In this review, we will focus on concreteness and not on generality.

1. <u>Symmetric bilinear form.</u>

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (or any field)

Let $V = \mathbb{K}^n$, $e_1, \cdots, e_n$ the standard basis

A vector $v \in V$ can be written as

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad \text{meaning } v_1 e_1 + \cdots + v_n e_n. \quad \text{where } x_i \in \mathbb{K}.$$

A symm bilinear form Q on V is something of the form

$$Q(v, w) = v^t \cdot Q \cdot w$$

$$= (v_1, \cdots, v_n) \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ \vdots & & & \vdots \\ Q_{n1} & \cdots & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

where symmetric means

$$Q_{ij} = Q_{ji}$$

<u>Main Question</u>: can one find a change of basis, such that Q be come diagonal?

The question can be formulated in <u>two ways</u>:

① Find new basis $\tilde{e}_1, \cdots, \tilde{e}_n, \in V$, such that

$$Q(\tilde{e}_i, \tilde{e}_j) = 0 \quad \text{if } i \neq j$$

② Find an invertible $\overset{n\times n}{\vee}$ matrix C, so that

$$C^t \cdot Q \cdot C = \tilde{Q} \text{ is a diagonal matrix.}$$

The relation between the two approaches is that,

$$C = \left(\begin{pmatrix} \\ \tilde{e}_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \tilde{e}_2 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ \tilde{e}_n \\ \\ \end{pmatrix} \right),$$ where we put the $\tilde{e}_i$ as column vectors of C.

<u>Main Result</u>: Yes, one can always diagonalize a symmetric bilinear form Q.

<u>Recipe</u>: we will construct a sequence of row & column operations that will take the symmetric matrix Q to a diagonal matrix.

<u>big steps</u>: Let $Q_0 = Q$.

① If $Q_0 \neq 0$
We want to find invertible matrix C_0. so that

$$C_0^t \cdot Q_0 \cdot C_0 = \begin{pmatrix} q_1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{\quad Q_1 \quad} \\ 0 & \end{pmatrix}, \quad \text{where } q_1 \neq 0$$

then Q_1 is a size $(n-1)\times(n-1)$ matrix

② If $Q_1 \neq 0$
We want to find invertible matrix C_1. so that

$$C_1^t \cdot Q_1 \cdot C_1 = \begin{pmatrix} q_2 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{\quad Q_2 \quad} \\ 0 & \end{pmatrix}, \quad \text{where } q_2 \neq 0$$

then Q_2 is a size $(n-2)\times(n-2)$ matrix

③ keep going.

If at certain step, $Q_K = 0$ is the zero matrix, then stop early, and set $q_{K+1} = \cdots = q_n = 0$

④ Let $C = \begin{bmatrix} C_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{C_1} \\ 0 & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 \cdots 0 \\ 0 & 0 & \\ \vdots & \vdots & \boxed{C_2} \\ 0 & 0 & \end{bmatrix} \cdots \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \boxed{C_3} \end{bmatrix} \cdots$

then

$$C^t \cdot Q \cdot C = \begin{pmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & \ddots & \\ 0 & & & q_n \end{pmatrix}$$

<u>How do we perform each step?</u>

Notice that, each step is the same, except the size of the matrix Q_k is different. So we just do the step ①

(1.1) Find a vector V, such that

$$Q_0(V, V) \neq 0$$

Let $\tilde{e}_1 = V$.

For example, if $Q_0(e_1, e_1) \neq 0$, then let $\tilde{e}_1 = e_1$.

