MATH 135: INTRODUCTION TO THE THEORY OF SETS
AUTUMN 2011
SOLUTIONS TO MIDTERM I

1. (20 points) Define the following terms. When referring to another term introduced in this course, completely define that term as well.

(1) Given a set x the set y is equal to the *union of* x, $\bigcup x$ if and only if

$$\underline{(\forall s)(s \in y \longleftrightarrow (\exists t)(t \in x \ \& \ s \in t))}$$

(2) A set x is an *ordered pair* if and only if

$$\underline{(\exists a)(\exists b) x = \langle a, b \rangle}$$

Here we say for sets a, b, and c we say that

$$c = \{a, b\} \Longleftrightarrow \underline{(\forall s)(s \in c \longleftrightarrow (s = a \text{ or } s = b))}$$

For sets a and b we say

$$b = \{a\} \Longleftrightarrow \underline{b = \{a, a\}}$$

For sets a, b and c we say

$$c = \langle a, b \rangle \Longleftrightarrow \underline{c = \{\{a\}, \{a, b\}\}}$$

(3) A set f is a *function* if and only if

$$\underline{(\forall t)(t \in f \to t \text{ is an ordered pair}) \text{ and } (\forall x)(\forall y)(\forall z)[(\langle x, y \rangle \in f \ \& \ \langle x, z \rangle \in f) \to y = z]}$$

(4) A set R is a *transitive relation* if and only if

$$\underline{(\forall t)(t \in R \to t \text{ is an ordered pair }) \text{ and } (\forall x)(\forall y)(\forall z)[(\langle x, y \rangle \in R \ \& \ \langle y, z \rangle \in R) \to \langle x, z \rangle \in R]}$$

2. (10 points) State the following axioms in the formal language. You may write the axiom in mathematical English also to explain the formal sentence. As in question **1**, if you refer to a term introduced in this course, you must define that term.

(1) Empty Set Axiom

$$(\exists x)(\forall t) \neg (t \in x)$$

(2) Extensionality Axiom

$$(\forall x)(\forall y)([(\forall s)(s \in x \leftrightarrow s \in y)] \leftrightarrow x = y)$$

3. (15 points) Prove or disprove: If $a \in b$, the $\mathscr{P}a \in \mathscr{P}b$.
Disproof: Consider $a = \{\varnothing\}$ and $b = \{\{\varnothing\}\}$. Then $\mathscr{P}a = \{\varnothing, \{\varnothing\}\}$ while $\mathscr{P}b = \{\varnothing, \{\{\varnothing\}\}\}$. Visibly, $a \in b$, but $\mathscr{P}a \notin \mathscr{P}b$ as there are two members of $\mathscr{P}a$ while every member of $\mathscr{P}b$ has at most one element. $\square$

4. (15 points) Prove or disprove: If $\mathscr{F}$ is a nonempty set of functions, then $\bigcap \mathscr{F}$ is a function.
Proof: Since $\mathscr{F} \neq \varnothing$, we can find some $f \in \mathscr{F}$. It follows that that $\bigcap \mathscr{F} \subseteq f$ is indeed a set as shown in class. Moreover, $\bigcap \mathscr{F}$ is indeed a relation as if $t \in \bigcap \mathscr{F}$, then in particular $t \in f$, which is a relation, so that t is an ordered pair. Finally, suppose that for sets x, y and z we have $\langle x, y \rangle \in \bigcap \mathscr{F}$ and $\langle x, z \rangle \in \bigcap \mathscr{F}$. Then, in particular, $\langle x, y \rangle \in f$ and $\langle x, z \rangle \in f$. Since f is a function, we have $y = z$. $\square$

Date: 20 September 2011.

5. (15 points) Show that if $f: A \to B$ is a function which is onto B but is not one-to-one, then there are at least two distinct functions $g: B \to A$ and $h: B \to A$ for which $f \circ g = I_B = f \circ h$.
Proof: We have already proven that there is at least one right inverse $g: B \to A$. Since f is not one-to-one, there are sets x, y, and z so that $\langle x, y \rangle \in f$ and $\langle z, y \rangle \in f$ but $x \neq z$. In particular, either $x \neq g(y)$ or $z \neq g(y)$. Without loss of generality, $x \neq g(y)$. Define

$$h := \{t \in B \times A \ : \ (\exists u)[u \in B \ \& \ u \neq y \ \& \ t = \langle u, g(u) \rangle] \text{ or } t = \langle y, x \rangle\}$$

