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EE181: Stochastic Processes

Computer Assignment #2

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Table of Contents

1 2-D Random Walk

This question simulates a random walk in the x-y plane where an object moves in random directions. The length of each step, denoted r, is a random variable drawn from a uniform distribution over the interval $[0, 1]$. Additionally, the direction θ of each step follows a uniform distribution over $[0, 2\pi]$ radians.

Part A. Mean and Variance

To theoretically determine the mean and variance of the object's displacement, we model its motion in the complex plane. Specifically, we represent each step as a complex number with a random magnitude r and a random phase angle θ corresponding to the step direction. After N steps, the object's displacement Z is the summation of N such complex exponentials, each with random amplitude r_i and phase θ_i.

$$Z = \sum_{i=1}^{N} r_i e^{j\theta_i} \rightarrow$$

$$\mathbb{E}\left[Z\right] = \mathbb{E}\left[\sum_{i=1}^{N} r_i e^{j\theta_i}\right] = \sum_{i=1}^{N} \mathbb{E}\left[r_i\right]\mathbb{E}\left[e^{j\theta_i}\right] = \sum_{i=1}^{N} \frac{1}{2}\left(\mathbb{E}\left[\cos\theta_i\right] + j\mathbb{E}\left[\sin\theta_i\right]\right) = \boxed{0}$$

$$\mathbb{E}\left[|Z|^2\right] = \mathbb{E}\left[ZZ^*\right] = \mathbb{E}\left[\sum_{i=1}^{N}\sum_{k=1}^{N} r_i r_k e^{j(\theta_i-\theta_k)}\right] = \sum_{i=1}^{N}\sum_{k=1}^{N} \mathbb{E}\left[r_i r_k\right]\mathbb{E}\left[e^{j(\theta_i-\theta_k)}\right] \Rightarrow$$

$$\mathbb{E}\left[e^{j(\theta_i-\theta_k)}\right] = \begin{cases} \mathbb{E}\left[e^{j\theta_i}\right]\mathbb{E}\left[e^{-j\theta_k}\right] = 0 & i \neq k \\ 1 & i = k \end{cases} \Rightarrow \mathbb{E}\left[|Z|^2\right] = \sum_{i=1}^{N} \mathbb{E}\left[r_i^2\right] = N\int_0^1 r^2 dr = \boxed{\frac{N}{3}}$$

$$\Longrightarrow \mathbb{E}\left[Z\right] = 0,\ Var(Z) = \frac{N}{3}$$

To validate the theoretical predictions, simulations of the random walk are conducted for $N = 100, 1000,$ and 10000 steps. The mean and variance of the object's displacement is computed over many trials for each N. The figures show the simulated mean and variance of displacement for varying N. We observe close matching between the theory and simulation averages, confirming the accuracy of the derived equations.

```
N = 100.0, Varinace of movement: 33.3212, Mean value of movement: -0.0024 + j(-0.0069)
```

Figure 1: $N = 100$

```
N = 1000.0, Varinace of movement: 333.5034, Mean value of movement: 0.0141 + j(-0.0071)
```

Figure 2: $N = 1000$

```
N = 10000.0, Varinace of movement: 3326.3555, Mean value of movement: 0.0518 + j(-0.0077)
```

Figure 3: $N = 10000$

Part B. Disk!

We can model the end position of the random walk using the Central Limit Theorem (CLT). After a large number of steps N, the distributions of the total displacements X and Y along the x and y directions can be approximated as independent Gaussian random variables with mean *0* and variance $N\sigma^2$, where σ^2 is the variance of displacement per step. Therefore:

$$X = \sum_{i=1}^{N} r_i \cos\theta_i, \ Y = \sum_{i=1}^{N} r_i \sin\theta_i \Rightarrow$$

