we have $B(x_a, y_b) \leq B(x_a, x_b) - 1 \leq n$, hence $(x_a, y_b) \in (P, P)_n$. Thus the inductive hypothesis gives $(x_a, y_b) \in P$. We have shown that there is some legal move from (x_a, x_b) which leads to a state in P, hence $(x_a, x_b) \in N$. That is, we have proven $(P, N)_{n+1} \subseteq N$.
- The argument from the previous bullet point, after interchanging the roles of a and b, shows that under the inductive hypothesis we have $(N, P)_{n+1} \subseteq N$.

This completes the proof by induction, and hence shows that we have $(P, P)_n \subseteq P$, $(P, N)_n \subseteq N$, and $(N, P)_n \subseteq N$ for all $n \in \mathbb{Z}_{\geq 0}$. Taking the union over all n, this shows that we have

$$\begin{aligned}
(P, P) &= \bigcup_{n \in \mathbb{Z}_{\geq 0}} (P, P)_n \subseteq P, \\
(P, N) &= \bigcup_{n \in \mathbb{Z}_{\geq 0}} (P, N)_n \subseteq N, \text{ and} \\
(N, P) &= \bigcup_{n \in \mathbb{Z}_{\geq 0}} (N, P)_n \subseteq N,
\end{aligned}$$

as desired.

(3) We'll construct examples to show that (N, N) states can be in either N or P in G. To do this, let G_a and G_b both be the subtraction game with the allowed subtractions $1, 2, 3$, and 4.

- First consider the state $(1, 1)$, which is in (N, N). The only legal moves from $(1, 1)$ are to $(1, 0)$ and $(0, 1)$, which are in (N, P) and (P, N), respectively. By parts (1) and (2) of this problem, we get $(1, 0), (0, 1) \in N$, hence $(1, 1) \in P$.
- Next consider the state $(1, 2)$ which is (N, N). There is a legal move from $(1, 2)$ to $(1, 1)$, which, by the above, is in P. Therefore, $(1, 2) \in N$.

4. Consider the following game: The game starts with two piles of chips say with n_1 and n_2 chips denoted by (n_1, n_2). The players alternate moves, and each move consists of throwing away one of the piles and then dividing the contents of the other pile into two piles (each of which has at least one chip). For instance, if the piles had 15 and 9 chips in them, a legal move would be to throw away the pile with 15 chips and split the other pile of 9 chips into piles of 4 and 5. The game ends when no legal moves can be made, which happens when there is 1 chip in each pile. As usual, the first player who cannot make a legal move loses.

(1) Show that the game is progressively bounded.

(2) Classify the starting positions into N or P states.

Solution: (1) Note that the legal moves from (n_1, n_2) are to $(n_1 - k_1, k_1)$ for $1 \le k_1 \le n_1 - 1$ and to $(k_2, n_2 - k_2)$ for $1 \le k_2 \le n_2 - 1$. In particular, the total number of chips goes from $n_1 + n_2$ to n_1 or n_2. Since $n_1, n_2 \ge 1$, this implies that the total number of chips must decrease by at least one at each step. Therefore, the game is progressively bounded with $B(n_1, n_2) \le n_1 + n_2$.

(2) Define the sets $P' = \{(n_1, n_2) : n_1 \text{ and } n_2 \text{ are both odd}\}$ and $N' = \{(n_1, n_2) : \text{ at least one of } n_1 \text{ or } n_2 \text{ is even}\}$. Now we make the following observations:

- Suppose that $(n_1, n_2) \in P'$. One type of legal move is to $(n_1 - k_1, k_1)$ for $1 \le k_1 \le n_1 - 1$. But $(n_1 - k_1) + k_1$ equals an odd number n_1, so at least one of $n_1 - k_1$ or k_1 must be even, hence $(n_1 - k_1, k_1) \in N'$. The other type of legal move is to $(k_2, n_2 - k_2)$ for $1 \le k_2 \le n_2 - 1$, and the same argument shows that at least one of k_2 or $n_2 - k_2$ must be even, hence $(k_2, n_2 - k_2) \in N'$. That is, every move from a state in P' leads to a state in N'.
- Suppose that $(n_1, n_2) \in N$. If n_1 is even, then we can make a legal move to $(n_1 - 1, 1)$, which is in P'. Otherwise n_2 is even, and we can make a legal move to $(1, n_2 - 1)$, which is in P'. That is, from any state in N', there is a legal move leading to a state in P'.

