Foundations of Machine Learning

Support Vector Machines

Sep 2022

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Classification Methods

- k-Nearest Neighbors
- Decision Trees
- Naïve Bayes
- Support Vector Machines
- Logistic Regression
- Neural Networks
- Ensemble Methods (Boosting, Random Forests)

SVM: Overview and History

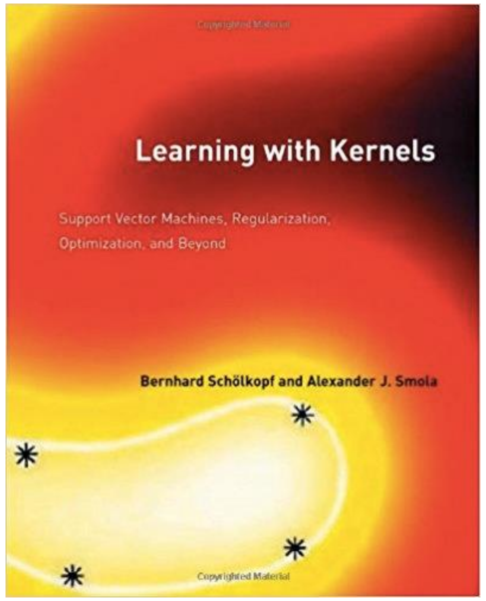
- A discriminative classifier
  - Parametric, Inductive
- Inspired from statistical learning theory
- Developed in 1992 by Vapnik, Guyon, Boser
- Became popular because of its success in handwritten digit recognition
- Was one of the go-to methods in ML since mid-1990s (only recently displaced by deep learning)

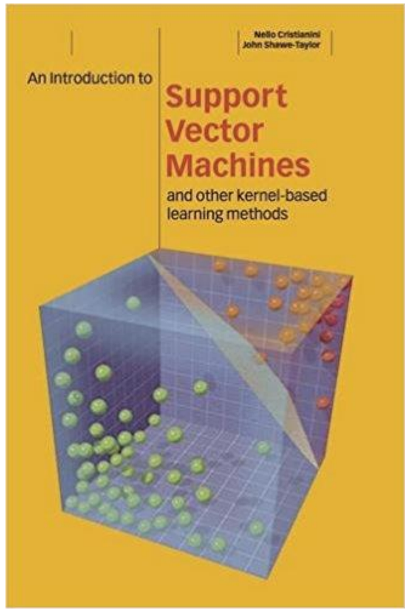
Papers that introduced SVM in its current form

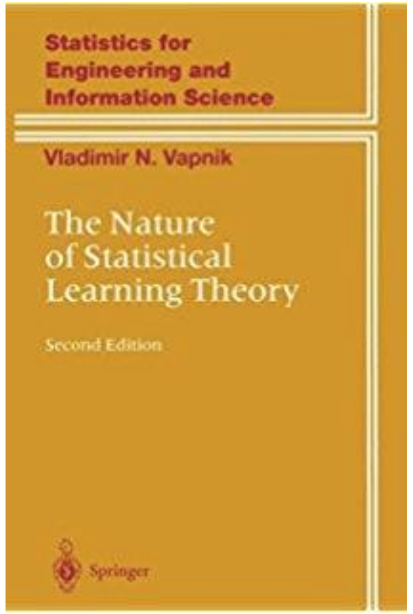
- Boser, B. E.; Guyon, I. M.; Vapnik, V. N. (1992). "A training algorithm for optimal margin classifiers". Proceedings of the fifth annual workshop on Computational learning theory – COLT '92.
- Cortes, C.; Vapnik, V. (1995). "Support-vector networks". Machine Learning. 20 (3): 273–297.

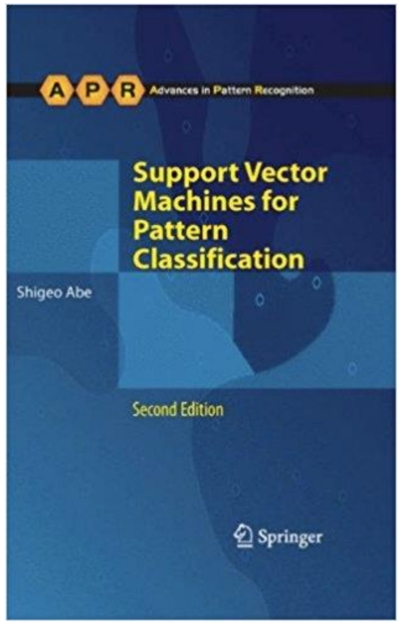
SVM: Overview and History

- Associated key words
  - Large-margin classifier, Max-margin classifier, Kernel methods, Reproducing kernel Hibert space, Statistical learning theory

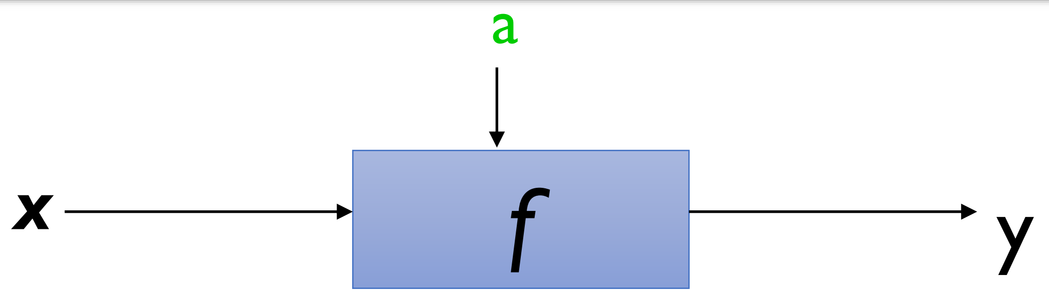

Learning with Kernels
Support Vector Machines, Regularization, Optimization, and Beyond
Bernhard Schölkopf and Alexander J. Smola



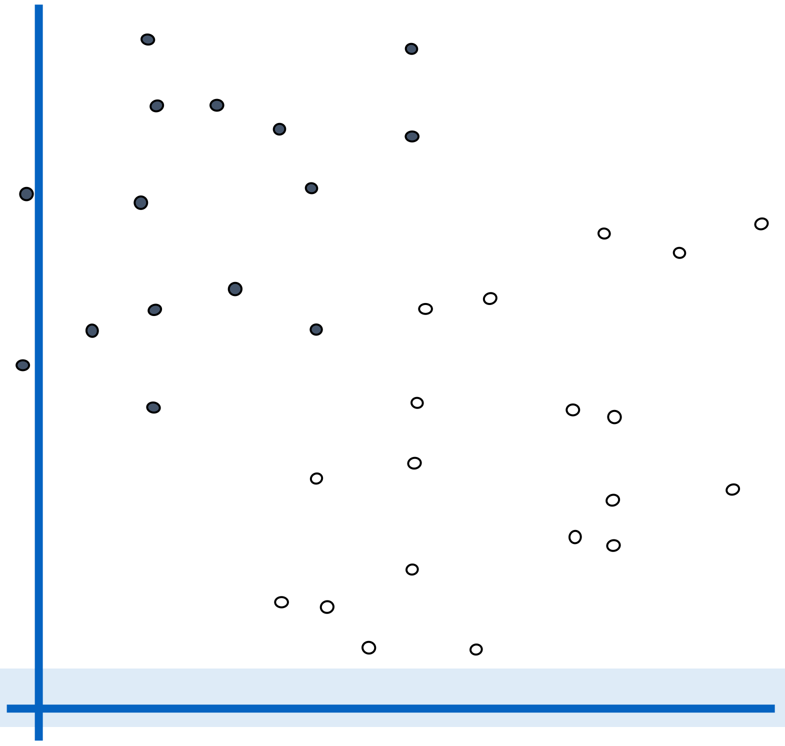

Statistics for Engineering and Information Science
Vladimir N. Vapnik
The Nature of Statistical Learning Theory
Second Edition
Springer


APR Advances in Pattern Recognition
Support Vector Machines for Pattern Classification
Shigeo Abe
Second Edition
Springer

Linear Classifiers



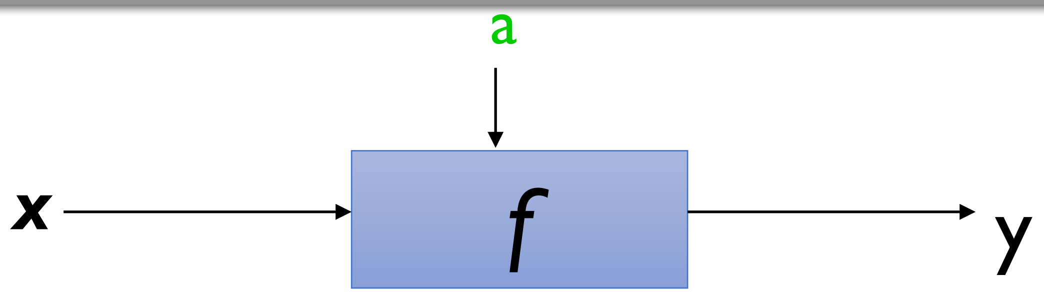
● denotes +1
○ denotes -1



$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} + b)$

How would you classify this data?

Linear Classifiers



$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} + b)$

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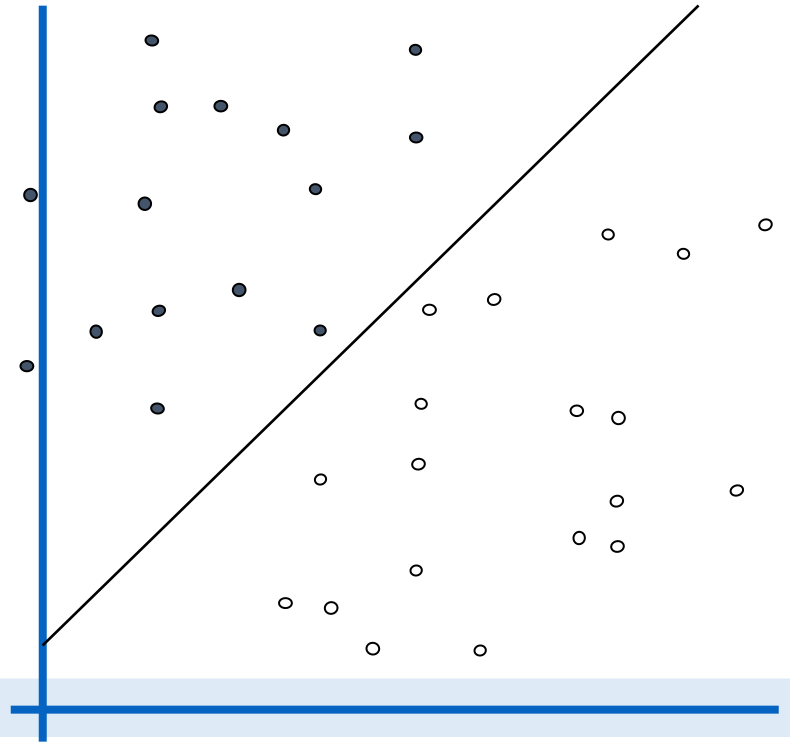
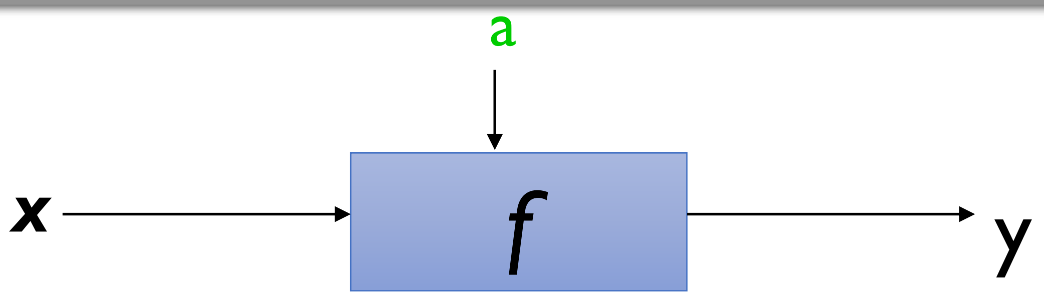


How would you classify this data?