$$\begin{aligned}
\mathbb{E}\left[X\right] &= \mathbb{E}\left[\sum_{i=1}^{N} r_i \cos\theta_i\right] = \sum_{i=1}^{N} \mathbb{E}\left[r_i\right]\mathbb{E}\left[\cos\theta_i\right] = 0 \\
\mathbb{E}\left[X^2\right] &= \mathbb{E}\left[\sum_{i=1}^{N}\sum_{k=1}^{N} r_i r_k \cos\theta_i \cos\theta_k\right] = \sum_{i=1}^{N} \mathbb{E}\left[r_i^2\right]\mathbb{E}\left[\cos^2\theta_i\right] = \frac{N}{3}\int_0^{2\pi} \frac{1}{2\pi}\cos^2\theta \, d\theta = \frac{N}{6} \\
&\Longrightarrow X \sim \mathcal{N}(0, N/6), \ Y \sim \mathcal{N}(0, N/6)
\end{aligned}$$

$$\begin{aligned}
Z = \sqrt{X^2+Y^2} \Rightarrow F_Z(z) &= \mathbb{P}(Z < z) = \mathbb{P}(\sqrt{X^2+Y^2} < z) = \mathbb{P}(X^2+Y^2 < z^2) \\
&= \int_0^{2\pi}\int_0^{z} \frac{1}{2\pi\frac{N}{6}} r e^{-\frac{r^2}{2\times\frac{N}{6}}} dr d\theta = 1 - e^{-\frac{3z^2}{N}}
\end{aligned}$$

$$\Longrightarrow \mathbb{P}(8 < z < 12) = F_Z(12) - F_Z(8) = (1 - e^{-\frac{3\times 12^2}{N}}) - (1 - e^{-\frac{3\times 8^2}{N}}) = \begin{cases} 0.1333 & ; N = 100 \\ 0.1761 & ; N = 1000 \\ 0.0233 & ; N = 10000 \end{cases}$$

The simulation results further validate the Gaussian theoretical model. The first figure shows the mean and variance of the x and y displacements numerically computed over 1,000,000 simulation runs for different step counts N. The close match of these simulated moments to the analytical predictions of mean and variance highlights the accuracy of our Gaussian approximation even for finite N.

```
N = 100.0, Probability of end point falling inside the disk: 0.1334
```

Figure 4: $N = 100$

```
N = 1000.0, Probability of end point falling inside the disk: 0.1754
```

Figure 5: $N = 1000$

```
N = 10000.0, Probability of end point falling inside the disk: 0.0236
```

Figure 6: $N = 10000$

Additionally, the following figure provides a histogram of simulated x-positions after 1000 steps compared to a Gaussian distribution with the theoretically expected variance of 1000/3. The strong overlap demonstrates empirically the convergence of the total displacement distribution to a Gaussian based on the Central Limit Theorem.

