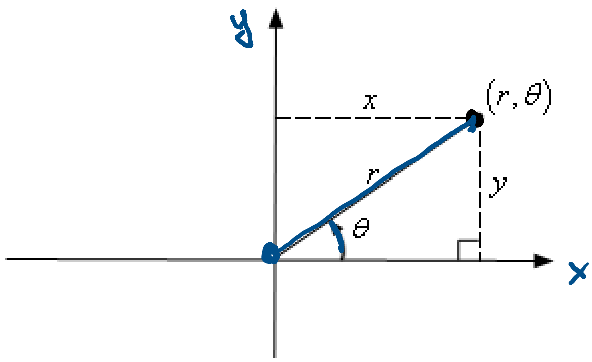
Polar Coordinates

To define polar coordinate we first fix an origin O and an initial ray from O. Then each point P can be assigned Polar Coordinates (r, θ), in which r gives the directed distance from O to P, and θ gives the directed angle from the initial ray to the segment OP.



Polar to Rectangular (only one correct answer)	*Rectangular to Polar* (infinitely many answers)
$x = r\cos\theta$ $y = r\sin\theta$	$x^2 + y^2 = r^2$ $\frac{y}{x} = \tan\theta$

$(x,y) \longleftrightarrow (r,\theta)$

$(r,\theta) \rightarrow \underbrace{(x,y)}$

only one correct outcom

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$

$(x,y) \rightarrow (r,\theta)$

$x^2 + y^2 = r^2 \Rightarrow$

$r = \pm\sqrt{x^2+y^2}$

θ: $\tan\theta = \frac{y}{x}$

$$\underbrace{\theta = \arctan\left(\frac{y}{x}\right)}_{\text{infinitely many correct answers}}$$

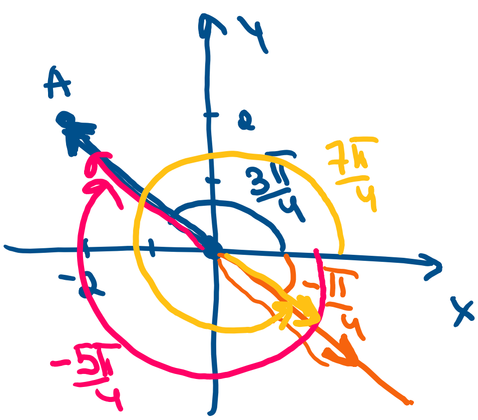
Example#1

a. Find the all rectangular coordinates of the point $(r,\theta) = (4, \frac{\pi}{6}) \longrightarrow (x, y) = ?$

$$\begin{cases} \boxed{x = r\cos\theta} = 4\cos\left(\frac{\pi}{6}\right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} \\ \boxed{y = r\sin\theta} = 4\sin\left(\frac{\pi}{6}\right) = 4 \cdot \frac{1}{2} = 2 \end{cases}$$

$$(r,\theta) = \left(4, \frac{\pi}{6}\right) \longrightarrow (x, y) = \left(2\sqrt{3}, 2\right)$$

b. Find all the polar coordinates of the point $(-2, 2) = (x, y) \Rightarrow (r, \theta) = ?$



$r^2 = x^2 + y^2$

$r^2 = (-2)^2 + (2)^2$

$r^2 = 8 \Rightarrow r = \pm 2\sqrt{2}$

θ: $\tan\theta = \frac{y}{x} = \frac{2}{-2} = -1 \Rightarrow$ 1) $\theta = \frac{3\pi}{4}$

(r,θ): $(+,+) \rightarrow \left(2\sqrt{2}, \frac{3\pi}{4} + 2\pi k\right)$

$$\begin{cases} x = 2\sqrt{2}\cdot\cos\left(\frac{3\pi}{4}\right) = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = -2 \\ y = 2\sqrt{2}\sin\left(\frac{3\pi}{4}\right) = 2\sqrt{2}\left(+\frac{\sqrt{2}}{2}\right) = 2 \end{cases} \quad (-2, 2)$$