We have shown that $B \times A$ is a set. Hence, from the subset axiom we know that h is a set. Moreover, h is clearly relation being a subset of $B \times A$. The relation h is a function as if $\langle a, b \rangle \in h$ and $\langle a, c \rangle \in h$, then either $a \neq y$ in which case $b = g(a) = c$ or $a = y$ in which case $b = x = c$. We see that $f \circ h = I_B$ as if $a \in B$, then either $a \neq y$ in which case $f \circ h(a) = f(h(a)) = f(g(a)) = a$ (as $f \circ g = I_B$) or $a = y$ in which case $f \circ h(a) = f(h(y)) = f(x) = y = a$. Finally, since $\langle y, x \rangle \in h$ but $\langle y, x \rangle \notin g$ we see that $h \neq g$. $\square$

6. (15 points) Prove that relative to the other axioms of set theory, the following assertion is equivalent to the Axiom of Choice:

*: For every partition Π of some set x, there is a set $y \subseteq x$ such that for each $\rho \in \Pi$ the set $\rho \cap y$ is a singleton.

Proof: Let us first show that ACI implies *. We know that ACI implies that if F is any function, $I = \mathrm{dom}(F)$ and $(\forall i \in I) F(i) \neq \varnothing$, then $\prod_{i \in I} F(i) \neq \varnothing$. Let $F := I_\Pi$ be the identity function of Π. By definition of a partition, for every $\rho \in \Pi$ we have $\rho \neq \varnothing$. Hence, $\prod_{\rho \in \Pi} \rho \neq \varnothing$. Let $g: \Pi \to \bigcup \Pi = x$ be some element of $\prod_{\rho \in \Pi} \rho$. Set $y := \mathrm{ran}(g)$, which is a subset of x as the target of g is x. Let now $\rho \in \Pi$. Then $g(\rho) \in \rho$ (by definition of the product) and $g(\rho) \in \mathrm{ran}(g) = y$ (by definition of the range). Hence, $g(\rho) \in \rho \cap y$. On the other hand, if $z \in y \cap \rho$, then there is some $\sigma \in \mathrm{dom}(g) = \Pi$ for which $z = g(\sigma)$. We know that $g(\sigma) \in \sigma$ (by definition of the product) so that $z \in \sigma \cap \rho$. As Π is a partition, having $\sigma \cap \rho \neq \varnothing$ implies that $\sigma = \rho$. As g is a function, this implies that $z = g(\rho)$. Thus, $y \cap \rho = \{g(\rho)\}$ is a singleton, as desired.

In the other direction, let R be any relation. Let $x := R$. For $a \in \mathrm{dom}(R)$, we define

$$R_a := \{s \in R \ : \ (\exists b) s = \langle a, b \rangle\}$$

Let

$$\Pi := \{t \in \mathscr{P}R \ : \ (\exists a \in \mathrm{dom}(R)) t = R_a\}$$

Then Π is a partition of R as if $u \in R$, then because R is a relation, u is an ordered pair so that $u = \langle a, b \rangle$ for some a and b. By definition of the domain, $a \in \mathrm{dom}(R)$ and then $u \in R_a$. Thus, $\bigcup \Pi = R$. By definition of $\mathrm{dom}(R)$, each set R_a for $a \in \mathrm{dom}(R)$ is nonempty. Finally, if $R_a \cap R_c \neq \varnothing$ for some a and c, then there are u and v for which $\langle a, u \rangle = \langle c, v \rangle$ which by the characteristic property of the ordered pair implies that $a = c$ so that $R_a = R_c$. Thus, Π is a partition of R. By $*$, there is a set $y \subseteq x = R$ so that for each $\rho \in \Pi$, $y \cap \rho$ is a singleton. I claim that y is a function whose domain is $\mathrm{dom}(R)$. Indeed, as y is a subset of R, it is a relation. If $\langle a, b \rangle \in y$ and $\langle a, c \rangle \in y$. Then $\langle a, b \rangle \in y \cap R_a$ and $\langle a, c \rangle \in y \cap R_a$. As $y \cap R_a$ is a singleton, $\langle a, b \rangle = \langle a, c \rangle$ so that $b = c$ as desired. Finally, if $a \in \mathrm{dom}(R)$, then as $y \cap R_a$ is a singleton, which in particular is nonempty, there is some $t \in y \cap R_a$. Every element of R_a has the form $\langle a, b \rangle$ for some b. Hence, $t = \langle a, b \rangle$ for some b. That is, $a \in \mathrm{dom}(y)$. Thus, $\mathrm{dom}(R) \subseteq \mathrm{dom}(y)$. As $y \subseteq R$, we have $\mathrm{dom}(y) \subseteq \mathrm{dom}(R)$. Therefore, $\mathrm{dom}(y) = \mathrm{dom}(R)$. $\square$