In this review, we will focus on concreteness and not on generality.

1. <u>Symmetric bilinear form.</u>

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (or any field)

Let $V = \mathbb{K}^n$, $e_1, \cdots, e_n$ the standard basis

A vector $v \in V$ can be written as

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad \text{meaning } v_1 e_1 + \cdots + v_n e_n. \quad \text{where } x_i \in \mathbb{K}.$$

A symm bilinear form Q on V is something of the form

$$Q(v, w) = v^t \cdot Q \cdot w$$
$$= (v_1, \cdots, v_n) \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ \vdots & & & \vdots \\ Q_{n1} & \cdots & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

where symmetric means

$$Q_{ij} = Q_{ji}$$

<u>Main Question</u>: can one find a change of basis, such that Q become diagonal?

The question can be formulated in <u>two ways</u>:

① Find new basis $\tilde{e}_1, \cdots, \tilde{e}_n, \in V$,

such that

$$Q(\tilde{e}_i, \tilde{e}_j) = 0 \quad \text{if } i \neq j$$

② Find an invertible $n\times n$ matrix C, so that

$$C^t \cdot Q \cdot C = \tilde{Q} \quad \text{is a diagonal matrix.}$$

The relation between the two approaches is that,

$$C = \begin{pmatrix} \begin{pmatrix} \tilde{e}_1 \end{pmatrix} & \begin{pmatrix} \tilde{e}_2 \end{pmatrix} & \cdots & \begin{pmatrix} \tilde{e}_n \end{pmatrix} \end{pmatrix},$$

where we put the $\tilde{e}_i$ as column vectors of C.

<u>Main Result</u>: Yes, one can always diagonalize a symmetric bilinear form Q.

<u>Recipe</u>: we will construct a sequence of row & column operations that will take the symmetric matrix Q to a diagonal matrix.

<u>big steps</u>: Let $Q_0 = Q$.

① If $Q_0 \neq 0$ We want to find invertible matrix C_0. so that

$$C_0^t \cdot Q_0 \cdot C_0 = \begin{pmatrix} q_1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{\quad Q_1 \quad} \\ 0 & \end{pmatrix}, \quad \text{where } q_1 \neq 0$$

then Q_1 is a size $(n-1)\times(n-1)$ matrix

② If $Q_1 \neq 0$ We want to find invertible matrix C_1. so that

$$C_1^t \cdot Q_1 \cdot C_1 = \begin{pmatrix} q_2 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{\quad Q_2 \quad} \\ 0 & \end{pmatrix}, \quad \text{where } q_2 \neq 0$$

then Q_2 is a size $(n-2)\times(n-2)$ matrix

③ keep going.

If at certain step, $Q_K = 0$ is the zero matrix, then stop early, and set $q_{K+1} = \cdots = q_n = 0$

④ Let $C = \begin{bmatrix} C_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{C_1} \\ 0 & \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 \cdots 0 \\ 0 & 0 & \\ \vdots & \vdots & \boxed{C_2} \\ 0 & 0 & \end{bmatrix} \cdot \cdots \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \boxed{C_3} \end{bmatrix} \cdots$

then

$$C^t \cdot Q \cdot C = \begin{pmatrix} q_1 & & & \\ & q_2 & & 0 \\ & 0 & \ddots & \\ & & & q_n \end{pmatrix}$$

How do we perform each step?

Notice that, each step is the same, except the size of the matrix Q_k is different. So we just do the step ①

(1.1) Find a vector v, such that

$$Q_0(v, v) \neq 0$$

Let $\tilde{e}_1 = v$.

For example, if $Q_0(e_1, e_1) \neq 0$, then let $\tilde{e}_1 = e_1$.

(1.2) Complete $\tilde{e}_1$ to a basis, call it $\{\tilde{e}_1, \hat{e}_2, \hat{e}_3, \cdots, \hat{e}_n\}$

then we will modify $\hat{e}_K$, so it is $\perp \tilde{e}_1$.

$$\tilde{e}_K = \hat{e}_K - \frac{Q(\tilde{e}_1, e_K)\, \tilde{e}_1}{Q(\tilde{e}_1, \tilde{e}_1)} \qquad k = 2, 3, \cdots, n.$$

Thus we have

$$C_0 = \left(\begin{pmatrix} \tilde{e}_1 \end{pmatrix} \begin{pmatrix} \tilde{e}_2 \end{pmatrix} \cdots \begin{pmatrix} \tilde{e}_n \end{pmatrix} \right), \quad \text{s.t.}$$

$$C_0^t \cdot Q_0 \cdot C_0 = \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{Q_1} & \\ 0 & & & \end{pmatrix}$$

Ex: Diagonalize the following symm bilinear form

$$Q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$