$(-,-) \rightarrow \left(-2\sqrt{2}, -\frac{\pi}{4} + 2\pi k\right)$

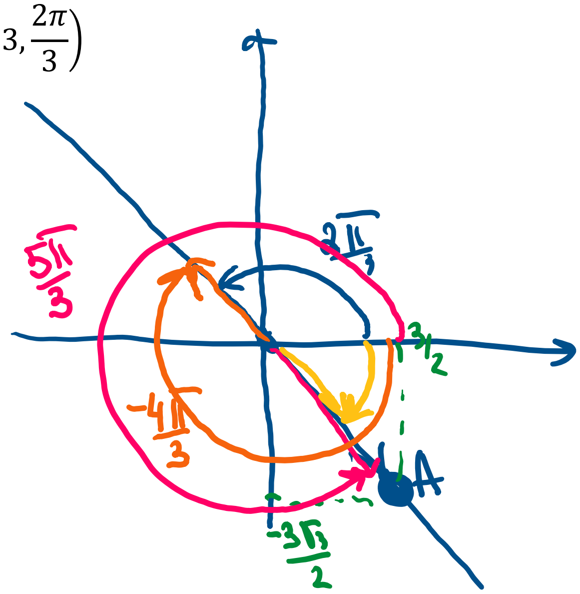
$$\begin{cases} x = (-2\sqrt{2})\cos\left(-\frac{\pi}{4}\right) = -2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = -2 \\ y = (-2\sqrt{2})\sin\left(-\frac{\pi}{4}\right) = -2\sqrt{2}\left(\frac{-\sqrt{2}}{2}\right) = 2 \end{cases} \quad (-2, 2)$$

$(+,-) \rightarrow \left(2\sqrt{2}, -\frac{5\pi}{4} + 2\pi k\right)$

$(-,+) \rightarrow \left(-2\sqrt{2}, \frac{7\pi}{4} + 2\pi k\right)$

Example#2

Find 3 other ways to represent the point $(r, \theta) = \left(-3, \frac{2\pi}{3}\right)$



$r = -3, \quad \theta = \frac{2\pi}{3} \qquad (-, +)$

$x = -3\cos\frac{2\pi}{3} = -3\left(-\frac{1}{2}\right) = \frac{3}{2}$

$y = -3\sin\frac{2\pi}{3} = -3\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$

$(+, +) \longrightarrow \left(3, \frac{5\pi}{3}\right)$

$(+, -) \longrightarrow \left(3, -\frac{\pi}{3}\right)$

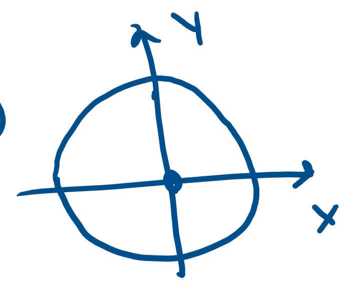
$(-, -) \longrightarrow \left(-3, -\frac{4\pi}{3}\right)$

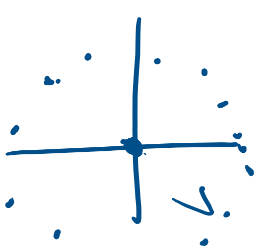
Example#3

Find a cartesian equivalent of the polar equation:

