Review on 12:00-1:00PM, Friday, Dec. 1

GEPP as LU factorization

Theorem: Let $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ be non-singular. Then GEPP computes an LU factorization with permutation matrix P such that

$$P \cdot A = L \cdot U = \begin{pmatrix} & \\ & \end{pmatrix} \cdot \begin{pmatrix} & \\ & \end{pmatrix}.$$

$P \cdot A = L \cdot U$, Proof by Induction

- GEPP for $n = 2, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.
  - **pivoting**:

$$\mathbf{piv}_1 \stackrel{def}{=} \mathbf{argmax}_{1 \le j \le 2} |a_{j1}|, \quad P = \begin{cases} I, & \text{if } \mathbf{piv}_1 = 1, \\ P_{1,2} & \text{if } \mathbf{piv}_1 = 2. \end{cases}$$

  - **elimination**:

$$P \cdot A = L \cdot U.$$

$P \cdot A = L \cdot U$, Proof by Induction

- GEPP for $n \geq 3$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$:
  - **pivoting**:

$$\mathbf{piv}_1 \overset{def}{=} \mathbf{argmax}_{1 \leq j \leq n} |a_{j1}|, \quad \overline{P} = \begin{cases} I, & \text{if } \mathbf{piv}_1 = 1, \\ P_{1,\mathbf{piv}_1} & \text{if } \mathbf{piv}_1 \geq 2. \end{cases}$$

  - **elimination**:

$$\overline{P} \cdot A = \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{array}\right), \quad \left(\mathbf{l} \overset{def}{=} \begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix}\right)$$

$P \cdot A = L \cdot U$, Proof by Induction

- GEPP for $n \geq 3$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$:
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  - **Induction hypothesis**:

$$\widehat{\mathbf{P}} \cdot \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \widehat{\mathbf{L}} \cdot \widehat{\mathbf{U}}.$$

- putting it together,

$$\underbrace{\begin{pmatrix} 1 & \\ & \widehat{\mathbf{P}} \end{pmatrix} \cdot \overline{P}}_{\mathbf{P}} \cdot A = \begin{pmatrix} 1 & \\ & \widehat{\mathbf{P}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ & \vdots & \ddots & \vdots \\ & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{array}\right)$$

$$= \left(\begin{array}{c|c} 1 & \\ \hline \widehat{\mathbf{P}} \cdot \mathbf{l} & \widehat{\mathbf{P}} \end{array}\right) \cdot \left(\begin{array}{c|c} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ \hline & \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \end{array}\right)$$

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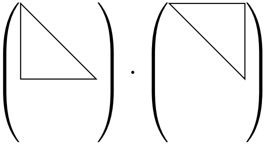
$$\stackrel{def}{=}$$



- putting it together,

$$
\begin{aligned}
\underbrace{\begin{pmatrix}1 & \\ & \widehat{\mathbf{P}}\end{pmatrix}\cdot\overline{P}}_{\mathbf{P}}\cdot A
&= \begin{pmatrix}1 & \\ & \widehat{\mathbf{P}}\end{pmatrix}\cdot\begin{pmatrix}1 & \\ \mathbf{l} & I_{n-1}\end{pmatrix}\cdot\left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ & \vdots & \ddots & \vdots \\ & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn}\end{array}\right) \\
&= \left(\begin{array}{c|c} 1 & \\ \hline \widehat{\mathbf{P}}\cdot\mathbf{l} & \widehat{\mathbf{P}}\end{array}\right)\cdot\left(\begin{array}{c|c} a_{11} & \begin{pmatrix}a_{12} & \cdots & a_{1n}\end{pmatrix} \\ \hline & \begin{pmatrix}\mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn}\end{pmatrix}\end{array}\right) \\
&= \left(\begin{array}{c|c} 1 & \\ \hline \widehat{\mathbf{P}}\cdot\mathbf{l} & I_{n-1}\end{array}\right)\cdot\left(\begin{array}{c|c} a_{11} & \begin{pmatrix}a_{12} & \cdots & a_{1n}\end{pmatrix} \\ \hline & \widehat{\mathbf{P}}\cdot\begin{pmatrix}\mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn}\end{pmatrix}\end{array}\right) \\
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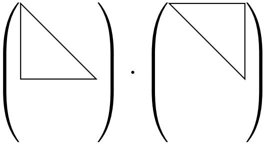
&= \underbrace{\left(\begin{array}{c|c} 1 & \\ \hline \widehat{\mathbf{P}}\cdot\mathbf{l} & \widehat{\mathbf{L}}\end{array}\right)}_{\mathbf{L}}\cdot\underbrace{\left(\begin{array}{c|c} a_{11} & \begin{pmatrix}a_{12} & \cdots & a_{1n}\end{pmatrix} \\ \hline & \widehat{\mathbf{U}}\end{array}\right)}_{\mathbf{U}}
\end{aligned}
$$

$\stackrel{def}{=}$ 

What if A is singular?

Solving general linear equations with GEPP

$$A\mathbf{x} = \mathbf{b}, \quad P \cdot A = L \cdot U$$

- interchanging components in **b**

$$P \cdot (A\mathbf{x}) = (P \cdot \mathbf{b}), \quad (L \cdot U)\, \mathbf{x} = (P \cdot \mathbf{b}).$$

- solving for **b** with forward and backward substitution

$$\begin{aligned} \mathbf{x} &= (L \cdot U)^{-1} (P \cdot \mathbf{b}) \\ &= \left(U^{-1} \left(L^{-1} (P \cdot \mathbf{b})\right)\right). \end{aligned}$$

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Cost Analysis

- computing $P \cdot A = L \cdot U$: about $2/3n^3$ operations.
- forward and backward substitution: about $2n^2$ operations.
- $\boxed{\text{most important to compute } P \cdot A = L \cdot U \textbf{ efficiently}}$

§6.6 Strictly Diagonally Dominant (**SDD**) Matrices

- **Definition**: Matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is **SDD** if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad \text{holds for each} \quad i = 1, 2, \cdots, n.$$

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$$|7| > |2| + |0|, \ |5| > |3| + |-1|, \ |-6| > |0| + |5|.$$

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$$|7| > |2| + |0|\,,\ |5| > |3| + |-1|\,,\ |-6| > |0| + |5|\,.$$