Linear Classifiers



$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} + b)$

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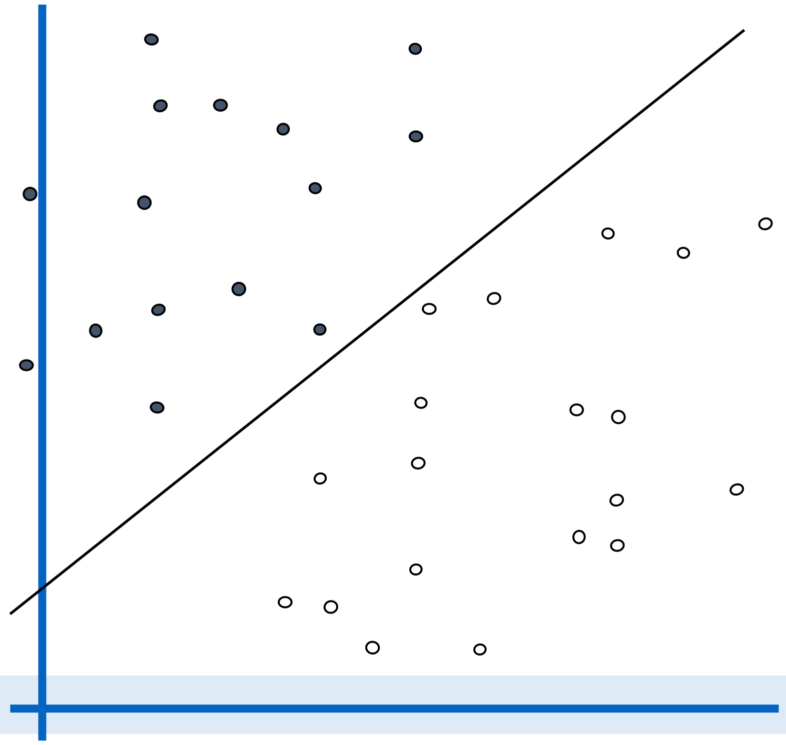
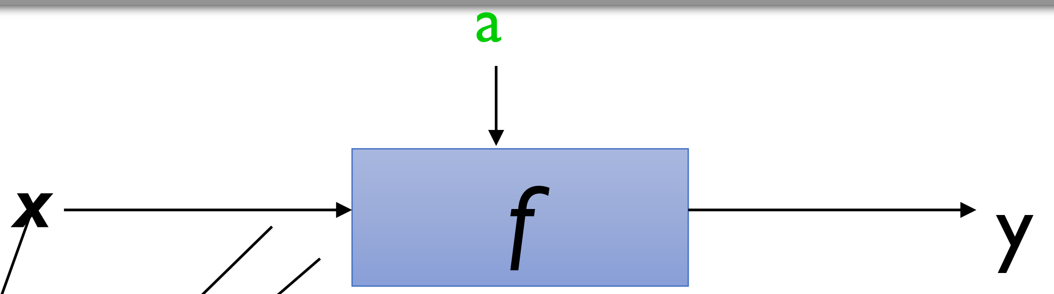


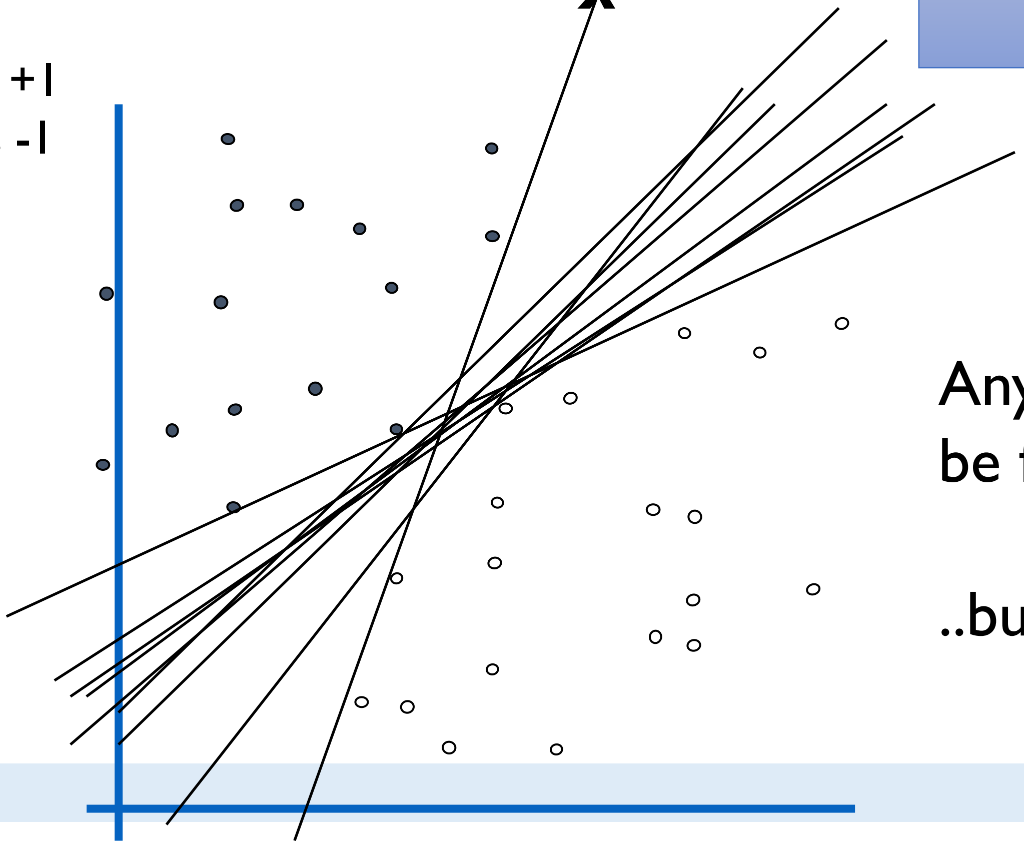
How would you classify this data?

Linear Classifiers



$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} + b)$

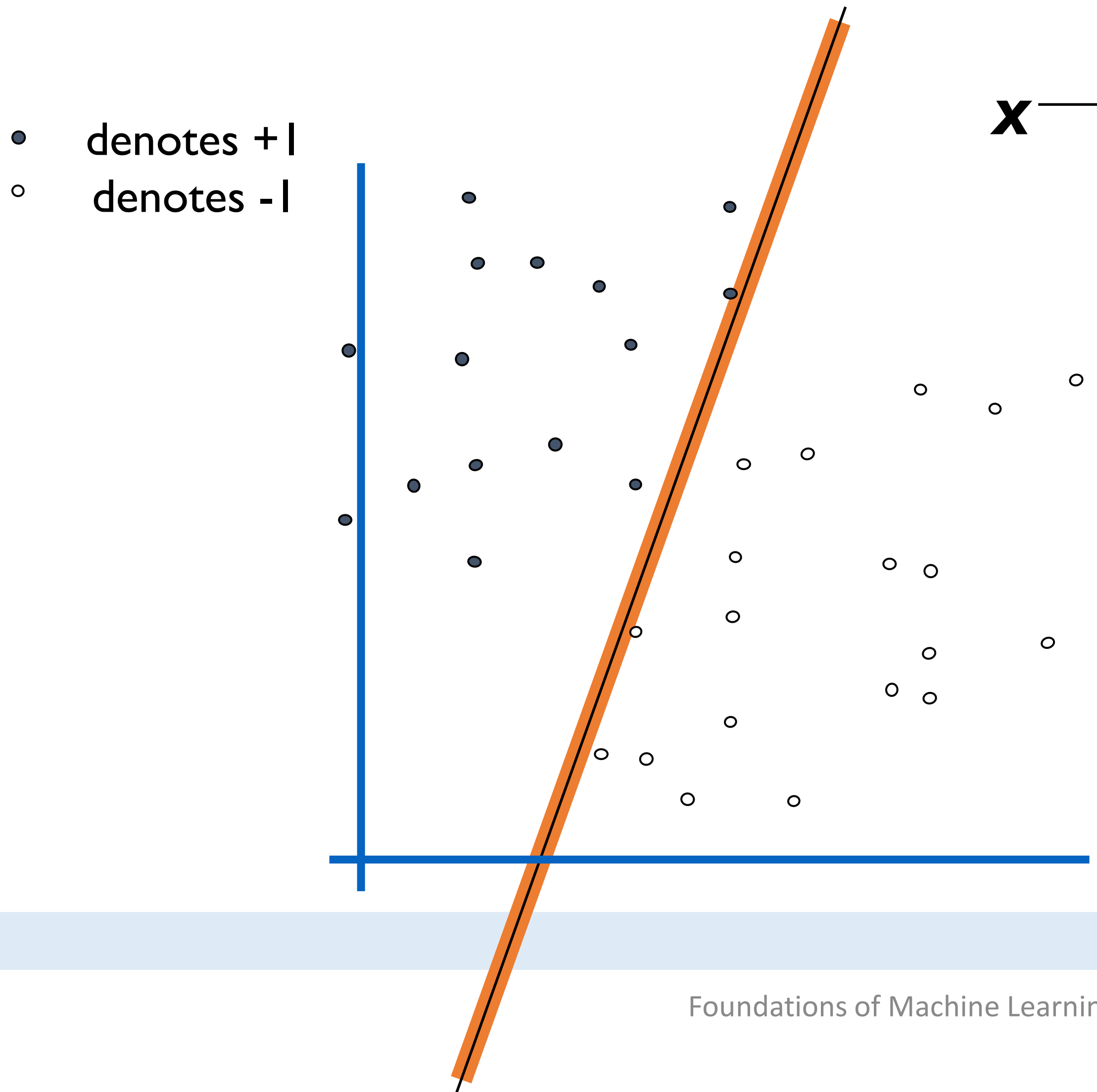
- ● denotes +1
- ○ denotes -1

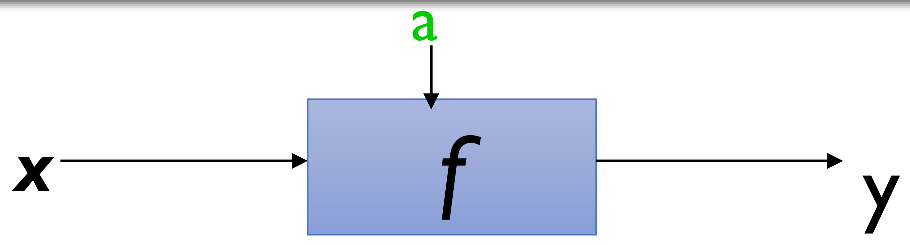


Any of these would be fine..

..but which is best?