a. $r = 3 \Rightarrow r^2 = 9 \Rightarrow x^2 + y^2 = 9$

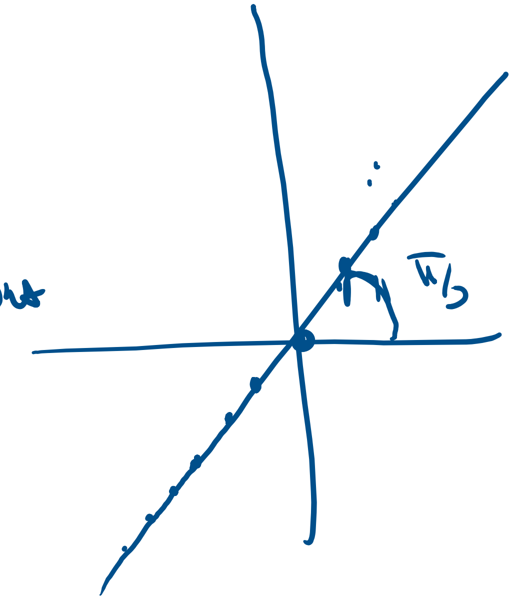
$r = -3 \Rightarrow r^2 = 9 \nearrow$





$\boxed{r = \text{Const} \Rightarrow \text{circle}}$

$b.\ \theta = \pi/3$ => r not given => means no restriction r anything



$$\frac{y}{x} = \tan\theta = \tan\frac{\pi}{3} = \sqrt{3} \Rightarrow$$

$$\boxed{y = \sqrt{3}x}$$

$c.\ r = 3sin\theta$

$$r = 3\sin\theta \quad | \cdot r$$

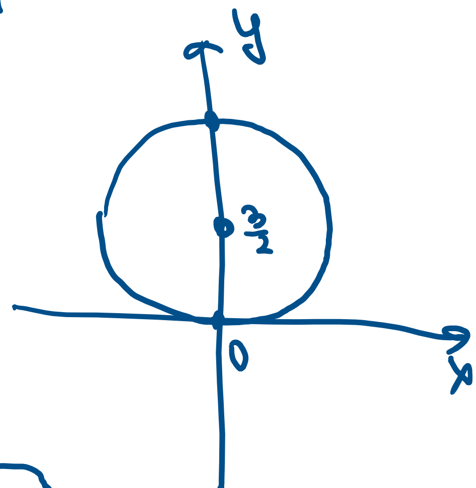
$$\underbrace{r^2} = 3\underbrace{r \cdot \sin\theta}$$

$$x^2 + y^2 = 3y$$

$$x^2 + \underbrace{y^2 - 3y} = 0$$

$$x^2 + \left(y - \frac{3}{2}\right)^2 - \frac{9}{4} = 0$$

$$\boxed{x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2}$$



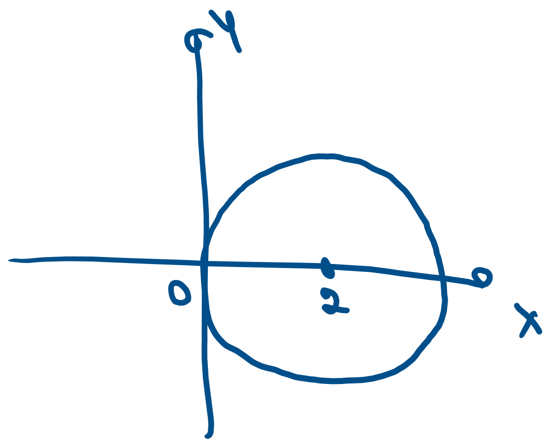
$d.\ r = 4cos\theta$

$$\underbrace{r^2} = 4\underbrace{r\cos\theta}$$

$$x^2 + y^2 = 4x$$

$$x^2 - 4x + y^2 = 0$$

$$(x-2)^2 + y^2 = 4$$

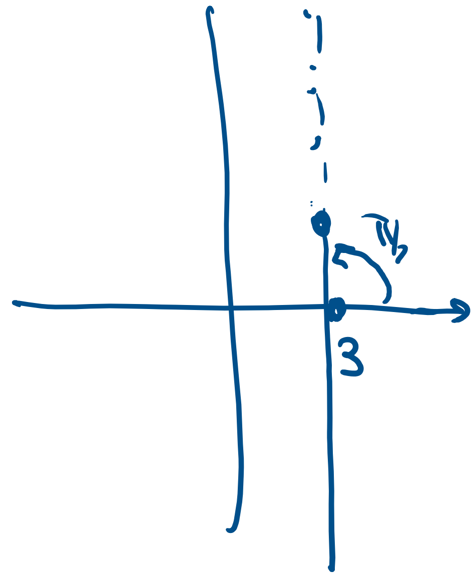


Example#4

Find a polar equivalent of the cartesian equation:

$\boldsymbol{x = 3}$

$r\cos\theta = 3 \Rightarrow \boxed{r = \dfrac{3}{\cos\theta}}$



$\boldsymbol{y = x}$

$r\sin\theta = r\cos\theta$

$r(\sin\theta - \cos\theta) = 0$

$\rightarrow r = 0 \Rightarrow (0,0)$

$\rightarrow \sin\theta = \cos\theta \Rightarrow \boxed{\theta = \dfrac{\pi}{4}}$ ✓

or $\boxed{\theta = \dfrac{5\pi}{4}}$ ✓

$\boldsymbol{y = 2x}$

$r\sin\theta = 2r\cos\theta$

$r(\sin\theta - 2\cos\theta) = 0 \Rightarrow \tan\theta = 2$

$\boxed{\theta = \arctan(2)}$

$\boldsymbol{x - y = 3}$

$r(\cos\theta - \sin\theta) = 3$

$\boxed{r = \dfrac{3}{\cos\theta - \sin\theta}}$

$x^2 + y^2 = 25$ $\Rightarrow$ $r^2 = 25$ $\rightarrow$ $r = 5$

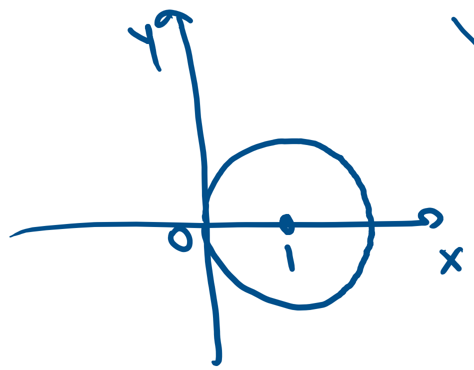
$\rightarrow$ $r = -5$

***Example*#5**

Plot the graph of $r^2 = 2r\cos\theta$ $\Rightarrow$ $x^2 + y^2 = 2x$

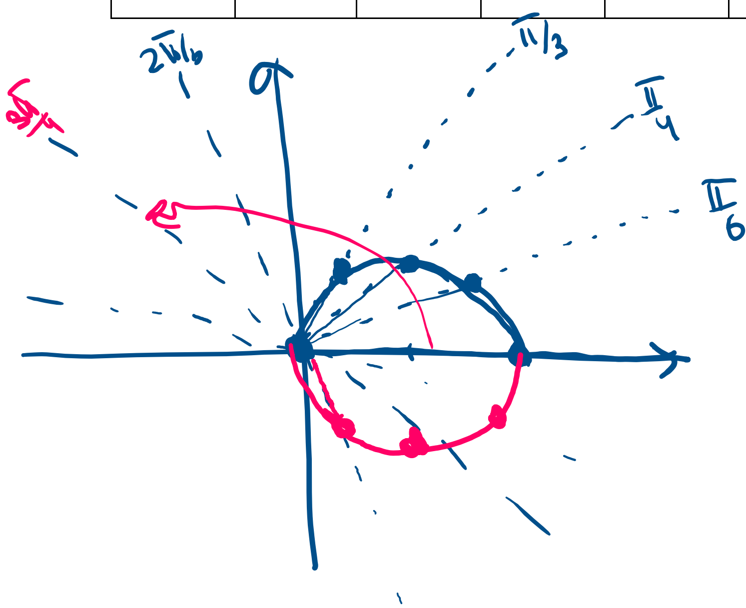
$x^2 - 2x + y^2 = 0$

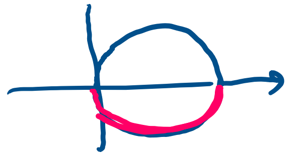
$(x-1)^2 + y^2 = 1$



$r = 2\cos\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2





$r = 2\cos\theta$

$\theta \to -\theta$ $\quad 2\cos(-\theta) = 2\cos\theta = r$

Test for Symmetry

Replace θ by $-\theta$	*symmetric with respect to $x-axis$*
Replace θ by $-\theta$ and r by $-r$ or θ by $\pi-\theta$	*symmetric with respect to $y-axis$*
Replace r by $-r$ or θ by $\pi+\theta$	*symmetric to the origin*

