CFRM 546
Factor Model Risk Analysis

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Outline

- Factor Model Specification
- Factor Risk Budgeting
- Factor Model Risk Reports
- Simulating Returns from Factor Models: Factor Model Monte Carlo
- Factor Model Scenario Analysis

Introduction

Factor models for asset returns (equity, fixed income, hedge funds, multi-asset portfolios etc.) are used to

- Decompose risk and return into explanable and unexplainable components
- Generate estimates of abnormal return
- Describe the covariance structure of returns
- Predict returns in specified stress scenarios
- Provide a framework for portfolio risk analysis

Three Types of Asset Return Factor Models

1. Macroeconomic factor model
   (a) Factors are observable economic and financial time series

2. Fundamental factor model
   (a) Factors are created from observerable asset characteristics

3. Statistical factor model
   (a) Factors are unobservable and extracted from asset returns

Factor Model Specification

The three types of multifactor models for asset returns have the general form

$$\begin{aligned} R_{it} &= \alpha_i + \beta_{1i} f_{1t} + \beta_{2i} f_{2t} + \cdots + \beta_{Ki} f_{Kt} + \varepsilon_{it} \qquad (1) \\ &= \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it} \end{aligned}$$

- R_{it} is the simple return (typically in excess of the risk-free rate) on asset i ($i = 1, \ldots, N$) in time period t ($t = 1, \ldots, T$),

- f_{kt} is the k^{th} *common factor* ($k = 1, \ldots, K$) (also typically in excess of the risk-free rate)

- β_{ki} is the *factor loading, beta* or *exposure* for asset i on the k^{th} factor,

- ε_{it} is the asset *specific factor*.

Assumptions for Unconditional Factor Model

1. The factor realizations, $\mathbf{f}_t$, are stationary with unconditional moments

$$\begin{aligned} E[\mathbf{f}_t] &= \boldsymbol{\mu}_f \\ \text{cov}(\mathbf{f}_t) &= E[(\mathbf{f}_t - \boldsymbol{\mu}_f)(\mathbf{f}_t - \boldsymbol{\mu}_f)'] = \underset{K \times K}{\boldsymbol{\Omega}_f} \end{aligned}$$

2. Asset specific error terms, ε_{it}, are uncorrelated with each of the common factors, f_{kt},

$$\text{cov}(f_{kt}, \varepsilon_{it}) = 0, \text{ for all } k, \ i \text{ and } t.$$

3. Error terms ε_{it} are serially uncorrelated and contemporaneously uncorrelated across assets

$$
\begin{aligned}
\text{cov}(\varepsilon_{it}, \varepsilon_{js}) &= \sigma_i^2 \text{ for all } i = j \text{ and } t = s \\
&= 0, \text{ otherwise}
\end{aligned}
$$

Remarks:

- Statistical modeling of returns involves statistical modeling of factors and residuals

- Typical factor models have a small number of factors (e.g., $K < 10$)

- Multivariate modeling of factors is a relatively low dimensional problem
  - Multivariate distribution and copula models are feasible for factors
  - EWMA and DCC is feasible for factor covariances

- $\text{cov}(\varepsilon_{it}, \varepsilon_{js}) = 0$ $(i \neq j) \Rightarrow$ only need univariate statistical models for ε_{it}

- The assumption $\text{cov}(\varepsilon_{it}, \varepsilon_{js}) = 0$ for $t \neq s$ is crucial because under a well specified factor model all of the cross asset correlation should be explained by the factors.

Assumptions for Conditional Factor Model

Let I_t denote the information available at time t. Then

$$\begin{aligned}
E[\mathbf{f}_t|I_{t-1}] &= \boldsymbol{\mu}_{f,t|t-1} = \boldsymbol{\mu}_{f,t} \\
\text{cov}(\mathbf{f}_t|I_{t-1}) &= E[(\mathbf{f}_t - \boldsymbol{\mu}_{f,t})(\mathbf{f}_t - \boldsymbol{\mu}_{f,t})'|I_{t-1}] = \underset{K\times K}{\boldsymbol{\Omega}_{f,t}} \\
\text{cov}(\varepsilon_{it}, \varepsilon_{js}|I_{t-1}) &= \sigma^2_{i,t} \text{ for all } i = j \text{ and } t = s \\
&= 0 \text{ otherwise}
\end{aligned}$$

Note: It is also possible to allow the other factor model parameters to be conditionally time varying so that $\alpha_i = \alpha_{i,t}$ and $\boldsymbol{\beta}_i = \boldsymbol{\beta}_{i,t}$.

