CSE380 HW04

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1 Introduction

Here, we will be solving the steady-state heat equation:

$$-k\nabla^2 T(x,y) = q(x,y) \tag{1}$$

where k is the thermal conductivity, T is the material temperature, and q is a heat source term. The expanded form of (1) in 1 dimension is:

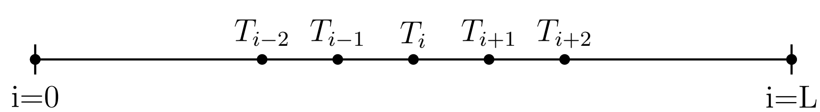
$$-k\frac{\delta^2 T(x)}{\delta x^2} = q(x) \tag{2}$$

The expanded form of (1) in 2 dimensions is:

$$-k\left(\frac{\delta^2 T(x)}{\delta x^2} + \frac{\delta^2 T(y)}{\delta y^2}\right) = q(x,y) \tag{3}$$

To solve the steady-state heat equation in 1 and 2 dimensions, we will first need to approximate the value of the second derivative terms in these equations. To do this, we will use 2nd- and 4th-order central finite-difference approximations. The resulting linear system from these numerical methods will be solved using two iterative solution mechanisms: Jacobi and Gauss-Seidel.

2 1 Dimension Approximations



Here, we are going to approximate the steady-state heat equation in 1 dimension. In the above diagram, T_i is the temperature at the point $x = i$, where $x \in [0, L]$. In this node-based scheme, we will be using constant mesh spacing of N points, such that the distance between a point, T_i, and the previous point, T_{i-1} or $T(x - \Delta x)$, or the next point, T_{i+1} or $T(x + \Delta x)$, is the same distance $\Delta x = L/N$.

We are assuming:

1. The mesh spacing is constant
2. Dirichlet boundary conditions are applied for any incomplete stencil
3. The values at the boundaries will be provided through manufactured solutions.
4. $T(x)$ is smooth and continuous for all $x \in [0, L]$
5. The values T_i and q_i are not changing over time

The 2nd-order approximation for the second derivative in (2) is:

$$\frac{\delta^2 T(x)}{\delta x^2} \approx \frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} \tag{4}$$

Substituting this approximation into (2) gives us:

$$T_{i-1} - 2T_i + T_{i+1} \approx -\frac{q_i \Delta x^2}{k} \tag{5}$$

where the truncation error of this approximation is on the order of Δx^2. The resulting linear system looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ \vdots \\ \vdots \\ T_{L-1} \\ T_L \end{bmatrix} = \begin{bmatrix} -\frac{q_0 \Delta x^2}{k} \\ -\frac{q_1 \Delta x^2}{k} \\ -\frac{q_1 \Delta x^2}{k} \\ \vdots \\ -\frac{q_{L-1} \Delta x^2}{k} \\ -\frac{q_L \Delta x^2}{k} \end{bmatrix}$$

Each interior (non-boundary) row contains 3 non-zero entries and $N - 3$ zeros.

The 4th-order approximation for the second derivative in (2) is:

$$\frac{\delta^2 T(x)}{\delta x^2} \approx \frac{-T_{i-2} + 16T_{i-1} - 30T_i + 16T_{i+1} - T_{i+2}}{12\Delta x^2} \tag{6}$$

Substituting this approximation into (2) gives us:

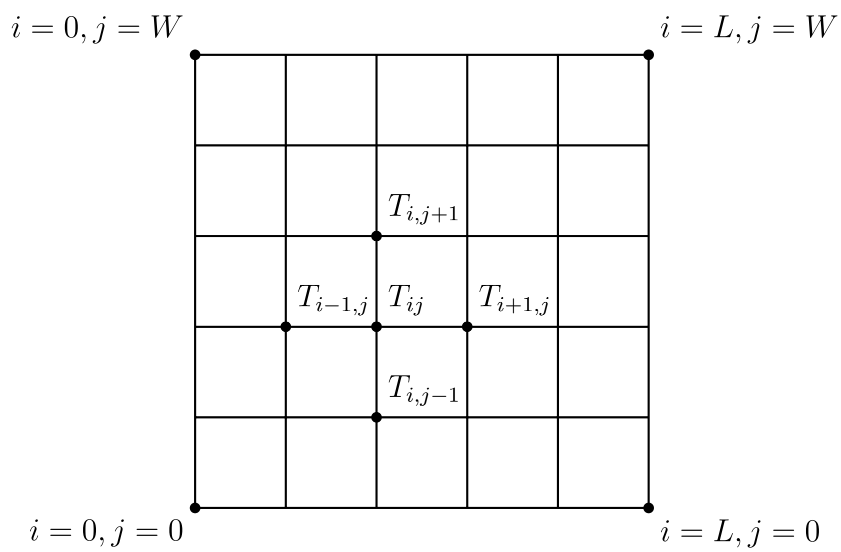
$$-\frac{1}{12}T_{i-2} + \frac{4}{3}T_{i-1} - \frac{5}{2}T_i + \frac{4}{3}T_{i+1} - \frac{1}{12}T_{i+2} \approx -\frac{q_i \Delta x^2}{k} \tag{7}$$

where the truncation error of this approximation is on the order of Δx^4. The resulting linear system looks like this:

