$\boxed{\text{Skip §4.9: Improper Integrals}}$

Gaussian Quadrature by Legendre polynomials

Gaussian Quadrature by Legendre polynomials: $P_0(x) = 1, P_1(x) = x$,

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad \text{for } n \geq 1.$$

- $P_n(x)$ has degree n, <u>always</u> with n $\boxed{\text{distinct}}$ roots $x_1, x_2, \cdots, x_n \in (-1, \ 1)$

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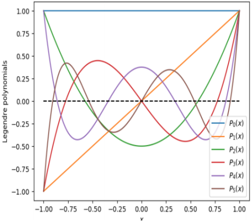
$$\int_{-1}^{1} P_n(x)\, P_j(x)\, dx = 0, \quad \text{for} \quad j < n$$



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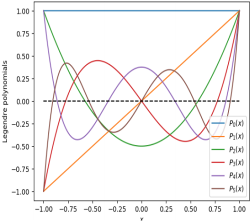

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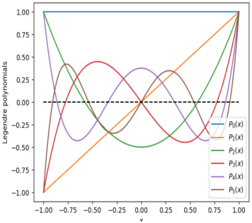

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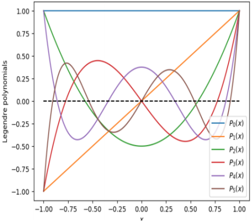

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$\boxed{\text{quadrature exact for polynomial with degree} \leq n-1}$

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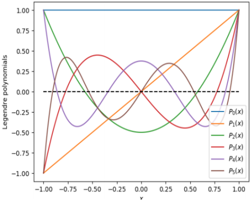

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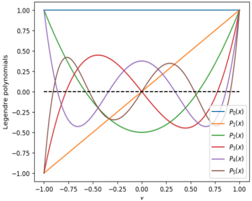

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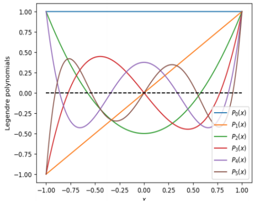

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Let $P(x)$ have deg $\leq 2n-1$,

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$$\int_{-1}^{1} P(x)\, dx \;\; \stackrel{\text{by (2)}}{=\!=} \;\; \int_{-1}^{1} \left(P_n(x)\, Q(x) + R(x)\right)\, dx \stackrel{\text{by (1)}}{=\!=} \int_{-1}^{1} R(x)\, dx$$

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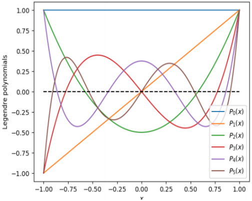

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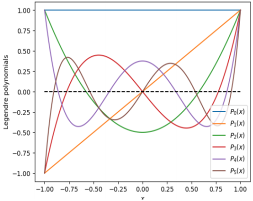

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\end{aligned}$$

Hermite Interpolation on roots of Legendre polynomials

- Given Legendre roots $x_1, x_2, \cdots, x_n$ with

$$(x_1, f(x_1), f'(x_1)), (x_2, f(x_2), f'(x_2)), \cdots, (x_n, f(x_n), f'(x_n)),$$

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- Interpolating polynomial $H(x)$ of degree $\leq 2n-1$ satisfies

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- (recall) **Thm**: For each $x \in [a, b]$, there exists $\xi(x) \in (a, b)$,

$$f(x) = H(x) + R(x), \quad R(x) = \frac{f^{(2n)}(\xi(x))}{(2n)!}(x-x_1)^2(x-x_2)^2 \cdots (x-x_n)^2.$$

Hermite Interpolation, with Gaussian Quadrature

$$\int_{-1}^{1} f(x)\,dx \quad = \quad \int_{-1}^{1} H(x)\,dx + \int_{-1}^{1} R(x)\,dx$$

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&\overset{def}{=\!=} c_1 f(x_1) + c_2 f(x_2) + \cdots + c_n f(x_n) + \mathbf{R}
\end{aligned}
$$

Gaussian Quadrature Error Estimate, `matlab` code

$$\mathbf{R} \;=\; \frac{f^{(2n)}(\xi)}{(2n)!} \int_{-1}^{1} (x-x_1)^2 (x-x_2)^2 \cdots (x-x_n)^2 \, dx$$