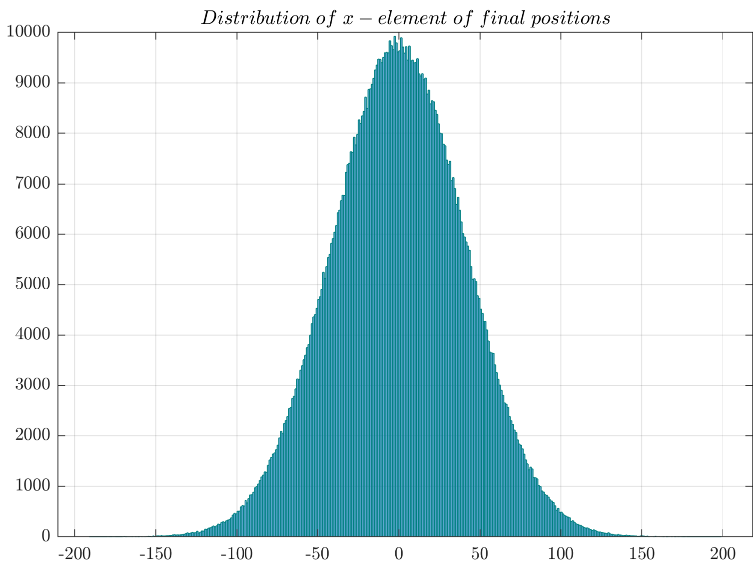

Figure 7: Actual Distribution

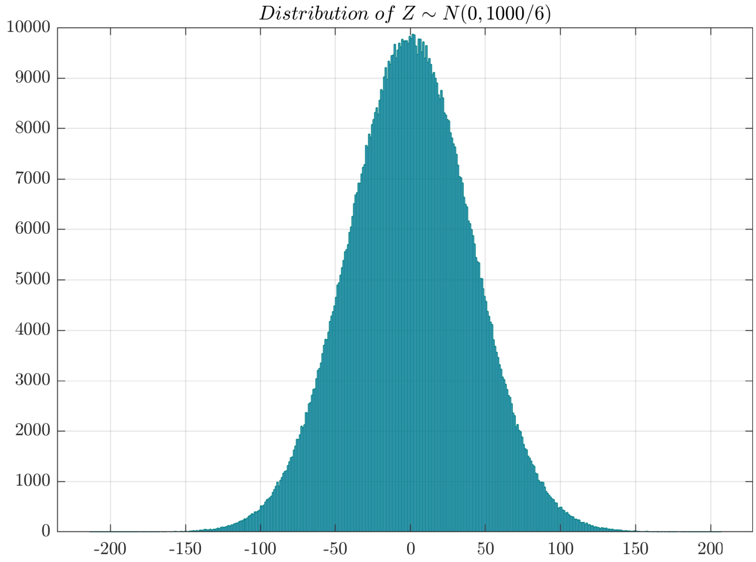

Figure 8: Approximated Distribution

Part C. Final Points Distribution

The simulated distributions visually confirm the theoretical dependence on number of steps N. The figures show the end positions from 2D random walks for various N values. As N increases, the cloud of points spreads outwards, reflecting the larger variance of the total displacement predicted by our Gaussian model. At the same time, the distributions remain centered on the origin, with an empirical mean of zero regardless of N. This aligns with the analytical solution of a zero-mean Gaussian for any N.

Together, the visualization of expanding variance and invariant zero-mean demonstrate how our Gaussian Central Limit Theorem solution accurately captures key distribution attributes. The simulated random walk patterns match the quantified theoretical expectations for how the end positions spread out from the origin but do not drift on average. This provides an intuitive visual validation to complement the mathematical derivations and histogram agreements.

