Linear Algebra

AIET 2022

Source: http://vmls-book.stanford.edu/vmls.pdf

Inner product, Outer product and Norms

1) *Word count and word count histogram vectors.* Suppose the n-vector w is the word count vector associated with a document and a dictionary of n words. For simplicity we will assume that all words in the document appear in the dictionary.

(a) What is $\mathbf{1}^T w$?

(b) What does $w_{282} = 0$ mean?

(c) Let h be the n-vector that gives the histogram of the word counts, *i.e.*, h_i is the fraction of the words in the document that are word i. Use vector notation to express h in terms of w. (You can assume that the document contains at least one word.)

(a) $$1^T w = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = w_1 + w_2 + \cdots w_n.$$

Total number of words in the document.

(b) $w_{282} = 0$.

282^{nd} word in the dictionary does not appear in the document.

(c) $h \in \mathbb{R}^n$

$$h = \frac{w}{1^T w}$$

2) *Average age in a population.* Suppose the 100-vector x represents the distribution of ages in some population of people, with x_i being the number of $i-1$ year olds, for $i = 1, \ldots, 100$. (You can assume that $x \neq 0$, and that there is no one in the population over age 99.) Find expressions, using vector notation, for the following quantities.

(a) The total number of people in the population.

(b) The total number of people in the population age 65 and over.

(c) The average age of the population. (You can use ordinary division of numbers in your expression.)

(a) $\underline{x} \in \mathbb{R}^{100}$.

total number of people $= \underline{1}^T \underline{x}$

(b) $y^T x$

$$y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} \text{65 entries.} \\ \rightarrow 66^{th} \text{ entry.} \\ \\ \\ \\ \\ \end{matrix}$$

(c) Average age of population $= \dfrac{\underline{v}^T \underline{x}}{\underline{1}^T \underline{x}}$

$$\underline{v} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 99 \end{bmatrix}$$

3) *Auto-regressive model.* Suppose that $z_1, z_2, \ldots$ is a time series, with the number z_t giving the value in period or time t. For example z_t could be the gross sales at a particular store on day t. An *auto-regressive* (AR) model is used to predict z_{t+1} from the previous M values, $z_t, z_{t-1}, \ldots, z_{t-M+1}$:

$$\hat{z}_{t+1} = (z_t, z_{t-1}, \ldots, z_{t-M+1})^T \beta, \quad t = M, M+1, \ldots.$$

Here $\hat{z}_{t+1}$ denotes the AR model's prediction of z_{t+1}, M is the memory length of the AR model, and the M-vector β is the AR model coefficient vector. For this problem we will assume that the time period is daily, and $M = 10$. Thus, the AR model predicts tomorrow's value, given the values over the last 10 days.

For each of the following cases, give a short interpretation or description of the AR model in English, without referring to mathematical concepts like vectors, inner product, and so on. You can use words like 'yesterday' or 'today'.

(a) $\beta \approx e_1$.

(b) $\beta \approx 2e_1 - e_2$.

(c) $\beta \approx e_6$.

(d) $\beta \approx 0.5e_1 + 0.5e_2$.

(a) $\beta \approx e_1$

Z_t = Today.

Z_{t-1} = Yesterday.

$$Z_{t+1} = Z_t.$$

(b) $\beta \approx 2e_1 - e_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$Z_{t+1} = 2Z_t - Z_{t-1}$$

(c) $\beta \approx e_6 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \rightarrow$ 6th entry.

$$Z_{t+1} = Z_{t-5}$$

(d) $\beta \approx 0.5e_1 + 0.5e_2 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$Z_{t+1} = \frac{Z_t + Z_{t-1}}{2}.$$

4) *Industry or sector exposure.* Consider a set of n assets or stocks that we invest in. Let f be an n-vector that encodes whether each asset is in some specific industry or sector, *e.g.*, pharmaceuticals or consumer electronics. Specifically, we take $f_i = 1$ if asset i is in the sector, and $f_i = 0$ if it is not. Let the n-vector h denote a portfolio, with h_i the dollar value held in asset i (with negative meaning a short position). The inner product $f^T h$ is called the (dollar value) *exposure* of our portfolio to the sector. It gives the net dollar value of the portfolio that is invested in assets from the sector. A portfolio h is called *neutral* (to a sector or industry) if $f^T h = 0$.

