An Introduction to Passivity–Based Control with Applications

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Introduction

Today we will briefly cover the following topics:

- Passive systems and their properties
- Port-Hamiltonian systems & passivity–based control

The lecture will be divided into two parts:

1. Theory (with examples)
2. State-of-the-art applications

Note: all the MATLAB code to reproduce the simulations shown in this lecture are available at `https://github.com/massastrello/PBCLecture`

Premises

Notation:

The set $\mathbb{R}$ ($\mathbb{R}^+$) is the the set of real (non negative real) numbers. The set of squared–integrable functions $z : \mathbb{R} \to \mathbb{R}^m$ is $\mathcal{L}_2^m$ while the set of d–times continuously differentiable functions is $\mathcal{C}^d$. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ denote the inner product on $\mathbb{R}^m$. The origin of $\mathbb{R}^n$ is $\mathbb{0}_n$. Let $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^2$ and let $\nabla \mathcal{H} \in \mathbb{R}^n$ be its gradient. The Hessian of $\mathcal{H}$ is $\nabla^2 \mathcal{H} \in \mathbb{R}^{n \times n}$.

Consider the following (nonlinear) dynamical system

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad x \in \mathcal{X} \subset \mathbb{R}^n, u \in \mathcal{U} \subset \mathbb{R}^m, y \in \mathcal{Y} \subset \mathbb{R}^m$$

$f : \mathcal{X} \to \mathbb{R}^n$, $g : \mathcal{X} \to \mathbb{R}^{m \times n}$ ($\text{rank}(g) = m \leq n$) and $h : \mathcal{X} \to \mathbb{R}^m$ are assumed smooth enough such that solutions are forward–complete for all initial conditions $x_0 \in \mathcal{X}$ and all inputs $u(t) \in \mathcal{L}_2^m$.
Let $\Phi(t, x_0, u)$ denote the state trajectory at time $t \geq 0$.

Supply Rate and Dissipativity

Definition: Supply Rate

Any real–valued function

$$\omega : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$$

is called *supply rate*.

Definition: Dissipative System

A system Σ is dissipative with respect to the supply rate $\omega(u, y)$ if there exists a continuous function

$$\mathcal{H} : \mathcal{X} \rightarrow \mathbb{R}^+ \quad \textit{(storage function)}$$

such that

$$\mathcal{H}(x(t)) - \mathcal{H}(x_0) \leq \int_0^t \omega(u(s), y(s))ds \quad \forall u \in \mathcal{U}, x_0 \in \mathcal{X}, t \geq 0$$

$\omega(u, y) \rightarrow$ *generalised power*, $\mathcal{H}(x) \rightarrow$ *generalised energy*

Passivity

Definition: Passive System

A system Σ is *passive* if it is dissipative w.r.t the supply rate

$$\omega(u, y) \triangleq \langle u, y \rangle \triangleq y^\top u$$

If a system is passive,

$$\mathcal{H}(x(t)) - \mathcal{H}(x_0) \leq \int_0^t y(s)^\top u(s) ds \quad \forall u \in \mathcal{U}, x_0 \in \mathcal{X}, t \geq 0$$

Remark

If $u = 0$, it holds

$$\dot{\mathcal{H}} = \nabla \mathcal{H}(x)\dot{x} \leq 0 \quad \forall x_0 \in \mathcal{X}, t \geq 0$$

$\Rightarrow$ The storage function **decreases** along any trajectory.
$\Rightarrow$ Any strict minimum x^* of $\mathcal{H}(x)$ is a Lyapunov stable equilibrium point of the system.

Passivity

Intuitions on the concept of passivity:

- it may be thought to be the "will" of the (autonomous) system to stabilize itself;
- the energy injected by the external input is not less than the variation of the stored energy:
  $\Rightarrow$ only part of the injected energy is stored, some energy is dissipated.

Definition: Kalman-Yakubovich-Popov (KYP) Property

A system Σ is said to enjoy the KYP property if $\exists\, \mathcal{H} : \mathcal{X} \to \mathbb{R}^+$, $\mathcal{H}(x) \in \mathcal{C}^1$, $\mathcal{H}(\mathbb{0}_n) = 0$ such that:

$$\nabla \mathcal{H}(x) f(x) \leq 0$$

$$\nabla \mathcal{H}(x) g(x) = h^\top(x)$$

Proposition (Brynes)

A system enjoys the KYP property **iff** it is passive.

Passive LTI Systems

Let us consider a **linear time invariant** (LTI) system

$$\Sigma_{lin}: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C = \mathbb{R}^{m\times n}$ and $u, y \in \mathbb{R}^m$. The *autonomous system* (u $= 0$) is **stable** iff $\mathfrak{Re}[\lambda_i(A)] \leq 0$ for all $i = 1, \dots, n$.

