FUNCTIONS * RELATIONS
OPERATIONS * OPERATIONS
RELATIONS * FUNCTIONS

Mariusz Wodzicki

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1 Preliminaries

1.1 The language of functions

1.1.1 Mathematical structures

Modern Mathematics is concerned with *mathematical structures*. A "mathematical structure" consists of one or more sets equipped with data of certain type.

This informal initial definition already covers practically all fundamental types of structures that a mathematician encounters on a daily basis.

1.1.2 The concept of a function

An example of a mathematical structure is provided by the familiar concept of a function. A function of n variables consists of

- a list of n sets
$$X_1, \dots, X_n \tag{1}$$
- a set Y
- an assignment
$$x_1, \dots, x_n \longmapsto y \tag{2}$$
that assigns a *single* element y of set Y to *every* list $x_1, \dots, x_n$ such that
$$x_1 \in X_1, \dots, x_n \in X_n . \tag{3}$$

1.1.3 The domain of a function

The list of sets, (1), is called the *domain* of the function. We shall also call it the *source-list* and will refer to n as the *length* of that list.

1.1.4 The antidomain of a function

The set Y is called the *antidomain* of the function. We shall also refer to it as the *target*.

1.1.5 The argument-list and the value of a function

We shall refer to $x_1, \dots, x_n$ satisfying Condition (3) as the *argument-list*. The single element $y \in Y$ that is assigned to it is then called the *value* of the function on that particular argument-list.

If the *name* of the function is, say, f, its value on the list $x_1, \dots, x_n$ is denoted
$$f(x_1, \dots, x_n) \tag{4}$$

1.1.6 The arrow representation of a function

The symbolic representation of a function

$$f : X_1, \dots, X_n \longrightarrow Y \tag{5}$$

at a glance supplies the following information: *the function's name*, often represented by a symbol, its domain, and its target. In (5) the name of the function is 'f', the domain is the list of sets $X_1, \dots, X_n$, and the target is the set denoted Y.

It is often more convenient to place the name of a function above the arrow representating the function

$$X_1, \dots, X_n \xrightarrow{\quad f \quad} Y \ .$$

1.1.7 Equality of functions

Two functions are declared to be *equal* if

- *their domains are equal,*
- *their targets are equal,*
- *and their assignments are equal.*

In particular, a function

$$V_1, \dots, V_m \xrightarrow{\quad f \quad} W$$

can be equal to a function

$$X_1, \dots, X_n \xrightarrow{\quad g \quad} Y$$

only when

$$m = n, \qquad V_1 = X_1, \ \dots, \ V_m = X_m \ , \qquad \text{and} \qquad W = Y.$$

1.1.8 Functions of zero variables

When $n = 0$, the domain of a function is the empty list of sets. The arrow representation of such a function would be thus

$$\xrightarrow{\quad f \quad} Y \tag{6}$$

There is only one argument list in this case, namely the empty list. The function assigns to it a single element $y \in Y$. In particular,

$$f \quad \longleftrightarrow \quad \text{the value of } f \text{ on the empty argument-list}$$

defines a canonical identification between functions (6) and elements of the target-set Y.

1.1.9 Functions constant in the i-th variable

If the value (4) does not depend on x_i, we say that f is *constant in i-th variable.*

1.1.10

We shall denote the set of all functions (5) by

$$\mathrm{Funct}(X_1, \dots, X_n\,; Y) \tag{7}$$

or

$$Y^{X_1,\dots,X_n}. \tag{8}$$

1.1.11 Lists with omitted entries

Since lists with certain entries having been omitted are frequently encountered in Mathemtics, we have the notation to denote such lists. For example,

$$x_1, \dots, \hat{x}_i, \dots, x_n \tag{9}$$

stands for the list of length $n-1$

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$$

while

$$x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n \tag{10}$$

stands for the list of length $n-2$

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n,$$

and so on.

1.1.12 Freezing a variable in a function of n-variables

For any $1 \leq i \leq n$ and any $a \in X_i$, assignment

$$x_1, \dots, \hat{x}_i, \dots, x_n \;\longmapsto\; f(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n)$$

defines a function of $n-1$ variables

$$X_1, \dots, \hat{X}_i, \dots, X_n \;\longrightarrow\; Y. \tag{11}$$

We shall denote function (11) by $\mathrm{ev}^i_a f$.

1.1.13 The associated evaluation functions of one variable

Assignment

$$x_i \;\mapsto\; \mathrm{ev}^i_{x_i} f$$

defines a function of a single variable

$$X_i \;\longrightarrow\; \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,; Y) \tag{12}$$

We shall denote function (12) by $\mathrm{ev}^i f$ and call it the i-th *evaluation function* associated with a function f.

1.1.14 Adjunction correspondence

Assignment

$$f \longmapsto \mathrm{ev}^i f$$

defines a canonical bijection

$$\mathrm{Funct}(X_1, \dots, X_n\,;\, Y) \longleftrightarrow \mathrm{Funct}(X_i, \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,;\, Y)) \tag{13}$$

whose inverse is given by sending a function

$$\phi \in \mathrm{Funct}(X_i, \mathrm{Funct}(X_1, \dots, \hat{X}_i, \dots, X_n\,;\, Y))$$

to the function

$$X_1, \dots, X_n \longrightarrow Y, \qquad x_1, \dots, x_n \longmapsto \big(\phi(x_i)\big)(x_1, \dots, \hat{x}_i, \dots, x_n)\,.$$

Correspondence (13) is a manifestation of what is perhaps the single most important phenomenon in Modern Mathematics known as a *pair of adjoint functors*. This is not your first encounter with this phenomenon—you encountered it in some fundamental theorems of basic Mathematical curriculum, but it is the first time that you are expressly told about it.

1.1.15

In order to describe the conceptual mechanics behind the concept of *adjoint* functors, one needs to introduce a proper language, the language of *morphisms* and *categories*, cf. Chapter 3 and Section 3.6.

1.1.16 Adjunction correspondence in exponential notation

Canonical identification (13) in exponential notation (8) acquires particularly suggestive form

$$Y^{X_1,\dots,X_n} \longleftrightarrow \left(Y^{X_1,\dots,\hat{X}_i,\dots,X_n}\right)^{X_i}\,. \tag{14}$$

1.1.17 Surjective functions

A function (5) is said to be *surjective* if

$$\text{for every } y \in Y \text{ there exists an argument-list } x_1, \dots, x_n \text{ such that } f(x_1, \dots, x_n) = y\,. \tag{15}$$

You are likely to be familiar with an informal expression "a function f is onto" instead of being surjective. I encourage you to use the term surjective.

1.1.18 Injective functions

A function (5) is said to be *injective* if it has the property

$$\text{if } f(x_1, \dots, x_n) = f(x_1', \dots, x_n'), \text{ for two argument-lists, then the two argument-lists are equal.} \tag{16}$$

You are likely to be familiar with an informal expression "a function f is one-to-one" instead of of being injective.

1.1.19 Bijective functions

A function is said to be *bijective* if it is both surjective and injective. This terminology is used primarily for functions of a single variable.

1.2 Composition of functions

1.2.1 Postcomposition

Given a function (5) and a function $g : Y \to Y'$, their composition yields the function

$$g \circ f : X_1, \dots, X_n \longrightarrow Y', \qquad x_1, \dots, x_n \longmapsto g(f(x_1, \dots, x_n)). \tag{17}$$

1.2.2

Postcomposition with a function g is itself a function between the function sets

$$g_* \,:\, \mathrm{Funct}(X_1, \dots, X_n; Y) \longrightarrow \mathrm{Funct}(X_1, \dots, X_n; Y'), \qquad f \longmapsto g \circ f. \tag{18}$$

1.2.3 Precomposition

Given a function (5) and a function list $h_1, \dots, h_n$,

$$X_1', \dots, X_m' \xrightarrow{h_1} X_1 \quad , \quad \dots \quad , \quad X_1', \dots, X_m' \xrightarrow{h_n} X_n \tag{19}$$

their composition yields the function

$$f \circ (h_1, \dots, h_n) \,:\, X_1', \dots, X_m' \longrightarrow Y, \qquad x_1', \dots, x_m' \longmapsto f(\,h_1(x_1', \dots, x_m'), \dots, h_n(x_1', \dots, x_m')\,). \tag{20}$$

1.2.4

Precomposition with a function list $h_1, \dots, h_n$ is itself a function between the function sets

$$(h_1, \dots, h_n)^* \,:\, \mathrm{Funct}(X_1, \dots, X_n\,;\, Y) \longrightarrow \mathrm{Funct}(X_1', \dots, X_m'\,;\, Y), \qquad f \longmapsto f \circ (h_1, \dots, h_n). \tag{21}$$

1.2.5 Invertible functions of a single variable

Composition of functions of a single variable produces a function of a single variable. We say that $f : \mathrm{X} \to Y$ is a *left-invertible* function, if there exists a function $g : Y \to \mathrm{X}$ such that

$$g \circ f = id_{\mathrm{X}}\,. \tag{22}$$

We say that $f : \mathrm{X} \to Y$ is a *right-invertible* function, if there exists a function $g : Y \to \mathrm{X}$ such that

$$f \circ g = id_Y\,. \tag{23}$$

Exercise 1 *Show that, if g is a left inverse of f and h is a right inverse of f, then $g = h$.*

1.2.6

We denote that unique left- and right-inverse by f^{-1}.

Exercise 2 *Show that a left-invertible function f is injective and a right-invertible function is surjective.*

In particular, an invertible function is bijective.

Exercise 3 *Show that a bijective function is invertible.*

Lemma 1.1 *Suppose that $f : \mathrm{X} \to Y$ is injective. Then there is a natural correspondence between left inverses of f and functions $h : Y \setminus f_* \mathrm{X} \longrightarrow \mathrm{X}$.*

Proof. The target of a function f is the union of disjoint sets

$$Y' := f_* \mathrm{X} \qquad \text{and} \qquad Y'' := Y \setminus f_* \mathrm{X}.$$

Exercise 4 *Show that $g : Y \to \mathrm{X}$ is a left inverse of f if and only if the restriction of g to Y' is the function*

$$y \longmapsto \textit{the unique } x \in \mathrm{X} \textit{ such that } f(x) = y.$$

Thus, the set of left inverses of f is in bijective correspondence with the set of functions $Y'' \to X$,

$$\text{Left Inverses}(f) \longleftrightarrow \text{Funct}(Y'', \mathrm{X}), \qquad g \longmapsto g_{|Y''}.$$

□

Since the function set $\text{Funct}(Y'', \mathrm{X})$ is not empty as long as either X is not empty or Y'' is empty, we obtain the following two corollaries.

Corollary 1.2 *A function $f : \mathrm{X} \to Y$ with $\mathrm{X} \neq \emptyset$ is left-invertible if and only if f is injective. A function $f : \emptyset \to Y$ is left invertible if and only $Y = \emptyset$, i.e., if and only if f is bijective.*

Corollary 1.3 *A function $f : \mathrm{X} \to Y$ with $\mathrm{X} \neq \emptyset$ is bijective if and only if it has a unique left-inverse. That unique left-inverse is also a right-inverse.*

1.2.7 Finite sets

We say that a set is *finite* if every left-invertible function $f : \mathrm{X} \to \mathrm{X}$ is invertible.

1.2.8 Infinite sets

Accordingly, we say that a set X is *infinite*, if it admits a left-invertible function $f : \mathrm{X} \to \mathrm{X}$ that is not right-invertible.

1.2.9 Axiom of Infinity

The so called *Axiom of Infinity* of Set Theory asserts existence of an infinite set.

Existence of an infinite set cannot be proven using the remaining axioms of Set Theory. In fact, the remaining axioms of Set Theory are consistent with the assertion that every set is finite.

We shall prove later that Axiom of Infinity is equivalent to existence of the semiring $(\mathbf{N}, 0, 1, +, \cdot)$ of natural numbers.

1.3 The language of relations

1.3.1 Statements

A *statement* is a well-formed sentence that is either true or false. Any human language whose vocabulary is extended by adding various, previously defined, mathematical terms, is acceptable.

1.3.2 A relation is a function whose values are statements

A *relation* on sets $X_1, \dots, X_n$ is a function of n variables

$$\rho : X_1, \dots, X_n \longrightarrow \text{Statements}, \qquad x_1, \dots, x_n \longmapsto \rho(x_1, \dots, x_n). \tag{24}$$

We say in this case that ρ is an n*-ary* relation. We also say that the relation is *between* elements of sets $X_1, \dots, X_n$.

1.3.3 Nullary, unary, binary, ternary, ... relations

For small values of n, instead of speaking about 0-ary, 1-ary, 2-ary, 3-ary, ..., relations, we speak of *nullary*, *unary*, *binary*, *ternary*, ..., relations.

1.3.4 {nullary relations} $\longleftrightarrow$ {statements}

According to Section 1.1.8, there is a canonical identification between *nullary relations* and *statements*.

1.3.5 Relations *on* a set

When all sets X_i in the domain coincide with a set X, we speak of an n-ary relation *on* X.

The statement $\rho(x_1, \dots, x_n)$ needs not refer to some or even to anyone of the element variables x_i.

1.3.6 Total relations

The statement $\rho(x_1, \dots, x_n)$ may hold for every list of arguments. Such a relation is sometimes referred to as *a total* relation.

1.3.7 Void relations

The statement $\rho(x_1, \dots, x_n)$ may fail for every list of arguments. Such a relation is sometimes referred to as *a void* relation.

1.3.8

Since a nullary relation reduces to a single statement, and since every statement either holds or fails, a nullary relation is either total or void.

1.4 Operations on sets

1.4.1

An *n-ary operation* on a set Y is a function

$$\mu : X_1, \dots, X_n \longrightarrow Y \tag{25}$$

where all the sets $X_1, \dots, X_n$ are equal to Y.

1.4.2 {nullary operations on Y} $\longleftrightarrow$ Y

To declare a nullary operation on a set Y is equivalent to supplying a single element of Y. For this reason, nullary operations on Y are thought of as "distinguished" elements of Y. In particular, there is a canonical bijection between the set of nullary operations on Y and the set Y itself.

1.4.3 Induced operations

Given a list of n functions of m variables,

$$f_1, \dots, f_n \in \mathrm{Funct}(X_1, \dots, X_m; Y),$$

let us assign to the argument list

$$x_1, \dots, x_m$$

the list of values

$$f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)$$

and then apply the operation μ. Composite assignment

$$x_1, \dots, x_m \longmapsto f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m) \longmapsto \mu\big(f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big)$$

defines a function $X_1, \dots, X_m \longrightarrow Y$. We shall denote this function by $\mu_*(f_1, \dots, f_n)$.

Assignment

$$f_1, \dots, f_n \longmapsto \mu_*(f_1, \dots, f_n) \tag{26}$$

defines then an n-ary operation μ_* on the set of functions $\mathrm{Funct}(X_1, \dots, X_m; Y)$. We refer to it as the operation *induced by* μ.

1.4.4

Operations on sets of Y-valued functions induced by operations defined on Y have been playing an essential role in Mathematics since the time when the foundations of Differential and Integral Calculus had been laid down nearly 400 years ago.

1.5 Canonical operations on $\mathscr{P}X$

1.5.1 Canonical operations

A general set X has no distinguished elements, hence it is not equipped with any distinguished nullary operation. Similarly, there are no distinguished binary, ternary, etc., operations on a general set. The identity function

$$\mathrm{id}_X : X \longrightarrow X, \qquad x \longmapsto x, \tag{27}$$

is the only distinguished unary operation.

Certain sets, however, are *naturally* equipped with various operations. We refer to such operations as *canonical*. An example of prime importance is provided by the set of all subsets, $\mathscr{P}X$, of an arbitrary set X. A shorter designation for $\mathscr{P}X$ is the *power-set of* X.

1.5.2 Canonical nullary operations on $\mathscr{P}X$

The power-set of a general nonempty set has exactly two distinguished elements: the empty subset $\emptyset$ and X. In other words, $\mathscr{P}X$ is equipped with exactly two canonical nullary operations.

1.5.3 The complement of a subset

The power-set of a general set has a canonical unary operation

$$\complement : \mathscr{P}X \longrightarrow \mathscr{P}X, \qquad A \longmapsto \complement A := \{x \in X \mid x \notin A\}, \tag{28}$$

that sends a subset $A \subseteq X$ to its *complement*. We shall usually denote the complement of a subset $A \subseteq X$ by A^c and use symbol $\complement$ to denote the complement operation.

1.5.4 Involutions on a set

Note that $\complement^2 := \complement \circ \complement$ is the identity operation. A unary operation $\mu : X \to X$ with this property is called an *involution* (on a set X). The identity operation id_X is a *trivial* involution.

1.5.5 Canonical unary operations on $\mathscr{P}X$

The power-set $\mathscr{P}X$ of a nonempty set is equipped with exactly two unary operations, both of them involutions on $\mathscr{P}X$: the identity operation $\mathrm{id}_{\mathscr{P}X}$ and the complement operation $\complement$.

1.5.6 Canonical binary operations on $\mathscr{P}X$

Union of two sets,

$$A, B \longmapsto A \cup B,$$

intersection of two sets,

$$A, B \longmapsto A \cap B,$$

difference of two sets,

$$A, B \longmapsto A \setminus B,$$

are examples of canonical binary operations on the power-set.

1.5.7

Any one of the above three operations can be expressed in terms of any of the remaining two and of the complement operation. For example, the union and the intersection operations are linked to each other by the following pair of identities

$$A \cap B = (A^c \cup B^c)^c \qquad \text{and} \qquad A \cup B = (A^c \cap B^c)^c \qquad (A, B \subseteq X) \tag{29}$$

called *de Morgan laws*.

Note also the following identities

$$A \cup A^c = X, \qquad A \cap A^c = \emptyset \qquad \text{and} \qquad A \setminus B = A \cap B^c = (A^c \cup B)^c \qquad (A, B \subseteq X).$$

Exercise 5 *Find the identity expressing $\cap$ in terms of $\setminus$ and $\complement$, and prove it.*

1.6 Operations on Statements

1.6.1 Basic binary operations on sentences

The following table contains the list of basic binary operations on sentences (symbols P and Q stand for arbitrary sentences).

Read:	Symbolic notation	Name
P *and* Q	$P \wedge Q$	Conjunction
P *or* Q	$P \vee Q$	Alternative
if P, *then* Q	$P \Rightarrow Q$	Implication
P *if and only if* Q	$P \Leftrightarrow Q$	Equivalence

1.6.2 Negation

The negated sentence P will be symbolically denoted $\neg P$. In many languages, negating a sentence is performed according to rules that depend on the syntactical structure of that sentence. For this reason, it is difficult or impossible to provide one single reading of the negated sentence $\neg P$. We can circumvent this difficulty by saying, instead, "the negation of P" or "P negated", when we need to refer to $\neg P$.

1.6.3 Validity of the corresponding statements

Assuming that P and Q are statements,

- $P \wedge Q$ holds precisely when P and Q hold;
- $P \vee Q$ holds precisely when P or Q holds;
- $P \Rightarrow Q$ fails if P holds and Q fails, otherwise it holds;
- $P \Leftrightarrow Q$ holds precisely when P and Q both hold or both fail;
- $\neg P$ holds precisely when P fails.

In particular, Conjunction, Alternative, Implication, Equivalence, define binary operations on the set of Statements, while Negation defines a unary operation.

1.6.4 Operations on Statements = Relations on Statements

On the set of statements the concepts of an n-ary operation and of an n-ary relation coincide.

1.6.5 Operations on relations

Any operation on Statements induces the corresponding operations on the sets of relations, $\mathrm{Rel}(X_1, \dots, X_n)$, between elements of sets $X_1, \dots, X_n$.

1.6.6

Thus, given relations $\rho, \sigma \in \mathrm{Rel}(X_1, \dots, X_n)$, we can form the relations $\neg\rho$, $\rho\vee\sigma$, $\rho\wedge\sigma$, $\rho\Rightarrow\sigma$ and $\rho\Leftrightarrow\sigma$. They assign to an argument list $x_1, \dots, x_n$ the statements

$$\neg\rho(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \vee \sigma(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \wedge \sigma(x_1, \dots, x_n), \quad \rho(x_1, \dots, x_n) \Rightarrow \sigma(x_1, \dots, x_n)$$

and, respectively,

$$\rho(x_1, \dots, x_n) \Leftrightarrow \sigma(x_1, \dots, x_n).$$

1.7 Quantification

1.7.1 Universal quantification

Given a relation ρ between elements of sets $X_1, \dots, X_n$, assigning to a list $x_1, \dots, \hat{x}_i, \dots, x_n$ the statement

$$\textit{for all } x_i \in X_i\,,\ \rho(x_1, \dots, x_n) \tag{30}$$

defines an $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$. Instead of "for all", we can also say "for every" with the same meaning.

Symbolically, statement (30) is represented

$$\forall_{x_i \in X_i}\, \rho(x_1, \dots, x_n).$$

1.7.2 " Statement S is a special case of statement T "

Suppose $\rho : X \longrightarrow \text{Statements}$ is a (unary) relation on a set X. Consider the statements obtained by universal quantification of relation ρ over subsets $A \subseteq X$ and $B \subseteq X$

$$S := \text{“ } \forall_{x \in A}\, \rho(x) \text{ ”} \qquad \text{and} \qquad T := \text{“ } \forall_{x \in B}\, \rho(x) \text{ ”}. \tag{31}$$

Note that, if $A \subseteq B$, then

$$S \Longrightarrow T\,. \tag{32}$$

If so, we shall say that *statement S is a special case of statement T*.

In general, given two statements S and T, we shall say that S is a special case of T if there exist

a unary relation ρ on a certain set X and subsets $A \subseteq B \subseteq X$

such that S and T have the form as in (31).

1.7.3 " Statement S trivially implies statement T "

Note that in order to establish implication (32), one deoes not need to know anything about a set X, a relation ρ on X, or subsets A and B. One only needs to know that both statements are obtained by *universal* quantification of the *same* certain unary relation over two subsets $A \subseteq B$ of X.

This is one of those situations when mathematicians are likely to say that a statement S *trivially* implies a statement T.

1.7.4 Existential quantification

Assigning to a list $x_1, \dots, \hat{x}_i, \dots, x_n$ the statement

$$\textit{there exists } x_i \in X_i \textit{ such that } \rho(x_1, \dots, x_n) \tag{33}$$

defines another an $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$.

Symbolically, statement (33) is represented

$$\exists_{x_i \in X_i} \rho(x_1, \dots, x_n).$$

1.7.5

Suppose $\rho : X \longrightarrow \text{Statements}$ is a (unary) relation on a set X. Consider the statements obtained by existential quantification of relation ρ over subsets $A \subseteq X$ and $B \subseteq X$

$$S := \text{“ } \exists_{x \in A} \rho(x) \text{ ”} \qquad \text{and} \qquad T := \text{“ } \exists_{x \in B} \rho(x) \text{ ”}. \tag{34}$$

Note that, if $A \subseteq B$, then

$$T \Longrightarrow S. \tag{35}$$

Also in this case we say that statement T *trivially implies* statement S.

