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EE181: Stochastic Processes

Computer Assignment #1

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1 Probability Density Function

Part A.

$$
\begin{aligned}
Y = X^2 \rightarrow F_Y(y) &= \mathbb{P}(Y < y) = \mathbb{P}(X^2 < y) \overset{y \geq 0}{=} \mathbb{P}(-\sqrt{y} < X < \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\
\rightarrow f_Y(y) &= \frac{\partial}{\partial y} F_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) \\
&= \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y}));\ 0 < y < \infty
\end{aligned}
$$

Part B.

$$
X \sim \mathcal{N}(0,1) \rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \right)
$$

PDF of Chi-squared distribution with k degrees of freedom: $f_W(w;k) = \begin{cases} \frac{w^{k/2-1} e^{-w/2}}{2^{k/2}\Gamma\left(\frac{k}{2}\right)}, & w > 0 \\ 0, & \text{otherwise} \end{cases}$

$$
\Gamma(\frac{1}{2}) = \sqrt{\pi} \rightarrow f_Y(y) = \frac{y^{-1/2} e^{-y/2}}{2^{1/2}\Gamma(\frac{1}{2})};\ y > 0 = f_W(w,1)
$$

$$
\rightarrow Y \sim \chi^2(1)
$$

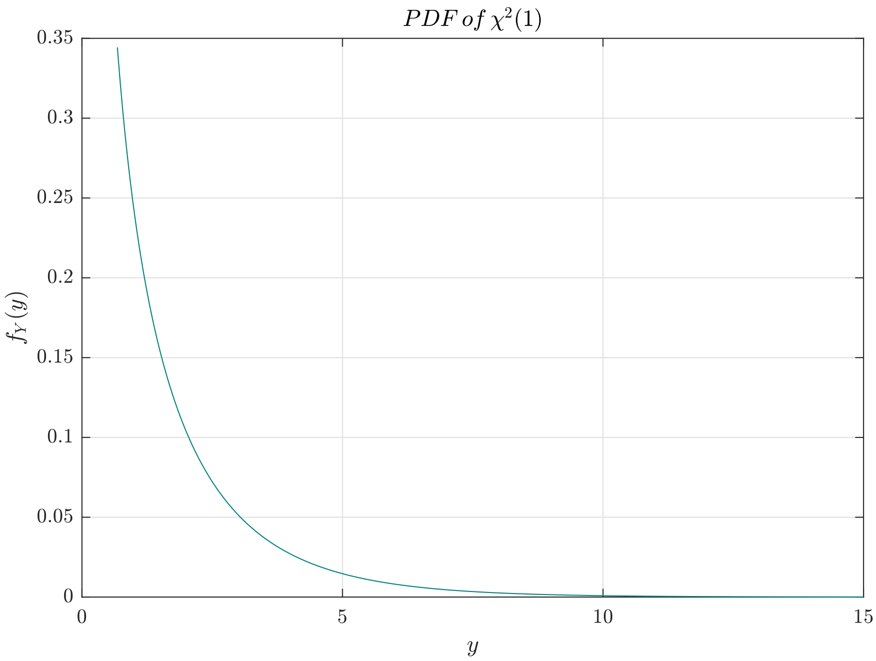


Figure 1: PDF of $\chi^2(1)$ - Theoretical

Part C.