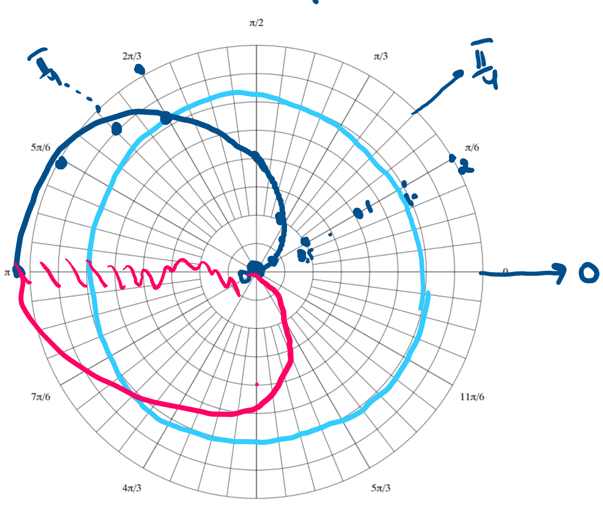
(The graph of polar equation $r = r(\theta)$ can be symmetric without satisfying the preceding tests)

Example#6

Plot the graph of $\boxed{r = 1 - \cos\theta}$

$\theta \to -\theta \Rightarrow 1 - \cos(-\theta) = \boxed{1 - \cos\theta = r}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	$1-\frac{\sqrt{3}}{2}$	$1-\frac{\sqrt{2}}{2}$	$1-\frac{1}{2}$	1	$1+\frac{1}{2}$	$1+\frac{\sqrt{2}}{2}$	$1+\frac{\sqrt{3}}{2}$	2
		≈ 0.1	≈ 0.3	0.5	1	1.5	≈ 1.7	≈ 1.9	



Snail - limacon

$r = a \pm b\cos\theta$

$r = a \pm b\sin\theta$

Example#7

Plot the graph of $\boxed{r = 2(1 - \sin\theta)}$ $= 2 - 2\sin\theta$

$\theta \to -\theta \Rightarrow$

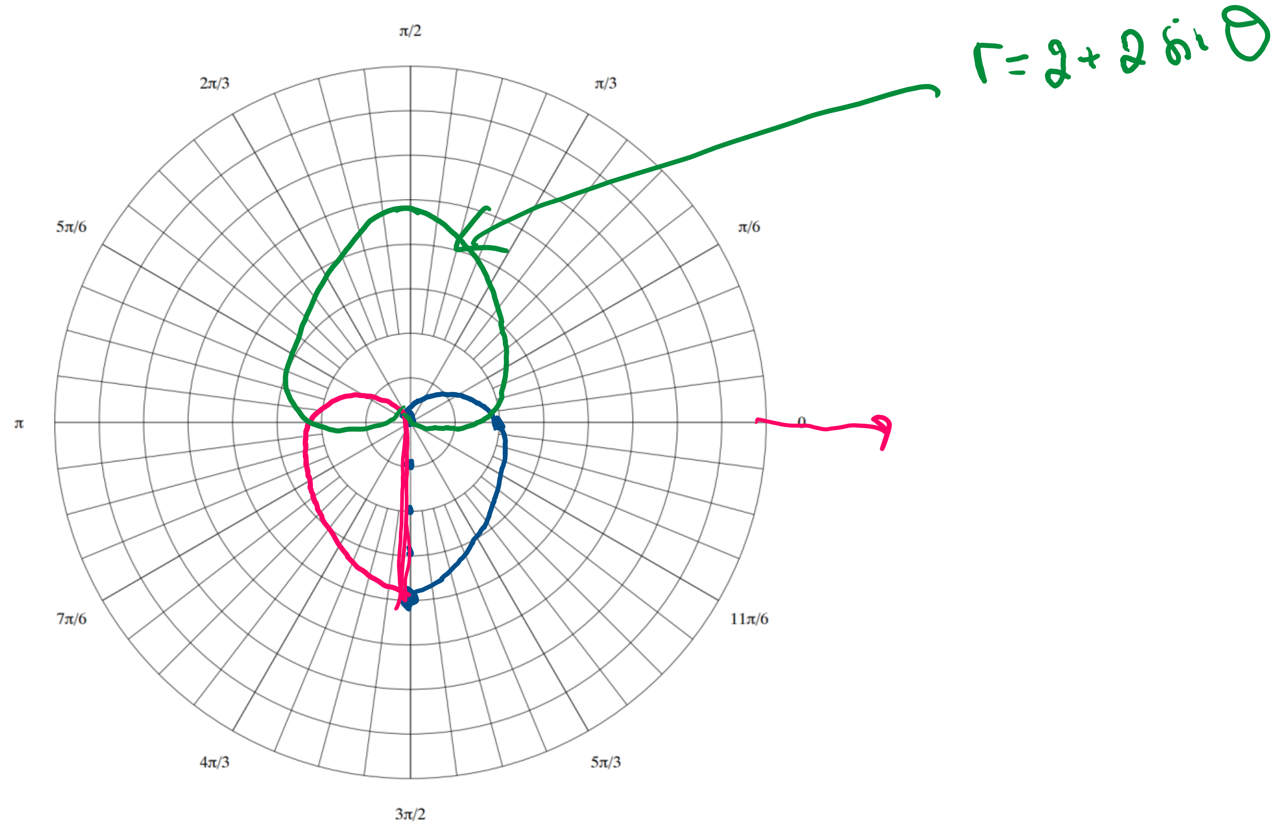
$2 - 2(\sin(-\theta)) = 2 + 2\sin\theta \neq r$