Gaussian Quadrature Error Estimate, `matlab` code

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\begin{aligned}
\mathbf{R} &= \frac{f^{(2n)}(\xi)}{(2n)!} \int_{-1}^{1} (x-x_1)^2 (x-x_2)^2 \cdots (x-x_n)^2 \, dx \\
&= \frac{2^{2n}\ (n!)^3\ (n-1)!}{(2n+1)!\,(2n)!\,(2n-1)!}\, f^{(2n)}(\xi) = O\left(\frac{4^{-n}\ \left|f^{(2n)}(\xi)\right|}{(2n)!}\right).
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$\boxed{\text{Rapid convergence for smooth functions}}$

Gaussian Quadrature Error Estimate, `matlab` code

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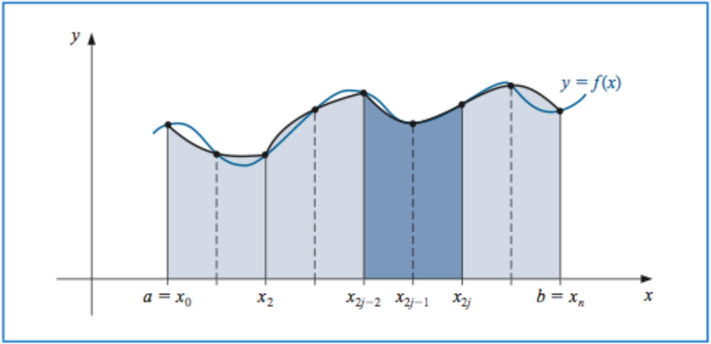
$$

$\boxed{\text{Rapid convergence for smooth functions}}$

```
function [c,x] = Legendre(n)
%
b     = transpose((1:n-1));
b     = b./sqrt((2*b-1).*(2*b+1));
B     = diag(b,1)+diag(b,-1);
[Q,D] = eig(B);
x     = diag(D);
x(abs(x)<1e-15) = 0;
c     = 2*transpose(Q(1,:).^2);
```

In contrast: Composite Simpson's Rule $\left(n = 2m, x_j = a + j\,h, h = \frac{b-a}{n}, 0 \le j \le n\right)$

$$\int_a^b f(x)dx = \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{m-1} f(x_{2j}) + 4\sum_{j=1}^{m} f(x_{2j-1}) + f(b)\right) - \frac{(b-a)h^4}{180} f^{(4)}(\mu)$$



Simpson/Trapezoidal vs. Gaussian Quadratures

Simpson/Trapezoidal:

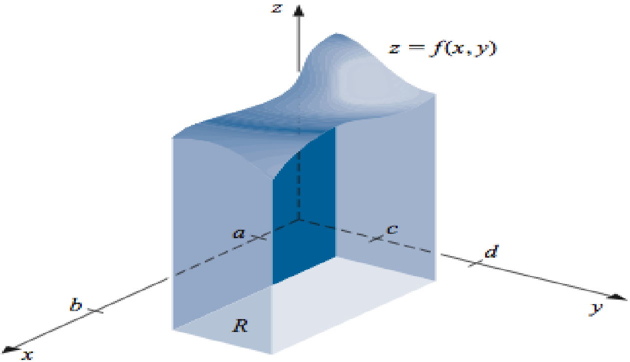
- Composite rules:
  - Adding more EQUI-SPACED points.
- Romberg extrapolation:
  - Obtaining higher order rules from lower order rules.
- Adaptive quadratures:
  - Adding more points ONLY WHEN NECESSARY.

Gaussian Quadrature:

- points different for different n.

Gaussian Quadrature good for given n,
Simpson good for given tolerance.

§4.8 Double Integral $\int\int_R f(x,y)\,dA$



$$R = \{(x,y) \mid \quad a \le x \le b, \quad c \le y \le d.\}$$

Double Integral = Integral of Integral function

$$\begin{aligned}\int\int_R f(x,y)\,dA &= \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx \\ &= \int_a^b g(x)\,dx, \quad \text{with } \boxed{g(x) \overset{def}{=\!=} \int_c^d f(x,y)\,dy}\end{aligned}$$

Double Integral = Integral of Integral function

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$$= \int_a^b g(x)\,dx, \quad \text{with } \boxed{g(x) \stackrel{def}{=\!=} \int_c^d f(x,y)\,dy}$$