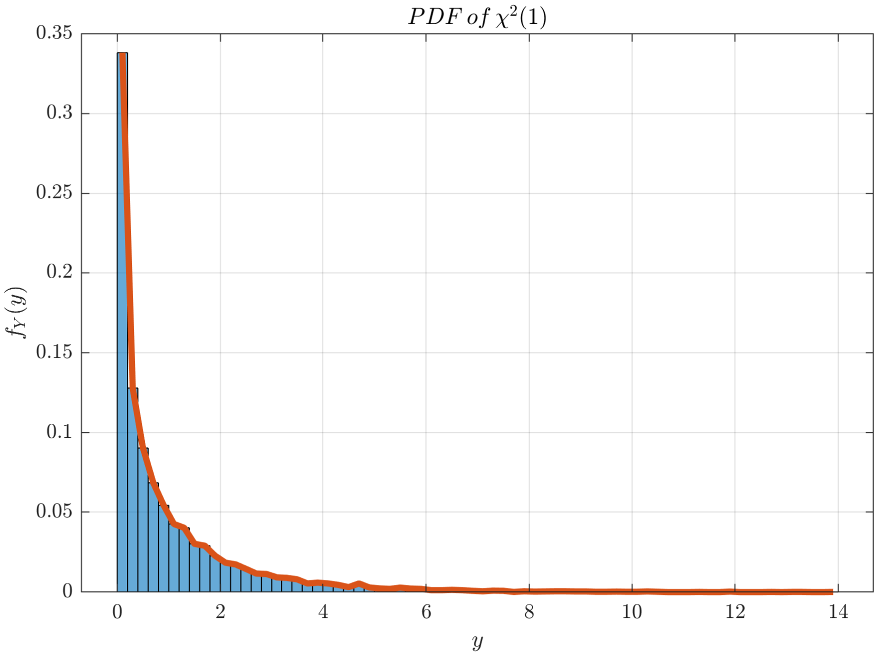


Figure 2: PDF of $\chi^2(1)$ - MATLAB simulation

Clearly, the simulation results in Figure 2, employing the histogram envelope to estimate the Probability Density Function (PDF) of random variable Y, verify the theoretical formula illustrated in Figure 1.

2 Auto correlation and Power Spectral Density Function

Part B.

The autocorrelation function of a Gaussian random process is given by: $R_x(\tau) = \frac{N_0}{2}\delta(\tau)$. Since samples are from $\mathcal{N}(0, 5)$, then $\frac{N_0}{2} = 5$. Therefore, $R_x(\tau) = 5\delta(\tau)$ As illustrated in Figure 3, the simulation results confirm this relationship.

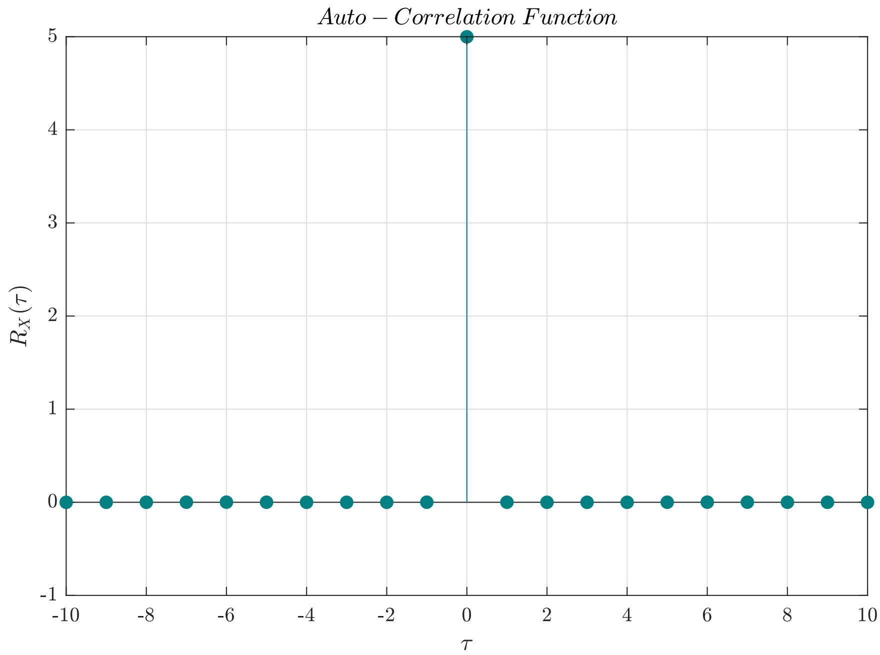


Figure 3: Auto correlation function of Gaussian process - MATLAB simulation

Part C.