Let us consider a quadratic storage function $\mathcal{H}(x) = \frac{1}{2}x^\top Px$, $P = P^\top > 0$
Indeed, $\mathcal{H} \in \mathcal{C}^1$, $\mathcal{H}(\mathbb{0}_m) = 0$.
$\Rightarrow$ Let us try to apply the KYP lemma to the LTI system. It holds,

$$f(x) = Ax, \;\; g(x) = B, \;\; h(x) = Cx$$

Therefore, the conditions on P for the KYP property become:

$$\begin{array}{lcl} x^\top PAx \leq 0 & \Leftrightarrow & A^\top P + PA \leq 0 \\ x^\top PB = x^\top C^\top & \Leftrightarrow & B^\top P = C \end{array}$$

Example: Mass–Spring–Damper

Consider a forced mass–spring–damper system with unitary mass

$$\ddot{\xi} + b\dot{\xi} + k\xi = \varphi \quad k, b \in \mathbb{R}^+$$

By choosing $u = \varphi$, and $x = [\xi, \dot{\xi}]^\top$, the state–space form of the system becomes:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

The autonomous system is exponentially stable if $b > 0$. Let P be

$$P = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad P = P^\top > 0,\ A^\top P + PA < 0$$

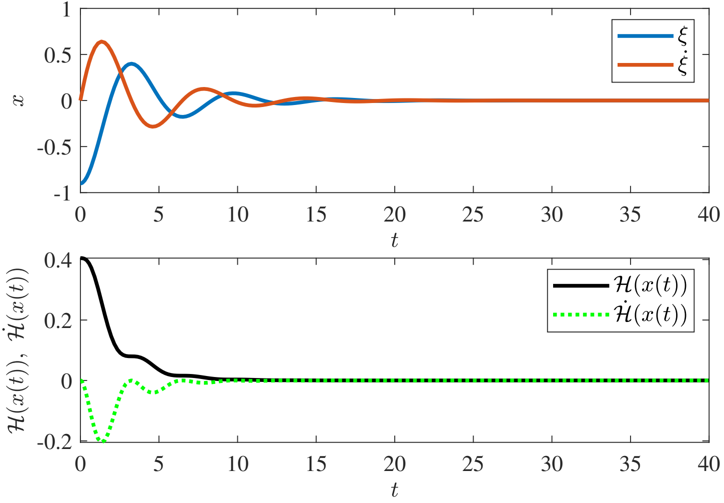
$\Rightarrow$ The system is passive (with storage function $\mathcal{H}(x) = \frac{1}{2}x^\top Px$) choosing

$$y = Cx \triangleq B^\top Px = \dot{\xi}$$

Note: $\mathcal{H}(x)$ is the total energy of the system.

Numerical Simulations

Numerical Simulation: $k = 1,\ b = 0.5,\ x_0 = [-0.9, 0]^\top$



Numerical Simulations

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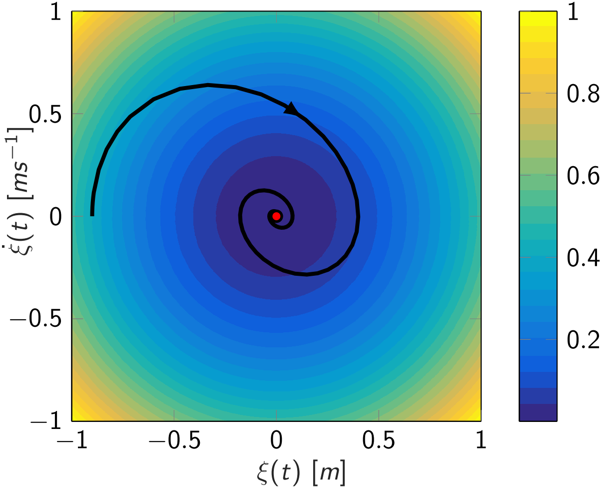


Output Feedback Asymptotic Stabilisation I

Definition: Detectability

A system Σ with $f(\mathbb{0}_n) = \mathbb{0}_n$ is *zero–state detectable* if

$$\forall x \in \mathcal{X} \quad y(t) = h(\Phi(t, x, \mathbb{0}_m)) = \mathbb{0}_m \; \forall t \geq 0 \Rightarrow \lim_{t \to \infty} \Phi(t, x, \mathbb{0}_m) = \mathbb{0}_n$$

Intuition: states should converge to zero when inputs and outputs are zero.