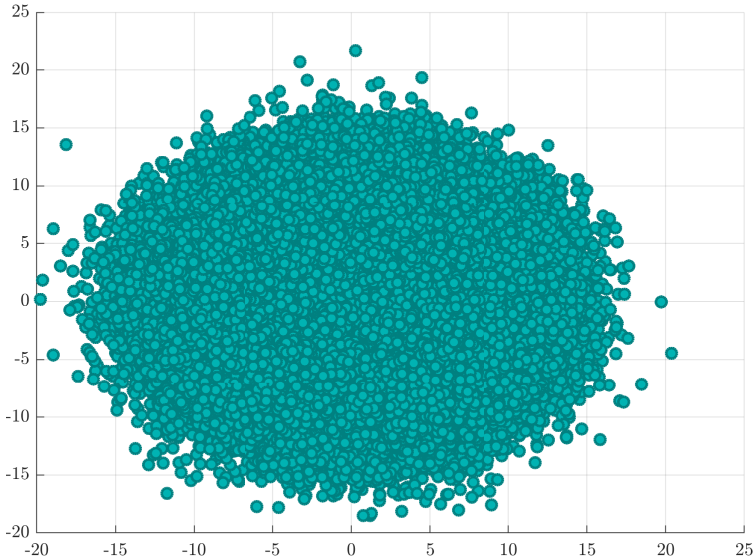

Figure 9: $N = 100$

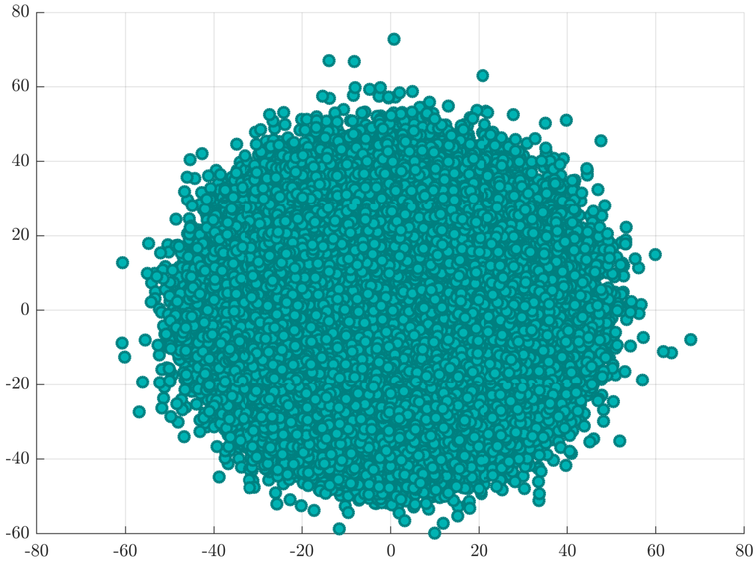

Figure 10: $N = 1000$

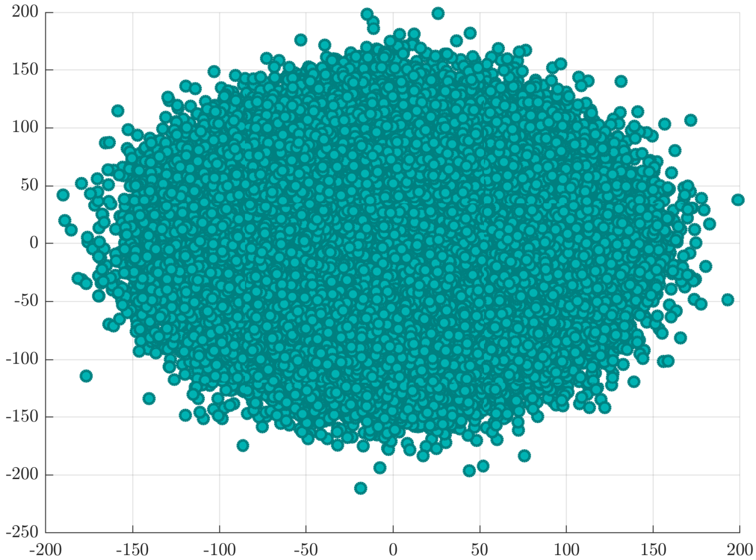

Figure 11: $N = 10000$

2 Observation of Celestial Bodies

Part A. λ Estimation

In this part, we are given a dataset containing the output voltage of a photon detector. Plotting this output shows sharp spikes fluctuating between positive and negative saturation values, indicating photon arrival events albeit with some noise. The detector output is nominally ± 1, changing value when a photon is received. However, the voltage values in the dataset are noisy and not precisely ± 1. We will assume the photon arrival times follow a Poisson process. We also assume this is a semi-random process, meaning we know $X(0^+) = +1$ initially. Our task is to determine the λ parameter for this Poisson process.

To estimate λ, we leverage the fact that the distance between events in a Poisson process follows an exponential distribution with rate parameter λ. We can thus find the pulse lengths in the data and take their mean. The expected value of an exponential distribution equals $1/\lambda$. Therefore, λ for our Poisson process equals the inverse of the mean pulse length. The following figure shows the estimated value of the λ parameter.

