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Acknowledgements

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Abstract

This thesis is an attemp to deal with multiobjective optimization problems (MOPs) with two types of special cone ordering, called natural ordering and lexicographical ordering. For natural ordering, another equivalent definition of proper efficiency is presented, and some new properties of properness are developed. A new quality index, named homogeneity, is introduced for proper efficient points.

With the aid of the new definition of properness, a transformation technique is proved to transform a multiobjective problem to a more convenient one. Some conditions are determined under which the original and the transformed problems have the same Pareto and properly efficient solutions. This transformation has some benefits. Firstly, the weighted version of the transformed problem is expected to generate more homogenous properly efficient solutions. Secondly, this transformation could be employed for the sake of convexification and simplification in order to improve the computational efficiency for solving the given problem. Finally, some existing results in the multiobjective optimization literature are generalized and strengthened using the special case of the proposed transformation scheme.

For lexicographical ordering, a transformation of a nonlinear lexicographic MOP (LMOP) into an equivalent single objective program is investigated. By some counter examples, it is illustrated that this transformation is not possible in general, and some conditions are determined under which the transformation would work. This transformation of linear LMOPs is employed to propose a new simplex based method to efficiently solve linear LMOPs. Moreover, if the priority of the objective functions is changed, it is shown that the optimal solution can be built by employing the preceding computations, without the need for solving a new problem.

Introduction

Life inevitably involves decision making, variety of choices and searching for compromises. The difficulty here lies in the (at least partial) conflict between our various objectives and goals. Most everyday decisions and compromises are made on the basis of intuition, common sense, chance or a combination of these. However, there are areas where mathematical modeling and programming are needed, such as engineering and economics. Here, the problems to be solved vary from designing spacecraft, bridges, robots or camera lenses to blending sausages, planning and pricing production systems or managing pollution problems in environmental control. The area of multiobjective programming has developed rapidly, as demonstrated by Steuer et al. (1996) [65]. Most of the theoretical and practical issues concerning multiobjective optimization are comprehensively treated in five books; Sawaragi et al. (1985) [61], Miettinen (1999) [51], Ehrgott (2005) [19], Chinchuluun et al. (2008) [15], Zopounidis and Pardalos (2010) [81]. This thesis is about multiobjective programming is organized in three chapters.

In chapter 1, some basic definitions and preliminaries of multiobjective programming are presented for general ordering cones. Since we are concerned with natural ordering and lexicographical ordering, special attention is paid to these two important cases of cone orderings.

When dealing with efficient solutions, it may be that only a subset of them should be considered, some being eliminated as being improper in some sense, in order to avoid some undesirable situations. The main drawback of improper efficient points is that they cannot be satisfactorily characterized by a scalar optimization problem, even if the decision set is convex. There are various definitions of properness in multiobjective literature (a comprehensive survey could be found in [30]). In chapter 2 of this thesis, a new equivalent definition of properness is presented based on the trade-off notion. The trade-off ratio has been defined in the literature as the amount of increment in one objective due to a decrement of one unit in another objective in order to establish the search direction for a satisfactory solution in interactive approaches. This concept is generalized in this chapter, though, our aim is not finding the search direction for a satisfactory solution. We aim to build an easier definition of proper efficiency. Within the new definition, properly efficient points are efficient points with bounded trade-offs. Moreover, by taking advantage of the new definition, some new properties of properness are proved which are useful in establishing the transformation technique for MOPs.

In radiotherapy optimization, there is a quality index for treatment plans that is called homogeneity. In the multiobjective framework, it means decreasing all objective functions at the same time. This concept is brought into multiobjective programming in chapter 2. In fact, between two properly efficient solutions, the more homogeneous one is preferred because it has better trade-offs among its objectives. It is shown that if the objective function imposes more penalty for the larger values than the smaller values, then generating more homogenous properly efficient solutions is expected by the weighted scalarization method.

The idea of transformation of MOPs has been introduced by Romeijn et al. (2004) [60] to transform some MOPs that have arisen in radiotherapy optimization for the sake of convexification and simplification and creating a unified framework for those problems. In chapter 2, we will point out two major theoretical and practical drawbacks of their transformation, and then the way in which these difficulties can be overcome will be discussed. In fact, under some conditions, a transformation technique is proved to transform a multiobjective problem to the more convenient one with the same Pareto and properly efficient solutions. This transformation has some benefits. Firstly, the weighted version of the transformed problem is expected to generate more homogenous properly efficient solutions. Secondly, the transformation could be applied to convexification and simplification in order to improve the computational efficiency of solving the given problem. Finally, some existing results in multiobjective optimization literature are generalized and strengthened by employing the special case of the proposed transformation scheme.

Chapter 3 is about lexicographic multiobjective optimization problems (LMOPs) which are MOPs for which the image space is equipped with lexicographical ordering. Lexicographic programs arise naturally when conflicting objectives exist in a decision problem, but for reasons outside the control of the decision maker, the objectives have to be considered in a hierarchical manner. There are two approaches to solve LMOPs, namely the sequential method and the transformation approach. Here, we address the latter. This approach has been introduced by Sherali and Soyster (1982)[62] to transform linear LMOPs and LMOPs with finite feasible solutions to the equivalent single objective problems. This technique is investigated for non-linear LMOPs in this chapter. First, by some counter examples, it is illustrated that this transformation is not possible in general, and then some conditions are determined under which the transformation works. Moreover, this transformation for linear LMOPs is employed to propose a new simplex based method to efficiently solve linear LMOPs. Furthermore, if the new algorithm is applied to solving a linear LMOP and the priority of the objective functions is changed, it is shown that the optimal solution can be built by employing the preceding computations, without the need for solving a new problem.

Chapter 1

Theoretical Basics of Multiple Objective Optimization Problems

1.1 Basic Concepts

Our aim is a minimization of a vector-valued objective function $f : \mathbb{R}^n \to \mathbb{R}^r$ over a nonempty feasible solution $\mathcal{X} \subset \mathbb{R}^n$. We consider optimization problems defined by

$$MOP \qquad \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_r(\mathbf{x})). \tag{1.1.1}$$

The image of $\mathcal{X}$ under f is denoted by $\mathcal{Y} = f(\mathcal{X})$ and referred to as the image of the feasible set, or image space. Sometimes, it is useful to consider the MOP problem in the image space:

$$\min_{\mathbf{y} \in \mathcal{Y}} \mathbf{y} = (y_1, y_2, ..., y_r). \tag{1.1.2}$$

For $r = 1$, the problem (MOP) reduces to a standard nonlinear optimization problem with a scalar-valued objective function. However, in multiobjective optimization we are interested in several competing objective functions and thus we assume $r \geq 2$. We will discuss only minimization problems. However, each maximization problem can be transformed to a minimization problem very easily by considering the negative objective function values.

We are looking for the minimal values of the function f over the set $\mathcal{X}$. In general, there is not one solution satisfying all the objectives at best at the same time. We assume that the decision maker specifies which alternative $\mathbf{x}$ he prefers to another point $\mathbf{x}'$ by declaring if he prefers $f(\mathbf{x})$ to $f(\mathbf{x}')$ or not. With these statements, a binary relation is defined in the image space $\mathbb{R}^r$ (and in the parameter space $\mathbb{R}^n$, respectively), also called preference order. Different types of preference orders are possible, but in practical applications a very common concept is partial ordering. The following definition recalls the concept of a binary relation and partial ordering.

Definition 1.1.1. (a) A nonempty subset $R \subset \mathbb{R}^r \times \mathbb{R}^r$ is called a binary relation on $\mathbb{R}^r$. We write $\mathbf{x} \preccurlyeq \mathbf{y}$ (or $\mathbf{y} \succcurlyeq \mathbf{x}$) for $(\mathbf{x}, \mathbf{y}) \in R$.

(b) A binary relation $\preccurlyeq$ on $\mathbb{R}^r$ is called a partial ordering if for arbitrary $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^r$:

(i) $\mathbf{x} \preccurlyeq \mathbf{x}$ (reflexivity),

(ii) $\mathbf{x} \preccurlyeq \mathbf{y}$, $\mathbf{y} \preccurlyeq \mathbf{z} \Rightarrow \mathbf{x} \preccurlyeq \mathbf{z}$ (transitivity),

(iii) $\mathbf{x} \preccurlyeq \mathbf{y}$, $\mathbf{w} \preccurlyeq \mathbf{z} \Rightarrow \mathbf{x} + \mathbf{w} \preccurlyeq \mathbf{y} + \mathbf{z}$ (compatibility with the addition),

(iv) $\mathbf{x} \preccurlyeq \mathbf{y}$, $\alpha \in \mathbb{R}^+ \Rightarrow \alpha\mathbf{x} \preccurlyeq \alpha\mathbf{y}$ (compatibility with the scalar multiplication).

(c) A partial ordering on $\mathbb{R}^r$ is called antisymmetric if for arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{R}^r$

$$\mathbf{x} \preccurlyeq \mathbf{y},\ \mathbf{y} \preccurlyeq \mathbf{x} \Rightarrow \mathbf{x} = \mathbf{y}.$$

(d) A partial ordering on $\mathbb{R}^r$ is called total if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^r$, either $\mathbf{x} \preccurlyeq \mathbf{y}$ or $\mathbf{y} \preccurlyeq \mathbf{x}$.

Partial orderings can be characterized by convex cones. Any partial ordering $\preccurlyeq$ in $\mathbb{R}^r$ defines a convex cone by

$$\mathcal{K} := \{\mathbf{x} \in \mathbb{R}^r |\ 0 \preccurlyeq \mathbf{x}\},$$

and any convex cone, hence also called ordering cone, defines a partial ordering on $\mathbb{R}^r$ by

$$\preccurlyeq_{\mathcal{K}} := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^r \times \mathbb{R}^r |\ \mathbf{y} - \mathbf{x} \in \mathcal{K}\}.$$

The following theorem shows the relationship between a partial ordering and its related convex cone.

Theorem 1.1.1. *Let $\mathcal{K}$ be a convex cone and let $\preccurlyeq_{\mathcal{K}}$ be its related partial ordering. Then, the following statements hold.*
1. $\preccurlyeq_{\mathcal{K}}$ is antisymmetric if and only if $\mathcal{K}$ is pointed, i.e., $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.
2. $\preccurlyeq_{\mathcal{K}}$ is total if and only if $\mathcal{K} \cup (-\mathcal{K}) = \mathbb{R}^r$.

With the help of orderings introduced by ordering cones $\mathcal{K}$ in $\mathbb{R}^r$ we can define minimal elements of sets in $\mathbb{R}^r$.

Definition 1.1.2. Let T be a nonempty subset of the linear space $\mathbb{R}^r$ partially ordered by the convex cone $\mathcal{K}$. A point $\bar{\mathbf{y}} \in T$ is a $\mathcal{K}$-minimal point of the set T if

$$(\bar{\mathbf{y}} - \mathcal{K}) \cap T \subset \bar{\mathbf{y}} + \mathcal{K}. \tag{1.1.3}$$

If the cone $\mathcal{K}$ is pointed, then (1.1.3) is equivalent to

$$(\bar{\mathbf{y}} - \mathcal{K}) \cap T = \bar{\mathbf{y}}.$$

If for $\mathbf{y}, \tilde{\mathbf{y}} \in T$ we have $\mathbf{y} - \tilde{\mathbf{y}} \in \mathcal{K} \backslash \{0\}$, then we say $\tilde{\mathbf{y}}$ dominates $\mathbf{y}$. Figure 1.1 illustrates the concepts of $\mathcal{K}$-minimality and domination.

Using this concept, we can define minimality for the multiobjective optimization problem.

Definition 1.1.3. A point $\bar{\mathbf{x}} \in \mathcal{X}$ is a minimal point (or efficient or $\mathcal{K}$-minimal) of MOP w.r.t. the ordering cone $\mathcal{K}$ if $f(\bar{\mathbf{x}})$ is a $\mathcal{K}$-minimal point of the set $f(\mathcal{X})$. The set of all minimal points w.r.t. the cone $\mathcal{K}$ is denoted by $\mathcal{X}_E(\mathcal{K})$.
When $\bar{\mathbf{x}} \in \mathcal{X}_E(\mathcal{K})$, a point $\bar{\mathbf{y}} = f(\bar{\mathbf{x}})$ is called non-dominated w.r.t. cone $\mathcal{K}$. The set of all non-dominated points w.r.t. the cone $\mathcal{K}$ is denoted by $\mathcal{Y}_E(\mathcal{K})$.

Figure 1.1: $\mathcal{K}$-minimal point $\bar{\mathbf{y}}$. A point $\mathbf{y}$ dominated by the point $\tilde{\mathbf{y}}$.

There are weaker and stronger minimality notions in vector optimization. The following definition provides the weaker minimality concept and Definition 1.1.6 introduces the stronger one.

Definition 1.1.4. Let $\mathcal{K}$ be a pointed ordering cone with $int(\mathcal{K}) \neq \emptyset$. A point $\bar{\mathbf{x}} \in \mathcal{X}$ is a weakly minimal/efficient solution of the MOP w.r.t. $\mathcal{K}$ if

$$(f(\bar{\mathbf{x}}) - int(\mathcal{K})) \cap f(\mathcal{X}) = \emptyset.$$

The set of all weakly minimal solutions w.r.t. the cone $\mathcal{K}$ is denoted by $\mathcal{X}_{WE}(\mathcal{K})$. If $\bar{\mathbf{x}} \in \mathcal{X}_{WE}(\mathcal{K})$, then a point $\bar{\mathbf{y}} = f(\bar{\mathbf{x}})$ is called weakly non-dominated w.r.t. cone $\mathcal{K}$. The set of all weakly non-dominated points w.r.t. the cone $\mathcal{K}$ is denoted by $\mathcal{Y}_{WE}(\mathcal{K})$.

In other words, the weakly $\mathcal{K}$-minimal points are the minimal points w.r.t. the cone $int(\mathcal{K}) \cup \{0\}$.

Before we present the definition of proper efficiency, we need to introduce the closure and the conical hull of a particular set $\mathcal{Y}$.

Definition 1.1.5. 1. The closure of $\mathcal{Y}$ is

$$cl(\mathcal{Y}) := int(\mathcal{Y}) \cup bd(\mathcal{Y}).$$

2. The conical hull of $\mathcal{Y}$ is

$$cone(\mathcal{Y}) := \{\alpha \mathbf{y} : \ \alpha \geq 0, \ \mathbf{y} \in \mathcal{Y}\} = \bigcup_{\alpha \geq 0} \alpha \mathcal{Y}.$$

In fact, the conical hull of $\mathcal{Y}$ is the smallest cone that includes $\mathcal{Y}$. An example of conical hull is shown in Figure 1.2.

Figure 1.2: Conical hull of $\mathcal{Y}$.

Definition 1.1.6. (*Benson, 1979*): A feasible point $\bar{\mathbf{x}} \in \mathcal{X}$ is called properly minimal/efficient for the MOP w.r.t. cone $\mathcal{K}$ if

$$cl(cone(f(\mathcal{X}) - f(\bar{\mathbf{x}}) + \mathcal{K})) \cap (-\mathcal{K}) = \{0\}.$$

The set of all properly minimal points w.r.t. cone $\mathcal{K}$ is denoted by $\mathcal{X}_{PE}(\mathcal{K})$. In this case, $\bar{\mathbf{y}} = f(\bar{\mathbf{x}})$ is called properly non-dominated point and the set of all properly non-dominated points is denoted by $\mathcal{Y}_{PE}(\mathcal{K})$.

Proper minimality is a natural concept in vector optimization, and plays an important role in our work here. There are a great number of works devoted to the definition of proper efficiency and to the existence of proper efficient points; see for instance, [7, 10, 33, 80]. The following relationships can be directly obtained from the above definitions:

$$\mathcal{X}_{PE}(\mathcal{K}) \subset \mathcal{X}_E(\mathcal{K}) \subset \mathcal{X}_{WE}(\mathcal{K})$$
$$\mathcal{Y}_{PN}(\mathcal{K}) \subset \mathcal{Y}_N(\mathcal{K}) \subset \mathcal{Y}_{WN}(\mathcal{K})$$

Let $\mathcal{K}^*$ denote the dual of the cone $\mathcal{K}$, which is defined by

$$\mathcal{K}^* := \{k^* \in \mathbb{R}^r | \ < k^*, k > \geq 0 \ \forall k \in \mathcal{K}\},$$

where, $<, >$ denotes the scalar product in $\mathbb{R}^r$.

Theorem 1.1.2 expresses the main results about properly efficient solutions.

Theorem 1.1.2. *Assume $\mathcal{K}$ is a nontrivial, closed convex cone.*
1. If for some $k^* \in int(\mathcal{K}^*)$, $\bar{\mathbf{x}}$ is an optimal solution for Problem (1.1.4), then $\bar{\mathbf{x}}$ is properly efficient:

$$\min_{\mathbf{x} \in \mathcal{X}} < k^*, f(\mathbf{x}) > \tag{1.1.4}$$

2. Suppose that $\bar{\mathbf{x}} \in \mathcal{X}_{PE}(\mathcal{K})$, $int(\mathcal{K}^*) \neq \emptyset$ and f is a convex function with respect to $\mathcal{K}$ on the convex set $\mathcal{X}$. Then, $\bar{\mathbf{x}}$ is an optimal solution of Problem (1.1.4) for some $k^* \in int(\mathcal{K}^*)$.

The above theorem states that properly efficient solutions can be obtained by minimizing a weighted sum of the objective functions. For convex problems, optimality for the weighted sum scalarization is a necessary and sufficient condition for proper efficiency.

Theorem 1.1.3 (see [28], Theorem 2.9]) shows that it is sufficient to consider only the boundary $\partial f(\mathcal{X})$ of the set $f(\mathcal{X})$ for determining all efficient points:

Theorem 1.1.3. *Let $\mathcal{K}$ be a nonempty ordering cone with $\mathcal{K} \neq \{0\}$. Then*

$$\mathcal{X}_E(\mathcal{K}) \subset \partial f(\mathcal{X}).$$

This is also true for the weakly efficient points.

1.2 Natural Ordering

The most useful partial ordering on $\mathbb{R}^r$ is the natural (or componentwise) ordering $\leqq$ defined by

$$\leqq := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^r \times \mathbb{R}^r |\ \mathbf{x}_i \leq \mathbf{y}_i \text{ for all } i = 1, 2, ..., r\}.$$

In this thesis we combine both relations $\mathbf{x} \leqq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ by $\mathbf{x} \leq \mathbf{y}$. The ordering cone representing the natural ordering in $\mathbb{R}^r$ is the nonnegative orthant $\mathbb{R}^r_{\geqq}$. In this case, the $\mathcal{K}$-minimal points are also called Pareto efficient/optimal points. Natural ordering is not a total ordering and so there can exist points which are not comparable; e.g., the points $(1, 2)$ and $(2, 1)$ in $\mathbb{R}^2$ w.r.t. the natural ordering. That is the reason why, in general, multiobjective optimization problems have an infinite number of solutions. For natural ordering, we can simplify the minimality definitions.

Definition 1.2.1. A feasible solution $\bar{\mathbf{x}} \in \mathcal{X}$ is efficient or Pareto optimal, if there is no other $\mathbf{x} \in \mathcal{X}$ such that $f(\mathbf{x}) \leq f(\bar{\mathbf{x}})$.

Definition 1.2.2. A feasible solution $\bar{\mathbf{x}} \in \mathcal{X}$ is weakly efficient, if there is no other $\mathbf{x} \in \mathcal{X}$ such that $f(\mathbf{x}) < f(\bar{\mathbf{x}})$.

For notational simplicity in the case of natural ordering, we denote the set of Pareto and weakly efficient points by $\mathcal{X}_E$ and $\mathcal{X}_{WE}$ respectively. Similarly, we denote the set of Pareto and weakly non-dominated points by $\mathcal{Y}_N$ and $\mathcal{Y}_{WN}$. In fact, based on the previous notations, we have

$$\mathcal{X}_E = \mathcal{X}_E(\mathbb{R}^r_{\geqq}),\ \mathcal{X}_{WE} = \mathcal{X}_{WE}(\mathbb{R}^r_{\geqq}),\ \mathcal{Y}_N = \mathcal{Y}_N(\mathbb{R}^r_{\geqq}),\ \mathcal{Y}_{WN} = \mathcal{Y}_{WN}(\mathbb{R}^r_{\geqq}).$$

For proper minimality, we can also have an equivalent definition in the case of natural ordering, but we will consider this subject thoroughly in the next chapter.

The next theorem shows that the optimal solutions of the weighted sum problem (1.2.1) are weakly/Pareto efficient for nonnegative/positive weights:

$$\min_{\mathbf{x} \in \mathcal{X}} < \lambda, f(\mathbf{x}) > . \tag{1.2.1}$$

Theorem 1.2.1. *Suppose that $\bar{\mathbf{x}}$ is an optimal solution of the weighted sum optimization problem (1.2.1). Then, the following statements hold:*
1. If $\lambda \geq 0$, then $\bar{\mathbf{x}} \in \mathcal{X}_{WE}$.
2. If $\lambda > 0$, then $\bar{\mathbf{x}} \in \mathcal{X}_{PE} \subset \mathcal{X}_E$.