Now the result of problem 5 below implies that $N = N'$ and $P = P'$, as desired. (Alternatively, we could have proved this directly by induction instead of citing another problem.)

5. In class we showed the following theorem:

Theorem 0.1. *In a progressively bounded impartial combinatorial game under normal play, $X = N \cup P$. That is, from any initial position, one of the players has a winning strategy. Moreover,*
- *P: Every move leads to N.*
- *N: Some move leads to P (hence cannot contain terminal positions).*

Let now P' and N' be disjoint subsets of X (set of positions) such that $X = N' \cup P'$ and:
- P': Every move leads to N'.
- N': Some move leads to P'.

Show that $N' = N$ and $P' = P$.

Solution: Define the sets $P'_n = \{x \in X : x \in P', B(x) \leq n\}$ and $N'_n = \{x \in X : x \in N', B(x) \leq n\}$. Let's prove by induction on $n \in \mathbb{Z}_{\geq 0}$ that we have $P'_n \subseteq P$ and $N'_n \subseteq N$. Consider the base case of $n = 0$. Since P'_0 contains only terminal positions, we have $P'_0 \subseteq P$. Also note that, from every state $x \in N'$, there exists some legal move to P', hence $B(x) \geq 1$. Therefore, $N'_0 = \emptyset$. Since the empty set is a subset of any set, we have $N'_0 \subseteq N$, and this establishes the base case.

For the inductive step, suppose that we have $P'_n \subseteq P$ and $N'_n \subseteq N$, and let us use this to prove that we have $P'_{n+1} \subseteq P$ and $N'_{n+1} \subseteq N$. Note the following:

- If $x \in P'_{n+1}$, then any legal move to y implies $y \in N'$. Also, we have $B(y) \leq B(x) - 1 \leq n$. Therefore, $y \in N'_n$, so the inductive hypothesis gives $y \in N$. We have shown that any legal move from x leads to N, hence $x \in P$. That is, we have shown $P'_{n+1} \subseteq P$, as needed.
- If $x \in N'_{n+1}$, then there is a legal move to y with $y \in P'$. Since we have $B(y) \leq B(x) - 1 \leq n$, this implies $y \in P'_n$, hence the inductive hypothesis gives $y \in P$. We have shown that there exists a legal move from x to a state in P, hence $x \in N$. That is, we have shown $N'_{n+1} \subseteq N$.

This completes the induction and shows that we have $P'_n \subseteq P$ and $N'_n \subseteq N$ for all $n \in \mathbb{Z}_{\geq 0}$. Taking unions gives

$$P' = \bigcup_{n \in \mathbb{Z}_{\geq 0}} P'_n \subseteq P, \text{ and}$$
$$N' = \bigcup_{n \in \mathbb{Z}_{\geq 0}} N'_n \subseteq N.$$

Now assume that P' were strictly smaller than P. That is, assume that there exists some $x \in P$ with $x \notin P'$. Since P' and N' cover X, this implies $x \in N'$. But then our result $N' \subseteq N$ implies $x \in N$. However, x cannot be in both N and P, so we have reached a contradiction. Therefore, $P' = P$. The same argument shows that $N' = N$: If there were some $x \in N$ but not in N', then this would imply $x \in P'$, hence $x \in P$, which contradicts $x \in N$. Therefore, we have $P = P'$ and $N = N'$ as claimed.