Math Preliminaries
(Ch.2 of text book)

Set concepts

- Set definitions

  Enumeration : {2,5,8}

  Property definition (useful for infinite sets):

  e.g {x | x % 2 = 0} – set of even integers
- Membership : $x \in S$, x is in S

  $x \notin S$, x is not in S
- Subset relation : $S \subseteq T \rightarrow$ for all $x \in S$, $x \in T$

  $S \subset T \rightarrow S \subseteq T$ and $S \neq T$
- Subset operations: (union, intersection,complement)

  $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$

  $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$

  $S - T$ (or $S \setminus T$) $= \{x \mid x \in S \text{ and } x \notin T\}$

Set operations

- $\emptyset$ - *empty set*
- ∪ and ∩ are associative

$S \cup (T \cup V) = (S \cup T) \cup V$

$S \cap (T \cap V) = (S \cap T) \cap V$

- ∪ and ∩ are commutative

$S \cup T = T \cup S$ and $S \cap T = T \cap S$

- $S \cup \emptyset = S = \emptyset \cup S$ and $S \cap \emptyset = \emptyset = \emptyset \cap S$
- ∪ distributes over ∩; ∩ distributes over ∩

$S \cap (T \cup V) = (S \cap T) \cup (S \cap V)$

$S \cup (T \cap V) = (S \cup T) \cap (S \cup V)$

- De Morgan's laws : $\overline{S} = U - S$ for universal set U

  $\overline{S \cup T} = \overline{S} \cap \overline{T}$ and $\overline{S \cap T} = \overline{S} \cup \overline{T}$

Common numerical functions

- Polynomial functions:

A *polynomial of degree* d is a function p of the form

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

$$= a_0 + a_1 n + a_2 n^2 + a_3 n^3 + \cdots + a_d n^d,$$

where $a_d \neq 0$.

The a_i's are the *coefficients* of the polynomial.

For large values of n the polynomial $p(n)$ does not differ much from $a_d n^d$, since n^d dominates the lower-order terms n^i for $i < n$.

Common numerical functions

- Exponents

For any $a > 0$ and integers m, n,

$$a^0 = 1$$

$$a^1 = a$$

$$a^{-1} = \frac{1}{a}$$

$$(a^m)^n = a^{mn}$$

$$a^m a^n = a^{m+n}$$

Common numerical functions

- Logarithms

If $a = b^c$ for b > 0 , then $c = log_b\ a$

For any positive a, b, c and n,

$$a = b^{\log_b a}$$

$$\log_c(ab) = \log_c a + \log_c b$$

$$\log_b a^n = n \log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b(1/a) = -\log_b a$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b n} = n^{\log_b a}$$

Series and summations

$$, \sum_{i=0}^{n}(ai + b) = (\mathrm{n} + 1)\mathrm{b} + \frac{n(n+1)}{2}\,\mathrm{a}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} a^i = \frac{a - a^{n+1}}{1-a}, \quad a \neq 1, \sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$$

$$\sum_{i=1}^{n} ia^i = \frac{a - a^{n+1}}{(1-a)^2} - \frac{na^{n+1}}{1-a}, \quad a \neq 1$$

In particular (take $a = 1/2$):

$$\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$$

What is a mathematical proof ?

- Prove an assertion or a predicate from a given set of facts or axioms assumed to be true
- Sequence of logical implications

A (axioms) → P1 → P2 → P3 ……. → Assertion

- Theorem : Major mathematical assertion
- Lemma : Minor assertion useful for proving theorems
- Conjecture : Assertion without a proof based on some observations. Can be true or false.

Proof techniques

1. Direct proofs
2. Contra-positive proofs
3. Proof by contradiction
4. Proof by induction