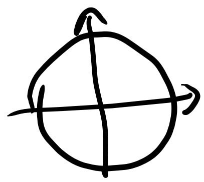
Parametric Equations

Imagine a particle moving along a curve C in the plane and we want to describe the particle's motion.

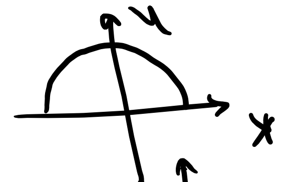
What to do if the curve C is not the graph of a function $y = f(x)$?
(and not even $x = g(y)$*)*

$x^2 + y^2 = 1 \Rightarrow$

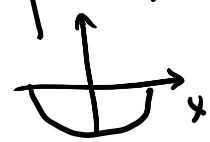


$y^2 = 1 - x^2 \nearrow$

$y = \sqrt{1 - x^2}$



$y = -\sqrt{1 - x^2}$



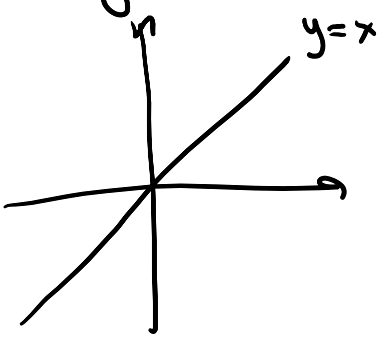
We can describe the particle's motion by specifying its coordinates as functions of time t:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad \textit{parametric equations}$$

so, at time t, the particle is located at the point $c(t) = \big(x(t), y(t)\big)$

A key feature of parametrizations is that they are not unique.
In fact, every curve can be parametrized in infinitely many different ways

$y = x$

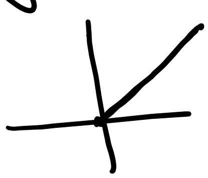


1) $\begin{cases} x = x(t) = t \\ y = y(t) = t \end{cases}$ $-\infty < t < \infty$, $t=1 \Rightarrow (1,1)$, $t=0 \Rightarrow (0,0)$

2) $\begin{cases} x = (3t) \\ y = (3t) \end{cases}$ $t=0$. $(0,0)$. $t=1$, $(3,3)$.

3) $\begin{cases} x = t^3 \\ y = t^3 \end{cases}$? $-\infty \le t < \infty$

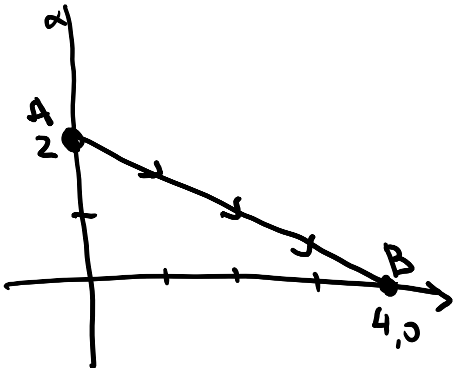
4) $\begin{cases} \cancel{x = t^2} \\ \cancel{y = t^2} \end{cases}$



$-\infty < t < \infty$

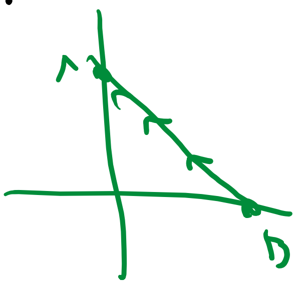
Example#1

a. Find a parametrizations of the line segment connecting the points $(0,2)$ *and* $(4,0)$



$m = -\frac{1}{2} \Rightarrow \boxed{y = -\frac{1}{2}x + 2}$

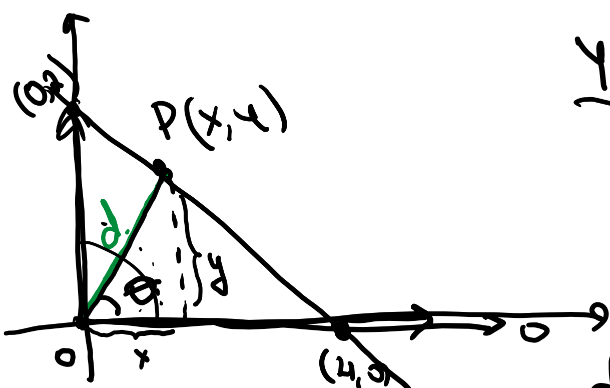
1) $\begin{cases} x = x(t) = t \\ y = -\frac{1}{2}t + 2 \end{cases}$ $\quad t = 0 \to t = 4$ $\quad (0,2) \to (4,0)$