General approach in a nutshell

- Approximate $\int_a^b g(x)\,dx$ with ANY quadrature

$$\int_a^b g(x)\,dx \approx c_1 g(x_1) + c_2 g(x_2) + \cdots + c_n g(x_n)$$

- For each node x_i, approximate $g(x_i) = \displaystyle\int_c^d f(x_i, y)\,dy$ with ANY quadrature.
- Rest is book keeping: work out 2D quadrature / error estimate

$\int\int_R f(x,y)\,dA = \int_a^b g(x)\,dx, \quad g(x) = \int_c^d f(x,y)\,dy$

- Approximate $\int_a^b g(x)\,dx$ with n-point quadrature:

$$\int_a^b g(x)\,dx = c_1 g(x_1) + c_2 g(x_2) + \cdots + c_n g(x_n) + \mathbf{R}(g)$$

$\int\int_R f(x,y)\,dA = \int_a^b g(x)\,dx, \quad g(x) = \int_c^d f(x,y)\,dy$

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- For $1 \le i \le n$, approximate $g(x_i)$ with m-point quadrature:

$$\int_c^d f(x_i,y)\,dy = \widehat{c}_1 f(x_1,y_1) + \widehat{c}_2 f(x_i,y_2) + \cdots + \widehat{c}_m f(x_i,y_m) + \widehat{\mathbf{R}}(f(x_i,\cdot)).$$

$\int\int_R f(x,y)\,dA = \int_a^b g(x)\,dx, \quad g(x) = \int_c^d f(x,y)\,dy$

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$$\begin{aligned}\int\int_R f(x,y)\,dA &= \left(\sum_{i=1}^n c_i g(x_i)\right) + \mathbf{R}(g)\\ &= \left(\sum_{i=1}^n c_i \left(\left(\sum_{j=1}^m \widehat{c}_j f(x_i,y_j)\right) + \widehat{\mathbf{R}}(f(x_i,\cdot))\right)\right) + \mathbf{R}(g)\end{aligned}$$

$\int\int_R f(x,y)\,dA = \int_a^b g(x)\,dx, \quad g(x) = \int_c^d f(x,y)\,dy$

- Approximate $\int_a^b g(x)\,dx$ with n-point quadrature:

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$$\begin{aligned}
\int\int_R f(x,y)\,dA &= \left(\sum_{i=1}^n c_i g(x_i)\right) + \mathbf{R}(g) \\
&= \left(\sum_{i=1}^n c_i \left(\left(\sum_{j=1}^m \widehat{c}_j f(x_i,y_j)\right) + \widehat{\mathbf{R}}(f(x_i,\cdot))\right)\right) + \mathbf{R}(g) \\
&= \boxed{\sum_{i=1}^n \sum_{j=1}^m c_i \widehat{c}_j f(x_i,y_j)} + \boxed{\left(\sum_{i=1}^n c_i \widehat{\mathbf{R}}(f(x_i,\cdot))\right) + \mathbf{R}(g)}
\end{aligned}$$

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&= \boxed{\sum_{i=1}^n \sum_{j=1}^m c_i \widehat{c}_j f(x_i,y_j)} + \boxed{\left(\sum_{i=1}^n c_i \widehat{\mathbf{R}}(f(x_i,\cdot))\right) + \mathbf{R}(g)} \\
&= \boxed{\text{2D quadrature}} + \boxed{\text{error estimate}}
\end{aligned}$$

$$\int\int_R f(x,y)\,dA = \sum_{i=1}^n \sum_{j=1}^m c_i \widehat{c}_j f(x_i, y_j) + \sum_{i=1}^n c_i \widehat{\mathbf{R}}(f(x_i,\cdot)) + \mathbf{R}(g)$$

$$R = \{(x,y)\,|\quad a \le x \le b, \quad c \le y \le d.\}$$

Example, Simpson's Rule with $m = n = 3$:

- Simpson's Rule on $[a,b]$: $\quad (x_1, x_2, x_3) = \left(a, \frac{a+b}{2}, b\right)$.
- Simpson's Rule on $[c,d]$: $\quad (y_1, y_2, y_3) = \left(c, \frac{c+d}{2}, d\right)$.

