A Novel Linear Recursive Estimator Based on the Frisch Scheme

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slides and code: http://github.com/massastrello/Recursive-Estimation

Search for **connections** between observations ("**Laws of Nature**")
⇒ At the basis of the development of **scientific knowledge**.
e.g. *Babylonian astronomical diaries* ($\approx$ 747 BC)

Given:	Variables	Observations
	$x_1,\ x_2,\ \ldots,\ x_n$	$x_{i1},\ x_{i2},\ldots,\ x_{in}$

Search for a relation describing the process generating the data:

$$f(x_1,\ x_2,\ \ldots,\ x_n) = 0$$

satisfied by **every set** of observations, i.e.

$$\forall i \quad f(x_{i1},\ x_{i2},\ \ldots,\ x_{in}) = 0$$

Problem: In ALL practical situations, data will NEVER satisfy the relation.
⇒ **Observations are affected by noise.**

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Problem setting of this paper:

Extract **linear relations** from data affected by **additive noise**.

Approach:

We introduce a **novel estimator**: a **recursive** version of the *Frisch Scheme*.

Contribution:

- **Reduced size** of the solutions space w.r.t. standard methods;
- Improved computational efficiency

Originality:

Approximate the intersection of n–dimensional simplexes by means of bounding boxes.

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Linear Relation:

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$$

with

$$x_i = \hat{x}_i + \tilde{x}_i$$

Observation matrix

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} \in \mathbb{R}^{m\times n}$$

Parameters vector

$$\mathbf{A} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n$$

It holds:

$$\mathbf{X} = \hat{\mathbf{X}} + \tilde{\mathbf{X}}$$

$$\boxed{\hat{\mathbf{X}}\mathbf{A} = 0}$$

Linear Relation:

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$$

with

Practical Examples:

- n–DOF robot: $\boldsymbol{\tau} = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\boldsymbol{\theta}$
- Linear ODEs: $L\ddot{V} + R\dot{V} - \frac{1}{C}(V + u) = 0$
- Autoregressive models
- dynamical systems linear w.r.t. some parameters

$\mathbf{X} =$ $x_{m1} \quad x_{m2} \quad \dots \quad x_{mn}$ α_n

It holds:

$$\mathbf{X} = \hat{\mathbf{X}} + \tilde{\mathbf{X}}$$

$$\boxed{\hat{\mathbf{X}}\mathbf{A} = 0}$$

We define the *sample covariance matrix* of the data:

$$\Sigma = \frac{\mathbf{X}^\top \mathbf{X}}{m} \in \mathbb{R}^{n\times n}$$

Assumption: Noise and noiseless samples are orthogonal:

$$\sum_{t=1}^{m} \hat{x}_{it}\tilde{x}_{jt} = 0 \quad \forall i, j$$

We have

$$\Sigma = \hat{\Sigma} + \tilde{\Sigma} \quad , \quad \Sigma \succ 0, \quad \tilde{\Sigma} \succeq 0$$

$$\hat{\Sigma}\mathbf{A} = 0$$

$$\hat{\Sigma} \succeq 0 \quad \text{and} \quad \det(\hat{\Sigma}) = 0$$

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Problem: Estimation Scheme [Kalmann, 1982]
Given a sample covariance matrix of noisy observations, Σ, determine positive definite or semidefinite noise covariance matrices $\tilde{\Sigma}$ such that

$$\hat{\Sigma} = \Sigma - \tilde{\Sigma} \succeq 0 \quad \text{and} \quad \det(\hat{\Sigma}) = 0$$

$\Rightarrow$ Any basis of $\mathrm{null}(\hat{\Sigma})$ will describe a set of linear relations compatible with the data

$$\mathbf{A} = \mathrm{null}(\hat{\Sigma})$$

Frisch Scheme's Assumption: The noise variables are mutually independent:

$$\sum_{t=1}^{m} \tilde{x}_{it}\tilde{x}_{jt} = 0 \quad \forall i \neq j \quad \Rightarrow \quad \tilde{\Sigma} \text{ is diagonal}$$

Frisch Scheme Solution:

Every positive definite or semidefinite diagonal matrix $\tilde{\Sigma}$

$$\tilde{\Sigma} = \mathrm{diag}\begin{pmatrix}\tilde{\sigma}_1^2 & \tilde{\sigma}_2^2 & \cdots & \tilde{\sigma}_n^2\end{pmatrix}$$

such that

$$\hat{\Sigma} = \Sigma - \tilde{\Sigma} \geq 0 \quad \text{and} \quad \det \hat{\Sigma} = 0$$

is a *solution* of the Frisch scheme.

