Game Theory Worksheet #5, 02/22/2021

1. Find the safety strategies and a Nash equilibrium in the following general-sum games:

(a) $\begin{pmatrix} (1,1) & (4,2) \\ (2,4) & (3,3) \end{pmatrix}$

Solution. Let's first find the safety strategies for both players. Note that the matrices describing the payoffs to Players I and II are A and B respectively, where

$$A = B^T = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

Note that, in A, the cell containing a 2 is a saddle point, so the strategy $(0,1)^{\mathrm{T}}$ is a safety strategy for Player I. Simiarly, the cell containing a 2 is a saddle point in B (note that, now, it is the top-right cell), so $(0,1)^{\mathrm{T}}$ is also a safety strategy for Player II.

For the Nash equilibria, let's consider equalization: That is, let's assume that a pair of strategies in Nash equilibrium is $(x, 1-x)$ and $(y, 1-y)$ for $0 < x < 1$ and $0 < y < 1$. (We will later check if the solution we come to is fully mixed or not.) If Player II plays the fixed strategy $(y, 1-y)$, what are the possible outcomes for Player I? If she plays the top row, her payoff is $y + 4(1-y)$, and if she pays the bottom row, her payoff is $2y + 3(1-y)$. Now we notice that Player I has no incentive to deviate from this strategy as long as these values are equal, which occurs exactly when $y = \frac{1}{2}$. Similarly we can ask, if Player I plays the fixed strategy $(x, 1-x)$ what are the possible outcomes for Player II? If he plays the left column, his payoff is $x + 4(1-x)$, and if he plays the right column, his payoff is $2x + 3(1-x)$. Again, he has no incentive to deviate as long as these are equal which occurs at $x = \frac{1}{2}$. Since these in fact satisfy $0 < x < 1$ and $0 < y < 1$, we see that $(\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$ and $(\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$ are a pair of Nash equilibria.

Alternatively, we could identify a pair of pure strategies which constitute a Nash equilibrium. Indeed, consider the bottom-left cell $(2,4)$. If Player II is committed to playing his current strategy of the left column, then Player I has no incentive to change her strategy by switching from the bottom row to the top row, since this will reduce her payoff from 2 to 1. Simiarly, if Player I is committed to playing her current strategy of the bottom row, then Player II has no incentive to change his strategy by switching from the left column to the right column, since this will redice his payoff from 4 to 3. Therefore, the pair $(0,1)^{\mathrm{T}}$ and $(1,0)^{\mathrm{T}}$ is a Nash equilibrium. (Symmetrically, $(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$ is also a Nash equilibrium.) □

(b) $\begin{pmatrix} (1,2) & (4,3) & (4,1) \\ (3,3) & (5,4) & (4,5) \\ (2,5) & (3,5) & (5,6) \end{pmatrix}$

Solution. Write the matrix A of payoffs to Player I as

$$\begin{pmatrix} 1 & 4 & 4 \\ \underline{3} & 5 & 4 \\ 2 & 3 & 5 \end{pmatrix},$$

and notice that the underlined cell is a saddle point. Thus, $(0,1,0)^{\mathrm{T}}$ is a safety strategy for Player I.

For Player II, we also write the payoff matrix and identify a saddle point:

$$\begin{pmatrix} 2 & \underline{3} & 1 \\ 3 & 4 & 5 \\ 5 & 5 & 6 \end{pmatrix}.$$

Therefore, $(0,1,0)^{\mathrm{T}}$ is also a safety strategy for Player II.

In the full matrix, we notice that the bottom-right cell is a Nash equilibrium:

$$\begin{pmatrix} (1,2) & (4,3) & (4,1) \\ (3,3) & (5,4) & (4,5) \\ (2,5) & (3,5) & \underline{(5,6)} \end{pmatrix}.$$

If Player II commits to playing his strategy of the third column, then Player I will have no incentive to change her strategy to another row, since her payoff would reduce from 6 to 5 or 1. Similarly, if Player I commits to playing her strategy of the third row, then Player II will have no incentive to change his strategy to another column, since his payoff would reduce from 6 to 5. Therefore, the pair of $(0,0,1)^{\mathrm{T}}$ and $(0,0,1)^{\mathrm{T}}$ is a Nash equilibrium. □

(c) $\begin{pmatrix} (1,1) & (3,0) & (2,1) \\ (2,0) & (1,3) & (3,0) \\ (3,1) & (2,0) & (1,1) \end{pmatrix}$

Solution. Let's first look for safety strategies in the matrix A, which is

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

There are no saddle points and no row or column is dominated by any other, so maybe it's a good idea to look for a pair of fully mixed strategies via equalization. This leads us to the equalization equations $x_1 + 2x_2 + 3x_3 = 3x_1 + x_2 + 2x_3 = 2x_1 + 3x_2 + x_1$ for Player I, which we see can be satisfied by $x_1 = x_2 = x_3 = \frac{1}{3}$. Setting up equalization for Player II yields the same equations, and hence to the solution $y_1 = y_2 = y_3 = \frac{1}{3}$ as well. Since these are both fully mixed, equalization indeed yields a safety strategy. Thus, a safety strategy for Player I is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{\mathrm{T}}$.