- Example II: matrix $B = \begin{pmatrix} 7 & 5 & 0 \\ 3 & 5 & -1 \\ 0 & -3 & 3 \end{pmatrix}$ is NOT **SDD**.

$$|3| \leq |0| + |-3|\,.$$

GE on **SDD**: succeeds without pivoting (I)

- Let $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ be **SDD**, so

$$|a_{11}| > \sum_{j=1, j \neq 1}^{n} |a_{1j}| \geq 0.$$

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- elimination with $a_{11} \neq 0$:

$$A \;=\; \left(\begin{array}{c|ccc} 1 & & & \\ \hline l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & \widehat{\mathbf{a}}_{22} & \cdots & \widehat{\mathbf{a}}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widehat{\mathbf{a}}_{n2} & \cdots & \widehat{\mathbf{a}}_{nn} \end{array}\right),$$

$$l_{j1} = \frac{a_{j1}}{a_{11}}, \quad \widehat{\mathbf{a}}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}}\, a_{1k}, \quad 2 \leq j \leq n, \quad 2 \leq k \leq n.$$

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- **only do**: show $\widehat{\mathbf{A}} \stackrel{def}{=} \begin{pmatrix} \widehat{\mathbf{a}}_{22} & \cdots & \widehat{\mathbf{a}}_{2n} \\ \vdots & \ddots & \vdots \\ \widehat{\mathbf{a}}_{n2} & \cdots & \widehat{\mathbf{a}}_{nn} \end{pmatrix}$ remains **SDD**.

GE on **SDD**: only need to show $\widehat{\mathbf{A}}$ remains **SDD**

$$
\begin{aligned}
|a_{11}| &> \sum_{j=1, j\neq 1}^{n} |a_{1j}|, \quad |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 2, \cdots n, \\
\widehat{\mathbf{A}} &= \begin{pmatrix} \widehat{\mathbf{a}}_{22} & \cdots & \widehat{\mathbf{a}}_{2n} \\ \vdots & \ddots & \vdots \\ \widehat{\mathbf{a}}_{n2} & \cdots & \widehat{\mathbf{a}}_{nn} \end{pmatrix}, \quad \widehat{\mathbf{a}}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}.
\end{aligned}
$$

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$$\widehat{\mathbf{A}} \;=\; \begin{pmatrix} \widehat{\mathbf{a}}_{22} & \cdots & \widehat{\mathbf{a}}_{2n} \\ \vdots & \ddots & \vdots \\ \widehat{\mathbf{a}}_{n2} & \cdots & \widehat{\mathbf{a}}_{nn} \end{pmatrix}, \quad \widehat{\mathbf{a}}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}\, a_{1j}.$$

- for $i = 2, \cdots n$,

$$\sum_{j=\mathbf{2}, j\neq i}^{n} |\widehat{\mathbf{a}}_{ij}| \;= \sum_{j=\mathbf{2}, j\neq i}^{n} \left| a_{ij} - \frac{a_{i1}}{a_{11}}\, a_{1j} \right| \leq \left(\sum_{j=\mathbf{2}, j\neq i}^{n} |a_{ij}| \right) + \left| \frac{a_{i1}}{a_{11}} \right| \left(\sum_{j=\mathbf{2}, j\neq i}^{n} |a_{1j}| \right)$$

GE on **SDD**: only need to show $\widehat{\mathbf{A}}$ remains **SDD**

$$|a_{11}| \quad > \quad \sum_{j=1, j\neq 1}^{n} |a_{1j}|\,, \quad |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|\,, \quad i = 2, \cdots n,$$

$$\widehat{\mathbf{A}} \quad = \quad \begin{pmatrix} \widehat{\mathbf{a}}_{22} & \cdots & \widehat{\mathbf{a}}_{2n} \\ \vdots & \ddots & \vdots \\ \widehat{\mathbf{a}}_{n2} & \cdots & \widehat{\mathbf{a}}_{nn} \end{pmatrix}, \quad \widehat{\mathbf{a}}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}\, a_{1j}.$$

- for $i = 2, \cdots n$,

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$$\overset{\mathbf{SDD}}{<} \quad \left(|a_{ii}| - |a_{i1}|\right) + \left| \frac{a_{i1}}{a_{11}} \right| \left(|a_{11}| - |a_{1i}|\right) = |a_{ii}| - \left| \frac{a_{i1}}{a_{11}} \right| |a_{1i}|$$

GE on **SDD**: only need to show $\widehat{\mathbf{A}}$ remains **SDD**

$$|a_{11}| \;>\; \sum_{j=1,j\neq 1}^{n} |a_{1j}|\,, \quad |a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|\,, \quad i = 2,\cdots n,$$

$$\widehat{\mathbf{A}} \;=\; \begin{pmatrix} \widehat{\mathbf{a}}_{22} & \cdots & \widehat{\mathbf{a}}_{2n} \\ \vdots & \ddots & \vdots \\ \widehat{\mathbf{a}}_{n2} & \cdots & \widehat{\mathbf{a}}_{nn} \end{pmatrix}, \quad \widehat{\mathbf{a}}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}\, a_{1j}.$$

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$$\overset{\mathbf{SDD}}{<} \;\; (|a_{ii}| - |a_{i1}|) + \left| \frac{a_{i1}}{a_{11}} \right| (|a_{11}| - |a_{1i}|) = |a_{ii}| - \left| \frac{a_{i1}}{a_{11}} \right| |a_{1i}|$$

$$\le \;\; \left| a_{ii} - \frac{a_{i1}}{a_{11}}\, a_{1i} \right| = |\widehat{\mathbf{a}}_{ii}|\,.$$

GE on **SDD**: example

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{pmatrix} \quad \text{is } \textbf{SDD}.$$

$$A \;=\; \begin{pmatrix} 1 & & \\ \frac{3}{7} & 1 & \\ 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 2 & 0 \\ & \frac{29}{7} & -1 \\ & 5 & -6 \end{pmatrix} \quad \left[\begin{pmatrix} \frac{29}{7} & -1 \\ 5 & -6 \end{pmatrix} \text{ is } \textbf{SDD} \right]$$

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Symmetric Positive Definite Matrices

- **Definition**: Matrix $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ is **SPD** if

$$A = A^T, \quad \mathbf{x}^T A \mathbf{x} > 0 \quad \text{for any non-zero} \quad \mathbf{x} \in \mathbf{R}^n.$$

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$$\widehat{A} = \begin{pmatrix} \boxed{0} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \widehat{A}^T$$