2) $2y = -x + 4 \Rightarrow x = 4 - 2y$

$\begin{cases} x = 4 - 2t \\ y = t \end{cases}$ $\quad 0 \le t \le 2$ $\quad t=0 \to (4,0)$, $t = 2 \to (0,2)$

b. Find parametrization of this segment using angle θ *as a parameter*



$y = -\frac{1}{2}x + 2$

$x = d\cos\theta \,,\quad y = d\cdot\sin\theta$

$d\sin\theta = -\frac{1}{2}d\cos\theta + 2$

$d\left(\sin\theta + \frac{1}{2}\cos\theta\right) = 2$

$d = \frac{2}{\sin\theta + \frac{1}{2}\cos\theta}$

$$\boxed{\begin{cases} x = \dfrac{2\cos\theta}{\sin\theta + \frac{1}{2}\cos\theta} = x(\theta) \\[2ex] y = \dfrac{2\sin\theta}{\sin\theta + \frac{1}{2}\cos\theta} = y(\theta) \end{cases} \qquad 0 \le \theta \le \pi/2}$$

$\theta = 0$

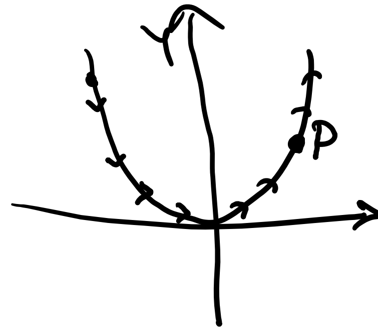
$x = \frac{2}{1/2} = 4$

$y = 0$

$\theta = \pi/2$

$x = 0$

$y = 2.$



Example#2

The point P travels on the parabola $y = x^2$.

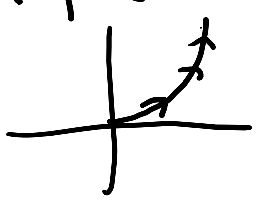
1. *Give parametric formulas for the location of P where the parameter is the first coordinate of the point P.*

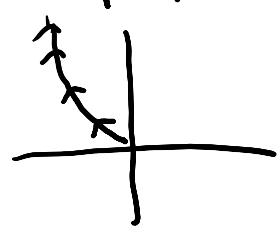
$$P(x,y) \Rightarrow x = t \Rightarrow \begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \qquad -\infty \le t < \infty$$

2. *Give parametric formulas for the location of P where the parameter is the second coordinate of the point P.*

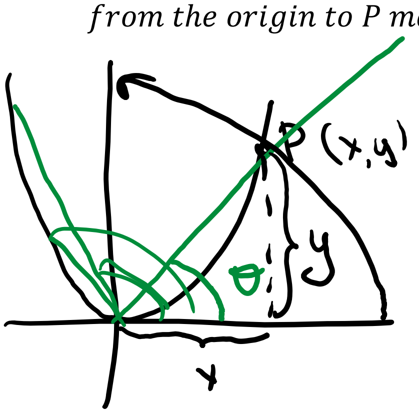
$$P(x,y) \Rightarrow \begin{cases} y = x^2 \Rightarrow t = x^2 \\ y(t) = t \end{cases} \qquad \begin{matrix} \nearrow x = \sqrt{t} \\ \searrow x = -\sqrt{t} \end{matrix}$$

$$\begin{cases} x = \sqrt{t} \\ y = t \end{cases} \quad \text{and} \quad \begin{cases} x = -\sqrt{t} \\ y = t \end{cases} \qquad 0 \le t < \infty$$





3. *Give parametric formulas for the location of P where the parameter is the angle that the ray from the origin to P makes with the positive x − axis.*



$$\frac{y}{x} = \tan\theta$$

$$y = x\tan\theta$$

$$(y = x^2) \Rightarrow x^2 = x\tan\theta \Rightarrow \begin{cases} x = \tan\theta = x(\theta) \\ y = \tan^2\theta \end{cases}$$

$$0 \le \theta < \frac{\pi}{2}, \quad \frac{\pi}{2} < \theta \le \pi$$

Example#3
Let C be the given parametric curve
Obtain the equation of the curve in the form $y = f(x)$
Sketch the graph C