Theorem 1.2.2 below shows that under convexity assumptions all weakly efficient solutions are optimal solutions of Problem (1.2.1) with nonnegative weights. Before that we introduce the notion of $\mathbb{R}^r_{\geqq}$-convexity.

Definition 1.2.3. A set $\mathcal{Y} \subset \mathbb{R}^r$ is called $\mathbb{R}^r_{\geqq}$-convex (closed) if $\mathcal{Y} + \mathbb{R}^r_{\geqq}$ is convex (closed).

It can be readily shown that if the MOP is convex problem, i.e., f is a convex function over the convex set $\mathcal{X}$, then $f(\mathcal{X})$ is $\mathbb{R}^r_{\geqq}$-convex, but the converse is not necessarily true.

Theorem 1.2.2. *Let $f(\mathcal{X})$ be $\mathbb{R}^r_{\geqq}$-convex. If $\bar{\mathbf{x}} \in \mathcal{X}$ is weakly efficient, then there exist some nonnegative weights λ for which the weighted sum problem yields $\bar{\mathbf{x}}$ as an optimal solution.*

At the end of this section, we define the ideal and Nadir points as lower and upper bounds on non-dominated points. These points give an indication of the range of the values which non-dominated points can attain. They are often used as reference points in weighted sum problems or in interactive methods, the aim of which is to find a most preferred solution for a decision maker.

Definition 1.2.4. 1. The point $\mathbf{y}^I = (y_1^I, y_2^I, ..., y_r^I)$ given by

$$y_k^I := \min_{\mathbf{x} \in \mathcal{X}} f_k(\mathbf{x}) = \min_{\mathbf{y} \in \mathcal{Y}} y_k$$

is called an ideal point of MOP.
2. The point $\mathbf{y}^N = (y_1^N, y_2^N, ..., y_r^N)$ given by

$$y_k^N := \max_{\mathbf{x} \in \mathcal{X}_E} f_k(\mathbf{x}) = \max_{\mathbf{y} \in \mathcal{Y}_N} y_k$$

is called a Nadir point of MOP.

1.3 Lexicographical Ordering

Lexicographical ordering (LO) is one of the important total orderings in multiobjective optimization, and is defined as follows.

Definition 1.3.1. A vector $\mathbf{a} = (a_1, ..., a_n)$ is called lexicographically positive (denoted by $\mathbf{a} \succ_l 0$) if the first non-zero component of a is positive, and is called lexicographically non-negative (denoted by $\mathbf{a} \succcurlyeq_l 0$) if $\mathbf{a} \succ_l 0$ or $\mathbf{a} = 0$.

The convex cone of LO is shown by

$$\mathcal{K}_l := \{\mathbf{y} \in \mathbb{R}^r |\ \mathbf{y} \succcurlyeq_l 0\},$$

Figure 1.3 depicts the LO cone in two dimension. As seen, the LO cone is not closed.

Figure 1.3: Ordering cone of the lexicographical ordering for $r = 2$.

Generally, it can be proved that a pointed convex cone cannot be closed and satisfy the total ordering property at the same time; see [22] .

When LO is taken into consideration as an ordering cone, MOP is called a lexicographic multiobjective optimization problem (LMOP), and is defined by

$$\text{LMOP} \qquad \qquad lexmin_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_r(\mathbf{x})). \qquad (1.3.1)$$

Like the natural ordering, we can simplify the definition of minimality for the lexicographic order. In this case, the $\mathcal{K}_l$-minimal points are also called lexicographic/preemptive optimal points.

Definition 1.3.2. A feasible solution $\bar{\mathbf{x}} \in \mathcal{X}$ is called lexicographic/preemptive optimal or an optimal solution for problem (1.3.1) if

$$f(\mathbf{x}) - f(\bar{\mathbf{x}}) \succeq_l 0, \text{ for all } \mathbf{x} \in \mathcal{X}.$$

For notational simplicity, the set of all lexicographic optimal solutions is denoted by $\overline{\mathcal{X}}_E$, instead of $\mathcal{X}_E(\mathcal{K}_l)$.

As opposed to the natural ordering, LO is total, and hence, if LMOP has a solution then there is only one lexicographic optimal point in the image space.

The unboundedness for LMOP could be defined by the next definition.

Definition 1.3.3. LMOP is unbounded if for every $\bar{\mathbf{x}} \in \mathcal{X}$ there exists $\mathbf{x} \in \mathcal{X}$ such that $f(\bar{\mathbf{x}}) \succ_l f(\mathbf{x})$.

It can be easily shown that LMOP is unbounded if either $f_1(\mathbf{x})$ is unbounded on the feasible region or there exists a $t \in \{2, ..., r\}$ such that $f_t(\mathbf{x})$ is unbounded over the alternative optimal solutions of the preceding objective functions.

Chapter 2

Transformation of MOP and Another Definition of Proper Efficiency

2.1 Introduction

When dealing with efficient solutions, it may be that only a subset of these should be considered, some being eliminated as being improper in some sense, in order to avoid undesirable situations. The starting point for this kind of studies is the classical paper by Kuhn and Tucker [45], where the faults of the notion of efficiency are made apparent and where proper efficiency is immediately related to scalarization problems. After the Kuhn-Tucker paper, several authors [4, 7, 8, 9, 10] have proposed various modified versions of this notion. A comprehensive survey of proper efficiencies can be found in [30]. The main drawback of improperly efficient points is that they cannot be satisfactorily characterized by a scalar minimization problem, even if the decision set is convex.

Multiobjective optimization problems typically have conflicting objectives wherein an improvement in one objective is at the expense of another. This concept of a decision maker, trading off an increment in one objective for a decrement in another is fundamental to multiobjective decision making. Several approaches in interactive multiobjective programming use trade-offs to establish the search direction for a satisfactory solution. The reader is referred to Miettinen (1999) [51] for a review of interactive solution methods and the role of trade-offs in multiobjective optimization problems. The trade-off ratio has been defined in the literature as the amount of increment in one objective due to a decrement of one unit in another objective. This concept is generalized in the next section. However, our aim is not finding the search direction for a satisfactory solution. We aim to build an easier definition for proper efficiency. By a new definition, properly efficient points are efficient points with bounded trade-offs. Moreover, by taking advantage of the new definition, some new properties of properness are proved which are useful in establishing a transformation technique for MOP.

In radiotherapy optimization, there is a quality index for treatment plans that is called homogeneity. In multiobjective framework, it means decreasing all objective functions at the same time. This concept is brought into multiobjective programming in Section 2.3.

The idea of transformation of a MOP has been introduced by Romeijn et al. (2004) [60] to transform some MOPs that have emerged in radiotherapy optimization for the sake of convexification and simplification and creating a unified framework for those problems. In Section 2.4, two major theoretical and practical drawbacks of their transformation are brought, and then the way these difficulties can be overcome is demonstrated. Some applications of this transformation are introduced in Subsection 2.4.1.

2.2 A New Definition of Proper Efficiency

The Kuhn and Tucker's [45] notion of properness requires differentiability and appears to be too broad for a satisfactory analysis. Geoffrion defined properness with respect to the nonnegative orthant. Later, this was generalized by Benson [4] and Henig [34] with respect to the general closed convex cone.

Definition 2.2.1. (*Geoffrion, 1968*): A feasible solution $\bar{\mathbf{x}} \in \mathcal{X}$ is properly efficient if there is a real number $M > 0$ such that for all $i \in \{1, 2, ..., r\}$ and $\mathbf{x} \in \mathcal{X}$ satisfying $f_i(\mathbf{x}) < f_i(\bar{\mathbf{x}})$ there exists an index $j \in \{1, 2, ..., r\}$ with $f_j(\bar{\mathbf{x}}) < f_j(\mathbf{x})$ and

$$\frac{f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}})} \leq M.$$

Definition 2.2.2. (*Benson, 1979*): A feasible point $\bar{\mathbf{x}} \in \mathcal{X}$ is called properly efficient if

$$cl(cone(f(\mathcal{X}) - f(\bar{\mathbf{x}}) + \mathbb{R}^r_{\geq})) \cap (-\mathbb{R}^r_{\geq}) = \{0\}.$$

Definition 2.2.3. (*Henig, 1982*): A feasible point $\bar{\mathbf{x}} \in \mathcal{X}$ is called properly efficient if $(f(\bar{\mathbf{x}}) - \mathcal{K}) \cap f(\mathcal{X}) = \{f(\bar{\mathbf{x}})\}$, for some convex cone $\mathcal{K}$ with $\mathbb{R}^r_{\geqq} \backslash \{0\} \subset int(\mathcal{K})$.

Benson and Henig's definitions of proper efficiency are not restricted to the natural order. They are therefore applicable in the more general context of orders defined by cones. Geoffrions definition, on the other hand, explicitly uses the natural order. In the case of natural ordering, the definitions of Geoffrion, Benson and Henig actually coincide, so that they are proper generalization of Geoffrion's.

Similar to the previous chapter, for notational simplicity in the case of natural ordering, we denote the set of properly efficient and non-dominated points by $\mathcal{X}_{PE}$ and $\mathcal{Y}_{PN}$ respectively. Therefore, based on the previous notations we have used for the general ordering cone, we have

$$\mathcal{X}_E = \mathcal{X}_E(\mathbb{R}^r_{\geqq}),\ \mathcal{Y}_{PN} = \mathcal{Y}_{PN}(\mathbb{R}^r_{\geqq}).$$

The following relationships can be directly obtained from the definitions of Pareto, weakly and properly efficient points:

$$\mathcal{X}_{PE} \subset \mathcal{X}_E \subset \mathcal{X}_{WE},$$
$$\mathcal{Y}_{PN} \subset \mathcal{Y}_N \subset \mathcal{Y}_{WN}.$$

Figure 2.1 illustrates that the gap between $\mathcal{Y}_{WN}$ and $\mathcal{Y}_N$ might be quite large, even for the convex cases. This is not possible for the gap between $\mathcal{Y}_{PN}$ and $\mathcal{Y}_N$. Hartley (1978) [33] proved that under the $\mathbb{R}^r_{\geqq}$-convexity and $\mathbb{R}^r_{\geqq}$-closedness assumptions on $\mathcal{Y}$, the set of properly non-dominated points is dense in the set of efficient points.

Figure 2.1: $\mathcal{Y}_N$ is empty, $\mathcal{Y}_{WN}$ is not.

Theorem 2.2.1. *If $\mathcal{Y}$ is $\mathbb{R}^r_{\geqq}$-convex and $\mathbb{R}^r_{\geqq}$-closed,. Then, the following inclusions hold:*

$$\mathcal{Y}_{PN} \subset \mathcal{Y}_N \subset cl(\mathcal{Y}_{PN}).$$

At this point, we intend to provide a new definition of proper efficiency for natural ordering. This definition coincides with Geoffrion's definition, and hence, Benson and Henig's definitions. This definition is based on the trade-off notion.

According to Definition 1.2.1, an efficient solution does not allow for the improvement of one objective function while retaining the same values on the others. Improvement of some criterion can only be obtained at the expense of the deterioration of at least one other criterion. So, we can define the trade-off between an efficient point $\bar{\mathbf{x}}$ and an arbitrary feasible point $\mathbf{x} \in \mathcal{X}$ by

$$T_{\mathbf{x}}(\bar{\mathbf{x}}) = \frac{\|(f(\bar{\mathbf{x}}) - f(\mathbf{x}))^+\|}{\|(f(\mathbf{x}) - f(\bar{\mathbf{x}}))^+\|}, \tag{2.2.1}$$

where, $\|.\|$ is an arbitrary norm on $\mathbb{R}^r$, and ζ^+ shows the positive entries of ζ, defined by

$$\zeta_i^+ = \begin{cases} \zeta_i, & \zeta_i \geq 0 \\ 0, & \zeta_i < 0 \end{cases}$$

In the definition of $T_{\mathbf{x}}(\bar{\mathbf{x}})$, the numerator measures how $\mathbf{x}$ is better than $\bar{\mathbf{x}}$, and the denominator gauges how $\mathbf{x}$ is worse than $\bar{\mathbf{x}}$. So, $T_{\mathbf{x}}(\bar{\mathbf{x}})$ shows the amount of decrement due to the amount of increment, and could be considered as the trade-off between $\mathbf{x}$ and $\bar{\mathbf{x}}$. We define the trade-off at $\bar{\mathbf{x}}$ as the maximal trade-off between $\bar{\mathbf{x}}$ and all the feasible points as follows:

$$T_{\mathcal{X}}(\bar{\mathbf{x}}) = \sup_{\mathbf{x}\in\mathcal{X}} T_{\mathbf{x}}(\bar{\mathbf{x}}).$$

We define the properness based on the above trade-off definition.

Definition 2.2.4. An efficient point $\bar{\mathbf{x}} \in \mathcal{X}$ is called properly efficient in the trade-off sense if it has a finite trade-off, i.e., $T_{\mathcal{X}}(\bar{\mathbf{x}}) < \infty$.

From real analysis we know that all norms on a finite dimensional space are equivalent. It means, for two norms $\|.\|_\alpha$ and $\|.\|_\beta$ there exist positive real numbers C and D such that

$$C\|.\|_\alpha \leq \|.\|_\beta \leq D\|.\|_\alpha$$

By the equivalency of norms on $\mathbb{R}^r$, it can be concluded that Definition 2.2.4 is regardless of the norm, employed to define the trade-off. The following theorems show that our definition of properness is equivalent to the definitions of Geoffrion and Benson. Since Geoffrion and Benson's definitions are equivalent, it is sufficient to prove the equivalency to one of them. However, we provide the equivalency to both of them because the proofs might be useful for future research and possible extensions of the trade-off definition for the general cone ordering. Moreover, the two theorems below can be considered as another proof for the equivalency of Geoffrion and Benson's definitions.

Theorem 2.2.2. *Trade-off definition of properness and Geoffrion's definition are equivalent.*

Proof. If we consider $\|.\|_\infty$ in the definition of trade-off, then $T_{\mathcal{X}}(\bar{\mathbf{x}})$ can be rewritten as

$$T_{\mathcal{X}}(\bar{\mathbf{x}}) = \sup_{\mathbf{x}\in\mathcal{X}} \frac{\max\limits_{i:\ f_i(\mathbf{x})<f_i(\bar{\mathbf{x}})} (f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x}))}{\max\limits_{j:\ f_j(\mathbf{x})>f_j(\bar{\mathbf{x}})} (f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}}))}. \tag{2.2.2}$$

$\Rightarrow$) Suppose that $\bar{\mathbf{x}}$ is properly efficient in the trade-off sense, and $T_{\mathcal{X}}(\bar{\mathbf{x}}) = M$. For an arbitrary $\mathbf{x} \in \mathcal{X}$ assume that $f_i(\mathbf{x}) < f_i(\bar{\mathbf{x}})$ for some $i \in \{1, 2, ..., r\}$. To prove $\bar{\mathbf{x}}$ is properly efficient with respect to the Geoffrion's definition, it is sufficient to show that there exists $j \in \{1, 2, ..., r\}$ such that $f_j(\bar{\mathbf{x}}) < f_j(\mathbf{x})$ and $\frac{f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}})} \leq M$. If we define

$$j^* = \underset{j}{\operatorname{argmax}}\{f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}})\},$$

then we have

$$\frac{f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x})}{f_{j^*}(\mathbf{x}) - f_{j^*}(\bar{\mathbf{x}})} \leq \frac{\max\limits_{i:\ f_i(\mathbf{x}) < f_i(\bar{\mathbf{x}})} (f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x}))}{\max\limits_{j:\ f_j(\mathbf{x}) > f_j(\bar{\mathbf{x}})} (f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}}))} \leq T_{\mathcal{X}}(\bar{\mathbf{x}}) = M$$

$\Leftarrow$) Suppose that $\bar{\mathbf{x}}$ is properly efficient in the Geoffrion's sense. For an arbitrary $\mathbf{x} \in \mathcal{X}$ if we define

$$i^* = \underset{i}{\operatorname{argmax}}\{f_i(\bar{\mathbf{x}}) - f_j(\mathbf{x})\},$$

then by Geoffrion's definition there exists $j \in \{1, 2, ..., r\}$ such that $\frac{f_{i^*}(\bar{\mathbf{x}}) - f_{i^*}(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}})} \leq M$. Therefore,

$$\frac{\max\limits_{i:\ f_i(\mathbf{x}) < f_i(\bar{\mathbf{x}})} (f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x}))}{\max\limits_{j:\ f_j(\mathbf{x}) > f_j(\bar{\mathbf{x}})} (f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}}))} \leq \frac{f_{i^*}(\bar{\mathbf{x}}) - f_{i^*}(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}})} \leq M$$

Since in the relation above the value of M is independent of $\mathbf{x}$, we conclude that $T_{\mathcal{X}}(\bar{\mathbf{x}}) \leq M$, which completes the proof. $\square$

Theorem 2.2.3. *Trade-off definition of properness and Benson's definition are equivalent.*

Proof. $\Rightarrow$) Let $\bar{\mathbf{x}}$ be an improperly efficient point in the trade-off sense. So, there exists a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathcal{X}$ such that $\frac{\|(f(\bar{\mathbf{x}}) - f(\mathbf{x}_n))^+\|}{\|(f(\mathbf{x}_n) - f(\bar{\mathbf{x}}))^+\|}$ tends towards infinity. Denote $t_n := \frac{1}{\|(f(\bar{\mathbf{x}}) - f(\mathbf{x}_n))^+\|}$. If we show that $t_n(f(\mathbf{x}_n) - f(\bar{\mathbf{x}}))$ has a subsequence which tends towards $-d$, for some $d \in \mathbb{R}^r_{\geqq}$, and $\|d\| = 1$, then $\bar{\mathbf{x}}$ is improper in Benson's sense. To this end, we rewrite $t_n(f(\mathbf{x}_n) - f(\bar{\mathbf{x}}))$ as

$$t_n(f(\mathbf{x}_n) - f(\bar{\mathbf{x}})) = \frac{(f(\mathbf{x}_n) - f(\bar{\mathbf{x}}))^+}{\|(f(\bar{\mathbf{x}}) - f(\mathbf{x}_n))^+\|} - \frac{(f(\bar{\mathbf{x}}) - f(\mathbf{x}_n))^+}{\|(f(\bar{\mathbf{x}}) - f(\mathbf{x}_n))^+\|}.$$

Since $\frac{\|(f(\bar{\mathbf{x}})-f(\mathbf{x}_n))^+\|}{\|(f(\mathbf{x}_n)-f(\bar{\mathbf{x}}))^+\|}$ tends towards infinity, $\frac{(f(\mathbf{x}_n)-f(\bar{\mathbf{x}}))^+}{\|(f(\bar{\mathbf{x}})-f(\mathbf{x}_n))^+\|}$ tends towards zero. On the other hand, if we denote $d_n := \frac{(f(\bar{\mathbf{x}})-f(\mathbf{x}_n))^+}{\|(f(\bar{\mathbf{x}})-f(\mathbf{x}_n))^+\|}$, then $d_n \geq 0$ and $\|d_n\| = 1$, and so we have a subsequence which tends towards some $d \in \mathbb{R}^r_{\geqq}$ and $\|d\| = 1$. Therefore, $t_n(f(\mathbf{x}_n) - f(\bar{\mathbf{x}}))$ has a subsequence which tends towards $-d$, which completes the proof.

$\Leftarrow$) Suppose that $\bar{\mathbf{x}}$ is improperly efficient in the Benson's sense. Thus, there exist $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathcal{X}$, $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}_{\geqq}$ and $\{d_n\}_{n=1}^{\infty} \subset \mathbb{R}^r_{\geqq}$ such that $t_n(f(\mathbf{x}_n) - f(\bar{\mathbf{x}}) + d_n)$ tends towards $-d$, where, d is a nonzero and non-negative vector. Therefore, the positive entries of $t_n(f(\mathbf{x}_n) - f(\bar{\mathbf{x}}) + d_n)$, which are shown by $t_n(f(\mathbf{x}_n) - f(\bar{\mathbf{x}}) + d_n)^+$, tend towards zero and the negative entries, which are shown by $t_n(f(\bar{\mathbf{x}}) - f(\mathbf{x}_n) - d_n)^+$, tend towards $-d$. So, $\frac{\|(f(\bar{\mathbf{x}})-f(\mathbf{x}_n)-d_n)^+\|}{\|(f(\mathbf{x}_n)-f(\bar{\mathbf{x}})+d_n)^+\|}$ tends towards infinity. In this case, $\frac{\|(f(\bar{\mathbf{x}})-f(\mathbf{x}_n))^+\|}{\|(f(\mathbf{x}_n)-f(\bar{\mathbf{x}}))^+\|}$ tends towards infinity as well because of $\frac{\|(f(\bar{\mathbf{x}})-f(\mathbf{x}_n))^+\|}{\|(f(\mathbf{x}_n)-f(\bar{\mathbf{x}}))^+\|} \geq \frac{\|(f(\bar{\mathbf{x}})-f(\mathbf{x}_n)-d_n)^+\|}{\|(f(\mathbf{x}_n)-f(\bar{\mathbf{x}})+d_n)^+\|}$. This means $\bar{\mathbf{x}}$ is improperly efficient in the trade-off sense.