$$\int\int_R f(x,y)\,dA = \sum_{i=1}^n \sum_{j=1}^m c_i \widehat{c}_j f(x_i, y_j) + \sum_{i=1}^n c_i \widehat{\mathbf{R}}(f(x_i, \cdot)) + \mathbf{R}(g)$$

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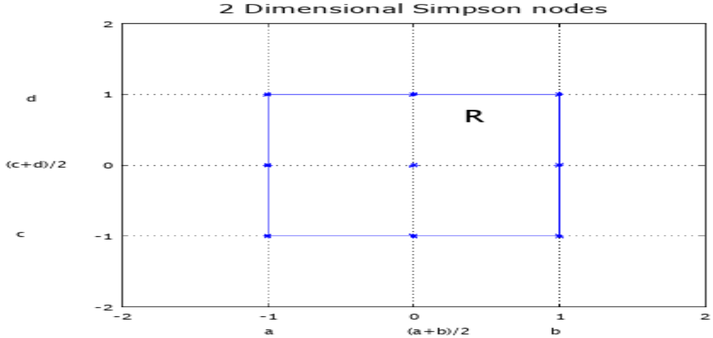

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Example, $m = n = 3$, $g(x) \stackrel{def}{=} \int_c^d f(x,y)\,dy$

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$$\int_a^b g(x)\,dx = c_1 g(x_1) + c_2 g(x_2) + c_3 g(x_3) - \frac{h^5}{90}\,g^{(4)}(\xi),$$

$$h = \frac{b-a}{2}, \quad (c_1,c_2,c_3) = \frac{h}{3}(1,4,1).$$

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$$g(x_i) = \int_c^d f(x_i,y)\,dy = \widehat{c}_1 f(x_i,y_1) + \widehat{c}_2 f(x_i,y_2) + \widehat{c}_3 f(x_i,y_3) - \frac{k^5}{90}\,\frac{\partial^4 f}{\partial^4 y}(x_i,\eta_i),$$

$$k = \frac{d-c}{2}, \quad (\widehat{c}_1,\widehat{c}_2,\widehat{c}_3) = \frac{k}{3}(1,4,1).$$

Composite Simpson Rules, in general

$$\int\int_R f(x,y)\,dA = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} c_i \widehat{c}_j f(x_i, y_j)\right)$$

$$-\frac{k^4\,(d-c)}{180}\left(\sum_{i=1}^{m} c_i \frac{\partial^4 f}{\partial^4 y}(x_i,\eta_i)\right) - \frac{h^4\,(b-a)}{180}\int_c^d \frac{\partial^4 f}{\partial^4 x}(\xi, y)dy$$

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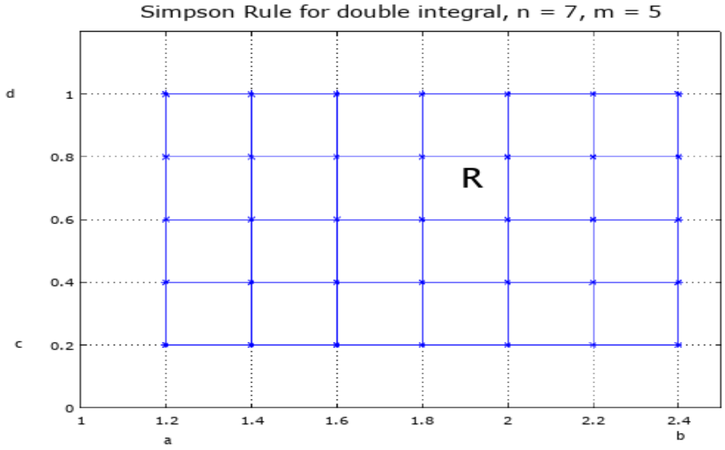
$$= \sum_{i=1}^{n}\sum_{j=1}^{m} c_i \widehat{c}_j f(x_i, y_j)$$

$$- \frac{(b-a)(d-c)}{180}\left(k^4 \frac{\partial^4 f}{\partial^4 y}(\widehat{\xi},\widehat{\eta}) + h^4 \frac{\partial^4 f}{\partial^4 x}(\xi,\eta)\right).$$