Output Feedback Asymptotic Stabilisation II

Theorem: Output feedback asymptotic stabilisation

Let Σ be passive with $\mathcal{H}(x) : \mathcal{X} \to \mathbb{R}^{+}$ and $f(\mathbb{0}_n) = \mathbb{0}_n$ that is zero–state detectable and let $\varphi : \mathcal{Y} \to \mathcal{U}$, $\varphi \in \mathcal{C}^1$ such that

- $\varphi(\mathbb{0}_m) = \mathbb{0}_m$
- $y^\top \varphi(y) > 0 \quad \forall y \neq \mathbb{0}_m$

Then the control law

$$u = -\varphi(y)$$

asymptotically stabilises the equilibrium point.

Note: A possible choice of the input is $u = -\alpha y$ $(\alpha > 0)$

Output Feedback Asymptotic Stabilisation III

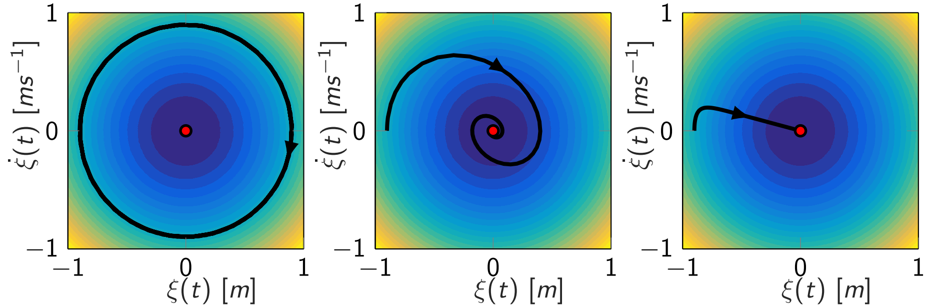
Example: Mass–spring–damper

system with $u = -\alpha y = -\alpha\dot{\xi}$ $(\alpha > 0)$.
Closed–loop system:

$$\dot{x} = \begin{bmatrix} \dot{\xi} \\ -k\xi - (b+\alpha)\dot{\xi} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \alpha\dot{\xi} = \begin{bmatrix} \dot{\xi} \\ -k\xi - (b+\alpha)\dot{\xi} \end{bmatrix}$$
$$y = \dot{\xi}$$

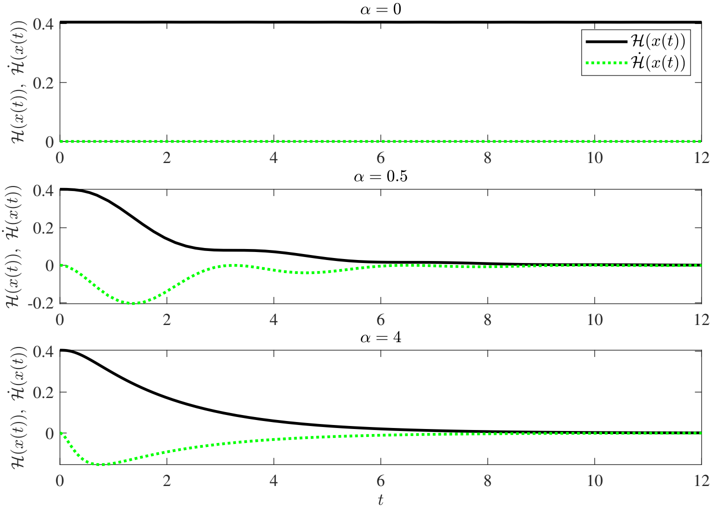
Numerical Simulations

Numerical Simulations: $k = 1$ and $b = 0$ and various values of α.



Numerical Simulations

Numerical Simulations: $k = 1$ and $b = 0$ and various values of α.



Passivity–Based Control

Question: Can we exploit systems' passivity for control?

$\Rightarrow$ **of course we can!**

Classic control design

Design of state/output feedback (or even dynamic controller) based on *reference tracking* $\Rightarrow$ systems are thought as *signal processors*.

Passivity–based control (PBC)

Aims to modify the energetic properties of the system to achieve the desired behavior $\Rightarrow$ particular attention on performance and not only in asymptotic stability even in nonlinear systems.

Port–Hamiltonian systems: framework in which PBC has been consistently developed

Port–Hamiltonian Systems: Origins

The theory of Port–Hamiltonian systems brings together different traditions of physical systems modeling and analysis:

1. **Port-Based Modeling** ($\sim$ 50-60', Paynter): unified framework for modeling systems belonging to different physical domains.
   Achieved by recognizing **energy** as only common factor between physical domains.
2. **Geometric Mechanics** ($\sim$ 70', Arnold): branch of *analytical mechanics* where the classical Hamiltonian mechanics is formalized in a "geometric way".

$\Rightarrow$ **Port–Hamiltonian (PH) systems** theory intrinsically merges *geometry* with *network theory* on a passivity substrate.

