Game Theory Worksheet #2, 09/04/2018 and 09/07/2018

1. Consider the following game: there are two piles of chips, two players make moves in turns. At each turn the player can take some chips from one of the piles: either from 1 to 2 chips from the first pile, or from 1 to 3 chips from the second one. Denote the number of chips in the first pile as m, and in the second — as n. For every pair (m, n) s.t. $m \leq 6, n \leq 6$ determine, is this position in P or in N.
   Solution:
   We need to fill the 7×7 table:

m \ n	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

$(0, 0)$ is terminal, so it is a P-position. Thus, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 0)$, $(2, 0)$ are N-positions (there are moves to $(0, 0) \in P$ from each of them).

m \ n	0	1	2	3	4	5	6
0	P	N	N	N			
1	N						
2	N						
3							
4							
5							
6							

Now we see that $(0, 4), (1, 1)$, and $(3, 0)$ are in P (all the moves from those positions lead to N). Thus, the postions from where we can go there are in N:

m \ n	0	1	2	3	4	5	6
0	P	N	N	N	P	N	N
1	N	P	N	N	N		
2	N	N			N		
3	P	N	N	N			
4	N						
5	N						
6							

We can fill the whole table like that. The result is:

m \ n	0	1	2	3	4	5	6
0	P	N	N	N	P	N	N
1	N	P	N	N	N	P	N
2	N	N	P	N	N	N	P
3	P	N	N	N	P	N	N
4	N	P	N	N	N	P	N
5	N	N	P	N	N	N	P
6	P	N	N	N	P	N	N

2. For each of the following positions in Nim determine if that is a P- or N-position. If it is a N-position, find all the winning first moves.

   (a) (1, 2, 3);
   Solution:

   $$1 \oplus 2 \oplus 3 = 01_2 \oplus 10_2 \oplus 11_2 = 00_2.$$

   The Nim sum is zero, this is a P-position.

   (b) (5, 3, 2);
   Solution:

   $$5 \oplus 3 \oplus 2 = 101_2 \oplus 011_2 \oplus 010_2 = 100_2 = 4.$$

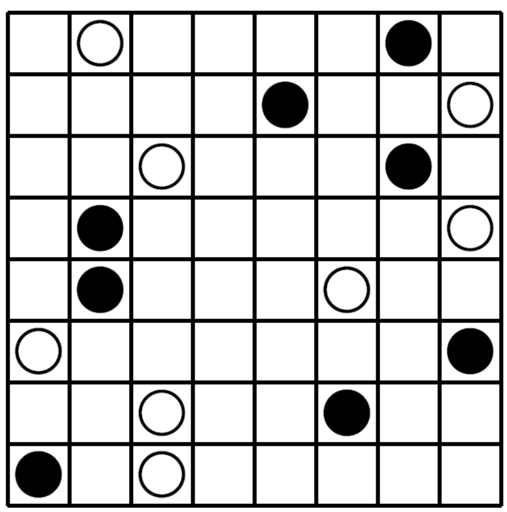
   The Nim sum is non-zero, this is a N-position. The lefttmost digit 1 in the binary representation of the Nim sum is on the third position from the right. Among $\{2, 3, 5\}$ only 5 has digit 1 in the same position in binary representation. Thus, the only winning move is to reduce the pile of 5 chips to the pile of $5 \oplus 4 = 1$ chips (take 4 chips from the pile of 5 chips).

   (c) (7, 12, 13, 11);
   Solution:
   Analogously to the previous part, this is a N-position, there are 3 winning moves:

   $$12 \to 1, \ 13 \to 0, \ \text{and} \ 11 \to 6.$$

3. **Northott's Game**. A position in Northcott's game is a checkerboard with one black and one white checker on each row. "White" moves the white checkers and "Black" moves the black checkers. A checker may move any number of squares along its (horizontal) row, but may not jump over or onto the other checker. Players move alternately and the last to move wins. Determine who wins if the game starts from the position below and "White" moves first.

   Hint: This game is Nim in disguise.



   Solution:

   The disguise is the following: we have distances between the checkers in each row instead of piles. Indeed, at each move exactly one of those distances is changed, and the player can arbitrarily decrease any one of those distances. Thus, the player A, who moves from a position with non-zero nim sum of the distances, can always make the Nim sum equal to 0 (by moving her checker closer to one of the opponent's checkers), and her opponent B will have to make it non-zero again. Thus, A will never lose. Moreover, the game will eventually end because A moves each checker only in one direction, and it cannot last forever (the board is finite in each direction). Thus, A will win.

   In the picture the distances are $4, 2, 3, 5, 3, 6, 2, 1$, and $4 \oplus 2 \oplus 3 \oplus 5 \oplus 3 \oplus 6 \oplus 2 \oplus 1 = 6 \neq 0$. Thus, "White" can win.