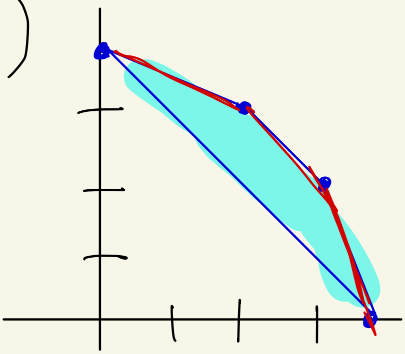
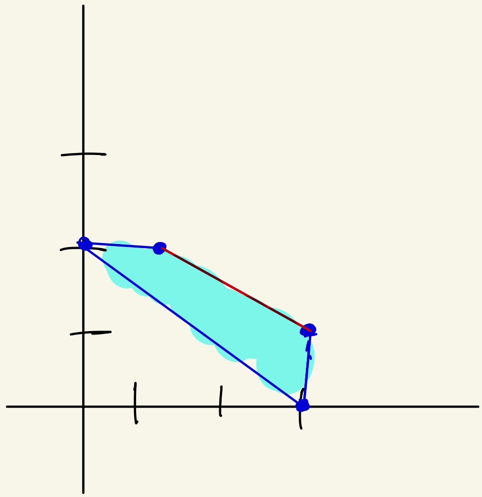
Mason McBride
303592541l
mason2@berkeley.edu
STAT 155

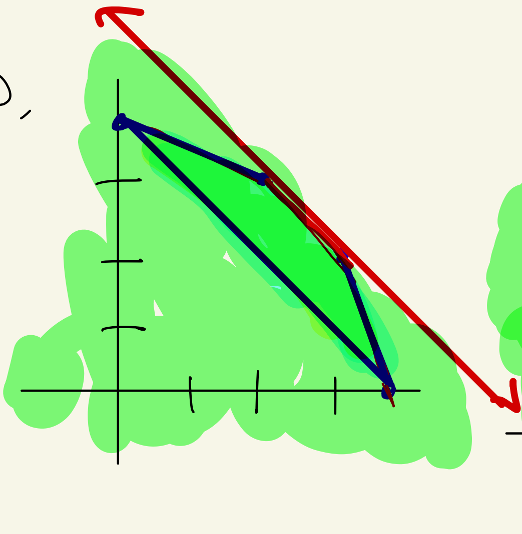
1.

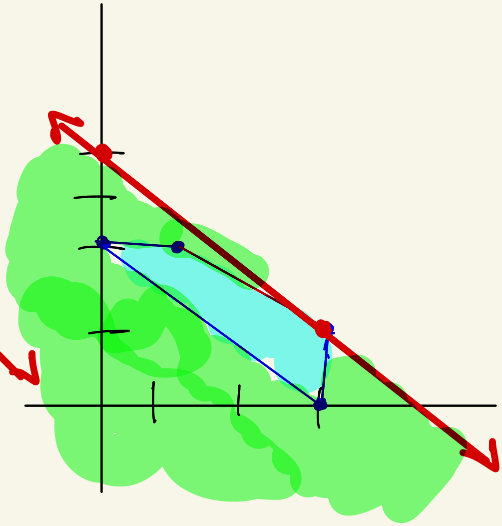
a)





b.





2. for the first game, we first compat $A+B$ and $A-B$

$$A+B = \begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix} \quad A-B = \begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}$$

$-4x + (1-x) = 4x-(1-x)$ $\quad -4x + 4(1-x) = x-(1-x)$

$-4x + 1 - x = 4x - 1 + x$ $\quad -4x + 4 - 4x = x - 1 + x$

$-5x + 1 = 5x - 1$ $\quad -8x + 4 = 2x - 1$

$2 = 10x$ $\quad 5 = 10x \quad x = \frac{1}{2}$

$y_* = \frac{1}{5}$ $\quad \boxed{y_* = (\frac{1}{5}, \frac{4}{5})}$ $\quad x = \frac{1}{2}$ $\quad \boxed{x_* = (\frac{1}{2}, \frac{1}{2})}$

thred strategies

disagreement point: $(x_*^T A y_*, x_*' B y_*)$

$\frac{1}{2} \cdot \frac{1}{5} \cdot 0 + \frac{1}{2} \cdot \frac{4}{5} \cdot 3 + \frac{1}{2} \cdot \frac{1}{5} \cdot 4 + \frac{1}{2} \cdot \frac{4}{5} \cdot 2$

