$\boldsymbol{Series\ (Part 4)}$

Suppose, that not all a_n are positive.

DEF: *A series* $\sum a_n$ ***converges absolutely*** *if* $\sum |a_n|$ *converges*

DEF: *A series* $\sum a_n$ ***converges conditionally*** *if* $\sum a_n$ ***converges*** $\sum |a_n|$ *diverges*

$$\boxed{If \ \sum |a_n| \ converges, then \sum a_n \ converges}$$

Ratio Test

Let $\sum \mathrm{a_n}$ *be any series and* $\lim\limits_{n\to\infty} \left\| \frac{\mathrm{a_{n+1}}}{\mathrm{a_n}} \right\| = \rho$
If $\boldsymbol{\rho < 1}$, then the series **converges absolutely**
If $\boldsymbol{\rho > 1}$, or $\boldsymbol{\rho = \infty}$ then the series **diverges**
If $\boldsymbol{\rho = 1}$, the test is **inconclusive**

Very useful with factorials!

Root Test

Let $\sum \mathrm{a_n}$ *be any series and* $\lim\limits_{n\to\infty} \sqrt[n]{\|a_n\|} = \rho$
If $\boldsymbol{\rho < 1}$, then the series **converges absolutely**
If $\boldsymbol{\rho > 1}$, or $\boldsymbol{\rho = \infty}$ then the series **diverges**
If $\boldsymbol{\rho = 1}$, the test is **inconclusive**

- $\boxed{\ln(\ln n) \ll \ln n \ll \underbrace{n^a \ll n}_{0<a<1} \underbrace{\ll n^b}_{1<b} \ll \underbrace{a^n}_{a>1} \ll n! \ll n^n}$

- $\lim_{n\to\infty} \sqrt[n]{a} = 1$

- $\lim_{n\to\infty} \sqrt[n]{n} = 1$

- $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$

- $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)(n)$

- $0! = 1, \quad 1! = 1$

Examples:

1. $\sum_{n=1}^{\infty} \underbrace{\frac{n!}{3^n}}_{a_n > 0} \longrightarrow$ Ratio ? $\left(\sqrt[n]{n!} \longrightarrow ? \right)$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\left(\frac{(n+1)!}{3^{n+1}}\right) \leftarrow a_{n+1}}{\frac{n!}{3^n}} = \lim_{n\to\infty} \frac{(n+1)!\ \cancel{3^n}}{n!\ \underbrace{\cancel{3^{n+1}}}_{3}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{1\cdot 2\cdot 3\cdots n}(n+1)}{\cancel{1\cdot 2\cdot 3\cdots n}\cdot 3} = \lim_{n\to\infty} \frac{n+1}{3} = \boxed{\infty = \rho}$$

$\sum_{n=1}^{\infty} \frac{n!}{3^n}$ diverges by Ratio Test.

no(!) $\longrightarrow$ and powers $\underline{n}$ $\rightarrow$ $\underline{\text{Root}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \lim \frac{(n^3)^{\frac{1}{n}}}{(3^n)^{\frac{1}{n}}} =$$

$$= \lim_{n\to\infty} \frac{(n^{\frac{1}{n}})^3}{3} = \lim_{n\to\infty} \frac{(\overset{1}{\sqrt[n]{n}})^3}{3} = \frac{1}{3} = \rho < 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges by Root Test.

(!) $\longrightarrow$ Ratio

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{\left(\frac{(n+1)!}{(n+1)^{n+1}}\right)}{\left(\frac{n!}{n^n}\right)} = \lim_{n\to\infty} \frac{(n+1)!\cdot n^n}{n!\,(n+1)^{n+1}} =$$

a_{n+1}

a_n

$$\lim(1+\frac{a}{n})^n = e^a$$

$$\lim_{n\to\infty} \frac{\cancel{1\cdot 2\cdots n}\cancel{(n+1)}\cdot n^n}{\cancel{1\cdot 2\cdots n}\,(n+1)^n\cdot\cancel{(n+1)}} = \lim_{n\to\infty} \left(\frac{\overset{1}{n}}{n+1}\right)^{\overset{\infty}{n}} =$$

$$\lim_{n\to\infty} \left(\frac{(n+1)-1}{n+1}\right)^{n+1-1} = \lim_{n\to\infty} \boxed{\left(1+\frac{-1}{n+1}\right)^{n+1}} \cdot \left(1+\frac{-1}{n+1}\right)^{-1} = e^{-1} = \frac{1}{e} < 1$$