<u>IS NOT SPD:</u>

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} \boxed{0} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

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- **Example II**: matrix $A = \begin{pmatrix} \boxed{2} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = A^T$

<u>IS SPD:</u>

PROOF: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \mathbf{0}$,

$\Downarrow$

$$\mathbf{x}^T A \mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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$$\begin{aligned}
\mathbf{x}^T A \mathbf{x} &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix} = 2x_1^2 - 2x_1\,x_2 + 2x_2^2 - 2x_2\,x_3 + 2x_3^2
\end{aligned}$$

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$$\widehat{A} = \begin{pmatrix} \boxed{0} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \widehat{A}^T$$

<u>IS NOT SPD:</u>

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} \boxed{0} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

- **Example II**: matrix $A = \begin{pmatrix} \boxed{2} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = A^T$

<u>IS SPD:</u>

PROOF: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \mathbf{0}$,

$\Downarrow$

$$\begin{aligned}
\mathbf{x}^T A \mathbf{x} &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix} = 2x_1^2 - 2x_1 x_2 + 2x_2^2 - 2x_2 x_3 + 2x_3^2 \\
&\overset{\text{rearranging}}{=\!=} x_1^2 + \left(x_1^2 - 2x_1 x_2 + x_2^2\right) + \left(x_2^2 - 2x_2 x_3 + x_3^2\right) + x_3^2 \\
&= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0
\end{aligned}$$

GE on **SPD**: faster and succeeds without pivoting (I)

- Let $A = \left(a_{ij}\right) \in \mathbf{R}^{n \times n}$ be **SPD**, then

$$
\begin{aligned}
A^T &= A, \quad \text{therefore} \quad a_{jk} = a_{kj} \quad \text{for all } 1 \leq j, k \leq n \\
a_{11} &= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot A \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} > 0
\end{aligned}
$$

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- elimination without pivoting

$$A = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{pmatrix} \cdot \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{array}\right)$$

- for $j \ge k = 2, \cdots, n$

$$\begin{aligned} l_{j1} &= \frac{a_{j1}}{\mathbf{a}_{11}} \\ \mathbf{a}_{j,k} &\mathrel{-}= l_{j1}\, a_{1k} \end{aligned}$$

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$$\text{Define } \mathbf{l}_1 \stackrel{def}{=} \begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix}, \quad \widehat{\mathbf{A}} \stackrel{def}{=} \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \widehat{\mathbf{A}}^T, \quad \text{then}$$

$$A \;=\; \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{11}\mathbf{l}_1^T \\ & \widehat{\mathbf{A}} \end{pmatrix} \quad \Longleftarrow \text{Gaussian elimination}$$

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$$\begin{aligned} A &= \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{11}\mathbf{l}_1^T \\ & \widehat{\mathbf{A}} \end{pmatrix} && \Longleftarrow \text{Gaussian elimination} \\ &= \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \widehat{\mathbf{A}} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{l}_1^T \\ & I \end{pmatrix} && \Longleftarrow \text{due to symmetry of } A \end{aligned}$$

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$$l_{j1} = \frac{a_{j1}}{\mathbf{a}_{11}}$$

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$$\begin{aligned} A &= \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{11}\mathbf{l}_1^T \\ & \widehat{\mathbf{A}} \end{pmatrix} && \Longleftarrow \text{Gaussian elimination} \\ &= \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \widehat{\mathbf{A}} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{l}_1^T \\ & I \end{pmatrix} && \Longleftarrow \text{due to symmetry of } A \end{aligned}$$

next step: show $\widehat{\mathbf{A}}$ remains **SPD**

$$A = \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \widehat{\mathbf{A}} \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix}^T$$

- Let $\widehat{\mathbf{x}} \in \mathbf{R}^{n-1}$ be any non-zero vector. Must show $\widehat{\mathbf{x}}^T \cdot \widehat{\mathbf{A}}\widehat{\mathbf{x}} > 0$ for $\widehat{\mathbf{A}}$ to be **SPD**.
- Note that

$$\mathbf{x} \stackrel{def}{=} \begin{pmatrix} -\widehat{\mathbf{x}}^T\mathbf{l}_1 \\ \widehat{\mathbf{x}} \end{pmatrix} \in \mathbf{R}^n \quad \text{is non-zero} \implies \mathbf{x}^T \cdot A \cdot \mathbf{x} > 0$$

$$\begin{pmatrix} 0 \\ \widehat{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix}^T \cdot \begin{pmatrix} -\mathbf{l}_1^T\widehat{\mathbf{x}} \\ \widehat{\mathbf{x}} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \widehat{\mathbf{A}} \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix}^T$$

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- consequently,

$$\begin{aligned} \widehat{\mathbf{x}}^T \cdot \widehat{\mathbf{A}} \cdot \widehat{\mathbf{x}} &= \begin{pmatrix} 0 \\ \widehat{\mathbf{x}} \end{pmatrix}^T \cdot \begin{pmatrix} a_{11} & \\ & \widehat{\mathbf{A}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \widehat{\mathbf{x}} \end{pmatrix} \\ &= \begin{pmatrix} -\mathbf{l}_1^T \widehat{\mathbf{x}} \\ \widehat{\mathbf{x}} \end{pmatrix}^T \cdot \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \widehat{\mathbf{A}} \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I \end{pmatrix}^T \cdot \begin{pmatrix} -\mathbf{l}_1^T \widehat{\mathbf{x}} \\ \widehat{\mathbf{x}} \end{pmatrix} \\ &= \mathbf{x}^T \cdot A \cdot \mathbf{x} > 0 \end{aligned}$$

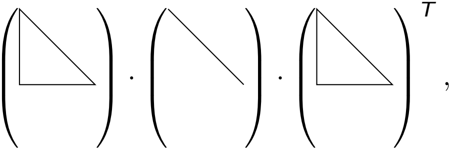
Major Cholesky, 1875 - 1918

- Invented Cholesky factorization for geodesic work
- Fell for his country (France) in WWI

Cholesky factorization for SPD matrix: $A = L\,D\,L^T$

- Cholesky for $n = 2$:

$$
\begin{aligned}
A &= \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix} \quad [A \text{ is symmetric}] \\
&= \begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{a}_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix}^T, \quad \left[l_{21} = \frac{a_{21}}{a_{11}} \right] \\
&\stackrel{def}{=} L \cdot D \cdot L^T =
\end{aligned}
$$



Cholesky factorization for SPD matrix: $A = L\,D\,L^T$

- Cholesky for $n \geq 3$:

$$A = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad [A \text{ is symmetric}]$$

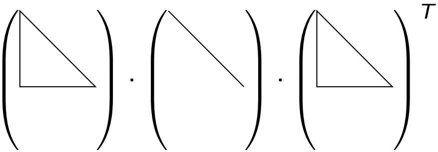
$$= \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & & & \\ & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ & \vdots & \ddots & \vdots \\ & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix}^T,$$

where $\mathbf{l} \stackrel{def}{=} \begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix}$.