They have been firstly introduced by Arjian Van Der Schaft and Bernard Maschke in the mid 90'

Port–Hamiltonian Systems I

The input–state–output model of a port-Hamiltonian system is:

$$\Sigma_{PH} : \begin{cases} \dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\,(\nabla\,\mathcal{H}(x))^\top + g\,(x)\,u \\ y = g^\top\,(\nabla\,\mathcal{H}(x))^\top \end{cases}$$

$\mathcal{H}$: energy (*Hamiltonian* function), $\mathcal{J}(x) = -\mathcal{J}^\top(x)$: *interconnection matrix*, $\mathcal{R}(x) = \mathcal{R}^\top(x) \geq 0$: *dissipation matrix* $g(x)$: input matrix (or *power port*).

Proposition:

A port–Hamiltonian system is passive with storage function $\mathcal{H}(x)$.

Power balance equation: $\dot{\mathcal{H}}(x) = y^\top u - \underbrace{\nabla\,\mathcal{H}(x)\mathcal{R}(\nabla\,\mathcal{H}(x))^\top}_{d(t)\geq 0}$

PH Formulation of LTI system I

$$\mathcal{H}(x) = \frac{1}{2}x^\top P \Rightarrow \nabla\mathcal{H}(x) = x^\top P$$

Therefore, let $u = 0$

$$Ax = [\mathcal{J} - \mathcal{R}]\, Px \overset{\ker(P)\subseteq\ker(A)}{\Longrightarrow} AP^{-1} = \mathcal{J} - \mathcal{R}$$

How to find $\mathcal{J}$ and $\mathcal{R}$?

Observe that

$$[\mathcal{J} - \mathcal{R}] + [\mathcal{J} - \mathcal{R}]^\top = -2\mathcal{R}$$

Therefore,

$$\mathcal{R} = -\frac{1}{2}\left[AP^{-1} + P^{-1}A^\top\right]$$

$$\mathcal{J} = AP^{-1} + \mathcal{R} = AP^{-1} - \frac{1}{2}\left[AP^{-1} + P^{-1}A^\top\right] = \frac{1}{2}\left[AP^{-1} - P^{-1}A^\top\right]$$

PH Formulation of LTI system II

The natural dissipation $d(t)$ of a LTI system is therefore

$$\begin{aligned} d(t) \triangleq \nabla \mathcal{H}(x) \mathcal{R} \nabla^\top \mathcal{H}(x) &= -\frac{1}{2} x^\top P \left[A P^{-1} + P^{-1} A^\top \right] P x \\ &= -\frac{1}{2} x^\top \left[P A + A^\top P \right] x \\ &= -x^\top P A x \end{aligned}$$

Any passive system Σ_{lin} with $\ker(P) \subseteq \ker(A)$ admits a port-Hamiltonian representation:

$$\begin{cases} \dot{x} = [\mathcal{J} - \mathcal{R}] P x + B u \\ y = B^\top P x \end{cases}$$

PH Formulation of LTI system III

Example: mass–spring–damper

$$AP^{-1} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} 1/k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix}$$

$$P^{-1}A^\top = \begin{bmatrix} 1/k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -k \\ 1 & -b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix}$$

$$\mathcal{R} = -\frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$$\mathcal{J} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$d(t) = -x^\top PAx = -\begin{bmatrix} \xi & \dot{\xi} \end{bmatrix} \begin{bmatrix} 0 & k \\ -k & -b \end{bmatrix} \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} = b\dot{\xi}^2 \geq 0$$

Port-Hamiltonian Systems: Control

In the PH framework, the plant is viewed as an **energy-transformation device**. Indeed, the control is based on **passivity**.

$$\underbrace{\mathcal{H}(x(t)) - \mathcal{H}(x(0))}_{stored\ \ energy} = \underbrace{\int_0^t y^\top(s)u(s)ds}_{supplied\ \ energy} - \underbrace{\int_0^t d(s)ds}_{dissipated\ \ energy}$$

Passivity Based Control (PBC): preserve the energy conservation property but with desired energy and dissipation functions:

PBC = Energy Shaping + Damping Injection

- **Energy–balancing PBC**: its final realization is static state and output feedbacks
- **Control by interconnection**: the controller is itself a PH system *interconnected* to the plant

Passivity–Based Control

Remark: We said the minima of $\mathcal{H}(x)$ are Lyapunov stable equilibrium points of the system.
$\Rightarrow$ Often, the configuration to be stabilised **is not** a strict minimum of $\mathcal{H}(x)$.

Passivity–Based Control Design:

1. **Energy Shaping**: Shape the energy of the system by means of a proper control law which assign a strict minimum in the desired set point;
2. **Damping Injection**: Add dissipation to the system in order to asymptotically stabilize the desired configuration.

$\Rightarrow$ We will now introduce the so–called *Energy–Balancing PBC* (EB–PBC).