$S_X(\omega)$ corresponds to the Fourier transform of $R_X(\tau)$. Figure 4 illustrates the result of applying the Fourier transform to the previously obtained $R_X(\tau)$.

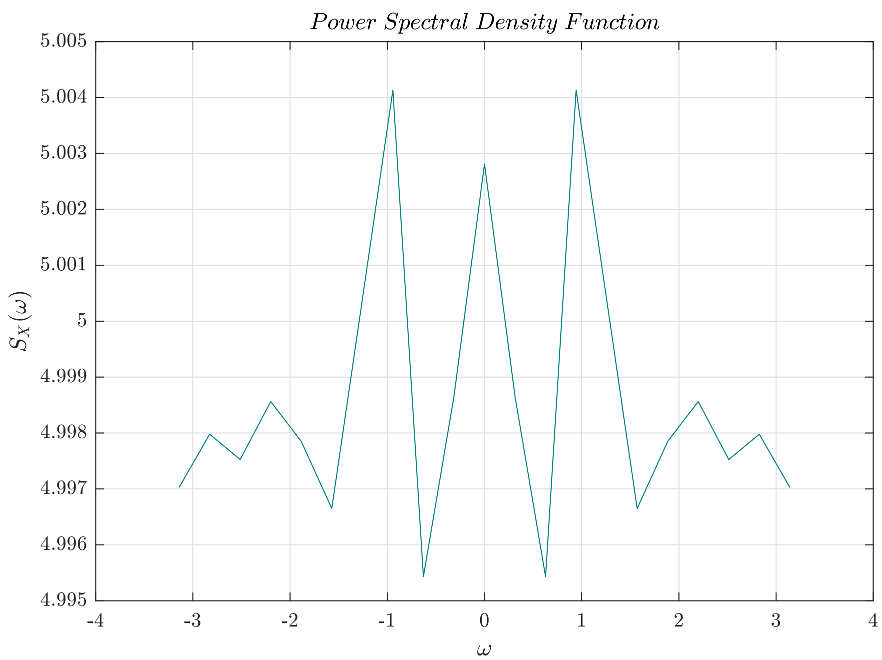


Figure 4: Power Spectral Density function of Gaussian process - MATLAB simulation

$$S_X(\omega) = \mathcal{F}\{R_X(\tau)\} = \mathcal{F}\{5\delta(\tau)\} = 5;\ -\pi < \omega < \pi$$

Evidently, the theoretical outcomes align with the simulation results.

Part D.

Theoretical formulations were presented in each relevant section, demonstrating the alignment of all simulation results with these theoretical constructs. Specifically, it was shown that the auto correlation function of a Gaussian process takes the form of a delta function, and consequently, the power spectral density of a Gaussian process manifests as a flat function. Both of these assertions were confirmed through both simulation and theoretical analysis.

3 ARMA Processes

Part A.

Auto-regressive Moving Average (ARMA-(p, q)) random process is defined as:

$$\sum_{k=0}^{p} a_k\ x[n-k] = \sum_{k=0}^{q} b_k\ v[n-k] \tag{1}$$

where $v[n]$ is a white noise (WSS) random process.

The given process is:

$$x[n] = \frac{1}{2}x[n-1] + v[n] - \frac{1}{3}v[n-1] \rightarrow x[n] - \frac{1}{2}x[n-1] = v[n] - \frac{1}{3}v[n-1] \tag{2}$$

The equation 2 evidently exhibits the identical form as 1, with the specific values $a_0 = 1, a_1 = -\frac{1}{2}$, $b_0 = 1$, and $b_1 = -\frac{1}{3}$. Consequently, the described process conforms to an ARMA(1,1) model.

Part B.