Cholesky factorization for SPD matrix: $A = L\,D\,L^T$

- induction hypothesis

$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \widehat{\mathbf{L}}\widehat{\mathbf{D}}\widehat{\mathbf{L}}^T =$$



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- Cholesky factorization

$$\begin{aligned}
A &= \begin{pmatrix} 1 & \\ \mathbf{l} & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & I \end{pmatrix}^T, \\
&= \begin{pmatrix} 1 & \\ \mathbf{l} & I \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \widehat{\mathbf{L}}\widehat{\mathbf{D}}\widehat{\mathbf{L}}^T \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & I \end{pmatrix}^T, \\
&= \begin{pmatrix} 1 & \\ \mathbf{l} & \widehat{\mathbf{L}} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \widehat{\mathbf{D}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & \widehat{\mathbf{L}} \end{pmatrix}^T = \begin{pmatrix} \quad \end{pmatrix} \cdot \begin{pmatrix} \quad \end{pmatrix} \cdot \begin{pmatrix} \quad \end{pmatrix}^T.
\end{aligned}$$

Cholesky factorization $A = L\,D\,L^T$: example

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{pmatrix} = A^T.$$

$$A \;=\; \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & & \\ & 4 & 3 \\ & 3 & \frac{13}{4} \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix}^T$$

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&= \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & \\ & \begin{pmatrix} 1 & \\ \frac{3}{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \frac{3}{4} & 1 \end{pmatrix}^T \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix}^T \\
&= \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & \frac{3}{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & & \\ & 4 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & \frac{3}{4} & 1 \end{pmatrix}^T.
\end{aligned}$$

Cholesky factorization is a special LU factorization

- $A = L\,D\,L^T = L\,U$, with $U = D\,L^T =$
- only L need to be computed, saving half of the work in factorization.
- total cost: about $\frac{1}{3}n^3$ operations.

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$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} = \left(D^{\frac{1}{2}}\right)^2, \quad \text{for } D^{\frac{1}{2}} \stackrel{def}{=} \begin{pmatrix} d_1^{\frac{1}{2}} & & \\ & \ddots & \\ & & d_n^{\frac{1}{2}} \end{pmatrix}$$

$$A = L\left(D^{\frac{1}{2}}\right)^2 L^T = \left(L\,D^{\frac{1}{2}}\right)\left(L\,D^{\frac{1}{2}}\right)^T = \begin{pmatrix} \text{◺} \end{pmatrix} \cdot \begin{pmatrix} \text{◺} \end{pmatrix}^T$$

Example:
$$A = \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & \frac{3}{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & & \\ & 4 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & \frac{3}{4} & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 2 & & \\ -\frac{1}{2} & 2 & \\ \frac{1}{2} & \frac{3}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & & \\ -\frac{1}{2} & 2 & \\ \frac{1}{2} & \frac{3}{2} & 1 \end{pmatrix}^T$$

Review: Natural Splines equations in matrix form

Equations for spline coefficients $\{c_j\}_{j=1}^{n-1}$,

$$A \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix} = \mathbf{rhs},$$

where

$$A \stackrel{def}{=} \begin{pmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix}$$

$$\boxed{A \text{ is } \mathbf{SDD}, \mathbf{SPD}, \text{ and } \mathbf{tri\text{-}diagonal}}$$

DEFINITION: $A \in \mathbf{R}^{n \times n}$ is **tri-diagonal** if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

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$\boxed{\text{LU should GREATLY simplify since } A \text{ has so many zero entries}}$

Tri-diagonal LU factorization with $a_{jj} \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

Tri-diagonal LU factorization with $a_{jj} \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix} \quad \left(\text{set } l_{21} = \frac{a_{21}}{a_{11}} \text{ and } \mathbf{a}_{22} = a_{22} - l_{21}\, a_{12}\right)$$

$$= \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \cdot \left(\begin{array}{c|cccc} a_{11} & a_{12} & & & \\ \hline & \mathbf{a}_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

Tri-diagonal LU factorization with $a_{jj} \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix} \quad \left(\text{set } l_{21} = \frac{a_{21}}{a_{11}} \text{ and } \mathbf{a}_{22} = a_{22} - l_{21}\, a_{12}\right)$$

$$= \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \cdot \left(\begin{array}{c|cccc} a_{11} & a_{12} & & & \\ \hline & \mathbf{a}_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

- elimination = computing l_{21} and $\mathbf{a}_{22}$ (3 operations)

Tri-diagonal LU factorization with $a_{jj} \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix} \quad \left(\text{set } l_{21} = \frac{a_{21}}{a_{11}} \text{ and } \mathbf{a}_{22} = a_{22} - l_{21}\, a_{12}\right)$$

$$= \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \cdot \left(\begin{array}{c|cccc} a_{11} & a_{12} & & & \\ \hline & \mathbf{a}_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

- elimination = computing l_{21} and $\mathbf{a}_{22}$ (3 operations)
- Trailing matrix remains **tri-diagonal**

Recursively on all trailing matrices,

$$A = \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ & \ddots & \ddots & & \\ & & l_{n-1,n-2} & 1 & \\ & & & l_{n,n-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_{11} & a_{12} & & & \\ & \mathbf{a}_{22} & a_{23} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{a}_{n-1,n-1} & a_{n-1,n} \\ & & & & \mathbf{a}_{n,n} \end{pmatrix}$$

- for $j = 1, \cdots, n{-}1$

$$l_{j+1,j} = \frac{a_{j+1,j}}{\mathbf{a}_{jj}}$$
$$\mathbf{a}_{j+1,j+1} \mathrel{-}= l_{j+1,j}\, a_{j,j+1}$$

grand total: $3n$ operations, **if** $\mathbf{a}_{jj} \neq 0$ for all j

(non-singular) **tri-diagonal** LU factorization with partial pivoting (I)