Energy–Balancing PBC I

Problem: Energy Shaping

Consider a PH system Σ_{PH}. A control action $u = \beta(x) + v$ solves the PBC problem if the closed-loop system satisfies a desired power-balance equation

$$\dot{\mathcal{H}}^* = z^\top v - d^*(t) \tag{1}$$

where $\mathcal{H}^*$ is the desired energy function, $d^*(t)$ the desired dissipation function and $z \in \mathbb{R}^m$ the new power conjugated (passive) output.

Solution: Energy–Balancing PBC (Ortega)

Design the controller solving the dissipation inequality (1) setting:

$$z \triangleq y$$
$$d^*(t) \triangleq d(t) = \nabla \mathcal{H} \mathcal{R} \nabla \mathcal{H}$$

Energy–Balancing PBC II

Proposition:

If it is possible to find a function $\beta(x)$ such that

$$\dot{\mathcal{H}}_a = -y^\top \beta(x)$$

then the control law $u = \beta(x) + v$ is such that

$$\dot{\mathcal{H}}^* = y^\top v - d^*$$

is satisfied for $\mathcal{H}^* \triangleq \mathcal{H} + \mathcal{H}_a$.

$\Rightarrow$ $\beta(x)$ is such that the *added energy* $\mathcal{H}_a$ equals the energy supplied to the system

$\Rightarrow$ $\mathcal{H}^*$ is the difference between the stored and supplied energy.

Closed-form solution of the EB–PBC controller:

$$\beta(x) = -g^+ [\mathcal{J} - \mathcal{R}]^\top (\nabla \mathcal{H}_a)^\top$$

Energy–Balancing PBC III

where g^+ is the left pseudoinverse of g and $\mathcal{H}_a$ satisfies the following matching equations

$$\begin{bmatrix} g^\perp [\mathcal{J} - \mathcal{R}]^\top \\ g^\top \end{bmatrix} (\nabla \mathcal{H}_a)^\top = \mathbb{0}_{n+m}$$

being $g^\perp$ a left full–rank annihilator of g.
Note: $g^+ \triangleq (g^\top g)^{-1} g^\top$, $g^\perp : g^\perp g = 0$

The closed–loop system is then equivalent to

$$\dot{x} = [\mathcal{J} - \mathcal{R}] (\nabla \mathcal{H}^*)^\top + gv$$
$$y = g^\top (\nabla \mathcal{H}^*)^\top = g^\top (\nabla \mathcal{H})^\top$$

Damping Injection

It is possible to asymptotically stabilise the equilibrium configuration (a local minimum of the *shaped* Hamiltonian function) by the control law:

$$v = -\alpha y, \quad \alpha > 0$$

The controlled system can be described by the following equation:

$$\begin{aligned} \dot{x} &= [\mathcal{J} - \mathcal{R}]\,(\nabla\,\mathcal{H}^*)^\top - \alpha g g^\top (\nabla\,\mathcal{H}^*)^\top \\ &= \left[\mathcal{J} - \left(\mathcal{R} + \alpha g g^\top]\right)\right](\nabla\,\mathcal{H}^*)^\top \end{aligned}$$

$\Rightarrow$ The damping injection adds to the system **power dissipation**.
$\Rightarrow$ Even in the *lossless* case, i.e., $\mathcal{R} = 0$ the closed–loop system would be asymp. stable

Dissipation Obstacle

The matching conditions of the EB–PBC requires that at the equilibrium the energy extracted by the controller is zero, i.e.

$$y^T \beta(x^*) = 0$$

This limits the implementation of the EB–PBC.
$\Rightarrow$ However, it can be overcome with more advanced PBC techniques, i.e. *control by interconnection*.

Propisition:

The EB–PBC is subjected to the dissipation obstacle:

$$\begin{bmatrix} g^{\perp}\left[\mathcal{J}-\mathcal{R}\right]^{\top} \\ g^{\top} \end{bmatrix} (\nabla \mathcal{H}_a)^{\top} = \mathbb{0}_{n+m} \Rightarrow \mathcal{R}(\nabla \mathcal{H}_a)^{\top} = 0$$

EB–PBC of LTI System

Unshaped energy: $\mathcal{H}(x) = \frac{1}{2}x^\top Px$
We want a desired *shaped* energy function which has a minimum in a configuration x^*, e.g.

$$\mathcal{H}^*(x) = \frac{1}{2}(x - x^*)^\top K(x - x^*), \;\; K = K^\top \geq 0$$