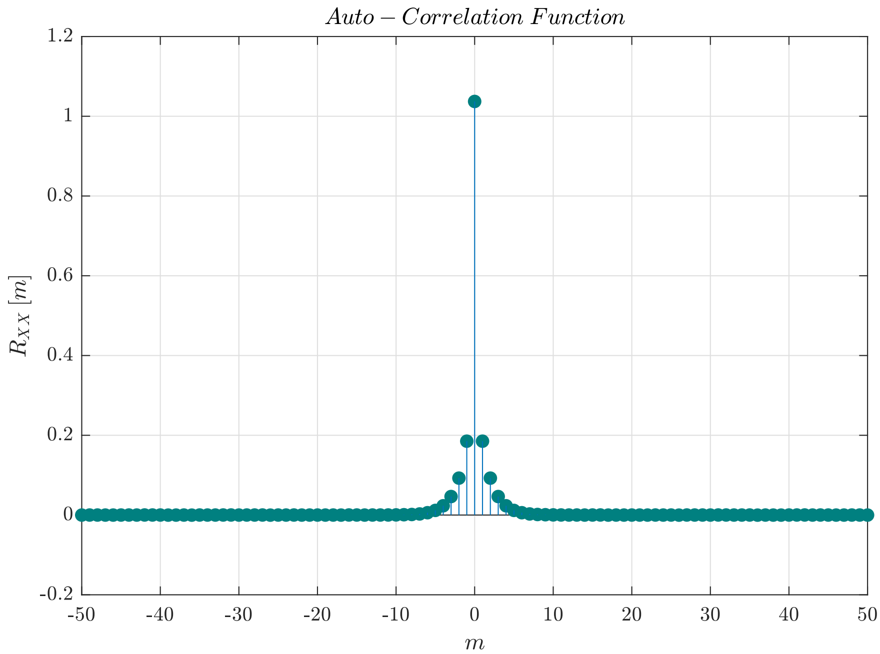


Figure 5: $R_{XX}[m]$

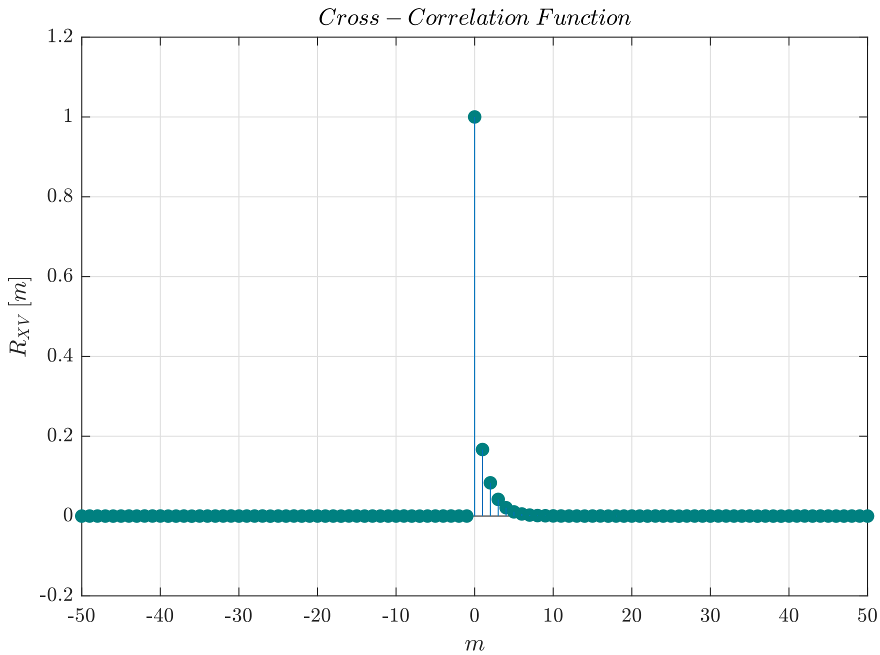


Figure 6: $R_{XV}[m]$