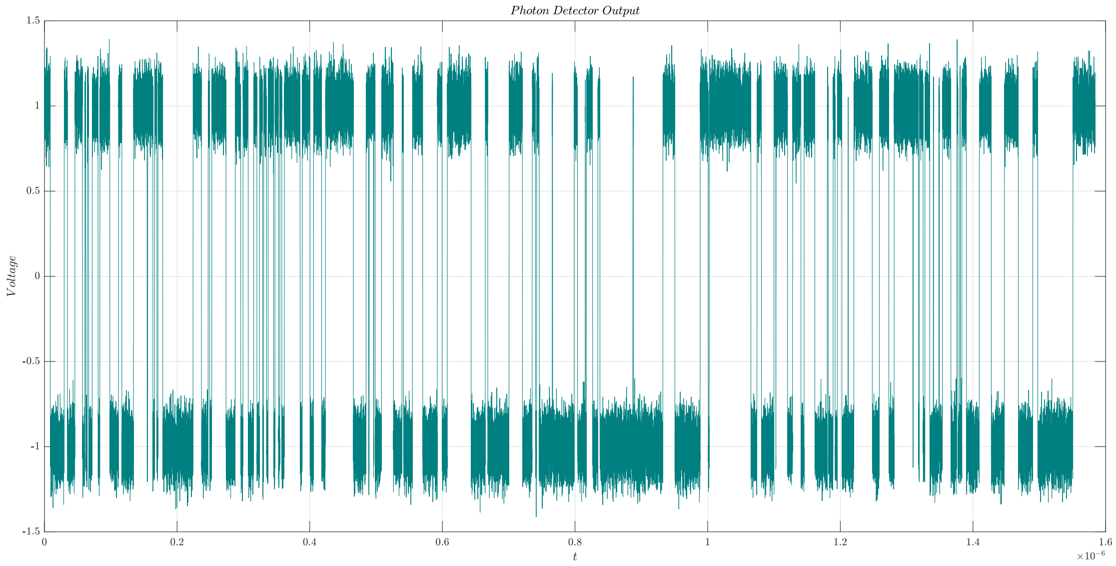

Figure 12: Detector's Output

```
The estimated value of λ : 9.30171e+07
```

Figure 13: Estimated value of λ

Part B. Theoretical Approach

$$I(t) = \frac{\int_{0^+}^{t} X(\tau)\, d\tau}{t} \Rightarrow \mathbb{E}\left[I(t)\right] = \frac{\int_{0^+}^{t} \mathbb{E}\left[X(\tau)\right]\, d\tau}{t}$$

$$\begin{aligned}
\Longrightarrow \mathbb{E}\left[X(\tau)\right] &= (+1) \times \mathbb{P}(X(\tau) = +1) + (-1) \times \mathbb{P}(X(\tau) = -1) \\
&= \mathbb{P}(\text{even number of Poisson points}) - \mathbb{P}(\text{odd number of Poisson points}) \\
&= \sum_{k=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^{2k}}{(2k)!} - \sum_{k=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^{2k+1}}{(2k+1)!} \\
&= e^{-\lambda\tau}\left(1 + \frac{(\lambda\tau)^2}{2!} + \frac{(\lambda\tau)^4}{4!} + \cdots\right) - e^{-\lambda\tau}\left(\lambda\tau + \frac{(\lambda\tau)^3}{3!} + \frac{(\lambda\tau)^5}{5!} + \cdots\right) \\
&= e^{-\lambda\tau}\left(1 - \lambda\tau + \frac{(\lambda\tau)^2}{2!} - \frac{(\lambda\tau)^3}{3!} + \frac{(\lambda\tau)^4}{4!} - \cdots\right) \\
&= e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda\tau)^n}{n!} = e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(-\lambda\tau)^n}{n!} \\
&= e^{-\lambda\tau} \times e^{-\lambda\tau} = e^{-2\lambda\tau}
\end{aligned}$$

$$\Longrightarrow \mathbb{E}\left[I(t)\right] = \frac{\int_{0^+}^{t} e^{-2\lambda\tau}\, d\tau}{t} = \boxed{\frac{1 - e^{-2\lambda t}}{2\lambda t}} \tag{1}$$

Part C. Another Estimation of λ

In this part, we leverage the process $I(t)$ defined in Part B, along with its derived expected value, to determine the λ parameter. Specifically, we will estimate λ based on the expected value $\mathbb{E}[I(t = 16.67ns)]$.

First, we must estimate $\mathbb{E}[I(t = 16.67ns)]$ from the data. To do this, we utilize the memoryless property of the Poisson process. We separate the time axis into $16.67ns$ slices and calculate the value of $I(t)$ within each slice. The overall mean of the $I(t)$ values across slices provides our data-based estimate of $\mathbb{E}[I(t = 16.67ns)]$. Note that because we have a semi-random process with $X(0^+) = +1$, we align the start point of each slice to begin with a positive voltage.

After numerically estimating $\mathbb{E}[I(t = 16.67ns)]$, we can substitute this, along with the known time value, into the derived equation 1 to solve for λ.