□

Example 2.2.1. *Consider the following feasible solution set:*

$$\mathcal{X} = \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2 |\ (\mathbf{x}_1 - 1)^2 + (\mathbf{x}_2 - 1)^2 \leq 1,\ 0 \leq \mathbf{x}_1, \mathbf{x}_2 \leq 1\}.$$

If we assume the identity function as the objective function, then we have the same image set, i.e., $\mathcal{Y} = \mathcal{X}$ (see Figure 2.2). Now, consider the points $\hat{\mathbf{y}}$ and $\bar{\mathbf{y}}$ in Figure 2.2. The trade-offs for these points can be computed by

$$T_{\mathcal{Y}}(\hat{\mathbf{y}}) = \frac{1 - \hat{\mathbf{y}}_1}{1 - \sqrt{1 - (\hat{\mathbf{y}}_1 - 1)^2}},$$
$$T_{\mathcal{Y}}(\bar{\mathbf{y}}) = \infty.$$

So, $\hat{y}$ has a bounded trade-off and is properly efficient, and $\bar{y}$ has unbounded trade-off and is improperly efficient.

2.3 Generating Properly Efficient Points

The notion of proper efficiency has always been strictly related to the scalarization of a vector problem. The weighted sum method is the most famous scalarization technique in multiobjective literature. Generally, the weighted sum problem could be written as follows:

$$\min_{\mathbf{x}\in\mathcal{X}} < \lambda, (f(\mathbf{x}) - \mathbf{y}^U)^p >, \tag{2.3.1}$$

where, $\mathbf{y}^U$ is a utopia point and $p \in \mathbb{R}$. In multiobjective literature, Problem (2.3.1) has also been referred to as the *weighted compromise programming/p−power/ p−norm/L_p-norm problem*. In order to facilitate reading, we will henceforth refer to the problem (2.3.1) as weighted problem and the optimal solutions as weighted solutions. Some authors apply the root $\frac{1}{p}$ to the whole of the objective function of the weighted problem, but formulations with and without the root theoretically provide the same solution. To some extent, the following problem called (*weighted*) *min-max/ min-norm/ ∞-norm/ Tchebycheff problem* in the literature could be considered as another version of the weighted problem for $p = \infty$:

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{i=1,\ldots,k} w_i(f_i(\mathbf{x}) - y_i^U). \tag{2.3.2}$$

Here, the following two natural and important questions may arise:

1. Are all optimal solutions of the weighted problem necessarily properly efficient?

2. Are all properly efficient solutions obtainable by solving the weighted problem?

Theorem 2.3.1 answers the first question by providing conditions under which the optimal solutions of the weighted problem are guaranteed to be properly efficient.

Theorem 2.3.1. *An optimal solution of weighted problem is properly efficient if one of the following conditions holds:*

Figure 2.2: $\bar{\mathbf{y}}$ is improperly and $\hat{\mathbf{y}}$ is properly efficient.

- *(Geoffrion, 1968) The weights are positive, and* $p = 1$.
- *(Gearhart, 1979)* $p = 2$, $\lambda > 0$, *and* $f(\mathcal{X})$ *is* $\mathbb{R}^r_{\geqq}-$*convex.*
- *(Sawaragi et al., 1985) The weights are positive,* $p \in [1, \infty)$, *and* $f(\mathcal{X})$ *is* $\mathbb{R}^r_{\geqq}-$*closed.*

The following theorem answers the second aforementioned question by providing conditions under which all properly efficient points could be captured by the weighted problem.

Theorem 2.3.2. *Let* $\bar{\mathbf{x}}$ *be a properly efficient point. Then,* $\bar{\mathbf{x}}$ *is an optimal solution of the weighted problem if*

- *(Geoffrion, 1968)* $p = 1$ *and* $f(\mathcal{X})$ *is* $\mathbb{R}^r_{\geqq}-$*convex.*
- *(Gearhart, 1979)* $p = 2$ *and* $f(\mathcal{X})$ *is* $\mathbb{R}^r_{\geqq}-$*convex.*
- *(Sawaragi et al., 1985)* $f(\mathcal{X})$ *is* $\mathbb{R}^r_{\geqq}-$*closed and* $p \in [1, \infty)$.
- *(White, 1988)* p *is sufficiently large and* $f(\mathcal{X})$ *is finite.*

It should be mentioned that the result of White (1988) was presented in the original work for Pareto efficient points. However, in the trade-off definition of proper efficiency it is evident that all Pareto efficient points are properly efficient for finite MOPs. Thus, this result is given in the fourth part of the above theorem for properly efficient points. In Corollary 2.4.8, it will be shown that we can generalize these existing results with the aid of our transformation technique.

Theorems 2.3.1 and 2.3.2 state that for $p = 1$, properly efficient points can be characterized in terms of the weighted solutions. Now, the question that arises here is: " Why do we solve the weighted problem for $p > 1$ when it increases the nonlinearity of the problem?". Answering this question clarifies one of our motivations to transform multiobjective problems. In what follows, we try to answer this question.

While being able to generate all properly efficient points for different parameters makes the scalarization problem more reliable, the decision maker usually is not interested in all of these points. In this case, it is important to generate more attractive efficient points with a few weights. In the weighted problem, for a specific weight, when p is set to 1, all distances from the goal are weighted equally, and if $p > 1$, larger distances from the goal are penalized more than smaller distances from the goal. So, as p increases, all objective functions tend to decrease simultaneously, and this is usually satisfactory from the decision maker's viewpoint. Generally, p is proportional to the amount of emphasis placed on minimizing the function with the largest difference between $f_i(\mathbf{x})$ and y_i^U (Koski and Silvennoinen, 1987 [44]). Selecting the value for the exponent p is treated in Ballestero (1997) [1] from the point of view of risk aversion. The conclusion is that for bigger risk aversion, we should use a larger value of p. Another guideline is that for a smaller number of objective functions we should select a larger p value.

The weighted optimization problem has also an application in cancer treatment by radiation therapy. Niemierko (1999) [54] introduced the usage of the weighted problem to deal with the normal and critical tissues in radiation therapy, and since its introduction, it has been used by many researchers; for example, refer to Romeijn et al. (2004) [60] and references therein. In this application, the value of p depends on the structure of the organ. For more sensitive organs like spinal cord, called *serial* organs, the value of p should be large enough to decrease all objectives at the same time (here, each objective function is the amount of radiation in a particle of body that is called voxel). For the less sensitive organs such as lung and kidney, called *parallel* organs, a small p usually works. In radiotherapy optimization, decreasing/increasing the amount of radiation in all voxels[1] at the same time is an important issue. A treatment plan is called *homogenous* if it decreases the amount of radiation in all voxels simultaneously. To measure the homogeneity of a treatment plan, Yoon et al. (2007) [74] introduced the S-Index based on the standard deviation. Standard

[1]In radiotherapy optimization, the patient's body is discretized into elements called voxels.

deviation measures the deviation of the amount of the dose in each voxel to the median dose, and hence, it reflects how close the amount of dose in all voxels is. Therefore, the smaller standard deviation a treatment plan has, the more homogenous it would be. Now, we want to bring the homogeneity concept into multiobjective optimization by the following definition.

Definition 2.3.1. Let $\bar{\mathbf{x}}$ be a properly efficient point. The amount of homogeneity at $\bar{\mathbf{x}}$ is denoted by $H(\bar{\mathbf{x}})$ and is defined as follows:

$$H(\bar{\mathbf{x}}) := \sqrt{\sum_{i=1}^{r}(f_i(\bar{\mathbf{x}}) - f_{mean}(\bar{\mathbf{x}}))^2},$$

where, $f_{mean}(\bar{\mathbf{x}}) := \frac{1}{r}\sum_{i=1}^{r} f_i(\bar{\mathbf{x}})$. A properly efficient $\hat{\mathbf{x}}$ is said to be more homogenous than $\bar{\mathbf{x}}$ if $H(\hat{\mathbf{x}}) < H(\bar{\mathbf{x}})$.

Remark 2.3.1. It is worthwhile to mention that we can generalize the above definition by defining the amount of homogeneity as

$$H(\bar{\mathbf{x}}) := \|f(\bar{\mathbf{x}}) - f_{mean}(\bar{\mathbf{x}})\|,$$

where, $\|.\|$ is an arbitrary norm. In this case, the above definition can be obtained by considering $\|x\|_2$ as a special norm.

Example 2.3.1 illustrates how the value of p in weighted problem is related to the homogeneity of the efficient points.

Example 2.3.1. *Consider the following MOP:*

$$\begin{aligned} \min\{x, y, z\} \\ s.t. \\ x + y + z \geq 10 \\ x, y, z \geq 0. \end{aligned}$$

Figure 2.3 shows the efficient surface of this problem. $\mathbf{y}^U = 0$ can be considered as a utopia point for this problem and so we can write the weighted problem as

$$\begin{aligned} \min \lambda_1 x^p + \lambda_2 y^p + \lambda_3 z^p \\ s.t. \\ x + y + z \geq 10 \\ x, y, z \geq 0. \end{aligned}$$

Now, take into account 66 uniformly distributed positive weights over $(0,1) \times (0,1) \times (0,1)$, and generate the weighted solutions for these weights and three choices of p. For $p = 2$, 3 and 4 the weighted solutions are denoted by the spots on the efficient surface in Figure 2.3. It can be seen that, as p increases, they are more likely to be placed on the center of the efficient frontier.

To discover the relationship between the exponent p and the homogeneity of weighted solutions, we compute the homogeneity of all generated weighted solutions. The results show that homogeneity is improved as p increases. In particular, for a specific weight, the weighted solution for $p = 3$ is more homogenous than the weighted solution for $p = 2$.

Another approach to comparing the homogeneity of the weighted solutions is comparing the average of the homogeneity of all the 66 generated weighted solutions for each p. These averages are 12.87, 7.03 and 4.78, for p equal to 2,3, and 4, respectively. Hence, the average homogeneity also confirms that the homogeneity is improved as p increases.

Here, it is worthwhile to mention that for a specific weight, the weighted solution for the bigger p is expected to be more homogeneous, and it is not guaranteed. To observe this, it is sufficient to consider $p = 1$ and $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. In this case, all points on the efficient surface are optimal for the weighted problem, and we cannot say that, for this weight and bigger value of p, we have more homogenous efficient solution.

□

In the light of above example and what was discussed about the impact of p on the

Figure 2.3: Weighted solutions are depicted by the spots on the efficient surface for different values of p.

weighted problem, in summary we can say that, for a specific weight, as p increases the bigger distances from the goal are penalized more than the smaller distances, and so, all objective functions try to decrease simultaneously, and hence, improvement in the homogeneity of the weighted solution is expected.

At the end of this section, we want to point out another scalarization technique to generate properly efficient points. The weighted problem provides a parametric scheme to generate the properly efficient solutions of MOP in the convex case. Choo and Atkins (1983) [16] proposed a similar parametric scheme for the non-convex case. They proved that proper efficiency can be characterized by a naturally extended form of the generalized Tchebycheff norm defined by Bowman [11]. This characterization leads to a parametric scheme for generating the properly efficient solutions of non-convex MOPs.

Theorem 2.3.3 states that properly efficient points can be characterized as the best approximation to the utopia point with respect to the augmented weighted Tchebycheff metric without any convexity assumption (see [16, 72, 41, 42]).

$\bar{\mathbf{x}}$ *is properly efficient if and only if there exist* $\lambda > 0$ *and* $\rho > 0$*, such that* $\bar{\mathbf{x}}$ *solves*

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{i=1,\dots,r} \lambda_i(f_i(\mathbf{x}) - y_i^U) + \rho\sum_{i=1}^{r}(f_i(\mathbf{x}) - y_i^U).$$

2.4 Transformation of Multiple Objective Problems

The idea of transformation of multiple objective problems has been introduced by Romeijn et al. (2004) [60] to transform some MOPs arisen in radiotherapy optimization for the sake of convexification, simplification and making unifying framework for those problems. Generally, we can transform our MOP (1.1.1) to the following MOP:

$$\min_{\mathbf{x} \in \mathcal{X}} gof(\mathbf{x}), \tag{2.4.1}$$

where, the transformation function g is defined by

$$g(\mathbf{x}) = (g_1(x_1), g_2(x_2), ..., g_r(x_r))^T : \mathbb{R}^r \to \mathbb{R}^r, \quad g_i(\theta) : \; \mathbb{R} \to \mathbb{R}.$$

In other words, the transformation function g includes the r scalar functions.

In order to facilitate reading, we will henceforth refer to Problem (1.1.1) as the original MOP and Problem (2.4.1) as the transformed MOP, and we denote the properly efficient points of the transformed MOP by $\mathcal{X}_{PE}^T$. Romeijn et al. [60] exploited increasing[2] functions to transform their MOPs because in this case both general and transformed MOPs have the same Pareto efficient solutions.

Remark 2.4.1. Romeijn et al.'s [60] idea is decomposing the objective function f into the GOF (G and F are vector valued functions) for an increasing function G, and then considering $\min_{\mathbf{x} \in \mathcal{X}} F(\mathbf{x})$ as a transformed MOP. Although their idea seems to be different from our transformation technique at first glance, they are the same. In fact, in our framework, their idea is like considering the inverse of the function G as a transformation function which is also increasing.

Now, we want to strengthen the existing transforming idea by overcoming the following two potential drawbacks:

1. Example 2.4.1 shows that the original and the transformed MOPs can have different properly efficient points.

2. Example 2.4.2 illustrates that the transformation function may deteriorate the homogeneity of the efficient solutions.

The first drawback is perhaps more important from the theoretical viewpoint, while the second one is more important from the practical perspective.

[2]Throughout this thesis, by increasing we mean strictly increasing.

Example 2.4.1. *Consider the multiobjective problem* $\min_{t<0}\{\frac{1}{t^2}, t^2\}$ *as the original MOP. As Figure 2.4 shows, all efficient points of this problem could be characterized as the weighted solutions and so they are properly efficient. If we employ* $h_1(\mathbf{x}) = h_2(\mathbf{x}) = \frac{-1}{\sqrt{\mathbf{x}}}$ *as an increasing transformation function, then the transformed problem is* $\min_{t<0}\{t, \frac{1}{t}\}$*. Since the transformation function is increasing, the original and the transformed problems have the same Pareto efficient solutions. However, it can be seen in the right picture in Figure 2.4 that the weighted problem of the transformed problem is unbounded, for all positive weights, meaning that all efficient points are improper. Therefore, the original and the transformed problem have different properly efficient points, while they have the same Pareto efficient solution sets.*

Figure 2.4: The original and the transformed MOPs have different properly efficient solutions.

Consider the following problem as the original MOP:

$$\begin{aligned} \min\{\exp(x), \exp(y), \exp(z)\} \\ s.t. \\ x + y + z \geq 10, \\ x, y, z \geq 0. \end{aligned} \tag{2.4.2}$$

Now, we want to investigate the impact of two increasing transformation functions $h_1(t) = h_2(t) = t^2$ *and* $h_1(t) = h_2(t) = \sqrt{t}$ *on the homogeneity of the weighted solutions. Like Example 2.3.1, we consider 66 positive uniformly distributed weights, and then we generate the weighted solutions for the original and the two transformed MOPs. The results show that when we use* t^2 *as the transformation function, the homogeneity is improved in 64 solutions out of 66, and the homogeneity is deteriorated in all of the solutions while we utilize* $\sqrt{x}$*. Moreover, the average of homogeneity of the solutions are 0.74 and 8.75, for the transformed MOPs with* t^2 *and* $\sqrt{x}$ *as the transformation functions, respectively, while it is 4.37 for the original MOP.*

The above example shows that improvement in the homogeneity of the weighted solutions is expected when we apply the square function as the transformation function, and deterioration is expected for the square root function. Now, the question is: "what property of the transformation function causes improvement or deterioration in the homogeneity of the weighted solutions?" We can answer this question by getting back to the main point raised in the previous section about the impact of exponent p on the homogeneity of the weighted solutions. In fact, the square function has an increasing derivative which leads to an increase in the rate of changes of the original objective function and causes more penalty to be imposed for the larger values than the smaller values, and hence, improve the homogeneity of the efficient solutions. On the other hand, the square root function has a decreasing derivative and deteriorates the homogeneity of the solutions.

The conclusion that we can draw here about the transformation of MOPs is that if the transformation function has an increasing derivative, then improvement in the homogeneity of the solutions is expected, and we can overcome the second potential drawback mentioned before. Theorem 2.4.1 reveals that this condition is almost sufficient to overcome the first potential difficulty as well. We need the following assumption in our transformation technique.

Assumption A. *Function f is bounded below on $\mathcal{X}$, i.e., ($\mathbf{y}^I > -\infty$), and for each $i \in \{1, 2, ..., r\}$.*

1. g_i is continuous on $[\min\limits_{\mathbf{x}\in\mathcal{X}} f_i(\mathbf{x}), \max\limits_{\mathbf{x}\in\mathcal{X}} f_i(\mathbf{x})]$.

2. g_i is differentiable and g_i' is positive on $(\min\limits_{\mathbf{x}\in\mathcal{X}} f_i(\mathbf{x}), \max\limits_{\mathbf{x}\in\mathcal{X}} f_i(\mathbf{x}))$.

3. Both g_i and its derivative g_i' are increasing on $(\min\limits_{\mathbf{x}\in\mathcal{X}} f_i(\mathbf{x}), \max\limits_{\mathbf{x}\in\mathcal{X}} f_i(\mathbf{x}))$.

Theorem 2.4.1. *If Assumption A holds, then the original and the transformed MOPs have the same properly efficient points, i.e., $\mathcal{X}_{PE} = \mathcal{X}_{PE}^T$.*

Proof. If we consider $\|.\|_1$ in the definition of the trade-off, then $\bar{\mathbf{x}} \in \mathcal{X}_{PE}$ iff there exists a real M such that for all $\mathbf{x} \in \mathcal{X}$,

$$\sum_{i\in I_{\mathbf{x}}} (f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x})) \leq M \sum_{j\in J_{\mathbf{x}}} (f_j(\mathbf{x}) - f_j(\bar{\mathbf{x}})), \tag{2.4.3}$$

where, $I_{\mathbf{x}} = \{i \in \{1,2,...,n\} | f_i(\mathbf{x}) \leq f_i(\bar{\mathbf{x}})\}$ and $J_{\mathbf{x}} = \{j \in \{1,2,...,n\} | f_j(\mathbf{x}) > f_j(\bar{\mathbf{x}})\}$. In the same manner, $\bar{\mathbf{x}} \in \mathcal{X}_{PE}^{T}$ iff there exists a real N such that for all $\mathbf{x} \in \mathcal{X}$,

$$\sum_{i \in I_{\mathbf{x}}} (g_i o f_i(\bar{\mathbf{x}}) - g_i o f_i(\mathbf{x})) \leq N \sum_{j \in J_{\mathbf{x}}} (g_j o f_j(\mathbf{x}) - g_j o f_j(\bar{\mathbf{x}})). \tag{2.4.4}$$

Since for each $t \in \{1,2,...,n\}$, g_t is increasing, we did use the same $I_{\mathbf{x}}$ and $J_{\mathbf{x}}$ in Relations (2.4.4) and (2.4.3). To prove the equivalence of the above relations, we exploit the following inequalities which are true for all $\delta < \alpha < \beta$ and $t \in \{1,2,...,n\}$:

$$g_t(\beta) - g_t(\alpha) \leq g_t'(\beta)(\beta - \alpha), \tag{2.4.5}$$

$$\beta - \alpha \leq \frac{g_t(\beta) - g_t(\alpha)}{g_t'(\alpha)}, \tag{2.4.6}$$

$$\beta - \alpha \leq \frac{\beta - \delta}{g_t(\beta) - g_t(\delta)} (g_t(\beta) - g_t(\alpha)). \tag{2.4.7}$$

Relations (2.4.5) and (2.4.6) could be directly obtained by employing the mean value theorem. To prove Relation (2.4.7), it is sufficient to consider the auxiliary function $h(\alpha) = \frac{g_t(\beta) - g_t(\alpha)}{\beta - \alpha}$ and show that it is increasing. Then, the relation can be obtained by the fact $h(\delta) \leq h(\alpha)$.