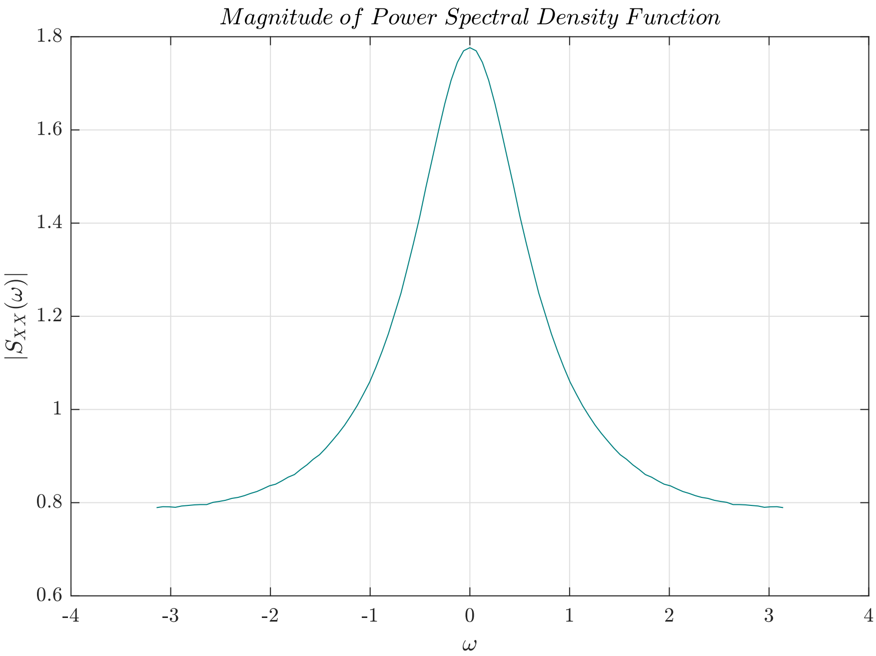


Figure 7: $S_{XX}(\omega)$

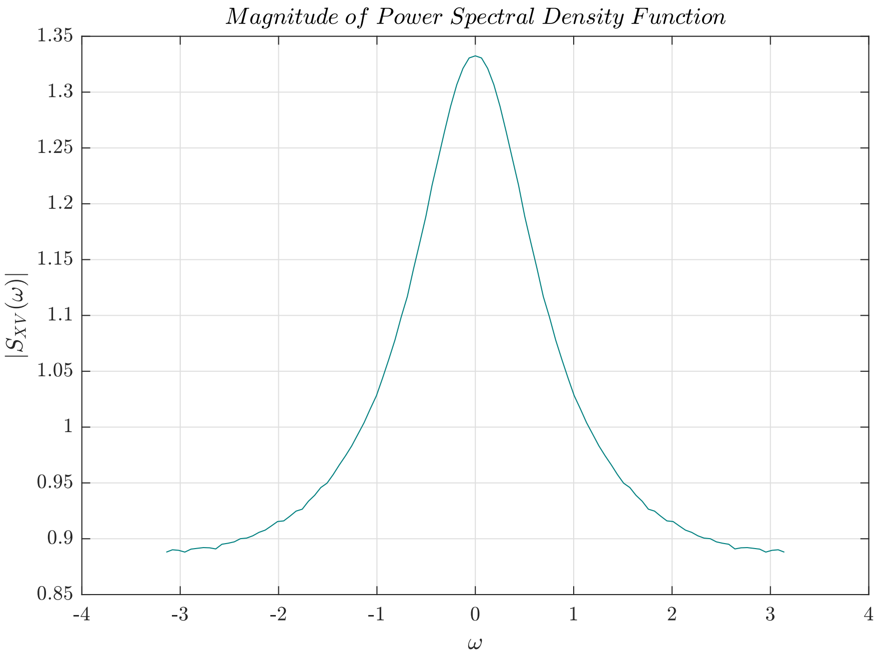


Figure 8: $S_{XV}(\omega)$

Part C.

