Georgia Institute of Technology

ECE 6254 - Assignment 4

Fall 2020 - v1.1

- There are 2 problems over 2 pages (including the cover page)
- The problems are not necessarily in order of difficulty.
- All problems are assigned the same weight in the overall grade.
- Each question is graded out of two points (0 for no meaningful work, 1 for partial work, 2 if correct)
- Unless otherwise specified, you should concisely indicate your reasoning and show all relevant work.
- The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written. If we can't read it, we can't grade it.
- Please use a pen and not a pencil if you handwrite your solution.
- **You must submit your exam on Gradescope. Make sure you allocate time for the submission**.

Problem 1: Maximum likelihood estimation for uniform distributions

Assume that you have access to i.i.d. realization of $\{x_i\}_{i=1}^N$ of a uniformly distributed random variable X.

[Q1] Show that the maximum likelihood estimator for the parameters $a < b$ when X is uniform on $[a; b]$ are

$$\hat{a} = \min_i x_i \qquad \hat{b} = \max_i x_i. \tag{1}$$

[Q2] Show that the maximum likelihood estimator for the parameters $a > 0$ when X is uniform on $[-a; a]$ is

$$\hat{a} = \max_i |x_i|. \tag{2}$$

[Q3] Now assume that X is uniform on $[a; 2a]$ with $a > 0$.

(a) Show that the estimator $\hat{a}_1 = \min_i x_i$ is well defined, in the sense that $\forall i \; \hat{a}_1 \leqslant x_i \leqslant 2\hat{a}_1$.

(b) Show that the estimator $\hat{a}_2 = \frac{\max_i x_i}{2}$ is well defined, in the sense that $\forall i \; \hat{a}_2 \leqslant x_i \leqslant 2\hat{a}_2$.

(c) Which one is the maximum likelihood estimator?

Problem 2: Estimation of parameters and bias variance tradeoff

Consider a sample of n real numbers $\{x_i\}_{i=1}^n$ that are drawn independently from the same distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The parameters mu and σ^2 are unknown. To assess how well an estimator recovers the underlying value of the parameter, we study its bias and variance. The bias is defined by the expectation of the deviation from the true value under the true distribution of the sample. Biased estimators systematically under-estimate or over-estimate the parameter. The variance of the estimator measures the anticipated uncertainty in the estimated value due to the particular selection of the sample. Estimators that minimize both bias and variance are preferred, but typically there is a trade-off between bias and variance.

[Q1] Show from first principles that the Maximum Likelihood Estimates (MLEs) of μ and σ are

$$\hat{\mu} \triangleq \frac{1}{n}\sum_{i=1}^n x_i \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \hat{\mu})^2.$$

[Q2] Show that $\hat{\mu}$ is unbiased.

[Q3] Show that $\hat{\sigma}^2$ is biased.

[Q4] Show that $\hat{\sigma}_1^2 \triangleq \frac{n}{n-1}\hat{\sigma}^2$ is unbiased.

[Q5] Show that the MSE of any estimator can always be written as bias squared plus variance.

[Q6] Show that the MSE of $\hat{\sigma}^2$ is lower than the MSE of $\hat{\sigma}_1^2$.