A portfolio h is called *long only* if each entry is nonnegative, *i.e.*, $h_i \geq 0$ for each i. This means the portfolio does not include any short positions.

What does it mean if a long-only portfolio is neutral to a sector, say, pharmaceuticals? Your answer should be in simple English, but you should back up your conclusion with an argument.

$f \in \mathbb{R}^n$

$$f_i = \begin{cases} 1 & \text{if asset } i \text{ is in sector} \\ 0 & \text{otherwise.} \end{cases}$$

$h \in \mathbb{R}^n$

h_i: dollar value for asset i.

$$h_i = \begin{cases} +ve & \text{long.} \\ -ve. & \text{short} \end{cases}$$

$$f^T h = 0 \quad , \quad h_i \geq 0 \;\; \forall i$$

no assets in pharmaceutical sector.

5) *Regression model.* Consider the regression model $\hat{y} = x^T\beta + v$, where $\hat{y}$ is the predicted response, x is an 8-vector of features, β is an 8-vector of coefficients, and v is the offset term. Determine whether each of the following statements is true or false.

(a) If $\beta_3 > 0$ and $x_3 > 0$, then $\hat{y} \geq 0$.

(b) If $\beta_2 = 0$ then the prediction $\hat{y}$ does not depend on the second feature x_2.

(c) If $\beta_6 = -0.8$, then increasing x_6 (keeping all other x_is the same) will decrease $\hat{y}$.

$$\hat{y} = x^T\beta + v$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_8 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_8 \end{bmatrix} + v$$

$$= x_1\beta_1 + x_2\beta_2 + \cdots + x_8\beta_8 + v..$$

(a) False

$$\hat{y} = \underbrace{x_1 \beta_1 + x_2\beta_2 + x_4\beta_4 + \cdots + x_8\beta_8 + v}_{<0.} + \underset{>0.}{x_3\beta_3}$$

$$|x_3\beta_3| < |x_1\beta_1 + x_2\beta_2 + x_4\beta_4 + \cdots + x_8\beta_8 + v|$$

(b) True.

(c) True.

6) *Linear combinations of linear combinations.* Suppose that each of the vectors $b_1, \ldots, b_k$ is a linear combination of the vectors $a_1, \ldots, a_m$, and c is a linear combination of $b_1, \ldots, b_k$. Then c is a linear combination of $a_1, \ldots, a_m$. Show this for the case with $m = k = 2$.

$m = k = 2.$

$\underline{b}_1 = \alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2$

$\underline{b}_2 = \alpha_3 \underline{a}_1 + \alpha_4 \underline{a}_2.$

$$\begin{aligned}
\underline{c} &= \beta_1 \underline{b}_1 + \beta_2 \underline{b}_2. \\
&= \beta_1(\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2) + \beta_2(\alpha_3 \underline{a}_1 + \alpha_4 \underline{a}_2) \\
&= (\beta_1\alpha_1 + \beta_2\alpha_3)\, \underline{a}_1 + (\beta_1\alpha_2 + \beta_2\alpha_4)\, \underline{a}_2 \\
&= \gamma_1 \underline{a}_1 + \gamma_2 \underline{a}_2
\end{aligned}$$

7) *Angle between two nonnegative vectors.* Let x and y be two nonzero n-vectors with nonnegative entries, *i.e.*, each $x_i \geq 0$ and each $y_i \geq 0$. Show that the angle between x and y lies between 0 and 90°. Draw a picture for the case when $n = 2$, and give a short geometric explanation. When are x and y orthogonal?

- Initialize two vectors(2-D) with nonnegative entries and plot them.(use matplotlib)
- Modify the entries of vectors and observe the angle between the vectors.

$\underline{x}, \underline{y} \in \mathbb{R}^n$.

$\underline{x}^T \underline{y} = \|x\| \|y\| \cos \theta$

θ : angle between $\underline{x}$ and $\underline{y}$

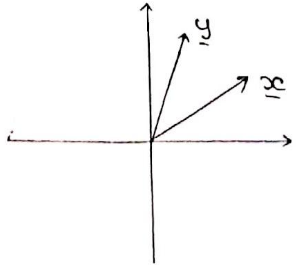
$$\cos \theta = \frac{x^T y}{\|x\| \|y\|}.$$

$x^T y \geq 0$ as $x_i \geq 0$, $y_i \geq 0$.

$\|x\| \geq 0$, $\|y\| \geq 0$.