For this section, we shall use the following relations.

$$\begin{cases} S_V(e^{j\omega}) = \dfrac{N_0}{2} & (3) \\ R_V[m] = \dfrac{N_0}{2}\delta[m] & (4) \\ S_{XV}(e^{j\omega}) = S^*_{VX}(e^{j\omega}) = \big(S_V(e^{j\omega})H^*(e^{j\omega})\big)^* = \dfrac{N_0}{2}H(e^{j\omega}) & (5) \\ S_X(e^{j\omega}) = S_V(e^{j\omega})|H(e^{j\omega})|^2 = \dfrac{N_0}{2}|H(e^{j\omega})|^2 & (6) \\ R_{XV}[m] = \mathcal{Z}^{-1}\{S_{XV}(z)\} & (7) \\ R_X[m] = \mathcal{Z}^{-1}\{S_X(z)\} & (8) \end{cases}$$

To obtain the impulse response of the system, we apply the z-transform in equation 2:

$$\rightarrow \mathcal{X}(z) - \frac{1}{2}z^{-1}\mathcal{X}(z) = \mathcal{V}(z) - \frac{1}{3}z^{-1}\mathcal{V}(z) \Rightarrow H(z) = \frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$\rightarrow H(e^{j\omega}) = \frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} \overset{N_0/2=1}{\Longrightarrow} S_{XV}(e^{j\omega}) = \underline{\frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}}$$

$$\rightarrow S_X(e^{j\omega}) = |H(e^{j\omega})|^2 = \frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} \times \frac{1-\frac{1}{3}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}} = \underline{\frac{\frac{10}{9}-\frac{1}{3}e^{j\omega}-\frac{1}{3}e^{-j\omega}}{\frac{5}{4}-\frac{1}{2}e^{j\omega}-\frac{1}{2}e^{-j\omega}}}$$

$$\rightarrow R_{XV}[m] = \mathcal{Z}^{-1}\Big\{\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\Big\} = \mathcal{Z}^{-1}\Big\{\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\Big\}$$

$$\Rightarrow R_{XV}[m] = (\frac{1}{2})^m u[m] - (\frac{1}{3})(\frac{1}{2})^{m-1}u[m-1]$$

$$\Rightarrow R_{XV}[m] = \underline{(\frac{1}{2})^m \left[u[m] - \frac{2}{3}u[m-1]\right]}$$

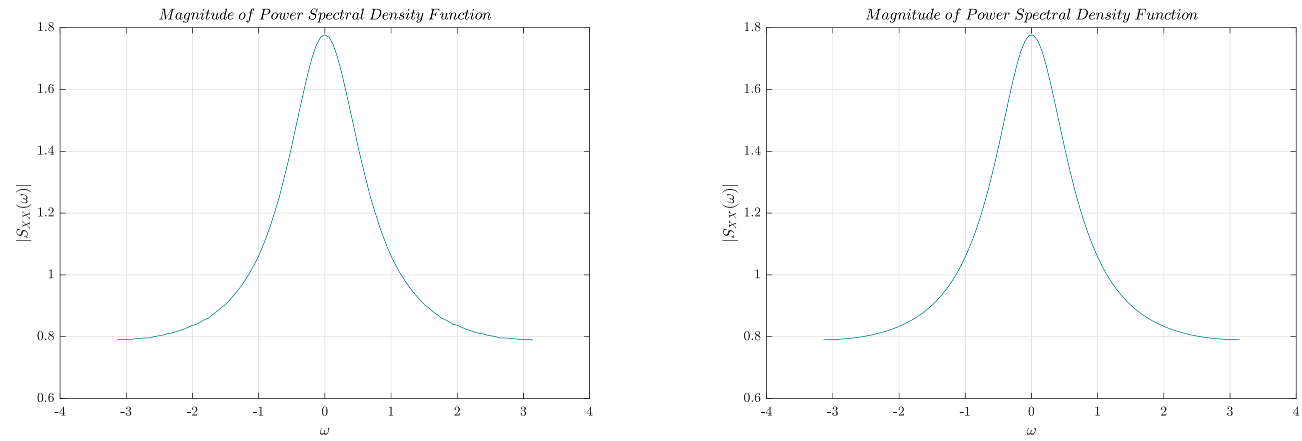
$$\rightarrow R_X[m] = \mathcal{Z}^{-1}\{S_X(z)\} = \mathcal{Z}^{-1}\Big\{\left(1+\frac{1}{6}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}\right) \times \left(1+\frac{1}{6}\frac{z}{1-\frac{1}{2}z}\right)\Big\}$$