(2.4.3) $\rightarrow$ (2.4.4): Suppose that Relation (2.4.3) holds, for all $\mathbf{x} \in \mathcal{X}$. By employing Relation (2.4.5), we have

$$\begin{aligned} \sum_{i \in I_{\mathbf{x}}} (g_i o f_i(\bar{\mathbf{x}}) - g_i o f_i(\mathbf{x})) &\leq \sum_{i \in I_{\mathbf{x}}} g_i' o f_i(\bar{\mathbf{x}})(f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x})) \\ &\leq \max_{t \in \{1,2,\ldots,n\}} g_t'(\bar{\mathbf{x}}_t) \sum_{i \in I_{\mathbf{x}}} (f_i(\bar{\mathbf{x}}) - f_i(\mathbf{x})). \end{aligned} \tag{2.4.8}$$

By exploiting (2.4.3) and (2.4.6), we have

$$\begin{aligned}
&\max_{t\in\{1,2,\dots,n\}} g_t'(\bar{\mathbf{x}}_t)\sum_{i\in I_{\mathbf{x}}}(f_i(\bar{\mathbf{x}})-f_i(\mathbf{x})) \qquad (2.4.9)\\
&\quad\le M\max_{t\in\{1,2,\dots,n\}} g_t'(\bar{\mathbf{x}}_t)\sum_{j\in J_{\mathbf{x}}}(f_j(\mathbf{x})-f_j(\bar{\mathbf{x}}))\\
&\quad\le M\max_{t\in\{1,2,\dots,n\}} g_t'(\bar{\mathbf{x}}_t)\sum_{j\in J_{\mathbf{x}}}\frac{g_jof_j(\mathbf{x})-g_jof_j(\bar{\mathbf{x}})}{g_j'of_j(\bar{\mathbf{x}})}\\
&\le M\max_{t\in\{1,2,\dots,n\}} g_t'(\bar{\mathbf{x}}_t)\max_{t\in\{1,2,\dots,n\}}\frac{1}{g_t'of_t(\bar{\mathbf{x}})}\sum_{j\in J_{\mathbf{x}}}(g_jof_j(\mathbf{x})-g_jof_j(\bar{\mathbf{x}})).
\end{aligned}$$

To complete the proof, it is sufficient to combine Relations (2.4.8) and (2.4.9) and let

$$N := M\max_{t\in\{1,2,\dots,n\}} g_t'(\bar{\mathbf{x}}_t)\max_{t\in\{1,2,\dots,n\}}\frac{1}{g_t'(\bar{\mathbf{x}}_t)}.$$

(2.4.4) $\rightarrow$ (2.4.3): For an arbitrary $\mathbf{x}\in\mathcal{X}$, the following two cases may occur.

Case 1: $\sum_{j\in J_{\mathbf{x}}}(f_j(\mathbf{x})-f_j(\bar{\mathbf{x}}))\ge 1$. In this case, we have

$$\frac{\sum_{i\in I_{\mathbf{x}}}(f_i(\bar{\mathbf{x}})-f_i(\mathbf{x}))}{\sum_{j\in J_{\mathbf{x}}}(f_j(\mathbf{x})-f_j(\bar{\mathbf{x}}))}\le\sum_{i\in I_{\mathbf{x}}}(f_i(\bar{\mathbf{x}})-f_i(\mathbf{x}))\le\sum_{t=1}^{r}(f_t(\bar{\mathbf{x}})-y_t^U). \qquad (2.4.10)$$

Case 2: $\sum_{j\in J_{\mathbf{x}}}(f_j(\mathbf{x})-f_j(\bar{\mathbf{x}}))<1$. In this case, we have $f_j(\mathbf{x})<f_j(\bar{\mathbf{x}})+1$, $j\in J_{\mathbf{x}}$. Since g_j' is increasing, then

$$g_j'of_j(\mathbf{x})\le g_j'(f_j(\bar{\mathbf{x}})+1),\ \text{for all } j\in J_{\mathbf{x}}. \qquad (2.4.11)$$

The following relation is obtained by applying (2.4.7) and definition of utopia point:

$$\begin{aligned}
\sum_{i\in I_{\mathbf{x}}}(f_i(\bar{\mathbf{x}})-f_i(\mathbf{x}))&\le\sum_{i\in I_{\mathbf{x}}}\frac{f_i(\bar{\mathbf{x}})-y_i^U}{g_iof_i(\bar{\mathbf{x}})-g_i(y_i^U)}(g_iof_i(\bar{\mathbf{x}})-g_iof_i(\mathbf{x})) \qquad (2.4.12)\\
&\le\max_{t\in\{1,2,\dots,n\}}\frac{f_t(\bar{\mathbf{x}})-y_t^U}{g_tof_t(\bar{\mathbf{x}})-g_t(y_t^U)}\sum_{i\in I_{\mathbf{x}}}(g_iof_i(\bar{\mathbf{x}})-g_iof_i(\mathbf{x})).
\end{aligned}$$

By taking into account (2.4.4), (2.4.5) and (2.4.11), respectively, we have

$$\max_{t\in\{1,2,\ldots,n\}} \frac{f_t(\bar{\mathbf{x}})-y_t^U}{g_t o f_t(\bar{\mathbf{x}})-g_t(y_t^U)} \sum_{i\in I_{\mathbf{x}}}(g_i o f_i(\bar{\mathbf{x}})-g_i o f_i(\mathbf{x})) \tag{2.4.13}$$

$$\leq N \max_{t\in\{1,2,\ldots,n\}} \frac{f_t(\bar{\mathbf{x}})-y_t^U}{g_t o f_t(\bar{\mathbf{x}})-g_t(y_t^U)} \sum_{j\in J_{\mathbf{x}}}(g_j o f_j(\mathbf{x})-g_j o f_j(\bar{\mathbf{x}}))$$

$$\leq N \max_{t\in\{1,2,\ldots,n\}} \frac{f_t(\bar{\mathbf{x}})-y_t^U}{g_t o f_t(\bar{\mathbf{x}})-g_t(y_t^U)} \sum_{j\in J_{\mathbf{x}}} g_j' o f_j(\mathbf{x})(f_j(\mathbf{x})-f_j(\bar{\mathbf{x}}))$$

$$\leq N \max_{t\in\{1,2,\ldots,n\}} \frac{f_t(\bar{\mathbf{x}})-y_t^U}{g_t o f_t(\bar{\mathbf{x}})-g_t(y_t^U)} \sum_{j\in J_{\mathbf{x}}} g_j'(f_j(\bar{\mathbf{x}})+1)(f_j(\mathbf{x})-f_j(\bar{\mathbf{x}}))$$

$$\leq N \max_{t\in\{1,2,\ldots,n\}} \frac{f_t(\bar{\mathbf{x}})-y_t^U}{g_t o f_t(\bar{\mathbf{x}})-g_t(y_t^U)} \max_{t\in\{1,2,\ldots,n\}} g_t'(f_j(\bar{\mathbf{x}})+1) \sum_{j\in J_{\mathbf{x}}}(f_j(\mathbf{x})-f_j(\bar{\mathbf{x}}))$$

Now, Relation (2.4.3) can be achieved by combining (2.4.10), (2.4.12), (2.4.13) and letting

$$M := \max\{\sum_{t=1}^{r}(f_t(\bar{\mathbf{x}})-y_t^U), N \max_{t\in\{1,2,\ldots,n\}} \frac{f_t(\bar{\mathbf{x}})-y_t^U}{g_t o f_t(\bar{\mathbf{x}})-g_t(y_t^U)} \max_{t\in\{1,2,\ldots,n\}} g_t'(f_j(\bar{\mathbf{x}})+1)\}.$$

□

The following proposition shows another advantage of our transformation. It says that, the transformation always preserves the convexity of the original problem, therefore, we do not need to worry about losing the convexity of the given problem. The proof of part 1 can be found in [12], and the proof of part 2 is omitted due to being trivial.

Proposition 2.4.2. *Let g_i and g_i' be increasing functions.*
1. If the original problem is convex, then the transformed problem is convex as well.
2. If $f(\mathcal{X})$ is $\mathbb{R}^r_{\geqq}$-convex, then $gof(\mathcal{X})$ is also $\mathbb{R}^r_{\geqq}$-convex.

The following corollary which can be obtained from Theorems 1.2.1, 2.3.3 and 2.4.1 and Proposition 2.4.2, shows how to capture the efficient points of the original problem by solving the scalarized version of the transformed MOP.

Corollary 2.4.3. *Let Assumption A be satisfied.*

1. If $\lambda > 0$ *(*$\lambda \geq 0$*), then an optimal solution of Problem (2.4.14) is properly (weakly) efficient.*

2. If $f(\mathcal{X})$ *is* $\mathbb{R}^r_{\geqq}$*-convex and* $\bar{\mathbf{x}}$ *is properly (weakly) efficient, then there exists* $\lambda > 0$ *(*$\lambda \geq 0$*) for which* $\bar{\mathbf{x}}$ *solves Problem (2.4.14) below:*

$$\min_{\mathbf{x}\in\mathcal{X}} < \lambda, gof(\mathbf{x}) > . \tag{2.4.14}$$

3. A feasible point $\bar{\mathbf{x}}$ *is properly efficient if and only if there exist* $\lambda > 0$ *and* $\rho > 0$*, such that* $\bar{\mathbf{x}}$ *solves*

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{i=1,\ldots,r} \lambda_i(g_i o f_i(\mathbf{x}) - g_i(y_i^U)) + \rho \sum_{i=1}^{r} (g_i o f_i(\mathbf{x}) - g_i(y_i^U)).$$

In the next subsection, some potential applications of the proposed transformation technique are presented.

2.4.1 Some Applications

Transformation technique has several applications besides improving the homogeneity of the weighted solutions. Here, we point out these applications as outlined below:

1. Example 2.4.3 depicts how we can convexify a given MOP by an appropriate transformation, and Corollary 2.4.7 provides a general scheme to convexify MOPs with the aid of p-power and exponential functions.

2. Simplifying some well-known MOPs in radiotherapy optimization by the proper transformations, in order to improve the computational efficiency, is presented in Examples 2.4.6.

3. Corollary 2.4.8 shows that even the special case of the transformation could strengthen and generalize the existing results for characterizing the properly efficient points in terms of the weighted solutions.

Transformation of MOPs is useful if the transformed MOP happens to be more efficiently solvable than the original MOP. Perhaps, more importantly, if the original problem is not convex and we could make it convex, then we can find the optimal solutions by solving an equivalent convex problem.

Proposition 2.4.2 guarantees preserving the convexity under the transformation. However, if the original problem is non-convex, the transformed problem could be either convex or non-convex. In this case, convexifying the original problem is important and challenging issue.

One approach to convexify the given problem is to find a possibly available transformation function based on the special structure of the problem. The following example clarifies this point by a simple bi-objective problem.

Example 2.4.3. *Consider the following non-linear and non-convex MOP:*

$$\begin{aligned} \min\{-\frac{x}{y}, -z\} \\ s.t. \\ y^2x^{-3} - z \geq 0 \\ x, y, z \geq 1. \end{aligned}$$

We can transform the above problem to the following linear problem with the aid of $(g_1(t), g_2(t)) := (-\ln(-t), -\ln(-t))$*, as a transformation function, and considering the fact that the inequality* $y^2x^{-3} \geq z$ *holds if and only if the inequality* $\ln(y^2x^{-3}) \geq \ln(z)$ *is satisfied.*

$$\begin{aligned} \min\{-\bar{x} + \bar{y}, -\bar{z}\} \\ s.t. \\ 2\bar{y} - 3\bar{x} \geq \bar{z} \\ \bar{x}, \bar{y}, \bar{z} \geq 0, \end{aligned}$$

where, $\bar{x} := \ln(x), \bar{y} := \ln(y), \bar{z} := \ln(z)$*. Since the transformation function satisfies assumption A, both MOPs are equivalent and have the same properly and Pareto efficient solutions. On the other hand, the transformed problem is linear and can be solved efficiently, while the original problem is non-convex and computationally* *intractable. Moreover, we can generate more homogeneous efficient solutions by the scalarized version of the transformed problem.*

Another approach for convexification has been introduced by Li (1996) [46] in order to convexify general MOPs by taking the p-power function of the objective functions. Subsequently, Li and Biswal (1998) [47] did the same by employing the exponential function. To prove their results, they utilized the following assumption on the efficient frontier function (here denoted by $\Phi(f_1, f_2, ..., f_{r-1}) = f_r$).

Assumption B. Φ is 2nd-order differentiable and each partial derivative $\frac{\partial f_r}{\partial f_i}$ is strictly negative, for all $\mathbf{x} \in \mathcal{X}_E$, and each 2nd-order derivative of Φ is bounded.

Theorem 2.4.4. *(Li, 1996): If Assumption B is satisfied and $f(\mathbf{x}) > 0$, for all feasible solutions, then the efficient frontier in the $(f_1^p, f_2^p, ..., f_r^p)(\mathcal{X})$ space is convex for large enough p.*

Theorem 2.4.5. *(Li and Biswal, 1998): If Assumption B is satisfied, then the efficient frontier in the $(exp(pf_1), exp(pf_2), ..., exp(pf_r))(\mathcal{X})$ space is convex for large enough p.*

These convexification results can be expressed in our transformation framework. It should be noted that these methods convexify the efficient frontier function, while in our framework we need to convexify the image space in order to characterize all properly efficient points. So, we need to make a connection between these contexts. Lemma 2.4.6 provides such a connection.

Definition 2.4.1. The set $f(\mathcal{X})$ is said to be externally stable if for any point $\mathbf{x} \in \mathcal{X}$ there exists an efficient solution $\bar{\mathbf{x}} \in \mathcal{X}_E$ such that $f(\bar{\mathbf{x}}) \leq f(\mathbf{x})$, i.e., $f(\mathcal{X}) \subset \mathcal{Y}_N + \mathbb{R}^r_{\geqq}$.

Lemma 2.4.6. *If $f(\mathcal{X})$ is externally stable and Φ is a convex function, then $f(\mathcal{X})$ is $\mathbb{R}^r_{\geqq}$-convex.*

Proof. Since Φ is a function that defines the efficient frontier we have

$$\mathcal{Y}_N = \{(y_1, y_2, ..., y_{r-1}, \Phi(y_1, y_2, ..., y_{r-1})) | (y_1, y_2, ..., y_{r-1}) \in dom(\Phi)\}.$$

Since $f(\mathcal{X})$ is externally stable, the following relation could easily be proved:

$$f(X) + \mathbb{R}^r_{\geqq} = \{(y_1, y_2, ..., y_{r-1}, \Phi(y_1, y_2, ..., y_{r-1}))|(y_1, y_2, ..., y_{r-1}) \in dom(\Phi)\} + \mathbb{R}^r_{\geqq}.$$

The proof is complete by considering the fact that the right hand side of the above relation is convex as long as Φ is a convex function. □

With the aid of the above lemma we can express Theorems 2.4.4 and 2.4.5 in our framework as follows.

Corollary 2.4.7. *Let Assumption B hold and $f(\mathcal{X})$ be externally stable. For large enough values of p, the original MOP can be convexified with the aid of $g_i(t) := (t - y_i^U)^p$ or $g_i(t) := exp(pt)$ ($i \in \{1, 2, ..., r\}$) as the transformation functions.*

The two following examples employ the above corollary. Example 2.4.4 utilizes the p-power as a transformation function, while the exponential transformation function is exploited in Example 2.4.5.

Example 2.4.4. *Consider the bi-objective optimization problem studied in [46],*

$$\begin{aligned} \min\{x, y\} \\ s.t. \\ x^2 + y^2 \geq 1 \\ 0 \leq x, y \leq 0.9 \end{aligned}$$

The shadow part in the left picture in Figure 2.5 demonstrates $f(\mathcal{X}) + \mathbb{R}^2_{\geqq}$ of the above problem which verifies its non-convexity. It can be shown that all the conditions of Corollary 2.4.7 hold, and therefore, we can apply t^2 as the transformation function. The right picture in Figure 2.5 confirms the convexity of the transformed problem.

Figure 2.5: The shadow parts depict $f(\mathcal{X}) + \mathbb{R}^2_{\geqq}$ in the original (left picture) and the transformed (right picture) MOPs.

Consider the bi-objective optimization problem studied in [47],

$$\min\{-xy, x+y\}$$
$$s.t.$$
$$1 \leq x, y \leq 10.$$

As the shadow part in the left picture in Figure 2.6 shows, this problem is non-convex. Since the conditions of Corollary 2.4.7 are satisfied, we can utilize $exp(0.25t)$ *as the transformation function. The right picture in Figure 2.6 verifies the convexity of the transformed problem.*

Figure 2.6: The shadow parts depict $f(\mathcal{X}) + \mathbb{R}^2_{\geqq}$ in the original (left picture) and the transformed (right picture) MOPs.

Another application of the transformation technique is improvement of the computational efficiency by simplification of the objective functions. Example 2.4.6 reveals how we can simplify some objective functions arisen in radiotherapy optimization. More details about these objective functions can be found in [60] and references therein.

The second column of Table 2.1 contains some well known objective functions in radiotherapy optimization. The names of these functions in the radiotherapy literature are provided in the first column. The third and the fourth columns include the transformation function and the transformed problem, α *and* k *are two parameters and there are also other parameters in the original versions of these objectives, which are excluded here because they do not affect our transformation. It can be verified that Assumption A is satisfied for all the transformation functions. Therefore, using these transformations, improvement in homogeneity of the optimal solutions is expected. On the other hand, to some extent, the transformed objectives are simpler than the original ones and so improvement in computational efficiency is expected as well. For a specific objective function, there might be several transformation functions satisfying Assumption A. In this case, one criterion that could be considered is choosing* *the transformation function for which more improvement in homogeneity is expected. For instance, look at the EUD case in Table 2.1. Although the function $e^{\alpha t}$ satisfies the desired assumptions, for all $\alpha > 0$, e^t is preferred as long as $\alpha \in (0, 1]$. It is because the function $h(t) = e^{\alpha t}$ can be expressed as a combination of two functions $h(t) = h_2 o h_1(t)$, where, $h_1(t) = e^t$ and $h_2(t) = t^\alpha$. Since the derivative of h_2 is decreasing, for $\alpha < 1$, and increasing, for $\alpha > 1$, h' has the faster (slower) increasing rate than h'_1, for $\alpha > 1$ ($\alpha < 1$). Therefore, $e^{\alpha t}$ (e^t) is expected to make more improvement, in the homogeneity sense, for $\alpha > 1$ ($\alpha < 1$). Generally, when there are more than one option for the transformation, the one whose derivative has the faster rate of change is preferred. The rate of change of derivative can also be examined by looking at the second derivative. In our example, the second derivative of $e^{\alpha t}$, is larger (smaller) than the second derivative of e^t for $\alpha > 1$ ($\alpha < 1$). Thus, $e^{\alpha t}$ is preferred, for $\alpha > 1$, while e^t is upgraded, for $\alpha < 1$.*

Table 2.1: Transformation of some well-known objective functions in radiotherapy optimization.

Now, we point out the last application of our transformation in this subsection. Corollary 2.4.8 can be obtained from parts 1 and 2 of Corollary 2.4.3 and through considering p-power as the transformation function, and taking into account the fact that the properly efficient points of MOP are not changed if we shift the points in the image space by $f(\mathcal{X}) \mapsto f(\mathcal{X}) - \mathbf{y}^U$.

Corollary 2.4.8. *1. If the weights are positive, then an optimal solution of weighted problem (2.3.1) is properly efficient.*
2. If $f(\mathcal{X})$ is $\mathbb{R}^r_{\geqq}$-convex and $\bar{\mathbf{x}}$ is properly efficient, then there exists $\lambda > 0$ for which $\bar{\mathbf{x}}$ is an optimal solution of the weighted problem (2.3.1).

The above corollary, which is a special case of our transformation result for p-power as the transformation function, can generalize and strengthen the existing results introduced in Theorems 2.3.1 and 2.3.2 for generating the properly efficient points by the weighted problem. In the following we compare the above corollary with the existing results for the weighted problem.

- Gearhart's theorem says that the weighted solutions are properly efficient for $p = 2$ and under the convexity assumption, while Corollary 2.4.8 states the same results, for all $p \geq 1$, and an arbitrary MOP.

- Gearhart proved that properly efficient points could be characterized in terms of the weighted solutions under the convexity assumption for $p = 2$, while Corollary 2.4.8 shows this fact for each $p \geq 1$.

- Sawaragi's theorem states that the weighted solutions are properly efficient, for each $p \geq 1$, and under the closedness assumption, while Corollary 2.4.8 expresses this fact without closedness assumption.

2.5 Conclusions

Here, another equivalent definition of proper efficiency has been presented. This definition is based on the generalized version of the trade-off notion. To some extent, this definition is easier and more flexible than the existing definitions. In particular, it is independent of the norm being used in its definition, making it convenient to develop new properties of properness. Looking for a possible extension of this definition for the general ordering cone provides good grounds for further research and interested researchers.

The notion of homogenous properly efficient solutions has been introduced. The original idea of homogeneity comes from radiotherapy optimization. To some extent, between two properly efficient solutions, the more homogeneous one is preferred because it has a better trade-off between its objectives. Moreover, it has been clarified that if the objective function imposes more penalty for the larger values than the smaller values, then generating more homogenous properly efficient solutions is expected by the weighted scalarization method.

Using the new definition of properness, we have managed to prove a transformation technique to transform a multiobjective problem to a more convenient problem. Some conditions have been determined under which the original and the transformed problems have the same Pareto and properly efficient solutions. This transformation has some benefits. Firstly, the weighted version of the transformed problem is expected to generate more homogenous properly efficient solutions. Secondly, this transformation can be employed for purposes of convexification and simplification in order to improve the computational efficiency for solving the given problem. Finally, some existing results in the multiobjective optimization literature are generalized and strengthened using the special case of the proposed transformation scheme.

To date, we know that between two transformation functions, the function with the larger second derivative is preferred because generating more homogenous properly efficient solutions is expected in this case. A research topic that needs to be pursued in future is developing an analytic relation between the homogeneity of the solutions and the second derivative of the transformation function.