Linear Classifiers

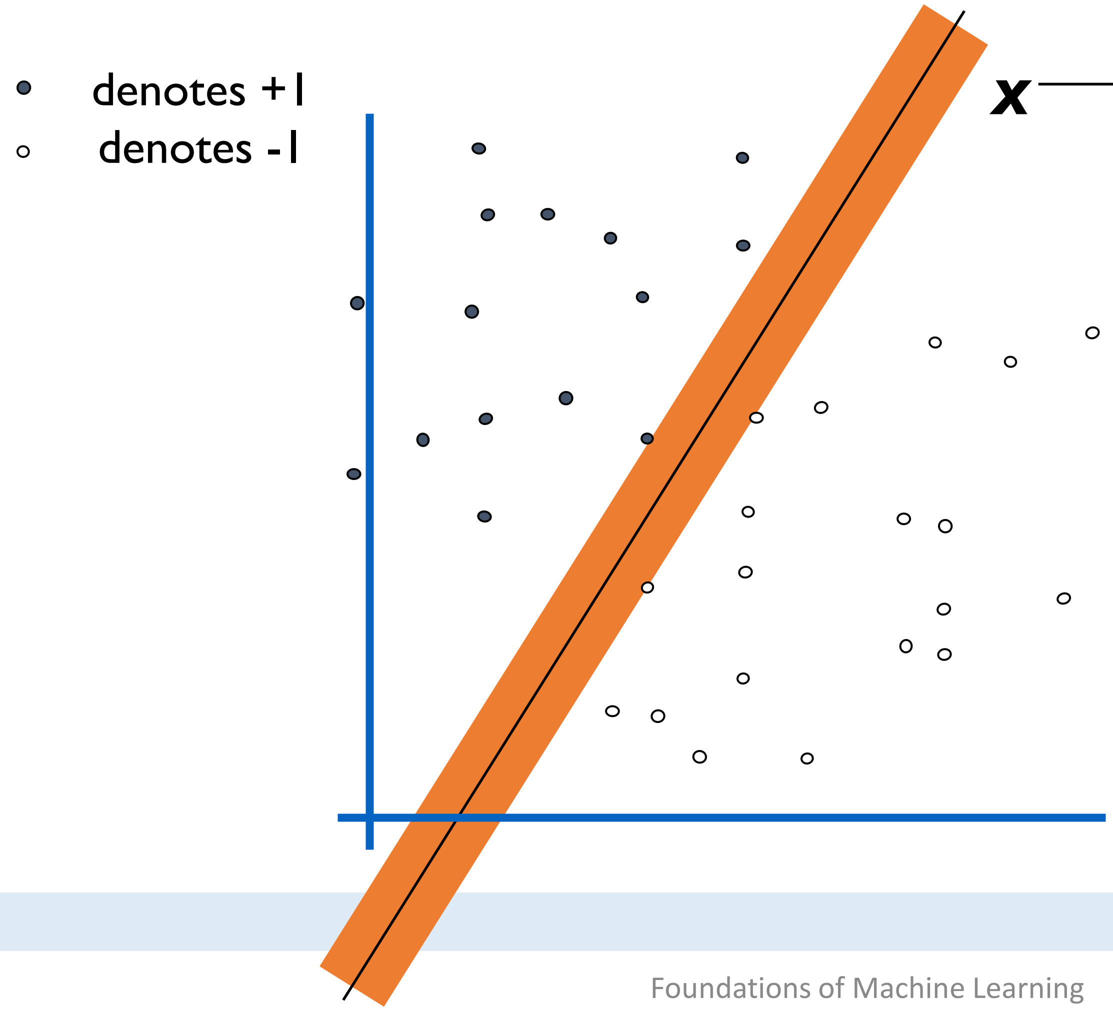


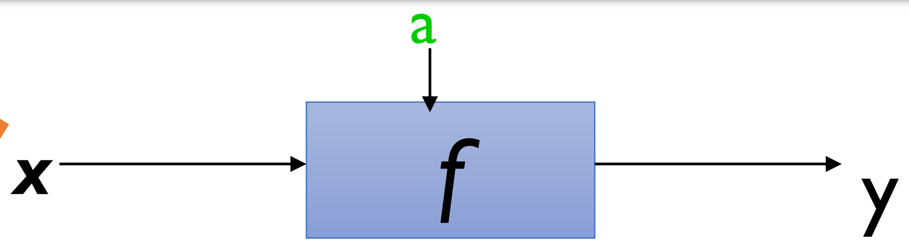


$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} + b)$

Define the margin of a linear classifier as the width that the boundary could be increased by before hitting a datapoint.

Linear Classifiers

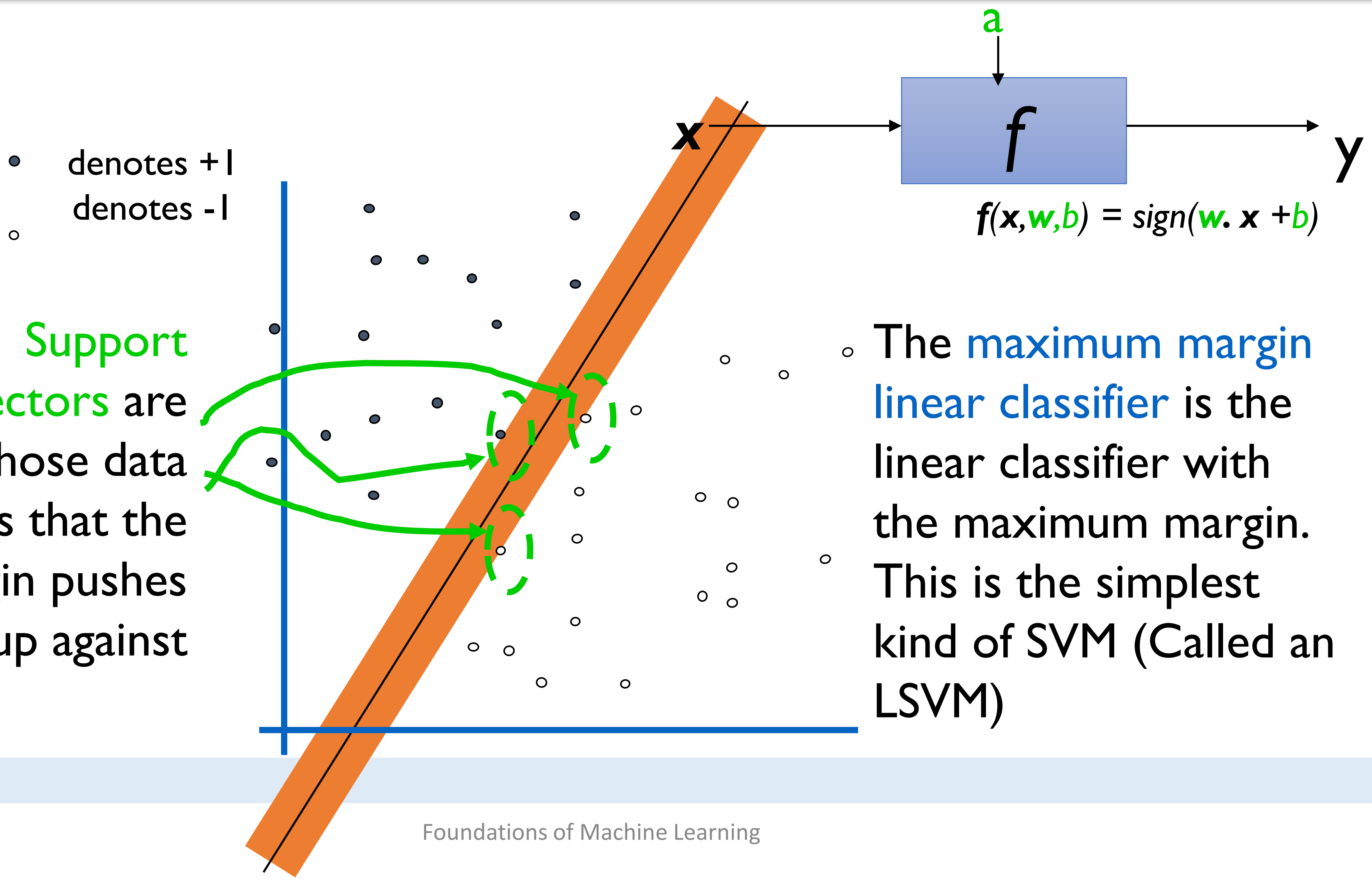




$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} + b)$

The maximum margin linear classifier is the linear classifier with the maximum margin.
This is the simplest kind of SVM (Called an LSVM)

Maximum Margin Classifier

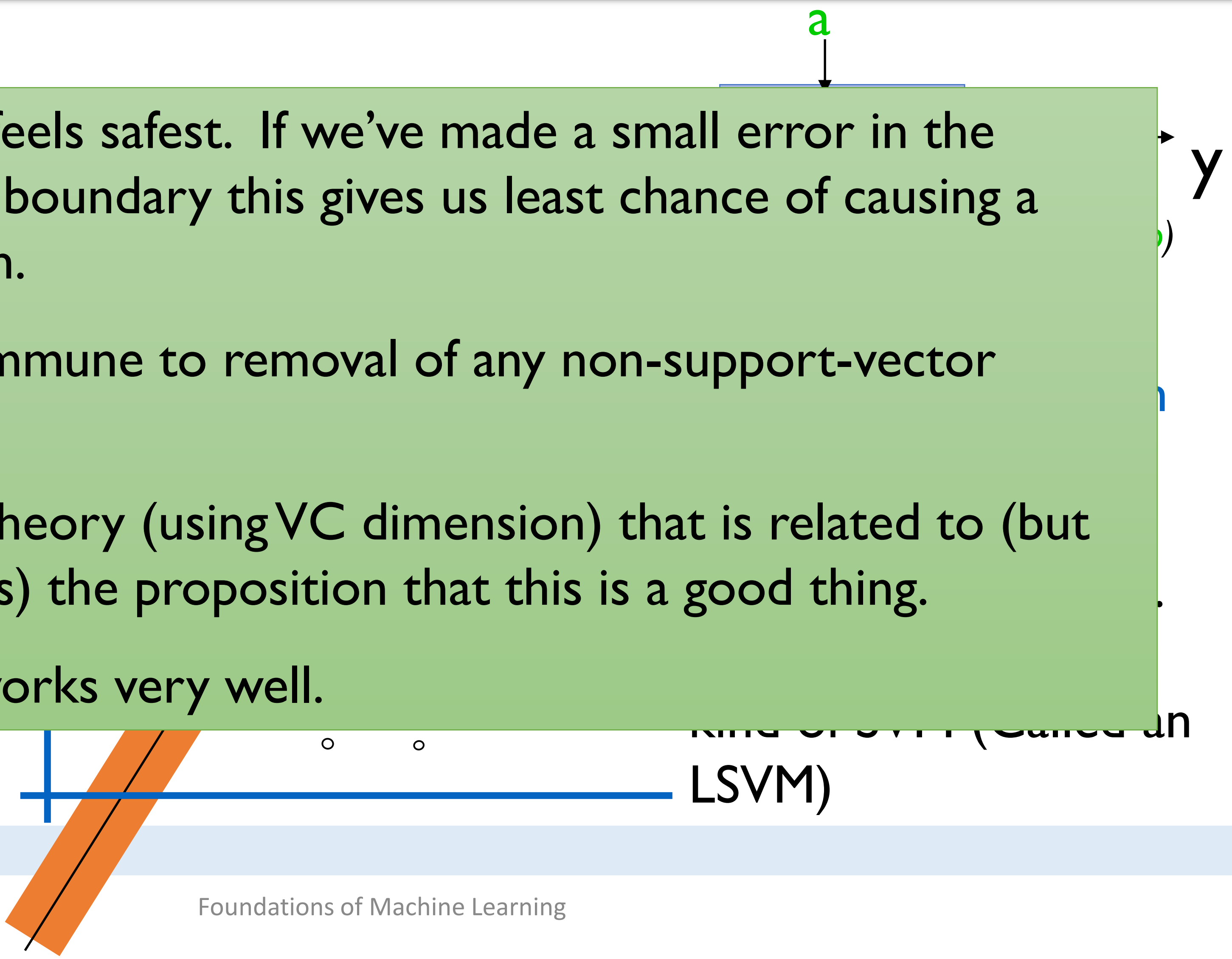


$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} + b)$

Support Vectors are those data points that the margin pushes up against

The maximum margin linear classifier is the linear classifier with the maximum margin. This is the simplest kind of SVM (Called an LSVM)

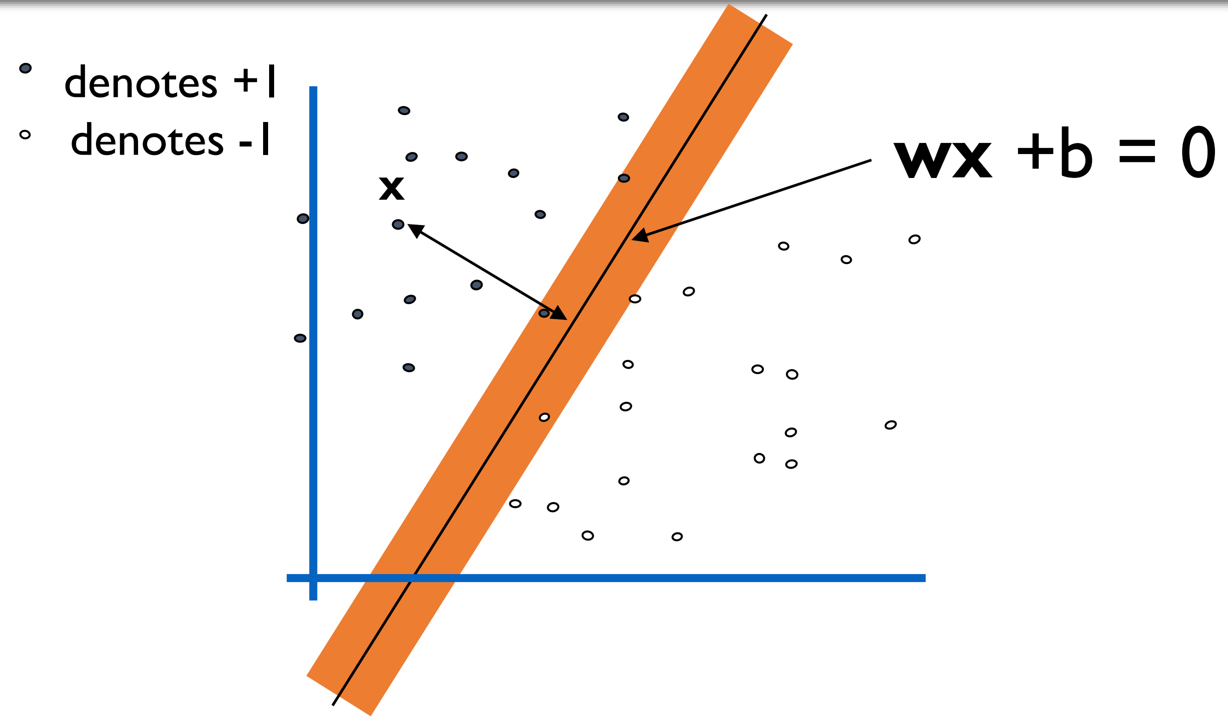
Why Maximum Margin?