1.7.6

Operations of quantification are frequently iterated. For example, given $i \neq j$,

$$\forall_{x_i \in X_i} \exists_{x_j \in X_j} \rho(x_1, \dots, x_n)$$

denotes the statement:

$$\textit{for all } x_i \in X_i, \textit{ there exists } x_j \in X_j \textit{ such that } \rho(x_1, \dots, x_n).$$

1.7.7

The statement

$$\forall_{\varepsilon \in \mathbf{R}^+} \exists_{i \in \mathbf{N}} \forall_{j \in \mathbf{N}} \left(i \leq j \Rightarrow |x_j - a| < \varepsilon\right) \tag{36}$$

describes the fact that a sequence of real numbers (x_n) converges to a point a of the real line. Here, $\mathbf{R}^+$ denotes the set of positive real numbers and $\mathbf{N}$ denotes the set of natural numbers. The statement is about sequences (x_n) of real numbers and points a of the real line. It defines a binary relation between elements of these two sets. The relation is the result of applying one-after-another universal and existential quantification to the statement that has the form of implication

$$i \leq j \;\Rightarrow\; |x_j - a| < \varepsilon\,. \tag{37}$$

Here x_j denotes the j-th term of the sequence (x_n). Statement (37) is a statement about natural numbers i and j, a sequence of real numbers (x_n), a point of the real line a, and a positive real number ε. As such, it is a 5-ary relation. Application of three consecutive quantifications yields the binary relation defined in (36).

What you see here are examples of typical statements encountered in Mathematical Analysis.

Exercise 6 *Let $\rho : X_1, X_2 \longrightarrow$ Statements be a binary relation. Consider the statements*

$$S := \text{“ } \exists_{x_1 \in A_1} \forall_{x_2 \in A_2}\, \rho(x_1, x_2) \text{ ”} \qquad \textit{and} \qquad T := \text{“ } \exists_{x_1 \in B_1} \forall_{x_2 \in B_2}\, \rho(x_1, x_2) \text{ ”}$$

where A_1 and B_1 are subsets of X_1 while A_2 and B_1 are subsets of X_2. Under what condition on A_1, A_2, B_1 and B_2, statement S implies statement T?

1.8 Comparing relations

1.8.1

Let ρ and σ be two relations between elements of sets $X_1, \dots, X_n$. Let us consider the nullary relation, i.e., the statement

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n} \;\big(\rho(x_1, \dots, x_n) \Rightarrow \sigma(x_1, \dots, x_n)\big)\;.$$

Here

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n}$$

is an abbreviation for

$$\forall_{x_1 \in X_1} \cdots \forall_{x_n \in X_n}\;.$$

1.8.2

We say that ρ is *weaker* than σ, and that σ is *stronger* than ρ, if that statement holds, i.e., if the relation

$$\rho \Rightarrow \sigma \tag{38}$$

is a *total* relation. It is common in this situation to represent this fact symbolically by writing

$$\rho \Longrightarrow \sigma \tag{39}$$

and to say that ρ *implies* σ.

1.8.3 Caveat

Make sure not to confuse (38) with (39). Symbol $\Rightarrow$ in (38) denotes a binary *operation* on the set of relations $\mathrm{Rel}(X_1, \dots, X_n)$ while symbol $\Longrightarrow$ in (39) denotes a binary *relation* on the same set.

1.8.4 Equipotent relations

The terms "weaker" and "stronger" is not an ideal terminology in view of the fact that a relation ρ can be both weaker and stronger than a relation σ. If this happens, we say that the two relations are *equipotent*. This happens, precisely, when the statement

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n} \; \big(\sigma(x_1, \dots, x_n) \Leftrightarrow \rho(x_1, \dots, x_n)\big)$$

holds, i.e., when

$$\rho \Leftrightarrow \sigma \tag{40}$$

is a *total* relation.

In other words, statement $\sigma(x_1, \dots, x_n)$ holds precisely for the same lists of arguments as statement $\rho(x_1, \dots, x_n)$ does.

It is common in this situation to represent this fact symbolically by writing

$$\rho \Longleftrightarrow \sigma \tag{41}$$

and to say that relations ρ and σ are *equipotent*.

1.8.5 Caveat

Though it is preferable to refer to such relations as equipotent, mathematicians keep saying that such relations are *equivalent*. Since the term "equivalence" is used also as a generic term for a binary relation that is *reflexive, symmetric* and *transitive*, cf. Section 4.2.4, I encourage you to use the term "equipotent".

1.9 Functions of n variables viewed as $(n+1)$-ary relations

1.9.1

Given sets X_1, ... , X_n and Y, and a function of n variables

$$f : X_1, \dots, X_n \longrightarrow Y, \tag{42}$$

we can associate with it an $(n+1)$-ary relation

$$\rho_f : X_1, \dots, X_n, Y \longrightarrow \text{Statements}, \qquad x_1, \dots, x_n, y \;\longmapsto\; \text{“}\; f(x_1, \dots, x_n) = y \;\text{”}. \tag{43}$$

1.9.2

The $(n+1)$-ary relation ρ_f has the following property:

$$\text{\textit{for every list of elements } } x_1 \in X_1, \dots, x_n \in X_n \text{\textit{, there exists a unique }} y \in Y \text{\textit{, such that }} \rho(x_1, \dots, x_n, y). \tag{44}$$

1.9.3

Given any $(n+1)$-ary relation satisfying property (44), we can define a function (42) where $f(x_1, \ldots, x_n)$ is defined to be that unique element $y \in Y$ such that

$$\rho(x_1, \ldots, x_n, y).$$

Let us denote this function f_ρ.

Exercise 7 *Show that $f_\sigma = f_\rho$ if and only if σ and ρ are equipotent.*

1.10 Composing relations

1.10.1

Suppose that two relations are given,

$$\text{an } (m+1)\text{-ary relation between elements of sets } X_0, \ldots, X_m ,$$

denoted σ, and

$$\text{an } (n+1)\text{-ary relation between elements of sets } X_m, \ldots, X_{m+n+1} ,$$

denoted ρ. Assigning to a list $x_1, \ldots, \hat{x}_m, \ldots, x_{m+n+1}$ the statement

$$\textit{there exists } x_m \in X_m \textit{ such that } \sigma(x_0, \ldots, x_m) \textit{ and } \rho(x_m, \ldots, x_{m+n+1}) \tag{45}$$

defines an $(m+n+1)$-ary relation between elements of sets

$$X_1, \ldots, \hat{X}_m, \ldots, X_{m+n+1} .$$

Symbolically, statement (45) is represented

$$\exists_{x_m \in X_m} \left(\sigma(x_0, \ldots, x_m) \wedge \rho(x_m, \ldots, x_{m+n+1})\right) .$$

1.10.2

We call the relation defined above, the *composite of ρ and σ* and denote it $\rho \circ \sigma$.

1.11 Cartesian product $X_1 \times \cdots \times X_n$

1.11.1

Given a list of sets $X_1, \ldots, X_n$, let us form its Cartesian product

$$X_1 \times \cdots \times X_n . \tag{46}$$

By definition, its elements are ordered n-tuples $(x_1, \ldots, x_n)$ of elements $x_1 \in X_1, \ldots, x_n \in X_n$.

1.11.2 The concept of an ordered n-tuple

What is an ordered n-tuple? There is not much difference between lists of length n and ordered n-tuples. When we speak of an ordered n-tuple, we always think of it being a *single* entity, while when we speak of a list of length n, we think of n separate entities.

1.11.3

To illustrate this further, the assignment

$$x, y \mapsto x + y \qquad (x, y \in \mathbf{N})$$

defines a function of 2 variables on the set of natural numbers $\mathbf{N}$, while the assignment

$$(x, y) \mapsto x + y \qquad (x, y \in \mathbf{N})$$

defines a function of a single variable on the Cartesian square $\mathbf{N} \times \mathbf{N}$ of $\mathbf{N}$. The targets of both functions are the same, namely the set of natural numbers.

1.11.4 The equality principle

The principal property built into the concept of an ordered n-tuple is the following equality principle

$$(x_1, \dots, x_m) = (y_1, \dots, y_n)$$

if and only if $m = n$ and $x_i = y_i$ for all $1 \le i \le n$.

1.11.5 The standard set-theoretic model of an ordered pair

The actual model of an ordered n-tuple is of little importance. It is possible to prove existence of such a model using only basic set theoretic concepts. For example, the axiom of Set Theory called Axiom of a Pair states that, for any x and y, the set $\{x, y\}$, whose elements are x and y, exists. Thus, $\{x\} = \{x, x\}$ and $\{x, y\}$ exist and therefore also the following set

$$\{\{x\}, \{x, y\}\} \tag{47}$$

exists. This set is a model of an *ordered pair*, i.e., of an ordered a 2-tuple.

Exercise 8 *Show that*

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$$

if and only if $x = x'$ *and* $y = y'$.

1.11.6

If $x \in X$ and $y \in Y$, then (47) is a *family* of subsets of $X \cup Y$, i.e., it is a subset of the power-set of $X \cup Y$,

$$\{\{x\}, \{x, y\}\} \subseteq \mathscr{P}(X \cup Y) .$$

Accordingly, the Cartesian product $X \times Y$ is realized as the appropriate subset of the power-set of the power-set of $X \cup Y$,

$$X \times Y := \{ P \in \mathscr{P}(\mathscr{P}(X \cup Y)) \mid \exists_{x \in X} \exists_{y \in Y} P = \{\{x\}, \{x, y\}\} \} ,$$

which demonstrates its existence.

1.11.7

Having a model of on ordered pair, the ordered pair

$$((x, y), z)$$

becomes a model of an ordered triple and the Cartesian product

$$(X \times Y) \times Z$$

becomes a model of $X \times Y \times Z$. By induction on n, one can construct a model of an ordered n-tuple

$$(x_1, \dots, x_n)$$

and of

$$X_1 \times \cdots \times X_n \,,$$

There are other, more convenient models.

1.11.8 An ordered n-tuple as a function

A convenient model of an ordered n-tuple $(x_1, \dots, x_n)$ is provided by a function

$$\xi : \{1, \dots, n\} \longrightarrow X_1 \cup \cdots \cup X_n \tag{48}$$

whose value at i is, for every $1 \leq i \leq n$, an element of X_i.

In this model, the Cartesian product $X_1 \times \cdots \times X_n$ is represented as a subset of the set of all functions (48).

1.11.9 Universal functions of n-variables

We shall say that a function

$$\tau : X_1, \dots, X_n \longrightarrow T \tag{49}$$

is a *universal* function with the domain list $X_1, \dots, X_n$, if *every* function (5) can be produced from τ by postcomposing τ with a *unique* function $\tilde{f} : T \to Y$,

$$f = \tilde{f} \circ \tau.$$

In that case, the bijective correspondence

$$\mathrm{Funct}(X_1, \dots, X_n; Y) \longleftrightarrow \mathrm{Funct}(T, Y), \qquad f \longleftrightarrow \tilde{f}, \tag{50}$$

identifies the set of Y-valued functions of n-variables, with the domain-list $X_1, \dots, X_n$, with the set of functions of a single variable $T \to Y$.

1.11.10 The canonical function of n-variables $X_1, \dots, X_n \longrightarrow X_1 \times \cdots \times X_n$

For every list of sets $X_1, \dots, X_n$, there exists a canonical function of n-variables with that list as its domain. It assigns to an argument list $x_1, \dots, x_n$ the corresponding ordered n-tuple,

$$X_1, \dots, X_n \longrightarrow X_1 \times \cdots \times X_n, \qquad x_1, \dots, x_n \longmapsto (x_1, \dots, x_n), \tag{51}$$

1.11.11

The canonical function has the universal property defined in Section 1.11.9. Indeed,

$$f \quad \longmapsto \quad \left(\tilde{f} : X_1 \times \cdots \times X_n \to Y, \quad (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)\right)$$

is a bijective correspondence and f is produced by postcomposing function (51) with $\tilde{f}$.

1.11.12 The case of functions of zero variables

When $n = 0$, Cartesian product of the empty list of sets consists of functions from the *empty* set of natural numbers to the union of the empty family of sets. The latter, as we already know, is the empty set. In other words, Cartesian product of the empty list of sets is the set of functions

$$\emptyset^{\emptyset} = \mathrm{Funct}(\emptyset, \emptyset) = \{\mathrm{id}_{\emptyset}\}, \tag{52}$$

and that set has a unique element, namely the identity function associated with the empty set. Exponential notation $\emptyset^{\emptyset}$, cf. (8) is particularly apt in this case. We observe that foundations of Set Theory themeselves are telling us that 0^0 is well defined and equals to 1.

1.11.13 Canonical identification $\mathrm{Op}_0(Y) \longleftrightarrow \mathrm{Funct}(\emptyset^{\emptyset}, Y)$

In particular, nullary operations on a set Y, i.e., Y-valued functions of zero of variables, are canonically identified with functions $\emptyset^{\emptyset} \to Y$.

1.11.14

Every statement containing references to functions of n-variables can be now replaced by an equivalent statement containing references exclusively to functions of a single variable.

This explains why the use of the concept of a function of n-variables has practically disappeared from modern mathematical language. This is also the reason why Cartesian product is today present everywhere where normally one would be mentioning functions of n-variables: Cartesian product

$$X_1 \times \cdots \times X_n$$

is the *target* of the universal function of n-variables (51).

1.11.15 Canonical projections $(\pi_i)_{i \in \{1,\dots,n\}}$

The Cartesian product is more than just a set, it is a *mathematical structure*, like a relation or a function. One should consider the Cartesian product to consist of a set $X_1 \times \cdots \times X_n$ equipped with a list of functions

$$\pi_1, \dots, \pi_n\ , \tag{53}$$

called the *canonical projections*, where π_i is defined as

$$\pi_i : X_1 \times \cdots \times X_n \longrightarrow X_i, \qquad (x_1, \dots, x_n) \mapsto x_i\ . \tag{54}$$

Having just set $X_1 \times \cdots \times X_n$ alone would not suffice to recover the list of sets $X_1, \dots, X_n$. For example, $X_1 \times \cdots \times X_n$ is the empty set whenever at least one set X_i is empty.

1.11.16 *Naturality* of Cartesian product

Cartesian product assigns to a list of sets $X_1, \ldots, X_n$ a single set $X_1 \times \cdots \times X_n$ equipped with the list of functions $\pi_1, \ldots, \pi_n$. A function list

$$X_1 \xrightarrow{f_1} X'_1 \,,\; \ldots \;,\; X_n \xrightarrow{f_n} X'_n \,, \tag{55}$$

induces a function between the corresponding Cartesian-product sets

$$f_1 \times \cdots \times f_n : X_1 \times \cdots \times X_n \longrightarrow X'_1 \times \cdots \times X'_n \,, \qquad (x_1, \ldots, x_n) \longmapsto (f_1(x_1), \ldots, f_n(x_n)) \,. \tag{56}$$

Moreover, the assignment

$$f_1, \ldots, f_n \longmapsto f_1 \times \cdots \times f_n$$

commutes with the operations of function composion.

Exercise 9 *Given a function list*

$$X'_1 \xrightarrow{f'_1} X''_1 \,,\; \ldots \;,\; X'_n \xrightarrow{f'_n} X''_n \,,$$

show that

$$(f'_1 \times \cdots \times f'_n) \circ (f_1 \times \cdots \times f_n) \;=\; (f'_1 \circ f_1, \ldots, f'_n \circ f_n) \,.$$

Mathematicians refer to such behavior as *naturality* of the assignment

$$X_1, \ldots, X_n \longmapsto X_1 \times \cdots \times X_n \,.$$

1.11.17 The graph of a relation

Given a relation ρ between elements of sets $X_1, \ldots, X_n$, the following subset of the Cartesian product,

$$\Gamma_\rho := \{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n \mid \rho(x_1, \ldots, x_n)\} \tag{57}$$

is guaranteed to exist by the axioms of Set Theory. This is the set of those ordered n-tuples for which statement $\rho(x_1, \ldots, x_n)$ holds. One calls it the *graph* of ρ.

Exercise 10 *Let ρ and σ be two relations between elements of sets $X_1, \ldots, X_n$. Show that ρ is weaker than σ if and only if*

$$\Gamma_\rho \subseteq \Gamma_\sigma \,.$$

1.11.18

In particular, relations ρ and σ are equipotent if and only if their graphs are equal

$$\Gamma_\rho = \Gamma_\sigma \,.$$

1.11.19 Correspondences

The graph of a relation provides another example of a mathematical structure. It involves the list of the following data:

- a list of sets $X_1, \dots, X_n$,
- a subset $C \subseteq X_1 \times \cdots \times X_n$.

Having just the set C alone would not suffice to recover the list of sets $X_1, \dots, X_n$.

A structure of this kind begs for a name. I propose to call it a *correspondence between elements of sets* $X_1, \dots, X_n$ or, an n*-correspondence*, in short.

1.11.20

When all sets X_i are one and the same set X, we shall speak of n-correspondences *on* X.

1.11.21 1-correspondences

In particular, 1-correspondences on X are the same as *subsets* of X.

1.11.22

In practice, we still be denoting a correspondence by the symbol denoting the subset C of $X_1 \times \cdots \times X_n$.

1.11.23 Caveat

In fact, a common practice among mathematicians is to call precisely this structure a *relation.* This approach to the concept of a relation, while being much less intuitive than the 'statements-valued function' approach, it allows one to place theory of relations entirely within the realm of Set Theory. For example, relations with a given domain (1) form a well defined set.

1.11.24

The main advantage of such a restrictive notion of a relation is that it frees a mathematician from any concerns about what is and what is not a *statement* while still being sufficient for studying the whole of Mathematics.

Indeed, given a correspondence C between elements of sets $X_1, \dots, X_n$, let $\rho_C(x_1, \dots, x_n)$ be the statement

$$(x_1, \dots, x_n) \in C \, .$$

This defines a relation between elements of sets $X_1, \dots, X_n$.

Exercise 11 *Show that any relation ρ is equipotent to the relation ρ_{Γ_ρ}.*

Exercise 12 *Show that, for any correspondence C, one has $C = \Gamma_{\rho_C}$.*

1.11.25

We shall express the operations on relations, introduced in Sections 1.6.5–1.7, in terms of their graph correspondences. For this we need to introduce some notation.

Exercise 13 *Given a relation ρ, show that*

$$\Gamma_{\neg\rho} = \complement\Gamma_{\rho} \,. \tag{58}$$

Exercise 14 *Given relations ρ and σ with the same domain, show that*

$$\Gamma_{\rho\vee\sigma} = \Gamma_{\rho} \cup \Gamma_{\sigma} \qquad \textit{and} \qquad \Gamma_{\rho\wedge\sigma} = \Gamma_{\rho} \cap \Gamma_{\sigma} \,. \tag{59}$$

1.11.26

The above two exercises demonstrate that the operations of negation, alternative and conjunction of relations translate into the operations of taking the complement, the union, and the intersection, of correspondences.

Exercise 15 *Given relations ρ and σ with the same domain, show that*

$$\Gamma_{\rho\Rightarrow\sigma} = \complement\Gamma_{\rho} \cup \Gamma_{\sigma} \,. \tag{60}$$

1.11.27 The function-list canonically associated with an n-correspondence

By post-composing the canonical inclusion $\iota : C \hookrightarrow X_1 \times \cdots \times X_n$ with the list of canonical projections $\pi_1, \dots, \pi_n$, we obtain a list of functions

$$\begin{array}{c} C \\ {\scriptstyle \partial_1}\big\downarrow \cdots \big\downarrow{\scriptstyle \partial_n} \\ X_1, \dots, X_n \end{array} \tag{61}$$

that is canonically associated with the correspondence. Here $\partial_i := \pi_i \circ \iota$, $1 \leq i \leq n$.

1.11.28 Oriented graphs

When $n = 2$ and X_1 and X_2 are the same set X, a list (61) is called an *oriented graph*. Elements of X are referred to, in this case, as *vertices* and elements of C are referred as *oriented edges*, or *arrows*, of the graph.

1.11.29 2-Correspondences as oriented graphs

In particular, 2-correspondences on a set X can be viewed as oriented graphs with vertices being elements of X, such that no two oriented edges have the same source and the same target.

1.12 The language of diagrams

1.12.1

Situations involving several functions are frequently expressed and analyzed in the language of oriented graphs, represented visually as diagrams drawn on a blackboard, or on a page. Arrows in a diagram represent functions. Vertices represent their domains and targets. *Oriented paths* in such graphs represent composable lists of functions.

1.12.2 Commutative diagrams

When composition of two paths with the same origin and the same terminus produces the same result, we call such a diagram *commutative*. Most common examples of commutative diagrams have a form of a *commutative sqaure*,

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & S \arrow[l, "\beta"'] \arrow[d, "\delta"] \\ T & Y \arrow[l, "\gamma"] \end{tikzcd} . \tag{62}$$

Commutativity of square diagram (62) expresses the equality

$$\alpha \circ \beta = \gamma \circ \delta .$$

1.12.3

Commutativity of a diagram is often signaled by placing a symbol ⟲, or its cousins: ⟳, ↺, or ↻, between two composable paths of arrows originating and terminating in a common vertex.

1.12.4

Diagrams are employed not only to illustrate situations that can be discussed without introducing diagrams. It has been long observed that employing diagrams can greatly clarify and enhance analysis of complex scenarios. We shall illustrate it here by considering one example. Later in these notes you will see many more appearances of commutative diagrams.

1.12.5 An example

Consider a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \tag{63}$$

We do not know whether it is possible to complete diagram (63) to a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} , \tag{64}$$

we observe, however, that diagram (63) defines in a canonical manner a binary relation between elements of Y_1 and Y_2,

$$\rho : Y_1, Y_2 \longrightarrow \text{Statements}, \qquad y_1, y_2 \longmapsto \text{“ } \exists_{x_1 \in X_1}\, y_1 = \chi_1(x_1) \wedge y_2 = (\chi_2 \circ \gamma)(x_1) \text{ ”}. \tag{65}$$

1.12.6

It is clear that

$$\forall_{y_1 \in Y_1} \exists_{y_2 \in Y_2} \rho(y_1, y_2)$$

if and only if function χ_1 is surjective.

1.12.7

Let $y_2, y_2' \in Y_2$ be two elements in relation with a given element $y_1 \in Y_1$. Then, there are elements $x_1, x_1' \in X_1$ such that

$$y_1 = \chi_1(x_1) = \chi_1(x_1'), \qquad y_2 = (\chi_2 \circ \gamma)(x_1) \qquad \text{and} \qquad y_2' = (\chi_2 \circ \gamma)(x_1').$$

By combining this with commutativity of diagram (63) we obtain a chain of equalities

$$\varphi_2(y_2) = (\varphi_2 \circ \chi_2 \circ \gamma)(x_1) = (\alpha \circ \varphi_1)(\chi_1(x_1)) = (\alpha \circ \varphi_1)(\chi_1(x_1')) = (\varphi_2 \circ \chi_2 \circ \gamma)(x_1') = \varphi_2(y_2').$$

If φ_2 is injective, then $y_2 = y_2'$ and relation (65) defines a function

$$\beta : Y_1 \longrightarrow Y_2, \qquad y_1 \longmapsto \text{the unique } y_2 \in Y_2 \text{ such that } \rho(y_1, y_2).$$

1.12.8 Diagram chasing

The method we used to construct relation (65) and then to prove that under suitable hypotheses (65) defines a function, is referred to as *diagram chasing*.

1.12.9

Let us represent surjective functions by two-headed arrows $\twoheadrightarrow$ and injective functions by tailed arrows $\rightarrowtail$. We established the following fact.

Lemma 1.4 *Every commutative diagram*

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, phantom, "\circlearrowleft"] & X_1 \arrow[l, two heads, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, tail, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \tag{66}$$

admits a completion to a commutative diagram (64)*. Moreover, function* β *that makes diagram* (64) *commutative is unique.* □

Exercise 16 *Prove uniqueness of* β.