Notation

Vectors with a subscript t represent the cross-section of all assets

$$\underset{(N\times 1)}{\mathbf{R}_t} = \begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix}, \ t = 1, \ldots, T$$

Vectors with a subscript i represent the time series of a given asset

$$\underset{(T\times 1)}{\mathbf{R}_i} = \begin{pmatrix} R_{i1} \\ \vdots \\ R_{iT} \end{pmatrix}, \ i = 1, \ldots, N$$

Matrix of all assets over all time periods (columns = assets, rows = time period)

$$\underset{(T\times N)}{\mathbf{R}} = \begin{pmatrix} R_{11} & \cdots & R_{N1} \\ \vdots & \ddots & \vdots \\ R_{1T} & \cdots & R_{NT} \end{pmatrix}$$

Cross Section Regression

The multifactor model (1) may be rewritten as a *cross-sectional* regression model at time t by stacking the equations for each asset to give

$$\begin{aligned}
\underset{(N\times 1)}{\mathbf{R}_t} &= \underset{(N\times 1)}{\boldsymbol{\alpha}} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times 1)}{\mathbf{f}_t} + \underset{(N\times 1)}{\boldsymbol{\varepsilon}_t},\ t = 1, \ldots, T \qquad (2) \\
\underset{(N\times K)}{\mathbf{B}} &= \begin{bmatrix} \boldsymbol{\beta}_1' \\ \vdots \\ \boldsymbol{\beta}_N' \end{bmatrix} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1K} \\ \vdots & \ddots & \vdots \\ \beta_{N1} & \cdots & \beta_{NK} \end{bmatrix} \\
E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' | \mathbf{f}_t] &= \mathbf{D} = \mathsf{diag}(\sigma_1^2, \ldots, \sigma_N^2)
\end{aligned}$$

Note: Cross-sectional heteroskedasticity

This representation is useful for risk analysis across assets.

Time Series Regression

The multifactor model (1) may also be rewritten as a *time-series* regression model for asset i by stacking observations for a given asset i to give

$$\underset{(T\times 1)}{\mathbf{R}_i} = \underset{(T\times 1)}{\mathbf{1}_T}\underset{(1\times 1)}{\alpha_i} + \underset{(T\times K)}{\mathbf{F}}\underset{(K\times 1)}{\boldsymbol{\beta}_i} + \underset{(T\times 1)}{\boldsymbol{\varepsilon}_i},\ i = 1,\ldots,N \tag{3}$$

$$\underset{(T\times K)}{\mathbf{F}} = \begin{bmatrix} \mathbf{f}_1' \\ \vdots \\ \mathbf{f}_T' \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{K1} \\ \vdots & \ddots & \vdots \\ f_{1T} & \cdots & f_{KT} \end{bmatrix}$$

$$E[\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'] = \sigma_i^2\mathbf{I}_T$$

Note: Time series homoskedasticity

This representation is useful for estimating α_i and β_i using linear regression

Multivariate Regression

Collecting data from $i = 1, \ldots, N$ allows the model (3) to be expressed as the multivariate regression

$$[\mathbf{R}_1, \ldots, \mathbf{R}_N] = \mathbf{1}_T[\alpha_1, \ldots, \alpha_N] + \mathbf{F}[\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N] + [\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_N]$$

or

$$\begin{aligned}
\underset{(T\times N)}{\mathbf{R}} &= \underset{(T\times 1)}{\mathbf{1}_T}\underset{(1\times N)}{\boldsymbol{\alpha}'} + \underset{(T\times K)}{\mathbf{F}}\underset{(K\times N)}{\mathbf{B}'} + \underset{(T\times N)}{\mathbf{E}} \\
&= \mathbf{X}\boldsymbol{\Gamma}' + \mathbf{E} \\
\underset{(T\times(K+1))}{\mathbf{X}} &= [\mathbf{1}_T \vdots \mathbf{F}], \quad \underset{((K+1)\times N)}{\boldsymbol{\Gamma}'} = \begin{bmatrix} \boldsymbol{\alpha}' \\ \mathbf{B}' \end{bmatrix},
\end{aligned}$$

Alternatively, collecting data from $t = 1, \ldots, T$ allows the model (2) to be expressed as the multivariate regression

$$[\mathbf{R}_1, \ldots, \mathbf{R}_T] = [\boldsymbol{\alpha}, \ldots, \boldsymbol{\alpha}] + \mathbf{B}[\mathbf{f}_1, \ldots, \mathbf{f}_T] + [\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_T]$$

or

$$\begin{aligned}
\underset{(N\times T)}{\mathbf{R}'} &= \underset{(N\times 1)}{\boldsymbol{\alpha}}\underset{(1\times T)}{\mathbf{1}'_T} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times T)}{\mathbf{F}'} + \underset{(N\times T)}{\mathbf{E}'} \\
&= \boldsymbol{\Gamma}\mathbf{X}' + \mathbf{E}' \\
\underset{((K+1)\times T)}{\mathbf{X}'} &= \left[\begin{array}{c} \mathbf{1}'_T \\ \mathbf{F}' \end{array}\right], \underset{(N\times(K+1))}{\boldsymbol{\Gamma}} = [\boldsymbol{\alpha} \vdots \mathbf{B}],
\end{aligned}$$