Example: $\int\int_R \log(x+2y)\,dA$ with $n = 7, m = 5$

$$R = \{(x,y) \mid \quad 1.2 \le x \le 2.4, \quad 0.2 \le y \le 1.\}$$



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\frac{\partial^4 f}{\partial^4 x} &= -\frac{6}{(x+2y)^4}, \qquad \frac{\partial^4 f}{\partial^4 y} = -\frac{96}{(x+2y)^4}. \\
\left|\frac{\partial^4 f}{\partial^4 x}\right| &\le \frac{6}{(1.2+2\times 0.2)^4} \qquad \approx 0.91553 \quad \text{for} \quad (x,y)\in R, \\
\left|\frac{\partial^4 f}{\partial^4 y}\right| &\le \frac{96}{(1.2+2\times 0.2)^4} \qquad \approx 14.648.
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\textbf{Quad\_Error} &= \frac{(b-a)(d-c)}{180}\left|k^4\frac{\partial^4 f}{\partial^4 y}(\widehat{\xi},\widehat{\eta}) + h^4\frac{\partial^4 f}{\partial^4 x}(\xi,\eta)\right| \\
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&\le \frac{(2.4-1.2)(1-0.2)}{180}\left(0.2^4\times 14.648 + 0.2^4\times 0.91553\right)\\
&\approx 1.328\times 10^{-4}.
\end{aligned}
$$

Example: $\int\int_R \mathbf{log}(x+2y)\,dA$ with $n=7, m=5$

$$\int\int_R \mathbf{log}(x+2y)\,dA \quad = \quad \underline{1.0360}481\cdots$$

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\int\int_R \log(x+2y)\,dA &= \underline{1.0360}481\cdots \\
\int\int_R \log(x+2y)\,dA &\approx \sum_{i=1}^{7}\sum_{j=1}^{5} c_i \widehat{c}_j f(x_i, y_j) \\
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\end{aligned}$$

Therefore

$$\left|\int\int_R \mathbf{log}(x+2y)\,dA - \left(\sum_{i=1}^{7}\sum_{j=1}^{5} c_i\widehat{c}_j f(x_i, y_j)\right)\right| = 1.546\times 10^{-5}$$

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&< 1.328\times 10^{-4}.
\end{aligned}$$

(Error Estimate)

2-Dimensional Gaussian Quadratures

$$
\begin{aligned}
R &= \{(x,y) \mid \quad a \le x \le b, \quad c \le y \le d.\} \\
\int\int_R f(x,y)\,dA &= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.
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\end{aligned}
$$

Perform change of variables

$$x = \frac{a+b}{2} + \frac{b-a}{2}\, u, \quad y = \frac{c+d}{2} + \frac{d-c}{2}\, v \quad \text{for} \quad u, v \in [-1, 1].$$

Double integral becomes

$$\int\int_R f(x, y)\, dA = \frac{(b-a)(d-c)}{4} \int_{-1}^{1} \widehat{g}(u) du, \quad \text{where}$$

$$\widehat{g}(u) \stackrel{def}{=\!=} \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}\, u, \frac{c+d}{2} + \frac{d-c}{2}\, v \right) dv.$$

$\int\int_R f(x,y)\,dA = \frac{(b-a)(d-c)}{4}\int_{-1}^{1}\widehat{g}(u)du$

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- n-point Gaussian quadrature for $\int_{-1}^{1}\widehat{g}(u)du$:

$$\int_{-1}^{1}\widehat{g}(u)du \approx c_1\widehat{g}(u_1) + c_2\widehat{g}(u_2) + \cdots + c_n\widehat{g}(u_n).$$

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- m-point Gaussian quadrature for $\widehat{g}(u_i)$,

$$\begin{aligned}\widehat{g}(u_i) &\approx \widehat{c}_1 f\left(x_i, \frac{c+d}{2} + \frac{d-c}{2}\,v_1\right) + \cdots + \widehat{c}_m f\left(x_i, \frac{c+d}{2} + \frac{d-c}{2}\,v_m\right)\\ &= \widehat{c}_1 f\left(x_i, y_1\right) + \cdots + \widehat{c}_m f\left(x_i, y_m\right), \quad y_j \stackrel{def}{=} \frac{c+d}{2} + \frac{d-c}{2}\,v_j.\end{aligned}$$

$\int\int_R f(x,y)\,dA = \frac{(b-a)(d-c)}{4}\int_{-1}^{1}\widehat{g}(u)du$

$$
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\int_{-1}^{1}\widehat{g}(u)du &\approx c_1\widehat{g}(u_1) + c_2\widehat{g}(u_2) + \cdots + c_n\widehat{g}(u_n) \\
\widehat{g}(u_i) &\approx \widehat{c}_1 f\left(x_i, y_1\right) + \cdots + \widehat{c}_m f\left(x_i, y_m\right).
\end{aligned}
$$