1.12.10

Consider now an arbitrary commutative diagram

$$\begin{tikzcd} Z_1 & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] & X_1 \arrow[l, "\chi_1"'] \\ Z_2 \arrow[from=1-1, d, "\alpha"'] \arrow[from=1-1, to=2-2, phantom, "\circlearrowleft"] & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \tag{67}$$

If arrow χ_2 admits a right inverse $\xi : Y_2 \to X_2$, then

$$\gamma := \xi \circ \beta \circ \chi_1$$

obviously makes the diagram

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd}$$

commute.

1.12.11

We can sum our discussion up in the following lemma.

Lemma 1.5 *Consider a diagram*

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] & Y_1 \arrow[l, "\varphi_1"'] & X_1 \arrow[l, two heads, "\chi_1"'] \\ Z_2 & Y_2 \arrow[l, tail, "\varphi_2"] & X_2 \arrow[l, two heads, "\chi_2"] \end{tikzcd} \tag{68}$$

The following three properties of diagram (68) *are equivalent:*

(a) *it admits a completion to a commutative diagram* (63);

(b) *it admits a completion to a commutative diagram* (67).

(c) *it admits a completion to a commutative diagram*

$$\begin{tikzcd} Z_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_1 \arrow[l, "\varphi_1"'] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[l, "\chi_1"'] \arrow[d, "\gamma"] \\ Z_2 & Y_2 \arrow[l, "\varphi_2"] & X_2 \arrow[l, "\chi_2"] \end{tikzcd} . \tag{69}$$

Implications (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) rely on Axiom of Choice, cf. Section 1.14.14, or one has to add the hypothesis to the effect that χ_2 has a right inverse (such functions are said to be *split surjections*).

1.13 Power-set functions induced by a function $f : X \to Y$

1.13.1 The *image-of-a-subset* and the *preimage-of-a-subset* functions f_* and f^*

Given a function $f : X \longrightarrow Y$, there are two associated functions between the power-sets

$$\mathscr{P}X \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathscr{P}Y \ , \tag{70}$$

where the associated *image* function is defined by

$$f_*(A) := \{y \in Y \mid \exists_{x \in A} f(x) = y\} \qquad (A \subseteq X) \tag{71}$$

and the associated *preimage* function is defined by

$$f^*(B) := \{x \in X \mid \exists_{y \in B} f(x) = y\} \qquad (B \subseteq Y). \tag{72}$$

1.13.2 A comment about notation

What I here denote by $f_*(A)$ and $f^*(B)$ is usuallA comment about notationy denoted $f(A)$ and $f^{-1}(B)$. This is all right as long as there is no need to consider the assignments

$$A \mapsto f(A) \qquad \text{and} \qquad B \mapsto f^{-1}(B)$$

as functions between the corresponding power-sets. When such a need arises, one needs an appropriate notation to denote the image and the preimage functions associated with f. This is why I adopted the *lower-* and the *upper-star* notation that is universally used in Modern Mathematics to denote all sorts of functions that are naturally associated with a given function.

1.13.3

This has yet another advantage: it often allows us to skip parentheses around the arguments of functions f_* and f^* in the interest of keeping notation as simple as possible, without affecting the intended meaning. Thus, we shall, generally, write f_*A and f^*B instead of $f_*(A)$ and $f^*(B)$.

1.13.4

I will say later why in some cases we mark the associated function by placing $*$ as a *subscript* while in other cases—as a *superscript*.

1.13.5 The *fiber* of a function $f : X \to Y$ at $y \in Y$

The preimage f^*B of a singleton subset $B = \{y\}$ is referred to as the *fiber of f at y*. It is usually denoted $f^{-1}y$ or $f^{-1}(y)$. We shall denote it $f^*\{y\}$.

1.13.6 Caveat

One must be careful not to confuse notation $f^{-1}(y)$, when it is used to denote the *fiber* of f at y, with notation $f^{-1}(y)$ used to denote the *value* of the *inverse* function. The inverse function, denoted f^{-1}, is defined only when f is invertible. In that case, the fiber of f at $y \in Y$ is given by

$$f^*\{y\} = \{f^{-1}(y)\}\,.$$

1.13.7 The characteristic function of a subset

Given a subset $A \subset X$, its *characteristic function* is defined by

$$\chi_A : X \to \mathbf{F}_2, \qquad \chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}, \tag{73}$$

where $\mathbf{F}_2 = \{0, 1\}$ denotes the 2-element field.

Assignment

$$A \longmapsto \chi_A$$

yields a canonical identification

$$\chi : \mathscr{P}X \longleftrightarrow \mathrm{Funct}(X, \mathbf{F}_2)\,. \tag{74}$$

Exercise 17 *Prove that, given a function $f : X \to Y$ and a subset $B \subset Y$, one has*

$$f^*\chi_B = \chi_{f^*B}\,. \tag{75}$$

In other words, the preimage function $f^* : \mathscr{P}Y \to \mathscr{P}X$ can be viewed also as the precomposition function

$$f^* : \mathrm{Funct}(Y, \mathbf{F}_2) \longrightarrow \mathrm{Funct}(X, \mathbf{F}_2)\,.$$

1.13.8

Identity (75) can be also expressed by saying that the following square diagram of functions

$$\begin{tikzcd} \mathscr{P}X \arrow[r, "\chi"] & \mathrm{Funct}(X, \mathbf{F}_2) \\ \mathscr{P}Y \arrow[u, "f^*"] \arrow[r, "\chi"'] & \mathrm{Funct}(Y, \mathbf{F}_2) \arrow[u, "f^*"'] \end{tikzcd}$$

commutes.

1.13.9

Note how close the definitions of the image and of the preimage are to each other: they are both defined by *existential* quantification of the *same* binary relation

$$X, Y \longrightarrow \text{Statements}, \qquad x, y \longmapsto \rho(x, y) \;:=\; \text{“ } f(x) = y \text{ ”} \tag{76}$$

over the corresponding subsets $A \subseteq X$ and $B \subseteq Y$, respectively. We shall often refer to f_* as the *direct image map* and to f^* as the *inverse image map.*[1]

[1] The term “map” is very frequently used today as an alternative term for “function”. This use became established among Mathematical Analysts who preferred to reserve the term “function” for real or complex-valued functions. The word *map* is meant to be an abbreviated form of the word *mapping,* which is a calque from German word *Abbildung,* introduced early in the 20th Century by topologists, writing in German, to denote a function between *spaces.*

1.13.10

Note the equality of sets

$$f^*B = \{x \in X \mid f(x) \in B\}. \tag{77}$$

The right-hand-side of (77) is how the inverse image is usually defined. Such a definition, however, obfuscates the fact that f_* and f^* are "twin sisters".

1.13.11 The *conjugate image* function $f_!$

These two concepts or, if you wish, constructions, naturally associated with every function $f : X \longrightarrow Y$, are omnipresent. One encounters them nearly in every mathematical argument involving functions between sets. What remains a very little known fact is that f^* has yet another "sibling"

$$f_! : \mathscr{P}(X) \longrightarrow \mathscr{P}(Y), \qquad A \longmapsto (f_*(A^c))^c, \tag{78}$$

that I propose to call the *conjugate image* function.

The name, "conjugate image" stems from the fact that $f_!$ is the *conjugate* of f_* by the *complement operation*,

$$f_! = \complement \circ f_* \circ \complement. \tag{79}$$

Caveat: the *inner* complement operation is applied to a subset of X whereas the *outer* complement operation is applied to a subset of Y. When fully expanded the value of $f_!$ on a subset A of X equals

$$f_!A = Y \setminus f_*(X \setminus A).$$

Exercise 18 *Let $A \subseteq X$ and $B \subseteq Y$. Show that*

$$A \subseteq f^*B \quad \textit{if and only if} \quad f_*A \subseteq B\,. \tag{80}$$

Exercise 19 *Show that*

$$f^*(B^c) = (f^*B)^c. \tag{81}$$

1.13.12

Identities (79) and (81) can be expressed by a pair of commutative square diagrams

$$\begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{P}X \arrow[d, "f_!"] \\ \mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] & \mathscr{P}Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{P}X \\ \mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] \arrow[u, "f^*"] & \mathscr{P}Y \arrow[u, "f^*"'] \end{tikzcd} \tag{82}$$

that can be combined into a single diagram

$$\begin{tikzcd} \mathscr{P}X \arrow[r, leftrightarrow, "\complement"] \arrow[d, bend right, "f_*"'] \arrow[dr, phantom, "\circlearrowleft\circlearrowright"] & \mathscr{P}X \arrow[d, bend left, "f_!"] \\ \mathscr{P}Y \arrow[r, leftrightarrow, "\complement"'] \arrow[u, "f^*"'] & \mathscr{P}Y \arrow[u, "f^*"] \end{tikzcd} \tag{83}$$

in which both squares commute.

1.13.13

I used two different circle-arrows to make you aware that in the left diagram in (82), the composite arrows have their source at one of the *top* vertices and their target in the diagonally opposite *bottom* vertex. In the right diagram in (82) the roles are reversed: the composite arrows have their source at one of the *bottom* vertices and their target in the diagonally opposite *top* vertex.

Normally, I will be marking commutativity of any (portion of a) diagram by using the circle-arrow symbol that I consider the most appropriate.

Exercise 20 *Show that*

$$f^*B \subseteq A \quad \text{if and only if} \quad B \subseteq f_!A\,. \tag{84}$$

Exercise 21 *Given an n-ary relation ρ between elements of sets $X_1, \dots, X_n$, let ρ_i be the $(n-1)$-ary relation between elements of sets $X_1, \dots, \hat{X}_i, \dots, X_n$ defined in Section 1.7.4. Show that*

$$\Gamma_{\rho_i} = (\pi_{\hat{i}})_* \Gamma_\rho \tag{85}$$

where

$$\pi_{\hat{i}} : X_1 \times \cdots \times X_n \longrightarrow X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n \tag{86}$$

removes from an ordered n-tuple its i-th component,

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, \hat{x}_i, \dots, x_n).$$

Exercise 22 *Let $g : Y \to Z$ be a function. Show that*

$$(g \circ f)_* = g_* \circ f_*\,, \qquad (g \circ f)^* = f^* \circ g^* \qquad \text{and} \qquad (g \circ f)_! = g_! \circ f_!\,. \tag{87}$$

Exercise 23 *Show that all three functions*

$$(id_{\mathrm{X}})_*\,, \qquad (id_{\mathrm{X}})^* \qquad \text{and} \qquad (id_{\mathrm{X}})_!\,, \tag{88}$$

are equal to the identity function $\mathrm{id}_{\mathscr{P}\mathrm{X}}$ of power-set $\mathscr{P}\mathrm{X}$.

An immediate consequence of identities (87) and (88) is that, for every invertible function f, one has

$$(f^{-1})_* = (f_*)^{-1}\,. \tag{89}$$

Exercise 24 *Show that, for an invertible function f, one has*

$$f^* = (f^{-1})_*\,.$$

Exercise 25 *Let ρ^i be the $(n-1)$-ary relation defined in Section 1.7.1. Show that*

$$\Gamma_{\rho^i} = (\pi_{\hat{i}})_! \Gamma_\rho\,. \tag{90}$$

1.14 Families of sets

1.14.1

A *family of sets* is, by definition, a set whose elements are themselves sets. In a restrictive approach to Set Theory every set is requiered to be of this form. It is possible to develop all of Mathematics within such a restrictive framework.

1.14.2 Notation

A general practice is to denote *elements* of sets by lower case Latin alphabet letters :

$$a, b, c, d, e, f, g, h, i, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, x, z,$$

and to denote *sets* by capital letters :

$$A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, X, Z.$$

1.14.3 Families of sets

A set whose elements are sets is often referred to as a *family of sets*. We shall denote families of sets by capital calligraphic letters :

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I}, \mathscr{J}, \mathscr{K}, \mathscr{L}, \mathscr{M}, \mathscr{O}, \mathscr{P}, \mathscr{Q}, \mathscr{R}, \mathscr{S}, \mathscr{T}, \mathscr{U}, \mathscr{V}, \mathscr{W}, \mathscr{X}, \mathscr{Y}, \mathscr{Z}.$$

1.14.4 The union of a family of subsets of a set

Given a family of subsets $\mathscr{A}$ of a set X, the *union of* $\mathscr{A}$ is the set

$$\bigcup \mathscr{A} \;:=\; \{x \in X \mid \exists_{A \in \mathscr{A}}\, x \in A\} \,. \tag{91}$$

The existence of such a set is guaranteed by the axioms of Set Theory. It is the *smallest* subset of X *containing* each member set $A \in \mathscr{A}$. An alternative notation :

$$\bigcup_{x \in \mathscr{A}} A \,. \tag{92}$$

1.14.5 The intersection of a family of subsets of a set

The set

$$\bigcap \mathscr{A} \;:=\; \{x \in X \mid \forall_{A \in \mathscr{A}}\, x \in A\} \tag{93}$$

is called the *intersection* of (family) $\mathscr{A}$. It is the *greatest* subset of X *contained in* each member set $A \in \mathscr{A}$. An alternative notation

$$\bigcap_{x \in X} A \,. \tag{94}$$

1.14.6

Union and intersection define two canonical functions

$$\mathscr{P}X \overset{\bigcup}{\underset{\bigcap}{\longleftarrow\!\!\!\longleftarrow}} \mathscr{P}\mathscr{P}X\,. \tag{95}$$

Exercise 26 *Let* $\mathscr{A} \subseteq \mathscr{B}$ *(we say, in this case, that* $\mathscr{A}$ *is a* subfamily *of* $\mathscr{B}$. *Show that*

$$\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B} \qquad \textit{and} \qquad \bigcap \mathscr{A} \supseteq \bigcap \mathscr{B}. \tag{96}$$

1.14.7 Union and intersection of the *empty* family of subsets

If $\mathscr{A} = \{A\}$ consists of a single set A, then

$$\bigcup \mathscr{A} = A = \bigcap \mathscr{A}.$$

Since the empty family $\emptyset$ of subsets of X is contained in every family of subsets, in particular in the singleton family $\{\emptyset\}$, the union of the empty family is contained in set $\emptyset$,

$$\bigcup \emptyset \subseteq \bigcup \{\emptyset\} = \emptyset,$$

hence it is the empty set.

Since the empty family $\emptyset$ of subsets of X is contained in the singleton family $\{X\}$, the intersection of the empty family of subsets of X contains set X,

$$\bigcap \emptyset \supseteq \bigcap \{X\} = X,$$

hence it equals X.

1.14.8

The above argument demonstrates that the union of the empty family of subsets of X is the empty set independently of what set X is.

On the other hand, the intersection of the empty family of subsets of X equals X, hence it *does* depend on X.

1.14.9 Selectors of a family

A function $\xi : \mathscr{X} \longrightarrow \bigcup \mathscr{X}$ satisfying the property

$$\forall_{X \in \mathscr{X}}\ \xi(X) \in X \tag{97}$$

is called a *selector* or a *choice function* of family $\mathscr{X}$.

1.14.10 A comment about the use of the quantifier notation

Mathematicians, unless they are logicians or axiomatic-set-theorists, prefer to limit the use of the quantifier symbols in their formulae to those rare occasions when their use clarifies, not obfuscates, the meaning. The reason is partly a reflection of their habits, partly is related to the physiology of human brain perception of abstract symbolic expressions. The defining property of a selector (97) can be also written as:

$$\xi(X) \in X \quad \textit{for every} \quad X \in \mathscr{X}. \tag{98}$$

or, more tersely,

$$\xi(X) \in X \qquad (X \in \mathscr{X}). \tag{99}$$

Each expression (97)–(99) carries exactly the same meaning and can be read in the same way. From now on you will be frequently exposed to notation (99) that eliminates the need to use quantifier symbols in phrases involving only universal quantifiers.

1.14.11 Axiom of Choice

For obvious reasons, no selector exists if family $\mathscr{X}$ contains the empty set $\emptyset$. It is not obvious, however, that a selector exists *for every* family of nonempty sets. *Axiom of Choice* states just that. That statment was proven to be independent of other axioms of Set Theory. Some mathematicians do not accept it automatically while all mathematicians are, generally, cautious when they are forced to use it. Much of Mathematics can be developed without assuming its validity.

1.14.12 The product of a family of sets

The set of all selectors of family $\mathscr{X}$ forms the set

$$\prod \mathscr{X}, \qquad \text{alternately denoted} \qquad \prod_{X \in \mathscr{X}} X, \tag{100}$$

which is called the *product* of (family) $\mathscr{X}$.

1.14.13

Axiom of Choice says:

$$\textit{The product of a family of nonempty sets is nonempty.} \tag{101}$$

1.14.14 An equivalent form of Axiom of Choice

$$\textit{Every surjective function } f: \mathrm{X} \to Y \textit{ is right-invertible.} \tag{102}$$

1.14.15 Independence of Axiom of Choice

It was established long ago that Axiom of Choice is consistent with the remaining axioms of Set Theory. This means that if there are contradictory statements in Mathematics provable with the aid of Axiom of Choice, then there are contradictory statements provable without Axiom of Choice.

It took much longer to resolve the open question whether Axiom of Choice is, or is not, a consequence of the remaining axioms of Set Theory. This was finally resolved by a brilliant mathematician, Paul Cohen, whose demonstrated strength was in Harmonic and Functional Analysis, not in Set Theory or Mathematical Logic. He proved that Axiom of Choice is *not* a consequence of axioms of Set Theory. Statements in Mathematics that are consistent but not provable are said to be *independent* of axioms of Set Theory.

1.15 Canonical functions between the sets-of-families

1.15.1

As we saw in Sections 1.13.1 and 1.13.11, every function $f : X \to Y$ induces three functions between the corresponding power-sets

$$\begin{tikzcd} \mathscr{P}Y \arrow[d, "f^*"] \\ \mathscr{P}X \arrow[u, bend left, "f_*"] \arrow[u, bend right, "f_!"'] \end{tikzcd} \quad . \tag{103}$$

Families of subsets of X are elements of the power-set-of-the-power-set $\mathscr{P}\mathscr{P}X$ and similary for families of subsets of Y. In particular, each of the three functions in diagram (103) induces three functions between the corresponding sets of families of subsets:

$$\begin{matrix} (f_*)_* & (f_*)^* & (f_*)_! \\ (f^*)_* & (f^*)^* & (f^*)_! \\ (f_!)_* & (f_!)^* & (f_!)_! \, . \end{matrix} \tag{104}$$

One can omit parentheses provided one carefully observes the spacing that distinguishes between, e.g., $f_*{}^*$ and $f^*{}_*$.

$$\begin{matrix} f_{**} & f_*{}^* & f_{*!} \\ f^*{}_* & f^{**} & f^*{}_! \\ f_{!*} & f_!{}^* & f_{!!} \end{matrix} \quad . \tag{105}$$

Exercise 27 *Find all functions in diagram* (105) *that are functions from* $\mathscr{P}\mathscr{P}X$ *to* $\mathscr{P}\mathscr{P}Y$.

1.15.2

Of these nine canonical functions between sets of families of subsets, four play an important role in Topology, Measure Theory, Mathematical Analysis, where families of subsets are essential objects of study.

1.15.3

Let $\mathscr{A} \subset \mathscr{P}X$ be a family of subsets of X, let $\mathscr{B} \subset \mathscr{P}Y$ be a family of subsets of Y.

Exercise 28 *Show that*

$$f_*\left(\bigcup \mathscr{A}\right) = \bigcup f_{**}\mathscr{A} \qquad \text{and} \qquad f^*\left(\bigcup \mathscr{B}\right) = \bigcup f^*{}_*\mathscr{B} \tag{106}$$

and express each identity in the form of a commutative diagram.

Exercise 29 *Show that*

$$f^*\left(\bigcap \mathscr{B}\right) = \bigcap f^*{}_*\mathscr{B} \qquad \text{and} \qquad f_!\left(\bigcap \mathscr{A}\right) = \bigcap f_{!*}\mathscr{A} \tag{107}$$

and express each identity in the form of a commutative diagram.[2]

Exercise 30 *Show that*

$$f_*\left(\bigcap \mathscr{A}\right) \subseteq \bigcap f_{**}\mathscr{A} \qquad \text{and} \qquad f_!\left(\bigcup \mathscr{A}\right) \subseteq \bigcup f_{!*}\mathscr{A}\,. \tag{108}$$

In general, $\subseteq$ cannot be replaced by $=$ in (108).

1.16 Indexed families of sets

1.16.1

An indexed family of sets $(X_i)_{i\in I}$ is, by definition, a function from a certain set I to the power-set of a certain set U,

$$I \longrightarrow \mathscr{P}(U)\,, \qquad i \mapsto X_i\,.$$

The standard notation for the value at $i \in I$ is X_i. The set I is referred to as the *indexing set.*

1.16.2 The union and the intersection of an indexed family

Let us denote by $\mathscr{X}$ the *image* of this function in $\mathscr{P}(U)$. It is a family of sets. The union and the intersection of $\mathscr{X}$ are called, respectively, the *union* and the *intersection* of $(X_i)_{i\in I}$, and denoted

$$\bigcup_{i\in I} X_i \qquad \text{and} \qquad \bigcap_{i\in I} X_i\,.$$

Explicitly,

$$\bigcup_{i\in I} X_i \;:=\; \{x \mid \exists_{i\in I}\; x \in X_i\} \tag{109}$$

and

$$\bigcap_{i\in I} X_i \;:=\; \{x \mid \forall_{i\in I}\; x \in X_i\}\,. \tag{110}$$

[2] A hint for both exercises: recall that $\bigcup$ and $\bigcap$ define certain canonical functions, cf. (95).

1.16.3

When the indexing set I is empty, the comments made about the union and the intersection of an empty family of subsets apply, cf. 1.14.8.

1.16.4 Selectors of an indexed family

Functions

$$I \longrightarrow \bigcup_{i\in I} X_i\,, \qquad i \mapsto x_i\,, \tag{111}$$

satisfying

$$x_i \in X_i \qquad (i \in I)\,,$$

could be called *selectors* of indexed family $(X_i)_{i\in I}$. They are more frequently called *I-tuples* because in the case

$$I = \{1, \dots, n\}\,,$$

they correspond to ordered n-tuples of elements of $\bigcup_{i\in I} X_i$.

1.16.5 "Tuple" notation

Standard notation for an I-tuple is $(x_i)_{i\in I}$. The subscript $i \in I$ is usually omitted when the indexind set is understood from the context.

1.16.6 The product of an indexed family of sets

Predictably, the set of all I-tuples of $(X_i)_{i\in I}$ is called the *product* of $(X_i)_{i\in I}$ and is denoted

$$\prod_{i\in I} X_i\,. \tag{112}$$

1.16.7

For $I = \{1, 2\}$, the product is naturally identified with the Cartesian product

$$X_1 \times X_2\,,$$

and, for $I = \{1, \dots, n\}$, it provides the most convenient model of the Cartesian product

$$X_1 \times \cdots \times X_n\,.$$

1.16.8 Canonical projections (π_J)

Restricting a function (111) to a subset $J \subseteq I$ defines a function

$$\pi_J : \prod_{i\in I} X_i \longrightarrow \prod_{i\in J} X_i\,, \tag{113}$$

called the *canonical projection* (associated with a subset J of the indexing set. We have encountered these functions in Section 1.11.15 where $I = \{1, \dots n\}$ and $J = \{i\}$.

1.16.9 Notation

In the interest of keeping notation simple, when, e.g., $J = \{2, 5, 7\}$, we write

$$\pi_{2,5,7} \quad \text{instead} \quad \pi_{\{2,5,7\}}$$

or, even, as

$$\pi_{257}$$

when it it is clear from the context that the elements of J are natural numbers less than 10.

