Game Theory (STAT 155) Worksheet #3, 02/08/2021

1. For games with following payoff matrices find the value of the game and some safety strategies for both players:

   Solution.

   (a) $\begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix}$

   Let's do this calculation fully just to see how it works. Let Player I have possible moves T and B for "top" and "bottom", andet Player II have possible moves L and R for "left" and "right". A standard convention is to assume that Player I uses she/her pronouns and that Player II uses he/him pronouns, so that we can distinguish the players without writing out the names fully. (In cryptography we make this even more concrete by assigning the name "Alice" to the first player and "Bob" to the second player.)

   Let's first think about Player I's strategy. Her possible strategies are fully described by letting x be the probability of playing T and letting $1-x$ be the probability of playing B. What is her payoff under this strategy? If Player II plays L, then her payoff is $1 \cdot x + (-3)(1-x) = 4x - 3$, and if Player II plays R, then her payoff is $(-2)x + 5(1-x) = -7x + 5$. Thus, her worst-case payoff is the worse of these two vaues, which we is just the function $p(x) = \min\{4x - 3, -7x + 5\}$, since it depends on $0 \leq x \leq 1$. Now if Player I maximizes the function $p(x)$ over $0 \leq x \leq 1$, then she has found the strategy which gives her the best worst-case payoff. To solve this maximization problem, we notice that p is piece-wise linear, so its maximum must be achieved at the end point of its domain, or at the place where the two lines intersect. (This can be proven with some simple calculus, but we won't get into that here.) We can check that $4x - 3 = -7x + 5$ is solved at $x = \frac{8}{11}$ and its value here is $p(x) = -\frac{1}{11}$. Also, we can check $p(0) = -3$ and $p(1) = -2$. Therefore, Player I's optimal strategy is to play T with probability $\frac{8}{11}$ and B with probability $\frac{3}{11}$, which we encode by the vector $(\frac{8}{11}, \frac{3}{11})$. Upon playing this strategy she can guarantee herself a payoff of at least $-\frac{1}{11}$.

   Now let's think about Player II's strategy. His possibly strategies are fully described by letting y be the probability of playing L and letting $1-y$ be the probability of playing R. To understand this strategy, recall that, for Player II, the values in the matrix represent losses. So if Player I plays T then his loss is $1 \cdot y + (-2)(1-y) = 3y - 2$, and that if Player I plays B then his loss is $(-3)y + 5(1-y) = -8y + 5$. So, the worst-case loss is described by the function $q(y) = \max\{3y - 2, -8y + 5\}$, and we wish to minimize this over $0 \leq y \leq 1$. To do this, note that $3y - 2 = -8y + 5$ is satsified by $y = \frac{7}{11}$, for which we have $q(\frac{7}{11}) = -\frac{1}{11}$. Also, we have $q(0) = 5$ and $q(1) = 1$. Therefore, Player II's optimal strategy is to player L with probability $\frac{7}{11}$ and R with probability $\frac{4}{11}$. Upon playing this strategy he can guarantee himself a loss of no more than $-\frac{1}{11}$.

   Notice that $V = -\frac{1}{11}$ appears in both results. Player I has a strategy under which she can guarantee a payoff of at least V, and Player II has a strategy under which he can guarantee a loss of no more than V. Such a number V is called the *value* of the game, and it is somewhat surprising that this quantity exists. Von Neumann's minimax theorem asserts (among other things) that any two-person zero-sum game has a well-defined value.

   (b) $\begin{pmatrix} 3 & 1 \\ 5 & \underline{\mathbf{3}} \end{pmatrix}$

   Since this is a two-person zero-sum game, it has a well-defined value that we denote by V. Now notice the underlined and bolded 3 appearing in the matrix; this is interesting from the perspective of either player.

   From the perspective of Player I, it is interesting that this number is the minimum in its row. That's good because, if Player I plays B, then she can guarantee herself a payoff of at least 3. So $V \geq 3$.

   And from the perspective of Player II, it is interesting that this number is the maximum in its column. This is because, if Player II plays R, then he can guarantee himself a loss of no more than 3. So $V \leq 3$.

   Putting this all together shows that the value of the game is $V = 3$, that Player I's optimal strategy is is to always play B and that Player II's optimal strategy is to always play R.

   More generally, if a matrix is such that $a_{i,j}$ is the minimum in its row and the maximum in its column, then the value of the game is $V = a_{i,j}$, and the players have optimal strategies to purely play row i and column j, respectively. (An interesting question from section was, if there are two saddle points, then which one is the value of the game? Problem 2 below shows that, in fact, any two saddle points must be equal!)

   (c) $\begin{pmatrix} 1 & 2 & 3 & 5 & 8 \\ 8 & 5 & 3 & 2 & 1 \end{pmatrix}$

   In principle, we could solve this problem by setting up a large system of minimization and maxmization problems, but this would be slightly annoying. Instead, let's just "guess" the strategies and then verify that they are optimal.

   Notice that the top row and the bottom row are very similar, since one is just the reverse of the other. So, we might guess that Player I's optimal strategy is just to play $(\frac{1}{2}, \frac{1}{2})$. How much payoff can she guarantee under this strategy? If Player II plays column 1 or 5 she receives $\frac{9}{2}$, if player II plays column 2 or 4 she receives $\frac{7}{2}$, and if player II plays column 3 she receives 3. Since these are all at least 3, we see that Player I can guarantee a payoff of at least 3, hence $V \geq 3$.

But notice that Player II can guarantee himself a loss of at most 3 if he simply plays the non-random strategy of playing column 3, hence $V \leq 3$.