Chapter 3

Lexicographic Multiple Objective Optimization Problems

3.1 Introduction

Lexicographic (preemptive priority) multiobjective optimization programs (LMOPs) arise naturally when conflicting objectives exist in a decision problem, but for reasons outside the control of the decision maker, the objectives have to be considered in a hierarchical manner. We mention some applications of the LMOP here. One of the earliest examples is the two-phase method of linear programming. In the setting of linear LMOP to the general linear programming problem, two objective functions can be associated. The objective function with the highest priority is the sum of the artificial variables which has to be minimized, whereas the actual objective function has a lower priority; its optimization can be achieved only to the extent that the minimal value of the first objective function is not affected. Another example of LMOP is the determination of the critical path in a PERT which has two objective functions, the first representing the mean and the second representing the variance of the completion time. Hernández-Lerma and Hoyos-Reyes (2001) [35] introduced a lexicographic multiobjective control formulation of the priority assignment problem for a discrete-time single-server queueing system with q competing classes of customers, and a discounted cost criterion. Weber et al. (2002) [70] described the optimization of water resources planning for Lake Verbano (Lago Maggiore) in northern Italy. The goal was to determine an optimal policy for the management of the water supply over some planning horizon. The objectives were to maximize flood protection, minimize supply shortage for irrigation, and maximize electricity generation. This order of objectives is prescribed by law, so that the problem indeed has a lexicographic nature. Nijkamp (1980) [55] described an application of LMOP to a land-use problem for industrial activities in a newly created industrial area in the Rhine-delta region near Rotterdam (the so-called Meuse flat). In some cases, lexicographic optimization can be used as a technique for resolution of some difficult scalar valued optimization problems (Turnovec, 1985 [67]). Several mathematical and game-theoretic applications of nonlinear LMOP are reported in Behringer (1972) [3]. Furthermore, under simple requirements of convexity of the objectives, every Pareto point of a MOP is the lexicographic minimum of some prioritized sequence of weighted averages of the original objectives under consideration. Therefore, all Pareto optima, in general, can be satisfied by the basic lexicographic optimum characterization (Rentmeesters, 1998 [59]). To see other applications of LMOPs, refer to (Bhushan and Rengaswamy, 2004 [5]; Desaulniers, 2007 [17]; Erdoğan et al., 2010 [21]; Gascon et al., 2000 [25]; Isermann, 1982 [39]; Padhiyar and Bhartiya, 2009 [56] and Sun et al., 1999 [66]).

Generally, an LMOP is shown as follows:

$$lexmin_{\mathbf{x}\in\mathcal{X}}(f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_r(\mathbf{x})). \tag{3.1.1}$$

In order to solve (3.1.1), one typically adopts the following procedure known as the sequential method (Ignizio, 1976 [36]). First, one minimizes $f_1(\mathbf{x})$ subject to $\mathbf{x} \in \mathcal{X}$, and determines an optimal solution $\mathbf{x}^*$ with, say, $f_1(\mathbf{x}^*) = \beta_1$. Next, one solves the problem of minimizing $f_2(\mathbf{x})$ subject to $f_1(\mathbf{x}) \leq \beta_1$ and $\mathbf{x} \in \mathcal{X}$, and so on. In general, at the qth iteration, one solves

$$\min_{\mathbf{x}\in\mathcal{X}}\{f_q(\mathbf{x}) : f_i(\mathbf{x}) \leq \beta_i, i = 1, ..., q-1\}. \tag{3.1.2}$$

If either (3.1.2) has a unique optimum or q=r, then the optimal solution to (3.1.2) is a preemptive optimum. Otherwise, one proceeds to iteration *q+1*.

The second approach was proposed by Isermann (1982) [39] to solve the linear LMOPs based on the primal simplex algorithm. However, the proof of the correctness of the algorithm is rather involved, but the implementation is easy. In this method, one objective row is considered for each objective function. The lexicographic primal simplex method differs from the ordinary simplex method by the selection rule of the pivot column (see [39], for more discussion).

The third approach is the so-called transformation or weighted method. It could be considered as the special case of the weighted sum method for the general MOPs with the special weights structure. Section 3.2 presents the transformation method.

3.2 Transformation Method to Solve LMOPs

In this method, it is shown that if M is a big enough value, then the set of the optimal solutions of problem (3.1.1) is identical to the set of the optimal solutions of the following problem (denoted by $\mathcal{X}_E^M$):

$$P(M): \quad \min_{\mathbf{x}\in\mathcal{X}} \sum_{i=1}^{r} M^{r-i} f_i(\mathbf{x}). \tag{3.2.1}$$

The following theorem was introduced by Sherali and Soyster (1983).

Theorem 3.2.1. *If problem (3.1.1) is not unbounded and f_k, for $k = 1, ..., r$, are linear functions and $\mathcal{X}$ is as defined by (3.2.2) ($\mathcal{X}$ is a polyhedral set), or f_k, for $k = 1, ..., r$, are arbitrary functions and $\mathcal{X}$ is a finite discrete feasible region, then there exists an $\overline{M} \geq 0$ such that for each $M \geq \overline{M}$ the set of preemptive optimal solutions is precisely equal to the set of optimal solutions of $P(M)$, i.e., $\overline{\mathcal{X}}_E = \mathcal{X}_E^M$:*

$$\mathcal{X} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}. \tag{3.2.2}$$

Many authors employed this approach to solve their lexicographic optimization problems; see for instance (Bhushan and Rengaswamy, 2004 [5]; Calvete and Mateo, 1998 [13]; Desaulniers, 2007 [17]; Erdoğan et al., 2010 [21]; Pourkarimi and Zarepisheh, 2007 [57]; Khorram et al., 2010 [43] and Vada, 2001 [68]). This method is considered only to solve linear LMOPs and LMOPs with a finite discrete feasible region, and in Section 3.3 we extend this scheme to the nonlinear case.

The importance of this method is twofold. Mathematically, it asserts that the nonpreemptive approach subsumes the preemptive approach, and it allows one to investigate in some depth the properties of LMOPs. Practically, it permits the use of conventional, readily available, mathematical programming software. Moreover, in this method, we need to minimize one objective function over the original feasible region, while in the sequential method we usually solve r optimization problems, and in each iteration one constraint is added to the feasible region (especially in the nonlinear case, there is no easy way of determining whether or not (3.1.2) has a unique optimum and we should proceed until $q{=}r$). Calvete and Mateo (1998) [13] developed a network-based algorithm for lexicographic optimization of generalized network flow problems based on the optimality conditions proved on the basis of the above mentioned transformation. Their algorithm is also independent of M, and there is no need to compute M. Gascon et al. (2000) [25] utilized a mixture of the weighted method and the sequential method to prevent the numerical instability that the bigness of M may create. In general, the shortcoming of this transforation method is that it is not clear, in advance, how big M should be; however, Sherali (1982) [62] and Ignizio and Thomas (1984) [37] proposed some computational methods for computing M in special practical cases. Another technique for solving this problem, employed by Desaulniers (2007) [17], is to solve (3.2.1) for a small value of M and increase the value of M if it leads to a lexicographic improvement in the optimal solutions. Another solution is to solve (3.2.1) for a sequence of M, and choose the optimal solution being repeated for a sequence of big M. These methods can be practical especially for large scale LMOPs with many objective functions, while using the sequential method is impractical, because r large scale optimization problems should be solved in sequential method and adding one constraint in each step makes the original problem grow.

The rest of this chapter unfolds as follows: Section 3.3 contains the transformation of nonlinear LMOPs. In Section 3.4, by taking advantage of the transformation method, an algorithm based on the dual simplex method is presented for solving linear LMOPs. Next, in Section 3.5, by providing an efficient algorithm employing the preceding computations, it is shown how we can solve the linear LMOP if the priority of the objective functions is changed. In fact, the proposed algorithm is a kind of sensitivity analysis on the priority of the objective functions in linear LMOPs. Section 3.5 includes some conclusions and points out some future research.

3.3 Transformation of Nonlinear LMOPs

In this section, by some counter-examples, it is shown that Problems (3.1.1) and (3.2.1) are not equivalent, in general. Then, some conditions are determined under which Theorem 3.2.1 holds.

Theorem 3.4.1 in the next section will show that the linear LMOP is unbounded if and only if its transformed problem is unbounded for all big enough values of M. However, the next two examples illustrate that this fact is not necessarily true for the nonlinear case. Example 3.3.1 shows that problem (3.1.1) can be bounded, although problem $P(M)$ is unbounded, for each $M \geq 0$.

Example 3.3.1. *Assume* $f_1(x) = x,\ f_2(x) = -x^2$ *and* $\mathcal{X} = [0, \infty)$*. Then, the unique preemptive optimal solution of this problem is* $x = 0$*. Now, consider the following problem P(M):*

$$\min_{x \in [0,\infty)} Mx - x^2.$$

This problem is unbounded, for each $M \geq 0$*.*

Example 3.3.2 indicates that problem (3.1.1) can be unbounded, although problem $P(M)$ is bounded, for each $M \geq 0$.

Example 3.3.2. *Consider* $f_1(x) = -x,\ f_2(x) = x^2$ *and* $\mathcal{X} = [0, \infty)$. *In this case, the lexicographic multiobjective problem is unbounded, while problem P(M) has an optimal solution* $x = \frac{M}{2}$ *with the optimal objective value* $-\frac{M^2}{4}$.

The following example shows that Theorem 3.2.1 does not hold in general.

Example 3.3.3. *Let* $f_1(x) = x^2,\ f_2(x) = -x,\ f_3(x) = x$ *and* $\mathcal{X} = [0, \infty)$. *For this problem, it can be readily shown that* $\overline{\mathcal{X}}_E = \{0\}$ *and* $\mathcal{X}_E^M = \{\frac{M-1}{2M^2}\}$, *for each* $M \geq 0$. *So, neither* $\overline{\mathcal{X}}_E \subset \mathcal{X}_E^M$ *nor* $\mathcal{X}_E^M \subset \overline{\mathcal{X}}_E$ *is correct.*

Hereafter, we seek the conditions under which Theorem 3.2.1 holds for nonlinear LMOPs.

The proofs of Lemmas 3.3.1 and 3.3.2 are omitted due to being easy.

Lemma 3.3.1. *If* $\bar{\mathbf{x}}$ *is an optimal solution for problem (3.1.1), then* $\bar{\mathbf{x}}$ *is an optimal solution for the following problem:*

$$lexmin_{\mathbf{x} \in \mathcal{X}} (c_1 f_1(\mathbf{x}), c_2 f_2(\mathbf{x}), ..., c_p f_p(\mathbf{x})),$$

where, $p \in \{2, ..., r\}$ *and* $c_i > 0$.

Definition 3.3.1. A feasible solution $\hat{\mathbf{x}} \in \mathcal{X}$ is called strongly properly efficient for the MOP if it is properly efficient for problem (3.3.1), for each $p \in \{2, ..., r\}$. The set of all strongly properly efficient points is denoted by $\mathcal{X}_{SPE}$:

$$\min_{\mathbf{x} \in \mathcal{X}} (f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_p(\mathbf{x})). \tag{3.3.1}$$

Lemma 3.3.2. *If* $\mathbf{x}^*$ *is strongly properly efficient for the MOP, then* $\mathbf{x}^*$ *is strongly properly efficient for the following problem:*

$$\min_{\mathbf{x}\in\mathcal{X}} (c_1 f_1(\mathbf{x}), c_2 f_2(\mathbf{x}), ..., c_p f_p(\mathbf{x})),$$

where, $p \in \{2, ..., r\}$ *and* $c_i > 0$.

Lemma 3.3.3. *If* $\mathcal{X}$ *is a convex set and* f_k *(for k=1,...,r) are convex functions over* $\mathcal{X}$ *and* $\hat{\mathbf{x}}$ *is a strongly properly efficient solution and preemptive optimal solution, then there exist some* $\overline{M}_i \geq 0$ *(for i=1,...,r) such that for each* $M_i \geq \overline{M}_i$ *(i=1,...,r), the following inequality holds:*

$$\prod_{i=1}^{r-1} M_i(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) + \prod_{i=2}^{r-1} M_i(f_2(\mathbf{x}) - f_2(\hat{\mathbf{x}})) + \cdots + (f_r(\mathbf{x}) - f_r(\hat{\mathbf{x}})) \geq 0. \quad (3.3.2)$$

Proof. The proof is by induction on r, the number of objective functions. Assume that $r = 2$. Since $\hat{\mathbf{x}}$ is properly efficient, considering Theorem 1.2.2, there exist $\lambda_1,\ \lambda_2 > 0$ such that

$$\lambda_1 f_1(\hat{\mathbf{x}}) + \lambda_2 f_2(\hat{\mathbf{x}}) \leq \lambda_1 f_1(\mathbf{x}) + \lambda_2 f_2(\mathbf{x}),$$

for each $\mathbf{x} \in \mathcal{X}$. If we define $\overline{\lambda}_1 = \frac{\lambda_1}{\lambda_2}$, then for each $\mathbf{x} \in \mathcal{X}$,

$$\overline{\lambda}_1(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) + (f_2(\mathbf{x}) - f_2(\hat{\mathbf{x}})) \geq 0.$$

Since $\hat{\mathbf{x}}$ is a preemptive optimal solution, then $(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) \geq 0$, and the above inequality holds if $\overline{\lambda}_1$ is replaced by any greater real number. Thus, inequality (3.3.2) holds if we define $\overline{M}_1 = \overline{\lambda}_1$.

Now, assume the lemma to be true for $r = k$ and consider $r = k + 1$. According to the assumption of the lemma, $\hat{\mathbf{x}}$ is strongly properly efficient and preemptive optimal for $(f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_{k+1}(\mathbf{x}))$. Regarding Lemmas 3.3.1 and 3.3.2, $\hat{\mathbf{x}}$ is strongly properly efficient and preemptive optimal for $(f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_k(\mathbf{x}))$. Now, by the induction hypothesis, there exist $\overline{M}_i \geq 0$ ($i = 1$,..., k) such that for each $M_i \geq \overline{M}_i$ $(i = 1, ..., k)$ and $\mathbf{x} \in \mathcal{X}$

$$\prod_{i=1}^{k-1} M_i(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) + \prod_{i=2}^{k-1} M_i(f_2(\mathbf{x}) - f_2(\hat{\mathbf{x}})) + \cdots + (f_k(\mathbf{x}) - f_k(\hat{\mathbf{x}})) \geq 0.$$

In other words, for each $\mathbf{x} \in \mathcal{X}$ and $\delta_i \geq 0$ $(i = 1, 2, ..., k)$

$$\prod_{i=1}^{k-1}(\overline{M}_i + \delta_i)(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) + \prod_{i=2}^{k-1}(\overline{M}_i + \delta_i)(f_2(\mathbf{x}) - f_2(\hat{\mathbf{x}})) + \cdots + (f_k(\mathbf{x}) - f_k(\hat{\mathbf{x}})) \geq 0. \tag{3.3.3}$$

Since $\hat{\mathbf{x}}$ is a properly efficient solution for $(f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_{k+1}(\mathbf{x}))$, by Theorem 1.2.2 there exist $\lambda_i > 0$ $(i = 1, 2, ..., k + 1)$ such that for each $\mathbf{x} \in \mathcal{X}$

$$\lambda_1(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) + \lambda_2(f_2(\mathbf{x}) - f_2(\hat{\mathbf{x}})) + \cdots + \lambda_{k+1}(f_{k+1}(\mathbf{x}) - f_{k+1}(\hat{\mathbf{x}})) \geq 0.$$

If we define $\overline{\lambda}_i = \frac{\lambda_i}{\lambda_{k+1}}$ $(i = 1, ..., k)$ then for each $\mathbf{x} \in \mathcal{X}$

$$\overline{\lambda}_1(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) + \cdots + \overline{\lambda}_k(f_k(\mathbf{x}) - f_k(\hat{\mathbf{x}})) + (f_{k+1}(\mathbf{x}) - f_{k+1}(\hat{\mathbf{x}})) \geq 0. \tag{3.3.4}$$

If we multiply (3.3.3) by an arbitrary value $\delta_k \geq 0$ and add it to (3.3.4), then the following inequality is obtained:

$$(\prod_{i=1}^{k-1}(\overline{M}_i + \delta_i)\delta_k + \overline{\lambda}_1)(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) + (\prod_{i=2}^{k-1}(\overline{M}_i + \delta_i)\delta_k + \overline{\lambda}_2)(f_2(\mathbf{x}) - f_2(\hat{\mathbf{x}})) + \cdots + (\delta_k + \overline{\lambda}_k)(f_k(\mathbf{x}) - f_k(\hat{\mathbf{x}})) + (f_{k+1}(\mathbf{x}) - f_{k+1}(\hat{\mathbf{x}})) \geq 0, \tag{3.3.5}$$

for each $\mathbf{x} \in \mathcal{X}$ and $\delta_i \geq 0$, $(i = 1, ..., k)$. We define functions g_t, (t=1,...,k), as follows:

$g_t(\delta_t, \delta_{t+1}, ..., \delta_k) = \prod_{i=t}^{k-1}(\overline{M}_i + \delta_i)\delta_k + \overline{\lambda}_t, \quad \text{for } t \in \{1, ..., k-1\},$

$g_k(\delta_k) = \delta_k + \overline{\lambda}_k.$

The functions g_t, (t=1,...,k-1), can be defined by the following recursive relation:

$$g_t(\delta_t, \delta_{t+1}, ..., \delta_k) = (\overline{M}_t + \delta_t)(g_{t+1}(\delta_{t+1}, ..., \delta_k) - \overline{\lambda}_{t+1}) + \overline{\lambda}_t. \tag{3.3.6}$$

Relation (3.3.5) can be rewritten by the functions g_t as follows:

$$\begin{aligned} g_1(\delta_1, ..., \delta_k)(f_1(\mathbf{x}) - f_1(\hat{\mathbf{x}})) + g_2(\delta_2, ..., \delta_k)(f_2(\mathbf{x}) - f_2(\hat{\mathbf{x}})) + \cdots + \\ g_k(\delta_k)(f_k(\mathbf{x}) - f_k(\hat{\mathbf{x}})) + (f_{k+1}(\mathbf{x}) - f_{k+1}(\hat{\mathbf{x}})) \geq 0. \end{aligned}$$

Now, if we show that there exist $\widehat{M}_t \geq 0$ for (t=1,...,k) such that for each $M_t \geq \widehat{M}_t$, (t=1,...,k), there exist $\delta_t^M \geq 0$, (t=1,...,k), such that $g_t(\delta_t^M, \delta_{t+1}^M, ..., \delta_k^M) = \prod_{i=t}^{k} M_i$, (t=1,...,k), then the proof is complete. In other words, we want to show:

$$\begin{gathered} \exists \widehat{M}_t \geq 0, \ (t = 1, ..., k), \text{ such that } \forall M_t \geq \widehat{M}_t, \ (t = 1, ..., k), \\ \exists \ \delta_t^M \geq 0, \ (t = 1, ..., k) \text{ such that } \ g_t(\delta_t^M, \delta_{t+1}^M, ..., \delta_k^M) = \prod_{i=t}^{k} M_i. \end{gathered}$$

Let $\widehat{M}_k = \overline{\lambda}_k$ and $\widehat{M}_t$, $(t = 1, ..., k-1)$, be defined by the following recursive relation.

$$\widehat{M}_t = \frac{\overline{\lambda}_t}{\widehat{M}_{t+1} \cdots \widehat{M}_k} + \overline{M}_t. \tag{3.3.7}$$

Assume that $M_t \geq \widehat{M}_t$, $(t = 1, ..., k)$, is determined. If we let $\delta_k^M = M_k - \overline{\lambda}_k$, then $g_k(\delta_k^M) = M_k$. Now, assume that $(\delta_k^M, \delta_{k+1}^M, ..., \delta_{p-1}^M)$ are determined such that

$$g_t(\delta_t^M, \delta_{t+1}^M, ..., \delta_k^M) = \prod_{i=t}^{k} M_i, \tag{3.3.8}$$

for $(t = p-1, p, ..., k)$. Then, δ_p^M is determined as follows.