A general rule is to separate the items in a list of subscripts or superscripts by commas when notation is, otherwise, ambiguous, and to omit the commas when no ambiguity arises.

1.16.10 Composition of correspondences

Given correspondences

$$C \subseteq X_0 \times \cdots \times X_{m+1} \qquad \text{and} \qquad D \subseteq X_{m+1} \times \cdots \times X_{m+n+1} ,$$

their preimages under the canonical projections

$$\pi^*_{0,\dots,m+1}C \qquad \text{and} \qquad \pi^*_{m+1,\dots,m+n+1}D$$

are correspondences between elements of sets

$$X_0, \dots, X_{m+n+1} .$$

In particular, we can form their intersection

$$\pi^*_{0,\dots,m+1}C \,\cap\, \pi^*_{m+1,\dots,m+n+1}D$$

and project it into $X_0 \times \cdots \times \hat{X}_{m+1} \times \cdots \times X_{m+n+1}$,

$$(\pi_{\widehat{m+1}})_*(\pi^*_{0,\dots,m+1}C \,\cap\, \pi^*_{m+1,\dots,m+n+1}D) , \tag{114}$$

where

$$\pi_{\widehat{m+1}} \;=\; \pi_{0,\dots,\widehat{m+1},\dots,m+n+1} .$$

We shall denote (114) by $C{\circ}D$.

1.16.11

Explicitly, $C{\circ}D$ consists of $(m + n + 1)$-tuples

$$(x_0, \dots, \hat{x}_{m+1}, \dots, x_{m+n+1})$$

for which there exists $x_{m+1} \in X_{m+1}$ such that

$$(x_0, \dots, x_{m+1}) \in C \qquad \text{and} \qquad (x_{m+1}, \dots, x_{m+n+1}) \in D .$$

1.16.12

It follows that for $C = \Gamma_\rho$ and $D = \Gamma_\sigma$, one has

$$\Gamma_{\rho\circ\sigma} = \Gamma_\rho \circ \Gamma_\sigma . \tag{115}$$

2 The language of mathematical structures

2.1 Mathematical structures

2.1.1 The concept of a mathematical structure

A list of sets

$$X_1, \dots, X_n$$

equipped with some 'data' is what a mathematical structure is. As such, a mathematical structure can be thought of as an ordered pair

$$(X_1, \dots, X_n \,;\, \text{'data'})$$

2.1.2

This sinple concept became a focal point of modern Mathematics because it allows to view many apparently distant phenomena as manifestations of the same general laws.

2.1.3

Functions, operations, relations, are obvious examples of mathematical structures.

2.1.4 Structures of functional type

Sets X equipped with a family $\mathscr{O} \subset \mathrm{Funct}(X, \mathbf{R})$ of real-valued functions on X,

$$(X, \mathscr{O}),$$

are a backbone of Analysis. Think, for example, of a subset X of Euclidean space $\mathbf{R}^n$ and $\mathscr{O}$ being the set of all infinitely differentiable functions on X.

2.1.5 Structures of topological type

Sets X equipped with a family $\mathscr{A} \subset \mathscr{P}X$ of subsets

$$(X, \mathscr{A})$$

are the central objects in Topology, Geometry, Measure Theory, Combinatorics.

2.1.6 Example: topological spaces

A set X equipped with a family of subsets $\mathscr{T} \subset \mathscr{P}X$ closed under formation of *finite* intersections and arbitrary unions is called a *topological space*. Members of $\mathscr{T}$ are referred to as *open subsets*.

2.1.7 Example: measurable spaces

A set X equipped with a family of subsets $\mathscr{M} \subset \mathscr{P}X$ closed under formation of *countable* intersections and under the complement operation $\complement$, cf. Section 1.5.3, is called a *measurable space*. Members of $\mathscr{M}$ are referred to as *measurable subsets*.

2.2 Algebraic structures

2.2.1

Sets X equipped with an indexed family $(\mu_i)_{i\in I}$ of operations on X are called *algebraic structures*. Groups, rings, fields, vector spaces, etc., are all examples of algebraic structures.

2.2.2 Example: groups

A group is an algebraic structure

$$(X\,;\,\mu_0,\mu_1,\mu_2)$$

where μ_2 is a binary operation on X,

$$X,X\longrightarrow X,\qquad x,y\mapsto xy\,,$$

referred to as *the multiplication*, μ_0 is a nullary operation on X,

$$\longrightarrow X,\qquad \langle\text{the empty list}\rangle\longmapsto e\,,$$

referred to as *the identity* element, and μ_1 is a unary operation on X,

$$X\longrightarrow X,\qquad x\mapsto\bar{x}\,,$$

that assigns to an element $x\in X$ its inverse. This family of 3 operations is required to satisfy 3 properties: *Associativity*, (two-sided) *Identity Element Property*, and (two-sided) *Inverse-Element Property*.

2.2.3 Associativity: $\forall_{x,y,z\in X}\,(xy)z=x(yz)\,.$

Associativity of a binary operation μ_2 is equivalent to commutativity of the diagram

$$\begin{tikzcd}
 & X,X,X \arrow[dl, "\mu_2"'] \arrow[dr, "\mu_2"] \arrow[dd, phantom, "\circlearrowleft"] & \\
X,X \arrow[dr, "\mu_2"'] & & X,X \arrow[dl, "\mu_2"] \\
 & X &
\end{tikzcd}\;.\tag{116}$$

2.2.4 Identity Element Property: $\forall_{x\in X}\,ex=x=xe\,.$

The left and the right equalities in the Identity Element Property are equivalently described as commutativity of the left and, respectively, right triangles in the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\mu_0"'] \arrow[dr, "\mu_0"] \arrow[dd, equal] & \\
X,X \arrow[r, phantom, "\circlearrowleft"] \arrow[dr, "\mu_2"'] & {} & X,X \arrow[l, phantom, "\circlearrowright"] \arrow[dl, "\mu_2"] \\
 & X &
\end{tikzcd}\;.\tag{117}$$

2.2.5 Inverse-Element Property: $\forall_{x \in X}\, \bar{x}x = e = x\bar{x}$.

The left and the right equalities in the Inverse-Element Property are equivalently described as commutativity of the left and, respectively, right triangles in the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\mu_1"'] \arrow[dr, "\mu_1"] \arrow[dd, "e_X"] & \\
X,X \arrow[dr, "\mu_2"'] \arrow[rr, phantom, "\circlearrowleft\qquad\circlearrowright"] & & X,X \arrow[dl, "\mu_2"] \\
 & X &
\end{tikzcd} \tag{118}$$

where e_X denotes the *constant* function

$$X \longrightarrow X, \qquad x \longmapsto e \qquad (x \in X).$$

Note that $e_X : X \to X$ is the composite function

$$X \longrightarrow \emptyset^{\emptyset} \xrightarrow{\tilde{e}} X$$

where $X \longrightarrow \emptyset^{\emptyset}$ is the unique function from X to the singleton set $\emptyset^{\emptyset}$ and $\tilde{e}$ is the function of a single variable canonically corresponding to the function of zero variables $e : \longrightarrow X$, cf. Section 1.11.13.

2.2.6 Example: monoids

If we remove from the definition of a group unary operation μ_1 and the Inverse Element Property, we obtain the definition of a *monoid*.

2.2.7 The canonical monoid structure on $\mathrm{Op}_1(X)$

Composition $\circ$ is a canonical binary operation on the set of all unary operations $\mathrm{Op}_1(X)$ on an arbitrary set X. The identity operation id_X is a distinguished element of $\mathrm{Op}_1(X)$. Composition of functions is associative and id_X is an identity element for the operation of composition.

Thus, $(\mathrm{Op}_1(X), \mathrm{id}_X, \circ)$ is a monoid and $\mathrm{Op}_1(X)$ provides an example of a set that is equipped with a canonical structure of a monoid.

2.2.8 Example: semigroups

If we remove from the definition of a group unary operation μ_1, nullary operation μ_0, and the two properties in which these two operations occur, we are left with a set equipped with a single binary operation that satisfies the Associativity Property. Such a structure is called a *semigroup*.

2.2.9

Semigroups, monoids, groups, are encountered everywhere where mathematical considerations are involved.

2.2.10 Pointed sets

Several types of structures frequently encountered outside of College Algebra are also examples of algebraic structures. For example *pointed sets* (X, x_0), i.e., sets equipped with a single nullary operation, i.e., a disinguished element $x_0 \in X$.

2.2.11 Actions of sets on other sets

A set X, equipped with a family of unary operations $(\lambda_a)_{a \in A}$ indexed by a set A, is referred to as a set equipped with an *action of set A*. A short designation for this structure is an *A-set.*

An action of a set A on a set X is the same as a function

$$\lambda : A \longrightarrow \mathrm{Op}_1(X) . \tag{119}$$

We shall use, in general, notation (X, λ) to denote A-sets where λ is a function (119).

2.2.12 Standard multiplicative notation

The value of operation λ_a on an element $x \in X$ is frequently denoted ax.

2.3 Relational structures

2.3.1

Sets X equipped with an indexed family $(\rho_i)_{i \in I}$ of relations on X are called *relational structures.* Such structures are encountered in all areas of Mathematics and especially so in Mathematical Logic and in Incidence Geomtry.

2.3.2

Particularly important are *binary relational structures,* i.e., sets equipped with a single binary relation. We discuss them in Chapter 4 devoted to binary relations.

3 Morphisms

3.1 Interactions between mathematical structures

3.1.1

If mathematical structures are *objects* of mathematical theories, studying a given structure is nearly always executed by observing how that structure *interacts* with other structures of the same type. Binary interactions between structures are expressed in the language of *morphisms*.

3.1.2 The concept of a morphism

A *morphism*

$$(X, \mathrm{data}) \longrightarrow (X', \mathrm{data}') \tag{120}$$

is most commonly understood to be a function between the *underlying sets*

$$f : X \longrightarrow X'$$

that *respects* the corresponding data. It is assumed that the data must be of the same type. The term 'respects' can be replaced by : 'is compatible with'. The meaning of this term is nearly always natural for each type of data. We shall illustrate this for some types of mathematical structures mentioned above.

3.1.3 The arrow notation

Morphisms are represented graphically as arrows. Every arrow has its source and its target, each being a structure of the same type. They are referred to as the *source* and the *target* of a morphism.

3.2 Morphisms between algebraic structures

3.2.1 Homomorphisms

Suppose that a set X is equipped with an n-ary operation μ and a set X' is equipped with an n-ary operation μ'. We say that a function $f : X \to X'$ is *compatible* with the operations if

$$\forall_{x_1,\dots,x_n \in X}\; f\big(\mu(x_1, \dots, x_n)\big) = \mu\big(f(x_1), \dots, f(x_n)\big)\,. \tag{121}$$

Algebraists refer to such functions as *homomorphisms*.

3.2.2

The definition of a morphism between sets equipped with an n-ary operation can be also expressed as commutativity of the following square diagram

$$\begin{tikzcd} X', \dots, X' \arrow[r, "\mu'"] & X' \\ X, \dots, X \arrow[u, "f \,\cdots\, f"] \arrow[r, "\mu"'] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[u, "f"'] \end{tikzcd} \,. \tag{122}$$

3.2.3

The above definition can be easily extended to general algebraic structures. A morphism

$$(X, (\mu_i)_{i\in I}) \longrightarrow (X', (\mu'_i)_{i\in I})$$

is a function $f : X \to X'$ such that it is a homomorphism

$$(X, \mu_i) \longrightarrow (X', \mu'_i)$$

for *each* $i \in I$. Notice that μ_i and μ'_i must have the same 'arity' for every $i \in I$.

The concept of a homomorphism provides the most natural definition of a morphism between algebraic structures.

3.2.4 Example: morphisms between pointed sets

A morphism from a pointed set (X, x_0) to a pointed set (X', x'_0) is, by definition, a function $f : X \to X'$ such that

$$f(x_0) = x'_0 . \tag{123}$$

3.2.5 Example: morphisms between A-sets

A morphism from an A-set (X, λ) to an A-set (X', λ'), cf. Section 2.2.11, is, by definition, a function $f : X \to X'$ such that

$$\forall_{a\in A}\ f \circ \lambda_a = \lambda'_a \circ f \tag{124}$$

or, equivalently, in multiplicative notation,

$$\forall_{a\in A, x\in X}\ f(ax) = af(x) . \tag{125}$$

3.2.6

Condition (124) can be expressed as commutativity of square diagrams

$$\begin{tikzcd} X' \arrow[r, "\lambda'_a"] & X' \\ X \arrow[u, "f"] \arrow[r, "\lambda_a"'] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[u, "f"'] \end{tikzcd} \tag{126}$$

for all $a \in A$.

3.2.7 Actions of binary structures $(A, \cdot)$ on sets

The set of unary operations $\mathrm{Op}_1(X)$ of any set X is canonically equipped with a structure of a monoid. When a set A, that is equipped with a binary operation $\cdot$, is acting on a set X, it is usually assumed that the action function λ in (119) is a homomorphism of binary algebraic structures, i.e., that

$$\forall_{a,b\in A}\ \lambda_{a\cdot b} = \lambda_a \circ \lambda_b . \tag{127}$$

Condition (127) is equivalently expressed as the identity that closely resembles Associativity

$$\forall_{a,b\in A}\ \forall_{x\in X}\ (a \cdot b)x = a(bx) . \tag{128}$$

3.2.8

If the same generic multiplicative notation is used for the binary operation in A and for the action of A on X, then the requirement that λ be a homomorphism takes the form of the identity

$$\forall_{a,b\in A}\,\forall_{x\in X}\ (ab)x = a(bx) \tag{129}$$

that is indistinguishable from Associativity. And for a good reason: Associativity of a binary algebraic structure $(A,\cdot)$ expresses the fact that the structure acts on set A by left-multiplication.

Exercise 31 *Show that a binary algebraic structure $(A,\cdot)$ is associative if and only if the* left-multiplication function

$$\lambda : A \longrightarrow \mathrm{Op}_1(A)\,, \qquad a \longmapsto \lambda_a\,, \tag{130}$$

is a homomorphism of binary algebraic structures. Here $\lambda_a \in \mathrm{Op}_1(A)$ denotes the function that multiplies an arbitrary element of A on the left by a,

$$\lambda_a : A \longrightarrow A\,, \qquad b \longmapsto ab\,.$$

3.2.9 The Regular Action

The action of a semigroup on itself by left-multiplication is referred to as the (left) *regular action*. The regular action is particularly important in Group Theory and in Theory of Group Actions.

3.3 Morphisms between n-ary relations

3.3.1

Consider two n-ary relations

$$\rho : X_1, \dots, X_n \longrightarrow \text{Statements} \qquad \text{and} \qquad \rho' : X'_1, \dots, X'_n \longrightarrow \text{Statements}\,.$$

A natural definition is to declare a function list (55) a *morphism from ρ to ρ'* if

$$\forall_{x_1\in X_1,\dots,x_n\in X_n}\ \rho(x_1,\dots,x_n) \Rightarrow \rho'\big(f_1(x_1),\dots,f_n(x_n)\big)\,. \tag{131}$$

3.3.2 Pulling-back a relation

Given a function list (55) and a relation ρ', we shall refer to

$$(f_1,\dots,f_n)^*\rho' \tag{132}$$

as the *pull-back of ρ'* by $f_1,\dots,f_n$.

Condition (131) is equivalently stated as

$$\rho \Longrightarrow (f_1,\dots,f_n)^*\rho' \tag{133}$$

where $\Longrightarrow$ denotes the *implication* relation on the set $\mathrm{Rel}_n(X)$ of n-ary relations on a set X.

Exercise 32 *Show that $f_1,\dots,f_n$ is a morphism from ρ to ρ' if and only if*

$$\Gamma_\rho \subseteq (f_1\times\cdots\times f_n)^*\Gamma_{\rho'}\,. \tag{134}$$

3.3.3 Morphisms between relational structures

When

$$X_1 = \cdots = X_n = X\,, \qquad X'_1 = \cdots = X'_n = X' \qquad \text{and} \qquad f_1 = \cdots = f_n = f\,,$$

we shall denote the pulled-back relation (132) by $f^*\rho'$.

We say that $f : X \to X'$ is a *morphism from a relational structure* (X, ρ) *to a relational structure* (X', ρ') if ρ *implies* $f^*\rho'$.

3.3.4

Condition (131) can be also expressed in the form of the diagram

$$\begin{tikzcd}
X'_1, \dots, X'_n \arrow[dr, "\rho'"] & \\
\arrow[r, phantom, "\Uparrow"] & \text{Statements} \\
X_1, \dots, X_n \arrow[uu, "f_1 \,\cdots\, f_n"] \arrow[ur, "\rho"'] &
\end{tikzcd} \,. \tag{135}$$

3.3.5 Faithful morphisms between n-ary relations

By replacing *Implicationi* $\Rightarrow$ in Condition (131) by *Equivalence* $\Leftrightarrow$, we obtain a stronger condition

$$\forall_{x_1 \in X_1, \dots, x_n \in X_n}\ \rho(x_1, \dots, x_n) \Leftrightarrow \rho'\big(f_1(x_1), \dots, f_n(x_n)\big)\,. \tag{136}$$

We shall say in this case that the function list $f_1, \dots, f_n$ is a *faithful* morphism from ρ to ρ'.

3.4 Morphisms between structures of functional type

3.4.1

Suppose that a set X is equipped with a family of functions

$$\mathcal{O} \subset \mathrm{Funct}(X, \mathbf{R})$$

and a set X' is equipped with a family of functions

$$\mathcal{O}' \subset \mathrm{Funct}(X', \mathbf{R})\,.$$

We say that a function $f : X \to X'$ is a morphism if, for every $\phi' \in \mathcal{O}'$, the composite function $f^*\phi' = \phi' \circ f$ belongs to $\mathcal{O}$,

$$\forall_{\phi' \in \mathcal{O}'}\ f^*\phi' \in \mathcal{O}\,. \tag{137}$$

3.4.2

An equivalent form of condition (137) is

$$(f^*)_*\mathscr{O}' \subset \mathscr{O}\,. \tag{138}$$

This, in turn, can be expressed in the language of diagrams: a function $f : X \to X'$ is a morphism if the diagram

$$\begin{tikzcd}
\mathscr{O} \arrow[d, hook] & \mathscr{O}' \arrow[d, hook] \\
\mathrm{Funct}(X, \mathbf{R}) & \mathrm{Funct}(X', \mathbf{R}) \arrow[l, "f^*"']
\end{tikzcd}$$

admits a completion to a commutative square diagram

$$\begin{tikzcd}
\mathscr{O} \arrow[d, hook] & \mathscr{O}' \arrow[d, hook] \arrow[l] \\
\mathrm{Funct}(X, \mathbf{R}) & \mathrm{Funct}(X', \mathbf{R}) \arrow[l, "f^*"']
\end{tikzcd}$$

3.5 Morphisms between structures of topological type

3.5.1

Suppose that a set X is equipped with a family of subsets $\mathscr{A} \subset \mathscr{P}X$ and a set X' is equipped with a family of subsets $\mathscr{A}' \subset \mathscr{P}X'$. We say that a function $f : X \to X'$ is a morphism if the preimage under f of every member of family $\mathscr{A}'$ is a member of $\mathscr{A}$,

$$\forall_{A' \in \mathscr{A}'}\, f^*A' \in \mathscr{A}\,. \tag{139}$$

3.5.2

An equivalent form of condition (139) is

$$(f^*)_*\mathscr{A}' \subset \mathscr{A}\,. \tag{140}$$

Notice the similarity to condition (138).

3.5.3

Condition (140) can be expressed by saying that the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[d, hook] & \mathscr{A}' \arrow[d, hook] \\
\mathscr{P}Y & \mathscr{P}Y' \arrow[l, "f^*"']
\end{tikzcd}$$

admits a completion to a commutative square diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[d, hook] & \mathscr{A}' \arrow[d, hook] \arrow[l] \\
\mathscr{P}Y & \mathscr{P}Y' \arrow[l, "f^*"']
\end{tikzcd}$$

3.5.4 Continuous functions

When $\mathscr{A}$ and $\mathscr{A}'$ have the meaning of being the families of *open subsets* in a topological spaces, i.e., when $(X, \mathscr{A})$ and $(X', \mathscr{A}')$ are topological spaces, cf. Section 2.1.6, we obtain the definition of a morphism between topological spaces. This is precisely how a continuous function is defined.

3.5.5 Measurable functions

When $\mathscr{A}$ and $\mathscr{A}'$ have the meaning of being the families of *measurable subsets* in a measurable spaces, i.e., when $(X, \mathscr{A})$ and $(X', \mathscr{A}')$ are measurable spaces, cf. Section 2.1.7, we obtain the definition of a morphism between measurable spaces. This is precisely how a measurable function is defined.

3.5.6

Another condition that can be interpreted as saying that f respects distinguished families of subsets reads

$$\forall_{A \in \mathscr{A}}\; f_* A \in \mathscr{A}' \tag{141}$$

or, equivalently,

$$(f_*)_* \mathscr{A} \subset \mathscr{A}' . \tag{142}$$

Either condition can serve as a definition of a morphism between structures of topological type. It is however the former, (139), that plays a fundamental role in Topology and Measure Theory, not the latter, (141).

3.6 The language of categories

3.6.1

Whatever definition of a morphism between mathematical structures one adopts, it always has the following features

- any morphism α has a *source* and a *target* that are mathematical structures of the same type
- if the source $s(\alpha)$ of a morphism α coincides with the target $t(\beta)$ of a morphism β, then their composition $\alpha \circ \beta$ is defined and

  $$t(\alpha \circ \beta) = t(\alpha) \qquad \text{and} \qquad s(\alpha \circ \beta) = s(\beta)$$

- composition of morphisms is associative, i.e.,

  $$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

  for any composable list α, β, γ of morphisms

3.6.2

The above observations led to the introduction of the concept of a *category*. In a nutshell, a *category* $\mathscr{C}$ consists of two classes, a class $\mathscr{C}_0$ of *objects* and a class $\mathscr{C}_1$ of *morphisms*, equipped with an associative operation of composition of morphisms. You will be introduced to the language of categories gradually.

3.6.3

Various classes of mathematical structures equipped with appropriate classes of morphisms form natural categories. Studying the category of groups is what Group Theory does. Studying the category of rings is what Ring Theory does. Algebraic geometers study the category of *algebraic varieties* and the bigger category of *algebraic schemes*. Topologists study the category of *topological spaces*, and so on.

3.6.4

Every mathematical theory can be expressed in a categorical language. This usually provides an added degree of clarity to a theory and yields insights that are otherwise lost.

3.6.5 Endomorphisms

Morphisms whose source and target coincide with an object c are referred as *endomorphisms* of object c.

3.6.6 The semigroup of endomorphisms

Equipped with composition as its binary operation, the set of endomorphisms of an object c of any category becomes a semigroup, denoted

$$\mathrm{End}_{\mathscr{C}}\, c\,. \tag{143}$$

The semigroups of endomorphisms of various mathematical structures play a fundamental role in nearly every area of Mathematics and Mathematical Pysics.

3.6.7 The monoid of endomorphisms

In many categories objects have a distinguished endomorphism, referred to as the *identity endomorphism*, and denoted id_c or, sometimes, 1_c, which serves as the Identity Element for the binary operation of composition. In this case,

$$(\mathrm{End}_{\mathscr{C}}\, c, \mathrm{id}_c, \circ)$$

is a monoid. For example, the monoid of unary operations $\mathrm{Op}_1(X)$ on a set X is precisely the monoid of endomorphsisms of X viewed as an object of the category of sets.