The following figure shows the results of this simulation-based estimation process. The λ value obtained is similar to that from the previous approach, with small differences likely due to noise in data, discrepancies between the theoretical and empirical mean values, etc.

```
The estimated value of λ : 1.00917e+08
```

Figure 14: Estimated value of λ

3 Brownian Bridge

Part A. Step size

The Brownian Bridge process defines a stochastic process bridging intervals per the equation:

$$X(t) = b + (1 - \frac{t}{T}) \left(a - b + \int_0^t \frac{\sigma}{1 - \frac{t'}{T}} dW(t') \right)$$

Deriving this process over time determines the step size update function:

$$\frac{dX(t)}{dt} = -\frac{1}{T}\left(a - b + \int_0^t \frac{\sigma}{1-\frac{t'}{T}} dW(t')\right) + (1-\frac{t}{T})\left(\frac{d}{dt}\int_0^t \frac{\sigma}{1-\frac{t'}{T}} dW(t')\right)$$
$$= \frac{b - X(t)}{T-t} + (1-\frac{t}{T})\left(\underbrace{\frac{d}{dt}\int_0^t \frac{\sigma}{1-\frac{t'}{T}} dW(t')}_{I_1}\right)$$

I_1 represents the derivative of a stochastic integral. Thus, we utilize Itô's calculus, which states:

$$\text{If: } I(t) = \int_0^t H(s)dW(s)$$
$$\rightarrow dI(t) = H(t)dW(t)$$

$$\Rightarrow I_1 = \frac{\sigma}{1-\frac{t}{T}} dW(t) \Longrightarrow \frac{dX(t)}{dt} = \frac{b-X(t)}{T-t} + (1-\frac{t}{T})\left(\frac{\sigma}{1-\frac{t}{T}} dW(t)\right)$$
$$= \frac{b-X(t)}{T-t} + \sigma\frac{dW(t)}{dt}$$
$$\Longrightarrow dX(t) = \frac{b-X(t)}{T-t}dt + \sigma dW(t)$$

The Wiener property gives:

$$dW(t) = W(t+dt) - W(t) \Rightarrow dW(t) \sim \mathcal{N}(0, dt) \Rightarrow dW(t) \sim \sqrt{dt}\mathcal{N}(0,1)$$

Therefore:

$$X(t+dt) = X(t) + dX(t) = X(t) + \frac{b-X(t)}{T-t}dt + \sigma\sqrt{dt}\mathcal{N}(0,1)$$

Having derived the formula for the step size, we can now generate Brownian Bridge trajectories between designated start and end points. Specifically, 10,000 sample paths were simulated using the update equation. The following figure visualizes these stochastically generated bridges. Qualitatively, the random trajectories exhibit the expected behaviorthey diffuse from the starting point but are pulled towards the specified ending location. This characteristic shrinking dispersion mirrors our analytical understanding of the Brownian Bridge process derived earlier.