- Assume $|a_{11}| \geq |a_{21}|$. Let $l_{21} = \frac{a_{21}}{a_{11}}$ and $\bar{\mathbf{a}}_{22} = a_{22} - l_{21}\, a_{12}$, then

$$
A = \left(\begin{array}{c|cccc}
1 & & & & \\
\hline
l_{21} & 1 & & & \\
 & & \ddots & & \\
 & & & 1 & \\
 & & & & 1
\end{array}\right) \cdot \left(\begin{array}{c|cccc}
a_{11} & a_{12} & & & \\
\hline
 & \bar{\mathbf{a}}_{22} & \mathbf{a}_{23} & & \\
 & \ddots & \ddots & \ddots & \\
 & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\
 & & & a_{n,n-1} & a_{n,n}
\end{array}\right)
$$

$\boxed{\text{elimination only means computing } l_{21} \text{ and } \bar{\mathbf{a}}_{22}}$

(non-singular) **tri-diagonal** LU factorization with partial pivoting (I)

- Assume $|a_{11}| \geq |a_{21}|$. Let $l_{21} = \frac{a_{21}}{a_{11}}$ and $\bar{\mathbf{a}}_{22} = a_{22} - l_{21}\, a_{12}$, then

$$A = \left(\begin{array}{c|cccc} 1 & & & & \\ \hline l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|cccc} a_{11} & a_{12} & & & \\ \hline & \bar{\mathbf{a}}_{22} & \mathbf{a}_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

$$\boxed{\text{elimination only means computing } l_{21} \text{ and } \bar{\mathbf{a}}_{22}}$$

$$\text{Trailing matrix} \quad \begin{pmatrix} \bar{\mathbf{a}}_{22} & \mathbf{a}_{23} & & \\ \ddots & \ddots & \ddots & \\ & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & a_{n,n-1} & a_{n,n} \end{pmatrix} \quad \textbf{tri-diagonal}$$

(non-singular) **tri-diagonal** LU factorization with partial pivoting (I)

- Assume $|a_{11}| \geq |a_{21}|$. Let $l_{21} = \frac{a_{21}}{a_{11}}$ and $\bar{\mathbf{a}}_{22} = a_{22} - l_{21}\, a_{12}$, then

$$A \quad = \left(\begin{array}{c|cccc} 1 & & & & \\ \hline l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|cccc} a_{11} & a_{12} & & & \\ \hline & \bar{\mathbf{a}}_{22} & \mathbf{a}_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

$\boxed{\text{elimination only means computing } l_{21} \text{ and } \bar{\mathbf{a}}_{22}}$

Trailing matrix $\begin{pmatrix} \bar{\mathbf{a}}_{22} & \mathbf{a}_{23} & & \\ \ddots & \ddots & \ddots & \\ & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & a_{n,n-1} & a_{n,n} \end{pmatrix}$ **tri-diagonal**

- Otherwise $|a_{21}| > |a_{11}|$. Let $l_{21} = \frac{a_{11}}{a_{21}}$ and $(\bar{\mathbf{a}}_{22},\, \bar{\mathbf{a}}_{23}) = (a_{12} - l_{21}\, a_{22},\, -l_{21}\, a_{23})$

$$P_{2,1} \cdot A = \left(\begin{array}{c|cccc} 1 & & & & \\ \hline l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|cccc} a_{21} & a_{22} & a_{23} & & \\ \hline & \bar{\mathbf{a}}_{22} & \bar{\mathbf{a}}_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

$\boxed{\text{elimination only means computing } l_{21},\ \bar{\mathbf{a}}_{22} \text{ and } \bar{\mathbf{a}}_{23}.\ \text{Trailing matrix } \textbf{tri-diagonal}}$

(non-singular) **tri-diagonal** LU factorization with partial pivoting (I)

- Assume $|a_{11}| \geq |a_{21}|$. Let $l_{21} = \frac{a_{21}}{a_{11}}$ and $\bar{\mathbf{a}}_{22} = a_{22} - l_{21}\, a_{12}$, then

$$A \quad = \quad \left(\begin{array}{c|cccc} 1 & & & & \\ \hline l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|cccc} a_{11} & a_{12} & & & \\ \hline & \bar{\mathbf{a}}_{22} & \mathbf{a}_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

$$\boxed{\text{elimination only means computing } l_{21} \text{ and } \bar{\mathbf{a}}_{22}}$$

$$\text{Trailing matrix} \quad \begin{pmatrix} \bar{\mathbf{a}}_{22} & \mathbf{a}_{23} & & \\ \ddots & \ddots & \ddots & \\ & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & a_{n,n-1} & a_{n,n} \end{pmatrix} \quad \textbf{tri-diagonal}$$

- Otherwise $|a_{21}| > |a_{11}|$. Let $l_{21} = \frac{a_{11}}{a_{21}}$ and $(\bar{\mathbf{a}}_{22},\, \bar{\mathbf{a}}_{23}) = (a_{12} - l_{21}\, a_{22},\, -l_{21}\, a_{23})$

$$P_{2,1} \cdot A = \left(\begin{array}{c|cccc} 1 & & & & \\ \hline l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|cccc} a_{21} & a_{22} & a_{23} & & \\ \hline & \bar{\mathbf{a}}_{22} & \bar{\mathbf{a}}_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

$$\boxed{\text{elimination only means computing } l_{21},\ \bar{\mathbf{a}}_{22} \text{ and } \bar{\mathbf{a}}_{23}.\ \text{Trailing matrix } \textbf{tri-diagonal}}$$

tri-diagonal LU factorization with partial pivoting (II)

$$\text{Recap:} \quad A = \left(\begin{array}{c|ccccc} a_{11} & a_{12} & & & & \\ \hline a_{21} & a_{22} & a_{23} & & & \\ & \ddots & \ddots & \ddots & & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

tri-diagonal LU factorization with partial pivoting (II)

$$\text{Recap:} \quad A = \left(\begin{array}{c|cccc} a_{11} & a_{12} & & & \\ \hline a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

$$\mathbf{P}_1 \cdot A = \left(\begin{array}{c|cccc} 1 & & & & \\ \hline l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|cccc} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} & & \\ \hline & \mathbf{a}_{22} & \mathbf{a}_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{array}\right)$$

$$\boxed{\text{cost: at most 4 operations per elimination step, with or without pivoting}}$$

$$\text{Recursively:} \quad \widehat{\mathbf{P}} \cdot \begin{pmatrix} \mathbf{a}_{22} & \mathbf{a}_{23} & & \\ \ddots & \ddots & \ddots & \\ & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & a_{n,n-1} & a_{n,n} \end{pmatrix} = \widehat{\mathbf{L}} \cdot \widehat{\mathbf{U}}$$