(1.2) Complete $\tilde{e}_1$ to a basis, call it $\{\tilde{e}_1, \hat{e}_2, \hat{e}_3, \cdots, \hat{e}_n\}$

then we will modify $\hat{e}_K$, so it is $\perp \tilde{e}_1$.

$$\tilde{e}_K = \hat{e}_K - \frac{Q(\tilde{e}_1, e_K)\, \tilde{e}_1}{Q(\tilde{e}_1, \tilde{e}_1)} \qquad k = 2, 3, \cdots, n.$$

Thus we have

$$C_0 = \left(\begin{pmatrix} \tilde{e}_1 \end{pmatrix} \begin{pmatrix} \tilde{e}_2 \end{pmatrix} \cdots \begin{pmatrix} \tilde{e}_n \end{pmatrix} \right), \quad \text{s.t.}$$

$$C_0^t \cdot Q_0 \cdot C_0 = \begin{pmatrix} q_1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{Q_1} \\ 0 & \end{pmatrix}$$

Ex: Diagonalize the following symm bilinear form in $\mathbb{R}$

$$Q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Sol'n: $\sqrt{C^t Q C = \tilde{Q}}$. You may use other matrix C and get a different diagonal matrix $\tilde{Q}$

① $$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

② $$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

③ $$\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$$

④ $$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\text{Rescaling}}$:

If we change the matrix C, by multiplying the k-th column by c_k, then

$$\tilde{C} = C \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix}, \quad \tilde{C}^t = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} C$$

Thus

$$\tilde{C}^t Q \tilde{C} = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} \cdot C^t Q \cdot C \cdot \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix} \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix}$$

$$= \begin{pmatrix} q_1 \cdot c_1^2 & & & \\ & q_2 \cdot c_2^2 & & \\ & & \ddots & \\ & & & q_n \cdot c_n^2 \end{pmatrix}$$

2. Hermitian Form

- This is only for <u>complex</u> vector space
- $V = \mathbb{C}^n$. A hermitian form

$$H : V \times V \to \mathbb{C}$$

is given by

$$H(z, w) = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n) \underbrace{\begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & \\ \vdots & & \cdots & H_{nn} \end{pmatrix}}_{\text{matrix } H} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$= z^* \cdot H \cdot w.$$

where $H_{ij} = \overline{H_{ji}}$ (i.e. $H^* = H$ — transpose, then complex conjugate.)

Ex: $H = \begin{pmatrix} 2 & 1+i & 0 \\ 1-i & 4 & i \\ 0 & -i & 5 \end{pmatrix}$ is a Hermitian form.

If $z = \begin{pmatrix} 2 \\ 0 \\ i \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$, then

$$H(z,w) = \overset{z^*}{(2, 0, -i)} \begin{pmatrix} 2 & 1+i & 0 \\ 1-i & 4 & i \\ 0 & -i & 5 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$$

$$= \left(4, \underbrace{2(1+i) + (-i)(-i)}_{=1+2i}, 5\cdot(-i)\right) \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}$$

$$= 4 + (1+2i)\cdot i + (-5i)(-i)$$

$$= 4 + i - 2 - 5 = -3 + i$$

Main Question: Find an invertible matrix C, such that $C^* \cdot H \cdot C = \tilde{H}$ is a digonal matrix.

Equivalently, find basis $\tilde{e}_1, \cdots, \tilde{e}_n$ of $\mathbb{C}^n$, such that

$$H(\tilde{e}_i, \tilde{e}_j) = 0 \qquad \text{if } i \neq j$$

Approach: Exactly the same as in the symmetric bilinear form case.

Example: Diagonalize the following Hermitian form

$$H = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}$$

<u>Sol'n</u>:

- $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$

(same as the symm bilinear form. $\because$ C is real $\therefore$ $C^* = C^t$)

- $\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- $\begin{pmatrix} 1 & 1 \\ 1-i & -1-i \end{pmatrix}\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}\begin{pmatrix} 1 & 1+i \\ 1 & -1+i \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$

$$\begin{pmatrix} 1-i & -1-i \end{pmatrix}\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}\begin{pmatrix} 1+i \\ -1+i \end{pmatrix} = \begin{pmatrix} -2 & 2 \end{pmatrix}\begin{pmatrix} 1+i \\ -1+i \end{pmatrix} = -4$$

One can check if the result is correct or not, using Sylvester rule.