$\cos \theta \geq 0$. θ lies b/w 0 and 90°

$n = 2$.



$x^T y = 0$.

8) *Distance between Boolean vectors.* Suppose that x and y are Boolean n-vectors, which means that each of their entries is either 0 or 1. What is their distance $\|x - y\|$?

- Initialize two vectors with boolean entries(0s and 1s)
- Calculate the distance between them
- Calculate hamming distance (Use scipy)

eg: $x = [0\ 1\ 1\ 0\ 1\ 0\ 0]^T$

$y = [1\ 1\ 0\ 0\ 1\ 1\ 0]^T$

$x - y = [-1\ 0\ 1\ 0\ 0\ -1\ 0]$

$\|x - y\|^2 = (x - y)^T (x - y)$

$= [-1\ 0\ 1\ 0\ 0\ -1\ 0] \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

$= 1 + 0 + 1 + 0 + 0 + 1 + 0$

$= 3.$

$\|x - y\|^2 =$ hamming distance.

9) *Reverse triangle inequality.* Suppose a and b are vectors of the same size. The triangle inequality states that $\|a+b\| \leq \|a\| + \|b\|$. Show that we also have $\|a+b\| \geq \|a\| - \|b\|$.

Triangle equality. When does the triangle inequality hold with equality, *i.e.*, what are the conditions on a and b to have $\|a+b\| = \|a\| + \|b\|$?

$$\|a+b\| \leq \|a\| + \|b\|$$

On $\|(a+b)-b\|$. Apply triangle inequality.

$$\|(a+b)-b\| \leq \|a+b\| + \|-b\| \quad = \|a+b\| + \|b\|$$

$$\|a\| \leq \|a+b\| + \|b\|$$

$$\|a+b\| \geq \|a\| - \|b\|$$

$$\|a+b\| \le \|a\| + \|b\|.$$

$$\begin{aligned}\|a+b\|^2 &= (a+b)^T(a+b) \\ &= a^Ta + b^Tb + a^Tb + b^Ta \\ &= \|a\|^2 + \|b\|^2 + 2a^Tb.\end{aligned}$$

$|a^Tb| \le \|a\|\,\|b\|$. Cauchy Schwartz inequality.

$$a^Tb = \|a\|\,\|b\| \cos\theta$$

$$a^Tb = \|a\|\,\|b\|$$

$$\Rightarrow \cos\theta = 1.$$

$$\theta = 0 \text{ or } 180^\circ.$$

$$\begin{aligned}\|a+b\|^2 &= \|a\|^2 + \|b\|^2 + 2\|a\|\,\|b\| \\ &= (\|a\| + \|b\|)^2\end{aligned}$$

$$\Rightarrow \|a+b\|^2 = \|a\| + \|b\|.$$

when $\underline{a} = \alpha \underline{b}$, $\alpha \geq 0$.

or $\angle(\underline{a}, \underline{b}) = 0$.

10) *When is the outer product symmetric?* Let a and b be n-vectors. The inner product is symmetric, *i.e.*, we have $a^T b = b^T a$. The outer product of the two vectors is generally *not* symmetric; that is, we generally have $ab^T \neq ba^T$. What are the conditions on a and b under which $ab = ba^T$? You can assume that all the entries of a and b are nonzero. (The conclusion you come to will hold even when some entries of a or b are zero.)

$a, b \in \mathbb{R}^n$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$ab^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & & & \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{bmatrix}$$

$$ba^T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$= \begin{bmatrix} b_1a_1 & b_1a_2 & \cdots & b_1a_n \\ b_2a_1 & b_2a_2 & \cdots & b_2a_n \\ \vdots & & & \\ b_na_1 & b_na_2 & \cdots & b_na_n \end{bmatrix}$$

$$a_1b_2 = b_1a_2 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$a_1b_n = b_1a_n \Rightarrow \frac{a_1}{b_1} = \frac{a_n}{b_n}$$

generally $\frac{a_i}{b_i}$ = a constant.