The energy balancing control law becomes

$$\beta(x) = -B^+P^{-1}A^\top(\partial\mathcal{H}_a)^\top$$

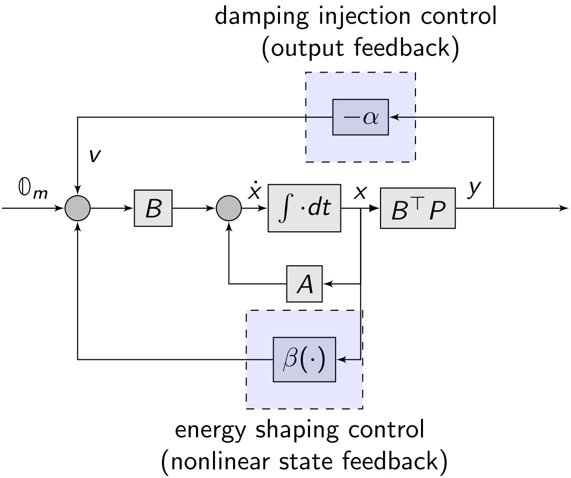
where

$$\mathcal{H}_a = \frac{1}{2}\left[(x - x^*)^\top K(x - x^*) - x^\top Px\right]$$

and with the matching conditions

$$\begin{bmatrix} B^\perp[\mathcal{J} - \mathcal{R}]^\top \\ B^\top \end{bmatrix}(\partial\mathcal{H}_a)^\top = \mathbb{0}_{n+m}$$

Final Control Scheme



EB–PBC of LTI System

Example: mass–spring–damper

Unshaped energy: $\mathcal{H}(x) = \frac{1}{2}k\xi^2 + \frac{1}{2}\dot{\xi}^2$
Desired fixed point: $x^* = [\xi^*, 0]^\top$
Desired Energy function: $\mathcal{H}^* = \frac{1}{2}k^*(\xi - \xi^*)^2 + \frac{1}{2}\dot{\xi}^2$
Added energy:
$\mathcal{H}_a = \mathcal{H}^* - \mathcal{H} = \frac{1}{2}k^*(\xi - \xi^*)^2 - k\xi^2 \Rightarrow \nabla \mathcal{H}_a = [k^*(\xi - \xi^*) - k\xi^2, 0]^\top$
Therefore, the energy shaping control law becomes

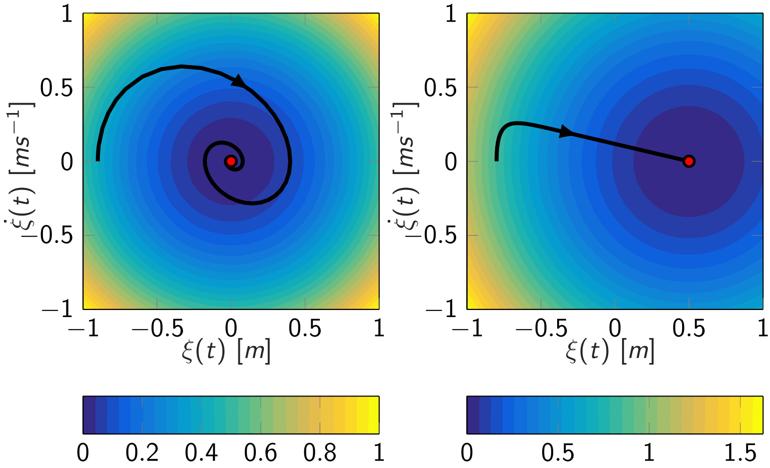
$$
\begin{aligned}
\beta(x) &= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix} \begin{bmatrix} K(\xi - \xi^*) \\ 0 \end{bmatrix} \\
&= -k^*(\xi - \xi^*) + k\xi
\end{aligned}
$$

Closed–loop system with also damping injection:

$$
\dot{x} = \begin{bmatrix} \dot{\xi} \\ -k^*(\xi - \xi^*) - (b + \alpha)\dot{\xi} \end{bmatrix}
$$

Numerical Simulation

Numerical simulation: $k = 1,\ b = 0.5,\ k^* = 1,\ \xi^* = 0.5,\ \alpha = 4,\ x_0 = [-0.8, 0]^\top$



Port-Hamiltonian Systems in Robotics

Port-Hamiltonian systems play an important role in robot control:

- **robot teleoperation** problem is often dealt with the PH systems theory;
- two widely used advanced robot control techniques correspond to special cases of PCB:
    - **PD + gravity compensation**
    - **Impedance control**
- Control of **flexible manipulators** (via distributed parameters PH systems)
- **robust robot control** with passive *variable structure* controllers

However, PH modeling is often applied in many more fields e.g.,

- power networks;
- multi–domain physical systems (e.g. electro–mechanical, thermo–electric etc.)

Application of Passivity–Based Control:

Multistable Energy Shaping of LTI Systems

Motivation

Claim:

Any autonomous exponentially stable LTI system admits only one equilibrium point. Furthermore, any linear controller only "moves" the equilibrium point.