$\theta \to \pi - \theta$ ✓

symmetric about y

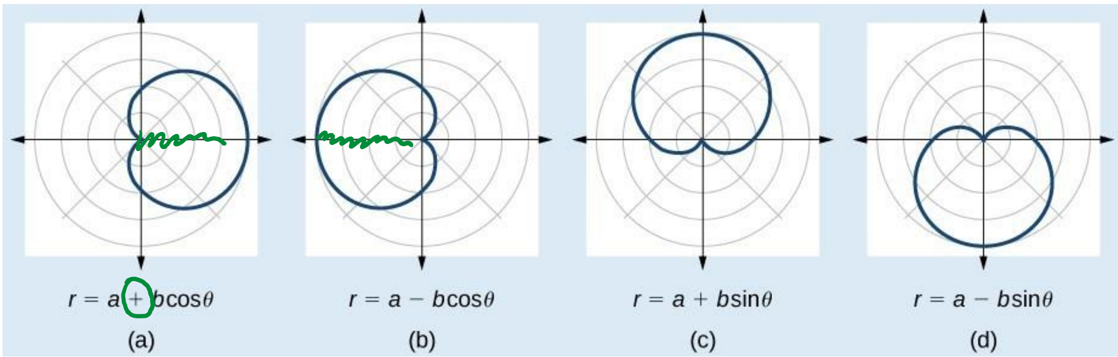
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	$2(1+1)$	$2(1+\frac{\sqrt{3}}{2})$	$2(1+\frac{\sqrt{2}}{2})$	$2(1+\frac{1}{2})$	$2(1+0)$	$2(1-\frac{1}{2})$	$2(1-\frac{\sqrt{2}}{2})$	$2(1-\frac{\sqrt{3}}{2})$	$2(1-1)$
	$\boxed{4}$				2				0