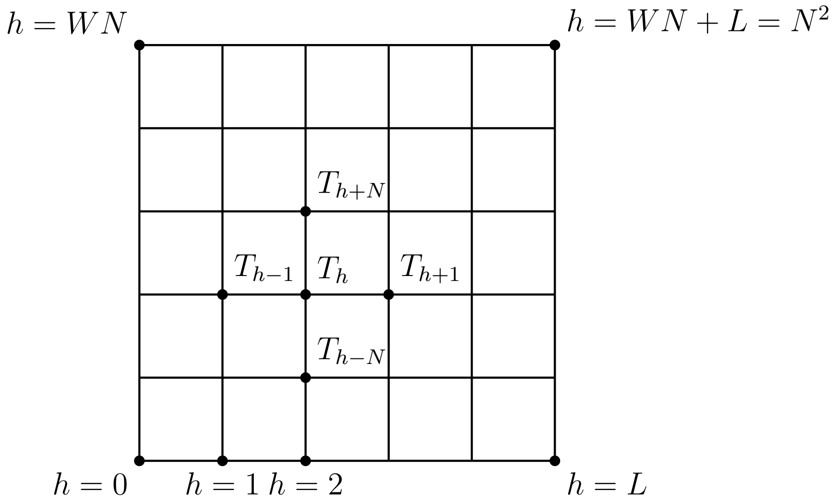
$$
\begin{bmatrix}
1 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\
0 & 1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\
-\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & \dots & 0 \\
0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \dots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \dots & \dots & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
T_0 \\ T_1 \\ T_2 \\ \vdots \\ \vdots \\ \vdots \\ T_{L-2} \\ T_{L-1} \\ T_L
\end{bmatrix}
=
\begin{bmatrix}
-\frac{q_0 \Delta x^2}{k} \\ -\frac{q_1 \Delta x^2}{k} \\ -\frac{q_2 \Delta x^2}{k} \\ \vdots \\ \vdots \\ \vdots \\ -\frac{q_{L-2} \Delta x^2}{k} \\ -\frac{q_{L-1} \Delta x^2}{k} \\ -\frac{q_L \Delta x^2}{k}
\end{bmatrix}
$$

Each interior row has 5 non-zero entries and $N-5$ zeros.

3 2 Dimension Approximations



Here, we are going to approximate the steady-state heat equation in 2 dimensions. In the above diagram of our node-based scheme, T_{ij} is the temperature at the point $x = i, y = j$ or $T(x = i, y = j)$, where $x \in [0, L]$ and $y \in [0, W]$. We are assuming a square domain, such that $L = W$. We will be using the same number of discretization points, N, for each dimension, such that $\Delta x = \Delta y = \Delta n$. However, we will need to make a mapping from this 2-D coordinate system to a 1-D vectored system. We will use the equation $h = i + j(N_x) = i + j(N_y)$ where N_x and N_y are the number of discretization points in the x- and y-directions, respectively. Again, they are the same value for the following solutions. Our new mesh grid will look like this:



We are assuming:

1. The mesh spacing is constant, such that $N_x = N_y$, therefore $\Delta x = \Delta y = \Delta n$
2. The domain is square, such that $L = W$
3. Dirichlet boundary conditions are applied for any incomplete stencil
4. The values at the boundaries will be provided through manufactured solutions.
5. $T(x, y)$ is smooth and continuous for all $x \in [0, L]$ and all $y \in [0, W]$
6. The values T_{ij} and q_{ij} are not changing over time

The 2nd-order approximation for the second derivative terms in (3) is:

$$\frac{\delta^2 T(x)}{\delta x^2} \approx \frac{T_{h-1} - 2T_h + T_{h+1}}{\Delta x^2} \tag{8}$$

$$\frac{\delta^2 T(y)}{\delta y^2} \approx \frac{T_{h-N} - 2T_h + T_{h+N}}{\Delta y^2} \tag{9}$$