The function $g_p(\delta_p, \delta_{p+1}^M, ..., \delta_k^M)$ is a linear and increasing function with respect to δ_p and its range is $[g_p(0, \delta_{p+1}^M, ..., \delta_k^M), \infty)$. Now, regarding (3.3.6), (3.3.7) and (3.3.8), we have:

$$\begin{aligned} g_p(0, \delta_{p+1}^M, ..., \delta_k^M) &= \overline{M}_p(g_{p+1}(\delta_{p+1}^M, ..., \delta_k^M) - \overline{\lambda}_{p+1}) + \overline{\lambda}_p \\ &= \overline{M}_p(\prod_{i=p+1}^{k} M_i - \overline{\lambda}_{p+1}) + \overline{\lambda}_p \\ &\leq \overline{M}_p \prod_{i=p+1}^{k} M_i + \overline{\lambda}_p \\ &\leq \prod_{i=p}^{k} M_i. \end{aligned}$$

So, $\prod_{i=p}^{k} M_i$ is in the range of the function $g_p(\delta_p, \delta_{p+1}^M, ..., \delta_k^M)$ and there is $\delta_p^M \geq 0$ such that $g_p(\delta_p^M, \delta_{p+1}^M, ..., \delta_k^M) = \prod_{i=p}^{k} M_i$. This completes the proof. □

Theorem 3.3.4. *If $\mathcal{X}$ is a convex set and f_k (for k=1,...,r) are convex functions over $\mathcal{X}$ and $\overline{\mathcal{X}}_E \cap \mathcal{X}_{SPE} \neq \emptyset$, then there exists an $\overline{M} \geq 0$ such that for each $M \geq \overline{M}$ every preemptive optimal solution is an optimal solution for P(M); i.e., $\overline{\mathcal{X}}_E \subset \mathcal{X}_E^M$.*

Proof. Let $\mathbf{x}^* \in \overline{\mathcal{X}}_E \cap \mathcal{X}_{SPE}$. Regarding Lemma 3.3.3, there exist $\overline{M}_i \geq 0$ $(i = 1, ..., r)$ such that for each $M_i \geq \overline{M}_i$ $(i = 1, ..., r)$ inequality (3.3.2) holds. Now, if we let $\overline{M} = \max_{i=1,...,r}\{\overline{M}_i\}$ and consider all M_i $(i = 1, ..., r)$ equal to M, then inequality (3.3.2) is transformed as follows:

$$M^{r-1}(f_1(\mathbf{x}) - f_1(\mathbf{x}^*)) + M^{r-2}(f_2(\mathbf{x}) - f_2(\mathbf{x}^*)) + \cdots + (f_r(\mathbf{x}) - f_r(\mathbf{x}^*)) \geq 0, \quad (3.3.9)$$

for each $\mathbf{x} \in \mathcal{X}$ and $M \geq \overline{M}$. This means that $\mathbf{x}^*$ is an optimal solution for $P(M)$, for each $M \geq \overline{M}$. Now, assume that $\bar{\mathbf{x}} \in \overline{\mathcal{X}}_E$. Since $\mathbf{x}^*$ and $\bar{\mathbf{x}}$ are preemptive optimal solutions, $f(\mathbf{x}^*) = f(\bar{\mathbf{x}})$. So, inequality (3.3.9) holds if $\mathbf{x}^*$ is replaced by any member of $\overline{\mathcal{X}}_E$. Thus,

$$\overline{\mathcal{X}}_E \subset \mathcal{X}_E^M, \quad \text{for each} \quad M \geq \overline{M}.$$

$\square$

Theorem 3.3.5. *Under the assumptions of Theorem 3.3.4, there exists an $\overline{M} \geq 0$ such that for each $M \geq \overline{M}$ every optimal solution of $P(M)$ is a preemptive optimal solution; i.e., $\mathcal{X}_E^M \subset \overline{\mathcal{X}}_E$.*

Proof. The proof is by induction on r, the number of objective functions. First, consider $r = 2$. According to Theorem 3.3.4,

$$\exists \widehat{M} \geq 0 \;\; s.t. \;\; \forall M \geq \widehat{M}, \quad \overline{\mathcal{X}}_E \subset \mathcal{X}_E^M. \tag{3.3.10}$$

Let $\overline{M} = \widehat{M} + 1$; we want to prove that for each $M \geq \overline{M}$, $\mathcal{X}_E^M \subset \overline{\mathcal{X}}_E$. Let $M \geq \overline{M}$ be an arbitrary real number and $\hat{\mathbf{x}} \in \mathcal{X}_E^M$ and $\mathbf{x}^* \in \overline{\mathcal{X}}_E \cap \mathcal{X}_{SPE}$ be arbitrary points. Regarding (3.3.10), $\mathbf{x}^* \in \mathcal{X}_E^M$, and so

$$M(f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*)) + (f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*)) = 0. \tag{3.3.11}$$

If $\hat{\mathbf{x}} \in \overline{\mathcal{X}}_E$, then the proof is complete for $r = 2$. By contradiction assume that $\hat{\mathbf{x}} \notin \overline{\mathcal{X}}_E$. Then,

$$(f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*), f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*)) \succ_l \mathbf{0}.$$

The following two cases may occur:

(1) $f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*) \geq 0 \quad$ and $\quad f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*) > 0$, which is in contradiction with (3.3.11).

(2) $f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*) > 0$ and $f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*) < 0$. In this case, regarding (3.3.11), we have

$$(M - \frac{1}{2})(f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*)) + (f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*)) < 0.$$

This means that $\mathbf{x}^* \notin \mathcal{X}_E(M - \frac{1}{2})$. Since $M - \frac{1}{2} > \widehat{M}$, this is in contradiction with (3.3.10).

So, the assumption is not correct and $\hat{\mathbf{x}} \in \overline{\mathcal{X}}_E$.

Now, assume the theorem to be true for $r = k$ and consider $r = k+1$. This means that we assume

$$\overline{\mathcal{X}}_E \cap \mathcal{X}_{SPE} \neq \emptyset \text{ for } (f_1, f_2, ..., f_{k+1}), \tag{3.3.12}$$

and we want to show that

$$\exists\ \overline{M} \geq 0\ s.t.\ \ \forall M \geq \overline{M},\quad \mathcal{X}_E^M \subset \overline{\mathcal{X}}_E \ \text{ for } (f_1, f_2, ..., f_{k+1}). \tag{3.3.13}$$

Regarding (3.3.12) and Lemmas 3.3.1, 3.3.2, we have

$$\overline{\mathcal{X}}_E \cap \mathcal{X}_{SPE} \neq \emptyset \ \text{ for }\ (kf_1, (k-1)f_2, ..., f_k). \tag{3.3.14}$$

So, by the induction hypothesis,

$$\exists \overline{M}_1 \geq 0\ \ s.t.\ \ \forall M \geq \overline{M}_1,\quad \mathcal{X}_E^M \subset \overline{\mathcal{X}}_E,\ \text{ for }\ (kf_1, (k-1)f_2, ..., f_k). \tag{3.3.15}$$

By (3.3.14) and Theorem 3.3.4

$$\exists \overline{M}_2 \geq 0\ \ s.t.\ \ \forall M \geq \overline{M}_2,\quad \overline{\mathcal{X}}_E \subset \mathcal{X}_E^M \text{ for }\ (kf_1, (k-1)f_2, ..., f_k). \tag{3.3.16}$$

Regarding (3.3.12) and Theorem 3.3.4

$$\exists \overline{M}_3 \geq 0\ \ s.t.\ \ \forall M \geq \overline{M}_3\ :\quad \overline{\mathcal{X}}_E \subset \mathcal{X}_E^M \text{ for }\ (f_1, f_2, ..., f_{k+1}). \tag{3.3.17}$$

Now, let $\overline{M} = \max\{\overline{M}_1, \overline{M}_2, \overline{M}_3\} + 1$. We want to prove that (3.3.13) holds. Assume that $M \geq \overline{M}$ is an arbitrary number and $\hat{\mathbf{x}} \in \mathcal{X}_E^M$, $\mathbf{x}^* \in \overline{\mathcal{X}}_E$, for $(f_1, f_2, ..., f_{k+1})$ are two arbitrary points. If we show that $f(\hat{\mathbf{x}}) = f(\mathbf{x}^*)$, then the proof is complete. Regarding (3.3.17), $\mathbf{x}^* \in \mathcal{X}_E^M$, for $(f_1, f_2, ..., f_{k+1})$. Since $\hat{\mathbf{x}} \in \mathcal{X}_E^M$ for, $(f_1, f_2, ..., f_{k+1})$, then

$$M^k(f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*)) + M^{k-1}(f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*)) + \cdots + (f_{k+1}(\hat{\mathbf{x}}) - f_{k+1}(\mathbf{x}^*)) = 0 \quad (3.3.18)$$

Define the function $h(\delta)$ as follows:

$$h(\delta) = (M+\delta)^k(f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*)) + (M+\delta)^{k-1}(f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*)) + \cdots + (f_{k+1}(\hat{\mathbf{x}}) - f_{k+1}(\mathbf{x}^*)). \quad (3.3.19)$$

Hence, $h(0) = 0$ by (3.3.18). Thus, $h(\delta)$ can be rewritten as follows:

$$\begin{aligned} h(\delta) = \sum_{i=1}^{k} \begin{pmatrix} k \\ i \end{pmatrix} M^{k-i}\delta^i(f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*)) + \sum_{i=1}^{k-1} \begin{pmatrix} k-1 \\ i \end{pmatrix} M^{k-1-i}\delta^i(f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*)) + \\ \cdots + \delta(f_k(\hat{\mathbf{x}}) - f_k(\mathbf{x}^*)) = \delta[kM^{k-1}(f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*)) + (k-1)M^{k-2}(f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*)) + \\ \cdots + (f_k(\hat{\mathbf{x}}) - f_k(\mathbf{x}^*)) + \delta q(\delta, M)] \end{aligned}$$

Now, consider

$$C = kM^{k-1}(f_1(\hat{\mathbf{x}}) - f_1(\mathbf{x}^*)) + (k-1)M^{k-2}(f_2(\hat{\mathbf{x}}) - f_2(\mathbf{x}^*)) + \cdots + (f_k(\hat{\mathbf{x}}) - f_k(\mathbf{x}^*)).$$

One of the following three cases may occur.

(1) $C > 0$: In this case it can be shown that there exists a $0 < \delta < 1$ such that $h(-\delta) < 0$. Regarding (3.3.19), this means that $\mathbf{x}^* \notin \mathcal{X}_E(M - \delta)$, for $(f_1, f_2, ..., f_{k+1})$. Since $M - \delta > M_3$, this is a contradiction to (3.3.17).

(2) $C < 0$: In this case, there exists a small enough $\delta > 0$ such that $h(\delta) < 0$,

meaning that $\mathbf{x}^* \notin \mathcal{X}_E(M+\delta)$, which is a contradiction to (3.3.17).

(3) $C = 0$: Since $\mathbf{x}^* \in \overline{\mathcal{X}}_E$, for $(f_1, f_2, ..., f_{k+1})$, regarding Lemma 3.3.1, $\mathbf{x}^* \in \overline{\mathcal{X}}_E$, for $(kf_1, (k-1)f_2, ..., f_k)$. Regarding (3.3.16), $\mathbf{x}^* \in \mathcal{X}_E^M$, for $(kf_1, (k-1)f_2, ..., f_k)$. Hence, $\hat{\mathbf{x}} \in \mathcal{X}_E^M$, for $(kf_1, (k-1)f_2, ..., f_k)$ regarding $C = 0$. According to (3.3.15), $\hat{\mathbf{x}} \in \overline{\mathcal{X}}_E$, for $(kf_1, (k-1)f_2, ..., f_k)$. Since $\hat{\mathbf{x}},\ \mathbf{x}^* \in \overline{\mathcal{X}}_E$, for $(kf_1, (k-1)f_2, ..., f_k)$, then

$$f_i(\hat{\mathbf{x}}) = f_i(\mathbf{x}^*) \quad \text{for} \quad (i = 1, 2, ..., k). \tag{3.3.20}$$

Now, regarding (3.3.18) and (3.3.20), we can conclude that $f_{k+1}(\hat{\mathbf{x}}) = f_{k+1}(\mathbf{x}^*)$. This means that $f(\hat{\mathbf{x}}) = f(\mathbf{x}^*)$, completing the proof. □

Example 3.3.3 shows that in Theorems 3.3.4 and 3.3.5, $\mathcal{X}_{SPE}$ can not be replaced by $\mathcal{X}_{PE}$. To see this, consider the unique preemptive optimal solution for this problem, $\hat{x} = 0$. This point is properly efficient (let $j = 3$ and $M = 1$ in Definition 2.2.1), but is not strongly properly efficient because it is not properly efficient, for the following problem

$$\min_{x \in [0,\infty)} (x^2, -x).$$

(In Definition 2.2.1, let $i = 2,\ j = 1,\ \{x_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty}$. Then, $\frac{f_i(\hat{x}) - f_i(x_n)}{f_j(x_n) - f_j(\hat{x})}$ is unbounded). So, the conditions of Theorems 3.3.4 and 3.3.5 are satisfied for this problem if $\mathcal{X}_{SPE}$ is replaced by $\mathcal{X}_{PE}$. But, as shown, neither $\overline{\mathcal{X}}_E \subset \mathcal{X}_E^M$ nor $\mathcal{X}_E^M \subset \overline{\mathcal{X}}_E$, for any $M \geq 0$, holds.

The following corollary which can be obtained directly from Theorems 3.3.4 and 3.3.5 shows that under the conditions of Theorem 3.3.4 the set of preemptive optimal solutions is precisely equal to the set of the optimal solutions of $P(M)$ for big enough values of M.

Corollary 3.3.6. *If $\mathcal{X}$ is a convex set and the f_k (for k=1,...,r) are convex functions over $\mathcal{X}$ and $\overline{\mathcal{X}}_E \cap \mathcal{X}_{SPE} \neq \emptyset$, then there exists an $\overline{M} \geq 0$ such that, for each $M \geq \overline{M}$, the set of preemptive optimal solutions is equal to the set of optimal solutions of $P(M)$; i.e., $\overline{\mathcal{X}}_E = \mathcal{X}_E^M$.*

Remark 3.3.1. Since the lexicographic order is a total order, all lexicographic optimal solutions have the same objective functions. Thus, either all lexicographic optimal solutions or non of them are strongly properly efficient. Hence, the condition $\overline{\mathcal{X}}_E \cap \mathcal{X}_{SPE} \neq \emptyset$ is equivalent to $\overline{\mathcal{X}}_E \subseteq \mathcal{X}_{SPE}$.

In the set of efficient solutions, a decision maker is usually interested in the solutions with finite tradeoffs among objective functions (properly efficient solutions). The condition $\overline{\mathcal{X}}_E \cap \mathcal{X}_{SPE} = \emptyset$ means that the preemptive optimal solutions have infinite tradeoffs between their objective functions which is not convenient from the decision maker's viewpoint. We just believe that this case happens rarely in practice, but it needs more empirical research.

Remark 3.3.2. For linear multiobjective programs every efficient solution is strongly properly efficient (Isermann 1974), and if $\mathcal{X}$ is a finite discrete feasible region then it can be easily shown that every efficient solution is strongly properly efficient. So, Theorem 3.2.1 can be obtained from Corollary 3.3.6.

The following theorem shows that, without any restriction, if $\mathbf{x}^*$ is an optimal solution of $P(M)$ for big enough values of M, then $\mathbf{x}^*$ is a preemptive optimal solution.

Theorem 3.3.7. *Let f_k (for k=1,...,r) be arbitrary objective functions and $\mathcal{X}$ be an arbitrary feasible region and $\mathbf{x}^* \in \mathcal{X}$. If there exists an $\overline{M} \geq 0$ such that $\mathbf{x}^* \in \mathcal{X}_E^M$, for each $M \geq \overline{M}$, then $\mathbf{x}^* \in \overline{\mathcal{X}}_E$.*

Proof. We argue by contradiction. Assume that $\mathbf{x}^* \notin \overline{\mathcal{X}}_E$. Since $\mathbf{x}^*$ is not a preemptive optimal solution, there is an $\mathbf{x} \in \mathcal{X}$ such that the first non-zero entry of $(f_1(\mathbf{x}) - f_1(\mathbf{x}^*), f_2(\mathbf{x}) - f_2(\mathbf{x}^*), ..., f_p(\mathbf{x}) - f_p(\mathbf{x}^*))$ is negative. Assume that $t \in \{1, 2, ..., r\}$ is the smallest index such that $(f_t(\mathbf{x}) - f_t(\mathbf{x}^*))$ is negative. It can be obviously shown that there exists a big enough $\widehat{M} \geq 0$ such that for each $M \geq \widehat{M}$,

$$M^{r-t}(f_t(\mathbf{x}) - f_t(\mathbf{x}^*)) + M^{r-t-1}(f_{t+1}(\mathbf{x}) - f_{t+1}(\mathbf{x}^*)) + \cdots + (f_r(\mathbf{x}) - f_r(\mathbf{x}^*)) < 0.$$

So, for each $M \geq \max\{\overline{M}, \widehat{M}\}$,

$$M^{r-1}f_1(\mathbf{x}) + M^{r-2}f_2(\mathbf{x}) + \cdots + f_r(\mathbf{x}) < M^{r-1}f_1(\mathbf{x}^*) + M^{r-2}f_2(\mathbf{x}^*) + \cdots + f_r(\mathbf{x}^*).$$

This means that $\mathbf{x}^* \notin \mathcal{X}_E^M$, which is a contradiction. □

From the practical point of view, the above theorem means that for an arbitrary LMOP if $\mathbf{x}^*$ is an optimal solution of $P(M)$ for a sequence of large values of M, then it is expected that $\mathbf{x}^*$ be a preemptive optimal solution.

3.4 Modified Simplex Algorithm for Solving Linear LMOPs

In this section, a modified simplex algorithm for solving linear lexicographic multi-objective optimization problems (LLMOPs) is introduced. The general LLMOP can be written as follows:

$$\operatorname{lex}\max_{\mathbf{w}\in\mathcal{W}}\{\mathbf{w}^t\mathbf{b}^1, \mathbf{w}^t\mathbf{b}^2, ..., \mathbf{w}^t\mathbf{b}^r\}, \tag{3.4.1}$$

where $\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^r | \ \mathbf{w}^t\mathbf{A} \leq \mathbf{c}, \ \mathbf{w}\}$, $\mathbf{A} \in \mathbb{R}^{r\times n}$ and $\mathbf{c}^T \in R^n$. Regarding Theorem 3.2.1, for big enough M, the set of the optimal solutions of problem (3.4.1) is identical to the set of the optimal solutions of the following problem:

$$\max\{M^{r-1}\mathbf{w}^t\mathbf{b}^1 + M^{r-2}\mathbf{w}^t\mathbf{b}^2 + \cdots + \mathbf{w}^t\mathbf{b}^r | \ \mathbf{w}^t\mathbf{A} \leq \mathbf{c}\} \tag{3.4.2}$$

The next theorem shows that for big enough M, LLMOP and its transformed version are equivalent in the unbounded case.

Theorem 3.4.1. *problem (3.4.1) is unbounded if and only if there exists an $\overline{M} > 0$ such that for each $M \geq \overline{M}$, problem (3.4.2) is unbounded.*

Proof. Regarding the unboundedness conditions for linear programs and Definition 1.3.3, it can be shown that problem (3.4.1) is unbounded if and only if there exists $d \in \mathbb{R}^r$ such that

$$d^t[b^1, b^2, ..., b^r] \succ_l 0 \tag{3.4.3}$$
$$d^t A \leq 0.$$

On the other hand, problem (3.4.2) is unbounded if and only if there exist $d \in \mathbb{R}^r$ and $\overline{M} > 0$ such that for each $M \geq \overline{M}$

$$M^{r-1}\mathbf{d}^t\mathbf{b}^1 + M^{r-2}\mathbf{d}^t\mathbf{b}^2 + \cdots + \mathbf{d}^t\mathbf{b}^r > 0 \tag{3.4.4}$$
$$d^t A \leq 0.$$

It can be easily shown that conditions (3.4.3) and (3.4.4) are equivalent. $\square$

According to Theorem 3.2.1, to find a preemptive optimal solution for problem (3.4.1), it is sufficient to find $\mathbf{w}^*$ such that it is an optimal solution for problem (3.4.2) for a big enough M. To this end, consider the dual of problem (3.4.2) as follows:

$$\min\{\mathbf{cx}|\mathbf{Ax} = M^{r-1}\mathbf{b}^1 + M^{r-2}\mathbf{b}^2 + \cdots + M\mathbf{b}^{r-1} + \mathbf{b}^r,\ \mathbf{x} \geq \mathbf{0}\}. \tag{3.4.5}$$

It is sufficient to find the matrix $\mathbf{B}$ as an optimal basis for problem (3.4.5), for any big enough M. Our algorithm is seeks such a basis.

Definition 3.4.1. Assume that $\mathbf{B}$ is a basic sub-matrix of $\mathbf{A}$. Then, $\mathbf{B}$ is a strongly feasible basis if $(\mathbf{B}^{-1}\mathbf{b}^1, \mathbf{B}^{-1}\mathbf{b}^2, ..., \mathbf{B}^{-1}\mathbf{b}^r) \succeq_l \mathbf{0}$.

Theorem 3.4.2. *If* $\mathbf{B}$ *is an optimal strongly feasible basis, then there exists an* $\overline{M} > 0$ *such that for each* $M \geq \overline{M}$, $\mathbf{B}$ *is an optimal feasible basis for problem (3.4.5) .*

Proof. Since optimality condition is independent of the right hand side vector, it is sufficient to show that feasibility condition holds. That is to say, there exists an $\overline{M} > 0$ such that for each $M \geq \overline{M}$, the following relation holds:

$$\mathbf{B^{-1}}(M^{r-1}\mathbf{b}^1 + M^{r-2}\mathbf{b}^2 + ... + M\mathbf{b}^{r-1} + \mathbf{b}^r) \geq \mathbf{0}. \tag{3.4.6}$$

To this end, let $i \in \{1, 2, .., r\}$ be an arbitrary index. We show that there exists an $\overline{M}_i > 0$ such that for each $M \geq \overline{M}_i$, we have:

$$(M^{r-1}\bar{b}_i^1 + M^{r-2}\bar{b}_i^2 + ... + M\bar{b}_i^{r-1} + \bar{b}_i^r) \geq 0, \tag{3.4.7}$$

where, $\bar{\mathbf{b}}^t = \mathbf{B}^{-1}\mathbf{b}^t, \quad t \in \{1, ..., r\}$.

Since $\mathbf{B}$ is a strongly feasible basis, $\Phi_i = (\bar{b}_i^1, \bar{b}_i^2, ..., \bar{b}_i^r) \succeq_l \mathbf{0}$.