11) *Nearest unit vector.* What is the nearest neighbor of the n-vector x among the unit vectors $e_1, \ldots, e_n$?

$$\|x - e_i\|^2 = (x - e_i)^T (x - e_i)$$

$$= x^T x + e_i^T e_i - x^T e_i - e_i^T x$$

$$= \|x\|^2 + \|e_i\|^2 - 2x^T e_i$$

$$\min_i \|x - e_i\|^2 = \max_i x^T e_i$$

12) Show that the following inequality holds

$$|w_1x_1y_1 + w_2x_2y_2 + \dots w_nx_ny_n| \leq ||x||_w ||y||_w$$

where

$$||x||_w = \sqrt{w_1x_1^2 + w_2x_2^2 + \cdots + w_nx_n^2}$$

$$|\tilde{x}^T\tilde{y}| \leq ||\tilde{x}|| \, ||\tilde{y}||.$$

$$\tilde{x} = \begin{bmatrix} \sqrt{w_1}\,x_1 \\ \sqrt{w_2}\,x_2 \\ \vdots \\ \sqrt{w_n}\,x_n \end{bmatrix} \qquad \tilde{y} = \begin{bmatrix} \sqrt{w_1}\,y_1 \\ \sqrt{w_2}\,y_2 \\ \vdots \\ \sqrt{w_n}\,y_n \end{bmatrix}$$

$$\tilde{x}^T\tilde{y} = \begin{bmatrix}\sqrt{\omega_1}x_1 & \sqrt{\omega_2}x_2 & \cdots & \sqrt{\omega_n}x_n\end{bmatrix}\begin{bmatrix}\sqrt{\omega_1}y_1 \\ \sqrt{\omega_2}y_2 \\ \vdots \\ \sqrt{\omega_n}y_n\end{bmatrix}$$

$$= \omega_1x_1y_1 + \omega_2x_2y_2 + \cdots + \omega_nx_ny_n.$$

$$|\tilde{x}^T\tilde{y}| = |\omega_1x_1y_1 + \omega_2x_2y_2 + \cdots + \omega_nx_ny_n|.$$

$$\|\tilde{x}\| = \sqrt{\omega_1x_1^2 + \omega_2x_2^2 + \cdots + \omega_nx_n^2} = \|x\|_\omega$$

$$\|\tilde{y}\| = \sqrt{\omega_1y_1^2 + \omega_2y_2^2 + \cdots + \omega_ny_n^2} = \|y\|_\omega.$$

$$|\tilde{x}^T\tilde{y}| \le \|\tilde{x}\|\|\tilde{y}\|.$$

$$\Rightarrow |\omega_1x_1y_1 + \omega_2x_2y_2 + \cdots + \omega_nx_ny_n| \le \|x\|_\omega\|y\|_\omega.$$

Matrices

13) *Block matrix.* Assuming the matrix

$$K = \begin{bmatrix} I & A^T \\ A & 0 \end{bmatrix}$$

makes sense, which of the following statements must be true? ('Must be true' means that it follows with no additional assumptions.)

(a) K is square.

(b) A is square or wide.

(c) K is symmetric, *i.e.*, $K^T = K$.

(d) The identity and zero submatrices in K have the same dimensions.

(e) The zero submatrix is square.

$$K = \begin{bmatrix} I & A^T \\ A & 0 \end{bmatrix}$$

let $A \in \mathbb{R}^{m \times n}$.

$$K = \begin{bmatrix} I_{n\times n} & A^T_{n\times m} \\ A_{m\times n} & 0_{m\times m} \end{bmatrix}_{(m+n)\times(m+n)}$$

(a) Must be true.

(b) A can be tall also.

(c) $K^T = \begin{bmatrix} I & A^T \\ A & 0 \end{bmatrix}^T = \begin{bmatrix} I^T & (A)^T \\ (A^T)^T & 0^T \end{bmatrix}$

$$= \begin{bmatrix} I & A^T \\ A & 0 \end{bmatrix} = K.$$

Must be true.

(d) $I \in \mathbb{R}^{n\times n}$

$0 \in \mathbb{R}^{m\times m}$

(e) Must be true.

14) *Matrix sizes.* Suppose A, B, and C are matrices that satisfy $A + BB^T = C$. Determine which of the following statements are necessarily true. (There may be more than one true statement.)

(a) A is square.

(b) A and B have the same dimensions.

(c) A, B, and C have the same number of rows.

(d) B is a tall matrix.

$A + BB^T = C$

Suppose. $B = m \times n$ then $B^T = n \times m$

Therefore BB^T is always a square.

and BB^T is $m \times m$.

(a) To perform addition BB^T and A should have same dimension. There fore A is square.

(b) A and B may not have the same dimension.

$B \rightarrow m \times n$.

$A \rightarrow m \times m$.

