MATH 135: INTRODUCTION TO THE THEORY OF SETS
AUTUMN 2011
FINAL EXAM

There are ten questions in total (including two extra credit questions). This exam will last three hours. Write your answers in your own blue (or green) book. With the exception of clarifications provided by the instructor, you may not refer to any external source in the course of completing this examination.

1. (20 points) Define the following terms. When referring to another term introduced in this course, completely define that term in the formal language of set theory as well unless with the exception of the term "function", and the defined formulas $\ulcorner y = \mathrm{dom}(f) \urcorner$, and $\ulcorner y = \mathrm{ran}(f) \urcorner$.

(1) A set x is *transitive* if and only if ____________.
(2) A set α is an *ordinal* if and only if ____________. (Note: Your definition should be expressed in the formal language of set theory. You may not take any notions other than "$\in$" as primitive.)
(3) A set x is a *Dedekind-finite* if and only if ____________.
(4) A set κ is a *cardinal* if and only if ____________.

2. (10 points) State the following axioms in the formal language. You may write the axiom in mathematical English also to explain the formal sentence. As in question **1**, if you refer to a term introduced in this course, you must define that term unless it was listed above.

(1) Regularity Axiom
(2) Replacement Axiom (Scheme)

3. (15 points) Relative to the other axioms of set theory give a direct proof that the Well Ordering Principle implies the Axiom of Choice (in our official formulation.)

4. (15 points) Show that for any two sets a and b, if $a^+ = b^+$, then $a = b$.

5. (15 points) Show that a set X is finite if and only if there is a well-ordering R of X for which the converse relation R^{-1} is also a well-ordering.

6. (15 points) Give a detailed proof from the transfinite recursive definition that for ordinals α, β, and γ we always have $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. (Hint: Work by transfinite induction on γ. Note that you will need to prove that ordinal addition is associative.)

7. (15 points) Prove that for any set x there is an inductive set I for which $x \in I$.

8. (15 points) Let X be any set and let $\mathscr{E}$ be a nonempty set of equivalence relations on X. Prove that $\bigcap \mathscr{E}$ is an equivalence relation on X.

Date: 16 December 2011.

Extra credit A. (5 points) Express the following ordinal in Cantor Normal Form. (Justify each step in your simplifications.)

$$(3\omega^2+\omega)^{\omega^2+1}+(\omega^\omega+\omega^3 4)(\omega^\omega+\omega^2)$$

Extra credit B. (5 points) For ordinals α and β and a function $f:\beta\to\alpha$ we define

$$\mathrm{supp}(f):=\{\gamma\in\beta\ :\ f(\gamma)\neq 0\}$$

We then define

$$\alpha\uparrow\beta:=\{f\in{}^{\beta}\alpha\ :\ \mathrm{supp}(f)\text{ is finite }\}$$

and a relation $f<g$ for f and g from $\alpha\uparrow\beta$ by

$$\begin{aligned} f<g \iff\ & f(\delta)<g(\delta)\\ & \text{where } \delta=\max\{\gamma\in\beta\ :\ f(\gamma)\neq g(\gamma)\} \end{aligned}$$

Prove that for ordinals α and β the ordered set $\alpha\uparrow\beta$ has epsilon-image α^β. [Note: in particular, you will how that $\alpha\uparrow\beta$ is well-ordered by the relation $<$ defined above. You will find it easiest to prove by transfinite induction on β that the set $\alpha\uparrow\beta$ with the relation $<$ is isomorphic to $(\alpha^\beta,\in)$ thereby establishing that $<$ is a well-ordering at the same time that you prove that its epsilon-image is α^β.]