3.6.8 An action of a set A on an object of a category

If A is a set and c is an object of a category $\mathscr{C}$, we have a ready definition of an *action of A on c* if we notice that $\mathrm{Op}_1(X)$ in (119) coincides with the monoid of endomorphisms of X in the category of sets. Thus, an action of a set A on an object c is defined to be a function

$$\lambda : A \longrightarrow \mathrm{End}_{\mathscr{C}}\, c\,. \tag{144}$$

3.6.9 An action of a binary structure $(A, \cdot)$ on an object of a category

We say that a binary structure $(A, \cdot)$ acts on an object c if the function in (144) is a homomorphism of binary structures.

3.6.10 An action of a monoid $(A, e, \cdot)$ on an object of a *unital* category

We say that a monoid $(A, \cdot)$ acts on an object c of a unital category if the function in (144) is a homomorphism of monoids.

3.6.11 Representation Theory of Groups

Classical Representation Theory studies group actions on the objects of the category of vector spaces over a field k. Such actions are referred to as *k-linear representations* of a given group. The cases $k = \mathbf{R}$ and $k = \mathbf{C}$ produce Real and, respectively, Complex Representation Theory.

3.6.12 Category of k-linear representations of a group

Given a group G, its k-linear representations form naturally objects of a category, and determination of the structure of that category is a central topic of Representation Theory.

3.6.13

Representation Theory has been, beginning from its roots in Linear Algebra in the latter part of 19th Century, an essential area of Mathematics, that had enormous impact on the development of Mathematical Physics in 20th Century. The shear wealth of the methods it employs and applications it produces is a reason why learning Representation Theory is simultaneously obligatory and takes several years of very intensive study.

4 Binary relations

4.1 Preliminaries

4.1.1 Canonical identification $\mathbf{Rel}(X,Y) \longleftrightarrow \mathbf{Rel}(Y,X)$

Given a binary relation

$$\rho : X, Y \longrightarrow \text{Statements}, \tag{145}$$

the *opposite* relation is defined by flipping the two arguments

$$\rho^{\mathrm{op}} : Y, X \longrightarrow \text{Statements}, \qquad y, x \longmapsto \rho(x,y). \tag{146}$$

Note that $(\rho^{\mathrm{op}})^{\mathrm{op}} = \rho$. In particular, assignment

$$\rho \longmapsto \rho^{\mathrm{op}}$$

defines a canonical identification of the sets of binary relations

$$\mathrm{Rel}(X,Y) \longleftrightarrow \mathrm{Rel}(Y,X).$$

4.1.2 A canonical involution on $\mathbf{Rel}_2(X)$

When $X = Y$, operation $(\ )^{\mathrm{op}}$ is a (canonical) involution on the set of binary relations on X.

4.1.3 The infix notation

In view of the fact that binary relations have been used by mathematicians long before the concept of a general relation was formulated and are still the most frequently encountered type of relation, special notation has been employed when binary relations are mentioned. The symbolic expression

$$x_1 \rho x_2$$

has the meaning:

$$\text{Statement } \rho(x_1, x_2) \text{ holds.}$$

4.1.4 The $\sim$ notation

More likely, however, you will see expressions like

$$x_1 \sim x_2, \tag{147}$$

since the symbol $\sim$ and its variants have been adopted as a generic symbol denoting a binary relation. The meaning of (147) is:

the binary relation, denoted $\sim$ *, holds for elements* $x_1 \in X_1$ *and* $x_2 \in X_2$ *.*

The difference between the *functional* notation and the *tilde* notation, when talking about binary relations, is similar to the difference between *direct speech* and *indirect speech*: compare the statements

$$3 < 5$$

and

$$\textit{inequality } 3 < 5 \textit{ holds}.$$

4.2 Binary relations *on* a set: a vocabulary of terms

4.2.1 Various types of binary relations on a set

A binary relation ρ on a set X is said to be:

reflexive if

$$\forall_{x\in X}\, \rho(x,x) \tag{148}$$

symmetric if

$$\forall_{x,y\in X}\, \big(\rho(x,y) \Rightarrow \rho(y,x)\big) \tag{149}$$

antisymmetric if

$$\forall_{x,y\in X}\, \big(\rho(x,y) \Rightarrow \neg\rho(y,x)\big) \tag{150}$$

weakly antisymmetric if

$$\forall_{x,y\in X}\, \big(\rho(x,y) \wedge \neg\rho(y,x) \;\Rightarrow\; x = y\big) \tag{151}$$

transitive if

$$\forall_{x,y,z\in X}\, \big(\rho(x,y) \wedge \rho(y,z) \;\Rightarrow\; \rho(x,z)\big) \tag{152}$$

Exercise 33 *Let $\phi : X' \to X$ be a function and $\phi^*\rho$ be the pulled-back binary relation on X'. Which properties from the list of properties 4.2.1 are inherited by $\phi^*\rho$?*

4.2.2

Of all the properties that a binary relation ρ on a set X may have, by far the most important is its *transitivity*.

4.2.3 Preorder relations

A transitive and reflexive relation is called a *preorder* or a *quasiorder*.

4.2.4 Equivalence relations

A symmetric preorder that is called an *equivalence relation*.

Exercise 34 *Suppose that ρ is a preorder relation on a set X. Show that the conjunction of ρ and its opposite relation ρ^{op}*

$$\rho \wedge \rho^{\mathrm{op}} : X, X \longrightarrow \text{Statements}, \qquad x, y \;\longmapsto\; \text{“}\; \rho(x,y) \wedge \rho(y,x) \;\text{”}, \tag{153}$$

is an equivalence relation on X.

We shall refer to a pair of elements satisfying $\rho \wedge \rho^{\mathrm{op}}$ as *ρ-equivalent*.

4.2.5 Order relations

A weakly antisymmetric preorder is called an *order relation*. A preorder is an order relation precisely when (153) is the *weakest equivalence relation* on X, i.e., when $\rho \wedge \rho^{\mathrm{op}}$ is equipotent with the equality realtion $=$.

4.2.6 Sharp order relations

An antisymmetric transitive relation is called a *sharp-order relation*.

4.2.7 Ordered sets

A set X equipped with an order relation will be called an *ordered set*. We shall use the generic symbol $\preccurlyeq$ to denote the order relation. When using the term "ordered set", remember that it is not a set, it is a binary relational structure $(X, \preccurlyeq)$.

4.2.8 Comments about terminology and notation

To emphasize that elements of an ordered set are not necessarily *comparable*, the adverb "partially" is often placed in front of "ordered". Those who insisted on using the term "partially ordered set" soon began to abbreviate it in typed texts as "p. o. sets." When the abbreviation dots got lost, a monstrous term "poset" was born. My advise: *do not use it*.

Bad habits have a tendency to spread like a virus. For example, the habit of using so called "blackboard-bold" letters in print where one should use upright boldface letters **N**, **Z**, **Q**, **R**, **C** as standard notation for the set of natural numbers, the set of integers, and so on.

4.2.9 Linearly ordered sets

Ordered sets whose elements are *comparable*, i.e., satisfy the condition

$$\forall_{x,y \in X}\; x \preccurlyeq y \vee y \preccurlyeq x, \tag{154}$$

are called *linearly*, or *totally*, ordered.

Naturally defined linear orders are scarce, unlike (partial) orders.

4.2.10 A canonical ordered-set structure on $\mathscr{P}X$

The set-containment relation equips the power-set of any set with a canonical ordered-set structure

$$\subseteq \,:\, \mathscr{P}X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad A, B \longmapsto \text{``}A \subseteq B\text{''}. \tag{155}$$

The ordered-set $(\mathscr{P}X, \subseteq)$ and its opposite, $(\mathscr{P}X, \supseteq)$, prove to play a central role in Mathematics.

4.3 Morphisms between binary relations

4.3.1

The definition of a morphism between n-ary relations was given in Section 3.3. For $n = 2$, it reads: a morphism from a binary relation

$$\rho : X, Y \longrightarrow \text{Statements}$$

to a binary relation

$$\rho' : X', Y' \longrightarrow \text{Statements}$$

consists of a pair of functions $f : X \to X'$ and $g : Y \to Y'$ such that relation ρ implies the relation $(f,g)^*\rho' = \rho' \circ (f,g)$. The definition of a morphism can be expresed by means of the diagram

$$\begin{tikzcd} X',Y' \arrow[dr, "\rho'"] & \\ & \text{Statements} \\ X,Y \arrow[uu, shift right, "f"] \arrow[uu, shift left, "g"'] \arrow[ur, "\rho"'] & \end{tikzcd} . \tag{156}$$

4.3.2 Morphisms between binary relational structures

When $X = Y$, $X' = Y'$ and $f = g$, we obtain the definition of a morphism from a binary relational structure (X, ρ) to a binary relational structure (X, ρ'), cf. Section 3.3.3.

4.3.3 $\sim$-commutative diagrams

A *$\sim$-commutative diagram* is a slight but very significant generalization of a commutative diagram, cf. 1.12.2. Whole areas of advanced modern Mathematics and Mathematical Physics are devoted to studying phenomena expressed in the language of $\sim$-commutative diagrams.

Commutativity of a diagram means that two composable paths of arrows (representating functions between sets), that have a common source and a common target, are equal. If that common target, call it T, is equipped with a binary relation $\sim$, then equality may be replaced by the condition that the corresponding composite functions satisfy the relation induced by $\sim$ on the set of T-valued functions.

Since $\sim$ is not necessarily symmetric, one needs to indicate which of the composite functions appears as the *left* argument and which appears as the *right* argument of the relation in question.

This can be represented in a diagram by placing a small arrow (ideally, a bent double arrow) near the common target of two composable paths of arrows, as is shown in the following simple example. A square-shaped diagram

$$\begin{tikzcd} X \arrow[d, "\upsilon"'] & S \arrow[l, "\varphi"'] \arrow[d, "\psi"] \arrow[dl, phantom, "\sim"] \\ T & Y \arrow[l, "\chi"] \end{tikzcd}$$

expresses the statement

$$\forall_{s \in S}\, \chi(\psi(s)) \sim \upsilon(\varphi(s)),$$

i.e., the composite arrow $\chi \circ \psi$ is in relation, induced by $\sim$, with the composite arrow $\upsilon \circ \varphi$.

We shall usually omit the label ($\sim$ here) when the binary relation on the target is clear from the context.

4.3.4

For the *equality* relation $=$, the class of $=$-comutative diagrams coincides with the class of commutative diagrams.

4.3.5

We encounter $\Longrightarrow$-commutative but, generally, not commutative, diagrams in the definition of a morphism between relations, cf. (135) and (156).

Exercise 35 *Suppose that (f, g) is a morphisms from ρ to ρ'. Show that the square*

$$\begin{tikzcd} \mathscr{P}X' \arrow[r, "R"] & \mathscr{P}Y' \\ \mathscr{P}X \arrow[u, "f_*"] \arrow[r, "R"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g_*"'] \end{tikzcd} \tag{157}$$

is $\supseteq$-commutative, while the square

$$\begin{tikzcd} \mathscr{P}X' & \mathscr{P}Y' \arrow[l, "L"'] \\ \mathscr{P}X \arrow[u, "f_*"] \arrow[ur, phantom, "\subseteq"] & \mathscr{P}Y \arrow[l, "L"] \arrow[u, "g_*"'] \end{tikzcd} \tag{158}$$

is $\subseteq$-commutative.

4.4 Functions naturally associated with a binary relation

4.4.1 Two *set-of-relatives* functions

Given a binary relation (145), we have two associated with it evaluation functions

$$\mathrm{ev}^1\rho : X \longrightarrow \mathrm{Rel}_1(Y) \qquad \text{and} \qquad \mathrm{ev}^2\rho : Y \longrightarrow \mathrm{Rel}_1(X)$$

cf. (12). By composing them with the graph functions

$$\mathrm{Rel}_1(Y) \xrightarrow{\ \Gamma\ } \mathscr{P}Y \qquad \text{and} \qquad \mathrm{Rel}_1(X) \xrightarrow{\ \Gamma\ } \mathscr{P}X\ ,$$

we obtain a pair of functions

$$X \longrightarrow \mathscr{P}Y, \qquad x \longmapsto [x\rangle_\rho := \{y \in Y \mid \rho(x, y)\}, \tag{159}$$

and, respectively,

$$Y \longrightarrow \mathscr{P}X, \qquad y \longmapsto {}_\rho\langle y] := \{x \in X \mid \rho(x, y)\}. \tag{160}$$

When the context allows that, we shall simplify notation by omitting the subscript denoting the relation. We shall refer to $[x\rangle$ as the set of *right relatives* of $x \in X$, and to $\langle y]$ as the set of *left relatives* of $y \in Y$.

Accordingly, we shall refer to (159) as the *right-relatives* function, and to (160) as the *left-relatives* function.

4.4.2

Let me pause before we make the next steps of our excursion into the land of binary relations, to make the following remark: even the *nearest* things we see when observing the conceptual territory surrounding the concept of a binary relation, reveal omnipresence of the fundamental mechanism of Modern Mathematics, that of a *pair of adjoint functors*, cf. Section 1.1.15. That mechanism will accompany us all the way to the end of our short inquiry, and will manifest itself both in *where* we get to and *how* we get there. This is one of the many miracles of Mathematics, perhaps its most important. In order to see it, however, one must be wearing "algebraic" glasses.

4.4.3 Initial and terminal elements

When $[x\rangle = Y$, we say that an element $x \in X$ is *initial* (for a given binary relation ρ). When $\langle y] = X$, we say that an element $y \in Y$ is *terminal*.

When ρ is an order relation on a set X, an initial element is unique when it exists. In Theory of Ordered Sets that unique initial element is said to be the *smallest element* of (X, ρ).

Similarly, a terminal element in an ordered set is unique when it exists. That unique element is said to be the *greatest element* of (X, ρ).

4.4.4

The pair of assignments

$$\begin{tikzcd} & \rho \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ [\;\rangle_\rho & & {}_\rho\langle\;] \end{tikzcd} \tag{161}$$

defines the pair of functions

$$\begin{tikzcd} & \mathrm{Rel}(X,Y) \arrow[dl] \arrow[dr] & \\ \mathrm{Funct}(X,\mathscr{P}Y) & & \mathrm{Funct}(Y,\mathscr{P}X) \end{tikzcd} \tag{162}$$

Exercise 36 *Show that*

$${}_{\rho^{\mathrm{op}}}\langle\;] = [\;\rangle_\rho \qquad \textit{and} \qquad [\;\rangle_{\rho^{\mathrm{op}}} = {}_\rho\langle\;].$$

Exercise 37 *Show that function* $\mathrm{Rel}(X,Y) \longrightarrow \mathrm{Funct}(X,\mathscr{P}Y)$ *in* (162) *is surjective.*

Exercise 38 *Show that, for any* $\rho, \sigma \in \mathrm{Rel}(X,Y)$, *one has*

$${}_\rho\langle\;] = {}_\sigma\langle\;] \qquad \textit{if and only if} \qquad \rho \Leftrightarrow \sigma.$$

4.4.5

It follows that the left- and the right-relatives functions induce a canonical identification of the set of *equipotence* classes of binary relations between elements of sets X and Y, with the sets of functions $\mathrm{Funct}(X,\mathscr{P}Y)$ and, respectively, $\mathrm{Funct}(Y,\mathscr{P}X)$.

4.4.6 The preorders on X and Y canonically associated with $\rho \in \mathbf{Rel}(X, Y)$

We shall denote by $\gtrsim$ the relation $\subseteq$ pulled-back to X from $\mathscr{P}Y$ by $[\ \rangle$,

$$x \gtrsim x' \quad \text{if} \quad [x\rangle \subseteq [x'\rangle \qquad (x, x' \in X)\,. \tag{163}$$

We shall denote by $\lesssim$ the relation $\subseteq$ pulled-back to Y from $\mathscr{P}X$ by $\langle\ ]$,

$$y \lesssim y' \quad \text{if} \quad \langle y] \subseteq \langle y'] \qquad (y, y' \in Y)\,. \tag{164}$$

4.4.7

By construction, $\gtrsim$ is the *strongest* binary relation on X such that $[\ \rangle : (X, \lesssim) \to (\mathscr{P}Y, \subseteq)$ is a morphism. Similarly, $\lesssim$ is the *strongest* binary relation on Y such that $\langle\ ] : (Y, \lesssim) \to (\mathscr{P}X, \subseteq)$ is a morphism.

Exercise 39 *Consider the relation ρ_f associated with a function $f : X \to Y$ between arbitrary sets, cf.* (43)*. Describe the relations $\gtrsim$ and $\lesssim$.*

4.4.8

Suppose $X = Y$. In that case all three relations, ρ, $\lesssim$ and $\gtrsim$, are members of the same set $\mathrm{Rel}_2(X)$ of binary relations on X, which is preordered by the $\Longrightarrow$ relation. This leads to the following natural questions that I am stating as exercises.

Exercise 40 *Characterize binary relations $\rho \in \mathrm{Rel}_2(X)$ such that $\rho \Longrightarrow \lesssim$.*[3]

Exercise 41 *Characterize binary relations $\rho \in \mathrm{Rel}_2(X)$ such that $\lesssim \Longrightarrow \rho$.*

Exercise 42 *State the analogs of the above two exercises for $\gtrsim$ instead of $\lesssim$.*

4.5 Functions $\mathscr{P}X \leftrightarrows \mathscr{P}Y$ canonically associated with a binary relation.

4.5.1 $R : \mathscr{P}X \longrightarrow \mathscr{P}Y$

Given a subset $A \subseteq X$ we obtain a family $([x\rangle)_{x \in A}$ of subsets of Y whose intersection,

$$RA := \bigcap_{x \in A} [x\rangle = \{y \in Y \mid \forall_{x \in A}\, \rho(x, y)\}\,, \tag{165}$$

consists of those elements of Y that are right relatives of *every* element of A.

Exercise 43 *Given a family $\mathscr{A} \subseteq \mathscr{P}X$ of subsets of X, show that*

$$\bigcap R_* \mathscr{A} = R\left(\bigcup \mathscr{A}\right)\,. \tag{166}$$

[3] *Characterize* means: 1° Find a property of ρ, that can be stated directly in terms of ρ and is as simple as possible, that holds precisely when $\rho \Longrightarrow \lesssim$; 2° then prove that.

4.5.2 The set of upper bounds of a subset of an ordered set

In Theory of Ordered Sets, RA is called the *set of upper bounds of a subset* $A \subseteq X$. A subset $A \subseteq X$ is said to be *bounded above* when RA is not empty.

4.5.3 Supremum of a subset

If $RA = [\alpha\rangle$, for some element $\alpha \in X$, we shall say that α is a *supremum* of a subset $A \subseteq X$. This term is yet another adoption from Theory of Ordered Sets. A supremum of A is unique when ρ is an order relation on X. In that case we denote it

$$\sup A. \tag{167}$$

For a general relation ρ, any two suprema[4] are $\gtrsim$-equivalent, cf. Section 4.4.6 and Exercise 34. One can still use notation (167) understanding, however, that the element of X denoted $\sup A$ is defined uniquely only up to a $\gtrsim$-equivalence.

4.5.4 $\mathscr{P}X \longleftarrow \mathscr{P}Y : L$

Given a subset $B \subseteq Y$ we obtain a family $\big(\langle y]\big)_{y \in B}$ of subsets of X whose intersection,

$$LB := \bigcap_{y \in B} \langle y] = \Big\{x \in X \mid \forall_{y \in B}\, \rho(x, y)\Big\}, \tag{168}$$

consists of those elements of X that are left relatives of *every* element of B.

4.5.5 The set of lower bounds of a subset of an ordered set

In Theory of Ordered Sets, LB is called the *set of lower bounds of a subset* $B \subseteq Y$. A subset $B \subseteq Y$ is said to be *bounded below* when LB is not empty.

4.5.6 Infimum of a subset

If $LB = \langle \beta]$, for some element $\beta \in Y$, we shall say that β is an *infimum* of a subset $B \subseteq Y$. That element is unique when ρ is an order relation on Y. In that case we denote it

$$\inf B. \tag{169}$$

For a general relation ρ, any two infima are $\lesssim$-equivalent and the use of notation (169) is subject to the same caveat as in the case of $\sup A$.

4.5.7 $R\emptyset = Y$ and $L\emptyset = X$

Note that $R\emptyset = Y$ and $L\emptyset = X$. In particular, $\emptyset \subseteq X$ is bounded above precisely when X is not empty. Suprema of $\emptyset \subseteq X$ are precisely initial elements of X, cf. Section 4.4.3.

Similarly, $\emptyset \subseteq Y$ is bounded below precisely when Y is not empty. Infima of $\emptyset \subseteq Y$ are precisely terminal elements of Y.

[4]The plural of the neuter noun *supremum* is *suprema*.

Exercise 44 (a) *Show that*

$$RX = \{y \in Y \mid y \text{ is a terminal element of } Y\} \tag{170}$$

and

$$LY = \{x \in X \mid x \text{ is an initial element of } X\}\,. \tag{171}$$

(b) *Show that*

$$\xi \in X \text{ is a supremum of } X \quad \Leftrightarrow \quad \xi \text{ is a smallest element of preordered set } (X, \succsim) \tag{172}$$

and

$$\upsilon \in Y \text{ is an infimum of } Y \quad \Leftrightarrow \quad \upsilon \text{ is a smallest element of preordered set } (Y, \precsim)\,. \tag{173}$$

4.5.8 Example: $(\mathscr{P}X, \subseteq)$

Given a subset $\mathscr{A} \subseteq \mathscr{P}X$, i.e., a family of subsets of a set X, the union of $\mathscr{A}$ is the smallest subset of X that contains every member of $\mathscr{A}$ and the intersection of $\mathscr{A}$ is the greatest subset of X that is contained in every member of $\mathscr{A}$. Thus, the supremum and infimum exist for every subset of the ordered set $(\mathscr{P}X, \subseteq)$ and they coincide with the union and, respectively, the intersection of family $\mathscr{A}$,

$$\sup \mathscr{A} = \bigcup \mathscr{A} \qquad \text{and} \qquad \inf \mathscr{A} = \bigcap \mathscr{A}\,. \tag{174}$$

4.5.9

Identity (166) and a similar identity for L can be both expressed in the form of a pair of commutative diagrams

$$\begin{tikzcd} \mathscr{P}X \arrow[d, "\cap"'] & \mathscr{P}Y \arrow[l, "L"'] \arrow[d, "\cup"] \arrow[dl, phantom, "\circlearrowleft"] \\ \mathscr{P}X & \mathscr{P}Y \arrow[l, "L"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathscr{P}X \arrow[r, "R"] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{P}Y \\ \mathscr{P}X \arrow[u, "\cup"] \arrow[r, "R"'] & \mathscr{P}Y \arrow[u, "\cap"'] \end{tikzcd}\,. \tag{175}$$

Exercise 45 *Consider the* membership *relation*

$$\in\, : X, \mathscr{P}X \longrightarrow \text{Statements}, \qquad x, A \longmapsto \text{“}x \in A\text{”}. \tag{176}$$

Determine RA and $L\mathscr{A}$ for $A \subseteq X$ and $\mathscr{A} \subseteq \mathscr{P}X$.