Substituting this approximation into (3) gives us:

$$T_{h-N} + T_{h-1} - 4T_h + T_{h+1} + T_{h+N} \approx -\frac{q_h \Delta n^2}{k} \tag{10}$$

where the truncation error of this approximation is on the order of Δn^2. The resulting linear system looks like this:

$$\begin{bmatrix} \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & Nzeros & 1 & -4 & 1 & Nzeros & 1 & 0 & \dots & \dots \\ \dots & 1 & Nzeros & 1 & -4 & 1 & Nzeros & 1 & 0 & \dots \\ \dots & 0 & 1 & Nzeros & 1 & -4 & 1 & Nzeros & 1 & 0 \\ \dots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & \dots & \dots & \dots & \dots & 0 & 1 & 0 & 0 \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ T_{h-1} \\ T_h \\ T_{h+1} \\ \vdots \\ \vdots \\ \vdots \\ T_{N^2-2} \\ T_{N^2-1} \\ T_{N^2} \end{bmatrix} = \begin{bmatrix} -\frac{q_0 \Delta n^2}{k} \\ -\frac{q_1 \Delta n^2}{k} \\ -\frac{q_2 \Delta n^2}{k} \\ \vdots \\ \vdots \\ \vdots \\ -\frac{q_{N^2-2} \Delta n^2}{k} \\ -\frac{q_{N^2-1} \Delta n^2}{k} \\ -\frac{q_{N^2} \Delta n^2}{k} \end{bmatrix}$$

Each interior row contains 5 non-zero entries and N-5 zeros.

The 4th-order approximation for the second derivative terms in (3) is:

$$\frac{\delta^2 T(x)}{\delta x^2} \approx \frac{-T_{h-2} + 16T_{h-1} - 30T_h + 16T_{h+1} - T_{h+2}}{12\Delta n^2} \tag{11}$$

$$\frac{\delta^2 T(y)}{\delta y^2} \approx \frac{-T_{h-2N} + 16T_{h-N} - 30T_h + 16T_{h+N} - T_{h+2N}}{12\Delta n^2} \tag{12}$$

Substituting this approximation into (3) gives us:

$$-\frac{1}{12}T_{h-2N} + \frac{4}{3}T_{h-N} - \frac{1}{12}T_{h-2} + \frac{4}{3}T_{h-1} + 5T_h + \frac{4}{3}T_{h+1} - \frac{1}{12}T_{h+2} + \frac{4}{3}T_{h+N} - \frac{1}{12}T_{h+2N} \approx -\frac{q_h \Delta n^2}{12k} \tag{13}$$

where the truncation error of this approximation is on the order of Δn^4. The resulting coefficient matrix looks like this:

$$A = \begin{bmatrix} \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & Nzeros & -\frac{1}{12} & \frac{4}{3} & 5 & \frac{4}{3} & -\frac{1}{12} & Nzeros & \frac{4}{3} & Nzeros \\ \dots & \frac{4}{3} & Nzeros & -\frac{1}{12} & \frac{4}{3} & 5 & \frac{4}{3} & -\frac{1}{12} & Nzeros & \dots \\ \dots & Nzeros & \frac{4}{3} & Nzeros & -\frac{1}{12} & \frac{4}{3} & 5 & \frac{4}{3} & -\frac{1}{12} & Nzeros \\ \dots & -\frac{1}{12} & Nzeros & \frac{4}{3} & Nzeros & -\frac{1}{12} & \frac{4}{3} & 5 & \frac{4}{3} & -\frac{1}{12} \\ \dots & 0 & -\frac{1}{12} & Nzeros & \frac{4}{3} & Nzeros & -\frac{1}{12} & \frac{4}{3} & 5 & \frac{4}{3} \\ \dots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Each interior row has 9 non-zero entries. The resulting linear system looks like this:

$$A \begin{bmatrix} \vdots \\ T_{h-1} \\ T_h \\ T_{h+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T_{N^2-1} \\ T_{N^2} \end{bmatrix} = \begin{bmatrix} -\frac{q_0 \Delta n^2}{k} \\ -\frac{q_1 \Delta n^2}{k} \\ -\frac{q_2 \Delta n^2}{k} \\ \vdots \\ \vdots \\ \vdots \\ -\frac{q_{N^2-2} \Delta n^2}{k} \\ -\frac{q_{N^2-1} \Delta n^2}{k} \\ -\frac{q_{N^2} \Delta n^2}{k} \end{bmatrix}$$

4 Jacobi Method

Initialize two, 1xN^2 T vectors, T_{old} and T_{new};
Set each value in T_{old} to 0;
while *currentError* $>$ *errorThreshold* **do**
    for *each h from 0 to* N^2 **do**
        solve for T_h with the values from T_{old} for the other components;
        store result T_h in T_{new};
    end
    currentError = $\|$T$_{old} - T_{new}\|$ / T$_{old}$;
    $T_{old} = T_{new}$
end

Estimated memory needed ≈8 bytes x 2N^2

5 Gauss-Seidel Method

Estimated memory needed ≈8 bytes x 2N^2

Initialize two, $1xN^2$ T vectors, T_{old} and T_{new};
Set each value in T_{old} to 0;
$T_{new} = T_{old}$;
while *currentError* $>$ *errorThreshold* **do**
    for *each h from 0 to* N^2 **do**
        solve for T_h *with the values from* T_{new} *for the other components;*
        store result T_h *in* T_{new}*;*
    end
    currentError $= \| T_{old} - T_{new} \| \ / \ T_{old}$;
    $T_{old} = T_{new}$;
end