In many practical situations, systems only operates in a finite number of *working modes* (voltages, positions, pressures, etc.).
$\Rightarrow$ A continuous exogenous reference signal must be constantly provided to the system in order to achieve the desired behavior.
$\Rightarrow$ The system itself has no "awareness" of the finiteness of the number of working modes.

Motivation

If it is possible to stabilise simultaneously all the working modes, we wouldn't need any external reference signal$\Rightarrow$ Use only an asynchronous mode selector.
A nonlinear controller is needed

Control Problem:

Simultaneously stabilise multiple fixed points by means of a proper **nonlinear feedback controller** to embed inside the system the information of the *working modes*;

Idea: Any stable LTI system could be passified with the right imput-output choice. $\Rightarrow$ Use the theory of passivity based control.

Multistable Energy Shaping

Multistable passivity-based control (MS-PBC) problem
Given a passive LTI system Σ_{lin}, find a control law $u = \beta(x) + v$ solving the PBC problem with $\mathcal{H}^* \in \mathcal{C}^2$ is a desired storage (energy) function with N minimum points x_i^* $(i = 1, \dots, N)$, i.e. such that:

$$\nabla \mathcal{H}^* |_{x=x_i^*} = 0,\ \nabla^2 \mathcal{H}^* |_{x=x_i^*} > 0 \quad \forall i = 1, \dots, N$$

$\Rightarrow$ The control problem can be just solved by the standard EB–PBC matching the requirements on the desired energy function.

$\Rightarrow$ The problem reduces in designing a proper $\mathcal{H}^*$ suitable for the task.

Example: Mass–Spring–Damper System

Consider the *mass–spring–damper* system and let the desired energy function have two minima symmetrically distrided on the displacement axes and that annihilates in the minima, e.g.,

$$\mathcal{H}^* = \lambda\xi^4 - \mu\xi^2 + \frac{1}{2}\dot{\xi}^2 - \frac{\mu^2}{2\lambda} \qquad \lambda, \mu > 0$$

which has two minima in $[\pm\sqrt{\mu/2\lambda}, 0]^\top$ and a local maximum in $[0, 0]^\top$. Thus,

$$\mathcal{H}_a = \mathcal{H}^* - \mathcal{H} = \lambda\xi^4 - (\mu + \frac{1}{2}k)\xi^2 + \frac{1}{2}\dot{\xi}^2 - \frac{\mu^2}{2\lambda}$$

and, therefore

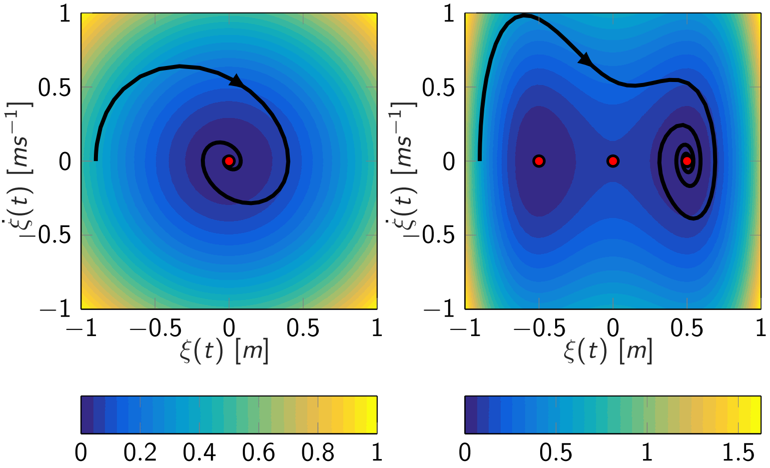
$$\partial\mathcal{H}_a = [4\lambda\xi^3 - (2\mu + k)\xi, 0]$$

It is straightforward to proof that the matching conditions of the EB–PBC are satisfied for $\mathcal{H}_a$ and the energy shaping control law becomes

$$\begin{aligned} \beta(x) &= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix} \begin{bmatrix} 4\lambda\xi^3 - (2\mu + k)\xi \\ 0 \end{bmatrix} \\ &= -4\lambda\xi^3 + (2\mu + k)\xi \end{aligned} \tag{2}$$

Numerical Simulation

Numerical simulation: $k = 1,\ b = 0.5,\ \lambda = 2,\ \mu = 1 \Rightarrow x^*_{1,2} = [\pm 0.5, 0]^\top$, $\alpha = 0,\ x_0 = [-0.9, 0]^\top$.



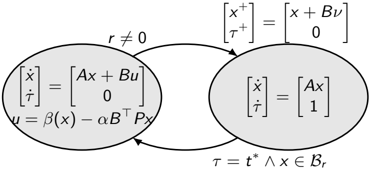
Mode Selector

Let us suppose:

- to perform the **multistable energy shaping control** on a LTI system.
- to have the system in steady state in one of those working modes, i.e. x_i^*, and we want the system to change mode to x_j^*.

