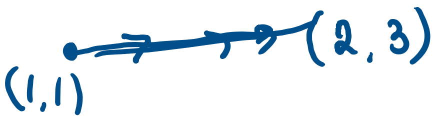
Calculus with Parametric Curves

Ex: Find a parametrization for the segment joining $(1,1)$ and $(2,3)$, $0 \le t \le 1$



$$\boxed{\begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}}$$

$x = x(t)$

$y = y(t)$

$t=0 \to (1,1)$ $\quad \begin{cases} 1 = x_0 + a \cdot 0 \\ 1 = y_0 + b \cdot 0 \end{cases} \Rightarrow \begin{matrix} x_0 = 1 \\ y_0 = 1 \end{matrix}$

$t=1 \to (2,3)$ $\quad \begin{cases} 2 = 1 + a \cdot 1 \\ 3 = 1 + b \cdot 1 \end{cases} \Rightarrow \begin{matrix} a = 1 \\ b = 2 \end{matrix}$

$$\begin{cases} x = 1 + t \\ y = 1 + 2t \end{cases} \quad 0 \le t \le 1$$

$t=0 \to (2,3)$ $\quad \begin{cases} 2 = x_0 + a \cdot 0 \\ 3 = y_0 + b \cdot 0 \end{cases} \quad \begin{matrix} x_0 = 2 \\ y_0 = 3 \end{matrix}$

$t=1 \to (1,1) \Rightarrow \begin{cases} 1 = 2 + a \cdot 1 \\ 1 = 3 + b \cdot 1 \end{cases} \quad \begin{matrix} a = -1 \\ b = -2 \end{matrix}$

$$\begin{cases} x = 2 - t \\ y = 3 - 2t \end{cases} \quad 0 \le t \le 1$$

- *Derivatives and Tangent lines* ✓
- *The Length of the Curve*
- *The Area under the Curve*
- *The Area of the Surface of Revolution*

Derivatives and Tangent lines

$$If\ C = \begin{cases} x = x(t) \\ y = y(t) \end{cases}\ then\ the\ slope\ \frac{dy}{dx}\ of\ the\ tangent\ line\ to\ C\ is\ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, provided\ \frac{dx}{dt} \neq 0\ , \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Arc Length

$$If\ C = \begin{cases} x = x(t) \\ y = y(t) \end{cases}\ a \le t \le b, \quad and\ C\ doesn't\ intersect\ itself, except\ possbly$$

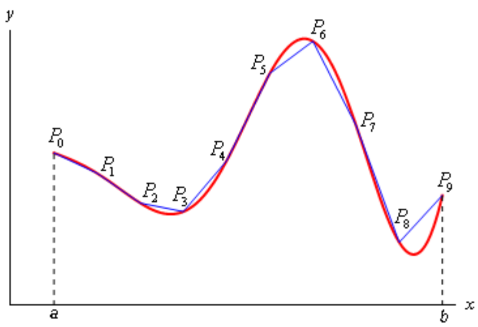
$for\ t = a\ and\ t = b$, *then the Length L of C is*

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$ ✓

$$\ell = \int \sqrt{1+(f'(x))^2}\, dx$$

$$(f'(x))^2 = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(y'(t))^2}{(x'(t))^2}$$

$$\sqrt{1+(f'(x))^2}\, dx = \sqrt{\frac{(x'(t))^2+(y'(t))^2}{(x'(t))^2}}\, dx = \frac{\sqrt{(x')^2+(y')^2}}{\cancel{x'(t)}}\, \cancel{x'(t)}\, dt$$



Let $\Delta t_k = t_k - t_{k-1}$

and let $P_k = (x(t_k), y(t_k))$ *and* $P_{k-1} = (x(t_{k-1}), y(t_{k-1}))$

be the points on C that correspond to t_k *and* t_{k-1}

If $d(P_k, P_{k-1})$ *is the length of of the line segment* $P_{k-1}P_k$

then the length of the broken line is $L_P = \sum_{k=1}^{n} d(P_k, P_{k-1})$

$$and\ L = \lim_{n\to\infty} L_P$$

$$d(P_k, P_{k-1}) = \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2} = \sqrt{(x'(w_k))^2 + (y'(v_k))b^2} \cdot \Delta t_k$$

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(w_k))^2 + (y'(v_k))b^2} \cdot \Delta t_k = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example#1

Find the length of a circle of radius R (Circumference => $2\pi R$)