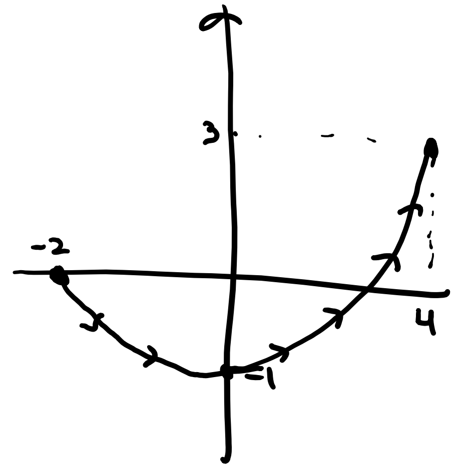
1. $C = \begin{cases} x = 2t \\ y = t^2 - 1 \end{cases} \quad -1 \le t \le 2$

$x = 2t \Rightarrow t = \frac{x}{2}$

$y = \left(\frac{x}{2}\right)^2 - 1 \Rightarrow$ parabola

$t = -1 \quad (x, y) = (-2, 0)$

$t = 2 \quad (x, y) = (4, 3)$



2. $C = \begin{cases} x = \sqrt{t-3} \\ y = \sqrt{4-t} \end{cases} \quad 3 \le t \le 4$

$x = \sqrt{t-3} \Rightarrow x \geq 0 \Rightarrow x^2 = t - 3 \Rightarrow t = x^2 + 3$

$y = \sqrt{4-t} \Rightarrow y \geq 0 \Rightarrow y^2 = 4 - t$

$y^2 = 4 - x^2 - 3$

$x^2 + y^2 = 1$

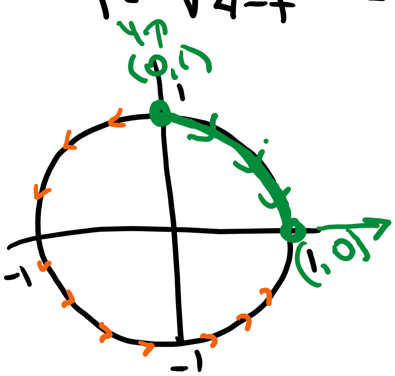
$t = 3 \quad (x, y) \to (0, 1)$

$t = 4 \quad (x, y) = (1, 0)$

$y = \sqrt{1 - x^2}$

x change from $0 \to 1$

$0 \le x \le 1$



3. $C = \begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases} \quad 0 \le t \le \pi$

$\Rightarrow \begin{cases} \frac{x}{3} = \cos t \\ \frac{y}{3} = \sin t \end{cases} \Rightarrow$

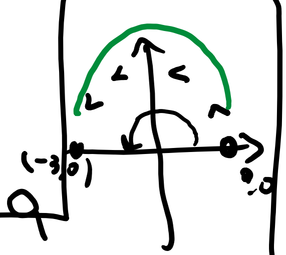
$\boxed{\cos^2 t + \sin^2 t = 1}$

$\Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = \ldots \quad x^2 + y^2 = 9$

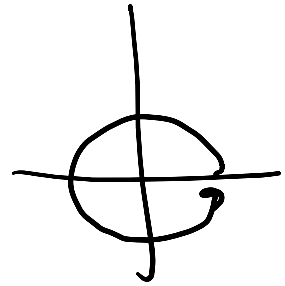
$y = \sqrt{9 - x^2}$

$t = 0 \Rightarrow (3, 0) \rightarrow$

$t = \pi \Rightarrow (-3, 0)$

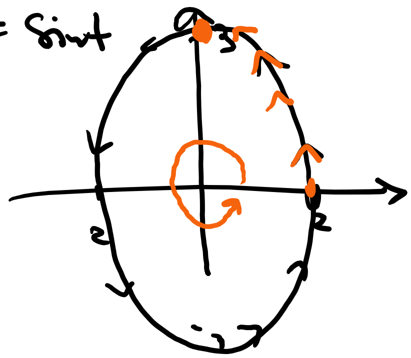


4. $C = \begin{cases} x = 2cost \\ y = 3sint \end{cases}$ $\underline{0 \leq t \leq 2\pi}$