So we have Gaussian quadrature for double integral:

$$
\int\int_R f(x,y)\,dA \approx \frac{(b-a)(d-c)}{4}\sum_{i=1}^{n}\sum_{j=1}^{m} c_i\widehat{c}_j f\left(x_i, y_j\right).
$$

Example: $\int\int_R \log(x+2y)\,dA = 1.036\cdots$, $n=7$, $m=5$

- Gaussian quadrature approximation

$$\int\int_R f(x,y)\,dA \approx \frac{(b-a)(d-c)}{4}\sum_{i=1}^{n}\sum_{j=1}^{m} c_i \widehat{c}_j f\left(x_i, y_j\right) \approx \underline{1.0360481706}5\cdots$$

$$\left|\int\int_R f(x,y)\,dA - \frac{(b-a)(d-c)}{4}\sum_{i=1}^{n}\sum_{j=1}^{m} c_i \widehat{c}_j f\left(x_i, y_j\right)\right| \approx 6.4\times 10^{-10}.$$

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- Simpson rule approximation

$$\int\int_R f(x,y)\,dA \approx \sum_{i=1}^{n}\sum_{j=1}^{m} c_i\widehat{c}_j f\left(x_i, y_j\right) \approx \underline{1.03603}270963\cdots.$$

$$\left|\int\int_R f(x,y)\,dA - \sum_{i=1}^{n}\sum_{j=1}^{m} c_i\widehat{c}_j f\left(x_i, y_j\right)\right| \approx 1.5\times 10^{-5}.$$

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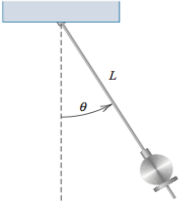
$\boxed{\text{Gaussian quadrature much more accurate}}$.

§5.1 Initial Value ODE

- The motion of a swinging pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0,$$



- Initial value conditions critical

$$\theta(t_0) = \theta_0, \quad \text{and} \quad \theta'(t_0) = \theta'_0.$$

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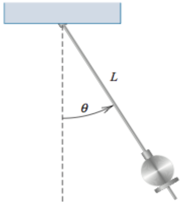
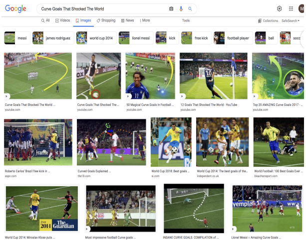


- Initial value conditions critical

$$\theta(t_0) = \theta_0, \quad \text{and} \quad \theta'(t_0) = \theta_0'.$$

- Curveball/knuckleball in soccer



- Knuckleball pitch

Lipschitz condition

Definition: function $f(t, y)$ satisfies a **Lipschitz condition** in the variable y on a set $D \subset \mathbf{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L\,|y_1 - y_2|\,,$$

whenever $(t, y_1), (t, y_2)$ are in D. L is Lipschitz constant.

- **Example 1**: Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the region

  $$D = \{(t, y) \quad | \quad 0 \leq t \leq T\}\,.$$

  Solution: For any $(t, y_1), (t, y_2)$ in D,

  $$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| \leq t\,|y_1 - y_2| \leq L\,|y_1 - y_2|\,,$$

  for $L = T$.

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- **Example 2**: Show that $f(t, y) = t\,y^2$ does not satisfy any Lipschitz condition on the region

$$D = \{(t, y) \quad | \quad 0 \leq t \leq T\}\,.$$

Solution: Choose $(T, y_1), (T, y_2)$ in D with $y_1 = 0, y_2 > 0$,

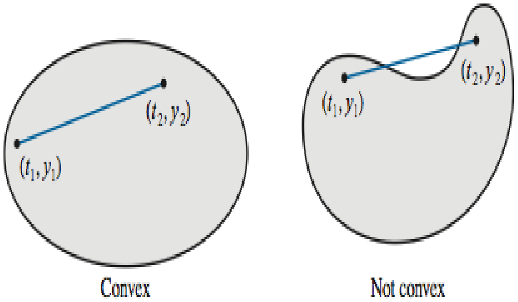
$$\frac{|f(T, y_1) - f(T, y_2)|}{|y_1 - y_2|} = T\,y_2,$$

which can be larger than L for any fixed $L > 0$.

