Power ***Series*** **(*part*1)**

$$\sum_{n=0}^{\infty} c_n(\underline{\underline{x}} - a)^n = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Where

x − *is a variable*,

a − *is a constant*(*given* "center"),

c_n − *are all constants depending on n only*

For each fixed x, this series is a series of constantsthat we can test for convergency.

A Power Series may converge for some values of x and diverge for another values

There are three possible types of convergence for a power series:

1. *Converges only when* $x = a$ $\quad \sum c_n(a-a)^n = \sum c_n(0)^n = \sum 0 = 0+0+\ldots$

2. *Converges for all x* $\quad x \in (-\infty, \infty)$

3. *There is a positive R such that*:

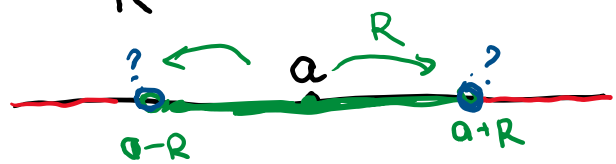
if $|x-a| < R$, $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges absolutely*.

$|x-a| < R$

$-R < x-a < R$

$a-R < x < a+R$

If $|x-a| > R$, *than* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *diveges.*



if $|x-a| = R$ we need to check

The number R is called the ***Radius of Convergence***.

*To find it we will use the r**atio or the root test***:

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = |x-a|L < 1 \to |x-a| < \frac{1}{L} \,(if L \neq 0), \qquad \boxed{R = \frac{1}{L}}$$

The intervals of convergence will be centered around $x = a$.

$(a-R, \ a+R)$

You will ***have to check the endpoints*** *of the interval of convergence separately to determine if the series converges on the closed interval.*

Find the radius and interval of convergence for the following power series:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2 x^n}{2^n}$

0

1

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n n^2 x^n}{2^n}\right|} = \lim \sqrt[n]{\frac{n^2}{2^n}|x|^n} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2 |x|}{2} =$$

$$= |x| \lim_{n\to\infty}\left(\frac{(\sqrt[n]{n})^2}{2}\right) = \underbrace{|x| \cdot \frac{1}{2} < 1}$$

$\sqrt[n]{n} \approx 1$

$(|x-a| < R)$

$|x| < 2 \longrightarrow \boxed{R=2}$

$-2 < x < 2 \longrightarrow (-2, 2)$

2

$\underline{x=-2} \longrightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (-2)^n}{2^n} = \sum \frac{n^2 \cdot 2^n}{2^n} = \sum n^2 \to$ diverges

$\lim n^2 = \infty \neq 0$

diverges by n-th term

$x=2 \to \displaystyle\sum \frac{(-1)^n n^2 2^n}{2^n} = \sum (-1)^n n^2$, diverges by n-th term

$\lim (-1)^n n^2 = (\pm\infty) = DNE$

$\boxed{R = 2, \quad Interval: (-2,2)}$

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$$

Ratio: $\lim_{n\to\infty} \left| \dfrac{\dfrac{(x+5)^{n+1}}{(n+1)((n+1)+1)}}{\dfrac{(x+5)^n}{n(n+1)}} \right| = \lim_{n\to\infty} \left| \dfrac{(x+5)^{\cancel{n+1}}\, n\,\cancel{(n+1)}}{\cancel{(n+1)}(n+2)\,\cancel{(x+5)^n}} \right| =$

$$= \lim_{n\to\infty} \frac{|x+5|\, n}{n+2} = |x+5| \underbrace{\lim_{n\to\infty} \frac{n}{n+2}}_{1} = |x+5| \cdot 1 < 1$$

$|x-a| < R$ $\qquad |x+5| < 1 \implies \boxed{R = 1}$

$-1 < x+5 < 1$

$-1-5 < x < 1-5$

$-6 < x < -4 \implies [-6, -4]$

End points

$x = -6$: $\sum \frac{(x+5)^n}{n(n+1)} = \sum \frac{(-1)^n}{n(n+1)}$ → ① $\sum \frac{1}{n(n+1)}$ conv. by Comp test $\sum \frac{1}{n^2}$, $p = 2 > 1$

at $x = -6$ $\sum \frac{(x+5)^n}{n(n+1)}$ conv. absolutely

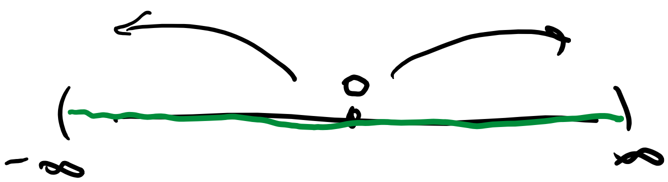
$x = 4$ $\quad \sum \frac{1^n}{n(n+1)} = \sum \frac{1}{n(n+1)}$ → – converges

$\boxed{R = 1, \quad Interval: [-6, -4]}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n)!}$