$x^2 + y^2 = R^2$

$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \quad 0 \le t \le 2\pi$

$$\ell = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^{2\pi} \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt =$$

$$= \int_0^{2\pi} \sqrt{R^2(\underbrace{\sin^2 t + \cos^2 t}_{1})}\, dt = R\int_0^{2\pi} dt = R \cdot t\Big|_0^{2\pi} = \underline{2\pi R}$$

Example#2

Consider the parametric equations given by $\underline{x(t) = 2sint}, \underline{y(t) = 2\ cost}$ $\quad x^2 + y^2 = 4$

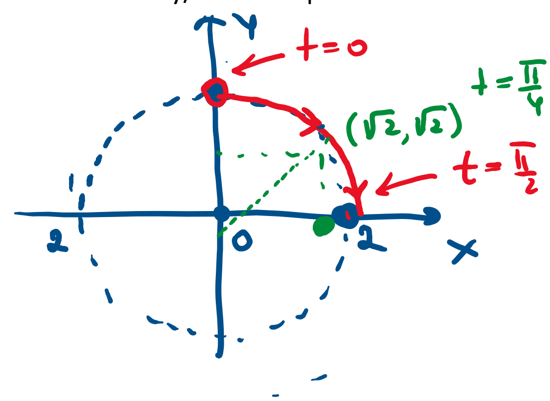
a. Find the length of the curve on the range $\underline{0 \le t \le \frac{\pi}{2}}$

$$\ell = \int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^{\pi/2} \sqrt{(2\cos t)^2 + (-2\sin t)^2}\, dt =$$

$$= \int_0^{\pi/2} \sqrt{4(\cos^2 t + \sin^2 t)} = 2 \cdot t\Big|_0^{\pi/2} = 2 \cdot \frac{\pi}{2} = \boxed{\pi}$$

b. Graph the portion of the curve from part (a) and find the slope of the curve dy/dx at the point $(\sqrt{2}, \sqrt{2})$

$$\begin{cases} x = 2\sin t \\ y = 2\cos t \end{cases} \Rightarrow x^2 + y^2 = 4$$



$t = 0 \Rightarrow (x, y) = (0, 2)$

$t = \frac{\pi}{2} \Rightarrow (x, y) = (2, 0)$

$0 \le t \le \frac{\pi}{2}$ $\quad x > 0$, $y > 0$

slope $\left(m = f'(x)\Big|_{x=x_0}\right) = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\Bigg|_{t_0=\frac{\pi}{4}} = \dfrac{-2\sin t}{2\cos t}\Bigg|_{t=\frac{\pi}{4}} = -\tan t\Big|_{t=\frac{\pi}{4}} = \boxed{-1}$

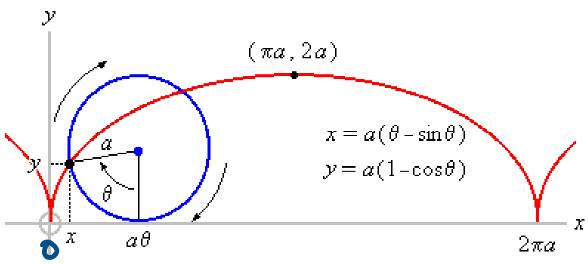
tangent line at $(\sqrt{2}, \sqrt{2})$

$y - y_0 = m(x - x_0)$

$y - \sqrt{2} = (-1)(x - \sqrt{2}) \Rightarrow \boxed{y = -x + 2\sqrt{2}}$

Example#3

Find the length of one arch of the cycloid $C = \begin{cases} x = t - sint \\ y = 1 - cost \end{cases}$



$a = 1, \quad 0 \le t \le 2\pi$

$$\ell = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt =$$

$$\left(\begin{aligned} x'(t) &= 1 - cost \\ y'(t) &= sint \\ (x')^2 + (y')^2 &= (1 - cost)^2 + sin^2 t = \\ &= 1 - 2cost + \underbrace{cos^2 t + sin^2 t}_{1} = \\ &= 2 - 2cost = 2(1 - cost) \end{aligned} \right)$$