Convex Set

Definition: A set $D \subset \mathbf{R}^2$ is convex if

$$\begin{aligned} &\text{whenever} \quad (t_1, y_1) \text{ and } (t_2, y_2) \in D \\ \longrightarrow \quad &\text{line segment} \quad (1-\lambda)(t_1, y_1) + \lambda(t_2, y_2) \in D \text{ for all } \lambda \in [0, 1]. \end{aligned}$$



Theorem: Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbf{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \quad \text{for all} \quad (t, y) \in D,$$

then f satisfies a Lipschitz condition with Lipschitz constant L.

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then f satisfies a Lipschitz condition with Lipschitz constant L.

- **Example 1**: Show that $f(t, y) = t\,y^2$ satisfies Lipschitz condition on the region

  $$D = \{(t, y) \quad | \quad 0 \leq t \leq T, \quad -Y \leq y \leq Y\}.$$

  Solution:

  $$\frac{\partial f}{\partial y}(t, y) = 2ty, \quad \left|\frac{\partial f}{\partial y}(t, y)\right| \leq 2T\,Y \quad \text{for all} \quad (t, y) \in D.$$

  so $f(t, y) = t\,y^2$ satisfies Lipschitz condition with $L = 2T\,Y$.

What is special with $f(t, y) = t\, y^2$?

- $f(t, y) = t\, y^2$ satisfies Lipschitz condition on the region

$$D = \{(t, y) \quad | \quad 0 \le t \le T, \quad -Y \le y \le Y\}.$$

- $f(t, y) = t\, y^2$ doesn't satisfy Lipschitz condition on region

$$D = \{(t, y) \quad | \quad 0 \le t \le T\}.$$

Initial value problem

$$y'(t) = t\, y^2(t), \quad y(t_0) = \alpha > 0$$

has unique, but unbounded solution

$$y(t) = \frac{2\,\alpha}{2 + \alpha(t_0^2 - t^2)},$$

the denominator vanishes at

$$t = \sqrt{\frac{2}{\alpha} + t_0^2}.$$

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- for $0 < t_0 < T < \sqrt{\frac{2}{\alpha} + t_0^2}$, ODE solution bounded on

$$D = \{(t, y) \quad | \quad t_0 \le t \le T\}.$$

- for $\sqrt{\frac{2}{\alpha} + t_0^2} < T$ ODE solution not defined at $t = \sqrt{\frac{2}{\alpha} + t_0^2}$.

Well-posed problem

Definition in English: ODE is well-posed if

- A unique ODE solution exists, and
- Small changes (perturbation) to ODE imply small changes to solution.

Well-posed problem

The initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha, \tag{5.2}$$

is said to be a **well-posed problem** if:

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε, with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0, \tag{5.3}$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b]. \qquad \blacksquare$$

Well-posed problem

Definition in English: ODE is well-posed if

- A unique ODE solution exists, and
- Small changes (perturbation) to ODE imply small changes to solution.

Theorem: Suppose

$$\mathcal{D} = \{(t,\, y) \;\mid\; a \leq t \leq b \;\text{ and }\; -\infty < y < \infty\}.$$

If f is continuous and satisfies a Lipschitz condition in the variable y on the set $\mathcal{D}$, then the initial-value problem

$$\frac{d\,y}{d\,t} = f\,(t,\, y)\,,\;\; a \leq t \leq b, \quad y(a) = \alpha$$

is well-posed.

Well-posed problem, example

Show that the initial-value problem

$$\frac{d\,y}{d\,x} = y - t^2 + 1, \quad 0 \leq t \leq 2,$$

is well posed on

$$\mathcal{D} \stackrel{def}{=} \{(t, y)\ |0 \leq t \leq 2 \ \text{ and } \ -\infty < y < \infty\}$$

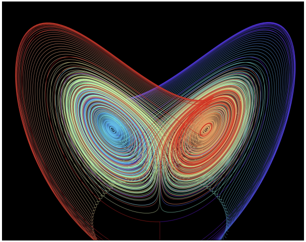
Solution: Because

$$\frac{\partial f}{\partial y}(t, y) = 1, \quad \left|\frac{\partial f}{\partial y}(t, y)\right| = 1.$$