$= \frac{6}{5} + \frac{2}{5} + \frac{4}{5} = \frac{12}{5}$

$\frac{1}{2} \cdot \frac{1}{5} \cdot 4 + \frac{1}{2} \cdot \frac{4}{5} \cdot 2 + \frac{1}{2} \cdot \frac{1}{5} \cdot 0 + \frac{1}{2} \cdot \frac{4}{5} \cdot 3$

$= \frac{2}{5} + \frac{4}{5} + \frac{6}{5} = \frac{12}{5}$

$\boxed{\text{the disagreement point is } \left(\frac{13}{5}, \frac{12}{5}\right)}$

value of the game is $\frac{13}{5} - \frac{13}{5} = 0$

$\left(\frac{5}{2}, \frac{5}{2}\right)$ is final payoff vector

$-x + 5 = x$

$2x = 5$

$x = \frac{3}{5}$

$\boxed{\text{the side-payment is } \frac{2}{5}}$

for the second game,

$$A+B = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \qquad A-B = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

max value is

4

threat strategies:

$$x^* = \left(\frac{5}{8}, \frac{3}{8}\right)$$

$$y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$2x - 2(1-x) = -x + 3 - 3x$$

$$2x - 2 + 2x = -4x + 3$$

$$8x = 5$$

$$x = \frac{5}{8}$$

$$2y - 1 + y = -2y + 3 - 3y$$

$$3y - 1 = -5y + 3$$

$$8y = 4$$

$$y = \frac{1}{2}$$

$$\frac{5}{8}\cdot\frac{1}{2}\cdot 3+\frac{5}{8}\cdot\frac{1}{2}\cdot 0+\frac{3}{8}\cdot\frac{1}{2}\cdot 1+\frac{3}{8}\cdot\frac{1}{2}\cdot 3=$$

$$\frac{15}{16}+\frac{3}{16}+\frac{9}{16}=\frac{27}{16}$$

$$\frac{5}{8}\cdot\frac{1}{2}\cdot 1+\frac{5}{8}\cdot\frac{1}{2}\cdot 2+\frac{3}{8}\cdot\frac{1}{2}\cdot 2+0=$$

$$\frac{5}{16}+\frac{10}{16}+\frac{5}{16}=\frac{20}{16}$$

disagreement point is $\left(\frac{27}{16},\frac{20}{16}\right)$

value of the game is $\frac{27}{16}-\frac{20}{16}=\boxed{\frac{7}{16}}$

$$\underbrace{\frac{\frac{64}{16}+\frac{7}{16}}{2}}, \quad \underbrace{\frac{\frac{64}{16}-\frac{7}{16}}{2}}$$

$\left(\frac{71}{32},\frac{57}{32}\right)$ is the payoff vector

$$\frac{20}{16} = \frac{27}{16} + b$$

$$-\frac{7}{16}$$

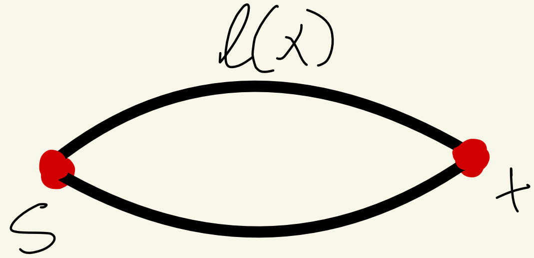
$$x - \frac{7}{16} = -x + 4$$

$$2x = \frac{71}{16}$$

$$x = \frac{71}{32}$$

the side payment is $\frac{71}{32}$

3.