(c)

$$\begin{array}{ccccc} A & + & BB^T & = & C \\ \downarrow & & \downarrow & & \downarrow \\ m \times m & & m \times m & & m \times m \end{array}$$

$A \rightarrow m \times m$

$B \rightarrow m \times n$

$C \rightarrow m \times m$

A, B, C has same no. of rows.

(d) B is in general any $m \times n$ matrix.

$m > n$ or $m = n$ or $m < n$.

It is not always true that B is tall.

15) *Multiplication by a diagonal matrix.* Suppose that A is an $m \times n$ matrix, D is a diagonal matrix, and $B = DA$. Describe B in terms of A and the entries of D. You can refer to the rows or columns or entries of A.

- Create a matrix A and diagonal matrix D
- Calculate B = DA
- Observe the entries of B

Let's take an example.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

$$B = DA = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} \end{bmatrix}$$

In general. The rows of A will be multiplied by the diagonal entries of D.

16) *Skew-symmetric matrices.* An $n \times n$ matrix A is called *skew-symmetric* if $A^T = -A$, *i.e.*, its transpose is its negative. (A symmetric matrix satisfies $A^T = A$.)

(a) Find all 2×2 skew-symmetric matrices.

(b) Explain why the diagonal entries of a skew-symmetric matrix must be zero.

(c) Show that for a skew-symmetric matrix A, and any n-vector x, $(Ax) \perp x$. This means that Ax and x are orthogonal. *Hint.* First show that for any $n \times n$ matrix A and n-vector x, $x^T(Ax) = \sum_{i,j=1}^n A_{ij}x_ix_j$.

(d) Now suppose A is any matrix for which $(Ax) \perp x$ for any n-vector x. Show that A must be skew-symmetric. *Hint.* You might find the formula

$$(e_i + e_j)^T(A(e_i + e_j)) = A_{ii} + A_{jj} + A_{ij} + A_{ji},$$

valid for any $n \times n$ matrix A, useful. For $i = j$, this reduces to $e_i^T(Ae_i) = A_{ii}$.

Note that A has to be square.

$$A^T = -A$$

If A is $m \times n$ then $A^T = n \times m$.

from equality $m = n$.

Now, $$-\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$\Rightarrow \quad -a_{11} = a_{11} \qquad \Rightarrow \quad a_{11} = 0$

$\Rightarrow \quad -a_{12} = a_{21} \qquad \Rightarrow \quad a_{12}, a_{21}$ has opposite sign

$\quad\ \ -a_{21} = a_{12} \qquad\qquad$ but same value.

$\Rightarrow \quad -a_{22} = a_{22} \qquad \Rightarrow \quad a_{22} = 0$

A will be in form $\begin{bmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{bmatrix}$

Hence any matrix $\alpha \begin{bmatrix} 0 & 1 \\ -1, & 0 \end{bmatrix}$ are skew-symmetric

(b) As $A^T = -A$.

therefore $a_{ii} = -a_{ii} \quad \forall i = 1, 2, \dots n$

$$\Rightarrow 2a_{ii} = 0$$

$$\Rightarrow a_{ii} = 0.$$

(c)

$x^T(Ax)$ $\qquad a_{ij} = -a_{ji} \ \forall i,j \quad . \ a_{ii} = 0 \ \forall i$

$$= (x_1 \ x_2 \ \cdots \ x_n)\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2$$

$$+ (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + \cdots$$

$$+ (a_{1n} + a_{n1})x_1x_n + (a_{23} + a_{32})x_2x_3 + \cdots + (a_{n-1} + a_n)x_{n-1}x_n$$

Now. $a_{11} = a_{22} = \cdots = a_{nn} = 0$

and $a_{ij} = -a_{ji}$

Hence. $x^T (Ax) = 0$.

(d) Consider a vector $x = (e_i + e_j)$

$$(e_i + e_j)^T A (e_i + e_j)$$

$$= (e_i^T + e_j^T) A (e_i + e_j) = e_i^T A e_i + e_j^T A e_i + e_i^T A e_j + e_j^T A e_j$$

$$= a_{ii} + a_{ji} + a_{ij} + a_{jj} = 0 \quad \text{—— (1)}$$

take $x = e_i$

$$e_i^T A e_i = 0$$

$\Rightarrow \; a_{ii} = 0 \quad \forall i$

from (1) $\quad a_{ij} + a_{ji} = 0 \;\Rightarrow\; a_{ij} = -a_{ji}$

$\Rightarrow$ A is skew-symmetric.