- Intuitively this feels safest. If we've made a small error in the location of the boundary this gives us least chance of causing a misclassification.
- The model is immune to removal of any non-support-vector datapoints.
- There's some theory (using VC dimension) that is related to (but not the same as) the proposition that this is a good thing.
- Empirically it works very well.

kind of SVM (Called an LSVM)

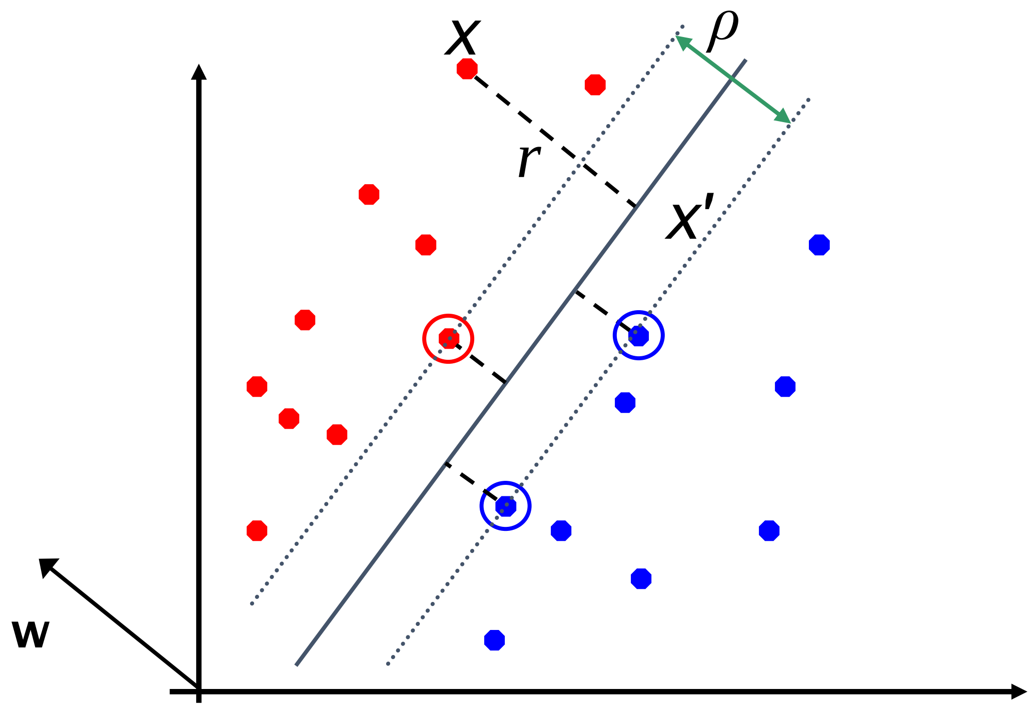
Estimating the Margin



- What is the distance expression for a point **x** to a line **wx**+b= 0?

Estimating the Margin

- Distance from example to the separator is $r = y\frac{\mathbf{w}^T\mathbf{x}+b}{\|\mathbf{w}\|}$

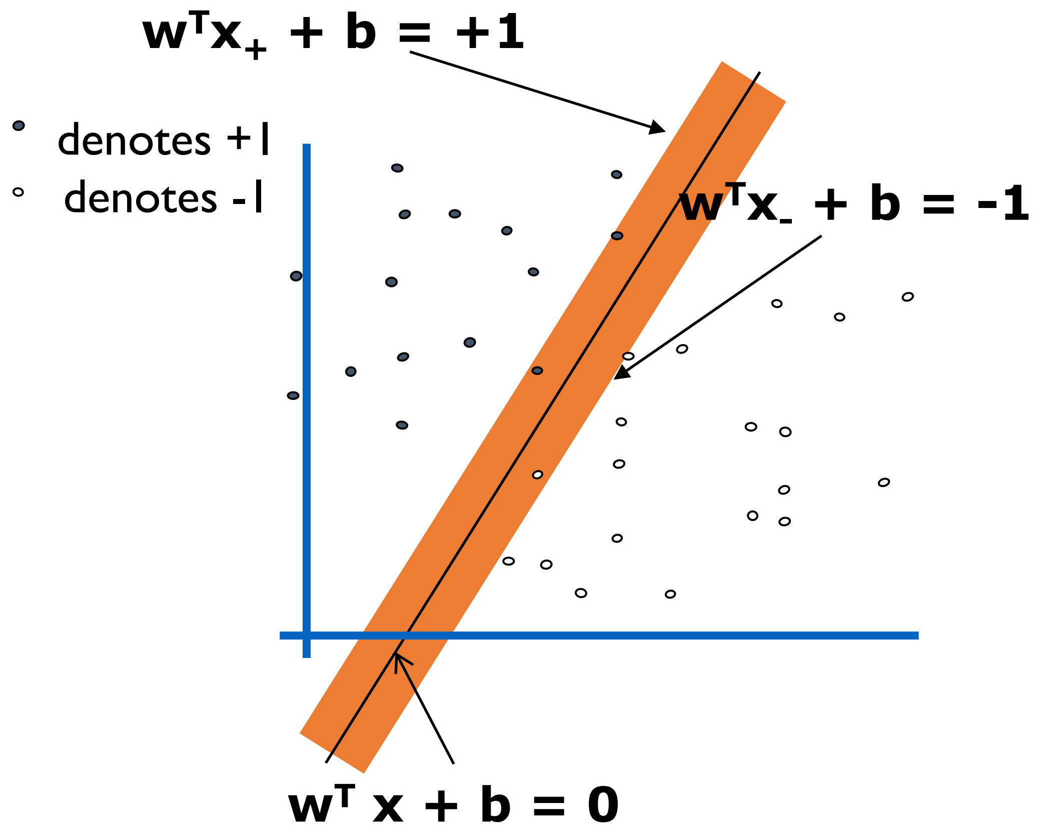


Derivation of finding *r*:

- Dotted line **x'** − **x** is perpendicular to decision boundary, so parallel to **w**.
- Unit vector is **w**/||**w**||, so line is r**w**/||**w**||.
- **x'** = **x** – yr**w**/||**w**||.
- **x'** satisfies $\mathbf{w}^T\mathbf{x}' + b = 0$.
- So $\mathbf{w}^T(\mathbf{x} - yr\mathbf{w}/\|\mathbf{w}\|) + b = 0$
- Recall that $\|\mathbf{w}\| = \text{sqrt}(\mathbf{w}^T\mathbf{w})$.
- So $\mathbf{w}^T\mathbf{x} - yr\|\mathbf{w}\| + b = 0$
- So, solving for r gives: $r = y(\mathbf{w}^T\mathbf{x} + b)/\|\mathbf{w}\|$

Estimating the Margin

- Since $\mathbf{w}^T\mathbf{x} + b = 0$ and $c(\mathbf{w}^T\mathbf{x} + b) = 0$ define the same plane, we have the freedom to choose the normalization of $\mathbf{w}$ (i.e. c)
- Let us choose normalization such that $\mathbf{w}^T\mathbf{x}_+ + b = +1$ and $\mathbf{w}^T\mathbf{x}_- + b = -1$ for the positive and negative support vectors respectively

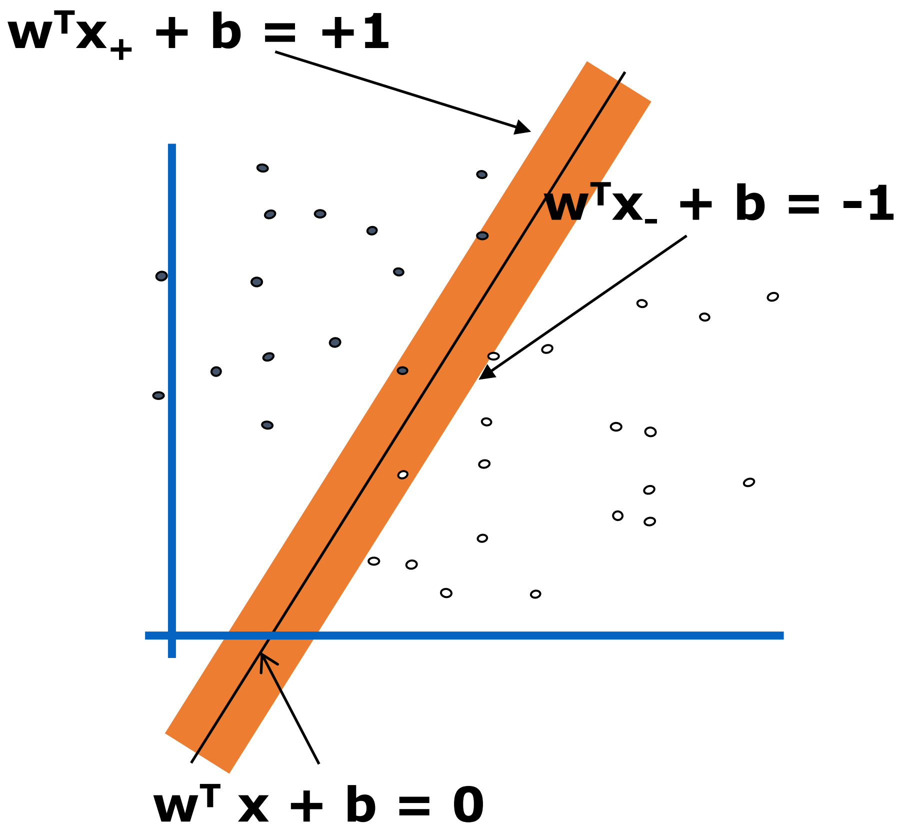


$\mathbf{w}^T\mathbf{x}_+ + \mathbf{b} = +1$
● denotes +1
○ denotes -1
$\mathbf{w}^T\mathbf{x}_- + \mathbf{b} = -1$
$\mathbf{w}^T\ \mathbf{x} + \mathbf{b} = 0$

