MATH 135: SET THEORY
SOLUTIONS TO MIDTERM # 2

1a. (5pts) Express the **Axiom of Infinity** in the formal language having nonlogical symbols for $\in$, $\cup$ (the binary union as a binary function symbol), $\{\cdot\}$ (the singleton set operator as a unary function symbol), $\emptyset$ (as a constant symbol).

Solution:

$$(\exists I)[\emptyset \in I \ \& \ (\forall x)(x \in I \to x \cup \{x\} \in I)]$$

1b. (5 pts) Express the **Power Set Axiom** using only the signature of set theory.

Solution:

$$(\forall x)(\exists y)(\forall t)[t \in y \leftrightarrow (\forall s)(s \in t \to s \in x)]$$

1c. (5 pts) Give a precise formal definition of the relation $\mathrm{card}(M) = \mathrm{card}(K)^{\mathrm{card}(L)}$. You may use as primitives the relation f `is a function`, a binary function symbols for the ordered pair set $\langle \cdot, \cdot \rangle$ and $\times$, and unary function symbols ran, dom, and $\mathcal{P}$. [**Hint:** It may be easier to first define the relations $X \approx Y$ and $Z = {}^{Y}X$, and then to use these to define $\mathrm{card}(M) = \mathrm{card}(K)^{\mathrm{card}(L)}$.]

Solution:

For sets X and Y we define:

$$\begin{aligned} X \approx Y \iff & (\exists f)[f \texttt{ is a function} \ \& \ \mathrm{dom}(f) = X \ \& \ \mathrm{ran}(f) = Y \\ & \& \ (\forall x)(\forall y)(\forall z)[(\langle x, y\rangle \in f \ \& \ \langle x, z \rangle \in f) \to y = z] \end{aligned}$$

Likewise for sets X, Y, and Z we define

$$\begin{aligned} Z = {}^{Y}X \iff & (\forall t)[t \in X \leftrightarrow t \texttt{ is a function} \\ & \& \ \mathrm{dom}(f) = Y \ \& \ \mathrm{ran}(f) \subseteq X] \end{aligned}$$

Finally, we define

$$\mathrm{card}(M) = \mathrm{card}(K)^{\mathrm{card}(L)} \iff M \approx {}^{L}K$$

1d. (5 pts) Give a precise formal definition of the relation $x = \mathbb{R}$. You may use as primitives a constant symbol for $\mathbb{Q}$, the binary relation symbol $<$ for the order relation on $\mathbb{Q}$, a constant symbol $\emptyset$ for the empty set, the power set $\mathcal{P}$ as a function symbol, and binary relation symbols $\subseteq$ for the subset relation and $\neq$ for inequality.

Solution: We define

$$\begin{aligned} x = \mathbb{R} \iff\ & (\forall t)[t \in x \longleftrightarrow t \in \mathcal{P}\mathbb{Q} \;\&\; t \neq \varnothing \;\&\; t \neq \mathbb{Q} \\ & \&\ (\forall p)(p \in t \to (\exists q)[q \in t \;\&\; p < q]) \\ & \&\ (\forall p)(\forall q)([q \in t \;\&\; p \in \mathbb{Q} \;\&\; p < q] \to p \in t)] \end{aligned}$$

1e. (5 pts) Give a precise formal definition of the condition $x = \mathbb{Z}$. You may use the following primitives: $\in$, ω, E `is an equivalence relation`, X/E (as a binary function symbol), $+$ (as a binary function symbol on ω), $\langle \cdot, \cdot \rangle$, and $\times$.