17. Trimming a vector. Find a matrix A for which $Ax = (x_2, x_3, \ldots, x_n)$, where x is an n-vector. (Be sure to specify the size of A, and describe all its entries.)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

$(n-1 \times n)$. $\quad (n \times 1) \quad (n-1) \times 1$

$A \rightarrow n-1 \times n.$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = x_2$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = x_3$$

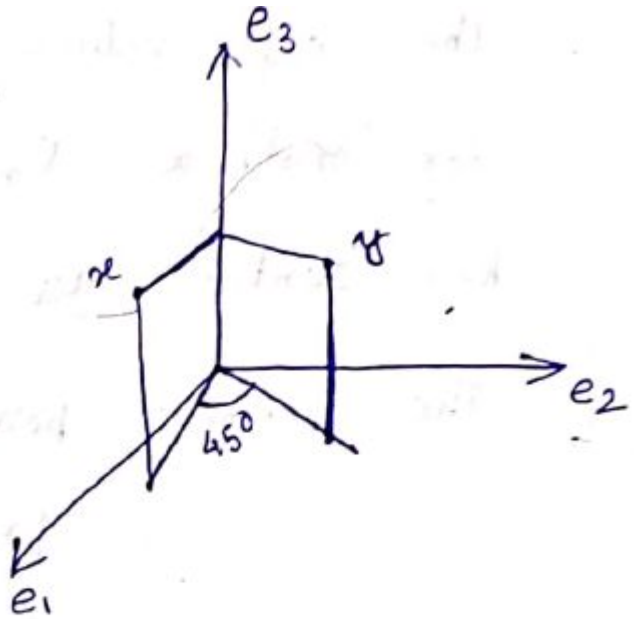
$$\vdots$$

$$a_{n-1,1}x_1 + a_{n-1,2}x_2 + \cdots + a_{n-1,n}x_n = x_n$$

$$a_{12} = 1, \quad a_{23} = 1, \quad \cdots \quad a_{n-1,n} = 1$$

$$A = \begin{array}{cc} & n \\ n-1 & \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \end{array}$$

18) *3-D rotation.* Let x and y be 3-vectors representing positions in 3-D. Suppose that the vector y is obtained by rotating the vector x about the vertical axis (*i.e.*, e_3) by 45° (counterclockwise, *i.e.*, from e_1 toward e_2). Find the 3×3 matrix A for which $y = Ax$.



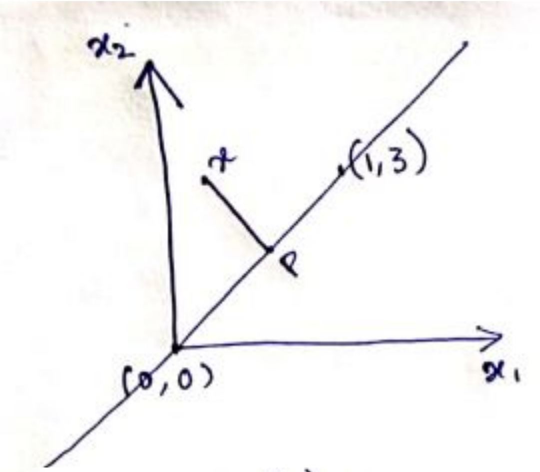
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x_3 = y_3$

Hence.
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

19. Projection on a line: Let P(x) denote the projection of the 2-D point (2-vector) x onto the line that passes through (0, 0) and (1, 3) (This means that P(x) is the point on the line that is closest to x) Give the matrix A for which P(x) = Ax for any x.



$$p = \theta \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The distance between x and p are ~~minim~~ minimum among all points on the line.