Similarly, the matrix B of payoffs to Player II is just

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Again, there are no saddle points and no pure strategy dominates any other, so we'll look for a fully mixed equalizing strategy. Setting up the equalization equations $y_1 + y_3 = 3y + 2$ and solving them yields the strategy $(\frac{3}{8}, \frac{2}{8}, \frac{3}{8})^{\mathrm{T}}$. Since the matrix B is symmetric, we know that the pair $x = y = (\frac{3}{8}, \frac{2}{8}, \frac{3}{8})^{\mathrm{T}}$ is optimal for the zero-sum game corresponding to B, hence this is a safety strategy for Player II.

Now let's check if there exists a fully mixed Nash equilibrium in the main game. By equalization, we know that, if Player I has a fully mixed strategy, then Player II must be indifferent between his possible moves. Hence, Player I's optimal strategy $\mathbf{x} = (x_1, x_2, x_3) \in \Delta_3$ must satisfy $x_1 + 2x_2 + 3x_3 = 3x_1 + x_2 + 2x_3 = 2x_1 + 3x_2 + x_1$. This was the same equation we solved in the first part, so we know it is satisfied by the fully mixed strategy $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{\mathrm{T}}$. Similarly, we know that, if Player II has a fully mixed strategy, then Player I must be indifferent between her possible moves. Hence Player II's optimal strategy $\mathbf{y} = (y_1, y_2, y_3) \in \Delta_3$ must satisfy $y_1 + y_3 = 3y_2$, which we know to be solved by $\mathbf{y} = (\frac{3}{8}, \frac{2}{8}, \frac{3}{8})^{\mathrm{T}}$.

Therefore, the pair $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{\mathrm{T}}, (\frac{3}{8}, \frac{2}{8}, \frac{3}{8})^{\mathrm{T}}$ is a Nash equilibrium. □

2. There are 3 tenants in a house. Each tenant can either participate in cleaning the kitchen or not. The following holds:

- If someone participates, then the kitchen becomes clean, which results in 5 units of utility for everyone.
- Cleaning takes 6 units of utility. This payment is distributed equally between all the participants.
- If a tenant doesn't participate, but both her housemates do, then the tenant feels ashamed, which results in her losing of 3 units of utility.

Find all the symmetric Nash equilibria in this game.

Solution. Usually, games with more than two players can be hard to understand, but our life is made easier by focusing only on symmetric Nash equilibria. That is, we assume that every tenant participates in the cleaning with probability p and does not participate with probability $1 - p$, and that tenant's choices are independent of the others'.

Without loss of generality, let's focus on Player I. If this triple of strategies is a Nash equilibrium and $0 < p < 1$, then we know that equalization guarantees that Player I will be indifferent about her two possible moves. So let's assume that these hypotheses are true and then check whether it yields a value of p with $0 < p < 1$.

In order to do this, let's form the following table which describes Player I's payoff in each case, and the probability of each case:

	# of other tenants who participate	0	1	2
Player I action	participate	$5 - 6 = -1$	$5 - \frac{6}{3} = 3$	$5 - \frac{6}{2} = 2$
	not participate	0	5	$5 - 3 = 2$
	probability	$(1-p)^2$	$2p(1-p)$	p^2

From here we see that Player I's payoff for participating is

$$-1 \cdot (1-p)^2 + 2 \cdot 2p(1-p) + 3 \cdot p^2 = -1 + 6p - 2p^2.$$

Similarly, Player I's payoff for not participating is

$$0 \cdot (1-p)^2 + 5 \cdot 2p(1-p) + 2 \cdot p^2 = 10p - 8p^2.$$

Setting these equal and rearranging slightly yields the quadratic equation $6p^2 - 4p - 1 = 0$, which has solutions

$$p = \frac{4 \pm \sqrt{(-4)^2 - (-4)6}}{2 \cdot 6} = \frac{1}{3} \pm \frac{\sqrt{10}}{6}.$$

Now check that one of these solutions is negative and hence cannot correspond to a probability. The other solution is

$$p_* = \frac{1}{3} + \frac{\sqrt{10}}{6} \approx 0.8604$$

which indeed lies between 0 and 1. $\square$