Exercise 46 *Consider the set-containment relation* (155). *Determine $R\mathscr{A}$ and $L\mathscr{B}$ for $\mathscr{A} \subseteq \mathscr{P}X$ and $\mathscr{B} \subseteq \mathscr{P}X$.*

Exercise 47 *Consider the relation ρ_f associated with a function $f : X \to Y$ between arbitrary sets, cf.* (43). *Determine $R\mathscr{A}$ and $L\mathscr{B}$ for $\mathscr{A} \subseteq \mathscr{P}X$ and $\mathscr{B} \subseteq \mathscr{P}X$.*

Exercise 48 *Show that R is a morphism of ordered sets $(\mathscr{P}X, \subseteq) \to (\mathscr{P}Y, \supseteq)$ and L is a morphism of ordered sets $(\mathscr{P}Y, \subseteq) \to (\mathscr{P}X, \supseteq)$.*

4.5.10 The preorders on $\mathscr{P}X$ and $\mathscr{P}Y$ canonically associated with $\rho \in \mathbf{Rel}(X,Y)$

Let us denote by $\overset{\mathrm{R}}{\subseteq}$ the relation $\supseteq$ pulled-back from $\mathscr{P}Y$ to $\mathscr{P}X$ by function R,

$$A \overset{\mathrm{R}}{\subseteq} A' \quad \text{if} \quad RA \supseteq RA' \qquad (A, A' \subseteq X)\,, \tag{177}$$

and let us denote by $\overset{\mathrm{L}}{\subseteq}$ the relation $\supseteq$ pulled-back from $\mathscr{P}X$ to $\mathscr{P}Y$ by function L,

$$B \overset{\mathrm{L}}{\subseteq} B' \quad \text{if} \quad LB \supseteq LB' \qquad (B, B' \subseteq Y)\,. \tag{178}$$

In view of the universal property of the pulled-back relation, cf. Section 3.3.3, relation $\overset{\mathrm{R}}{\subseteq}$ is a strongest relation on $\mathscr{P}X$ for which R is a morphism into $(\mathscr{P}Y, \supseteq)$. Similarly, $\overset{\mathrm{L}}{\subseteq}$ is a strongest relation on $\mathscr{P}Y$ for which L is a morphism into $(\mathscr{P}X, \supseteq)$. By combining this remark with Exercise 48, we deduce that both $\overset{\mathrm{R}}{\subseteq}$ and $\overset{\mathrm{L}}{\subseteq}$ are stronger than the containment relation,

$$\subseteq \implies \overset{\mathrm{R}}{\subseteq} \qquad \text{on } \mathscr{P}X \tag{179}$$

and

$$\subseteq \implies \overset{\mathrm{L}}{\subseteq} \qquad \text{on } \mathscr{P}Y\,. \tag{180}$$

In the following exercises, A denotes an arbitrary subset of X and B denotes an arbitrary subset of Y.

Exercise 49 *Show that*

$$A \subseteq LB \quad \textit{if and only if} \quad RA \supseteq B\,. \tag{181}$$

Exercise 50 *Show that*

$$A \subseteq LRA \qquad \textit{and} \qquad RLB \supseteq B. \tag{182}$$

Exercise 51 *Show that*

$$LRLB = LB \qquad \textit{and} \qquad RA = RLRA. \tag{183}$$

4.5.11

In other words, the pair of functions (L, R) satisfies the following identities

$$LRL = L \qquad \text{and} \qquad RLR = R\,, \tag{184}$$

and the unary operation LR on X, as well as the unary operation RL on Y, are *idempotent*, i.e.,

$$(LR)^2 = LR \qquad \text{and} \qquad (RL)^2 = RL\,. \tag{185}$$

Exercise 52 *Show that*

$$RL[x\rangle = [x\rangle\,. \tag{186}$$

Exercise 53 *Show that*

$$\rho(x,y) \Leftrightarrow L[x\rangle \subseteq \langle y] \Leftrightarrow [x\rangle \supseteq R\langle y]\,. \tag{187}$$

Solution. One has the following sequence of equivalences and implications

$$\rho(x,y) \Leftrightarrow [x\rangle \ni y \Leftrightarrow [x\rangle \supseteq \{y\} \Rightarrow L[x\rangle \subseteq \langle y] \Rightarrow RL[x\rangle \supseteq R\langle y] \overset{(186)}{\Rightarrow} [x\rangle \supseteq R\langle y] \Rightarrow [x\rangle \ni y \Leftrightarrow \rho(x,y)\,.$$

□

4.5.12 Ordered sets $\mathscr{L}(\rho)$ and $\mathscr{R}(\rho)$

Consider the images of power-sets $\mathscr{P}Y$ and $\mathscr{P}X$ under L and, respectively, R,

$$\mathscr{L}(\rho) := L_*\mathscr{P}Y \qquad \text{and} \qquad \mathscr{R}(\rho) := R_*\mathscr{P}X\,. \tag{188}$$

Exercise 54 *Show that, for any subsets B_1 and B_2 of Y, one has*

$$LB_1 \subseteq LB_2 \quad \textit{if and only if} \quad LB_1 \overset{\mathrm{R}}{\subseteq} LB_2 \tag{189}$$

and, for any subsets A_1 and A_2 of X, one has

$$RA_1 \subseteq RA_2 \quad \textit{if and only if} \quad RA_1 \overset{\mathrm{L}}{\subseteq} RA_2\,. \tag{190}$$

It follows that on $\mathscr{L}(\rho) \subseteq \mathscr{P}X$ the preorder relation $\overset{\mathrm{R}}{\subseteq}$ is equipotent with the order relation $\subseteq$, and similarly for the preorder relation $\overset{\mathrm{L}}{\subseteq}$ on $\mathscr{R}(\rho) \subseteq \mathscr{P}Y$. In particular, both preorders are order relations on those two subsets of the corresponding power-sets.

4.5.13 A canonical isomorphism $(\mathscr{L}(\rho), \subseteq) \simeq (\mathscr{R}(\rho), \supseteq)$

Restriction of $R : \mathscr{P}X \to \mathscr{P}Y$ to $\mathscr{L}(\rho)$ induces a morphism of ordered sets

$$(\mathscr{L}(\rho), \subseteq) \longrightarrow (\mathscr{R}(\rho), \supseteq)\,.$$

Similarly, restriction of $L : \mathscr{P}Y \to \mathscr{P}X$ to $\mathscr{R}(\rho)$ induces a morphism

$$(\mathscr{R}(\rho), \supseteq) \longrightarrow (\mathscr{L}(\rho), \subseteq)\,.$$

In view of identities (183), those two morphisms are inverse to each other.

4.5.14 Case: ρ a symmetric relation

For a symmetric relation on a set X, one has $R = L$ and Identities (184) become a single identity

$$R^3 = R\,. \tag{191}$$

Moreover, $\mathscr{R}(\rho) = \mathscr{L}(\rho)$ and R induces a canonical *antiinvolution* of $(\mathscr{L}(\rho), \subseteq)$, i.e., a morphism

$$(\mathscr{L}(\rho), \subseteq) \xrightarrow{\;R\;} (\mathscr{L}(\rho), \subseteq)^{\mathrm{op}} \qquad \text{such that} \qquad R \circ R = \mathrm{id} \tag{192}$$

where $(\mathscr{L}(\rho), \subseteq)^{\mathrm{op}}$ coincides with $(\mathscr{L}(\rho), \supseteq)$.

4.5.15 $(\mathscr{P}X, \subseteq)$ represented as $(\mathscr{L}(\rho), \subseteq)$

When ρ is the *nonequality* relation $\neq$ on a set X, we obtain

$$(\mathscr{P}X, \subseteq) = (\mathscr{L}(\neq), \subseteq) \qquad \text{and} \qquad R = \complement\,. \tag{193}$$

Exercise 55 *Prove* (193).

4.6 Right- and left-complete binary relations

4.6.1

We shall say that a binary relation ρ is *right-complete* when every subset $A \subseteq X$ has a supremum, cf. Section 4.5.3, i.e.,

$$[\ \rangle_* X = \mathscr{R}(\rho)\,. \tag{194}$$

We shall say that a binary relation ρ is *left-complete* when every subset $B \subseteq Y$ has an infimum, cf. Section 4.5.6, i.e.,

$$\langle\ ]_* Y = \mathscr{L}(\rho)\,. \tag{195}$$

4.6.2 Bicomplete binary relations

We shall say that a binary relation ρ is *bicomplete* if it is both right- and left-complete. A central fact of our theory is comprised by the following proposition.

Proposition 4.1 *For every binary relation ρ, ordered set $(\mathscr{L}(\rho), \subseteq)$ is bicomplete.*

Proof. Given a family $\mathscr{B}$ of subsets of Y, intersection of family $L_*\mathscr{B} = \{LB \mid B \in \mathscr{B}\}$ is the greatest subset of X contained in each member of family $L_*\mathscr{B}$ and, according to the left commutative square in (175), is a member of $\mathscr{L}(\rho)$:

$$\bigcap L_*\mathscr{B} = L\left(\bigcup \mathscr{B}\right)\,.$$

Every subset of $\mathscr{E} \subseteq \mathscr{L}(\rho)$ is a family of subsets of X of the form $L_*\mathscr{B}$ for some $\mathscr{B} \subseteq \mathscr{P}Y$. Indeed,

$$\mathscr{E} = L_*(R_*\mathscr{E})\,.$$

It follows that every subset of $\mathscr{L}(\rho)$ has infimum in $(\mathscr{L}(\rho), \subseteq)$. A similar argument demonstrates that every subset of $\mathscr{R}(\rho)$ has infimum in $(\mathscr{R}(\rho), \subseteq)$ and, henceforth, has supremum in $(\mathscr{R}(\rho), \supseteq)$.

Since $(\mathscr{L}(\rho), \subseteq)$ is isomorphic to $(\mathscr{R}(\rho), \supseteq)$, cf. Section 4.5.13, every subset of $\mathscr{L}(\rho)$ has supremum in $(\mathscr{L}(\rho), \subseteq)$. $\square$

Note how elegant is the above argument.

Exercise 56 *Given $\mathscr{E} \subseteq \mathscr{P}X$, find a formula for* $\sup \mathscr{E}$ *in* $(\mathscr{L}(\rho), \subseteq)$.

Exercise 57 *Determine $[\ \rangle_* X$, $\langle\ ]_* Y$, $\mathscr{R}(\rho_f)$, and $\mathscr{L}(\rho_f)$, for the relation associated with a function $f : X \to Y$ between arbitrary sets, cf.* (43).

4.6.3 Restricted notions of completeness

Given a family of subsets $\mathscr{A} \subseteq \mathscr{P}X$, we could say that a binary relation ρ is (right) *$\mathscr{A}$-complete* if

$$[\ \rangle_* X \supseteq R_*\mathscr{A} \tag{196}$$

and, similarly, given a family of subsets $\mathscr{B} \subseteq \mathscr{P}Y$, we could say that ρ is (left) *$\mathscr{B}$-complete* if

$$\langle\ ]_* Y \supseteq L_*\mathscr{B}\,. \tag{197}$$

4.6.4 Completness restricted to nonempty *bounded* subsets

Following the practice of Theory of Ordered Sets, let us say that a subset $A \subseteq X$ is *right-bounded* if $RA \neq \emptyset$, and a subset $B \subseteq Y$ is *left-bounded* if $LA \neq \emptyset$. Let us denote the family of all nonempty right-bounded subsets of X by

$$\mathscr{P}_{\triangleright} X := \{A \subseteq X \mid A \neq \emptyset \wedge RA \neq \emptyset\} \tag{198}$$

and the family of all nonempty right-bounded subsets of X by

$$\mathscr{P}_{\triangleleft} Y := \{B \subseteq Y \mid B \neq \emptyset \wedge LB \neq \emptyset\}\,. \tag{199}$$

Exercise 58 *Show that the functions L and R induce mutually inverse functions*

$$R_* \mathscr{P}_{\triangleright} X \underset{R}{\overset{L}{\rightleftarrows}} L_* \mathscr{P}_{\triangleleft} Y \quad . \tag{200}$$

Lemma 4.2 *If a binary relation ρ is right $\mathscr{P}_{\triangleright} X$-complete and*

$$L_* \{[x\rangle \mid x \in X \wedge [x\rangle \neq \emptyset\} \;\subseteq\; \langle\;]_* Y\,, \tag{201}$$

then ρ is left $\mathscr{P}_{\triangleright} Y$-complete.

Proof. If $B \in \mathscr{P}_{\triangleleft} Y$ and ρ is right $\mathscr{P}_{\triangleright} X$-complete, then

$$LB = RLB = L[\alpha\rangle$$

where $\alpha \in X$ is a supremum of RLB and $[\alpha\rangle \supseteq B \neq \emptyset$. Condition (201) implies that $LB = L[\alpha\rangle = \langle\beta]$ for a certain $\beta \in Y$. ☐

Condition (201) is obviously satisfied when ρ is a preorder relation on a set X. In that case, one has

$$\forall_{x \in X}\;\langle x] = L[x\rangle \wedge R\langle x] = [x\rangle\,.$$

Corollary 4.3 *A preorder relation on a set X is right $\mathscr{P}_{\triangleright} X$-complete if and only if it is left $\mathscr{P}_{\triangleleft} X$-complete.* ☐

4.6.5 Terminology: *complete* (pre)ordered sets

Such sets, in Theory of Ordered Sets, are said to be *complete*.

4.6.6 Finite completeness

The case of the family of all *finite* subsets is of particular importance. We shall refer to the corresponding binary relations as being *finitely* right-, or left-complete.

4.7 Semilattices

4.7.1 $\vee$-semilattices

An ordered set $(X, \preccurlyeq)$ is finitely right-complete precisely when it has the *smallest* element (frequently denoted 0), that coincides with supremum of $\emptyset \subseteq X$, and any two-element subset has a supremum which in this case becomes a commutative binary operation on X,

$$a, b \longmapsto a \vee b := \sup\{a, b\} \qquad (a, b \in X), \tag{202}$$

referred to as the *join* operation. Join is, by definition, commutative and every element is an *idempotent*,

$$\forall_{a \in X} \; a \vee a = a. \tag{203}$$

Exercise 59 *Show that join is associative and that*

$$\forall_{a \in X} \; 0 \vee a = a,$$

i.e., 0 is an identity element for binary operation $\vee$.

4.7.2 The associated monoid $(X, 0, \vee)$

In Theory of Ordered Sets, finitely right-complete ordered sets are called $\vee$- or *join semilattices*. The associated commutative monoid $(X, 0, \vee)$ satisfies Identity (203). In Algebra, any comutative monoid $(A, e, \cdot)$ such that every element is idempotent, is called a *semilattice*.

4.7.3

Connection between $\vee$-semilattices and semilattices in the sense of Algebra goes either way.

Exercise 60 *Given a semilattice $(A, e, \cdot)$, define the binary relation on X,*

$$\rho : A, A \longrightarrow \text{Statements}, \qquad a, b \longmapsto \text{“ } ab = b \text{ ”}. \tag{204}$$

Show that ρ is an order relation, e is the smallest element, and

$$\forall_{a, b \in A} \; \sup\{a, b\} = ab.$$

4.7.4

The monoid canonically associated to a $\vee$-semilattice, and the $\vee$-semilattice canonically associated to a semilattice in the sense of Algebra are constructions that are *natural*, i.e., they extend in a unique manner to a pair of *functors* between the corresponding categories.

For algebraic structures, homomorphisms provide a natural definition of a *morphism*, cf. Section 3.2.1. This does not correspond, however, to the standard definition of a morphism between ordered sets. Homomorphisms between semilattices correspond to, what we shall call below, *finitely* sup *-exact* morphisms.

4.7.5 $\wedge$-semilattices

An ordered set $(X, \preccurlyeq)$ is finitely left-complete precisely when it has the *greatest* element (frequently denoted 1), that coincides with infimum of $\emptyset \subseteq X$, and any two-element subset has a infimum which in this case becomes a commutative binary operation on X,

$$a, b \longmapsto a \wedge b := \inf\{a, b\} \qquad (a, b \in X), \tag{205}$$

referred to as the *meet* operation. Like join, meet is, by definition, commutative and every element is an *idempotent*,

$$\forall_{a \in X} \; a \wedge a = a .$$

If $(X, \preccurlyeq)$ is a meet semilattice, then $(X, \preccurlyeq)^{\mathrm{op}}$ is a join semilattice, and vice-versa. In particular, meet is an associative operation and 1 serves as its identity element.

In Theory of Ordered Sets, finitely left-complete ordered sets are called $\wedge$- or *meet semilattices*.

4.7.6 Terminology: lattices

In Theory of Ordered Sets, finitely bicomplete ordered sets are called *lattices*.

4.8 Complete lattices

4.8.1

In Theory of Ordered Sets, bicomplete ordered sets are called *complete lattices*.

In Section 4.5.8 we saw that power-set $(\mathscr{P}X, \subseteq)$ is a complete lattice. Above we established that $(\mathscr{L}(\rho), \subseteq)$ and $(\mathscr{R}(\rho), \subseteq)$ are complete lattices for any binary relation ρ.

4.8.2 The complete lattice of algebraic substructures

Let $\big(X, (\mu_i)_{i \in I}\big)$, be an algebraic structure, cf. Section 2.2.1. A subset $A \subseteq X$ that is *closed* under every operation μ_i, $i \in I$, inherits those operations from X and is an algebraic structure of the same type. We call such "inherited" structure a *substructure* of $\big(X, (\mu_i)_{i \in I}\big)$.

The family of all subsets of X that are closed under operations μ_i forms an ordered subset of power-set $(\mathscr{P}X, \subseteq)$. We shall denote it $\mathrm{Substr}\big(X, (\mu_i)_{i \in I}\big)$.

It follows immediately from the definition that intersection of any family $\mathscr{A} \subseteq \mathscr{P}X$ of subsets that are closed under operations μ_i is itself closed under those operations. Thus, every family of substructures has infimum.

4.8.3 An algebraic substructure generated by a subset

Given any subset $E \subseteq X$, the intersection of the family of subsets that are closed under the operations and containg E, is the *smallest* subset of X with that property. We call this the *substructure generated by a subset* $E \subseteq X$ and often denote it

$$\langle E \rangle .$$

Note that the above mentioned family is not empty: E is contained in X and X is closed under the operations by definition.

4.8.4

Given a family $\mathscr{A} \subseteq \mathrm{Substr}\big(X, (\mu_i)_{i \in I}\big)$, any substructure that contains every member of $\mathscr{A}$ will contain also their union,

$$\bigcup \mathscr{A}\,. \tag{206}$$

The union of even two subsets closed under an algebraic operation is closed under that operation only in exceptional circumstances (consider for example, the union of two lines passing through the origin in the 2-dimensional vector space). The substructure generated by subset (206) of X is, obviously, the smallest substructure containing every member of family $\mathscr{A}$.

In conclusion, for every $\mathscr{A} \subseteq \mathrm{Substr}\big(X, (\mu_i)_{i \in I}\big)$, supremum and infimum exist and are given by

$$\sup \mathscr{A} = \Big\langle \bigcup \mathscr{A} \Big\rangle \qquad \text{and} \qquad \inf \mathscr{A} = \bigcap \mathscr{A}\,. \tag{207}$$

Thus, the ordered set $\big(\mathrm{Substr}\big(X, (\mu_i)_{i \in I}\big), \subseteq\big)$ is a complete lattice, irrespective of what family of algebraic operations we consider.

4.9 Right- and left-exact functions

4.9.1

Given binary relations ρ and ρ' as in Section 4.3.1, and an arbitrary function

$$f : X \to X'\,,$$

we shall say that f is *right-exact* if

$$f_* : \big(\mathscr{P}X, \overset{\mathrm{R}}{\subseteq}\big) \longrightarrow \big(\mathscr{P}X', \overset{\mathrm{R}}{\subseteq}\big) \tag{208}$$

is a morphism of preordered sets, cf. Section 4.5.10. Similarly, given an arbitrary function

$$g : Y \to Y'\,,$$

we shall say that g is *left-exact* if

$$g_* : \big(\mathscr{P}Y, \overset{\mathrm{L}}{\subseteq}\big) \longrightarrow \big(\mathscr{P}Y', \overset{\mathrm{L}}{\subseteq}\big) \tag{209}$$

is a morphism of preordered sets.

4.9.2 Exact functions between binary relational structures

When $X = Y$, $X' = Y'$ and a function f is both right- and left-exact, we shall say that f is *exact*.

4.9.3 $\sup$- and $\inf$-exact functions

We shall say that f is $\sup$*-exact* if, for every subset $A \subseteq X$ that has a supremum, say $\alpha \in X$, its image under f has $f(\alpha)$ as a supremum. Similarly, we shall say that g is $\inf$*-exact* if, for every subset $B \subseteq Y$ that has an infimum, say $\beta \in Y$, its image under g has $g(\beta)$ as a supremum.

Exercise 61 *Show that*

$$\forall_{x_1,x_2\in X}\ \sup\{x_1,x_2\}=x_1 \Leftrightarrow x_1 \succsim x_2 \tag{210}$$

and

$$\forall_{y_1,y_2\in Y}\ \inf\{y_1,y_2\}=y_1 \Leftrightarrow y_1 \precsim y_2\,. \tag{211}$$

Exercise 62 *Show that a* $\sup$ *-exact function* f *is a morphism of preorded sets*

$$f:(X,\succsim)\longrightarrow (X',\succsim)\,. \tag{212}$$

Show that an $\inf$ *-exact function* g *is a morphism of preorded sets*

$$g:(Y,\precsim)\longrightarrow (Y',\precsim)\,. \tag{213}$$

Proposition 4.4 *A right-exact function* $f:X\to X'$ *is* $\sup$ *-exact. If* ρ *is right-complete and* f *is* $\sup$ *-exact, then* f *is right-exact.*

Similarly, a left-exact function $g:Y\to Y'$ *is* $\inf$ *-exact. If* ρ *is left-complete and* g *is* $\inf$ *-exact, then* g *is left-exact.*

Proof. Given subsets $A_1 \overset{\mathrm{R}}{\subseteq} A_2$ of X, with suprema α_1 and α_2, respectively, we observe that

$$[\alpha_1\rangle = RA_1 \supseteq RA_2 = [\alpha_2\rangle\,.$$

If f is $\sup$-exact, then

$$Rf_*A_1 = [f(\alpha_1)\rangle \qquad \text{and} \qquad [f(\alpha_2)\rangle = Rf_*A_2\,.$$

Since f is a morphism (212), cf. Exercise 62 one has

$$[f(\alpha_1)\rangle \supseteq [f(\alpha_2)\rangle\,.$$

It follows that $f_*A_1 \overset{\mathrm{R}}{\subseteq} f_*A_2$.

If ρ is right-complete, then the above argument applies to any pair A_1, A_2 of subsets of X.

The other case is proved similarly, or is reduced to the one already proven by passing to the opposite relations ρ^{op} and $(\rho')^{\mathrm{op}}$. □

4.9.4 Example: *supremum* as a right-exact function

Let $E \subseteq S$ be a subset of an ordered set $(S,\preccurlyeq)$ such that every subset $A \subseteq E$ has supremum in $(S,\preccurlyeq)$. Recall that in an ordered set a supremum element is unique when it exists. Consider the corresponding morphism of ordered sets

$$\mathbf{sup}:(\mathscr{P}E,\subseteq)\longrightarrow (S,\preccurlyeq)\,, \qquad A\longmapsto \sup A\,. \tag{214}$$

Lemma 4.5 *Supremum morphism* (214) *is right-exact.*

Proof. Recall that, for an order relation ρ on a set S, the associated preorder $\gtrsim$ coincides with ρ^{op}. Any function $f : S \to S'$ between sets equipped with a binary relation is a morphism $(S, \rho) \to (S', \rho')$ if and only if it is a morphism for the opposite relations, $(S, \rho^{\mathrm{op}}) \to (S', (\rho')^{\mathrm{op}})$. In particular, for functions between ordered sets, we can replace condition stating that f in (212) is a morphism by the equivalent condition that $f : (S, \rho) \to (S', \rho')$ is a morphism.