Estimating the Margin

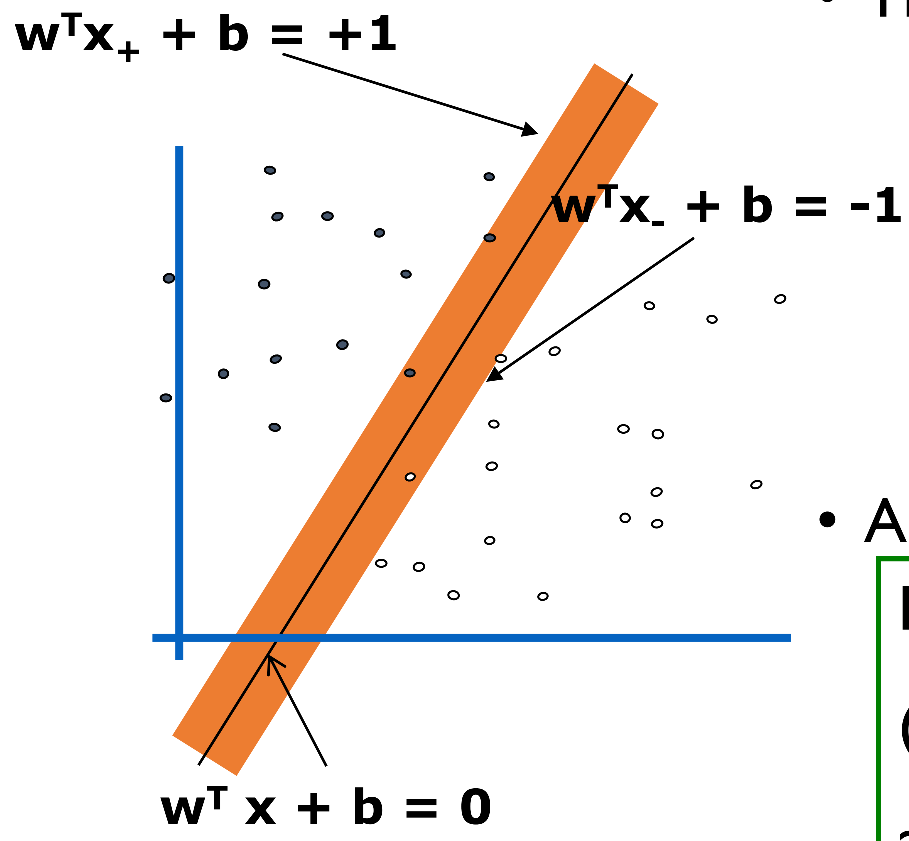
- Since $\mathbf{w}^T\mathbf{x} + \mathbf{b} = 0$ and $c(\mathbf{w}^T\mathbf{x} + \mathbf{b}) = 0$ define the same plane, we have the freedom to choose the normalization of **w** (i.e. c)
- Let us choose normalization such that $\mathbf{w}^T\mathbf{x}_+ + \mathbf{b} = +1$ and $\mathbf{w}^T\mathbf{x}_- + \mathbf{b} = -1$ for the positive and negative support vectors respectively
- Hence, margin now is:

$$(+1)*\frac{\mathbf{w}^T\mathbf{x}_+ + b}{\|\mathbf{w}\|} + (-1).\frac{\mathbf{w}^T\mathbf{x}_- + b}{\|\mathbf{w}\|} = \frac{2}{\|\mathbf{w}\|}$$



$\mathbf{w}^T\mathbf{x}_+ + \mathbf{b} = +1$
$\mathbf{w}^T\mathbf{x}_- + \mathbf{b} = -1$
$\mathbf{w}^T\mathbf{x} + \mathbf{b} = 0$

Maximizing the Margin



$\mathbf{w}^T\mathbf{x}_+ + b = +1$
$\mathbf{w}^T\mathbf{x}_- + b = -1$
$\mathbf{w}^T \mathbf{x} + b = 0$

- Then we can formulate the *quadratic optimization problem:*

Find **w** and *b* such that

$$r = \frac{2}{\|\mathbf{w}\|}$$ is maximized; and for all $\{(\mathbf{x_i}, y_i)\}$

$\mathbf{w}^T\mathbf{x_i} + b \geq 1$ if $y_i=+1$; $\mathbf{w}^T\mathbf{x_i} + b \leq -1$ if $y_i = -1$

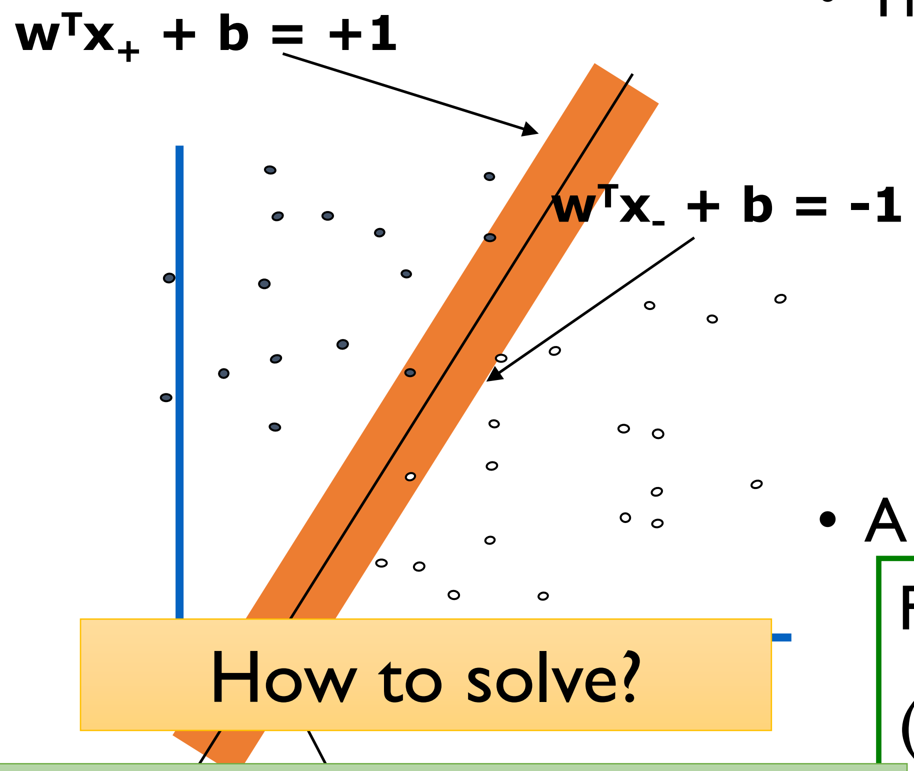
- A better formulation (min $||\mathbf{w}||$ = max $1/ ||\mathbf{w}||$):

Find **w** and *b* such that

$(\frac{1}{2}\, \mathbf{w}^T\mathbf{w})$ is minimized

and for all $\{(\mathbf{x_i}, y_i)\}$: $y_i\, (\mathbf{w}^T\mathbf{x_i} + b) \geq 1$

Maximizing the Margin



How to solve?

Quadratic Programming

- Then we can formulate the *quadratic optimization problem:*

Find $\mathbf{w}$ and b such that

$$r = \frac{2}{\|\mathbf{w}\|}$$ is maximized; and for all $\{(\mathbf{x_i}, y_i)\}$

$\mathbf{w^Tx_i} + b \geq 1$ if $y_i = +1$; $\mathbf{w^Tx_i} + b \leq -1$ if $y_i = -1$

- A better formulation (min $\|\mathbf{w}\|$ = max $1/\|\mathbf{w}\|$):

Find $\mathbf{w}$ and b such that

$(\frac{1}{2}\,\mathbf{w^Tw})$ is minimized

and for all $\{(\mathbf{x_i}, y_i)\}$: $y_i(\mathbf{w^Tx_i} + b) \geq 1$

Using Lagrange Multipliers

Optimization problem:

Minimize: $f(\vec{x})$

Such that: $g_i(\vec{x}) \leq 0$
(for all i)

Consider the augmented function:

$$\underbrace{L(\vec{x}, \vec{\lambda})}_{\text{(Lagrange function)}} := f(\vec{x}) + \sum_{i=1}^{n} \underbrace{\lambda_i}_{\text{(Lagrange variables, or dual variables)}} g(\vec{x})$$

Observation:

For ***any*** feasible *x* and ***all*** $\lambda_i \geq 0$, we have $L(\vec{x}, \vec{\lambda}) \leq f(\vec{x})$

$$\Longrightarrow \max_{\lambda_i \geq 0} L(\vec{x}, \vec{\lambda}) \leq f(\vec{x})$$

So, the optimal value to the constrained optimization:

$$p^* := \min_{\vec{x}} \max_{\lambda_i \geq 0} L(\vec{x}, \vec{\lambda})$$

The problem becomes unconstrained in x!

Slide credit: Nakul Verma, Columbia University

Duality

Optimal value: $p^* = \min_{\vec{x}} \max_{\lambda_i \geq 0} L(\vec{x}, \vec{\lambda})$

(also called the primal)

Now, consider the function: $\min_{\vec{x}} L(\vec{x}, \vec{\lambda})$

Observation:

Since, for ***any*** feasible *x* and ***all*** $\lambda_i \geq 0$:

$$p^* \geq \min_{\vec{x}} L(\vec{x}, \vec{\lambda})$$

Thus:

$$d^* := \max_{\lambda_i \geq 0} \min_{\vec{x}'} L(\vec{x}', \vec{\lambda}) \leq p^*$$

(also called the dual)

Optimization problem:

Minimize: $f(\vec{x})$

Such that: $g_i(\vec{x}) \leq 0$
(for all i)

Lagrange function:

$$L(\vec{x}, \vec{\lambda}) := f(\vec{x}) + \sum_{i=1}^{n} \lambda_i g_i(\vec{x})$$

Slide credit: Nakul Verma, Columbia University

Duality

Theorem (weak Lagrangian duality):

$$d^* \leq p^*$$

(also called the minimax inequality)

$p^* - d^*$ *(called the duality gap)*

Under what conditions can we achieve equality?

Optimization problem:

Minimize: $f(\vec{x})$

Such that: $g_i(\vec{x}) \leq 0$
(for all i)

Lagrange function:

$$L(\vec{x}, \vec{\lambda}) := f(\vec{x}) + \sum_{i=1}^{n} \lambda_i g_i(\vec{x})$$

Primal:

$$p^* = \min_{\vec{x}} \max_{\lambda_i \geq 0} L(\vec{x}, \vec{\lambda})$$

Dual:

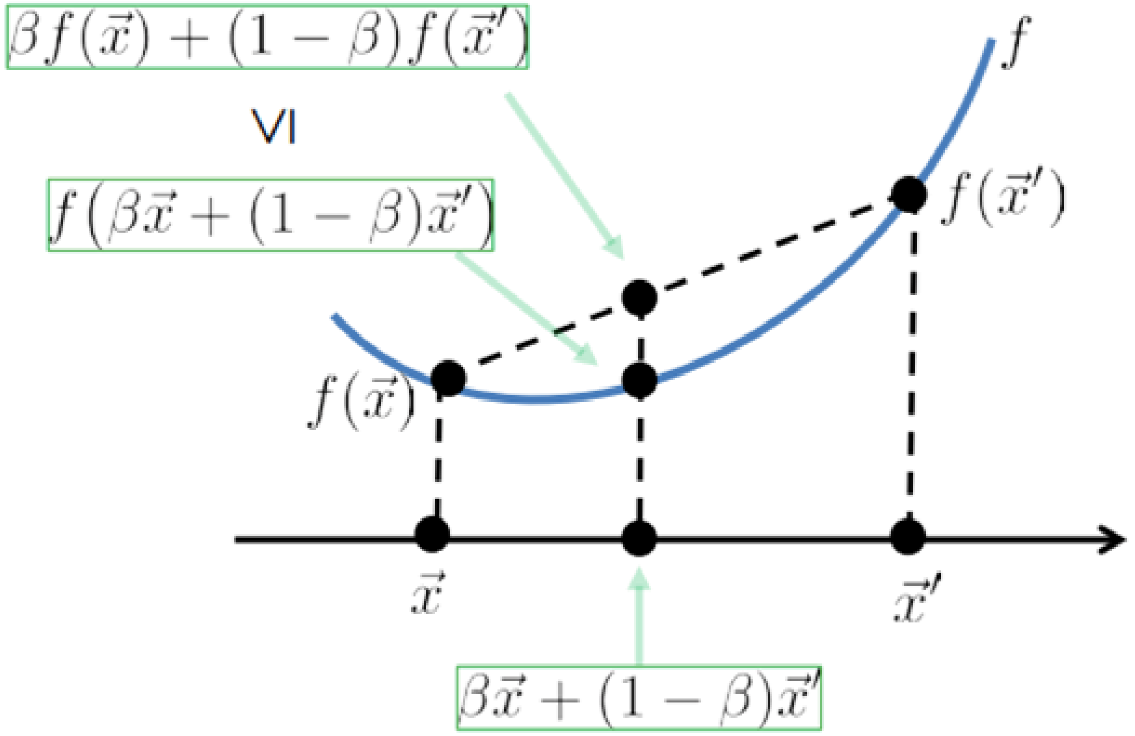
$$d^* := \max_{\lambda_i \geq 0} \min_{\vec{x}} L(\vec{x}, \vec{\lambda})$$

