Georgia Institute of Technology

ECE 6254 - Assignment 1

Fall 2020 - v1.0

- There are 7 problems over 3 pages (including the cover page)
- The problems are not necessarily in order of difficulty.
- All problems are assigned the same weight in the overall grade.
- Each question is graded out of two points (0 for no meaningful work, 1 for partial work, 2 if correct)
- Unless otherwise specified, you should concisely indicate your reasoning and show all relevant work.
- The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written. If we can't read it, we can't grade it.
- Please use a pen and not a pencil if you handwrite your solution.
- **You must submit your exam on Gradescope. Make sure you allocate time for the submission**.

Problem 1: Playing with 2D distributions

Consider variables X and Y with joint Probability Density Function (PDF)

$$p_{XY}(x,y) = \begin{cases} \frac{2}{\pi}(x^2+y^2) \text{ for } x^2+y^2<1 \\ 0 \text{ else} \end{cases}$$

[Q1] Show that p_{XY} in a valid PDF. (*Hint*: a change of variable will be useful)

[Q2] Are X and Y independent? Justify your answer.

[Q3] Find $\mathbb{E}\left(\frac{1}{X^2+Y^2}\right)$

[Q4] Find $\mathbb{P}(|X|<Y)$

[Q5] Find $\mathbb{P}(X>0|Y>1/2)$

Problem 2: Coverage for Bernoulli trials

Let $X_1,\dots,X_n$ be independent and identically distributed (i.i.d.) distributed according to a Bernoulli distribution with parameter $p \in (0,1)$,

[Q1] Let $\alpha > 0$ and define

$$\epsilon_n = \sqrt{\frac{1}{2n}\log\frac{2}{\alpha}}$$

Let $\hat{p}_n \triangleq \frac{1}{n}\sum_{i=1}^n X_i$ and define the interval $C_n = (\hat{p}_n - \epsilon_n, \hat{p}_n + \epsilon_n)$ (note that C_n is a random variable since $\hat{p}_n$ is. Use Hoeffding's inequality (look it up on Wikipedia!) to show that

$$\mathbb{P}(C_n \text{ contains } p) \geqslant 1-\alpha$$

[Q2] This part requires a computer. Let $\alpha = 0.05$ and $p = 0.4$. Conduct a simulation to see how often the intervals contain p, which is called the *coverage*, for various values of n between 1 and 10000. Plot the coverage versus n

[Q3] Plot the length of the interval versus n for $\alpha = 0.05$ as before. Suppose we want the length of the interval to be no more than 0.05. How large should n be?

Problem 3: Mill's inequality

Let $Z \sim \mathcal{N}(0,1)$. Prove that for any $t > 0$

$$\mathbb{P}(|Z|>t) \leqslant \sqrt{\frac{2}{\pi}}\frac{e^{-t^2/2}}{t}.$$

Hint: Prove that $\mathbb{P}(|Z|>t) = 2\mathbb{P}(Z>t)$. Now, write what $\mathbb{P}(Z>t)$ means and note that $x/t > 1$ when $x > t$.

Problem 4: Conditional probabilities with joint Gaussian

Let $[X\,Y]^\intercal$ be a Gaussian random vector with zero mean and covariance matrix

$$K = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \text{ with } |\rho| < 1.$$

[Q1] Express $\mathbb{P}(X \leqslant 1|Y)$ in terms of ρ, Y, and the standard normal Cumulative Distribution Function (CDF) Φ.

[Q2] Find $\mathbb{E}\big((X - Y)^2|Y = y\big)$ for $y \in \mathbb{R}$

Problem 5: Pythagora's theorem

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

[Q1] Under what condition is it true that

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2$$

[Q2] Under what condition is it true that

$$\|\mathbf{x} + \mathbf{y}\|_2 = \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$$

Problem 6: Eigenvalues and eigenvectors

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square symmetric non-singular matrices in $\mathbb{R}^{n\times n}$.

[Q1] Show that if $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvectors, then they commute, i.e., $\mathbf{AB} = \mathbf{BA}$.

[Q2] Show that if $\mathbf{AB} = \mathbf{BA}$ and $\mathbf{A}$ has no repeated and zero eigenvalues, then $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvectors. Clearly show where you use the assumption that there are not repeated eigenvalues.

Problem 7: Orthogonal projections

Define

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 9 \\ -2 \\ -6 \\ -7 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 7 \end{bmatrix}$$

Find a decomposition of $\mathbf{x}$ into $\mathbf{x} = \mathbf{x}^* + \mathbf{x}_e$ where $\mathbf{x}^*$ is in the span of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{x}_e$ is orthogonal to $\mathbf{x}^*$. Make sure to give both $\mathbf{x}^*$ and $\mathbf{x}_e$, and show your work and describe your method, even if you use a computer to help with the calculations.