Solution: We define the equivalence relation $\sim$ as follows.

$$\begin{aligned} x =\sim\ \iff\ & (\forall t)[t \in x \longleftrightarrow (\exists a)(\exists b)(\exists c)(\exists d)(a \in \omega \;\&\; b \in \omega \\ & \&\ c \in \omega \;\&\; d \in \omega \;\&\; a + b = c + d \\ & \&\ t = \langle \langle a, b \rangle, \langle c, d \rangle \rangle)] \end{aligned}$$

Then we define

$$\mathbb{Z} = (\omega \times \omega) / \sim$$

2. (15 pts) **Prove:** If $A \subseteq \mathbb{R}$, $A \neq \emptyset$, and A is bounded from below: that is

$$(\exists b \in \mathbb{R})(\forall a \in A)\ b \leq a\ ,$$

then A has a greatest lower bound, that is, there us some

$$(\exists c \in \mathbb{R})([(\forall a \in A)\ c \leq a] \;\&\; (\forall d \in \mathbb{R})[((\forall a \in A)\ d \leq a) \to d \leq c])\ .$$

Solution: Let $B := \{x \in \mathbb{R} \;:\; (\forall a \in A)\ x \leq a\}$. By the hypothesis that A is bounded from below, $B \neq \emptyset$. Since $A \neq \emptyset$, there is some $a \in A$. Then from the very definition of B as the set of lower bounds of A, a is an upper bound of B. Since $\mathbb{R}$ has the least upper bound property and the set B is nonempty and bounded from above, it has a least upper bound, which we will call c. Let us check that c is the greatest lower bound of A. First, if d is any other lower bound of A, then $d \in B$. Since c is the least upper bound of B, *a fortiori*, it is an upper bound of B and $d \leq c$. Secondly, we check that c is a lower bound of A. Let $a \in A$ be any element. If $d \in B$ is arbitrary, then from the definition of B, we have $d \leq a$. Thus,

a is an upper bound of B. Since c is the *least* upper bound of B, $c \leq a$. Thus, c is a lower bound of A and we have checked that c is the greatest lower bound.

3. (15 pts) **Prove (without using the Axiom of Choice):** If X is a finite set and $f : X \twoheadrightarrow Y$ is onto, then Y is finite.

Solution: We argue by induction on $\text{card}(X)$. (Recall that we have checked that if the set A is finite, then there is a unique natural number $n \in \omega$ with $A \approx n$ and have defined $\text{card}(A) := n$.) If $\text{card}(X) = 0$, then $X = \emptyset$, which implies that $f = \emptyset$ is the empty function and that $Y = \text{ran}(f) = \emptyset$ as well, which is a finite set. Consider now the inductive case that $\text{card}(X) = n^+$ and that we know the result for sets of cardinality n. Fix $g : n^+ \to X$ a bijection and let $X' := g[[n]]$ and $x := g(n)$. So $X = X' \cup \{x\}$ and $X' \approx n$. Let $Y' := f[[X']]$. By the inductive hypothesis, Y' is finite. Let $m = \text{card}(Y')$ and fix $h : m \to Y'$ some bijection. If $f(x) \in Y'$, then $Y = \text{ran}(f) = f[[X']] \cup \{f(x)\} = Y'$ is finite. If $f(x) \notin Y'$, then $Y = Y' \cup \{f(x)\}$ and $k := h \cup \{\langle m, f(x) \rangle\}$ is a bijection between m^+ and Y since k is the union of two bijections with disjoint domains and disjoint ranges. In either case, Y is finite.

4. (15 pts) **Prove (without using the Axiom of Choice):** If $X \subseteq \omega$ is infinite, then $X \approx \omega$.

Solution: Since X is infinite, *a fortiori* it is not empty. Let $a \in X$ be the least element of X. Since a subset of a finite set is finite, for every $n \in \omega$ we have that $X \neq X \cap n$. Define $h : X \to X$ by $x \mapsto$ the least element of $X \smallsetminus x^+$. Define $f : \omega \to X$ by recursion via $f(0) := a$ and $f(n^+) = h(f(n))$. For any $n \in \omega$ we have $f(n) \in f(n)^+ \in h(f(n)) = f(n^+)$. It follows that f is an increasing function. Indeed, we argue by induction on m that if $n < m$, then $f(n) < f(m)$. The starting point if $m = m^+$ where we have already checked that $f(n) < f(m)$. More generally, if we know that $f(n) < f(m)$, then because $f(m) < f(m^+)$, it follows that $f(n) < f(m^+)$. An increasing function is necessarily one-to-one. Thus, $\omega \preceq X$. The inclusion map $\iota : X \to \omega$ witnesses that $X \preceq \omega$. By Schröder-Bernstein, $X \approx \omega$.