We first prove that morphism (214) is sup-exact, then recall that $(\mathscr{P}E, \subseteq)$ is a complete lattice and, finally, invoke Proposition 4.4.

Let $\mathscr{A} \subseteq \mathscr{P}E$ be a family of subsets of E. The following calculation[5]

$$R\,\mathbf{sup}_*\mathscr{A} = R\left(\bigcup_{A\in\mathscr{A}}\{\sup A\}\right) = \bigcap_{A\in\mathscr{A}} R\{\sup A\} = \bigcap_{A\in\mathscr{A}}[\sup A\rangle = \bigcap_{A\in\mathscr{A}} RA = R\left(\bigcup\mathscr{A}\right) = \left[\sup\left(\bigcup\mathscr{A}\right)\right\rangle \tag{215}$$

demonstrates that $\mathbf{sup}_*\mathscr{A}$ has supremum in $(S, \preccurlyeq)$ and that

$$\sup \mathbf{sup}_*\mathscr{A} = \mathbf{sup}\left(\bigcup\mathscr{A}\right). \tag{216}$$

Thus, morphism (214) is sup-exact. $\square$

4.9.5 A basis of a complete lattice

When supremum morphism (214) is surjective, we say that subset $E \subseteq S$ is sup-*dense* in $(S, \preccurlyeq)$. Morphism (214) is an isomorphism precisely when every element $s \in S$ is the supremum of a unique subset of E. In this case we shall refer to E as a *basis* of complete lattice $(S, \preccurlyeq)$.

4.9.6 Description of right-exact morphisms *from* a complete lattice with a basis

If

$$f : (S, \preccurlyeq) \longrightarrow (S', \preccurlyeq') \tag{217}$$

is a right-exact morphism and φ is its restriction to E, then

$$f(s) = \sup \varphi_* A \qquad \text{where } A \subseteq E \text{ is the unique subset such that } s = \sup A$$

which shows that the restriction function

$$\mathrm{Hom}^{\mathrm{rex}}_{\mathrm{OrdSet}}\left((S, \preccurlyeq), (S', \preccurlyeq')\right) \longrightarrow \mathrm{Funct}(E, S'), \qquad f \longmapsto f_{|E}, \tag{218}$$

from the set of right-exact morphisms $(S, \preccurlyeq) \to (S', \preccurlyeq')$ to the set of arbitrary functions $E \to S'$, is injective.

Any function $\varphi : E \to S'$ canonically induces a right-exact morphism

$$\tilde{\varphi} : (\mathscr{P}E, \subseteq) \longrightarrow (S', \preccurlyeq'), \qquad \tilde{\varphi} := \sup \circ \varphi_*,$$

provided

$$\textit{every member of family } \varphi_*\mathscr{P}E \textit{ has supremum in } (S', \preccurlyeq'). \tag{219}$$

[5] For added clarity, when sup stands for the morphism between ordered sets, (214), I make it stand out in the above calculation by using the boldface font.

Then, by precomposing $\tilde{\varphi}$ with the inverse isomorphism

$$\mathbf{sup}^{-1} : (S, \preccurlyeq) \longrightarrow (\mathscr{P}E, \subseteq)$$

we obtain a right-exact morphism $f_\varphi : (S, \preccurlyeq) \longrightarrow (S', \preccurlyeq')$ whose restriction to E coincides with φ. It follows that (218) defines a bijective correspondence between right-exact morphisms (217) and functions $\varphi : E \to S'$ that satisfy Condition (219).

In particular, we establish the following lemma.

Lemma 4.6 *If $E \subseteq S$ is a basis of a complete lattice $(S, \preccurlyeq)$, then restriction to E defines a canonical bijection between the set of right-exact morphisms from $(S, \preccurlyeq)$ to any right-complete ordered set $(S', \preccurlyeq')$, and the set of arbitrary functions $E \to S'$.*

□

4.9.7 Right-exact morphisms $(\mathscr{P}X, \subseteq) \longrightarrow (\mathscr{P}Y, \supseteq)$

As a corollary we obtain the following important result.

Theorem 4.7 *Every right exact morphism*

$$f : (\mathscr{P}X, \subseteq) \longrightarrow (\mathscr{P}Y, \supseteq) \tag{220}$$

between the power-sets of arbitrary sets X and Y has the form of the canonical R-function associated with some binary relation $\rho : X, Y \longrightarrow \text{Statements}$, cf. Section 4.5.1.

More precisely, correspondence

$$R \longleftarrow \rho$$

defines a canonical bijection

$$\mathrm{Hom}^{\mathrm{rex}}_{\mathrm{OrdSet}}\left((\mathscr{P}X, \subseteq), (\mathscr{P}Y, \supseteq)\right) \longleftrightarrow \left\{ \begin{array}{c} \textit{Equipotence classes of binary relations} \\ \rho : X, Y \longrightarrow \text{Statements} \end{array} \right\}. \tag{221}$$

Proof. Every function

$$\varphi : X \longrightarrow \mathscr{P}Y \tag{222}$$

has the form of the right-relatives function

$$[\ \rangle : X \longrightarrow \mathscr{P}Y$$

for the binary relation

$$\rho_\varphi : X, Y \longrightarrow \text{Statements}, \qquad x, y \longmapsto \text{“}\ \varphi(x) \ni y\ \text{”}. \tag{223}$$

According to the proof of Lemma 4.6, for any function (222),

$$f \coloneqq \sup_{(\mathscr{P}Y, \supseteq)} \circ\, \varphi$$

is a right-exact morphism (220). Explicitly, for any $A \subseteq X$, one has

$$f(A) = \sup_{(\mathscr{P}Y,\supseteq)} \varphi_* A = \bigcap_{x \in A} \varphi_*(x) = \bigcap_{x \in A} [x\rangle = RA$$

where R is the canonical R-function associated with relation ρ_φ.

Moreover, according to Lemma 4.6, any right-exact morphism $f : (\mathscr{P}X, \subseteq) \longrightarrow (\mathscr{P}Y, \supseteq)$ has this form for a unique function (222).

Finally, equipotence classes of binary relations $\rho : X, Y \longrightarrow \text{Statements}$ are in bijective correspondence with the associated right-relatives functions $[\ \rangle : X \to \mathscr{P}Y$. $\square$

4.9.8 Complete description of right- and left-exact morphisms between power-sets

By pre- or post-composing the canonical R-functions of binary relations $\rho : X, Y \longrightarrow \text{Statements}$,

$$\complement \circ R, \qquad R \circ \complement, \qquad \complement \circ R \circ \complement,$$

with the complement antiinvolution $\complement$ that is both right- and left-exact, we obtain also complete description of right-exact morphisms between power-sets ordered by any combination of $\subseteq$ and $\supseteq$ relations. As a consequence, we also obtain a complete decription of left-exact morphisms between power-sets.

We can make the corresponding descriptions even more explicit.

Corollary 4.8 *Every right exact morphism*

$$f : (\mathscr{P}X, \subseteq) \longrightarrow (\mathscr{P}Y, \subseteq) \tag{224}$$

between the power-sets of arbitrary sets X and Y has the form

$$\rho_* : A \longmapsto \bigcup_{x \in A} [x\rangle \tag{225}$$

for some binary relation $\rho : X, Y \longrightarrow$ Statements*, and that relation is unique up to equipotence.*

Exercise 63 *Show that*

$$\rho_* = \complement \circ R_\neg \tag{226}$$

where $R_\neg$ is the R-function associated with the negated relation $\neg\rho$.

Corollary 4.9 *Every left exact morphism*

$$g : (\mathscr{P}Y, \subseteq) \longrightarrow (\mathscr{P}X, \subseteq) \tag{227}$$

between the power-sets of arbitrary sets X and Y has the form

$$_*\rho : B \longmapsto \bigcup_{y \in B} \langle y] \tag{228}$$

for some binary relation $\rho : X, Y \longrightarrow$ Statements*, and that relation is unique up to equipotence.*

4.9.9 Finitely exact functions

By analogy with our discussion of restricted notions of completeness of a binary relation, we can also consider similarly restricted notions of exactness of functions. Below we shall focus on the case of *finitely* right-, left-, sup-, and inf-, exact functions.

We begin from a pair of simple observations.

Exercise 64 *Show that*

$$\forall_{x_1,x_2\in X}\ x_1 \gtrsim x_2 \Leftrightarrow \sup\{x_1,x_2\}=x_1 . \tag{229}$$

Lemma 4.10 *A finitely* sup *-exact function* $f : X \to X'$ *is a morphism* $(X,\gtrsim) \to (X',\gtrsim')$.

Proof. If f is finitely sup-exact, then

$$x_1 \gtrsim x_2 \ \Leftrightarrow\ \sup\{x_1,x_2\} = x_1 \ \Rightarrow\ i\sup f_*\{x_1,x_2\} = x_1 = \sup\{f(x_1),f(x_2)\} = x_1 .$$

□

It follows that all four types of finitely exact functions are necessarily morphisms $(X,\gtrsim) \to (X',\gtrsim')$ or, respectively, $(Y,\lesssim) \to (Y',\lesssim')$.

4.9.10 Finitely biexact functions between power-sets

Note that a morphism $f : (\mathscr{P}X,\subseteq) \to (\mathscr{P}Y,\subseteq)$ is finitely right-exact if and only if

$$f(\emptyset) = \emptyset \qquad \text{and} \qquad \forall_{A_1,A_2\subseteq X}\ f(A_1\cup A_2) = f(A_1)\cup f(A_2) . \tag{230}$$

It is finitely left-exact if and only if

$$f(X) = Y \qquad \text{and} \qquad \forall_{A_1,A_2\subseteq X}\ f(A_1\cap A_2) = f(A_1)\cap f(A_2) . \tag{231}$$

For any subsets E and F of any set S the pair of conditions

$$E\cap F = \emptyset \qquad \text{and} \qquad E\cup F = S \tag{232}$$

is equivalent to the sigle condition

$$\complement E = F .$$

Lemma 4.11 *A finitely biexact morphism* $f : (\mathscr{P}X,\subseteq) \to (\mathscr{P}Y,\subseteq)$ *is a homomorphism of algebraic unary structures*

$$f : (\mathscr{P}X,\complement) \longrightarrow (\mathscr{P}Y,\complement) ,$$

i.e., $\complement\circ f = f\circ\complement$.

Exercise 65 *Prove Lemma 4.11.*

4.9.11 Complete description of exact morphisms between power-sets

We are ready to prove that every biexact function from $(\mathscr{P}X, \subseteq)$ to $(\mathscr{P}Y, \subseteq)$ coincides with the preimage morphism ϕ^* of a certain function $\phi : Y \to X$.

Theorem 4.12 *If $f : (\mathscr{P}X, \subseteq) \to (\mathscr{P}Y, \subseteq)$ is right-exact and finitely left-exact then it is left-exact.*
Any such function coincides with the preimage morphism

$$f = \phi^* \tag{233}$$

for a unique function

$$\phi : Y \longrightarrow X\,.$$

Proof. The first assertion of the theorem is an immediate corollary of Lemma 4.11. By combining sup-exactness of f with its finite inf-exactness we obtain the following chain of equalities

$$Y = f(X) = f\left(\bigcup_{x \in X} \{x\}\right) = \bigcup_{x \in X} f\big(\{x\}\big)\,.$$

Finite inf-exactness implies that sets $f\big(\{x\}\big)$ are disjoint for different $x \in X$,

$$\forall_{x_1, x_2 \in X}\; x_1 \neq x_2 \Rightarrow f\big(\{x_1\}\big) \cap f\big(\{x_2\}\big) = \emptyset\,.$$

Thus, f coincides with the *fiber* function

$$X \longrightarrow \mathscr{P}Y\,, \qquad x \longmapsto \phi^*\{x\}\,,$$

of the function

$$\phi : Y \longrightarrow X\,, \qquad y \;\longmapsto\; \text{the unique } x \in X \text{ such that } f\big(\{x\}\big) \ni y\,.$$

□

4.10 The canonical tower of morphisms

4.10.1

Every binary relation defines a canonical morphism to the binary relation

$${}_L\!\subseteq\, : \mathscr{P}Y, \mathscr{P}X \longrightarrow \text{Statements}\,, \qquad B, A \;\longmapsto\; \text{“ } LB \subseteq A \text{ ”}\,. \tag{234}$$

Exercise 66 *Show that the diagram*

$$\begin{tikzcd}
\mathscr{P}Y, \mathscr{P}X \arrow[dr, "{}_L\subseteq"] & \\
& \text{Statements} \\
X, Y \arrow[uu, shift left, "\langle\,\,\text{]}"'] \arrow[uu, shift right, "\text{[}\,\,\rangle"] \arrow[ur, "\rho"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}\;, \tag{235}$$

is $\Longrightarrow$-commutative, i.e., $[\;\rangle, \langle\;] : (X, Y, \rho) \to (\mathscr{P}Y, \mathscr{P}X, {}_L\subseteq)$ is a morphism.

4.10.2

Diagram (235) admits completion to the diagram

$$
\begin{tikzcd}
\mathscr{P}Y, \mathscr{P}X \arrow[ddr, "{}_L\subseteq"] & \\
\mathscr{R}(\rho), \mathscr{L}(\rho) \arrow[u, hook] \arrow[dr, dotted] & \\
[\,\rangle_* X, \langle\,]_* Y \arrow[u, hook] \arrow[r, dotted] & \text{Statements} \\
X, Y \arrow[u, two heads, "{[\,\rangle}", "{\langle\,]}"'] \arrow[ur, "\rho"'] \arrow[ur, phantom, bend left, "\Uparrow"] &
\end{tikzcd}
\qquad . \tag{236}
$$

The dotted arrows are the corresponding restrictions of relation ${}_L\subseteq$. In particular, the middle and the top triangles in (236) strictly commute.

We shall refer to diagram (236) as the *canonical tower associated with a binary relation* ρ.

4.10.3

Consider the bottom triangle of the canonical tower:

$$
\begin{tikzcd}
[\,\rangle_* X, \langle\,]_* Y \arrow[dr, "{}_L\subseteq"] & \\
& \text{Statements} \\
X, Y \arrow[uu, "{[\,\rangle}", "{\langle\,]}"'] \arrow[ur, "\rho"'] \arrow[ur, phantom, bend left, "\Uparrow"] &
\end{tikzcd}
\qquad . \tag{237}
$$

Lemma 4.13 *Let us denote by R' and L' the corresponding R- and, respectively, L-functions of the relation*

$$
{}_L\subseteq\, : [\,\rangle_* Y, \langle\,]_* X \longrightarrow \text{Statements}\,. \tag{238}
$$

One has the following commutative diagrams

$$
\begin{tikzcd}
\mathscr{P}\big([\,\rangle_* Y\big) \arrow[r, "R'"] & \mathscr{P}\big(\langle\,]_* X\big) \\
\mathscr{P}X \arrow[u, "{[\,\rangle_*}"] \arrow[r, "R"'] \arrow[ur, phantom, "\circlearrowright"] & \mathscr{P}Y \arrow[u, "{\langle\,]_*}"']
\end{tikzcd}
\qquad \textit{and} \qquad
\begin{tikzcd}
\mathscr{P}\big([\,\rangle_* Y\big) & \mathscr{P}\big(\langle\,]_* X\big) \arrow[l, "L'"'] \\
\mathscr{P}X \arrow[u, "{[\,\rangle_*}"] \arrow[ur, phantom, "\circlearrowleft"] & \mathscr{P}Y \arrow[u, "{\langle\,]_*}"'] \arrow[l, "L"]
\end{tikzcd}
\qquad . \tag{239}
$$

Proof. In view of the definition of relation (238), one has, for $A \subseteq X$,

$$
R'\big([\,\rangle_* A\big) = \{\langle y] \in \langle\,]_* Y \mid \forall_{x \in A}\, L[x\rangle \subseteq \langle y]\}\,.
$$

In view of (187), one has

$$
RA = \{y \in Y \mid L[x\rangle \subseteq \langle y]\}\,.
$$

Hence $\langle\,]_* RA = R'[\,\rangle_* A$. The commutativity of the second square in (239) has a similar proood. $\square$

4.10.4 Biexact morphisms

We shall say that a morphism between binary relations $f, g : (X, Y, \rho) \to (X', Y', \rho')$ is *biexact* if f is right-exact and g is left-exact.

Corollary 4.14 *Canonical morphism* (237) *is biexact.*

Proof. Given subsets $A_1, A_2 \subseteq X$, one has the following sequence of equivalences and implications

$$A_1 \overset{R}{\subseteq} A_2 \Leftrightarrow RA_1 \supseteq RA_2 \Rightarrow R'([\,\rangle_* A_1) = \langle\,]_* RA_1 \supseteq \langle\,]_* RA_2 = R'([\,\rangle_* A_2) \Leftrightarrow [\,\rangle_* A_1 \overset{R'}{\subseteq} [\,\rangle_* A_2 .$$

Right-exactness of $[\,\rangle$ follows. Left-exactness of $\langle\,]$ is a corollary of the second commutative square in (239). □

4.10.5 Right- and left-dense functions

Given binary relations ρ and ρ' as in Section 4.3.1, and an arbitrary function $f : X \to X'$, we shall say that f is *right-dense* if

$$R \circ L = R \circ f_* \circ f^* \circ L . \tag{240}$$

Similarly, given an arbitrary function $g : Y \to Y'$, we shall say that g is *left-dense* if

$$L \circ R = L \circ g_* \circ g^* \circ R . \tag{241}$$

Lemma 4.15 *Let $f, g : (X, Y, \rho) \to (X', Y', \rho')$ be a morphism between binary relations. The following conditions are equivalent:*

(a) *Morphism f, g is faithful, cf. Section 3.3.5.*

(b) $g^* \circ R \circ f_* = R$.

(c) $f^* \circ L \circ g_* = L$.

Proof. Let $A \subseteq X$ and $y \in Y$. One has

$$g^* R f_* A \ni y \;\Leftrightarrow\; \forall_{a \in A}\, \rho'(f(a), g(y)) . \tag{242}$$

By definition, f, g is a faithful morphism if

$$\forall_{a \in A}\, \rho(a, y) \Leftrightarrow \rho'(f(a), g(y)) . \tag{243}$$

By combining (242) with (243) we obtain

$$\forall_{A \subseteq X} \forall_{y \in Y}\; g^* R f_* A \ni y \Leftrightarrow RA \ni y ,$$

if f, g is faithful. This demonstrates implication (a)$\Rightarrow$(b). I leave the proof of the reverse implication (b)$\Rightarrow$(a) as a simple exercise.

Exercise 67 *Show that, for a morphism f, g and $x \in X$, one has*

$$g^*[f(x)\rangle = [x\rangle \quad \Leftrightarrow \quad \left(\forall_{y \in Y}\, \rho'(f(x), g(y)) \Leftrightarrow \rho(x,y) \right). \tag{244}$$

Equivalence (a)$\Leftrightarrow$(c) is the same as equivalence (a)$\Leftrightarrow$(b) for the morphism $g, f : \rho^{\mathrm{op}} \longrightarrow (\rho')^{\mathrm{op}}$. The latter is a faithful morphism if and only if $f, g : \rho \to \rho'$ is faithful. □

Lemma 4.16 *Let $f, g : (X, Y, \rho) \to (X', Y', \rho')$ be a faithful morphism between binary relations. If g is left-dense, then f is right-exact and vice-versa: if f is right-dense, then g is left-exact.*

Proof. Let $A_1, A_2 \subseteq X$. Faithfulness of f, g yields, in view of Lemma 4.15, the sequence of equivalences and implications

$$A_1 \overset{R}{\subseteq} A_2 \Leftrightarrow RA_1 \supseteq RA_2 \Leftrightarrow g^*Rf_*A_1 \supseteq g^*Rf_*A_2 \Rightarrow g_*g^*Rf_*A_1 \supseteq g_*g^*Rf_*A_2 \Rightarrow Lg_*g^*Rf_*A_1 \subseteq Lg_*g^*Rf_*A_2 \,. \tag{245}$$

Left-density of g yields the sequence of equivalences

$$Lg_*g^*Rf_*A_1 \subseteq Lg_*g^*Rf_*A_2 \Leftrightarrow LRf_*A_1 \subseteq LRf_*A_2 \Leftrightarrow Rf_*A_1 \supseteq Rf_*A_2 \Leftrightarrow f_*A_1 \overset{R}{\subseteq} f_*A_2 \,. \tag{246}$$

By combining (245) and (246), we deduce that f is right-exact. Left-exactness of g follows in a similar manner from faithfulness of f, g and right-density of f. □

4.10.6 Bidense morphisms

We shall say that a morphism between binary relations $f, g : (X, Y, \rho) \to (X', Y', \rho')$ is *bidense* if f is right-dense and g is left-dense.

Corollary 4.17 *A faithful bidense morphism between binary relations is biexact.*

□

4.10.7

Consider the middle triangle of the canonical tower:

$$\begin{tikzcd}
\mathscr{R}(\rho), \mathscr{L}(\rho) \arrow[dr, "L^{\subseteq}"] & \\
 & \text{Statements} \\
[\;\rangle_* X, \langle\;]_* Y \arrow[uu, shift left] \arrow[uu, shift right] \arrow[ur, "L^{\subseteq}"'] &
\end{tikzcd} \tag{247}$$

4.10.8 Calculations in $(\mathscr{R}(\rho), \mathscr{L}(\rho), {}_L\!\subseteq)$

Let us denote by $[\;\rangle''$ and $\langle\;]''$ the right-relatives function and, respectively, the left-relatives function of the relation

$$ {}_L\!\subseteq \,:\, \mathscr{R}(\rho), \mathscr{L}(\rho) \longrightarrow \text{Statements}\,, \tag{248}$$

and let us denote by R'' and by L'' the associated R- and L-functions between the power-sets of $\mathscr{R}(\rho)$ and $\mathscr{L}(\rho)$.

Lemma 4.18 *One has the following commutative diagrams*

$$\begin{tikzcd} \mathscr{P}\mathscr{L} \arrow[r, "L''"] & \mathscr{P}\mathscr{R} \\ \mathscr{R} \arrow[u, "{[\;\rangle''}"] \arrow[r, "L"'] \arrow[ur, phantom, "\circlearrowright"] & \mathscr{L} \arrow[u, "{\langle\;]''}"'] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathscr{P}\mathscr{L} & \mathscr{P}\mathscr{R} \arrow[l, "R''"'] \\ \mathscr{R} \arrow[u, "{[\;\rangle''}"] \arrow[ur, phantom, "\circlearrowleft"] & \mathscr{L} \arrow[l, "R"] \arrow[u, "{\langle\;]''}"'] \end{tikzcd} \tag{249}$$

where $\mathscr{R} = \mathscr{R}(\rho)$ *and* $\mathscr{L} = \mathscr{L}(\rho)$.