$\ast \downarrow e^{-1}$ $\quad$ $\downarrow 0$ $\quad$ 1

$\rho < 1$

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by Ratio Test

$n! \longrightarrow$ Ratio

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

$$\lim_{n\to\infty} \frac{\left(\frac{(n+1)^2}{(2(n+1)+1)!}\right)}{\frac{n^2}{(2n+1)!}} = \lim_{n\to\infty} \frac{\boxed{(n+1)^2}}{\boxed{n^2}} \cdot \frac{(2n+1)!}{(2n+3)!} =$$

$$\lim \left(\frac{n+1}{n}\right)^2 \cdot \frac{\cancel{1\cdot 2 \cdots (2n+1)}}{\cancel{1\cdot 2 \cdots (2n+1)}(2n+2)(2n+3)} = 0 < 1$$

$\left(\frac{n+1}{n}\right) \to 1$, $\quad \frac{1}{(2n+2)(2n+3)} \to 0$

$\rho = 0 < 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$ converges by Ratio Test

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$\lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{\overbrace{(n+1)!\cdot(n+1)!}^{((n+1)!)^2}\cdot(2n)!}{\underbrace{n!\cdot n!}_{(n!)^2}\cdot(2n+2)!} =$$

$$\lim_{n\to\infty} \frac{(n+1)(n+1)\cdot\cancel{(2n)!}}{\cancel{(2n)!}\cdot(2n+1)(2n+2)}$$

$$= \lim \frac{(n+1)}{(2n+1)} \frac{(n+1)}{(2n+2)} = \frac{1}{4} < 1$$

$\frac{n+1}{2n+1} \to \frac{1}{2}$, $\quad \frac{n+1}{2n+2} = \frac{1}{2}$

$\boxed{\rho = \frac{1}{4} < 1}$

$\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges by Ratio Test

6. $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$

Ratio: $\lim_{n\to\infty} \frac{\frac{(3(n+1))!}{((n+1)!)^3}}{\frac{(3n)!}{(n!)^3}} = \lim_{n\to\infty} \frac{(3(n+1))! \cdot (n!)^3}{(3n)! \, ((n+1)!)^3} =$

$= \lim_{n\to\infty} \frac{(3n+3)!}{(3n)!} \cdot \frac{(n!)\cdot(n!)(n!)}{((n+1)!)(n+1)!\,(n+1)!} =$

$(n+1)! = \underbrace{1\cdot 2 \cdots n}\cdot(n+1) \Rightarrow \boxed{(n+1)! = n!(n+1)}$

$\lim \frac{\cancel{1\cdot 2 \cdots (3n)}(3n+1)(3n+2)(3n+3)}{\cancel{1\cdot 2 \cdots (3n)}} \cdot \frac{\cancel{n!} \; \cancel{n!} \; \cancel{n!}}{\cancel{n!}(n+1)\,\cancel{n!}(n+1)\,\cancel{n!}(n+1)}$

$= \lim_{n\to\infty} \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+1)(n+1)} = 3\cdot 3\cdot 3 > 1$

$\left(\frac{3+\frac{1}{n}}{1+\frac{1}{n}}\right) \to 3$; $\to 3$ the same ; $= 3$

$\rho = 27 > 1$

$\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}$ diverges by Ratio Test.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{n^{2n}}\right)^{n}} = \lim_{n\to\infty} \frac{(2n+1)}{n^{2n\cdot\frac{1}{n}}} = \lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{n\to\infty}\Big(\underbrace{\frac{2}{n}}_{\downarrow\, 0} + \underbrace{\frac{1}{n^2}}_{\downarrow\, 0}\Big) = 0$$

$$\rho = 0 < 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ converges by Root Test.

8. $\displaystyle\sum_{n=1}^{\infty} n^{100} 2^{-2n}$

$$\lim_{n\to\infty} \sqrt[n]{n^{100}\cdot 2^{-2n}} = \lim_{n\to\infty} \Big(\underbrace{\sqrt[n]{n}}_{\downarrow\, 1}\Big)^{100} \cdot \underbrace{2^{-2}}_{=\frac{1}{4}} = \frac{1}{4} < 1$$

$\displaystyle\sum_{n=1}^{\infty} n^{100} 2^{-2n}$ converges by Root Test.