5. (15 pts) **Prove:** For every nonempty set K, there does not exist a set $\mathbb{K}$ having the property that for all sets x, $x \in \mathbb{K}$ if and only if $x \approx K$.

Solution: Let us suppose that such a set $\mathbb{K}$ exists. We shall show that if it did exist, then every set would belong to $\bigcup \mathbb{K}$, but because there is no set of all sets, this would lead to a contradiction.

Since $K \neq \emptyset$, we may fix some $a \in K$.

Let t be any set. If $t \in K$, then $t \in \bigcup \mathbb{K}$ as witnessed by $t \in K$ and $K \in \mathbb{K}$ (for $\text{id}_K : K \to K$ is a bijection between K and itself). If $t \notin K$, then consider the set $L := \{t\} \cup (K \smallsetminus \{a\})$. Define $f : K \to L$ by $f := \text{id}_{K \smallsetminus \{a\}} \cup \{\langle a, t \rangle\}$. We see that $\text{ran}(f) = (K \smallsetminus \{a\}) \cup \{t\} = L$ and that $\text{dom}(f) = (K \smallsetminus \{a\}) \cup \{a\} = K$. Moreover, because f is the union of two one-to-one function with disjoint domains and disjoint ranges, it is itself a one-to-one function. Thus, $L \approx K$. So, $t \in \bigcup \mathbb{K}$ as witnessed by $t \in L \in \mathbb{K}$. Hence, if $\mathbb{K}$ were a set, so would be $\bigcup \mathbb{K}$, by the Union Axiom, but this would mean that there would be a set of all sets, which we have already shown to be absurd. Thus, no such set $\mathbb{K}$ exists.

6. (15 pts) **Prove:** The Axiom of Choice implies that if X is infinite then there is a function $f : X \hookrightarrow X$ which is one-to-one, but is not onto. [Hint: It will be easier to use a result we proved in class about the consequences of the Axiom of Choice than to attempt to apply the Axiom of Choice directly.]

Solution: We have shown that the Axiom of Choice implies that because X is infinite, $\omega \preceq X$. Fix $g : \omega \to X$ a one-to-one function. Let $S : \omega \to \omega$ be the successor function $x \mapsto x^+$. Define $f : X \to X$ as $f := g \circ S \circ g^{-1} \cup \mathrm{id}_{X \smallsetminus \mathrm{ran}(g)}$. Each of g, S, and g^{-1} is a one-to-one function, $\mathrm{ran}(g^{-1}) = \omega = \mathrm{dom}(S)$, and $\mathrm{ran}(S) \subseteq \omega = \mathrm{dom}(g)$, so that $g \circ S \circ g^{-1}$ is one-to-one. Morevoer, because $g \circ S \circ g^{-1}$ and $\mathrm{id}_{X \smallsetminus \mathrm{ran}(g)}$ are one-to-one functions with disjoint domains and disjoint ranges, f is a one-to-one function. We see that $g(0) \notin \mathrm{ran}(f)$, for if $f(x) = g(0)$, then either $x \in \mathrm{ran}(g)$ and $g(0) = f(x) = g \circ S \circ g^{-1}(x) = g(S(g^{-1}(x)) = g(g^{-1}(x)^+)$, which implies because g is one-to-one, that $0 = g^{-1}(x)^+$, which is absurd, or $x \in X \smallsetminus \mathrm{ran}(g)$, and $g(0) = f(x) = \mathrm{id}_{X \smallsetminus \mathrm{ran}(g)}(x) = x$, which is absurd as $g(0) \in \mathrm{ran}(g)$ and x was assumed to lie in $X \smallsetminus \mathrm{ran}(g)$. Thus, $f : X \hookrightarrow X$ is a one-to-one function from X to itself which is not onto.

[**Remark:** With our earlier theorem that if X is finite and $f : X \hookrightarrow X$ is one-to-one, then f is onto, we see that the Axiom of Choice implies that a set X is finite if and only if for every one-to-one function $f : X \hookrightarrow X$, f must be onto X.]