Expected Return ($\alpha - \beta$) Decomposition

$$E[R_{it}] = \alpha_i + \boldsymbol{\beta}_i' E[\mathbf{f}_t]$$

- $\boldsymbol{\beta}_i' E[\mathbf{f}_t] =$ explained expected return due to systematic risk factors

- $\alpha_i = E[R_{it}] - \boldsymbol{\beta}_i' E[\mathbf{f}_t] =$ unexplained expected return (abnormal return). That is, α_i is the portion of $E[R_{it}]$ not explained by the factors through $E[\mathbf{f}_t]$

- Equilibrium asset pricing models impose the restriction $\alpha_i = 0$ (no abnormal return) for all assets $i = 1, \ldots, N$, so that

$$E[R_{it}] = \boldsymbol{\beta}_i' E[\mathbf{f}_t]$$

  In these models, R_{it} and $\mathbf{f}_t$ are expressed as excess returns (e.g., in excess of US T-Bills or LIBOR) so that $E[R_{it}]$ represents the risk premium on asset i and $E[\mathbf{f}_t]$ is the vector of factor risk premia.

- Interpretation of $\alpha_i \neq 0$. If the estimated α_i is statistically different from zero, this could represent a mis-pricing of the asset: $\alpha_i > 0$ implies that the asset is under priced relative to factor exposures (so return is higher than expected); $\alpha_i < 0$ implies that the asset is over-priced relative to factor exposures (so return is lower than expected).

- In actively managed portfolios, $\alpha_i > 0$ is often interpreted as a measure of the portfolio manager's skill

Performance Attribution Report

Performance attribution reports (which are common in industry) give the decomposition

$$\begin{aligned}
R_{it} &= \alpha_i + \beta_{1i} f_{1t} + \beta_{2i} f_{2t} + \cdots + \beta_{Ki} f_{Kt} + \varepsilon_{it} \\
\beta_i f_{kt} &= \text{return contribution of } k^{th} \text{ factor at time } t \\
\alpha_i + \varepsilon_{it} &= \text{asset specific (unexplained) return contribution at time } t
\end{aligned}$$

Here,

$$\begin{aligned}
\boldsymbol{\beta}' \mathbf{f}_t &= \text{systemic return contribution at } t \\
\alpha_i + \varepsilon_{it} &= \text{unsystematic return contribution at } t
\end{aligned}$$

Covariance Structure

Using the cross-section regression

$$\underset{(N\times 1)}{\mathbf{R}_t} = \underset{(N\times 1)}{\boldsymbol{\alpha}} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times 1)}{\mathbf{f}_t} + \underset{(N\times 1)}{\boldsymbol{\varepsilon}_t},\ t = 1, \ldots, T$$

and the assumptions of the multifactor model, the $(N \times N)$ covariance matrix of asset returns has the form

$$\text{cov}(\mathbf{R}_t) = \boldsymbol{\Omega}_{FM} = \mathbf{B}\boldsymbol{\Omega}_f\mathbf{B}' + \mathbf{D} \tag{4}$$

Note, (4) implies that

$$\begin{aligned}
\text{var}(R_{it}) &= \boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_i + \sigma_i^2 \\
\text{cov}(R_{it}, R_{jt}) &= \boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_j \\
\text{cor}(R_{it}, R_{jt}) &= \frac{\boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_j}{\left[\left(\boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_i + \sigma_i^2\right)(\boldsymbol{\beta}_j'\boldsymbol{\Omega}_f\boldsymbol{\beta}_j + \sigma_j^2)\right]^{1/2}}
\end{aligned}$$

Remarks:

- $\boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_i + \sigma_i^2$ = total return variance; $\boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_i$ = systematic (factor) return variance; σ_i^2 = unsystematic (asset specific) return variance

- The factor model R^2 is defined as systematic return variance divided by total return variance
$$R_i^2 = \frac{\boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_i}{\text{var}(R_{it})}$$
and gives the fraction of return variance that is explained by the factors.

- $\text{cov}(R_{it}, R_{jt}) = \boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_j$ implies that all of the cross asset return correlation is explained by the factors.