If $\Phi_i = \mathbf{0}$, then it is obvious that (3.4.7) holds for any $M \geq 0$.

Otherwise, assume that $\bar{b}_i^t$ is the first non-zero component of Φ_i. Since $\Phi_i \succeq_l \mathbf{0}$, then $\bar{b}_i^t > 0$. It can be readily shown that if $\overline{M_i} = \max\{1, \frac{1}{|\bar{b}_i^t|}(\bar{b}_i^{t+1} + \bar{b}_i^{t+2} + ... + \bar{b}_i^r)\}$, then for each $M \geq \overline{M_i}$, (3.4.7) holds.

Now, it is sufficient to let $\overline{M} = \max\{\overline{M}_i | i = 1, ..., r\}$. □

By the above theorem, in order to reach the preemptive optimal solution it is sufficient to find an optimal strongly feasible basis for problem (3.4.5). To this end, first we find a strongly feasible basis, if it exists, with the modified phase-I algorithm (see Section 3.4.1). Now, suppose that $\mathbf{B}$ is a strongly feasible basis. Then, Table 3.1 corresponds to basis $\mathbf{B}$. The algorithm is similar to the simplex algorithm except that

Table 3.1: The table corresponding to basis B.

when a tie occurs in determining the leaving variable by the criterion of the simplex method, we make use of additional columns, and the elementary matrix operations are performed on the additional columns, as well.

For simplicity, we use the following notations in the rest of this chapter:

$$\bar{\mathbf{b}}^j = \mathbf{B}^{-1}\mathbf{b}^j \ , \mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j, \ z_j = \mathbf{c_B}\mathbf{B^{-1}}\mathbf{a}_j,$$

where, $\mathbf{a}_j$ is the jth column of matrix $\mathbf{A}$.

Assume that x_k is the entering variable that is chosen by the simplex criterion. Then, the leaving variable is a variable whose index is in I_r :
$I_1 = \{p|\frac{\bar{b}^1_p}{y_{pk}} = \min\{\frac{\bar{b}^1_j}{y_{jk}}|y_{jk} > 0\}\}$,
$I_t = \{p|\frac{\bar{b}^t_p}{y_{pk}} = \min\{\frac{\bar{b}^t_j}{y_{jk}}|j \in I_{t-1}\}\}, \quad t = 2, ..., r.$

Theorem 3.2 shows that if the entering and leaving variables are chosen by the above procedure, then the new basis is also a strongly feasible basis.

Theorem 3.4.3. *If* $\mathbf{B}$ *is a strongly feasible basis and the entering and leaving variables are chosen by the above procedure, then the new basis is also a strongly feasible basis.*

Proof. Assume that x_k and x_p are the entering and the leaving variables, respectively, and $\hat{\mathbf{B}}$ is the new basis. We want to prove that every row of matrix $(\hat{\mathbf{B}}^{-1}\mathbf{b}^1, \hat{\mathbf{B}}^{-1}\mathbf{b}^2, ..., \hat{\mathbf{B}}^{-1}\mathbf{b}^r)$ is lexicographically non-negative. The ith row of this matrix is:

$$\begin{cases} \frac{1}{y_{pk}}(\bar{b}^1_i, \bar{b}^2_i, ..., \bar{b}^r_i), & \text{if } i = p \\ (\bar{b}^1_i - y_{ik}\frac{\bar{b}^1_p}{y_{pk}}, \bar{b}^2_i - y_{ik}\frac{\bar{b}^2_p}{y_{pk}}, ..., \bar{b}^r_i - y_{ik}\frac{\bar{b}^r_p}{y_{pk}}), & \text{if } i \neq p. \end{cases} \tag{3.4.8}$$

If $i = p$, since $\mathbf{B}$ is a strongly feasible basis and $y_{pk} > 0$, the proof is obvious.
When $i \neq p$, if $y_{ik} \leq 0$ it can be readily shown that (3.4.8) is the sum of two lexicographically non-negative vectors, and thus, (3.4.8) itself is also lexicographically non-negative. On the other hand, when $y_{ik} > 0$, by contradiction assume that (3.4.8) is not lexicographically non-negative, so its first nonzero element is negative. Let t

be the index of the first non zero element.

If $t = 1$, then since $I_r \subseteq I_{r-1} \subseteq ... \subseteq I_1$ and $p \in I_r$, $p \in I_1$, which is contrary to $\frac{\bar{b}_i^1}{y_{ik}} < \frac{\bar{b}_p^1}{y_{pk}}$.

If $t \geq 2$, then

$$\bar{b}_i^j - y_{ik}\frac{\bar{b}_p^j}{y_{pk}} = 0,\ (j = 1, ..., t-1) \Rightarrow \frac{\bar{b}_i^j}{y_{ik}} = \frac{\bar{b}_p^j}{y_{pk}}\ \ (j = 1, ..., t-1) \Rightarrow i \in I_{t-1}.$$

Now, the relation $\frac{\bar{b}_i^t}{y_{ik}} < \frac{\bar{b}_p^t}{y_{pk}}$ $(p \in I_t)$ clearly shows the contradiction. □

Now, we continue the aforementioned procedure until one of the following cases occurs.

Case 1: There is no entering variable, i.e., $z_j - c_j \leq 0$, for each $j \in \{1, ..., n\}$.

Case 2: There is an entering variable but there is no leaving variable, i.e., there exists a k such that $z_k - c_k > 0$ and $\mathbf{y}_k \leq \mathbf{0}$.

In Case 1, there is an optimal strongly feasible basis, and thus $\mathbf{c_B B}^{-1}$ is an optimal solution of the dual problem. In other words, $\mathbf{c_B B}^{-1}$ is an optimal preemptive solution for LLMOP.

In Case 2, $\mathbf{d}_k = \begin{pmatrix} -\mathbf{B}^{-1}\mathbf{a}_k \\ \mathbf{e}_k \end{pmatrix}$ ($\mathbf{e}_k$ is a vector for which the kth component is one and the other components are zero) is a recession direction for problem (3.4.5), and so problem (3.4.5) is unbounded, for any $M \geq 0$. Therefore, its dual problem, and hence LLMOP, is infeasible.

3.4.1 Modified Phase-I Problem

Up to now, we have assumed that we have a strongly feasible basis. Now, we modify the phase-I algorithm such that it gives a strongly feasible basis or shows the infeasibility of problem (3.4.5). Without loss of generality, we assume that $(\mathbf{b}^1, \mathbf{b}^2, ..., \mathbf{b}^r) \succeq_l \mathbf{0}$, because, if necessary, we can multiply some constraints by -1. Adding artificial variables, we rewrite the phase-I problem as follows:

$$\min\{\mathbf{1x}_a |\ \mathbf{Ax} + \mathbf{1x}_a = M^{r-1}\mathbf{b}^1 + M^{r-2}\mathbf{b}^2 + ... + M\mathbf{b}^{r-1} + \mathbf{b}^r,\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x}_a \geq \mathbf{0}.\} \quad (3.4.9)$$

Since $(\mathbf{I}^{-1}\mathbf{b}^1, ..., \mathbf{I}^{-1}\mathbf{b}^r) = (\mathbf{b}^1, ..., \mathbf{b}^r) \succeq_l \mathbf{0}$, $\mathbf{I}$ is a strongly feasible basis for the phase-I problem. We initiate the algorithm by applying $\mathbf{I}$ as an initial strongly feasible basis and solve the phase-I problem. Assume that in the optimal table, z_t^* $(t = 1, ..., r)$ denotes the element in the objective row corresponding to the RHS($\mathbf{b}^t$). In this case, the optimal value of the objective function is $z^* = M^{r-1}z_1^* + M^{r-2}z_2^* + ... + Mz_{r-1}^* + z_r^*$.

For the optimal table of the phase-I problem, one of the following cases may occur.

Case 1: For each t $(t = 1, ..., r)$, $z_t^* = 0$ and there is no artificial variable in the basis. In this case, the optimal basis is a strongly feasible basis for problem (3.4.5) and after omitting the artificial variables and substituting the objective function of problem (3.4.5) ,the phase-II problem initiates.

Case 2: For each t $(t = 1, ..., r)$, $z_t^* = 0$ and there are some artificial variables in the basis.

Table 3.2 is a typical table (possibly after rearranging) at the end of phase-I. It

Table 3.2: The optimal table of modified phase-1.

can be shown that in this table, $\bar{b}_i^t = 0$ $(i = k+1, ..., r\ ;\ t = 1, ..., r)$, because if we let $\Phi_i = (\bar{b}_i^1, \bar{b}_i^2, ..., \bar{b}_i^r)$, then we have

$\Phi_i \succeq_l \mathbf{0}$ and $(z_1^*, ..., z_r^*) = \sum_{i=k+1}^{r} \Phi_i = \mathbf{0} \ \Rightarrow\ \Phi_i = \mathbf{0}$ $(i = k+1, ..., r)$

$\Rightarrow\ \bar{b}_i^t = 0$ $(i = k+1, ..., r\ ;\ t = 1, ..., r)$.

In this case, if $\mathbf{R}_2$ has a nonzero element, then we can pivot on this element in order for its corresponding artificial variables to leave and a non-basic legitimate variable to enter the basis. Since the right hand side elements corresponding to the artificial variables in all the right hand side columns are zero, the new basis is also a strongly feasible basis after pivoting, and the value of z_t^* $(t = 1, ..., r)$ remains zero. If it is possible to exit all the artificial variables, then Case 1 occurs. Otherwise, some artificial variables remain in the basis, because the corresponding row in $\mathbf{R}_2$ is zero. Considering the fact that the initial table, without artificial variables and the objective row, denotes the system "$\mathbf{Ax} = M^{r-1}\mathbf{b}^1 + M^{r-2}\mathbf{b}^2 + ... + M\mathbf{b}^{r-1} + \mathbf{b}^r$" and that the final table is obtained by some *matrix elementary operations* on the initial table, we conclude that the system corresponding to the final table (without artificial variables, and the objective row) gives a system equivalent to the system "$\mathbf{Ax} = M^{r-1}\mathbf{b}^1 + M^{r-2}\mathbf{b}^2 + ... + M\mathbf{b}^{r-1} + \mathbf{b}^r$". On the other hand, the final table denotes that constraints corresponding to artificial variables are redundant, because these constraints are in the form of "$0x_1 + 0x_2 + \ldots + 0x_n = M^{r-1}0 + M^{r-2}0 + ... + M0 + 0$", and so the corresponding constraints in the system "$\mathbf{Ax} = M^{r-1}\mathbf{b}^1 + M^{r-2}\mathbf{b}^2 + ... + M\mathbf{b}^{r-1} + \mathbf{b}^r$" are also redundant.

It should be noted that if a redundant constraint is omitted, then its corresponding variable in problem (3.4.2) will also be omitted.

Case 3: There exists a t such that $z_t^* > 0$.

Since $(z_1^*, ..., z_r^*) = \mathbf{c_B}(\bar{\mathbf{b}}^1, ..., \bar{\mathbf{b}}^r)$ and the components of $\mathbf{c_B}$ belong to $\{0, 1\}$, and also every row of $(\bar{\mathbf{b}}^1, ..., \bar{\mathbf{b}}^r)$ is lexicographically non-negative, thus $(z_1^*, ..., z_r^*)$ is a non-negative combination of some lexicographically non-negative vectors, and so it is lexicographically non-negative. Now, since $z_t^* > 0$, it can be readily shown that there exists an $\overline{M} > 0$ such that for any $M \geq \overline{M}$, problem (3.4.5) is infeasible.

For problem (3.4.2), we may encounter one of these two subcases: either problem (3.4.5) is infeasible or unbounded. To determine which subcase may occur, we use phase-I of problem (3.4.2). The following theorem shows that if LLMOP is feasible, then from Table 3.2 it can be recognized which objective function is unbounded.

Theorem 3.4.4. *Assume that in Table 3.2 t is the first index such that $z_t^* > 0$, i.e., $z_k^* = 0$, for $k = 1, 2, ..., t-1$. If LLMOP is feasible, then the tth objective function is unbounded over the optimal region of the preceding objective functions.*

Proof. Table 3.2 corresponds to problem (3.4.9). The dual form of problem (3.4.9) is

$$\max\{M^{r-1}\mathbf{w}^t\mathbf{b}^1 + M^{r-2}\mathbf{w}^t\mathbf{b}^2 + \cdots + \mathbf{w}^t\mathbf{b}^r |\ \mathbf{w}^t\mathbf{A} \leq \mathbf{0},\ \mathbf{w} \leq \mathbf{1}\} \tag{3.4.10}$$

To prove the theorem, it should be shown that problem (3.4.11) has an optimal solution for each $j = 1, 2, ..., t-1$, and is unbounded for $j = t$:

$$\max\{\mathbf{w}^t\mathbf{b}^j | \mathbf{w}^t\mathbf{A} \leq \mathbf{c},\ \mathbf{w}^t\mathbf{b}^i = \bar{z}_i \text{ for } i = 1, 2, ..., j-1\}, \tag{3.4.11}$$

where $\bar{z}_i$ is an optimal value of the ith objective function over the optimal region of the preceding objective functions.

To show that problem (3.4.11) has an optimal solution, by contradiction assume that there exists $j \in \{1, 2, ..., t-1\}$ such that problem (3.4.11) is unbounded. This means that there exists $\mathbf{w}$ such that

$$\mathbf{w}^t\mathbf{b}^j > 0,\ \mathbf{w}^t\mathbf{A} \leq \mathbf{0},\ \mathbf{w}^t\mathbf{b}^i = 0 \text{ for } i = 1, 2, ..., j-1. \tag{3.4.12}$$

If we let $\bar{\mathbf{w}} = \frac{\mathbf{w}}{\|\mathbf{w}\|_\infty}$, then $\bar{\mathbf{w}}$ is a feasible point for problem (3.4.10) with the objective value equal to $M^{r-1}\bar{\mathbf{w}}^t\mathbf{b}^1 + M^{r-2}\bar{\mathbf{w}}^t\mathbf{b}^2 + \cdots + \bar{\mathbf{w}}^t\mathbf{b}^r$. From relation (3.4.12) and the definition of $\bar{\mathbf{w}}$, we have $\bar{\mathbf{w}}^t\mathbf{b}^j > 0$ and $\bar{\mathbf{w}}^t\mathbf{b}^i = 0$, for $i = 1, 2, ..., j-1$. Furthermore, from Table 3.2, the optimal value of problem (3.4.9), and consequently that of problem (3.4.10), is $M^{r-1}z_1^* + M^{r-2}z_2^* + \cdots + z_r^*$. Since $z_i^* = 0$, for $i = 1, 2, ..., t-1$, $\bar{\mathbf{w}}^t\mathbf{b}^j > 0$, for $j < t$, it is obvious that

$$M^{r-1}\bar{\mathbf{w}}^t\mathbf{b}^1 + M^{r-2}\bar{\mathbf{w}}^t\mathbf{b}^2 + \cdots + \bar{\mathbf{w}}^t\mathbf{b}^r > M^{r-1}z_1^* + M^{r-2}z_2^* + \cdots + z_r^*,$$

for any big enough M, which is a contradiction.

To complete the proof, it is sufficient to show that problem (3.4.11) is unbounded for $j = t$. This means there exists $\mathbf{w}$ such that

$$\mathbf{w}^t\mathbf{b}^t > 0,\ \mathbf{w}^t\mathbf{A} \leq \mathbf{0},\ \mathbf{w}^t\mathbf{b}^i = 0, \quad \text{for } i = 1, 2, ..., t-1.$$

If the matrix $\mathbf{B}$ is an optimal basis for problem (3.4.9), and we let $\mathbf{w}^t = \mathbf{c_B}\mathbf{B}^{-1}$, then the result can be obtained by considering that $\mathbf{c_B}\mathbf{B}^{-1}$ is a feasible point for problem (3.4.10) and $\mathbf{c_B}\mathbf{B}^{-1}\mathbf{b}^k = z_k^*$, for all k. $\square$

3.5 Sensitivity Analysis on the Priority of the Objective Functions in Linear LMOPs

Now, suppose that problem (3.4.1) is solved, and we want to solve it with the new priority of the objective functions. In other words, let $\pi : \{1, ..., r\} \to \{1, ..., r\}$ be a permutation , and consider the permutation $(\mathbf{b}^{\pi(1)}, ..., \mathbf{b}^{\pi(r)})$ of the objective functions. We want to solve the following new LLMOP:

$$\operatorname{lex} \max_{\mathbf{w} \in \mathcal{W}} \{\mathbf{w}^t \mathbf{b}^{\pi(1)}, \mathbf{w}^t \mathbf{b}^{\pi(2)}, ..., \mathbf{w}^t \mathbf{b}^{\pi(r)}\}. \tag{3.5.1}$$

The importance of solving the new LLMOP is two-fold:

- With the passing of time, the priority of the objective functions may change due to the change in the management strategy.
- For any permutation π, an optimal solution of problem (3.5.1) is a Pareto optimal solution for the following MOP:

$$\max_{\mathbf{w} \in \mathcal{W}} \{\mathbf{w}^t \mathbf{b}^1, \mathbf{w}^t \mathbf{b}^2, ..., \mathbf{w}^t \mathbf{b}^r\}.$$

So, solving problem (3.5.1) with different permutations leads to obtaining some Pareto optimal solutions for the related MOP.

If we apply the sequential method to solve (3.4.1), then we should solve problem (3.5.1) as a new problem. In this section, we want to show how we can solve problem (3.5.1) more easily by employing the preceding computations if problem (3.4.1) is solved by the method proposed in the previous section. In fact, this section deals with a kind of sensitivity analysis on the priority of the objective functions in LLMOPs.

By considering the dual problem and what was said in Section 3.4, we should solve the following problem:

$$\min\{\mathbf{cx} | \mathbf{Ax} = M^{r-1}\mathbf{b}^{\pi(1)} + M^{r-2}\mathbf{b}^{\pi(2)} + \cdots + M\mathbf{b}^{\pi(r-1)} + \mathbf{b}^{\pi(r)},\ \mathbf{x} \geq \mathbf{0}\}. \tag{3.5.2}$$

So, by the assumption that problem (3.4.5) is solved, we want to solve the above problem. One of the three following cases may occur for problem (3.4.5).

Case 1: Problem (3.4.5) is unbounded for any big enough M. In this case, LLMOP is infeasible. Since changing the priority of the objective functions does not affect the feasible region, the new LLMOP is infeasible, too.

Case 2: Problem (3.4.5) has an optimal strongly feasible basis, and hence LLMOP has a preemptive optimal solution. In this case, the new LLMOP may be unbounded or have a preemptive optimal solution.

Case 3: Problem (3.4.5) is infeasible for any big enough M. In this case, LLMOP could be infeasible or unbounded. If LLMOP is infeasible, then the new LLMOP is infeasible, too; but if LLMOP is unbounded, then the new LLMOP may be unbounded or have a preemptive optimal solution.

In the rest of this section, cases 2 and 3 are considered in detail in Subsections (3.5.1) and (3.5.2), respectively.

3.5.1 Problem (3.4.5) has an optimal strongly feasible basis

Now, suppose that Case 2 occurs and $\mathbf{B}$ is an optimal strongly feasible basis for problem (3.4.5), and Table 3.1 is an optimal table. In order to solve problem (3.5.2) and by considering permutation π, we change the right-hand side (RHS) columns in Table 3.1, and Table 3.3 is obtained. Since changing the RHS columns does not affect

Table 3.3: The table after changing the RHS columns in the optimal table based on the related permutation.

the optimality conditions, if $\mathbf{B}$ is a strongly feasible basis for the above table, i.e.,

$$(\mathbf{B}^{-1}\mathbf{b}^{\pi(1)}, \mathbf{B}^{-1}\mathbf{b}^{\pi(2)}, ..., \mathbf{B}^{-1}\mathbf{b}^{\pi(r)}) = (\bar{\mathbf{b}}^{\pi(1)}, \bar{\mathbf{b}}^{\pi(2)}, ..., \bar{\mathbf{b}}^{\pi(r)}) \succeq_l \mathbf{0},$$

then $\mathbf{B}$ is an optimal strongly feasible basis for problem (3.5.2), and $\mathbf{c_B B^{-1}}$ is a preemptive optimal solution for the new LLMOP. So, in this case we can obtain the preemptive optimal solution without any further computation.

Remark 3.5.1. From Table 3.1, it can be easily recognized under which permutations of the objective functions the preemptive optimal solution of LLMOP does not change.

Now, consider the case in which $\mathbf{B}$ is not a strongly feasible basis for Table 3.3. This means that there exists a basic variable whose row in the RHS columns is not lexicographically non-negative. Suppose that $x_{\mathbf{B}_t}$ is a basic variable with the smallest row in the RHS columns with respect to the lexicographic ordering, i.e.,

$$(\bar{\mathbf{b}}_t^{\pi(1)}, \bar{\mathbf{b}}_t^{\pi(2)}, ..., \bar{\mathbf{b}}_t^{\pi(r)}) \preceq_l (\bar{\mathbf{b}}_j^{\pi(1)}, \bar{\mathbf{b}}_j^{\pi(2)}, ..., \bar{\mathbf{b}}_j^{\pi(r)}), \text{ for all } j. \tag{3.5.3}$$

We apply the dual simplex algorithm for this variable to leave. The entering variable x_k is determined by the following minimum ratio test:

$$\frac{z_k - c_k}{y_{tk}} = \min\{\frac{z_j - c_j}{y_{tj}} : \ y_{tj} < 0\}. \tag{3.5.4}$$

By determining the entering and leaving variables with the above procedure, the next basis remains optimal. We continue the above procedure until one of the following cases occurs.