9. $\sum_{n=1}^{\infty} \frac{(1+lnn)^n}{n^n}$

$$\lim \sqrt[n]{\frac{(1+\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{1+\ln n}{n} = \frac{\infty}{\infty} = \overset{L'H}{\lim_{x\to\infty}} \frac{1+\ln x}{x} =$$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 = \rho < 1$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(1+lnn)^n}{n^n} \text{ converges by Root Test.}}$$

10. $\sum_{n=1}^{\infty} \frac{lnn}{n}$

Ratio $\lim_{n\to\infty} \frac{\frac{\ln(n+1)}{(n+1)}}{\frac{\ln n}{n}} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{n}{(n+1)} = 1 \cdot 1 = 1$

$\frac{\infty}{\infty}$ $\quad$ $\boxed{\rho = 1}$

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln(x)} = L'H_{rule} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} = 1$$

This Test does not work and we have to try something different

For example, try DCT : $\frac{\ln n}{n} > \frac{1}{n}$, for $n \geq 3$, so

$$\sum_{n=1}^{\infty}\frac{lnn}{n} \text{ diverges by DCT with } \sum_{n=1}^{\infty}\frac{1}{n} - \text{divergent harmonic series}$$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{2n}{\sqrt{n^3+4n}}$

Try Root, for example: $\lim \sqrt[n]{\dfrac{2n}{\sqrt{n^3+4n}}} = \lim \dfrac{\overset{\nearrow 1}{\sqrt[n]{2}} \cdot \overset{\nearrow 1}{\sqrt[n]{n}}}{\left(n^{3/2}\sqrt{1+\frac{4}{n^2}}\right)^{1/n}} =$

$\lim \dfrac{\overset{\nearrow 1}{\sqrt[n]{2}} \cdot \overset{\nearrow 1}{\sqrt[n]{n}}}{\underset{\searrow 1}{(n^{1/n})^{3/2}} \underset{\searrow 1}{\left(\sqrt{1+\frac{4}{n^2}}\right)^{\frac{1}{n} \nearrow 0}}} = 1$

$\rho = 1 \Rightarrow$ Test did not work

$$\text{For example, DCT}: \quad \frac{2n}{\sqrt{n^3+4n}} > \frac{2n}{\sqrt{n^3+n^3}} = \frac{2n}{\sqrt{2}\cdot n^{3/2}} = \frac{\sqrt{2}}{n^{1/2}}$$

$$\left(\text{LCT with } \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \text{ will work too}\right).$$

$$\sum_{n=1}^{\infty}\frac{2n}{\sqrt{n^3+4n}} \text{ diverges by DCT with } \sqrt{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \quad \left(p=\frac{1}{2}<1 \text{ divergent}\right)$$

Mixed Problems

conv. by Root

div

1. $\sum_{n=1}^{\infty} 2^{\frac{1}{n}}$

div

$a_n = 2^{\frac{1}{n}} \xrightarrow[n\to\infty]{} 2^0 = 1$ ($\frac{1}{n} \to 0$)

$a_n \to 1 \neq 0$

D.Test,

2. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

Root

$\lim \sqrt[n]{\left(1+\frac{1}{n}\right)^{-n^2}} =$

$\lim \left(1+\frac{1}{n}\right)^{-n} =$

$= \lim \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$

3. $\sum_{n=1}^{\infty} \frac{1}{10+e^{\frac{2}{n}}}$

$a_n = \frac{1}{10+e^{2/n}} \to \frac{1}{11}$ ($2/n \to 0$)

$a_n \not\to 0$

Try the rest! (Can be on quiz..  ..)

4. $\sum_{n=1}^{\infty} \frac{n^2}{(n+2)(n-2)}$

5. $\sum_{n=1}^{\infty} \frac{4^n}{3(n+1)^{n-1}}$

6. $\sum_{n=1}^{\infty} \frac{\cos^2 n + 1}{n^2+1}$

7. $\sum_{n=1}^{\infty} \frac{n+2^{-n}}{n2^n - 1}$

8. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{\ln n}}$

9. $\sum_{n=1}^{\infty} \frac{1}{\ln\left(n^2\right)}$

10. $\sum_{n=1}^{\infty} \frac{n}{e^n}$

11. $\sum_{n=1}^{\infty} \frac{1}{e^n}$

12. $\sum_{n=1}^{\infty} \frac{e^n}{n}$

13. $\sum_{n=1}^{\infty} \frac{\sin(n) + 2^n}{n + 5^n}$

14. $\sum_{\mathrm{n}=1}^{\infty} \left(\frac{\mathrm{n} + 1}{3\mathrm{n}}\right)^{\mathrm{n}}$