- In $\mathbf{\Omega}_{FM} = \mathbf{B}\mathbf{\Omega}_f\mathbf{B}' + \mathbf{D}$, $\mathbf{B}\mathbf{\Omega}_f\mathbf{B}'$ is the systematic return covariance matrix, and $\mathbf{D} =$diag$(\sigma_1^2, \ldots, \sigma_n^2)$ is the unsystematic return covariance matrix

Conditional Covariance Structure

Let I_t denote the information available at time t. We can allow the factor covariances and residual variances to be time varying

$$
\begin{aligned}
\underset{k\times 1}{\mathbf{f}_t} &= \boldsymbol{\mu}_{f|t} + \boldsymbol{\varepsilon}_{f,t} \\
\boldsymbol{\varepsilon}_{f,t} &= \boldsymbol{\Omega}_{f,t}^{1/2}\mathbf{z}_{f,t} \Rightarrow \text{var}(\boldsymbol{\varepsilon}_{f,t}|I_{t-1}) = \underset{k\times k}{\boldsymbol{\Omega}_{f,t}} \\
\varepsilon_{it} &= \sigma_{i,t}z_{it} \Rightarrow \text{var}(\varepsilon_{it}|I_{t-1}) = \sigma_{i,t}^2,\ i = 1,\ldots,n
\end{aligned}
$$

Then the factor model conditional covariance matrix is

$$
\text{cov}(\mathbf{R}_t|I_{t-1}) = \boldsymbol{\Omega}_{FM,t} = \mathbf{B}\boldsymbol{\Omega}_{f,t}\mathbf{B}' + \mathbf{D}_t
$$

Note: We can also allow the factor betas to be conditionally time varying (i.e., $\mathbf{B} = \mathbf{B}_t$)

Remarks

- Here,

$$\text{var}(R_{it}|I_{t-1}) = \boldsymbol{\beta}_i'\boldsymbol{\Omega}_{f,t}\boldsymbol{\beta}_i + \sigma_{i,t}^2$$

  Common movements in $\text{var}(R_{it}|I_{t-1})$ and $\text{var}(R_{jt}|I_{t-1})$ can be explained by $\boldsymbol{\beta}_i'\boldsymbol{\Omega}_{f,t}\boldsymbol{\beta}_i$ provided $\sigma_{i,t}^2$ is not too large. Hence, a factor structure for returns can explain the stylized fact of common movements in conditional volatility.

- Here,

$$\text{cov}(R_{it}, R_{jt}|I_{t-1}) = \boldsymbol{\beta}_i'\boldsymbol{\Omega}_{f,t}\boldsymbol{\beta}_j$$

  This result can explain common movements in correlations across assets. Also, if $\beta_i = \beta_{i,t}$ and $\beta_j = \beta_{j,t}$ then time varying correlations can also be due to changes in the factor exposures over time.

Portfolio Analysis

Let $\mathbf{w} = (w_1, \ldots, w_n)$ be a vector of portfolio weights (w_i = fraction of wealth in asset i). If $\mathbf{R}_t$ is the $(N \times 1)$ vector of simple returns then

$$R_{p,t} = \mathbf{w}'\mathbf{R}_t = \sum_{i=1}^{N} w_i R_{it}$$

Portfolio Factor Model

$$\begin{aligned}
\mathbf{R}_t &= \boldsymbol{\alpha} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t \Rightarrow \\
R_{p,t} &= \mathbf{w}'\boldsymbol{\alpha} + \mathbf{w}'\mathbf{B}\mathbf{f}_t + \mathbf{w}'\boldsymbol{\varepsilon}_t \\
&= \alpha_p + \boldsymbol{\beta}_p'\mathbf{f}_t + \varepsilon_{p,t} \\
\alpha_p &= \mathbf{w}'\boldsymbol{\alpha},\ \boldsymbol{\beta}_p' = \mathbf{w}'\mathbf{B},\ \varepsilon_{p,t} = \mathbf{w}'\boldsymbol{\varepsilon}_t \\
\mathrm{var}(R_{p,t}) &= \boldsymbol{\beta}_p'\boldsymbol{\Omega}_f\boldsymbol{\beta}_p + \mathrm{var}(\varepsilon_{p,t}) = \mathbf{w}'\mathbf{B}\boldsymbol{\Omega}_f\mathbf{B}'\mathbf{w} + \mathbf{w}'\mathbf{D}\mathbf{w}
\end{aligned}$$

Remarks

- The linearity of the portfolio return (in the portfolio weights) and the linearity of the factor model (in the factor betas) allows the factor model for the portfolio to be aggregated from the factor models from the individual assets. This is incredibly convenient and powerful.

