Name: ____
SID: ____

MATH 135: SET THEORY
MIDTERM # 2
SOLUTIONS

1a. Define

$$I \texttt{ is inductive } :\Longleftrightarrow (\varnothing \in I \ \& \ (\forall x)[x \in I \to x \cup \{x\} \in I]$$

The Axiom of Infinity says:

$$(\exists I)[I \text{ is inductive }]$$

1b.

$$n \texttt{ is a natural number } :\Longleftrightarrow (\forall I)[I \texttt{ inductive} \to n \in I]$$

2. Let A be a set such that for every $x \in A$, x is a transitive set. **Show** that $\bigcup A$ is a transitive set.
Proof: Let $y \in \bigcup A$ and $z \in y$. By definition of the union, there is some $x \in A$ with $y \in x$. By hypothesis, x is transitive. Hence, $z \in x$. Again by definition of the union, $z \in \bigcup A$. Thus, we have shown that every member of a member of $\bigcup A$ is also a member of $\bigcup A$. That is, $\bigcup A$ is transitive.

3. Show that if a, b, and c are natural numbers and $a < b$, then $a + c < b + c$.
Proof: Let us recall that for x and y natural numbers, we have $x < y \Rightarrow x^+ < y^+$. Indeed, by the trichotomy, if the conclusion were to fail, we would have $y^+ \leq x^+$, but then $y < y^+ \leq x^+$ so that $y \leq x$ contrary to the hypothesis that $x < y$.

Now for the problem proper, we work by induction on c. If $c = 0$, then $a+c = a+0 = a < b = b+0 = b+c$. Assuming $a+c < b+c$, we compute $a+c^+ = (a+c)^+ < (b+c)^+ = b+c^+$. Thus, we have established by induction that $a < b$ implies $a + c < b + c$.

4. Let X be a set and suppose that $\omega \preceq X$. **Show** that X is Dedekind-infinite (that is, there is some one-to-one function $f : X \hookrightarrow X$ which is *not* onto).
Proof: Let $g : \omega \hookrightarrow X$ be a one-to-one function from ω to X. Let $f : \omega \to \omega$ be the function defined by $f(x) = x^+$. Let $A := \mathrm{ran}(g)$ and define $h := \mathrm{I}_{X \smallsetminus A} \cup g \circ f \circ g^{-1}$. Since $A = \mathrm{dom}(g \circ f \circ g^{-1})$ is disjoint from $X \smallsetminus A$, h is a function. Since each of the functions in the union is one-to-one but have disjoint ranges (the range of $I_{X \smallsetminus A}$ is $X \smallsetminus A$ and of $g \circ f \circ g^{-1}$ is contained in A), h is one-to-one. However, $g(0) \notin \mathrm{ran}(h)$. Hence, we have produced a one-to-one function from X to itself which is not onto.

5. Prove that

$$2^{\aleph_0} = 3^{\aleph_0} \ .$$

Proof: $2^{\aleph_0} \leq 3^{\aleph_0} \leq 4^{\aleph_0} = (2^2)^{\aleph_0} = 2^{2 \cdot \aleph_0} = 2^{\aleph_0}$. By Schröder-Bernstein, the inequalities are equalities.