- $\widehat{\mathbf{U}}$: upper triangular with bandwidth at most 3.
- $\widehat{\mathbf{L}}$: unit lower triangular with one non-zero in each column below diagonal.

tri-diagonal LU factorization with partial pivoting (III)

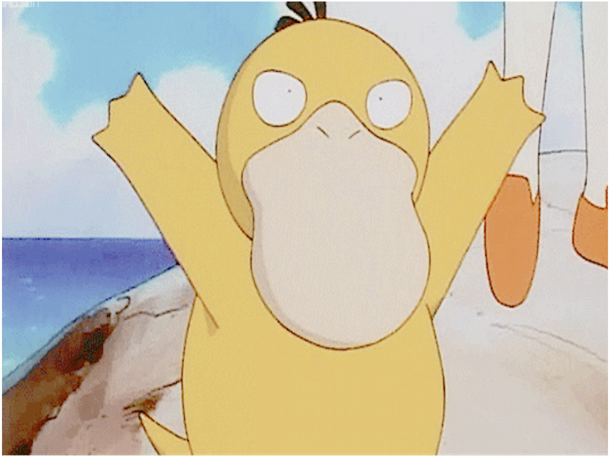
Together: $\mathbf{P}_1 \cdot A = \left(\begin{array}{c|cccc} 1 & & & & \\ \hline l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbf{a}_{11} & (\mathbf{a}_{12}\ \ \mathbf{a}_{13}\ \ ) \\ \hline & \begin{pmatrix} \mathbf{a}_{22} & \mathbf{a}_{23} & & \\ \ddots & \ddots & \ddots & \\ & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & a_{n,n-1} & a_{n,n} \end{pmatrix} \end{array}\right)$

$$\underbrace{\left(\begin{pmatrix} 1 & \\ & \widehat{\mathbf{P}} \end{pmatrix} \cdot \mathbf{P}_1\right)}_{P} \cdot A = \left(\begin{array}{c|c} 1 & \\ \hline \mathbf{l}_1 & I \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbf{a}_{11} & (\mathbf{a}_{12}\ \ \mathbf{a}_{13}\ \ ) \\ \hline & \widehat{\mathbf{L}} \cdot \widehat{\mathbf{U}} \end{array}\right), \quad \left(\mathbf{l}_1 = \widehat{\mathbf{P}} \begin{pmatrix} l_{21} \\ \\ \end{pmatrix}\right)$$

$$= \left(\begin{array}{c|c} 1 & \\ \hline \mathbf{l}_1 & \widehat{\mathbf{L}} \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbf{a}_{11} & (\mathbf{a}_{12}\ \ \mathbf{a}_{13}\ \ ) \\ \hline & \widehat{\mathbf{U}} \end{array}\right) \overset{def}{=} L \cdot U$$

$\boxed{\text{total cost: at most } 4\,n \text{ operations and } n \text{ comparisons}}$

And that is it.



Final Exam, Fri. Dec. 15, 11:30 am - 2:30 pm, Li Ka Shing 245

- Covers every section in Chapters 1-6, excluding
  - false position method in Section 2.3,
  - Section 5.8,
  - scaled partial or complete pivoting in Section 6.2.

Final Exam, Fri. Dec. 15, 11:30 am - 2:30 pm, Li Ka Shing 245

- Covers every section in Chapters 1-6, excluding
  - false position method in Section 2.3,
  - Section 5.8,
  - scaled partial or complete pivoting in Section 6.2.
- One question each for Chs. 2-3; two each for Chs. 4-6.

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- Problems mostly from, or are similar to those in the textbook.

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- Honor DSP requests.

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  - false position method in Section 2.3,
  - Section 5.8,
  - scaled partial or complete pivoting in Section 6.2.
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- Problems mostly from, or are similar to those in the textbook.
- No need to simplify answers unless told to.
- Sample exams will be posted.
- Honor DSP requests.
- Cheat sheet: one side on one page

Final Exam, Fri. Dec. 15, 11:30 am - 2:30 pm, Li Ka Shing 245

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  - false position method in Section 2.3,
  - Section 5.8,
  - scaled partial or complete pivoting in Section 6.2.
- One question each for Chs. 2-3; two each for Chs. 4-6.
- Problems mostly from, or are similar to those in the textbook.
- No need to simplify answers unless told to.
- Sample exams will be posted.
- Honor DSP requests.
- Cheat sheet: one side on one page
- OH: MW 1:30-3:00PM in 861 Evans or by appointment.

Ch. 2: Question

$$\textit{Let} \quad f(x) = x^2 - a \quad \textit{for} \quad a > 0$$

Apply Secant method to solve the equation $f(x) = 0$ *with initial guesses* $x_0 > x_1 > \sqrt{a}$*:*

$$x_{k+1} = x_k - \frac{f(x_k)\ (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad \textit{for} \quad k = 1, 2, \cdots .$$

Ch. 2: Question

$$\textit{Let} \quad f(x) = x^2 - a \quad \textit{for} \quad a > 0$$

Apply Secant method to solve the equation $f(x) = 0$ *with initial guesses* $x_0 > x_1 > \sqrt{a}$*:*

$x_{k+1} = x_k - \dfrac{f(x_k)\ (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad$ *for* $\quad k = 1, 2, \cdots.$ *Show that the iteration always converges*

Ch. 2: Question

$$\textit{Let} \quad f(x) = x^2 - a \quad \textit{for} \quad a > 0$$

Apply Secant method to solve the equation $f(x) = 0$ with initial guesses $x_0 > x_1 > \sqrt{a}$:

$$x_{k+1} = x_k - \frac{f(x_k)\ (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad \textit{for} \quad k = 1, 2, \cdots.$$ *Show that the iteration always converges*