Unconditional Asset Risk Measures: Factor Model and Normal Distribution

$$R_{it} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}$$

$$\mathbf{f}_t \sim iid\ N(\boldsymbol{\mu}_f, \boldsymbol{\Omega}_f),\ \varepsilon_{it} \sim iid\ N(0, \sigma_{\varepsilon,i}^2),\ \mathrm{cov}(f_{k,t}, \varepsilon_{is}) = 0 \text{ for all } k, t, s$$

Then $R_{it} \sim iid\ N(\mu_{FM,i}, \sigma_{FM,i}^2)$ and

$$\begin{aligned}
E[R_{it}] &= \mu_{FM,i} = \alpha_i + \boldsymbol{\beta}_i' \boldsymbol{\mu}_f \\
\mathrm{var}(R_{it}) &= \sigma_{FM,i}^2 = \boldsymbol{\beta}_i' \boldsymbol{\Omega}_f \boldsymbol{\beta}_i + \sigma_{\varepsilon,i}^2 \\
\sigma_{FM,i} &= \sqrt{\boldsymbol{\beta}_i' \boldsymbol{\Omega}_f \boldsymbol{\beta}_i + \sigma_{\varepsilon,i}^2} \\
\mathrm{VaR}_\alpha^{N,FM} &= \mu_{FM,i} + \sigma_{FM,i} \times q_p^Z,\ Z \sim N(0,1) \\
\mathrm{ES}_p^{N,FM} &= \mu_{FM,i} + \sigma_{FM,i} \frac{1}{p} \phi(q_p^Z)
\end{aligned}$$

Note: In practice, $\alpha_i = 0$ is typically imposed so that $\mu_{FM,i} = \boldsymbol{\beta}_i' \boldsymbol{\mu}_f$.

Conditional Asset Risk Measures: Factor Model and Normal Distribution

$$
\begin{aligned}
\mathrm{var}(R_{it}|I_{t-1}) &= \sigma^2_{FM,i,t} = \boldsymbol{\beta}_i'\boldsymbol{\Omega}_{f,t}\boldsymbol{\beta}_i + \sigma^2_{\varepsilon,i,t} \\
\sigma_{FM,i,t} &= \sqrt{\boldsymbol{\beta}_i'\boldsymbol{\Omega}_{f,t}\boldsymbol{\beta}_i + \sigma^2_{\varepsilon,i,t}} \\
\mathrm{VaR}^{N,FM}_{p,t} &= \mu_{FM,i,t} + \sigma_{FM,i,t} \times z_p \\
\mathrm{ES}^{N,FM}_{p,t} &= \mu_{FM,i,t} + \sigma_{FM,i,t}\frac{1}{p}\phi(z_p)
\end{aligned}
$$

where $\boldsymbol{\Omega}_{f,t}$ is modeled as an EWMA or DCC and $\sigma^2_{\varepsilon,i,t}$ is modeled as an EWMA or GARCH.

Note 1: For daily data it is typically assumed that $\mu_{FM,i,t} = 0$.

Note 2: We could also allow $\boldsymbol{\beta}_i = \boldsymbol{\beta}_{i,t}$ (e.g. estimate $\boldsymbol{\beta}_i$ over rolling windows for each t)

Factor Risk Budgeting

- Additively decompose (slice and dice) individual asset or portfolio return risk measures into factor contributions
- Allow portfolio manager to know sources of factor risk for allocation and hedging purposes
- Allow risk manager to evaluate portfolio from factor risk perspective

Factor Risk Decompositions

Assume asset or portfolio return R_t can be explained by a factor model

$$R_t = \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \varepsilon_t$$
$$\mathbf{f}_t \sim iid\ (\boldsymbol{\mu}_f, \boldsymbol{\Omega}_f),\ \varepsilon_t \sim iid\ (0, \sigma_\varepsilon^2),\ \text{cov}(f_{k,t}, \varepsilon_s) = 0 \text{ for all } k, t, s$$

Re-write the factor model as

$$\begin{aligned} R_t &= \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \varepsilon_t = \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \sigma_\varepsilon \times z_t \\ &= \alpha + \tilde{\boldsymbol{\beta}}'\tilde{\mathbf{f}}_t \\ \tilde{\boldsymbol{\beta}} &= (\boldsymbol{\beta}', \sigma_\varepsilon)',\ \tilde{\mathbf{f}}_t = (\mathbf{f}_t, z_t)',\ z_t = \frac{\varepsilon_t}{\sigma_\varepsilon} \sim iid\ (0, 1) \end{aligned}$$

Then

$$\sigma_{FM}^2 = \tilde{\boldsymbol{\beta}}'\boldsymbol{\Omega}_{\tilde{f}}\tilde{\boldsymbol{\beta}},\ \boldsymbol{\Omega}_{\tilde{f}} = \begin{pmatrix} \boldsymbol{\Omega}_f & 0 \\ 0 & 1 \end{pmatrix}$$

Linearly Homogenous Risk Functions

Let RM($\tilde{\boldsymbol{\beta}}$) denote the risk measures σ_{FM}, VaR$_{\alpha}^{FM}$ and ES$_{\alpha}^{FM}$ as functions of $\tilde{\boldsymbol{\beta}}$