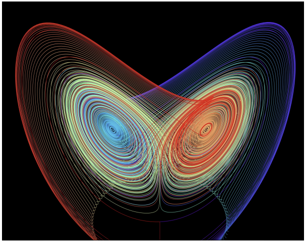
$f(t, y) = y - t^2 + 1$ satisfies a Lipschitz condition in y on $\mathcal{D}$ with Lipschitz constant 1.Therefore this ODE is well-posed. In fact,

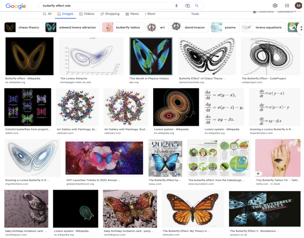
$$y(t) = 1 + t^2 + 2t - \frac{1}{2}e^t.$$

What is **not** well-posed?

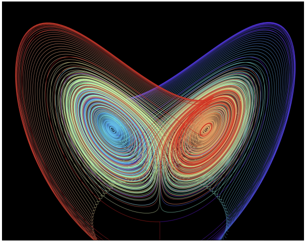


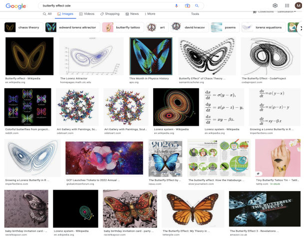
What is **not** well-posed?

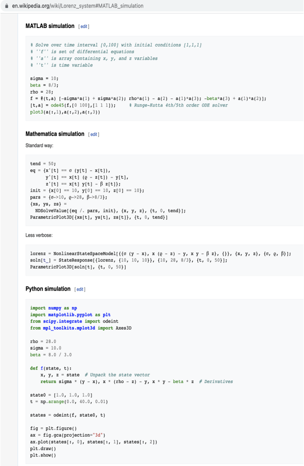




What is **not** well-posed?






en.wikipedia.org/wiki/Lorenz_system#MATLAB_simulation

MATLAB simulation [edit]

```
% Solve over time interval [0,100] with initial conditions [1,1,1]
% ''f'' is set of differential equations
% ''a'' is array containing x, y, and z variables
% ''t'' is time variable

sigma = 10;
beta = 8/3;
rho = 28;
f = @(t,a) [-sigma*a(1) + sigma*a(2); rho*a(1) - a(2) - a(1)*a(3); -beta*a(3) + a(1)*a(2)];
[t,a] = ode45(f,[0 100],[1 1 1]);     % Runge-Kutta 4th/5th order ODE solver
plot3(a(:,1),a(:,2),a(:,3))
```

Mathematica simulation [edit]

Standard way:

```
tend = 50;
eq = {x'[t] == σ (y[t] - x[t]),
      y'[t] == x[t] (ρ - z[t]) - y[t],
      z'[t] == x[t] y[t] - β z[t]};
init = {x[0] == 10, y[0] == 10, z[0] == 10};
pars = {σ->10, ρ->28, β->8/3};
{xs, ys, zs} =
  NDSolveValue[{eq /. pars, init}, {x, y, z}, {t, 0, tend}];
ParametricPlot3D[{xs[t], ys[t], zs[t]}, {t, 0, tend}]
```

Less verbose:

```
lorenz = NonlinearStateSpaceModel[{{σ (y - x), x (ρ - z) - y, x y - β z}, {}}, {x, y, z}, {σ, ρ, β}];
soln[t_] = StateResponse[{lorenz, {10, 10, 10}}, {10, 28, 8/3}, {t, 0, 50}];
ParametricPlot3D[soln[t], {t, 0, 50}]
```

Python simulation [edit]

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
from mpl_toolkits.mplot3d import Axes3D

rho = 28.0
sigma = 10.0
beta = 8.0 / 3.0

def f(state, t):
    x, y, z = state  # Unpack the state vector
    return sigma * (y - x), x * (rho - z) - y, x * y - beta * z  # Derivatives

state0 = [1.0, 1.0, 1.0]
t = np.arange(0.0, 40.0, 0.01)

states = odeint(f, state0, t)

fig = plt.figure()
ax = fig.gca(projection="3d")
ax.plot(states[:, 0], states[:, 1], states[:, 2])
plt.draw()
plt.show()
```