Slide credit: Nakul Verma, Columbia University

Convexity

A function f: $\mathbf{R}^d \rightarrow \mathbf{R}$ is called convex iff for any two points x, x' and $\beta \in [0,1]$

$$f\left(\beta\vec{x} + (1-\beta)\vec{x}'\right) \leq \beta f(\vec{x}) + (1-\beta) f(\vec{x}')$$



$\beta f(\vec{x}) + (1-\beta) f(\vec{x}')$
$\geq$
$f\left(\beta\vec{x} + (1-\beta)\vec{x}'\right)$
f
$f(\vec{x}')$
$f(\vec{x})$
$\vec{x}$
$\vec{x}'$
$\beta\vec{x} + (1-\beta)\vec{x}'$

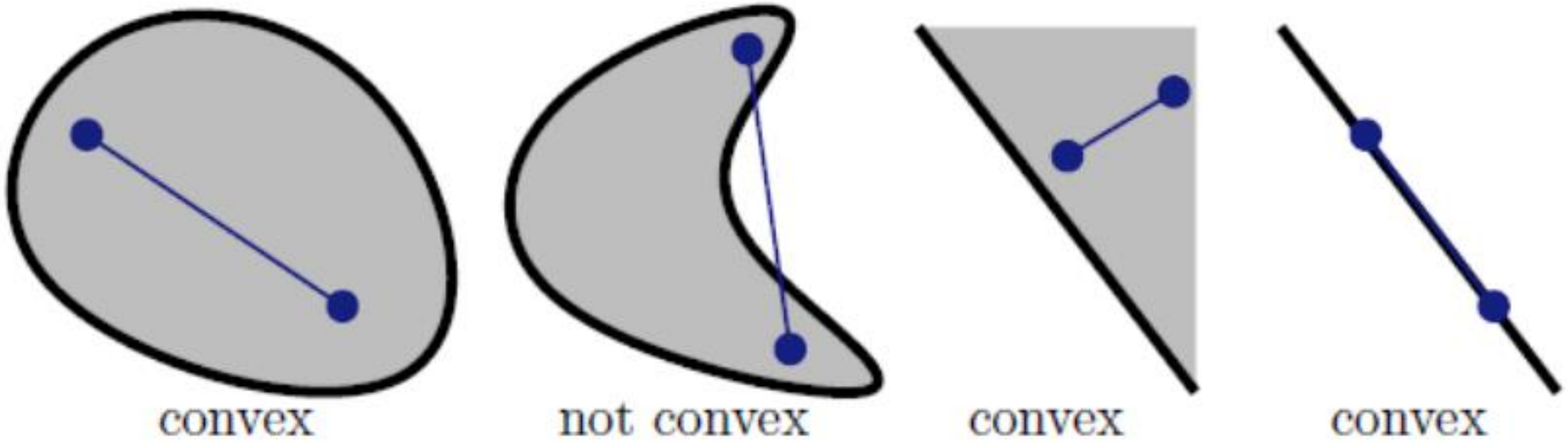
Slide credit: Nakul Verma, Columbia University

Convexity

A set $S \subset \mathbf{R}^d$ is called convex iff for any two points $x, x' \in S$ and any $\beta \in [0,1]$

$$\beta\vec{x} + (1-\beta)\vec{x}' \in S$$

Examples:



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Convex Optimization

A constrained optimization

$$\underset{\vec{x} \in \mathrm{R}^d}{\text{minimize}} \quad f(\vec{x}) \qquad \textit{(objective)}$$

$$\text{subject to:} \quad g_i(\vec{x}) \leq 0 \quad \text{for } 1 \leq i \leq n \qquad \textit{(constraints)}$$

is called convex a convex optimization problem

If:

the objective function $f(\vec{x})$ is convex function, and

the feasible set induced by the constraints g_i is a convex set

Why do we care?

*We and find the optimal solution for convex problems **efficiently**!*

Slide credit: Nakul Verma, Columbia University

Back to Duality

Theorem (weak Lagrangian duality):

$$d^* \leq p^*$$

Theorem (strong Lagrangian duality):

If *f* is convex and for a feasible point *x**

$g_i(\vec{x}^*) < 0$, or

$g_i(\vec{x}^*) \leq 0$ when *g* is affine

Then $d^* = p^*$

Slater's condition

Optimization problem:

Minimize: $f(\vec{x})$

Such that: $g_i(\vec{x}) \leq 0$
(for all i)

Lagrange function:

$$L(\vec{x}, \vec{\lambda}) := f(\vec{x}) + \sum_{i=1}^{n} \lambda_i g_i(\vec{x})$$

Primal:

$$p^* = \min_{\vec{x}} \max_{\lambda_i \geq 0} L(\vec{x}, \vec{\lambda})$$

Dual:

$$d^* := \max_{\lambda_i \geq 0} \min_{\vec{x}} L(\vec{x}, \vec{\lambda})$$

Slide credit: Nakul Verma, Columbia University

Back to Duality

SVM standard (primal) form:

Minimize: $\frac{1}{2}\|\vec{w}\|^2$
(w,b)

Such that: $y_i(\vec{w}\cdot\vec{x}_i + b) \geq 1$
(for all i)

Observations:

- object function is convex
- the constraints are affine, inducing a polytope constraint set.

So, SVM is a convex optimization problem (in fact a **quadratic program**)

Moreover, **strong duality holds**.

Let's examine the dual... the Lagrangian is:

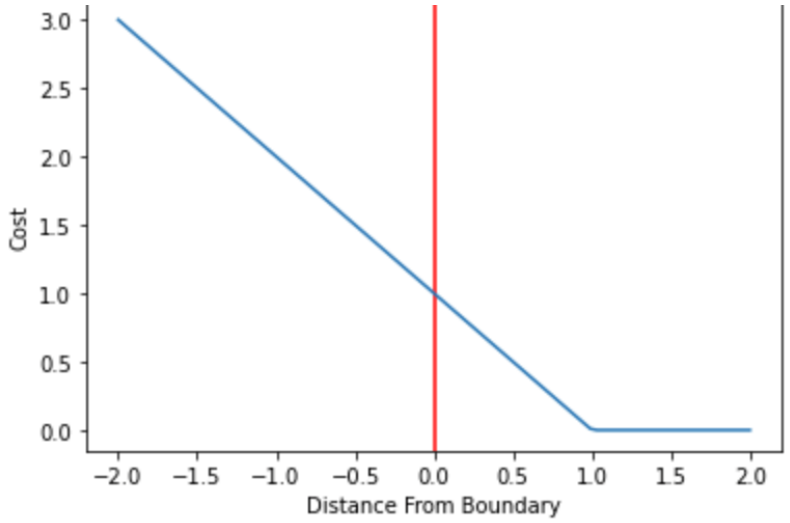
$$L(\vec{w}, b, \vec{\alpha}) = \frac{1}{2}\|\vec{w}\|^2 + \sum_{i=1}^{n} \alpha_i\big(1 - y_i(\vec{w}\cdot\vec{x}_i + b)\big)$$

Also known as hinge loss

Slide credit: Nakul Verma, Columbia University

Why hinge loss?

$$L(\vec{w}, b, \vec{\alpha}) = \frac{1}{2}\|\vec{w}\|^2 + \sum_{i=1}^{n} \alpha_i\left(1 - y_i(\vec{w}\cdot\vec{x}_i + b)\right)$$



Looks like a hinge

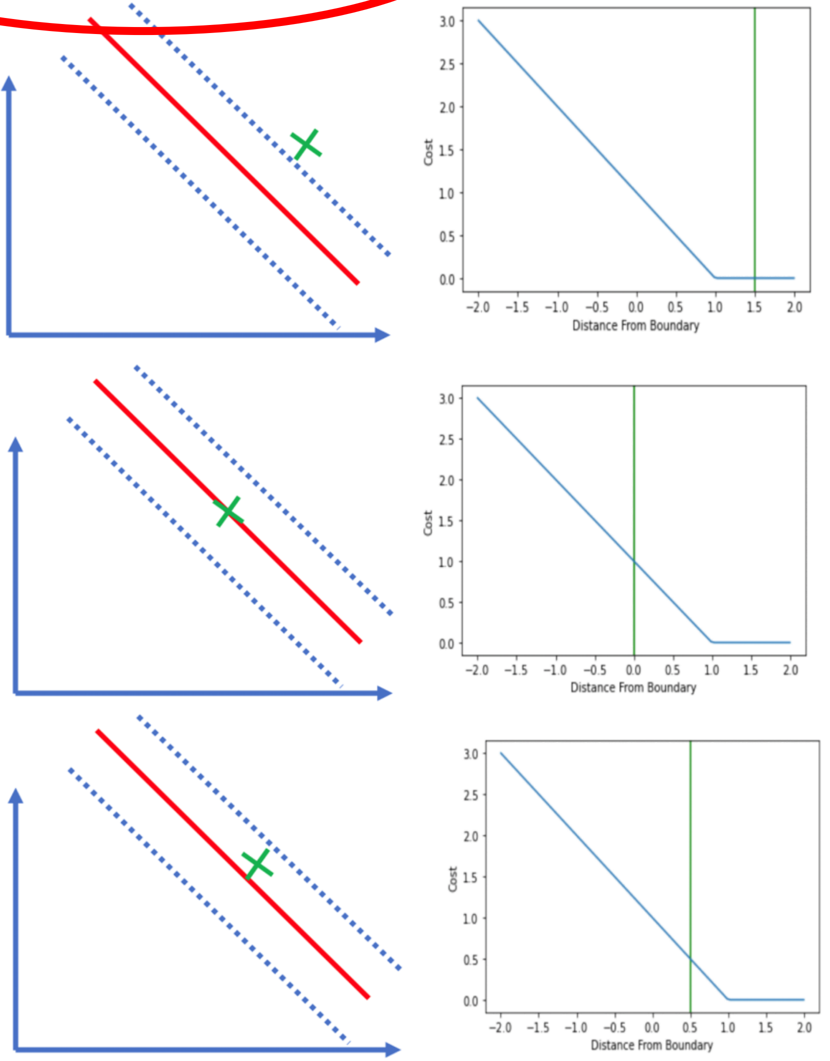
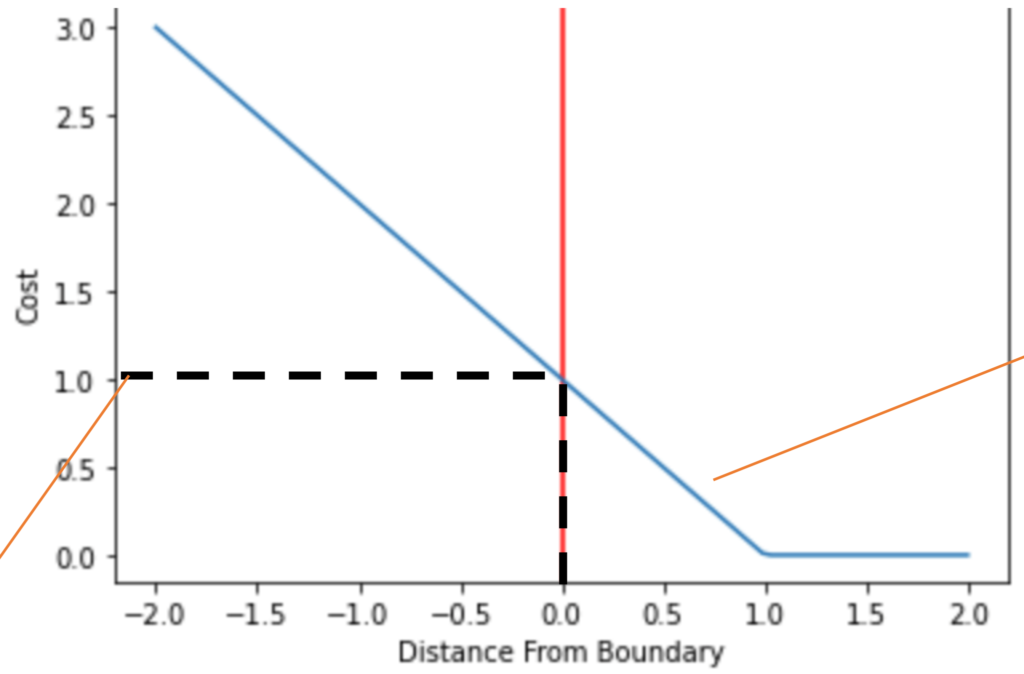


Image source: https://programmathically.com/understanding-hinge-loss-and-the-svm-cost-function/

Why hinge loss?