$$= \mathcal{Z}^{-1}\left\{1+\frac{1}{6}\frac{-2}{1-2z^{-1}} + \frac{1}{6}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}} + \frac{1}{36}\left(\frac{4/3}{1-\frac{1}{2}z^{-1}} + \frac{2/3z}{1-\frac{1}{2}z}\right)\right\}$$

$$= \mathcal{Z}^{-1}\left\{1-\frac{10}{27}\left(\frac{1}{1-2z^{-1}}\right) + \frac{1}{27}\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right) + \frac{1}{6}\left(\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}\right)\right\}$$

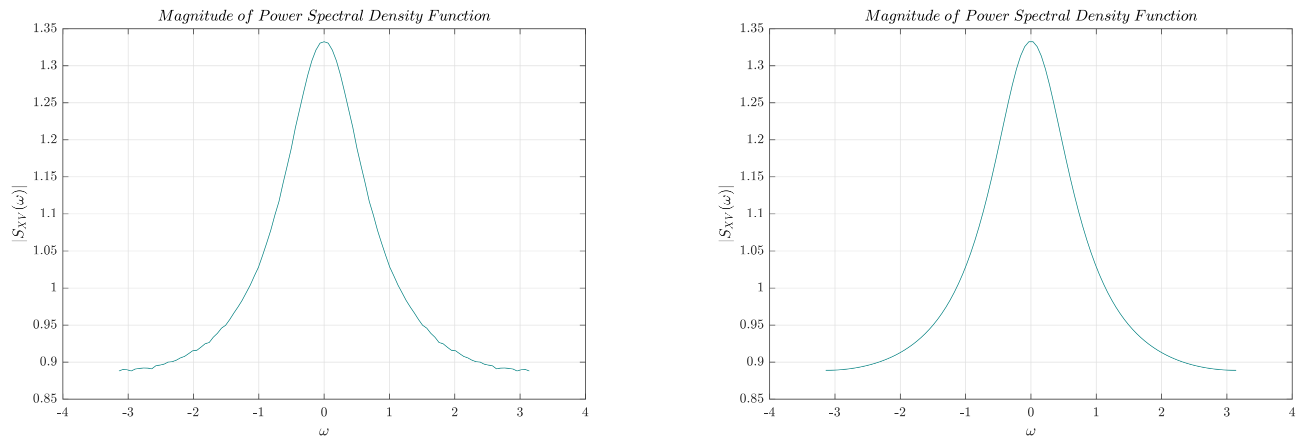
$$= \underline{\delta[m] + \frac{10}{27}2^m u[-m-1] + \frac{1}{27}(\frac{1}{2})^m u[m] + \frac{1}{6}(\frac{1}{2})^{n-1}u[m-1]}$$

The simulations and theoretical calculations presented below demonstrate close agreement, thus verifying the accuracy of the theoretical formulations. The simulation data points closely