Solution: Since $x_0 > x_1 > \sqrt{a} > 0$, the denominator for $k = 1$

$$f(x_k) - f(x_{k-1}) = x_k^2 - x_{k-1}^2 = (x_k - x_{k-1})\ (x_k + x_{k-1}) \neq 0$$

thus x_2 is defined. In general, for $k \geq 1$,

$$x_{k+1} = x_k - \frac{\left(x_k^2 - a\right)\ (x_k - x_{k-1})}{(x_k - x_{k-1})\ (x_k + x_{k-1})} = x_k - \frac{\left(x_k^2 - a\right)}{x_k + x_{k-1}} = \frac{x_k\, x_{k-1} + a}{x_k + x_{k-1}}$$

It follows that $$x_{k+1} - \sqrt{a} = \frac{x_k\, x_{k-1} + a - \sqrt{a}\,(x_k + x_{k-1})}{x_k + x_{k-1}} = \frac{(x_k - \sqrt{a})\ (x_{k-1} - \sqrt{a})}{x_k + x_{k-1}} > 0 \qquad (1)$$

To show Secant iteration is defined, we must show $x_{k+1} \neq x_k$ for all k. However, if $x_{k+1} = x_k$ for some k, then

$$x_k = \frac{x_k\, x_{k-1} + a}{x_k + x_{k-1}},$$ then

Ch. 2: Question

$$\textit{Let} \quad f(x) = x^2 - a \quad \textit{for} \quad a > 0$$

Apply Secant method to solve the equation $f(x) = 0$ with initial guesses $x_0 > x_1 > \sqrt{a}$:

$$x_{k+1} = x_k - \frac{f(x_k)\ (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad \textit{for} \quad k = 1, 2, \cdots. \quad \textit{Show that the iteration always converges}$$

Solution: Since $x_0 > x_1 > \sqrt{a} > 0$, the denominator for $k = 1$

$$f(x_k) - f(x_{k-1}) = x_k^2 - x_{k-1}^2 = (x_k - x_{k-1})\,(x_k + x_{k-1}) \neq 0$$

thus x_2 is defined. In general, for $k \geq 1$,

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It follows that $\quad x_{k+1} - \sqrt{a} = \dfrac{x_k\, x_{k-1} + a - \sqrt{a}\,(x_k + x_{k-1})}{x_k + x_{k-1}} = \dfrac{(x_k - \sqrt{a})\,(x_{k-1} - \sqrt{a})}{x_k + x_{k-1}} > 0 \qquad (1)$

To show Secant iteration is defined, we must show $x_{k+1} \neq x_k$ for all k. However, if $x_{k+1} = x_k$ for some k, then

$$x_k = \frac{x_k\, x_{k-1} + a}{x_k + x_{k-1}}, \quad \text{then} \quad x_k\ (x_k + x_{k-1}) = x_k\, x_{k-1} + a \implies x_k^2 = a$$

Ch. 2: Question

$$\textit{Let} \quad f(x) = x^2 - a \quad \textit{for} \quad a > 0$$

Apply Secant method to solve the equation $f(x) = 0$ with initial guesses $x_0 > x_1 > \sqrt{a}$:

$$x_{k+1} = x_k - \frac{f(x_k)\ (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad \textit{for} \quad k = 1, 2, \cdots. \quad \textit{Show that the iteration always converges}$$

Solution: Since $x_0 > x_1 > \sqrt{a} > 0$, the denominator for $k = 1$

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thus x_2 is defined. In general, for $k \geq 1$,

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It follows that $$x_{k+1} - \sqrt{a} = \frac{x_k\, x_{k-1} + a - \sqrt{a}\,(x_k + x_{k-1})}{x_k + x_{k-1}} = \frac{(x_k - \sqrt{a})\ (x_{k-1} - \sqrt{a})}{x_k + x_{k-1}} > 0 \qquad (1)$$

To show Secant iteration is defined, we must show $x_{k+1} \neq x_k$ for all k. However, if $x_{k+1} = x_k$ for some k, then

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Ch. 2: Question

$$\textit{Let} \quad f(x) = x^2 - a \quad \textit{for} \quad a > 0$$

Apply Secant method to solve the equation $f(x) = 0$ *with initial guesses* $x_0 > x_1 > \sqrt{a}$*:*

$$x_{k+1} = x_k - \frac{f(x_k)\ (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad \textit{for} \quad k = 1, 2, \cdots.$$ *Show that the iteration always converges*

Solution: Since $x_0 > x_1 > \sqrt{a} > 0$, the denominator for $k = 1$

$$f(x_k) - f(x_{k-1}) = x_k^2 - x_{k-1}^2 = (x_k - x_{k-1})\,(x_k + x_{k-1}) \neq 0$$

thus x_2 is defined. In general, for $k \geq 1$,

$$x_{k+1} = x_k - \frac{\left(x_k^2 - a\right)\ (x_k - x_{k-1})}{(x_k - x_{k-1})\,(x_k + x_{k-1})} = x_k - \frac{\left(x_k^2 - a\right)}{x_k + x_{k-1}} = \frac{x_k\, x_{k-1} + a}{x_k + x_{k-1}}$$

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From (1), $x_k > \sqrt{a}$ for all k, and therefore (1) implies for all k, $\quad 0 < x_{k+1} - \sqrt{a} < x_k - \sqrt{a}$

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From (1), $x_k > \sqrt{a}$ for all k, and therefore (1) implies for all k, $0 < x_{k+1} - \sqrt{a} < x_k - \sqrt{a}$

Therefore the sequence $\{x_k\}_{k=0}^{\infty}$ monotonically decreases with a lower bound. It must converge.

Ch. 3: Question

Let polynomial

$$P_n(x) \quad \stackrel{def}{=} \quad a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_{n+1}(x - x_0)^2 (x - x_1) \cdots (x - x_{n-1}) \quad (1)$$

interpolate function $f(x)$ at distinct points $x_0,\ x_1, \cdots, x_n$ such that

$P_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$ *and that* $P_n'(x_0) = f'(x_0)$. **Show that** $a_2 = f[x_0, x_0, x_1]$.

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interpolate function $f(x)$ *at distinct points* $x_0,\ x_1, \cdots, x_n$ *such that*

$P_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$ *and that* $P_n'(x_0) = f'(x_0)$. **Show that** $a_2 = f[x_0, x_0, x_1]$.