$\frac{x}{2} = \cos t$ $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \Rightarrow$

$\frac{y}{3} = \sin t$



$\frac{x^2}{4} + \frac{y^2}{9} = 1$ ellipce

$t = 0 \Rightarrow (2, 0)$

$t = \frac{\pi}{2} \Rightarrow (0, 3)$

$t = 2\pi \Rightarrow (2, 0)$

5. $C = \begin{cases} x = sint \\ y = 2cos^2(2t) \end{cases}$

$\sin^2 t = \frac{1 - \cos 2t}{2} \Rightarrow 2\sin^2 t = 1 - \cos 2t$

$\parallel$

$\underbrace{2(\cos 2t)^2}_{} = 2(1 - 2\sin^2 t)^2$

$\boxed{y = 2(1 - 2x^2)^2}$

Tangent Lines to Parametric Curves

Just as we use tangent lines to the graph of $y = f(x)$ *to determine the rate of change of the function* f*, we would like to be able to determine how* y *changes with* x *when the curve is described by parametric equations.*

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \iff y = f(x)$$

The slope of the tangent line is the derivative

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

$$\underbrace{f'(x)}_{\text{derivative with respect to } x} = \frac{y'(t)}{x'(t)}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \left(\frac{dy}{\cancel{dt}} \cdot \frac{\cancel{dt}}{dx} = \frac{dy}{dx} \right)$$

Example#1

Find the tangent line to the curve $\begin{cases} x = 2cost \\ y = 2sint \end{cases}$ *at* $t_0 = \frac{\pi}{4}$

$y = f(x) \Rightarrow$ tangent line at $(x_0, y_0) \Rightarrow$

$$y - y_0 = \underbrace{f'(x_0)}_{\text{slope}}(x - x_0) \; : \; y - y_0 = m(x - x_0)$$

? $t_0 = \frac{\pi}{4} \Rightarrow (x_0, y_0) = \begin{pmatrix} x_0 = 2\cos(\frac{\pi}{4}) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \\ y_0 = 2\sin(\frac{\pi}{4}) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2} \end{pmatrix}$

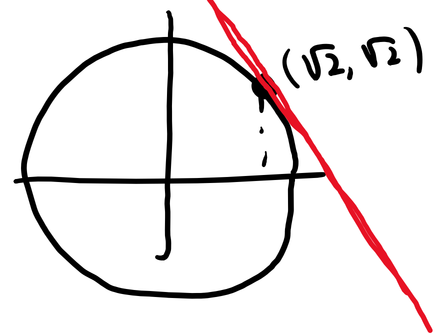
$$\boxed{(x_0, y_0) = (\sqrt{2}, \sqrt{2})}$$

? $m = f'(x_0) \Rightarrow f'(x_0) = \frac{y'(t_0)}{x'(t_0)} = \left. \frac{(2\sin t)'}{(2\cos t)'} \right|_{t_0 = \frac{\pi}{4}} =$

$$\Rightarrow f'(x_0) = \frac{2\cos(\frac{\pi}{4})}{-2\sin(\frac{\pi}{4})} = \boxed{-1}$$

$$y-\sqrt{2} = (-1)(x-\sqrt{2})$$
$$y-\sqrt{2} = -x+\sqrt{2}$$
$$y = -x+2\sqrt{2}$$

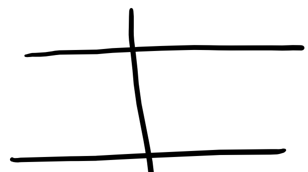
$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \qquad t_0 = \pi/4$$



Example#2

Find the points where the curve $\begin{cases} x = t^3 - 3t \\ y = 3t^2 - 9 \end{cases}$ *have horizontal and vertical tangents*