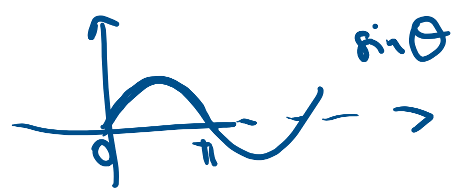
$$= \int_0^{2\pi} \sqrt{2(1 - cost)}\, dt =$$

$$sin^2 \theta = \frac{1 - cos 2\theta}{2} \qquad \theta = t/2, \; 2\theta = t$$

$$= \int_0^{2\pi} \sqrt{2\left(2 sin^2 \frac{t}{2}\right)}\, dt$$

$$= 2 \int_0^{2\pi} \left| sin \frac{t}{2} \right| dt$$

$$\left(\begin{aligned} 0 &\le t \le 2\pi \\ 0 &\le \frac{t}{2} \le \pi \\ sin \frac{t}{2} &\ge 0 \end{aligned} \right)$$



$$= 2 \int_0^{2\pi} sin \frac{t}{2}\, dt = 2\left(-2cos\frac{t}{2}\right)\Big|_0^{2\pi} = (-4)(-1 - 1) = \boxed{8}$$

Example#4 (final spring 2016)

$$C = \begin{cases} x = ln(1+t^2) \\ y = 2\arctan(t) \end{cases}$$

a. Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = y'(x) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+t^2}}{\frac{1}{1+t^2}\cdot 2t} = \boxed{\frac{1}{t}}$$

$$\frac{d^2y}{dx^2} = y''(x) = \frac{\frac{d}{dt}(y')}{\frac{dx}{dt}} = \frac{\left(\frac{1}{t}\right)'}{\frac{1}{1+t^2}\cdot 2t} = \frac{-\frac{1}{t^2}}{\frac{2t}{1+t^2}} = \frac{-(1+t^2)}{2\cdot t\cdot t^2} = \boxed{\frac{-(1+t^2)}{2t^3}}$$

b. Find the length of this curve for $0 \le t \le 1$

$$l = \int_0^1 \sqrt{\left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{2}{1+t^2}\right)^2}\,dt = \int_0^1 \sqrt{\frac{4t^2+4}{(1+t^2)^2}}\,dt =$$

$$= \int_0^1 2\sqrt{\frac{\cancel{1+t^2}}{(1+t^2)^{\cancel{2}}}}\,dt = \int_0^1 2\,\frac{1}{\sqrt{1+t^2}}\,dt$$

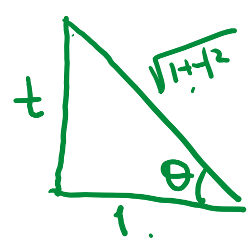
$$t = \tan\theta$$

$$1+t^2 = 1+\tan^2\theta = \sec^2\theta$$

$$dt = \sec^2\theta\,d\theta$$

$$\int \frac{2\cdot\sec^2\theta\,d\theta}{\sec\theta} = 2\int \sec\theta\,d\theta$$

$$= 2\ln|\sec\theta + \tan\theta| + C$$



$$\sec\theta = \sqrt{1+t^2}$$

$$= 2\ln\left|\sqrt{1+t^2} + t\right|\Big|_0^1 =$$

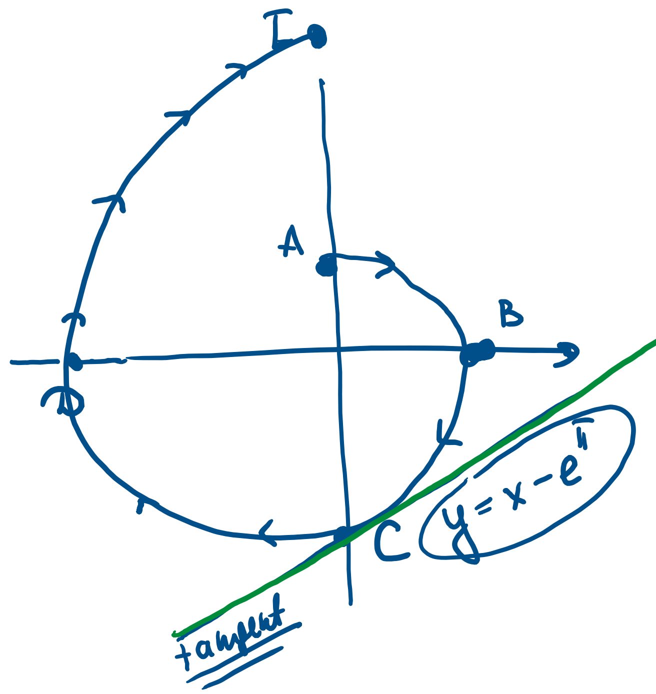
$$2\left(\ln|\sqrt{2}+1| - (\ln(1+0))\right) =$$