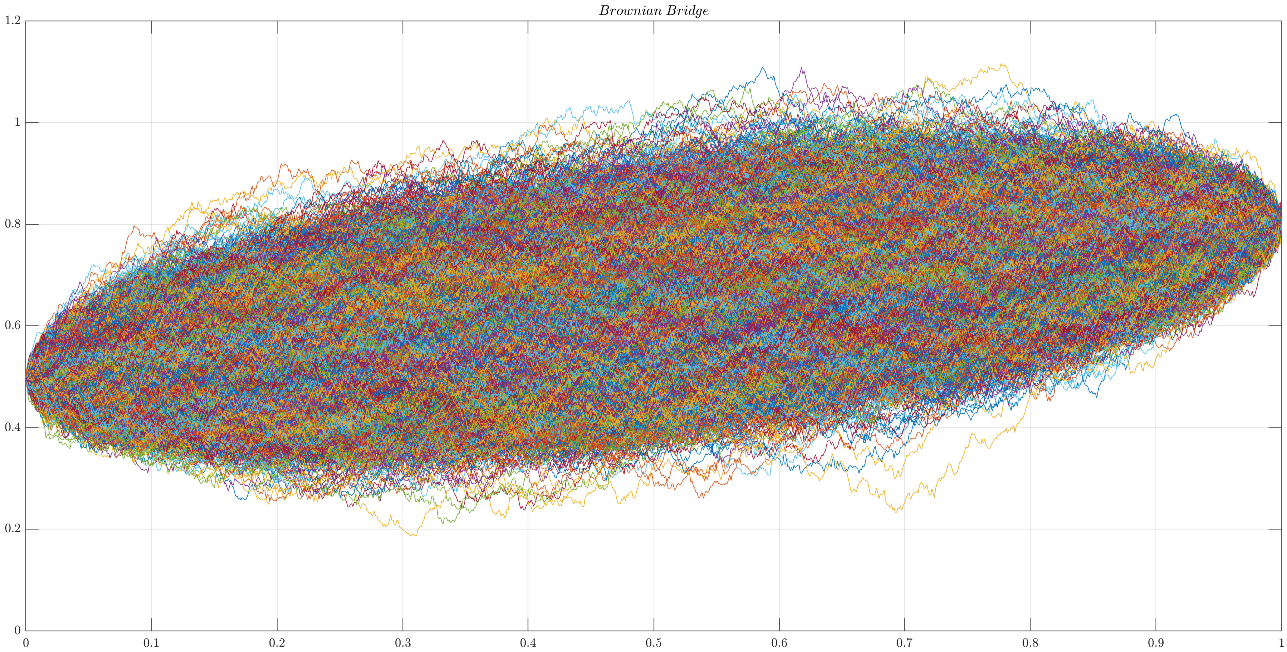

Figure 15: Brownian Bridges

Part B. Mean and Variance of Brownian Bridge

In this part, we will derive key statistics - specifically the mean and variance - for the Brownian Bridge process defined earlier. To begin our derivation, we leverage a fundamental property of stochastic integrals:

$$\int_0^T G(t')dW(t') = \sum_{k=0}^{m} G\left(\frac{t_{k+1}+t_k}{2}\right)(W(t_{k+1}) - W(t_{k+1})) \tag{2}$$

$$\begin{aligned}
\mathbb{E}\left[X(t)\right] &= b + (1-\frac{t}{T})\left(a - b + \mathbb{E}\left[\int_0^t \frac{\sigma}{1-\frac{t'}{T}} dW(t')\right]\right); G(t) = \frac{\sigma}{1-\frac{t}{T}} u(t-T) \\
&= b + (1-\frac{t}{T})\left(a - b + \mathbb{E}\left[\sum_{k=0}^{m} G\left(\frac{t_{k+1}+t_k}{2}\right)(W(t_{k+1}) - W(t_{k+1}))\right]\right) \\
&= b + (1-\frac{t}{T})\left(a - b + \sum_{k=0}^{m} G\left(\frac{t_{k+1}+t_k}{2}\right)(\mathbb{E}[W(t_{k+1})] - \mathbb{E}[W(t_{k+1})]\right) \\
&= b + (1-\frac{t}{T})(a-b) = \boxed{a + (b-a)\frac{t}{T}}
\end{aligned}$$

Note that we have leveraged the property of the Wiener process, which exhibits a zero mean across all time points, meaning that:

$$\mathbb{E}[W(t_{k+1})] = \mathbb{E}[W(t_k)] = 0$$

$$Var(X(t)) = (1-\frac{t}{T})^2 Var\left(\int_0^t \frac{\sigma}{1-\frac{t'}{T}} dW(t')\right) \Rightarrow Var\left(\int_0^t \frac{\sigma}{1-\frac{t'}{T}} dW(t')\right) = \mathbb{E}\left[(\int_0^t \frac{\sigma}{1-\frac{t'}{T}} dW(t'))^2\right]$$
$$= \mathbb{E}\left[\sum_{k=0}^{m}\sum_{n=0}^{m} G\left(\frac{t_{k+1}+t_k}{2}\right) G\left(\frac{t_{n+1}+t_n}{2}\right)(W(t_{k+1})-W(t_k))(W(t_{n+1})-W(t_n))\right]$$

For Wiener processes: $W(t_i) \perp W(t_j);\ i \neq j$

$$\Longrightarrow Var(X(t)) = (1-\frac{t}{T})^2 \mathbb{E}\left[\sum_{k=0}^{m} G^2\left(\frac{t_{k+1}+t_k}{2}\right)(W(t_{k+1})-W(t_k))^2\right]$$