Ratio $\lim \left| \frac{\frac{\cancel{(-1)^{n+1}}\, x^{n+1}}{(2(n+1))!}}{\frac{\cancel{(-1)^n}\, x^n}{(2n)!}} \right| = \lim \left| \frac{\cancel{x^{n+1}}\, (2n)!}{(2n+2)!\, \cancel{x^n}} \right| =$

$$= \lim_{n\to\infty} \frac{|x|}{(2n+1)(2n+2)} = |x| \cdot \lim_{n\to\infty} \underbrace{\frac{1}{(2n+1)(2n+2)}}_{0} = \underbrace{|x| \cdot 0 = 0 < 1}_{\text{always! for any } x}$$



$-\infty < x < \infty$

$(-\infty, \infty)$, $R = \infty$

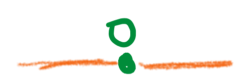
$\boxed{R = \infty, \quad Interval: \ (-\infty, \infty)}$

4. $\sum_{n=1}^{\infty} n!\, x^n$

$$\lim_{n\to\infty} \frac{|(n+1)!\, x^{n+1}|}{|n!\, x^n|} = \lim |x| \cdot (n+1) = |x| \underbrace{\lim (n+1)}_{\infty} = \underbrace{\infty < 1}_{\text{only when } |x| = 0}$$

$0 = |x| (\infty)$?

$\Rightarrow |x| = 0, \Rightarrow x = 0 \Rightarrow$



$\sum n!\, x^n$ conv. only for $x = 0$,

$R = 0$, no interval

$\boxed{R = 0, \quad Interval: \ NO\ INTERVAl, series\ converges\ for\ x = 0\ only.}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n(5x+2)^n}{3^{n+1}}$

Root

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n(5x+2)^n}{3^{n+1}}\right|} = \lim \frac{|5x+2|\overbrace{\sqrt[n]{n}}^{\to 1}}{3^{\frac{n+1}{n}}} = |5x+2| \lim \frac{\overbrace{\sqrt[n]{n}}^{\to 1}}{3^1 \cdot \underbrace{3^{1/n}}_{\to 3^0 = 1}} = ♡$$

$$♡ = |5x+2| \cdot \frac{1}{3} < 1$$

$|x - a| < R$

$$|5x+2| < 3$$

$$5\left|x + \frac{2}{5}\right| < 3$$

$$\left|x + \frac{2}{5}\right| < \frac{3}{5} \Longrightarrow \boxed{R = \frac{3}{5}}$$

$$-\frac{3}{5} < x + \frac{2}{5} < \frac{3}{5}$$

$$-\frac{2}{5} - \frac{3}{5} < x < -\frac{2}{5} + \frac{3}{5}$$

$$-1 < x < \frac{1}{5} \Longrightarrow \boxed{\left(-1, \frac{1}{5}\right)}$$

End points

$$x = -1 \to \sum \frac{n(5x+2)^n}{3^{n+1}} = \sum \frac{n\overbrace{(-3)^n}^{(-1)^n \cdot \cancel{3^n}}}{\cancel{3^{n+1}}} = \sum \frac{(-1)^n n}{3}$$ – diverges by n-th term test

$$x = \frac{1}{5} \to \sum \frac{n\overbrace{\left(5 \cdot \frac{1}{5} + 2\right)^n}^{\cancel{3^n}}}{\cancel{3^{n+1}}} = \sum \frac{n}{3}$$ same

$R = \frac{3}{5}$, $\quad Interval: (-1, \frac{1}{5})$

6. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n\sqrt{n}}$

Root $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{3^n \cdot \sqrt{n}}\right|} = \lim_{n\to\infty} \frac{|x-2|}{3\cdot(\sqrt[n]{\sqrt{n}})^{1/2}} = |x-2| \lim \frac{1}{3(\sqrt[n]{n})^{1/2}} =$

$$|x-2| \cdot \frac{1}{3} < 1$$

$|x-2| < 3 \longrightarrow \boxed{R=3}$

$-3 < x-2 < 3$

$-1 < x < 5 \longrightarrow [-1, 5)$

End-points $x=-1$ $\sum \frac{(x-2)^n}{3^n\sqrt{n}} = \sum \frac{(-3)^n}{3^n \cdot \sqrt{n}} = \sum \frac{(-1)^n \cancel{3^n}}{\cancel{3^n} \cdot \sqrt{n}} = \sum \frac{(-1)^n}{\sqrt{n}}$

① $\sum \frac{1}{\sqrt{n}}$ div as $p = \frac{1}{2} < 1 \Rightarrow$ no abs. conv

② AST: 1-2-3 ✓ $\begin{cases} \frac{1}{\sqrt{n}} > 0 \\ \frac{1}{\sqrt{n}} \searrow \\ \lim \frac{1}{\sqrt{n}} = 0 \end{cases}$ $\Rightarrow$ conv condition