Result 1: RM($\tilde{\boldsymbol{\beta}}$) is a linearly homogenous function of $\tilde{\boldsymbol{\beta}}$ for RM $= \sigma_{FM}$, VaR$_{\alpha}^{FM}$ and ES$_{\alpha}^{FM}$. That is, RM($c \cdot \tilde{\boldsymbol{\beta}}$) $=$$c \cdot$ RM($\tilde{\boldsymbol{\beta}}$) for any constant $c \geq 0$

Example: Consider RM($\tilde{\boldsymbol{\beta}}$) $= \sigma_{FM}(\tilde{\boldsymbol{\beta}})$. Then

$$\begin{aligned}
\sigma_{FM}(c \cdot \tilde{\boldsymbol{\beta}}) &= \left(c \cdot \tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_{\tilde{f}} c \cdot \tilde{\boldsymbol{\beta}}\right)^{1/2} = c \cdot \left(\tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_{\tilde{f}} \tilde{\boldsymbol{\beta}}\right)^{1/2} \\
&= c \cdot \sigma_{FM}(\tilde{\boldsymbol{\beta}})
\end{aligned}$$

Euler's Theorem and Additive Risk Decompositions

Result 2: Because $\text{RM}(\tilde{\boldsymbol{\beta}})$ is a linearly homogenous function of $\tilde{\boldsymbol{\beta}}$, by Euler's Theorem

$$\begin{aligned}
\text{RM}(\tilde{\boldsymbol{\beta}}) &= \sum_{j=1}^{k+1} \tilde{\beta}_j \frac{\partial \text{RM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} \\
&= \tilde{\beta}_1 \frac{\partial \text{RM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_1} + \cdots + \tilde{\beta}_{k+1} \frac{\partial \text{RM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_{k+1}} \\
&= \beta_1 \frac{\partial \text{RM}(\tilde{\boldsymbol{\beta}})}{\partial \beta_1} + \cdots + \beta_k \frac{\partial \text{RM}(\tilde{\boldsymbol{\beta}})}{\partial \beta_k} + \sigma_\varepsilon \frac{\partial \text{RM}(\tilde{\boldsymbol{\beta}})}{\partial \sigma_\varepsilon}
\end{aligned}$$

Terminology

Factor j *marginal contribution to risk*

$$\mathsf{FMCR}_j^{\mathsf{RM}} = \frac{\partial \mathsf{RM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j}$$

Factor j *contribution to risk*

$$\mathsf{FCR}_j^{\mathsf{RM}} = \tilde{\beta}_j \frac{\partial \mathsf{RM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} = \tilde{\beta}_j \times \mathsf{FMCR}_j^{\mathsf{RM}}$$

Factor j *percent contribution to risk*

$$\mathsf{FPCR}_j^{\mathsf{RM}} = \frac{\tilde{\beta}_j \frac{\partial \mathsf{RM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j}}{\mathsf{RM}(\tilde{\boldsymbol{\beta}})} = \frac{\mathsf{FCR}_j^{\mathsf{RM}}}{\mathsf{RM}(\tilde{\boldsymbol{\beta}})}$$

Analytic Results for $\mathrm{RM}(\tilde{\boldsymbol{\beta}}) = \sigma_{FM}(\tilde{\boldsymbol{\beta}})$

$$\begin{aligned} \sigma_{FM}(\tilde{\boldsymbol{\beta}}) &= \left(\tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_{\tilde{f}} \tilde{\boldsymbol{\beta}}\right)^{1/2} \\ \mathrm{FMCR}^{\mathrm{RM}} &= \frac{\partial \sigma_{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\boldsymbol{\beta}}} = \frac{1}{\sigma_{FM}(\tilde{\boldsymbol{\beta}})} \boldsymbol{\Omega}_{\tilde{f}} \tilde{\boldsymbol{\beta}} \end{aligned}$$

Factor $j = 1, \ldots, K$ percent contribution to $\sigma_{FM}(\tilde{\boldsymbol{\beta}})$

$$\frac{\beta_1 \beta_j \mathrm{cov}(f_{1t}, f_{jt}) + \cdots + \beta_j^2 \mathrm{var}(f_{jt}) + \cdots + \beta_K \beta_j \mathrm{cov}(f_{Kt}, f_{jt})}{\sigma_{FM}^2(\tilde{\boldsymbol{\beta}})},$$

Asset specific $(K+1)$ factor percent contribution to risk

$$\frac{\sigma_\varepsilon^2}{\sigma_{FM}^2(\tilde{\boldsymbol{\beta}})},\ j = K+1$$

Estimation of Factor Risk Contribution for $\mathsf{RM}(\tilde{\boldsymbol{\beta}}) = \sigma_{FM}(\tilde{\boldsymbol{\beta}})$