(a) MATLAB simulation (b) Theoretical

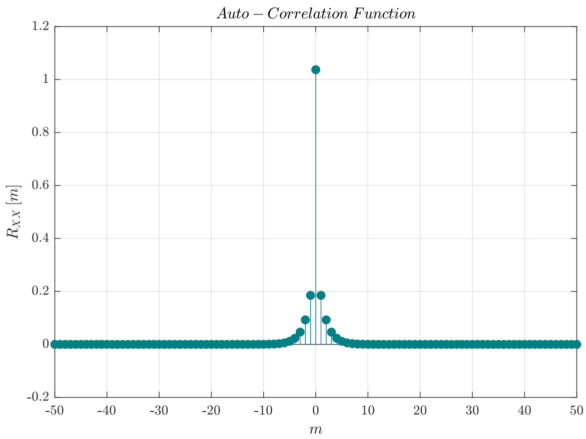
Figure 9: $|S_X(\omega)|$



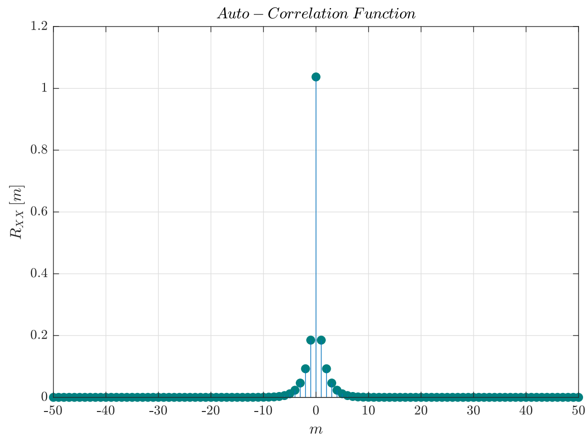
(a) MATLAB simulation (b) Theoretical

Figure 10: $|S_{XV}(\omega)|$

match the curves predicted by the equations, confirming that the calculations are correct. This consistency between simulation and theory validates the modeling approach.

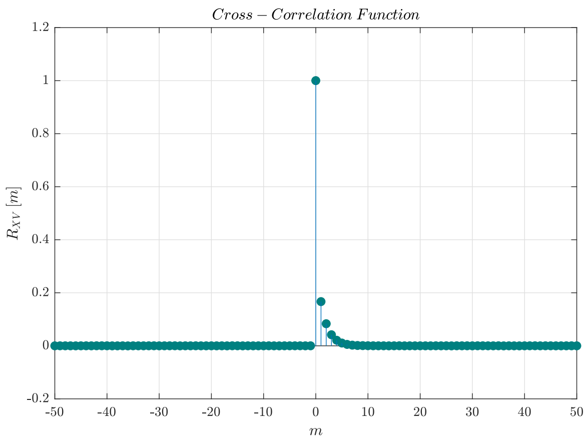


(a) MATLAB simulation

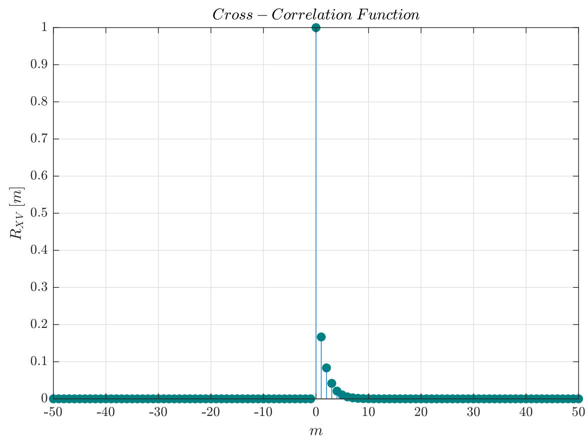


(b) Theoretical

Figure 11: $R_X[m]$



(a) MATLAB simulation



(b) Theoretical

Figure 12: $R_{XV}[m]$

Part D.

As demonstrated in the previous section:

$$H(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

The inverse system may have a non-unique impulse response unless we impose additional constraints, such as causality and stability, which specify the region of convergence (ROC). If we constrain the inverse system to be causal by choosing an ROC of $|z| > \frac{1}{2}$, it will have a unique stable impulse response. This is because the original system has a pole at $z = \frac{1}{2}$ inside the unit circle. According to the stability theorem, a linear time-invariant (LTI) system and its inverse are both causal and stable if and only if all poles and zeros of the original system lie inside the unit circle. Since the original system has its zero at $z = \frac{1}{3}$ inside the unit circle, the inverse system satisfying causality and stability will also be stable when the above ROC is selected.