$x=5$ $\sum \frac{3^n}{3^n\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ div. as $p = \frac{1}{2} < 1$

$\boxed{R = 3, \quad Interval: [-1, 5)}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^{3n}}{2^n lnn}$

$$\lim_{n\to\infty} \left| \frac{\frac{(x+3)^{3(n+1)}}{2^{n+1} \cdot \ln(n+1)}}{\frac{(x+3)^{3n}}{2^n \cdot \ln n}} \right| = \lim_{n\to\infty} \frac{|x+3|^{\cancel{3n}+3} \cdot \cancel{2^n} \cdot \ln n}{\cancel{|x+3|^{3n}} \cdot \cancel{2^{n+1}} \cdot \ln(n+1)}$$

$$= \lim |x+3|^3 \frac{1}{2} \left(\frac{\ln n}{\ln(n+1)}\right) = |x+3|^3 \cdot \frac{1}{2} < 1$$

$$\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \frac{\infty}{\infty} = \text{①}$$

$$= \lim \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim \frac{x+1}{x} = 1$$

$$|x+3|^3 < 2$$

$$|x+3| < \sqrt[3]{2} \implies \boxed{R = \sqrt[3]{2}}$$

$$-\sqrt[3]{2} < x+3 < \sqrt[3]{2}$$

$$-3-\sqrt[3]{2} = x < \sqrt[3]{2}-3 \implies \left(-3-\sqrt[3]{2}, -3+\sqrt[3]{2}\right)$$

$$x = (-3-\sqrt[3]{2}) \implies \sum \frac{(-1)^n (x+3)^{3n}}{2^n \cdot \ln n} = \sum \frac{(-1)^n (-\sqrt[3]{2})^{3n}}{2^n \cdot \ln n} = \sum \frac{(-1)^n (-1)^{3n} (\sqrt[3]{2})^{3n}}{2^n \ln n} =$$

$$= \sum \frac{2^n}{2^n \cdot \ln n} = \sum \frac{1}{\ln n},$$

$\ln n < n \implies$ DCT $\frac{1}{\ln n} > \frac{1}{n}$

diverges by DCT. $\sum \frac{1}{n}$ harm.

$$x = (-3+\sqrt[3]{2})$$

$$\implies \sum \frac{(-1)^n (\sqrt[3]{2})^{3n}}{2^n \cdot \ln n} = \sum \frac{(-1)^n}{\ln n}$$

① For abs / div

AST $\implies$ Check 1-2-3

$\implies$ conv. conditionally

$$\boxed{R = \sqrt[3]{2}, \quad \left(-3-\sqrt[3]{2}, -3+\sqrt[3]{2}\right]}$$

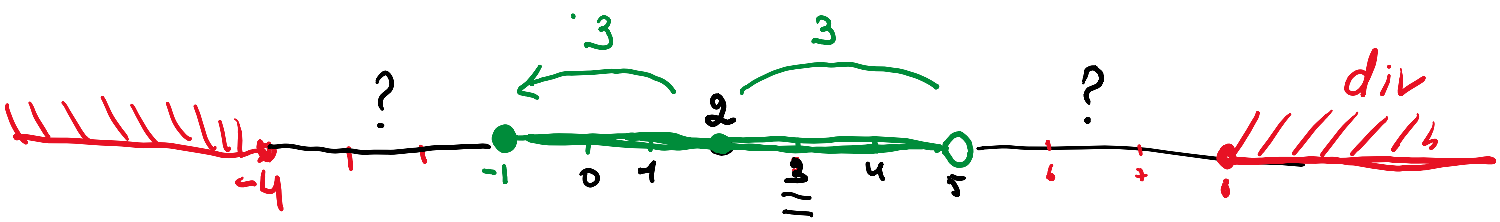
Example#8

Suppose $\sum_{n=1}^{\infty} c_n(x-2)^n$ *converges for* $x=-1$ *and diverges for* $x=8$

Circle the values of x for which the series MUST converge:

$x=2, x=3, x=5, x=6, x=8$

$x-2 \Rightarrow a=2$



for $x=5$, $x=6$ we can not know

$x=8 \rightarrow$

Example#9

Suppose $\sum_{n=1}^{\infty} c_n(x-1)^n$ *converges for* $x=-4$ *and diverges for* $x=8$

What can you say (if any) about the following series:

$$\sum_{n=1}^{\infty} c_n$$

$$\sum_{n=1}^{\infty} c_n (-1)^n \, 9^n$$

$$\sum_{n=1}^{\infty} c_n (-6)^n$$

$$\sum_{n=1}^{\infty} c_n n \, 3^{n-1} \; (\,later)$$

Extra practice

1. $Fall\ 2019 \quad \sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2 2^n}$

2. $Spring\ 2019 \quad \sum_{n=1}^{\infty} \frac{(x-1)^n}{n 3^n}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n (2x+3)^n}{3^n \ln{(2n-3)}}$