Proof. Given families $\mathscr{E} \subseteq \mathscr{L}$ and $\mathscr{F} \subseteq \mathscr{R}$, one has

$$L''\mathscr{E} = \{F \in \mathscr{R} \mid \forall_{E\in\mathscr{E}}\, LF \subseteq E\} = \{F \in \mathscr{R} \mid LF \subseteq \textstyle\bigcap \mathscr{E}\} = \big\langle \textstyle\bigcap \mathscr{E}\big]'' \tag{250}$$

and

$$\begin{aligned} R''\mathscr{F} &= \{E \in \mathscr{E} \mid \forall_{F\in\mathscr{F}}\, LF \subseteq E\} = \{E \in \mathscr{E} \mid \forall_{F\in\mathscr{F}}\, F \supseteq RE\} \\ &= \{E \in \mathscr{E} \mid \textstyle\bigcap \mathscr{F} \supseteq RE\} = \{E \in \mathscr{E} \mid L\textstyle\bigcap \mathscr{F} \subseteq E\} = \big[\textstyle\bigcap \mathscr{F}\big\rangle''\,. \end{aligned} \tag{251}$$

This shows that relation $(\mathscr{R}(\rho), \mathscr{L}(\rho), {}_L\!\subseteq)$ is complete which is not surprising in view of the fact that it is canonically isomorphic to the complete lattices $(\mathscr{R}(\rho), \mathscr{R}(\rho), \supseteq)$ and $(\mathscr{L}(\rho), \mathscr{L}(\rho), \subseteq)$.

Since

$$\textstyle\bigcap \{E \in \mathscr{L} \mid LF \subseteq E\} = LF\,,$$

we obtain the identity

$$L''[F\rangle'' = \big\langle \textstyle\bigcap [F\rangle''\big]'' = \langle LF]'' \tag{252}$$

that is equivalent to the commutativity of the left square in (249).

Exercise 68 *Prove commutativity of the right square is in* (249).

Lemma 4.19 *Canonical inclusion morphism* (247) *is bidense.*

Proof. Let $g : \langle\;]_*Y \hookrightarrow \mathscr{L}$ be the canonical inclusion. Given a family $\mathscr{E} \subseteq \mathscr{L}$, one has

$$g_* g^* \mathscr{E} = \mathscr{E} \cap \langle\;]_* Y\,.$$

Given $F \in \mathscr{R}$, one has

$$[F\rangle'' \cap \langle\;]_*Y = \{\langle y] \in \mathscr{L} \mid LF \subseteq \langle y]\}\,,$$

hence

$$\bigcap\big([F\rangle'' \cap \langle\ ]_* Y\big) = \bigcap\{\langle y] \in \mathscr{L} \mid LF \subseteq \langle y]\} = LF\,.$$

Combined with identities (250) and (252), this yields the identity

$$L''\left([F\rangle'' \cap \langle\ ]_* Y\right) = \langle LF] = L''[F\rangle''$$

and left-density of $g : \langle\ ]_* Y \hookrightarrow \mathscr{L}$ follows.

Exercise 69 *Prove right-density of the canonical inclusion $f : [\ \rangle_* X \hookrightarrow \mathscr{R}$.*

By combining Lemma 4.19 with Lemma 4.16, we obtain the following corollary.

Corollary 4.20 *Canonical inclusion morphism (247) is faithful, bidense, hence biexact.*

□

4.11 Completion of a binary relation

4.11.1

The bottom and the middle morphisms in the canonical tower (253),

$$\begin{tikzcd}
\mathscr{R}(\rho), \mathscr{L}(\rho) \arrow[drr, "L^{\subseteq}"] & & \\
{[\ \rangle_* X, \langle\ ]_* Y} \arrow[u, hook] \arrow[rr, "L^{\subseteq}"] \arrow[urr, phantom, "\circlearrowleft"] & & \text{Statements} \\
X, Y \arrow[u, two heads, "{[\ \rangle}", "{\langle\ ]}"'] \arrow[urr, "\rho"'] \arrow[urr, phantom, "\Uparrow"] & &
\end{tikzcd}, \tag{253}$$

are bidense and biexact. Composition preserves left- and right-density as well as left- and right-exactness, hence also their composite is both bidense and biexact.

4.11.2

The source of the composite morphism is the original binary relation ρ, the target is a complete relation that is canonically isomorphic to complete lattices $(\mathscr{R}(\rho), \mathscr{R}(\rho), \supseteq)$ and $(\mathscr{L}(\rho), \mathscr{L}(\rho), \subseteq)$.

4.11.3

In Theory of Ordered Sets those two lattices provide standard models for the biexact embedding of an arbitrary ordered set onto a bidense subset of a complete lattice.

4.11.4 The extended real line $(\overline{\mathbf{R}}, \leq)$

When applied to the standard linear order on the set of rational numbers $(\mathbf{Q}, \leq)$, we obtain a model for the *extended real line* $(\overline{\mathbf{R}}, \leq)$ where $\overline{\mathbf{R}}$ is the union of three disjoint sets

$$\overline{\mathbf{R}} = \{-\infty\} \cup \mathbf{R} \cup \{\infty\}\,. \tag{254}$$

If one uses complete lattice $(\mathscr{L}(\mathbf{Q}, \leq), \subseteq)$ as the model for the completion, then

(i) Real numbers $r \in \mathbf{R}$ correspond to the sets of lower bounds of nonempty bounded below subsets of $(\mathbf{Q}, \leq)$. In other words, $(\mathbf{R}, \leq)$ coincides with the ordered subset $(L_* \mathscr{P}_{\triangleright} \mathbf{Q}, \subseteq)$ of $(\mathscr{L}(\mathbf{Q}, \leq), \subseteq)$, cf. Section 4.6.4.

(ii) Extended real number $-\infty$ corresponds to $\emptyset$ viewed as the set of lower bounds of *not* bounded below subsets of $\mathbf{Q}$ (such subsets of $\mathbf{Q}$ are automatically not empty).

(iii) Extended real number ∞ corresponds to $\mathbf{Q}$ viewed as the set of lower bounds of the *empty* subset of $(\mathbf{Q}, \leq)$.

Exercise 70 *Make a similar description of* $\mathbf{R}$, $-\infty$, *and* ∞, *if one uses complete lattice* $(\mathscr{R}(\mathbf{Q}, \leq), \subseteq)$ *instead.*

You can take either of the above two descriptions as *the* definition of $-\infty$, $\mathbf{R}$ and ∞. It is wiser, however, to define *an* extended real line as *any* completion of ordered set $(\mathbf{Q}, \leq)$, and then to define $-\infty$ and ∞ as the smallest and, respectively, the greatest elements, and to define points of *a* real line to be the remaining elements.

4.12 Bimorphisms between binary relations

4.12.1

A binary relation provides an example of a mathematical structure where apart from the obvious definition of a morphism, there is yet another natural if less obvious definition.

4.12.2

We shall say that a pair of functions

$$f : X \to X' \qquad \text{and} \qquad Y \longleftarrow Y' : g \tag{255}$$

is a *bimorphism from* ρ *to* ρ' if relation $(\mathrm{id}_X, g)^* \rho$ implies relation $(f, \mathrm{id}_Y)^* \rho'$,

$$(\mathrm{id}_X, g)^* \rho \Longrightarrow (f, \mathrm{id}_Y)^* \rho'\,. \tag{256}$$

We shall represent Condition (256) by the following diagram

$$\begin{tikzcd}
X', Y' \arrow[dr, "\rho'"] & \\
X, Y' \arrow[u, shift right, "f"] \arrow[u, equal, shift left] \arrow[d, equal, shift right] \arrow[d, shift left, "g"'] \arrow[r, phantom, "\Uparrow"] & \text{Statements} \\
X, Y \arrow[ur, "\rho"'] &
\end{tikzcd} . \tag{257}$$

Exercise 71 *Suppose f, g is a bimorphism from ρ to ρ' and f', g' is a bimorphism from ρ' to ρ''. Show that $f' \circ f, g \circ g'$ is a bimorphism from ρ to ρ''.*

Lemma 4.21 *The following conditions are equivalent:*

(a) *A pair of functions* (255) *is a bimorphism.*

(b) $\forall_{y' \in Y'}\ f^*\langle y'] \supseteq \langle g(y')]$.

(c) $\forall_{x \in X}\ [f(x)\rangle \supseteq g^*[x\rangle$.

Exercise 72 *Express each identity in* (108) *in the form of a $\sim$-commutative diagram for an appropriate relation.*

4.12.3

Condition (b) of Lemma 4.21 says that the diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & Y' \arrow[l, "\langle\ ]"'] \arrow[d, "g"] \arrow[dl, phantom, "\subseteq"] \\ \mathscr{P}X & Y \arrow[l, "\langle\ ]"] \end{tikzcd} \tag{258}$$

is $\subseteq$-commutative, while Condition (c) says that the diagram

$$\begin{tikzcd} X' \arrow[r, "[\ \rangle"] & \mathscr{P}Y' \\ X \arrow[u, "f"] \arrow[r, "[\ \rangle"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g^*"'] \end{tikzcd} \tag{259}$$

is $\supseteq$-commutative.

Proof of Lemma 4.21. For every $x \in X$ and $y' \in Y'$, one has the following two sequences of equivalent statements

$$[f(x)\rangle \ni y' \iff \rho'(f(x), y') \iff f(x) \in \langle y'] \iff x \in f^*\langle y']$$

and

$$g^*[x\rangle \ni y' \iff [x\rangle \ni g(y') \iff \rho(x, g(y')) \iff x \in \langle g(y')].$$

If any of the statements in the bottom sequence implies any of the statements in the top sequence, then any other statement at the bottom implies any statement at the top.

The first statement in the formulation of Lemma 4.21 says that

$$\forall_{x \in X}\left(\forall_{y' \in Y'}\ \rho(x, g(y')) \Rightarrow \rho'(f(x), y')\right)$$

which is equivalent to the statement

$$\forall_{y' \in Y'}\left(\forall_{x \in X}\ \rho(x, g(y')) \Rightarrow \rho'(f(x), y')\right).$$

Condition (b) of Lemma 4.21 is equivalent to the statement

$$\forall_{x\in X}\left(\forall_{y'\in Y'}\ g^*[x\rangle \ni y' \Rightarrow [f(x)\rangle \ni y'\right).$$

Finally, Condition (c) of Lemma 4.21 is equivalent to the statement

$$\forall_{y'\in Y'}\left(\forall_{x\in X}\ x\in \langle g(y')] \Rightarrow x\in f^*\langle y']\right).$$

Thus, each of the conditions in Lemma 4.21 implies the remaining two. □

4.12.4

Note that the proof uses the fact that interchanging the order in which universal quantification is applied produces equivalent statements. The same holds for existential quantification of relations. This is an analogue of Fubini's Theorem stating that (under a mild integrability hypothesis) interchanging the order of integration in evaluation of an iterated double integral produces the same result.

Beware that interchanging the order in which universal and existential quantification are applied produces statements that are rarely equivalent.

Exercise 73 *Show that if f, g is a bimorphism from ρ to ρ', then g, f is a bimorphism from $(\rho')^{\mathrm{op}}$ to ρ^{op}.*

Later we shall see that this means that the category of binary relations and bimorphisms is equipped with a canonical $*$*-category* structure, i.e., it is a category with a canonical *anti-involution*. The exact meaning of these terms will be explained later. Here it is sufficient to say that $*$*-structures* play a fundamental role in Mathematics and, especially, in Mathematical Physics.

Corollary 4.22 *The following conditions are equivalent:*

(a) *A pair of functions* (255) *is a bimorphism.*

(b') $\forall_{B'\subseteq Y'}\ f^*LB' \supseteq L(g_*B')$.

(c') $\forall_{A\subseteq X}\ R(f_*A) \supseteq g^*RA$.

Proof. Condition (b) of Lemma 4.21 implies that

$$\forall_{y'\in B'}\ f^*\langle y'] \supseteq \langle g(y')]$$

which, in turn, implies that

$$\bigcap_{y'\in B'} f^*\langle y'] \supseteq \bigcap_{y'\in B'} \langle g(y')]. \tag{260}$$

The left-hand-side of (260) equals, in view of identity (107),

$$f^*\left(\bigcap_{y'\in B'}\langle y']\right)$$

while the right-hand-side equals

$$\bigcap_{x\in g_*B'}\langle x] = L(g_*B').$$

This demonstrates that statement (b) implies statement (b').

Exercise 74 *Write down two proofs of the equivalence*

$$\text{statement (c)} \quad \Longleftrightarrow \quad \text{statement (c}'\text{)}\,,$$

an explicit proof and a proof that deduces this from the already proven equivalence of statements (b) and (b').

The reverse implications (b′) $\Longrightarrow$ (b) and (c′) $\Longrightarrow$ (c) hold because

$$g_*\{y'\} = \{g(y')\}, \quad f_*\{x\} = \{f(x)\}, \quad L\{y\} = \langle y], \quad \text{and} \quad R\{x\} = [x\rangle\,,$$

hence statement (b) is a *special case* of statement (b′) and statement (c) is a *special case* of statement (c′), cf. Section 1.7.2. ☐

4.12.5

Condition (b′) of Corollary 4.22 says that the diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{P}Y' \arrow[l, "L"'] \arrow[d, "g_*"] \arrow[dl, phantom, "\subseteq"] \\ \mathscr{P}X & \mathscr{P}Y \arrow[l, "L"] \end{tikzcd} \tag{261}$$

is $\subseteq$-commutative, while Condition (c′) says that the diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[r, "R"] & \mathscr{P}Y' \\ \mathscr{P}X \arrow[u, "f_*"] \arrow[r, "R"'] \arrow[ur, phantom, "\supseteq"] & \mathscr{P}Y \arrow[u, "g^*"'] \end{tikzcd} \tag{262}$$

is $\supseteq$-commutative.

4.13 Galois connections

4.13.1 Faithful bimorphisms

By replacing, in the definition of a bimorphism, Implication $\Rightarrow$ by Equivalence $\Leftrightarrow$, we obtain the definition of a *faithful bimorphism*. Like for faithful morphisms, composition of faithful bimorphisms produces a faithful bimorphism. This follows immediately from the transitivity of the Equivalence relation on the set of statements.

The following characterization of faithful bimorphisms is an immediate corollary of Lemma 4.21.

Corollary 4.23 *The following conditions are equivalent:*

(a) *A pair of functions* (255) *is a faithful bimorphism.*

(b) $\forall_{y'\in Y'}\ f^*\langle y'] = \langle g(y')]\,.$

(c) $\forall_{x\in X}\ [f(x)\rangle = g^*[x\rangle\,.$

4.13.2

Condition (b) in Corollary 4.23 says that the square diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & Y' \arrow[l, "\langle\ \rbrack"'] \arrow[d, "g"] \arrow[dl, phantom, "\circlearrowleft"] \\ \mathscr{P}X & Y, \arrow[l, "\langle\ \rbrack"] \end{tikzcd} \tag{263}$$

is commutative, while Condition (c) says that the square diagram

$$\begin{tikzcd} X' \arrow[r, "\lbrack\ \rangle"] & \mathscr{P}Y' \\ X \arrow[u, "f"] \arrow[r, "\lbrack\ \rangle"'] \arrow[ur, phantom, "\circlearrowright"] & \mathscr{P}Y \arrow[u, "g^*"'] \end{tikzcd} \tag{264}$$

is commutative.

Exercise 75 *State the analogue of Corollary 4.22 for faithful bimorphisms and represent two out of three conditions as commutativity of appropriate diagrams.*

Exercise 76 *Show that, if f, g is a faithful bimorphism, then f is a morphism $(X, \succsim) \longrightarrow (X', \succsim)$ of preordered sets and, likewise, g is a morphism of preordered sets $(Y', \precsim) \longrightarrow (Y, \precsim)$. Here $\succsim$ and $\precsim$ are the corresponding canonical preorders associated with the binary relations ρ and ρ', cf. Section 4.4.6.*

4.13.3 Terminology

Faithful bimorphisms between ordered sets, i.e., when $X = Y$, $X' = Y'$, and the binary relations ρ and ρ' are order relations, are known as *Galois connections*.

Exercise 77 *Let $(X, \preccurlyeq)$ be an ordered set and*

$$\mathscr{P}_{\sup}X \ := \ \{A \subseteq X \mid \sup A \ \text{ exists}\}$$

be the set of those subsets of X that have supremum. Show that the pair of functions

$$\begin{tikzcd} (X, \preccurlyeq) \arrow[d, shift left, "\langle\ \rbrack"] \\ (\mathscr{P}_{\sup}X, \subseteq) \arrow[u, shift left, "\sup"] \end{tikzcd} \tag{265}$$

forms a Galois connection.

Exercise 78 *State the analogous assertion for the infimum function.*[6]

Exercise 79 *Peruse these notes from the beginning up to this point and identify all Galois connections that you can find.*

[6] Hint: if you do this mindlessly, you are likely to state it incorrectly and you will receive no credit.

4.13.4

Galois connections point towards one of the pillars of Modern Mathematics, the concept of a *pair of adjoint functors*. For this reason, I am tempted to extend the usage of the term *Galois connection* to arbitrary faithful bimorphisms.

4.14 Left and right adjoints: existence and exactness properties

4.14.1 Terminology

When f, g forms a faithful bimorphism, we say that f is a *left adjoint* of g and that g is a *right adjoint* of f.

4.14.2 Uniqueness up to an equivalence

If f_1 and f_2 are left adjoints of g, then

$$\forall_{x\in X}\, [f_1(x)\rangle = g^*[x\rangle = [f_2(x)\rangle\,,$$

i.e., f_1 is $\succsim$-equivalent to f_2.

Vice-versa, if f_2 is a left adjoint of g and f_1 is $\succsim$-equivalent to f_2, then also f_1 is a left adjoint of g.

Similarly for right adjoints of any function $f: X \to X'$: any two right adjoints $Y' \to Y$ of f are $\precsim$-equivalent and any function $Y' \to Y$ $\precsim$-equivalent to a right adjoint of f is a right adjoint of f.

4.14.3 Existence of left and right adjoints

It remains to characterize functions $f: X \to X'$ that admit a right adjoint, and functions $g: Y' \to Y$ that admit a left adjoint.

For a given function $f: X \to X'$ and binary relations ρ and ρ', let us consider the diagram

$$\begin{tikzcd}
\mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] & \langle\,]_* Y' \arrow[l, hook'] & Y' \arrow[l, two heads, "\langle\,]"'] \\
\mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\,]_* Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\,]"']
\end{tikzcd}\,. \tag{266}$$

The canonical inlusions are marked by hooked arrows, the canonical surjections—by two arrowheads.

Lemma 1.5 guarantees that the following three conditions are equivalent:

(i) One has

$$f^*{}_*\langle\,]_* Y' \subseteq \langle\,]_* Y\,,$$

i.e., diagram (266) can be completed to a commutative diagram

$$\begin{tikzcd}
\mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] & \langle\,]_* Y' \arrow[l, hook'] \arrow[d, "f^*"'] & Y' \arrow[l, two heads, "\langle\,]"'] \\
\mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\,]_* Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\,]"']
\end{tikzcd}\,. \tag{267}$$

(ii) Diagram (266) can be completed to a commutative diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] & \langle\ ]_* Y' \arrow[l, hook'] \arrow[d, "f^*"'] & Y' \arrow[l, two heads, "\langle\ ]"'] \arrow[d, "g"] \\ \mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\ ]_* Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\ ]"'] \end{tikzcd} \tag{268}$$

for some function $g : Y' \to Y$.

(iii) Diagram (266) can be completed to a commutative diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] & \langle\ ]_* Y' \arrow[l, hook'] & Y' \arrow[l, two heads, "\langle\ ]"'] \arrow[d, "g"] \\ \mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\ ]_* Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\ ]"'] \end{tikzcd} \tag{269}$$

for some function $g : Y' \to Y$.

4.14.4

Condition (i) means that f is a morphism of structures of "topological type",

$$(X, \mathscr{A}) \longrightarrow (X', \mathscr{A}'),$$

where $\mathscr{A} = \langle\ ]_* Y$ and $\mathscr{A}' = \langle\ ]_* Y'$.

4.14.5

Condition (iii) means that g is a right adjoint of f.

4.14.6

Given a subset $B' \subseteq Y'$, one has

$$f^* LB' = f^*\Big(\bigcap \{\langle y] \mid y \in B'\}\Big) = \bigcap_{y' \in B'} f^*[y'\rangle \tag{270}$$

and the right-hand-side of (270) belongs to $\mathscr{L}(\rho)$ if Condition (i) holds. In particular, each of the previous three conditions is equivalent to the condition:

(iv) Diagram (266) can be completed to a commutative diagram

$$\begin{tikzcd} \mathscr{P}X' \arrow[d, "f^*"'] & \mathscr{L}(\rho') \arrow[l, hook'] \arrow[d, "f^*"'] & \langle\ ]_* Y' \arrow[l, hook'] \arrow[d, "f^*"'] & Y' \arrow[l, two heads, "\langle\ ]"'] \arrow[d, "g"] \\ \mathscr{P}X & \mathscr{L}(\rho) \arrow[l, hook'] & \langle\ ]_* Y \arrow[l, hook'] & Y \arrow[l, two heads, "\langle\ ]"'] \end{tikzcd} \tag{271}$$

for some function $g : Y' \to Y$.

This concludes characterization of functions $f : X \to X'$ that admit a right adjoint. There is an analogous characterization of functions $g : Y' \to Y$ that admit a left adjoint.

Corollary 4.24 *If f, g is a faithful bimorphism, then f is right-exact and g is left-exact.*

In other words, if f admits a right adjoint, then f is right-exact. Similarly, if g admits a left adjoint, then g is left-exact.

Proof. Let A_1 and A_2 be subsets of X such that $RA_1 \supseteq RA_2$. Then,

$$Rf_*A_1 = g^*RA_1 \supseteq g^*RA_2 = Rf_*A_2 .$$

Similarly, if B_1 and B_2 are subsets of Y such that $LB_1 \supseteq LB_2$, then

$$Lg_*B_1 = f^*LB_1 \supseteq f^*LB_2 = Lg_*B_2 .$$

□

Note that Corollary 4.24 is a stronger version of the assertion of Exercise 76.

Corollary 4.25 *When $X = Y$ and $X' = Y'$, and $f : X \to X'$ admits both a right adjoint and a left adjoint, then f is exact, cf. Section 4.9.2.* □

Note that the right and the left adjoint of f, are, generally, not equal to each other.

4.14.7 Example: f^* and its two adjoints f_* and $f_!$

A fundamentally important example of this happening is provided by the direct image, inverse image, and conjugate image, associated with a function $f : X \to Y$, and operating between power-sets $(\mathscr{P}X, \subseteq)$ and $(\mathscr{P}Y, \subseteq)$, cf. (103). Of this trio, f_* is right-exact but, generally, not left-exact, $f_!$ is left-exact but, generally, not right-exact, whereas f^* is both left- and right-exact. And indeed,

$$f^* \text{ has both a left and a right adjoints, namely } f_* \text{ and } f_! \ . \tag{272}$$

According to Theorem 4.12 the preimage morphisms are the *only* exact functions from $(\mathscr{P}Y, \subseteq)$ to $(\mathscr{P}X, \subseteq)$.

4.14.8

A particularly interesting phenomenon, known in Category Theory as *equivalence of categories*, occurs when a function f admits a left adjoint g that is simultaneously a right adjoint of f. This concept has been a cornerstone of Mathematics for the last sixty years.

4.14.9 Example: Relations versus correspondences

Let X' be the set of n-ary relations $\mathrm{Rel}(S_1, \dots, S_n)$ between elements of sets $S_1, \dots, S_n$, preordered by implication $\Longrightarrow$. Let X be the power-set of the Cartesian product $S_1 \times \cdots \times S_n$, ordered by inclusion $\subseteq$. The function $X \to X'$ introduced in Section 1.11.24, that assigns in a canonical manner to a correspondence $C \subseteq S_1 \times \cdots \times S_n$, a relation $\rho_C \in \mathrm{Rel}(S_1, \dots, S_n)$, is both a left and a right adjoint of the graph function $\Gamma : X' \to X$, cf. Section 1.11.17.