SOLUTION: Let $x = x_0$ in equation (1) to reach $a_0 = P_n(x_0) = f(x_0)$. Additionally,

$$\frac{P_n(x) - f(x_0)}{x - x_0} = a_1 + a_2(x - x_0) + \cdots + a_{n+1}(x - x_0)\,(x - x_1)\cdots(x - x_{n-1})$$

Let $x \longrightarrow x_0$ gives $a_1 = P_n'(x_0) = f'(x_0)$. Now let $x = x_1$, we get

$$f[x_0, x_1] = f[x_0, x_0] + a_2(x_1 - x_0)$$

which implies

$$a_2 = f[x_0, x_0, x_1].$$

Ch. 4: Question

Let $0 < \alpha_1 < \alpha_2 < 1$, and let $N(h)$ be an approximation to a value M for every $h > 0$ such that

$$M = N(h) + k_1 h + k_2 h^2 + k_3 h^3 + \cdots .$$

Find μ_1 and μ_2 such that the expression $\frac{N(h)+\mu_1 N(\alpha_1 h)+\mu_2 N(\alpha_2 h)}{1+\mu_1+\mu_2}$ is an $O(h^3)$ approximation to M.

Ch. 4: Question

Let $0 < \alpha_1 < \alpha_2 < 1$, and let $N(h)$ be an approximation to a value M for every $h > 0$ such that

$$M = N(h) + k_1\, h + k_2\, h^2 + k_3\, h^3 + \cdots .$$

Find μ_1 and μ_2 such that the expression $\frac{N(h)+\mu_1\, N(\alpha_1\, h)+\mu_2\, N(\alpha_2\, h)}{1+\mu_1+\mu_2}$ is an $O(h^3)$ approximation to M.

SOLUTION: We write

$$\begin{aligned} M &= N(h) + k_1\, h + k_2\, h^2 + k_3\, h^3 + \cdots \\ M &= N(\alpha_1\, h) + k_1\, (\alpha_1\, h) + k_2\, (\alpha_1\, h)^2 + k_3\, (\alpha_1\, h)^3 + \cdots \\ M &= N(\alpha_2\, h) + k_1\, (\alpha_2\, h) + k_2\, (\alpha_2\, h)^2 + k_3\, (\alpha_2\, h)^3 + \cdots \end{aligned}$$

It follows that
$$\begin{aligned} M &= \frac{N(h) + \mu_1\, N(\alpha_1\, h) + \mu_2\, N(\alpha_2\, h)}{1 + \mu_1 + \mu_2} \\ &\quad + \frac{1 + \mu_1\, \alpha_1 + \mu_2\, \alpha_2}{1 + \mu_1 + \mu_2}\, k_1 h + \frac{1 + \mu_1\, \alpha_1^2 + \mu_2\, \alpha_2^2}{1 + \mu_1 + \mu_2}\, k_2 h^2 + O(h^3) \end{aligned}$$

Thus, we must choose μ_1 and μ_2 such that

$$1 + \mu_1\, \alpha_1 + \mu_2\, \alpha_2 = 0, \qquad 1 + \mu_1\, \alpha_1^2 + \mu_2\, \alpha_2^2 = 0$$

which leads to $\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = -\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Ch. 5: Question

Consider an initial value ODE

$$y' = f(t, y), \quad y(t_0) = Y_0, \quad (1)$$

and a method for solving (1)

$$w_{j+1} = w_j + h\left(\alpha\, f(t_{j+1}, w_{j+1}) + \beta\, f(t_j, w_j) + \gamma\, f(t_{j-1}, w_{j-1})\right), \quad (2)$$

for $j = 0,\ 1,\ 2, \cdots$.

Find relations on coefficients $\alpha,\ \beta,\ \gamma$ *so that (2) is a second-order method for solving (1).*

Ch. 5: Question

Consider an initial value ODE

$$y' = f(t, y), \quad y(t_0) = Y_0, \quad (1)$$

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for $j = 0,\ 1,\ 2, \cdots$.

Find relations on coefficients $\alpha,\ \beta,\ \gamma$ *so that (2) is a second-order method for solving (1).*

SOLUTION: By definition, $y'(t_j) = f(t_j, y(t_j))$, $y''(t_j) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\, f$

$$\begin{aligned}
\textbf{LTE} \quad &= \quad \frac{y(t_{j+1}) - y(t_j)}{h} - \left(\alpha\, f(t_{j+1}, y(t_{j+1})) + \beta\, f(t_j, y(t_j)) + \gamma\, f(t_{j-1}, y(t_{j-1}))\right) \\
&= \quad y'(t_j) + \frac{h}{2} y''(t_j) - (\alpha + \beta + \gamma)\, f(t_j, y(t_j)) - \left(\alpha\, h\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\, f\right) - \gamma\, h\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\, f\right)\right) + O(h^2)
\end{aligned}$$

Thus the relations are

$$\alpha + \beta + \gamma = 1, \qquad \alpha - \gamma = \frac{1}{2}$$

Ch. 6: Question

Let $h_j > 0$ *for* $j = 0, 1, \cdots, n$*, and let*

$$A \stackrel{def}{=} \begin{pmatrix} 2(h_0 + h_1) & h_1 & & & \\ h_1 & 2(h_1 + h_2) & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ & & & h_{n-1} & 2(h_{n-1} + h_n) \end{pmatrix}.$$

Show that A *is symmetric positive definite.*

Ch. 6: Question

Let $h_j > 0$ for $j = 0, 1, \cdots, n$, and let

$$A \stackrel{def}{=} \begin{pmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & h_{n-1} & 2(h_{n-1}+h_n) \end{pmatrix}.$$

Show that A is symmetric positive definite.

SOLUTION: A is symmetric. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ non-zero. Then

$$\begin{aligned} \mathbf{x}^T A \mathbf{x} &= 2\left(\sum_{j=1}^{n} x_j^2 (h_j + h_{j-1})\right) + 2\left(\sum_{j=2}^{n} x_j x_{j-1} h_{j-1}\right) \\ &= 2\left(\sum_{j=1}^{n} x_j^2 (h_j + h_{j-1})\right) + \left(\sum_{j=2}^{n} \left((x_j + x_{j-1})^2 - x_j^2 - x_{j-1}^2\right) h_{j-1}\right) \\ &= \left(\sum_{j=1}^{n} x_j^2 (h_j + h_{j-1})\right) + x_n^2\, h_n + x_1^2\, h_0 + \left(\sum_{j=2}^{n} (x_j + x_{j-1})^2\, h_{j-1}\right) \\ &> 0 \end{aligned}$$