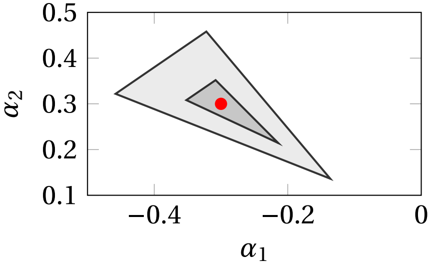
In general, the Frisch scheme has infinite solutions.

Define

$$\mathrm{Maxcor}_F(\Sigma) = \max_{\tilde{\Sigma}\in\mathcal{D}} \left\{ \dim\left[\mathrm{null}\left(\Sigma - \tilde{\Sigma} \right) \right] \right\}$$

Theorem [Kalmann, 1982]

If $\mathrm{Maxcor}_F(\Sigma) = 1$, the coefficients $\alpha_1, \ldots, \alpha_n$ of all linear relations compatible with the Frisch scheme lie (by normalizing one of the coefficients to 1) inside the simplex whose vertices are defined by the n *least squares solutions*.



Advantages

- post-identification degree of freedom in the choice of a single solution
  $\Rightarrow$ solve physical feasibility problems
  $\Rightarrow$ seek "optimal" solution
- closed form computation of the *simplex of solutions*

Disadvantages

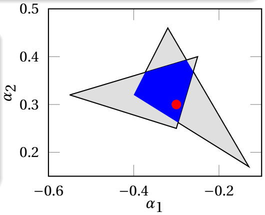
- To obtain a simplex of a useful size it is needed a *well-conditioned* experiment
  $\Rightarrow$ not suitable for diagnostics or online estimations
- If the solutions space is too wide, the post-identification process is meaningless

Research Aim:

- Reduce the size of the solutions space without introducing any further assumptions (*priors*)
- Develop a computationally efficient algorithm, suitable for online applications (control, diagnostics, etc.).

Recursive Frisch Scheme: Our Intuition

- If we compute the simplices corresponding to two different covariance matrices, **the true parameter must lie in both of them**, i.e. in their **intersection**.



Proposed Approach:

- Divide the data in several batches and compute the respetive simplexes;
- Intersect those simplexes to obtain a smaller solution space.

Remark:

By **successively intersecting** simplexes computed from different data sets it is possible to significantly **reduce the size** of the solutions space.

$$\Downarrow \text{size} \rightarrow \Uparrow \text{accuracy}$$

Main Issues:

⇒ No computationally efficient algorithms compute the intersection of convex objects only from the knowledge of their vertices;

⇒ The number of vertices of the intersection may increase with the iterations.

How can we approximate the intersection?

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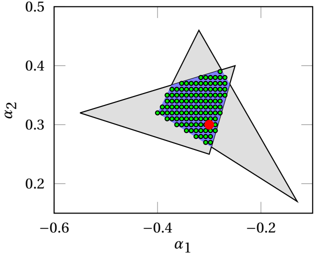
How can we approximate the intersection?

Current Method: Use particles!



[Massaroli, et al., 2018]

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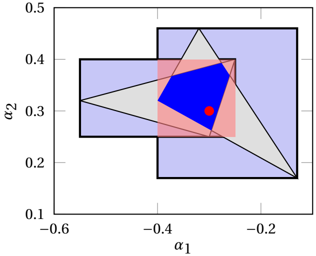
Proposed Method: Use bounding boxes



⇒**Bounding–box recursive Frisch scheme**

Geometry of simplexes and bounding boxes

Simplex Matrix:

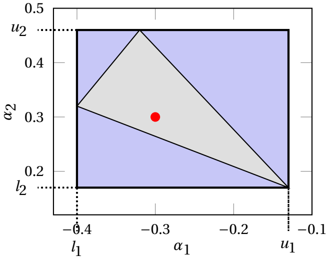
$$\mathbf{S}^p = (v_1, v_2, \ldots, v_{p+1}) \in \mathbb{R}^{p \times p+1}$$

$$\text{Simplex:} \quad \mathfrak{S}(\mathbf{S}^p) = \mathrm{conv}(\mathbf{S}^p)$$

Simplex Bounds:

$$l(\mathbf{S}^p) = (l_1, \ldots, l_p), \;\; u(\mathbf{S}^p) = (u_1, \ldots, u_p)$$

$$\text{with } \; l_i = \min_j \left(\mathbf{S}^p_{ij}\right) \; \text{ and } \; u_i = \max_j \left(\mathbf{S}^p_{ij}\right)$$



Simplex Bounding Box:

$$\mathfrak{B}(\mathbf{S}^p) = b_1 \times b_2 \times \cdots \times b_p = \prod_{i=1}^{p} b_i, \quad b_i = [l_i, u_i]$$