Case 1: An optimal strongly feasible basis is obtained. In this case, $\mathbf{c_B B^{-1}}$ is a preemptive optimal solution for the new LLMOP.

Case 2: There exists a basic variable $x_{\mathbf{B}_t}$ whose row in the RHS columns is not lexicographically non-negative, but it cannot leave the basis. This happens when $y_{tj} \geq 0$, for all j. In this case, the tth row in Table 3.3 reads as follows:

$$\sum_j y_{tj} x_j = M^{r-1}\bar{\mathbf{b}}_t^{\pi(1)} + M^{r-2}\bar{\mathbf{b}}_t^{\pi(2)} + \cdots + M\bar{\mathbf{b}}_t^{\pi(r-1)} + \bar{\mathbf{b}}_t^{\pi(r)} \tag{3.5.5}$$

Since the first non-zero component of $(\bar{\mathbf{b}}_t^{\pi(1)}, \bar{\mathbf{b}}_t^{\pi(2)}, ..., \bar{\mathbf{b}}_t^{\pi(r-1)}, \bar{\mathbf{b}}_t^{\pi(r)})$ is negative, it can be readily shown that the right side of equality (3.5.5) is negative for any big enough M. Hence, the equality (3.5.5) is infeasible for any big enough M due to the non-negativity condition of the variables and y_{tj}. So, problem (3.5.2) is infeasible for any big enough M, and hence the new LLMOP can be infeasible or unbounded. Moreover, in this subsection we assume that LLMOP has a preemptive optimal solution, and so the new LLMOP is not infeasible and hence is unbounded. The following theorem specifies how we can recognize from Table 3.3 which objective function in the new LLMOP is unbounded on the optimal region of the preceding objective functions.

Theorem 3.5.1. *Assume that the basis* $\mathbf{B}$ *is not a strongly feasible basis in Table 3.3, and in the above mentioned dual simplex procedure there exists a leaving variable* $x_{\mathbf{B}_t}$*, but there is no entering variable. If the first non-zero component of* $(\bar{\mathbf{b}}_t^{\pi(1)}, \bar{\mathbf{b}}_t^{\pi(2)}, ..., \bar{\mathbf{b}}_t^{\pi(r)})$ *is in the* k*th position, then the* k*th objective function in the new LLMOP is unbounded on the optimal region of the previous objective functions.*

Proof. To prove the theorem, it should be shown that problem (3.5.6) below has an optimal solution, for each $j = 1, 2, ..., k-1$, and is unbounded for $j = k$:

$$\max\{\mathbf{w}^t\mathbf{b}^{\pi(j)} | \mathbf{w}^t\mathbf{A} \leq \mathbf{c},\ \mathbf{w}^t\mathbf{b}^{\pi(i)} = \bar{z}_i \text{ for } i = 1, 2, ..., j-1\}, \tag{3.5.6}$$

where, $\bar{z}_i$ is an optimal value of $\mathbf{w}^t\mathbf{b}^{\pi(i)}$ over the optimal region of the preceding objective functions.

Regarding Relation (3.5.3) and being the first non-zero component of $(\bar{\mathbf{b}}_t^{\pi(1)}, \bar{\mathbf{b}}_t^{\pi(2)}, ..., \bar{\mathbf{b}}_t^{\pi(r)})$ in the kth position, we have

$$(0, 0, ..., 0) \preceq_l (\bar{\mathbf{b}}_j^{\pi(1)}, \bar{\mathbf{b}}_j^{\pi(2)}, ..., \bar{\mathbf{b}}_j^{\pi(k-1)}), \text{ for all } j. \tag{3.5.7}$$

Now, suppose that in Table 3.3, the columns RHS($\mathbf{b}^{\pi(k)}$) to RHS($\mathbf{b}^{\pi(r)}$) are omitted. Then, by Relation (3.5.7), the matrix $\mathbf{B}$ is an optimal strongly feasible basis for the table. This means that $\mathbf{c}_\mathbf{B}\mathbf{B}^{-1}$ is a preemptive optimal solution for the following problem:

$$\operatorname{lex}\max\{\{\mathbf{w}^t\mathbf{b}^{\pi(1)}, \mathbf{w}^t\mathbf{b}^{\pi(2)}, ..., \mathbf{w}^t\mathbf{b}^{\pi(k-1)}\} |\ \mathbf{w}^t\mathbf{A} \leq \mathbf{c}\} \tag{3.5.8}$$

Thus, problem (3.5.6) has an optimal solution, for each $j = 1, 2, ..., k-1$.

Now, it should be shown that problem (3.5.6) is unbounded for $j = k$. In terms of linear programming (see Bazaraa et al., 1990 [2]), this means there exists $\bar{\mathbf{w}}$ such that

$$\bar{\mathbf{w}}^t \mathbf{b}^{\pi(k)} > 0,\ \ \bar{\mathbf{w}}^t \mathbf{b}^{\pi(i)} = 0 \text{ for } i = 1, 2, ..., k-1,\ \ \bar{\mathbf{w}}^t \mathbf{A} \leq \mathbf{0}. \tag{3.5.9}$$

Let $\bar{\mathbf{w}}^t = -\mathbf{B}_{t.}^{-1}$, where, $\mathbf{B}_{t.}^{-1}$ is the tth row of $\mathbf{B}^{-1}$. Then

$$\bar{\mathbf{w}}^t \mathbf{b}^{\pi(j)} = -\mathbf{B}_{t.}^{-1} \mathbf{b}^{\pi(j)} = -\bar{\mathbf{b}}_t^{\pi(j)},$$

$$\bar{\mathbf{w}}^t \mathbf{A} = \bar{\mathbf{w}}^t [\mathbf{B}\ \ \mathbf{N}] = -[\mathbf{B}_{t.}^{-1}\mathbf{B}\ \ \mathbf{B}_{t.}^{-1}\mathbf{N}],$$

where, N is a matrix corresponding to the non-basic variables. Since $\mathbf{B}_{t.}^{-1}$ is the tth row of $\mathbf{B}^{-1}$, $\mathbf{B}_{t.}^{-1}\mathbf{B}$ is a vector of which the tth component is one and the others are zero, and $\mathbf{B}_{t.}^{-1}\mathbf{N} = (y_{t1}, y_{t2}, ..., y_{tr})$. Since $(y_{t1}, y_{t2}, ..., y_{tr}) \geq 0$, we have $\bar{\mathbf{w}}^t \mathbf{A} \leq \mathbf{0}$. Furthermore, by the assumption of the theorem, $\bar{\mathbf{b}}_t^{\pi(j)} = 0$, for $j = 1, 2, ..., k-1$, and $\bar{\mathbf{b}}_t^{\pi(k)} < 0$, which completes the proof.

□

3.5.2 Problem (3.4.5) is infeasible for any big enough M

This case occurs when in modified phase-I of problem (3.4.5) there are some artificial variables whose RHS vectors are not zero. As mentioned, in this case LLMOP could be infeasible or unbounded. If LLMOP is infeasible, then the new LLMOP is infeasible too, and there is nothing to obtain. So, in the sequel we suppose that the new LLMOP is feasible. Consider phase-I of problem (3.5.2) as follows:

$$\min\ \ \{\mathbf{1x}_a |\ \mathbf{Ax} + \mathbf{1x}_a = M^{r-1}\mathbf{b}^{\pi(1)} + M^{r-2}\mathbf{b}^{\pi(2)} + ... + M\mathbf{b}^{\pi(r-1)} + \mathbf{b}^{\pi(r)},\ \mathbf{x},\ \mathbf{x}_a \geq \mathbf{0}\}. \tag{3.5.10}$$

To solve the above problem, we change the RHS columns in the optimal table of phase-I of problem (3.4.5) (Table 3.2), and obtain Table 3.4. Assume that the matrix

B is a basis corresponding to the above table. Since B is an optimal strongly feasible basis for Table 3.2 and changing the RHS columns does not affect the optimality conditions, then B meets the optimality conditions in Table 3.4. But, matrix B may lose the strong feasibility conditions due to the change in the RHS columns. In this case, we apply the dual simplex pivoting strategy to make the basic variables whose RHS vectors are not lexicographically nonnegative leave, as explained in Subsection 3.5.1. One of the following three cases may occur.

Case 1: An optimal strongly feasible basis is obtained, and $z^*_{\pi(i)} = 0$ for all indices i. In this case, if there is no artificial variable in the basis, after omitting the artificial variables and substituting the objective function of problem (3.5.2) the phase-II problem initiates. But, if there are some artificial variables in the basis, first we should try to make them leave the basis and then initiate the Phase-II algorithm (see Case 2 in Subsection 3.4.1).

Case 2: An optimal strongly feasible basis is obtained, and there is $t \in \{1, 2, ..., r\}$ such that $z^*_{\pi(t)} > 0$. Suppose that t is the first index such that $z^*_{\pi(t)} > 0$. By applying Theorem 3.4.4 to Table 3.4 we can conclude that the tth objective function in the new LLMOP is unbounded on the optimal region of the preceding objective functions.

Case 3: There is a leaving variable but there is no an entering variable in the dual simplex procedure. In this case, problem (3.5.10) is infeasible for any big enough M, and hence problem (3.5.2) is infeasible for any big enough M, too. Since we have supposed that LLMOP is feasible, the new LLMOP is unbounded. To determine which objective function is unbounded, we can employ the modified phase-I algorithm and Theorem 3.4.4.

Table 3.4: The table after changing the RHS columns in Table 3.2.

3.6 Numerical Examples

In this section, some numerical examples are provided to demonstrate how the proposed algorithms in Sections 3.4 and 3.5 work in different cases.

The following example illustrates how the modified version of the simplex algorithm, introduced in Section 3.4, works.

Example 3.6.1. *Consider the following LLMOP:*

$$Lex \max\{x_1 + x_2, x_1 + \frac{1}{2}x_2\}$$
$$s.t. \begin{cases} x_1 - x_2 \leq 1 \\ x_1 + x_2 \leq 2 \\ -x_1 + x_2 \leq 2 \\ -x_1 - x_2 \leq -1 \\ x_1, x_2 \geq 0. \end{cases} \tag{3.6.1}$$

The standard form of the dual problem for problem (3.6.1) is:

$$\min \quad w_1 + 2w_2 + 2w_3 - w_4$$
$$s.t. \begin{cases} w_1 + w_2 - w_3 - w_4 - s_1 = M + 1 \\ -w_1 + w_2 + 2w_3 - w_4 - s_2 = M + \frac{1}{2} \\ w_1, w_2, w_3, w_4, s_1, s_2 \geq 0. \end{cases} \tag{3.6.2}$$

The phase-I problem is:

$$\min \quad w_{a_1} + w_{a_2}$$
$$s.t. \begin{cases} w_1 + w_2 - w_3 - w_4 - s_1 + w_{a_1} = M + 1 \\ -w_1 + w_2 + 2w_3 - w_4 - s_2 + w_{a_2} = M + \frac{1}{2} \\ w_1, w_2, w_3, w_4, s_1, s_2, w_{a1}, w_{a2} \geq 0. \end{cases}$$

The initial table of the phase-I problem (after updating the objective row) is shown in Table 3.5.

Table 3.5: The initial table for phase-I problem.

In this table, w_2 is an entering variable, and
$I_1 = \{a_1, a_2\}$, $I_2 = \{a_2\}$.
Thus, w_{a_2} is the leaving variable. Table 3.6 is the next table.

Table 3.6: The next table after w_2 entering and w_{a2} leaving.

In Table 3.6, the entering variable is w_1, and $I_1 = \{a_1\}$. Thus, w_{a_1} is the leaving variable. The next table, that is optimal, is shown as Table 3.7. Since $z_1^* = z_2^* = 0$

Table 3.7: Optimal table.

and there is no artificial variable in the basis, after omitting the non-basic artificial variables and substituting the objective function, the phase-II problem initiates. Table 3.8 shows the initial table before updating. After updating the objective row, we come up with Table 3.9. Since the vector under the surplus variables in the objective row is $-\mathbf{c_B B^{-1}}$, therefore $(x_1^*, x_2^*) = (\frac{3}{2}, \frac{1}{2})$ is the preemptive optimal solution of problem (3.6.1). This is shown in Figure 3.1. □

Example 3.6.2 shows the situation considered in Subsection 3.5.1.

Consider the following LLMOP:

$$Lex\max\{x_1 - x_2 + x_3, -x_2, x_1 + 2x_2\}$$
$$s.t. \begin{cases} x_1 + x_2 \leq 1 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1, x_2, x_3 \geq 0. \end{cases} \tag{3.6.3}$$

The standard form of the dual problem for problem (3.6.3) is:

$$\min \quad w_1 + 4w_2$$
$$s.t. \begin{cases} w_1 + w_2 - s_1 = M^2 + 1 \\ -w_1 + w_2 + s_2 = M^2 + M - 2 \\ w_2 - s_3 = M^2 \\ w_1, w_2, s_1, s_2, s_3 \geq 0 \end{cases} \tag{3.6.4}$$

Table 3.10 shows the optimal table of problem (3.6.4).

From Table 3.10, it can be seen that $(x_1^*, x_2^*, x_3^*) = (1, 0, 3)$ is the preemptive optimal solution of problem (3.6.3). Now, suppose that the priority of the objective functions in problem (3.6.3) is changed, and we want to solve the following new LLMOP (in fact, we have a permutation $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ such that $\pi(1) = 2, \pi(2) = 3, \pi(3) = 1$):

$$Lex \max\{x_1 + 2x_2, x_1 - x_2 + x_3, -x_2\}$$
$$s.t. \begin{cases} x_1 + x_2 \leq 1 \\ x_1 - x_2 + x_3 \leq 4 \\ x_1, x_2, x_3 \geq 0. \end{cases} \tag{3.6.5}$$

By what was stated in Subsection 3.5.1, in order to solve the above new LLMOP we should change the RHS columns in Table 3.10 with respect to the related permutation π. Then, Table 3.11 is obtained.

From Table 3.11, it is obvious that the RHS vectors are not lexicographically nonnegative. Regarding the proposed algorithm, to achieve a strongly feasible basis, the variable which has the smallest row in the RHS columns (with respect to the lexicographic ordering) should leave the basis (s_2, here), by dual simplex pivoting. The next table is shown as Table 3.12.

Table 3.8: The initial table for phase-II problem before updating.

Table 3.12 shows the optimal table for the dual problem of problem (3.6.5). Thus, $(x_1^*, x_2^*, x_3^*)_{new} = (0, 1, 5)$ is the preemptive optimal solution of Problem(3.6.5).

Now, suppose the priority of the objective functions in problem (3.6.3) is changed based on the permutation $\pi' : \{1, 2, 3\} \to \{1, 2, 3\}$ such that $\pi'(1) = 2, \pi'(2) = 1, \pi'(3) = 3$. In order to solve the new LLMOP, the first and second RHS columns in Table 3.10 should be exchanged. If the first and second RHS columns are swapped in Table 3.10, it is obvious that the RHS vectors remain lexicographically nonnegative. Thus, changing the priority of the first and second objective functions does not affect the preemptive optimal solution, and we can determine the preemptive optimal solution of the new LLMOP without any further computations.

$\square$

Examples 3.6.3 and 3.6.4 below deal with the situations which were depicted in Theorems 3.4.4 and 3.5.1, respectively.

Example 3.6.3. *Consider the following LLMOP and its dual:*

$$Lex \max\{2x_1 - 4x_2, x_1 + 3x_2\}$$
$$s.t. \begin{cases} x_1 - 2x_2 \leq 4 \\ -x_1 + x_2 \leq 3 \\ x_1, x_2 \geq 0 \end{cases} \tag{3.6.6}$$

$$\min \quad 4w_1 + 3w_2$$
$$s.t. \begin{cases} w_1 - w_2 - s_1 = 2M + 1 \\ 2w_1 - w_2 + s_2 = 4M - 3 \\ w_1, w_2, s_1, s_2 \geq 0 \end{cases} \tag{3.6.7}$$

Table 3.9: The initial table for phase-II problem after updating.

The modified phase-I problem of problem (3.6.7) is:

$$\min \quad w_{a_1}$$
$$s.t. \begin{cases} w_1 - w_2 - s_1 + w_{a_1} = 2M + 1 \\ 2w_1 - w_2 + s_2 = 4M - 3 \\ w_1, w_2, s_1, s_2 \geq 0. \end{cases} \tag{3.6.8}$$

The initial table of problem (3.6.8) (after updating the objective row) is shown in Table 3.13. Now, by employing the procedure explained in Section 3.4, w_1 is an entering variable, and s_2 is the leaving variable. Table 3.14 shows the next table which is an optimal table.

Inasmuch as $z_2^* > 0$ in Table 3.14, problem (3.6.6) is infeasible or unbounded. On the other hand, $(x_1, x_2) = (0, 0)$ is a feasible point of Problem(3.6.6) and hence Problem(3.6.6) is unbounded. By Theorem 3.4.4, since $z_2^* > 0$, the second objective function in problem (3.6.6) is unbounded on the optimal region of the first objective function.

□

Example 3.6.4. *Consider the following LLMOP and its standard form of the dual problem:*

$$Lex \max\{-4x_1 + x_2, -x_1 + 3x_2\}$$
$$s.t. \begin{cases} -x_1 + x_2 \leq 2 \\ -x_1 + 2x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{cases} \tag{3.6.9}$$

Figure 3.1: The optimal region of the first objective function is segment AB and the optimal value of the second objective function over AB is point B.

$$\min \quad 2w_1 + 6w_2$$
$$s.t. \begin{cases} w_1 + w_2 + s_1 = 4M + 1 \\ w_1 + 2w_2 - s_2 = M + 3 \\ w_1, w_2, s_1, s_2 \geq \end{cases} \tag{3.6.10}$$

Table 3.15 shows the optimal table of problem (3.6.10).

Now, take into account the case in which the priority of the objective functions in problem (3.6.9) is changed. To solve the new LLMOP, we should change the RHS columns in Table 3.15. After swapping the RHS columns in Table 3.15, the RHS vectors are not lexicographically nonnegative yet. To attain a strongly feasible basis, s_1 should leave the basis by employing the dual simplex algorithm. Table 3.16 is obtained after pivoting. The basis B in Table 3.16 is not strongly feasible, and w_1 should leave the basis. In view of the fact that there is no entering variable, the new LLMOP is unbounded. Theorem 3.5.1 shows how we can determine which objective function is unbounded on the optimal region of the preceding objective functions. In Table 3.16, w_1 has the smallest row in the RHS columns regarding the lexicographic ordering, and the first nonzero component of $(\overline{b}_2^{\pi(1)}, \overline{b}_2^{\pi(2)})$ is in its first position. By Theorem 3.5.1, the first objective function in the new LLMOP is unbounded over the feasible region.

$\square$

Table 3.10: The optimal table of problem (3.6.4).

Table 3.11: The table after changing the RHS columns in Table 3.10 with respect to the related permutation.

Table 3.12: The optimal table of problem (3.6.5).

Table 3.13: The initial table of phase-1 of problem (3.6.8).

Table 3.14: The optimal table of problem (3.6.8).

Table 3.15: The optimal table of problem (3.6.10).

Table 3.16: The table after swapping the RHS columns in Table 3.15 and applying the dual simplex method.

3.7 Conclusions

Through a survey of existing works, it is revealed that there are currently no generally applicable methods for solving nonlinear lexicographic multiple objective problems (LMOPs), and this is due to the lack of an adequate mathematical theory for such problems. A mathematical theory applicable to lexicographic optimization has been presented. We have studied the transformation of a nonlinear LMOP into an equivalent single objective program. By some examples, it has been shown that the transformation is not possible, in general, and some conditions were determined under which the transformation is possible. Since, for linear LMOPs and LMOPs with the finite discrete feasible region, every Pareto optimal solution is a strongly properly efficient solution, the results obtained by Sherali and Soyster could be deduced from the results of this chapter as a special case. However, finding a method for checking the settings in which the lexicographic optimal solutions are strongly properly efficient, and assessing the transformation results on an experimental basis are beyond the scope of this thesis and need further research.

We employed this transformation idea for linear LMOPs and proposed a new algorithm based on the simplex method for solving linear LMOPs. In this method, there is no need to compute M, and the pivoting procedure is modified, such that the bigness of M is automatically taken into account. Since in this method we make use of the dual problem, the method is more desirable for problems whose number of constraints is larger than the number of variables, while the algorithm proposed by Iserman [39] is based on the primal simplex method and is more suitable for problems whose number of variables is larger than the number of constraints, especially when we use the modified simplex method; as in the modified simplex method, the inverse of the matrix B is stored in the computer memory, and its dimension equals the number of constraints.

Moreover, it has been shown how we can solve LLMOP if the priority of the objective functions is changed, by providing an efficient algorithm which employs the preceding computations, and without solving a new problem. One of the applications of this algorithm is to determine some Pareto optimal solutions of the MOP, because any preemptive optimal solution of LLMOP is a Pareto optimal solution of the related multiobjective problem.

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