$$\boxed{2\ln(\sqrt{2}+1)}$$

***Example*#5**

Consider the planar parametric curve $\begin{cases} x = e^t \sin t \\ y = e^t \cos t \\ 0 \le t \le 2\pi \end{cases}$

a. Sketch the curve



$\underline{x^2 + y^2} = e^{2t}\sin^2 t + e^{2t}\cos^2 t = \underline{e^{2t}}$

$t = 0 \qquad (x, y) \to (0, 1) = A$

$t = \frac{\pi}{2} \qquad (x, y) \to (e^{\pi/2}, 0) = B$

$t = \pi \qquad (x, y) \to (0, -e^{\pi}) = C$

$t = \frac{3\pi}{2} \qquad (x, y) = (-\underline{e^{\frac{3\pi}{2}}}, 0) = D$

$t = 2\pi \qquad (x, y) \quad (0, e^{2\pi}) = E$

b. Find the length of the curve

$\begin{cases} x = e^t \cdot \sin t \\ y = e^t \cos t \end{cases}$

$x'(t) = e^t \sin t + e^t \cos t = \boxed{e^t(\sin t + \cos t)}$

$y'(t) = e^t \cos t - e^t \sin t = \boxed{e^t(\cos t - \sin t)}$

$(x')^2 + (y')^2 = e^{2t}(\sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + \cos^2 t - \cancel{2\sin t \cos t} + \sin^2 t)$

$= e^{2t}(1+1) = 2e^{2t}$

$l = \int_0^{2\pi} \sqrt{2e^{2t}}\, dt = \sqrt{2}\, e^t \Big|_0^{2\pi} = \sqrt{2}(e^{2\pi} - e^0) = \boxed{\sqrt{2}(e^{2\pi} - 1)}$

$c.$ *Find the tangent line to the curve at* $t = \pi$

$$m = \frac{y'(t)}{x'(t)} = \frac{\cancel{e^t}(\cos t - \sin t)}{\cancel{e^t}(\cos t + \sin t)}\Bigg|_{t=\pi} = \frac{-1-0}{-1+0} = \boxed{1}$$

$$t=\pi \rightarrow (x,y) = (0,-e^{\pi})$$

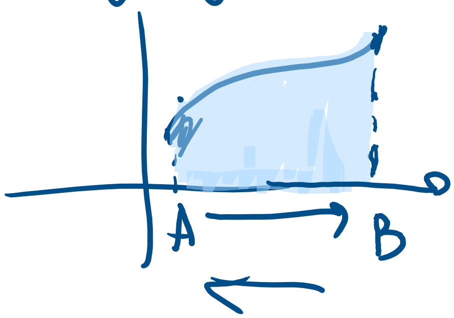
$$y - y_0 = m(x - x_0) \Rightarrow \quad y - (-e^{\pi}) = 1(x-0)$$

$$y + e^{\pi} = x$$

$$\boxed{y = x - e^{\pi}}$$

Area Under a Parametric curve

$y = y(x)$ $\quad$ $\text{Area} = \int_A^B y(x)\,dx$ $\quad$ $A \le x \le B$, $A < B$



$x = x(t)$

$dx = x'(t)\,dt$

$a \le t \le b$

$$\text{Area} = \left| \int_a^b y(t)x'(t)dt \right|$$

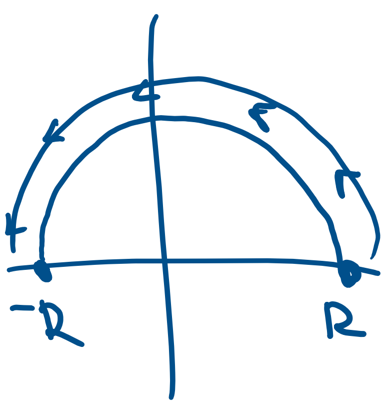
Example: *Find the area inside of the circle with radius R* $\quad (\pi R^2)$

$x = R\cos t$

$y = R\sin t$ $\quad$ $0 \le t \le 2\pi$

$$A = \left| \int_0^{2\pi} (R\sin t)\cdot(-R\sin t)\,dt \right| = \left| -R^2 \int_0^{2\pi} \sin^2 t\,dt \right| =$$

$$\left| -R^2 \int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt \right| =$$



