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Midterm 2.

1. T,
(a) ①
Because Hermitian forms commute with their adjoint and are normal, by the spectral theorem (concrete version), there exists a unitary matrix U s.t. U^*AU is a diagonal. Almost there, remember $U^* = U^{-1}$ because it is unitary

so choose $U^{-1}AU$ to answer the question ⊠

(b) T, this is the dream case because the eigenvectors form a diagonal matrix that is orthogonal.

(c) T, $TS = ST$ ①, $Tv = \lambda v$ ②, $v \in \mathbb{C}^n$ $\lambda \in \mathbb{C}$ are true.

Goal $T(Sv) = \lambda Sv$.

apply commuting rule ① to goal:

$S(Tv) = \lambda Sv$

apply eigenvector rule ② to goal:

$S(\lambda v) = \lambda Sv$

pull the scaler out and equal by reflection:

$\lambda Sv = \lambda Sv$ ⊠

2. $Te_1 = e_1 + Te_2 - e_3$

$= e_1 + Te_2 + e_1 - Te_3$

$T(e_1 - e_2 + e_3) = 2e_1$

(a) $\boxed{T(z)} = 2T(e_1) + 3T(e_2) + 1T(e_3)$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}}$$

(b) $\boxed{T(z)} = z_1 Te_1 + z_2 Te_2 + z_3 Te_3$

$$= \begin{pmatrix} z_1 \\ z_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ z_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} z_3 \\ 0 \\ z_3 \end{pmatrix} = \boxed{\begin{pmatrix} z_1 + z_3 \\ z_1 + z_2 \\ z_2 + z_3 \end{pmatrix}}$$

(c) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ F, T can not be diagonalized because you will only be able to find one eigenvector.

all of these transformation — independent on each other

3. (a) Proven is class, there is a lemma that every billinear form on fin. dim. v.s. admits a non-unique orthogonal basis.

$$[B] = \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & \ddots & \\ & & & b_{nn} \end{bmatrix}$$ re order the basis so everythings

on the diagonal. And diagonal is full because $\det([B]) \neq 0$.

The $[B]$ has signs $\begin{bmatrix} + & & & & \\ & + & & & \\ & & + & & \\ & & & - & \\ & & & & - \end{bmatrix}$ and since the field is $\mathbb{K}$,

we can even scale the -1's to 1, giving the indentity Matrix. $C^T[B]C = I_n$ because C digonalize and can scale the diag. entries of B to 1.

(b) Since $C^T[B]C = I_n$, any combination of basis vectors is not going to give 0 because each basis vector is at least positive. This implies $B(v,v) \neq 0$ because when you undo the transform on this number it could become negative because of the i's used for rescaling, but it must stay nonzero.

(c) $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2$$

$$\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = i$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} i/2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -i & 2 \end{pmatrix}\begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 0 & 2 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -i/2 \\ 1 \end{pmatrix} \sim \frac{i}{2}$$

$$\begin{pmatrix} 1 & 0 \\ -i & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ i & 5 \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 0 \\ -i & 2 \end{pmatrix}\begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}}$$

$$C^T [B] C = \tilde{B}$$