1: $\ell(x)$ being non-decreasing means

$\ell(x) \leq c$ for any $0 \leq x \leq 1$

This means there are two cases:

case 1: $\ell(x) = c$ case 2: $\ell(x) < c$

$x\,\ell(x) = c(1-x)$

$x\,\ell(x) = c - cx$

$x\,\ell(x) + cx = c$

$x(\ell(x) + c) = c$

$$\boxed{x = \frac{c}{\ell(x) + c}}$$

if $\ell(x)$ is worse than c for all $0 \leq x \leq 1$ then there is a pure

$\boxed{\text{NE of } x = 1}$

$x=1$ is the worst NE because there is only flow on the top edge.

The socially optimal outcome is a probability x that minimizes the total latency:

$$\min_{0 \le x \le 1} \left(x\,\ell(x) + c(1-x) \right)$$

$$POA = \frac{\text{cws worse NE}}{\text{socially opt. outcome}}$$

1. The POA is thus:

$$\boxed{POA = \frac{\ell(1)}{\min_{0 \le x \le 1} \left(x\,\ell(x) + c(1-x) \right)}}$$

and $\ell(1) = 1$ because the total flow is given to be 1

2. $\max_{0 \le x \le 1} (\ell(x)) = C$

we can be more specific because $\ell(x)$ is non decreasing so its max value must be $\max_{0 \le x \le 1} (\ell(x)) \le \ell(1)$

thus $C = \ell(1)$ is the minimum value and $\ell(1) = 1$

$\boxed{\text{so } C = 1}$

1

4. $POA = \dfrac{\ell(1) = 1}{\min\limits_{0 \le x \le 1} (x\,\ell(x) + c(1-x))}$

note that $c = 1$ and

$\ell(x) \le c \le x$ for any $0 \le x \le 1$

so $\ell(x)$ can be upper bounded by x

this means $\min\limits_{0 \le x \le 1} (x \cdot x + 1(1-x))$

can now be solved.

$f(x) = x^2 - x + 1$

$f'(x) = 2x - 1 \overset{set}{=} 0$

$x = \frac{1}{2}$

$f''(x) = 2 > 0$

$\boxed{\text{so } x = \frac{1}{2} \text{ is a min}}$

plugging into to the POA equation

$$POA = \frac{1}{\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \left(1 - \frac{1}{2}\right)}$$

$$\boxed{POA = \frac{4(1)}{4\left(\frac{1}{4} + \frac{1}{2}\right)} = \frac{4}{1+2} = \frac{4}{3}}$$

5. α is defined as where

$$\alpha = \frac{\max_{H' \in H_{equil}} L_{H'}(f)}{\min_{H'' \in H_{st}} L_{H''}(f)}$$

H_{Equil} is the set of all Nash Equilibria on H

We must prove:

$$\min_{G' \in G^{st}} L_{G'}(F) \geq \frac{1}{\alpha} \min_{H' \in H_{st}} L_{H'}(G)$$

or equivalently,

$$\alpha \cdot \min_{G' \in G^{st}} L_{G'}(F) \geq \min_{H' \in H_{st}} L_{H'}(G)$$

note that:

$$\min_{G' \in G_{st}} L_{G'}(f) \leq \min_{H' \in H_{st}} L_{H'}(f) \quad (1)$$

because the amount of edges increase from G to H but the flow is still I so the average must decrease

$$\frac{\max_{H \in H_{equil}} L_H(f)}{\min_{H \in H^{st}} L_H(f)} \cdot \min_{G' \in G_{st}} L_{G'}(t)$$

this value has a max value of I by (1)

So we can rewrite the inequality to be

$$\max_{H \in H_{equil}} L_H(f) \geq \min_{H' \in H_{st}} L_{H'}(6)$$

which is true because the RHS is the definition of a NE and the max NE on the LHS will always be at least the value of any NE