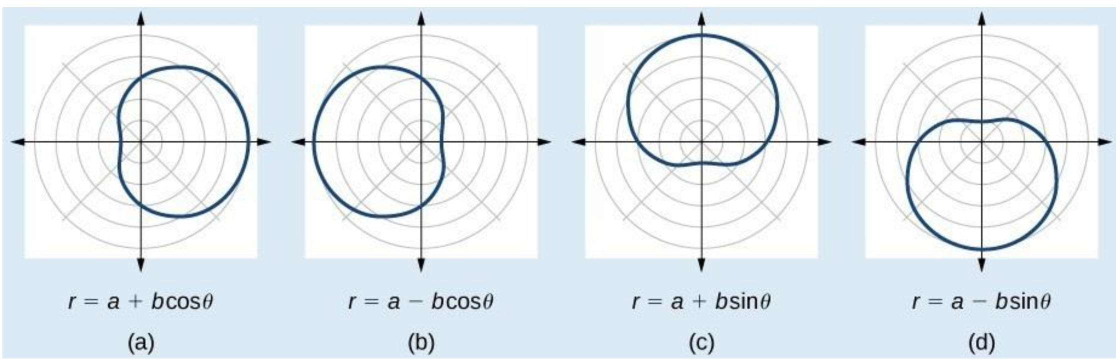


Formulas for One − Loop LimaÇons $\boxed{r = a \pm b\cos\theta}$ *or* $\boxed{r = a \pm b\cos\theta}$

1. $a = b$

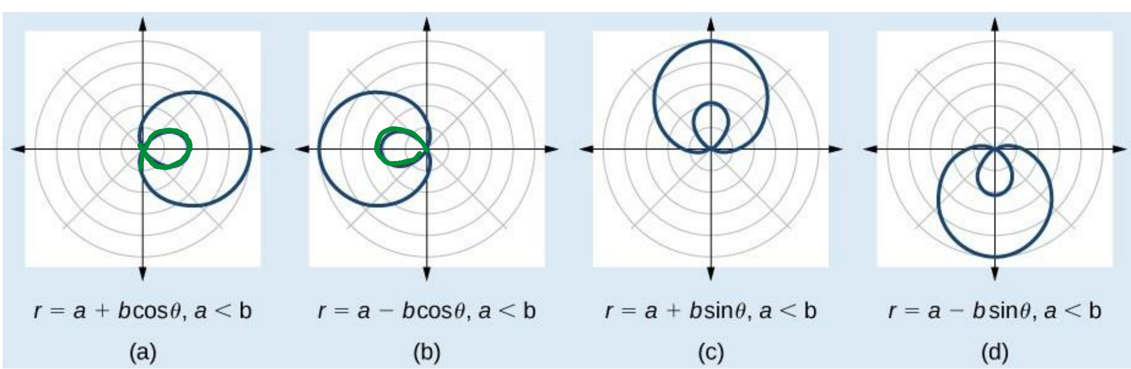


2. $1 < \dfrac{a}{b} < 2$



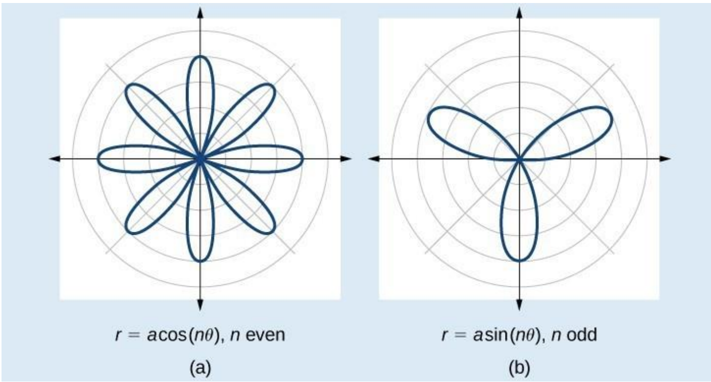
Formulas for Inner − Loop LimaÇons

3. $\dfrac{a}{b} < 1$



$\boldsymbol{Rose\ Curves : r = a\cos(n\theta), \quad r = a\sin(n\theta), \quad where\ a \neq 0,}$

$\boldsymbol{has\ (2n)\ petals\ if\ n\ is\ even\ and (n)\ petals\ if\ n\ is\ odd}$



$\boldsymbol{Formulas\ for\ Lemniscates}$

$r^2 = a^2\cos(2\theta)$ *is symmetric with respect to* everything: *pole*, $x-axis$, $y-axis$

$r^2 = a^2\sin(2\theta)$ *is symmetric with respect to* the *pole*