Hinge Loss

$\max(0, 1-y_i (wx+b))$

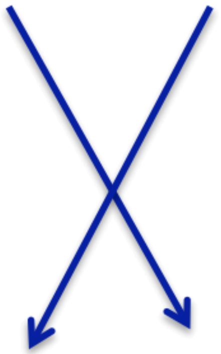
0-1 Loss

$1[y \neq (wx+b))$

Hinge loss upper bounds 0/1 loss!
It is the tightest convex upper bound on the 0/1 loss

SVM Dual

(Primal) $$\min_{\vec{w},b}\ \max_{\vec{\alpha}\geq 0}\ \frac{1}{2}||\vec{w}||^2 - \sum_j \alpha_j\left[(\vec{w}\cdot\vec{x}_j + b)\,y_j - 1\right]$$



Swap min and max

(Dual) $$\max_{\vec{\alpha}\geq 0}\ \min_{\vec{w},b}\ \frac{1}{2}||\vec{w}||^2 - \sum_j \alpha_j\left[(\vec{w}\cdot\vec{x}_j + b)\,y_j - 1\right]$$

Slater's condition from convex optimization guarantees that these two optimization problems are equivalent!

Slide credit: David Sontag, MIT

Solving using KKT conditions

$$\max_{\vec{\alpha}\geq 0}\ \min_{\vec{w},b}\ \frac{1}{2}||\vec{w}||^2 - \sum_j \alpha_j \left[(\vec{w}\cdot\vec{x}_j + b)\, y_j - 1\right]$$

Can solve for optimal w, b as function of α:

$$\frac{\partial L}{\partial w} = w - \sum_j \alpha_j y_j x_j \quad \rightarrow \quad \mathbf{w} = \sum_j \alpha_j y_j \mathbf{x}_j$$

$$\frac{\partial L}{\partial b} = -\sum_j \alpha_j y_j \quad \rightarrow \quad \sum_j \alpha_j y_j = 0$$

Karush-Kuhn-Tucker Conditions

Substituting these values back in (and simplifying), we obtain:

$$\max_{\vec{\alpha}\geq 0,\ \sum_j \alpha_j y_j = 0}\ \sum_j \alpha_j - \frac{1}{2}\sum_{i,j} \underbrace{y_i y_j \alpha_i \alpha_j}_{\text{scalars}} \underbrace{(\vec{x}_i \cdot \vec{x}_j)}_{\text{dot product}}$$

Sums over all training examples

Slide credit: David Sontag, MIT

Solving using KKT conditions

Maximize $\sum_{k=1}^{R} \alpha_k - \frac{1}{2} \sum_{k=1}^{R} \sum_{l=1}^{R} \alpha_k \alpha_l Q_{kl}$ where $Q_{kl} = y_k y_l (\mathbf{x}_k \cdot \mathbf{x}_l)$

subject to constraints: $\alpha_k \geq 0 \quad \forall k \qquad \sum_{k=1}^{R} \alpha_k y_k = 0$

Datapoints with $\alpha_k > 0$ will be the support vectors

Once solved, we obtain w and b using:

$$\mathbf{w} = \frac{1}{2} \sum_{k=1}^{R} \alpha_k y_k \mathbf{x}_k$$

$$y_i(x_i \bullet w + b) - 1 = 0$$

$$b = -y_i(y_i(x_i \bullet w) - 1)$$

..so this sum only needs to be over the support vectors.

Then classify with:

$f(\mathbf{x},\mathbf{w},b) = sign(\mathbf{w}.\ \mathbf{x} + b)$

Solving Convex Optimization Problems

- Every local optima is a global optima in a convex optimization problem.
- Example convex problems:
  - Linear programs, quadratic programs,
  - Conic programs, semi-definite program.
- Several solvers exist to find the optima:
  - CVX, SeDuMi, C-SALSA, …
- We can use a simple 'descent-type' algorithm for finding the minima!

Slide credit: Nakul Verma, Columbia University

Gradient Descent

Theorem (Gradient Descent):

Given a smooth function $f : \mathbf{R}^d \to \mathbf{R}$

Then, for any $\vec{x} \in \mathbf{R}^d$ and $\vec{x}' := \vec{x} - \eta \nabla_x f(\vec{x})$

For sufficiently small $\eta > 0$, we have: $f(\vec{x}') \leq f(\vec{x})$

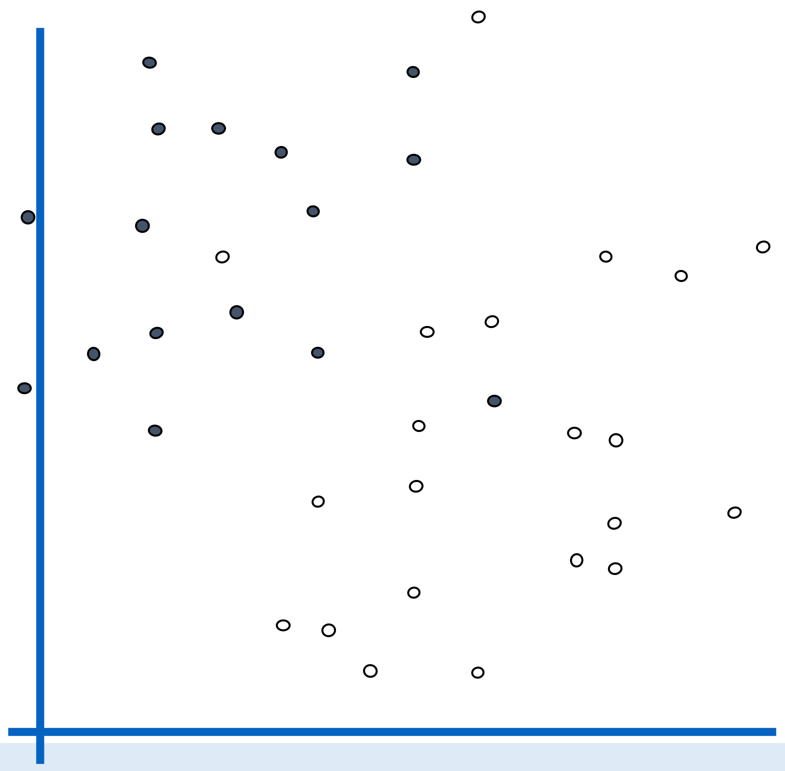
Can derive a **simple algorithm** (the projected Gradient Descent):

```
Initialize $\vec{x}^0$
for t = 1,2,…do
    $\vec{x}'^t := \vec{x}^{t-1} - \eta \nabla_x f(\vec{x}^{t-1})$    *(step in the gradient direction)*
    $\vec{x}^t := \Pi_{g_i}(\vec{x}'^t)$    *(project back onto the constraints)*
    terminate when no progress can be made, ie, $|f(\vec{x}^t) - f(\vec{x}^{t-1})| \leq \epsilon$
```

Slide credit: Nakul Verma, Columbia University

Non-separable Data

●	denotes +1
○	denotes -1



This is going to be a problem!
What should we do?

From hard-margin SVMs to soft-margin SVMs…

SVM for Noisy Data (C-SVM, Soft-Margin SVM)

$$\{\vec{w}^*, b^*\} = \min_{\vec{w},b} \sum_{i=1}^{d} w_i^2 + c\sum_{j=1}^{N} \varepsilon_j$$

$$y_1\left(\vec{w}\cdot\vec{x}_1 + b\right) \ge 1 - \varepsilon_1, \varepsilon_1 \ge 0$$

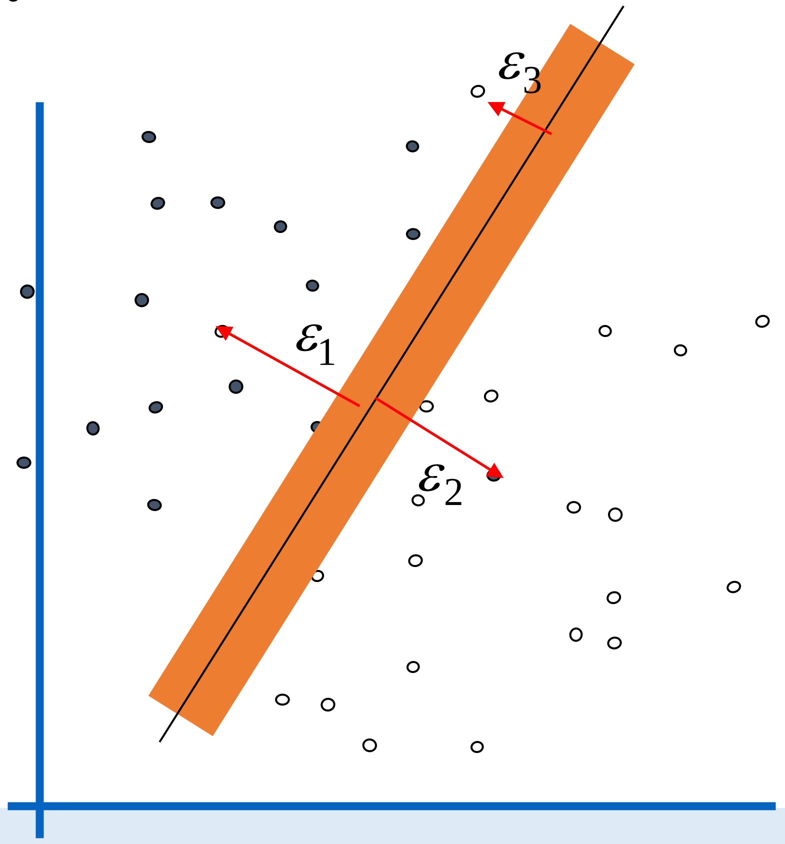
$$y_2\left(\vec{w}\cdot\vec{x}_2 + b\right) \ge 1 - \varepsilon_2, \varepsilon_2 \ge 0$$

$$\dots$$

$$y_N\left(\vec{w}\cdot\vec{x}_N + b\right) \ge 1 - \varepsilon_N, \varepsilon_N \ge 0$$

Balance the trade off between margin and classification errors

●	denotes +1
○	denotes -1

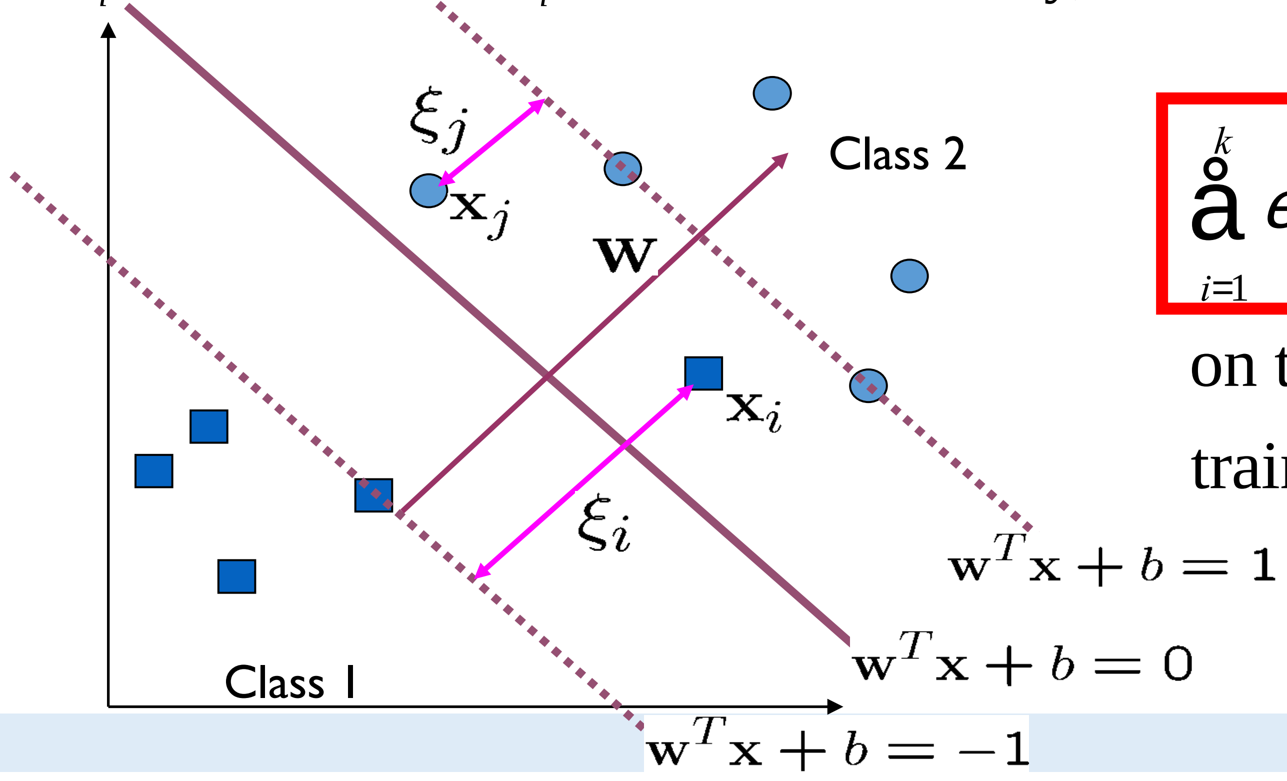


SVM for Noisy Data (C-SVM, Soft-Margin SVM)