So we have shown that $V = 3$, that an optimal strategy for Player I is $(\frac{1}{2}, \frac{1}{2})$, and that an optimal strategy for Player II is $(0, 0, 1, 0, 0)$.

(d) $\begin{pmatrix} 1 & 6 \\ 5 & 4 \\ 3 & 5 \end{pmatrix}$

Notice that the third row is the average of the first two rows. What does this mean for Player I? It means that if she has any strategy of the form $(x_1, x_2, 1 - x_1 - x_2)$, then the strategy $(\frac{x_1}{x_1+x_2}, \frac{x_2}{x_1+x_2}, 0)$ has exactly the same payoff. Thus, Player I can restrict herself only to stragies that assign zero probability to row three. This means we are now playing the reduced game

$\begin{pmatrix} 1 & 6 \\ 5 & 4 \end{pmatrix}.$

Since Player I's best worst-case gain did not change when we remove row three, we see that the value of of the game did not change either. Therefore, we can search for the value and the optimal strategies in the reduced game, and these correspond to the value and the optimal strategies in the original game.

We won't go through the details here, but an explicit calculation like in part (a) shows that the value of the game is $V = \frac{13}{3}$, and that optimal strategies in the reduced game are $(\frac{1}{6}, \frac{5}{6})$ and $(\frac{1}{3}, \frac{2}{3})$. Therefore, in the original game, the value is $V = \frac{13}{3}$, and optimal strategies are $(\frac{1}{6}, \frac{5}{6}, 0)$ and $(\frac{1}{3}, \frac{2}{3})$.

In this example, we had that one strategy was equivalent to a mixture of some other strategies. More generally, whenever one strategy has a worse payoff than a mixture of some other strategies, then it can also be removed from the game without changing its value. This is the technique of "domination".

(e) $\begin{pmatrix} 2 & 4 & 4 \\ 3 & 2 & 1 \\ 2 & 2 & 4 \end{pmatrix}$

Notice that, no matter what Player II does, it is always better for Player I to play the first row than to play the third row. That is, the third row is dominated by the first row, so the game can be reduced to

$\begin{pmatrix} 2 & 4 & 4 \\ 3 & 2 & 1 \end{pmatrix}.$

Now notice that, no matter what Player I does, it is always better for Player II to play the third column than the second column. That is, the second column is dominated by the third column, so the game can be further reduced to

$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}.$

Now we can solve this game explicitly (the details of which we omit) and find that its value is $V = \frac{5}{2}$, that the optimal strategy for Player I is $(\frac{1}{2}, \frac{1}{2})$, and that the optimal strategy for Player II is $(\frac{3}{4}, \frac{1}{4})$. So, in the original game, its value is $V = \frac{5}{2}$, an optimal strategy for Player I is $(\frac{1}{2}, \frac{1}{2}, 0)$, and an optimal strategy for Player II is $(\frac{3}{4}, 0, \frac{1}{4})$.

Notice that, in the original game, the second column is not dominated by the third column! It is only after removing the third row that this becomes the case.

(f) $\begin{pmatrix} 2 & 4 & 5 \\ 4 & \underline{4} & 4 \\ 8 & 4 & 2 \end{pmatrix}$

Notice that the underlined 4 is a saddle point. (Equality is okay in the definition of saddle point.) Hence, the value of the game is $V = 4$, and an optimal strategy for either player is $(0, 1, 0)$.

(g) $\begin{pmatrix} 11 & 3 & 9 \\ 12 & 5 & 5 \\ 13 & 7 & 1 \end{pmatrix}$

It is clear that the first column is dominated by both the second column and the third column, so the game reduces to

$\begin{pmatrix} 3 & 9 \\ 5 & 5 \\ 7 & 1 \end{pmatrix}.$

Now we notice that the second row is the average of the first row and the third row, so it can be removed, yielding

$\begin{pmatrix} 3 & 9 \\ 7 & 1 \end{pmatrix}.$

Now we solve the reduced game by hand, and interpret this in the setting of the original game. We find that the value of the original game is $V = 5$ and that some optimal strategies are given by $(\frac{1}{2}, 0, \frac{1}{2})$ for Player I and $(0, \frac{2}{3}, \frac{1}{3})$ for Player II.

2. Show that, if a zero-sum game $A = (a_{i,j})_{i,j}$ has two saddle points (i_1, j_1) and (i_2, j_2), then $a_{i_1,j_1} = a_{i_2,j_2}$.

Solution.

Consider the point a_{i_1,j_2}. It appears in the same row as a_{i_1,j_1}, hence we have $a_{i_1,j_1} \leq a_{i_1,j_2}$. It also appears in the same column as a_{i_2,j_2}, so we have $a_{i_2,j_2} \geq a_{i_1,j_2}$. Hence by transitivity we have $a_{i_1,j_1} \leq a_{i_2,j_2}$.

Next consider the point a_{i_2,j_1}. It appears in the same row as a_{i_2,j_2}, so we have $a_{i_2,j_2} \leq a_{i_2,j_1}$, and it appears in the same column as a_{i_1,j_1}, so we have $a_{i_2,j_1} \leq a_{i_1,j_1}$. Again by transitivity we have $a_{i_2,j_2} \leq a_{i_1,j_1}$.

Since we have $a_{i_1,j_1} \leq a_{i_2,j_2}$ and $a_{i_2,j_2} \leq a_{i_1,j_1}$, this implies $a_{i_2,j_2} = a_{i_1,j_1}$, as claimed.