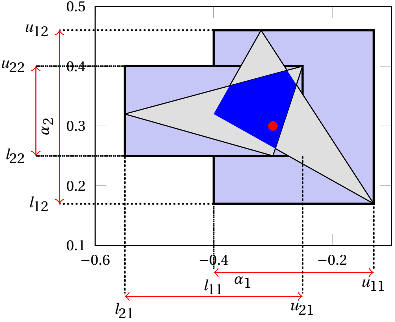
Proposition:

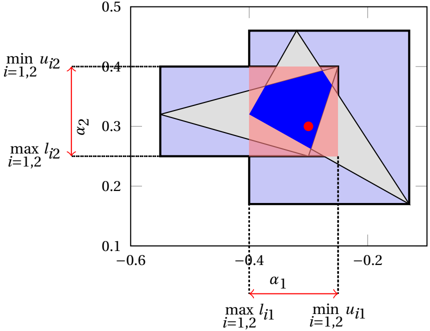
Given N simplex matrices $S_1^n, \dots, S_N^n$ it holds

$$\bigcap_{i=1}^{N} \mathfrak{S}(\mathbf{S}_i^n) \subseteq \bigcap_{i=1}^{N} \mathfrak{B}(\mathbf{S}_i^n)$$

Proposition:

$$\bigcap_{i=1}^{N} \mathfrak{B}(\mathbf{S}_i^n) = \prod_{j=1}^{n} [\max_i l_{ij}, \min_i u_{ij}]$$





BBRF Iteration

On–line estimation perspective: After the first m measurements, let's keep observing the system.

The state of the system at the time t_k is: $\mathbf{x}(t_k) = \begin{pmatrix} x_1(t_k) & x_2(t_k) & \cdots & x_n(t_k) \end{pmatrix}$

observation matrix	$\mathbf{X}(t_k) = \begin{pmatrix} \mathbf{x}(t_{k-m})^\top & \mathbf{x}(t_{k-m+1})^\top & \cdots & \mathbf{x}(t_k)^\top \end{pmatrix}^\top \in \mathbb{R}^{m\times n}$
covariance matrix	$\Sigma(t_k) = \dfrac{\mathbf{X}(t_k)^\top \mathbf{X}(t_k)}{m} \in \mathbb{R}^{n\times n}$
simplex matrix	$\mathbf{S}^{n-1}(t_k) = \begin{pmatrix} \mathbf{A}_1(t_k) & \mathbf{A}_2(t_k) & \cdots & \mathbf{A}_n(t_k) \end{pmatrix} \in \mathbb{R}^{n-1\times n}$
simplex bounding box	$\mathfrak{B}(\mathbf{S}^{n-1}(t_k)) = \prod_{i=1}^{n-1} [l_i(t_k), u_i(t_k)]$

It holds:

$$l_i(t_k) \le \alpha_i \le u_i(t_k) \quad \forall i, k \in \mathbb{N}$$

Solution of Bounding Box Recursive Frisch Scheme:

Approximate the intersection of simplexes by intersecting their bounding boxes.

BBRF scheme:

Lower Solution Bounds

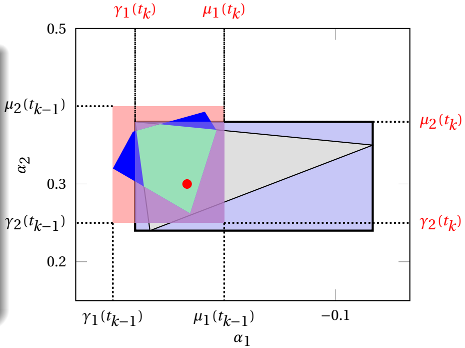
$$\gamma_i(t_k) = \max\{\gamma_i(t_{k-1}), l_i(t_k)\}$$

Upper Solution Bounds

$$\mu_i(t_k) = \min\{\mu_i(t_{k-1}), u_i(t_k)\}$$

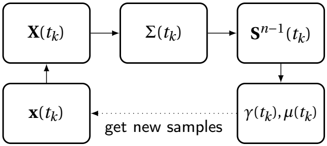
Solution Box

$$T(t_k) = \bigcap_{j=1}^{k} \mathfrak{B}(\mathbf{S}^{n-1}(t_j)) = \prod_{i=1}^{n-1} [\gamma_i(t_k), \mu_i(t_k)]$$



Proposition:

$$\lambda(T(t_k)) \leq \lambda(T(t_{k-1}))$$



Simulations

System:

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \quad \Rightarrow \quad \text{2D Simplex}$$

where

$$x_1, x_2 \sim \mathcal{N}(0,1) \qquad x_3 = -\alpha_1 x_1 - \alpha_2 x_2$$

and, thus

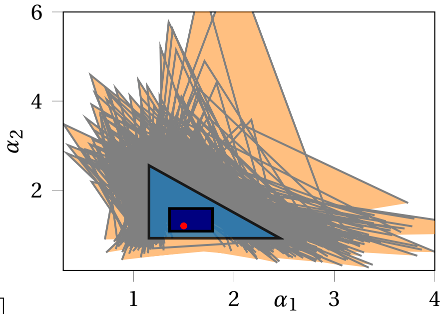
$$\mathbf{A} = \begin{pmatrix} 1.5 & 1.2 & 1 \end{pmatrix}^\top$$

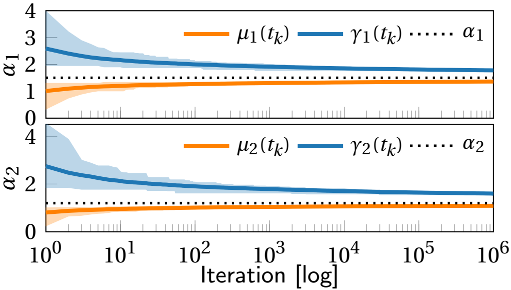
- Independent noise has been added to x_1, x_2, x_3 ($\sigma_{\%} = 30$);
- Observation matrix size: $m = 20$.

Results I.

"Smaller" solutions set w.r.t the simplex obtained with the whole dataset:

⇒ More efficient way to use data in the context of the Frisch scheme as alternative to perform a unique batch identification





Results II.

The solution bounds converge to constant values.

⇒ The uncertainty intervals at convergence are considerably smaller than the initial ones.

Results III.

Volume of solutions sets:

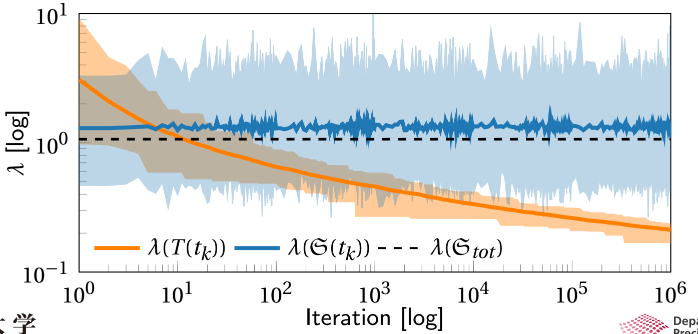
$$\lambda(T(t_k)) = \prod_{i=1}^{n-1} \left|\mu_i(t_k) - \gamma_i(t_k)\right|$$

$$\lambda(\mathfrak{S}(t_k)) = \frac{1}{(n-1)!} \left|\det\begin{pmatrix}\mathbf{S}^{n-1}(t_k) \\ 1\dots 1\end{pmatrix}\right|$$

$\lambda(T(t_k))$ decreases very fast in time $\Rightarrow$ Quick convergence of the algorithm.

In average, for $k > 10$,

$$\lambda(T(t_k)) \le \lambda(\mathfrak{S}(t_k)) \quad \wedge \quad \lambda(T(t_k)) \le \lambda(\mathfrak{S}_{tot})$$



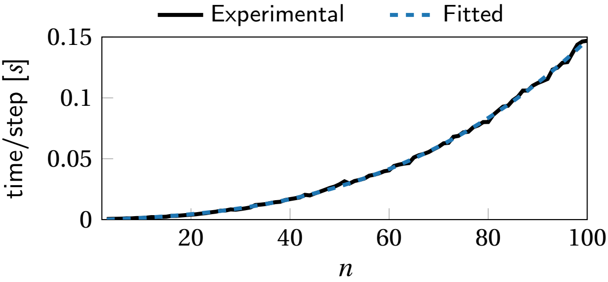
Computational Complexity Analysis

The computational complexity results to be $O(n^3)$ due to the extraction of null spaces bases, which is needed to compute the simplex vertices and performed in each iteration, i.e.

$$\mathbf{A}_i(t_k) = \mathrm{null}\left[\Sigma(t_k) - \mathrm{diag}\left(0, \dots, \tilde{\sigma}_i^2(t_k), \dots, 0\right)\right] \quad \forall i = 1, \dots, n$$

computed via SVD.

⇒ Suitable for online applications.



[Results obtained with Intel® Xeon E3-1240v5]

Conclusions and Future Works

Approach:

We proposed a **novel estimator**: the BBRF scheme.

Contribution:

- **Reduced size** of the solutions space w.r.t. standard methods;
- Improved computational efficiency

- It is always more convenient to perform the BBRF scheme rather than use the whole data set for a single Frisch estimation;
- The convergence speed and computational complexity of the proposed algorithm make it useful for online applications (diagnostics, control, etc);
- This approach can be modified to treat *time-varying systems*.
  ⇒ Use dynamic bounding boxes instead of simple intersections.