For Wiener processes: $W(t_{k+1}) - W(t_k) \sim \mathcal{N}(0, t_{k+1}-t_k)$

$$\Longrightarrow Var(X(t)) = (1-\frac{t}{T})^2 \left[\sum_{k=0}^{m} G^2\left(\frac{t_{k+1}+t_k}{2}\right)(t_{k+1}-t_k)\right]$$
$$= (1-\frac{t}{T})^2 \int_0^t G^2(t')dt' = (1-\frac{t}{T})^2 \int_0^t \left(\frac{\sigma}{1-\frac{t'}{T}}\right)^2 dt'$$
$$= (1-\frac{t}{T})^2 \times \sigma^2 \times \left(\frac{T}{1-\frac{t}{T}} - T\right) = \boxed{\sigma^2 \frac{t}{T}(T-t)}$$

Having theoretically derived the mean and variance of the Brownian Bridge, we proceeded to compute these metrics through simulation. The subsequent figures depict the outcomes. Notably, the simulation aligns closely with the theoretical approach, affirming the accuracy of our calculations.

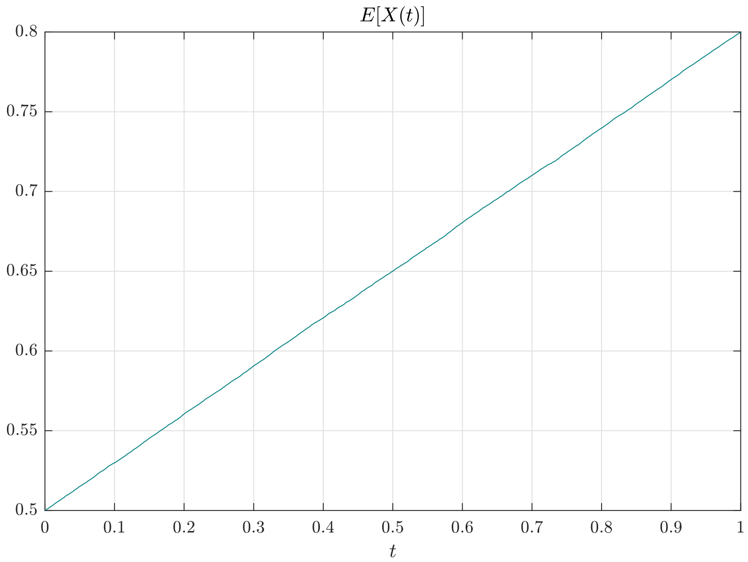

Figure 16: Mean

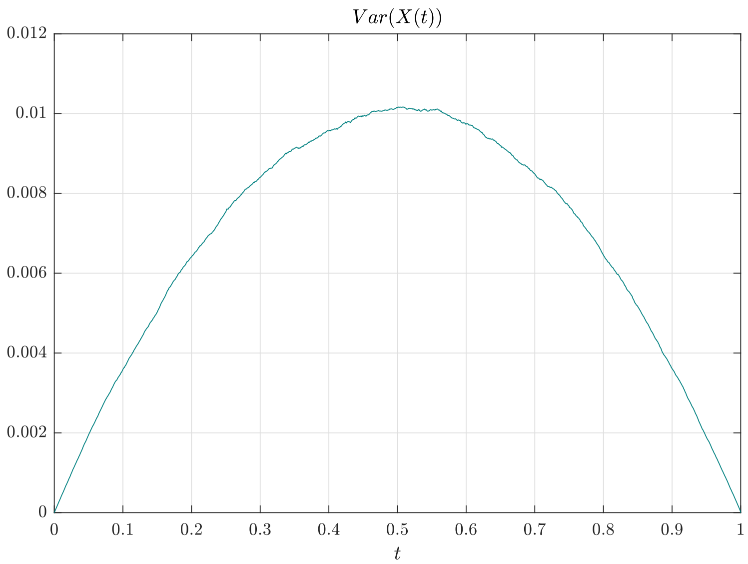

Figure 17: Variance