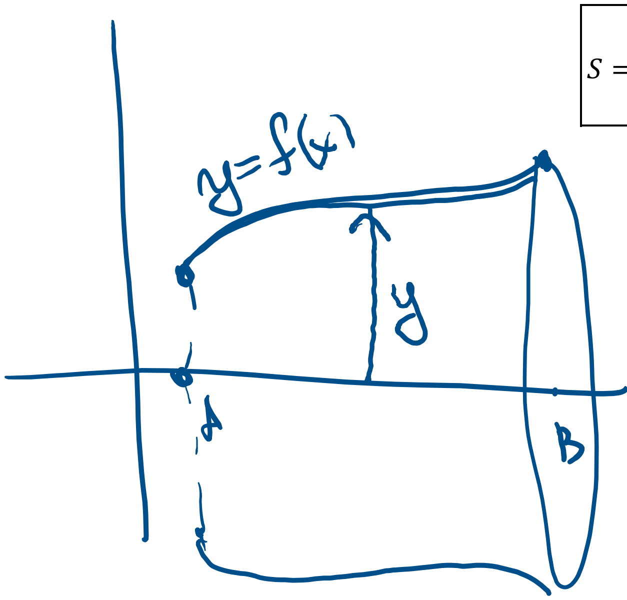
$-R \le x \le R$

$$\left| -\frac{R^2}{2}\left(t\Big|_0^{2\pi} - \frac{1}{2}\sin 2t \Big|_0^{2\pi} \right) \right| =$$

$$= \left| -\frac{R^2}{2}\left(2\pi - \frac{1}{2}(0-0) \right) \right| =$$

$$= \left| -\pi R^2 \right| = \boxed{\pi R^2}$$

Area of Surface of Revolution about $x-axis$

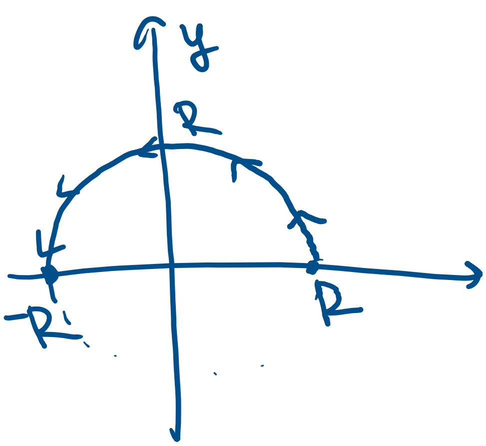


$$S = 2\pi\int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$S = 2\pi\int_A^B y(x)\sqrt{\boxed{1+f'(x)^2}}\,dx$$

Example#6

Verify that the surface area of a sphere of radius R is $4\pi R^2$



$y = R\sin t$

$x = R\cos t$

$0 \le t \le \pi$

$$2\pi\int_0^\pi \underbrace{R\sin t}_{y}\sqrt{R^2(\cos^2 t+\sin^2 t)}\,dt =$$

$$= 2\pi R^2\int_0^\pi \sin t = 2\pi R^2\underbrace{(-\cos t)\Big|_0^\pi}_{2} = 4\pi R^2$$

$\begin{cases} x = 1 - t^2 \\ y = 2t \end{cases}$, $0 \le t \le 1$ $\Rightarrow$ Find the area of the surface of revolution

Example#5

Find the s area of the surface generated by revolving $\begin{cases} x = 1 - t^2 \\ y = 2t \end{cases}$ *on the* $t-$ *interval* $[0,1]$

about $x-axis$

$$S = 2\pi \int_0^1 (2t) \sqrt{\left((1-t^2)'\right)^2 + \left((2t)'\right)^2}\, dt =$$

$$= 2\pi \int_0^1 2t \sqrt{(-2t)^2 + 2^2}\, dt =$$

$$= 4\pi \int_0^1 t \sqrt{4(t^2+1)}\, dt = 8\pi \int_0^1 t \sqrt{t^2+1}\, dt$$

$u = t^2 + 1$

$du = 2t\,dt$

$\frac{1}{2} du = t\,dt$

$\int t \sqrt{t^2+1}\, dt =$

$\frac{1}{2} \int \sqrt{u}\, du =$

$\frac{1}{2} \frac{u^{3/2}}{3/2}$

$$= 8\pi \cdot \underbrace{\frac{1}{2} \frac{(t^2+1)^{3/2}}{3/2}}\Bigg|_{t=0}^{1} = 4\pi \cdot \frac{2}{3}\left(\sqrt{8} - 1\right)$$

$$= \boxed{\frac{8\pi}{3}\left(2\sqrt{2} - 1\right)}$$