Unconditional estimates use estimates or values of $\tilde{\boldsymbol{\beta}}$ and the sample covariance matrix of $\boldsymbol{\Omega}_f$:

$$
\begin{aligned}
\mathsf{FMCR}^{\mathsf{RM}} &= \frac{1}{\hat{\sigma}_{FM}(\tilde{\boldsymbol{\beta}})} \hat{\boldsymbol{\Omega}}_{\tilde{f}} \tilde{\boldsymbol{\beta}},\ \hat{\sigma}_{FM}(\tilde{\boldsymbol{\beta}}) = \left(\tilde{\boldsymbol{\beta}}' \hat{\boldsymbol{\Omega}}_{\tilde{f}} \tilde{\boldsymbol{\beta}} \right)^{1/2}, \\
\hat{\boldsymbol{\Omega}}_f &= \frac{1}{T} \sum_{t=1}^{T} (\mathbf{f}_t - \hat{\boldsymbol{\mu}}_f)(\mathbf{f}_t - \hat{\boldsymbol{\mu}}_f)',\ \hat{\boldsymbol{\Omega}}_{\tilde{f}} = \begin{pmatrix} \hat{\boldsymbol{\Omega}}_f & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Conditional estimates use estimates or values of $\tilde{\boldsymbol{\beta}}$ and an EWMA or DCC covariance matrix estimate of $\boldsymbol{\Omega}_f$

$$
\mathsf{FMCR}_t^{\mathsf{RM}} = \frac{1}{\sigma_{FM,t}(\tilde{\boldsymbol{\beta}})} \hat{\boldsymbol{\Omega}}_{\tilde{f},t} \tilde{\boldsymbol{\beta}},\ \hat{\sigma}_{FM,t}(\tilde{\boldsymbol{\beta}}) = \left(\tilde{\boldsymbol{\beta}}' \hat{\boldsymbol{\Omega}}_{\tilde{f},t} \tilde{\boldsymbol{\beta}} \right)^{1/2}
$$

Results for $RM(\tilde{\boldsymbol{\beta}}) = VaR_{\alpha}^{FM}(\tilde{\boldsymbol{\beta}})$, $ES_{\alpha}^{FM}(\tilde{\boldsymbol{\beta}})$

Based on arguments in Scaillet (2002), Meucci (2007) showed that

$$\begin{aligned}
\frac{\partial \mathsf{VaR}_{\alpha}^{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} &= E[\tilde{f}_{jt} | R_t = \mathsf{VaR}_{\alpha}^{FM}(\tilde{\boldsymbol{\beta}})],\ j = 1, \ldots, K+1 \\
\frac{\partial \mathsf{ES}_{\alpha}^{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} &= E[\tilde{f}_{jt} | R_t \leq \mathsf{VaR}_{\alpha}^{FM}(\tilde{\boldsymbol{\beta}})],\ j = 1, \ldots, K+1
\end{aligned}$$

Remarks

- Intuitive interpretation as stress loss scenario
- Analytic results are available under normality; otherwise need simulation

	RM	Factor 1	$\cdots$	Factor K
Portfolio	RM_p	FCR_1^{RM}	$\cdots$	FCR_K^{RM}
Asset 1	RM_1	FCR_1^{RM}	$\cdots$	FCR_K^{RM}
Asset 2	RM_2	FCR_1^{RM}	$\cdots$	FCR_K^{RM}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Asset N	RM_N	FCR_1^{RM}	$\cdots$	FCR_K^{RM}

Table 1: Factor Contributions to Asset and Portfolio Risk

Factor Risk Reports

A common factor risk report summarizes factor contributions to asset and portfolio risk.

Here, the sum of the FCR_j^{RM} values across columns adds up to the risk measure given in the second column

	Factor 1	$\cdots$	Factor K
Portfolio	FCR_1^{RM}	$\cdots$	FCR_K^{RM}
Asset 1	FCR_1^{RM}	$\cdots$	FCR_K^{RM}
Asset 2	FCR_1^{RM}	$\cdots$	FCR_K^{RM}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Asset N	FCR_1^{RM}	$\cdots$	FCR_K^{RM}

Table 2: Factor Percent Contributions to Asset and Portfolio Risk

Factor Risk Reports

Another common factor risk report summarizes factor percent contributions to asset and portfolio risk.