$\varepsilon_i \geq 1 \quad \Leftrightarrow \quad y_i(wx_i + b) < 0,$ i.e., misclassification

$0 \prec \varepsilon_i \prec 1 \quad \Leftrightarrow \quad x_i$ is correctly classified, but lies inside the margin

$\varepsilon_i = 0 \quad \Leftrightarrow \quad x_i$ is classified correctly, and lies outside the margin



ξ_j
$\mathbf{x}_j$
$\mathbf{w}$
Class 2
$\mathbf{x}_i$
ξ_i
$\mathbf{w}^T\mathbf{x} + b = 1$
$\mathbf{w}^T\mathbf{x} + b = 0$
Class 1
$\mathbf{w}^T\mathbf{x} + b = -1$

$\boxed{\sum_{i=1}^{k} e_i}$ is an upper bound on the number of training errors.

SVM for Noisy Data (C-SVM, Soft-Margin SVM)

- Use the Lagrangian formulation for the optimization problem.
- Introduce a positive Lagrangian multiplier for each inequality constraint.

$$y_i(x_i \bullet w + b) - 1 + \varepsilon_i \geq 0, \text{ for all i.}$$

α_i Lagrangian multipliers

$$\varepsilon_i \geq 0, \text{ for all i.}$$

β_i

Get the following Lagrangian: $L_p = \|w\|^2 + c\sum_i \varepsilon_i - \sum_i \alpha_i\{y_i(x_i \bullet w + b) - 1 + \varepsilon_i\} - \sum_i \beta_i \varepsilon_i$

SVM for Noisy Data (C-SVM, Soft-Margin SVM)

$$L_p = \|w\|^2 + c\sum_i \varepsilon_i - \sum_i \alpha_i \{y_i(x_i \bullet w + b) - 1 + \varepsilon_i\} - \sum_i \beta_i \varepsilon_i$$

$$\frac{\partial L_p}{\partial w} = 2w - \sum_i \alpha_i y_i x_i = 0 \quad \Rightarrow \quad w = \frac{1}{2}\sum_i \alpha_i y_i x_i$$

$$\frac{\partial L_p}{\partial b} = -\frac{1}{2}\sum_i \alpha_i y_i = 0 \quad \Rightarrow \quad \sum_i \alpha_i y_i = 0$$

$$\frac{\partial L_p}{\partial \varepsilon_i} = c - \beta_i - \alpha_i = 0 \quad \Rightarrow \quad c = \beta_i + \alpha_i$$

Take the derivatives of L_p with respect to w, b, and ε_i.

Karush-Kuhn-Tucker Conditions

$$0 \le \alpha_i \le c \quad \forall i$$

$$L_D = \sum_i \alpha_i - \frac{1}{2}\sum_{i,j} \alpha_i \alpha_j y_i y_j (x_i \bullet x_j)$$

Both ε_i and its multiplier β_i are not involved in the function.

SVM for Noisy Data (C-SVM, Soft-Margin SVM)

Maximize $\sum_{k=1}^{R} \alpha_k - \frac{1}{2} \sum_{k=1}^{R} \sum_{l=1}^{R} \alpha_k \alpha_l Q_{kl}$ where $Q_{kl} = y_k y_l (\mathbf{x}_k \cdot \mathbf{x}_l)$

subject to constraints: $\boxed{0 \le \alpha_k \le c}$ $\forall k$ $\sum_{k=1}^{R} \alpha_k y_k = 0$

Compare this with the hard-margin SVM dual – what is different?

Maximize $\sum_{k=1}^{R} \alpha_k - \frac{1}{2} \sum_{k=1}^{R} \sum_{l=1}^{R} \alpha_k \alpha_l Q_{kl}$ where $Q_{kl} = y_k y_l (\mathbf{x}_k \cdot \mathbf{x}_l)$

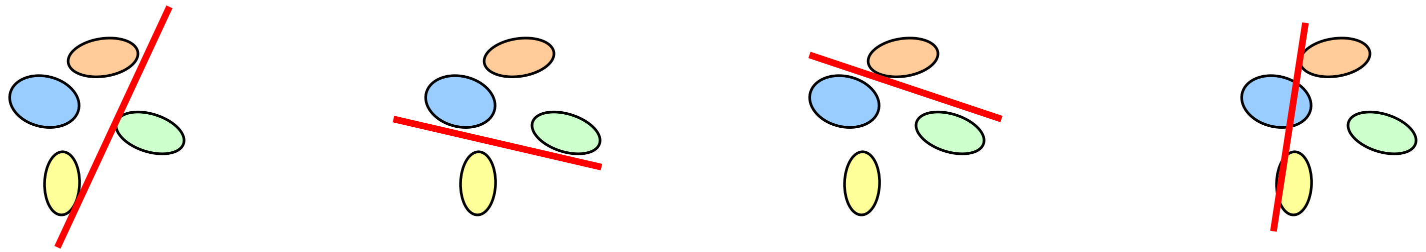
subject to constraints: $\boxed{\alpha_k \ge 0}$ $\forall k$ $\sum_{k=1}^{R} \alpha_k y_k = 0$

Multi-class Classification with SVMs

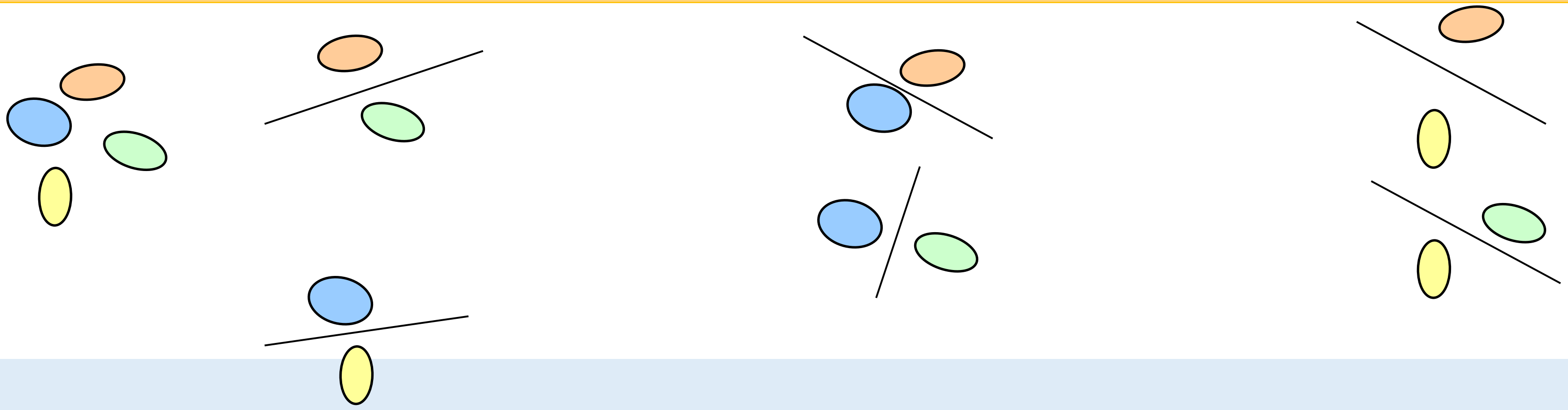
- SVMs can only handle two-class outputs.
- What can be done?
- Answer: with output arity N, learn N SVM's
  - SVM 1 learns "Output==1" vs "Output != 1"
  - SVM 2 learns "Output==2" vs "Output != 2"
  - :
  - SVM N learns "Output==N" vs "Output != N"
- Then to predict the output for a new input, just predict with each SVM and find out which one puts the prediction the furthest into the positive region.
- Other approaches
  - Pair-wise SVM, Tree-structured SVM

Multi-class Classification using SVM

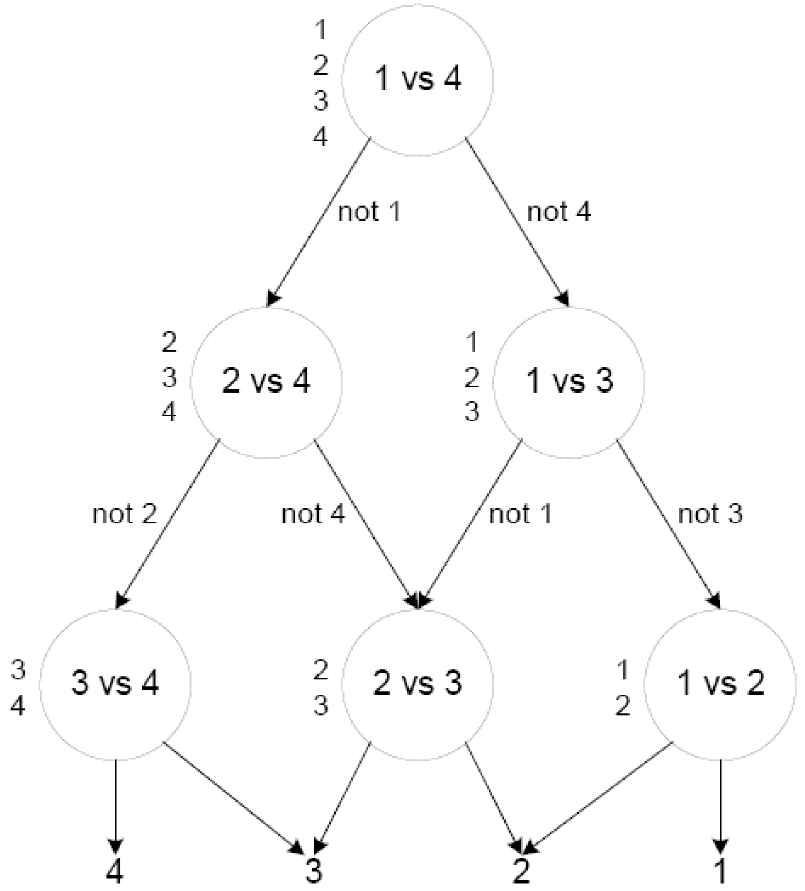
One- versus-all



One- versus-one



Tree-Structured SVM



Also called DAG-SVM (DAG = Directed Acyclic Graph)

Readings

- PRML, Bishop, Chapter 7 (7.1-7.3)
- "Introduction to Machine Learning" by Ethem Alpaydin, 2nd edition, Chapters 3 (3.1-3.4), Chapter 13 (13.1-13.9)
- Do read these!
  - https://www.svm-tutorial.com/2017/02/svms-overview-support-vector-machines/
  - https://www.svm-tutorial.com/2016/09/duality-lagrange-multipliers/
  - https://www.svm-tutorial.com/2017/10/support-vector-machines-succinctly-released/