So, θ can be found such that

$\|p - x\|^2$ is minimum

$$\|p - x\|^2 = (\theta - x_1)^2 + (3\theta - x_2)^2$$

$$= x_1^2 + x_2^2 - 2\theta x_1 - 6\theta x_2 + 10\theta^2$$

$$\frac{d}{d\theta}\left(\|p - x\|^2\right) = 0$$

$$\Rightarrow \quad -2x_1 - 6x_2 + 20\theta = 0$$

$$\Rightarrow \quad \theta = \frac{1}{10}\left(x_1 + 3x_2\right)$$

$$\therefore \quad p = \begin{pmatrix} \frac{1}{10}\left(x_1 + 3x_2\right) \\ \frac{3}{10}\left(x_1 + 3x_2\right) \end{pmatrix}$$

$$\therefore \quad A = \begin{pmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{pmatrix}$$

Eigenvalues and Eigenvectors

20) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

$$Ax = \lambda x$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad x_1 - x_2 = \lambda x_1$$

$$2x_1 + 4x_2 = \lambda x_2$$

$$\Rightarrow \quad x_1 = \frac{1}{(1-\lambda)} x_2$$

$$\Rightarrow \quad \frac{2x_2}{1-\lambda} + (4-\lambda)x_2 = 0$$

$$\Rightarrow \quad \frac{2}{1-\lambda} + (4-\lambda) = 0 \quad [x_2 \neq 0]$$

$$\Rightarrow \quad (1-\lambda)(4-\lambda) + 2 = 0$$

$$\Rightarrow \quad \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow \quad \lambda = 3, 2.$$

$$\text{Trace}(A) = 1 + 4 = 5 = \lambda_1 + \lambda_2$$

$$\text{Det}(A) = 4 + 2 = 6 = \lambda_1 \lambda_2.$$

21) Suppose that λ is an eigenvalue of A, and x is its eigenvector: $Ax = \lambda x$.

(a) Show that this same x is an eigenvector of $B = A - 7I$, and find the eigenvalue.

(b) Assuming $\lambda \neq 0$, show that x is also an eigenvector of A^{-1}—and find the eigenvalue.

Check for the A used in Q.20

$$B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

(a)

$$(A - 7I)\underline{x}$$

$$= A\underline{x} - 7\underline{x}$$

$$= \lambda\underline{x} - 7\underline{x}$$

$$= (\lambda - 7)\underline{x}$$

Hence, $\underline{x}$ is a eigen vector of $(A - 7I)$ and eigen value $= (\lambda - 7)$

(b) $(A^{-1}A)\underline{x} = I\underline{x} = \underline{x}$

$\Rightarrow A^{-1}(A\underline{x}) = \underline{x}$

$\Rightarrow A^{-1}\lambda\underline{x} = \underline{x}$

$\Rightarrow A^{-1}\underline{x} = \lambda^{-1}\underline{x}$

Ex:

$$\begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow (-6-\lambda)x_1 = x_2$

$2x_1 = (3+\lambda)x_2$

$\Rightarrow -\dfrac{(\lambda+6)(\lambda+3)}{2}x_2 = x_2$

$\Rightarrow (\lambda+6)(\lambda+3)+2 = 0$

$\Rightarrow \lambda^2 + 9\lambda + 20 = 0$

$\Rightarrow \lambda = -5, -4$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \qquad A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\frac{1}{6}\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad (4 - 6\lambda)x_1 = -x_2$$

$$-2x_1 + (1-6\lambda)x_2 = 0$$

$$\Rightarrow \quad -2x_1 - (1-6\lambda)(4-6\lambda)x_1 = 0$$

$$\Rightarrow \quad (1-6\lambda)(4-6\lambda) + 2 = 0$$

$$\Rightarrow \quad 36\lambda^2 - 30\lambda + 6 = 0$$

$$\Rightarrow \quad 6\lambda^2 - 5\lambda + 1 = 0. \quad \Rightarrow \lambda = \frac{1}{2}, \frac{1}{3}$$

22. Find the eigen value decomposition for the following matrix. Find the rotated axis on which A behaves like a diagonal matrix.

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow (1-\lambda) x_1 + \frac{1}{2} x_2 = 0$$

$$\frac{1}{2} x_1 + (1-\lambda) x_2 = 0$$

$$2(1-\lambda) x_1$$

$\lambda = \frac{1}{2}$, $\quad x_2 = -x_1$

$\lambda = \frac{3}{2}$ $\quad x_2 = x_1$

$\Rightarrow \lambda = \frac{1}{2}$ eigen vector $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$\lambda = \frac{3}{2}$ eigen vector $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} 1/2 & 0 \\ 0 & 3/2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{V^T}$$

- Initialize A.
- Calculate B=A-7I
- Calculate eigenvalues and eigenvectors of A and B
- Calculate sum and product of eigenvalues and compare them to trace and determinant of A.
- Compare eigenvalues of B with eigenvalues of A.