Marginal Contributions to Tail Risk: Non-Parametric Estimates

Assume R_t and $\tilde{\mathbf{f}}_t$ are iid but make no distributional assumptions:

$$\{(R_1, \tilde{\mathbf{f}}_1), \ldots, (R_T, \tilde{\mathbf{f}}_T)\} = \text{observed iid sample}$$

Estimate marginal contributions to risk using *historical simulation*

$$\begin{aligned}
\hat{E}^{HS}[\tilde{f}_{jt}|R_t &= \mathsf{VaR}_\alpha] = \\
&\frac{1}{m}\sum_{t=1}^{T} \tilde{f}_{jt} \cdot \mathbf{1}\left\{\widehat{\mathsf{VaR}}_\alpha^{HS} - \varepsilon \leq R_t \leq \widehat{\mathsf{VaR}}_\alpha^{HS} + \varepsilon\right\} \\
\hat{E}^{HS}[\tilde{f}_{jt}|R_t &\leq VaR_\alpha] = \frac{1}{[T\alpha]}\sum_{t=1}^{T} \tilde{f}_{jt} \cdot \mathbf{1}\left\{\widehat{VaR}_\alpha^{HS} \leq R_t\right\}
\end{aligned}$$

Problem: Not reliable with small samples or with unequal histories for R_t

Simulating Returns: Factor Model Monte Carlo

Assume asset or portfolio return R_{it} can be explained by a factor model

$$R_{it} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}$$

$$\mathbf{f}_t \sim iid\ (\boldsymbol{\mu}_f, \boldsymbol{\Omega}_f),\ \varepsilon_{it} \sim iid\ (0, \sigma^2_{\varepsilon,i}),\ \text{cov}(f_{k,t}, \varepsilon_{is}) = 0 \text{ for all } i, k, t, s$$

To simulate returns R_t

- Simulate from the pdf of $\mathbf{f}_t$

- Simulate from the pdf of ε_{it} (independent of $\mathbf{f}_t$)

This method is often called Factor Model Monte Carlo (FMMC)

Advantages of FMMC

- Number of factors is typically much smaller than the number of assets (e.g. 5 factors vs. 1000 assets)

- Multivariate modeling of $\mathbf{f}_t$ is feasible with a small number of factors

- Univariate models can be used for residuals ε_{it} because of independence across assets

- Dependence structure across assets is defined by factor loadings and dependence structure of factors

- Can deal with unequal histories for asset returns (e.g. hedge fund data)

Short History for Returns but Long History for Factors

$$
\begin{array}{cccc}
f_{1T} & \cdots & f_{KT} & R_{iT} \\
\vdots & \vdots & \vdots & \vdots \\
f_{1,T-T_i+1} & \cdots & f_{1,T-T_i+1} & R_{i,T-T_i+1} \\
\vdots & \vdots & \vdots & NA \\
f_{11} & \cdots & f_{1K} & NA
\end{array}
$$

- Observe full history for factors $\{\mathbf{f}_1, \ldots, \mathbf{f}_T\}$

- Observe partial history for assets (monotone missing data)

$$
\begin{aligned}
& \{R_{i,T-T_i+1}, \ldots, R_{iT}\}, \\
i \; &= \; 1, \ldots, n; \; t = T - T_i + 1, \ldots, T
\end{aligned}
$$

Simulation Algorithm

- Estimate factor models for each asset using partial history for assets and risk factors

$$R_{it} = \hat{\alpha}_i + \hat{\boldsymbol{\beta}}_i' \mathbf{f}_t + \hat{\varepsilon}_{it},\ t = T - T_i + \mathbf{1}, \ldots, T$$

- Simulate B values of the risk factors from the pdf of $\mathbf{f}_t$:

$$\{\mathbf{f}_1^*, \ldots, \mathbf{f}_B^*\}$$

- Simulate B values of the factor model residuals from the pdf of ε_{it}

$$\{\hat{\varepsilon}_{i1}^*, \ldots, \hat{\varepsilon}_{iB}^*\}$$

- Create pseudo factor model returns from fitted factor model parameters, simulated factor variables and simulated residuals:

$$
\begin{aligned}
& \{R_1^*, \ldots, R_B^*\} \\
R_{it}^* &= \hat{\boldsymbol{\beta}}_i' \mathbf{f}_t^* + \hat{\varepsilon}_{it}^*, \ t = 1, \ldots, B
\end{aligned}
$$

Simulating Factor Realizations: Distribution choices

- Multivariate distributions (e.g., multivariate normal, t, copula distributions etc) (parametric, unconditional)

- Conditional multivariate distributions (e.g. normal DCC model)

- Empirical distribution (non-parametric, unconditional)
  - Resample with replacement from observed history of factors

- Filtered historical simulation (semi-parametric, conditional)
  - use local time-varying factor covariance matrices to standardize factors prior to re-sampling and then re-transform with covariance matrices after re-sampling

Simulating Residuals: Distribution choices

- Normal distribution (parametric, unconditional)
- Non-normal: Student's t, Skewed Student's t etc. (parametric, unconditional)
- Empirical (resample with replacement from observed residuals) (nonparametric, unconditional)
- GARCH(1,1) (parametric, conditional)
- Filtered historical simulation (semi-parametric, conditional)