$\Rightarrow$ We can design an **hybrid controller** to be working as mode selector:

1. Switch–off energy shaping controller (the system is back linear);
2. Give an impulse to the system to bring the state close to x_j^*;
3. Switch–on again the energy shaping controller.



Impulse Control Design

When we switch-off the nonlinear controller the system is again LTI.
Problem: Bring the system from x_i^* to x_j^* in a time t^* with an impulse weights vector ν.
Impulse input:

$$u = \nu\delta(t), \ \nu \in \mathbb{R}^m$$

Impulse Response:

$$x(t^*) = e^{t^* A}(x_i^* + B\nu)$$

The design of the impulse controller $u(x_i^*, x_j^*)$ can be achieved by solving the following optimisation problem:

Find t^*, ν such that

$$(t^*, \gamma) = \arg\min_{t^*, \nu} \rho \| x_j^* - e^{t^* A}\left(x_i^* + B\nu\right) \|_2^2 + \gamma \|\nu\|_2^2$$

subject to $\Phi(t^*, x_i^*, u(x_i^*, x_j^*)) \in \mathcal{B}_j$

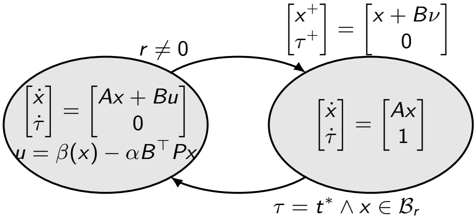
where $\rho, \nu \in \mathbb{R}^+$ and $\mathcal{B}_j$ is the basin of attraction of x_j^*.

Overall Hybrid System

The impulse control is equivalent to an abrupt change of initial condition (**jump**):

$$x^+ = x + B\nu \quad (x^+ \text{ denotes the value of } x \text{ after the impulse})$$

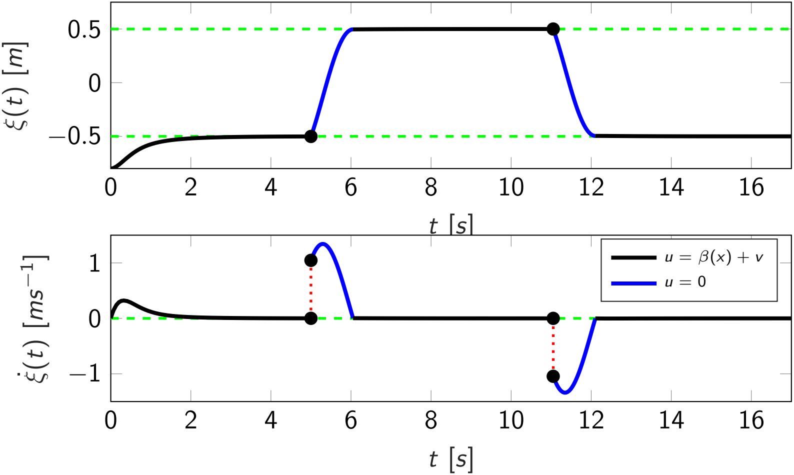
Then we leave the system uncontrolled for a time t^* before switching state again (A timer τ has been introduced).



Note: There is NO external continous reference signal $\Rightarrow$ only a "logic" signal r.

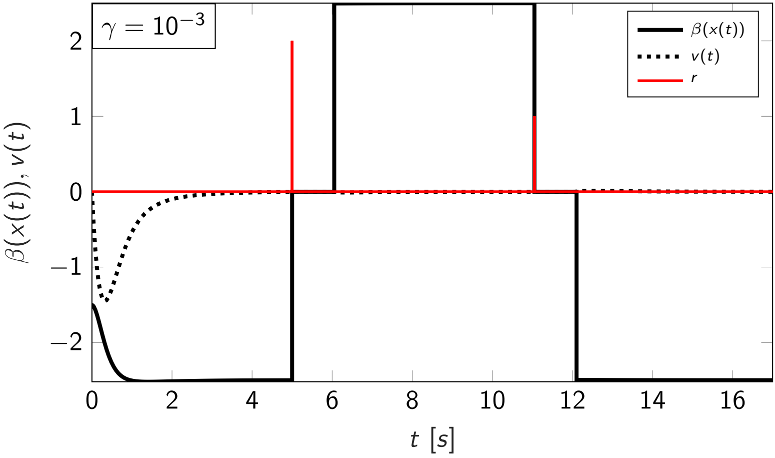
Numerical Simulation

Numerical Simulation: $\rho = 1$, $\gamma = 10^-3$



Numerical Simulation

Numerical Simulation: $\rho = 1$, $\gamma = 10^-3$



Numerical Simulation

Numerical Simulation: $\rho = 1,\ \gamma = 10^{-}3$