Horizontal tangent



$y = \text{const} \Rightarrow y'(x) = 0$

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{(3t^2-9)'}{(t^3-3t)'} = \frac{6t}{3t^2-3} = \frac{2t}{t^2-1}$$

$$y'(x) = 0 \Rightarrow \begin{cases} 2t = 0 \\ t^2-1 \neq 0 \end{cases} \Rightarrow t = 0 \Rightarrow \begin{array}{l} x = 0^3 - 3\cdot 0 = 0 \\ y = 3\cdot 0^2 - 9 = -9 \end{array}$$

$$\boxed{(0,-9)}$$

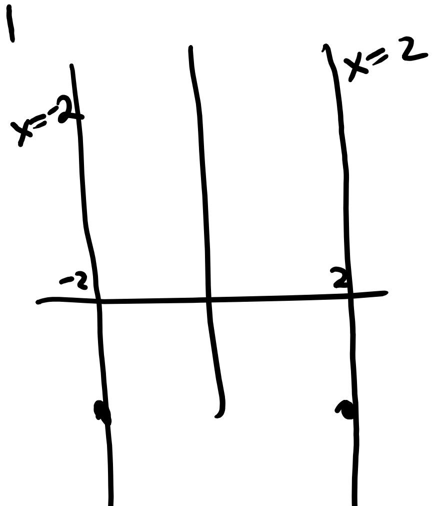
Vertical Tang $\Rightarrow$ $y'(x)$ does not exist.

$$t^2-1=0 \Rightarrow t = \pm 1$$

$\begin{cases} x = t^3 - 3t \\ y = 3t^2 - 9 \end{cases}$

$$t = 1 \Rightarrow \begin{array}{l} x = 1-3 = -2 \\ y = 3-9 = -6 \end{array} \quad (-2,-6)$$

$$t = -1 \Rightarrow \begin{array}{l} x = -1+3 = 2 \\ y = 3-9 = -6 \end{array} \quad (2,-6)$$



Example#3

a. Find the ~~*slope*~~ *of the tangent line to the curve* $\begin{cases} x = t^3 - 4t \\ y = t^2 \end{cases}$ *at the point* $(15,9)$

$(x_0, y_0) = (15, 9)$

$\Downarrow$

$t_0 = ?$

$y - y_0 = m(x - x_0)$, $y_0 = 9$, $x_0 = 15$

$\begin{cases} 15 = t^3 - 4t \\ 9 = t^2 \end{cases} \Rightarrow t = 3, \cancel{-3} \Rightarrow 15 = 27 - 4 \cdot 3 = 15$ ✓

$t = -3$: $15 = -27 + 12 = -15$ ✗

$\boxed{t_0 = 3} \longleftrightarrow (15, 9)$

$$m = y'(x)\Big|_{x=15} = \frac{y'(t)}{x'(t)}\Big|_{t=3} = \frac{2t}{3t^2 - 4}\Big|_{t=3} = \frac{6}{27-4} = \frac{6}{23}$$

($h(t)$)

the slope $m = \frac{6}{23}$

$$\boxed{y - 9 = \frac{6}{23}(x - 15)}$$

tangent line at $(15,9)$

$b.$ *Find the derivative* $\frac{d^2y}{dx^2}$ *at the same point.*

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad y'(x) = \boxed{\frac{y'(t)}{x'(t)} = h(t)}$$

$$\Rightarrow \begin{cases} y' = h(t) \\ x = x(t) \end{cases} \Rightarrow \quad y''(x) = \frac{(y'(t))'}{x'(t)} = \frac{y''(t)}{x'(t)}$$

$$h(t) = \frac{2t}{3t^2-4}$$

$$f''(x) = \frac{\left(\frac{2t}{3t^2-4}\right)'}{x'(t)} = \frac{\left(\frac{2t}{3t^2-4}\right)'}{(t^3-4t)'} = \frac{\frac{2(3t^2-4)-2t(6t)}{(3t^2-4)^2}}{(3t^2-4)} =$$

$$= \frac{-8-6t^2}{(3t^2-4)^3}\Bigg|_{(15,9) \to t=3} = \frac{-8-6\cdot 9}{(27-4)^3} = \boxed{\frac{-62}{(23)^3}}$$

Example#4

Find an equation of the horizontal tangent line to the curve $\begin{cases} x = t^6 - t + 3 \\ y = t^4 - 4t \end{cases}$

***Example*#5** $(Final\ Spring\ 2019)$

$Consider\ the\ parametris\ equation\ given\ by\ x(t) = 2\sin(t)\,, y(t) = 2\cos(t)$

1. $Graph\ the\ portion\ of\ the\ curve\ for\ on\ the\ range\ 0 \leq t \leq \frac{\pi}{2}$
   $and\ find\ the\ slope\ at\ the\ point\ \left(\sqrt{2}, \sqrt{2}\right)$