$Series(\ Part1)$

$Consider\ a\ sequense\ \{a_n\} = a_1, a_2, a_3, \dots a_n, \dots$

$From\ now\ on\ we\ will\ be\ interested\ in\ one\ question\ -$

$"\ What's\ happening\ with\ the\ sum\text{:}\ \underbrace{a_1 + a_2 + a_3 + \cdots + a_n + \cdots} = \sum_{n=1}^{\infty} a_n\ ?$ = finite ? or not

$We\ will\ call\ this\ sum\ \sum_{k=1}^{\infty} a_k - an\ infinite\ series.$

1) $The\ sum\ of\ the\ first\ n -\ terms\ it\ is\ called\ the\ nth$ <u>Partial Sum</u>:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

$\{S_n\}_{n=1}^{\infty}$:

$S_1, S_2, \dots S_n \dots$

$\{S_n\} \to ?$

$$\begin{cases} S_1 = a_1 & (n=1) \\ S_2 = a_1 + a_2 & (n=2) \\ S_3 = a_1 + a_2 + a_3 & (n=3) \\ \vdots & \\ S_n = a_1 + a_2 + a_3 + \dots a_n & \end{cases}$$

$If\ \lim_{n\to\infty} S_n = S\ (exists\ and\ finite)\ we\ \ call$ ***S* the SUM** $of\ the\ infinite\ series\ \ and$

$the\ series\ is\ called$ ***convergent***.

$Otherwise\ the\ series\ is\ called$ ***divergent***.

$$a_1 + a_2 = (\ln 2 - \ln 3) + (\ln 3 - \ln 4)$$

$A = 1 \qquad B = -1$

$$a_n = \frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1}$$

Example #1

$$\sum_{k=1}^{\infty} \frac{1}{n^2+n} \Rightarrow \qquad \{S_n\} = ? = S_1,\ S_2 \ldots\ S_n,$$

$$S_1 = a_1 = (n=1) = \frac{1}{1} - \frac{1}{1+1} = 1 - \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$n = 3$

$$S_4 = \ldots \qquad = 1 - \frac{1}{5} \quad (\text{check!})$$

$$S_n = 1 - \frac{1}{n} \quad \therefore \quad \lim_{n\to\infty}(S_n) = \lim_{n\to\infty}\left(1 - \underset{\downarrow\, 0}{\frac{1}{n}}\right) = \boxed{1} = S$$

$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges to $S = 1$ (by def. of Partial sum)

2. $\sum_{k=1}^{\infty} \ln(n+1) - \ln(n+2)$

$$a_n = \ln(n+1) - \ln(n+2)$$

$$S_n = ?$$

$$S_1 = a_1 = \ln(1+1) - \ln(1+2) = \ln 2 - \ln 3$$

$$S_2 = a_1 + a_2 = (\ln 2 - \cancel{\ln 3}) + (\cancel{\ln 3} - \ln 4) = \ln 2 - \ln 4$$

$S_3 = a_1 + a_2 + a_3 = (\ln 2 - \cancel{\ln 3}) + (\cancel{\ln 3} - \cancel{\ln 4}) + (\cancel{\ln 4} - \ln 5) = \ln 2 - \ln 5$

$\vdots$

$S_n = \ln 2 - \ln(n+2)$

$\lim_{n\to\infty} S_n = \lim (\ln 2 - \underbrace{\ln(n+2)}_{\infty}) = (\ln 2 - \infty) = -\infty$

Series diverges.

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3. $\sum_{k=1}^{\infty} \frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}$ $\qquad a_n = \frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}$

$S_n = ?$

$S_1 = \underbrace{\frac{1}{\ln(1+2)} - \frac{1}{\ln(1+1)}}_{a_1} = -\frac{1}{\ln 2} + \frac{1}{\ln 3}$

$S_2 = \underbrace{\left(\cancel{\frac{1}{\ln(1+2)}} - \frac{1}{\ln(1+1)}\right)}_{a_1} + \underbrace{\left(\frac{1}{\ln(2+2)} - \cancel{\frac{1}{\ln(2+1)}}\right)}_{a_2} = -\frac{1}{\ln 2} + \frac{1}{\ln 4}$

$S_3 = \underbrace{\left(\cancel{\frac{1}{\ln(1+2)}} - \frac{1}{\ln(1+1)}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{\ln(2+2)}} - \cancel{\frac{1}{\ln(2+1)}}\right)}_{a_2} + \underbrace{\left(\frac{1}{\ln(3+2)} - \cancel{\frac{1}{\ln(3+1)}}\right)}_{a_3} = -\frac{1}{\ln 2} + \frac{1}{\ln 5}$

$S_n = -\frac{1}{\ln 2} + \frac{1}{\ln(n+2)}$ for any n

$S_n \longrightarrow ?$

So, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(-\frac{1}{ln2}+\underbrace{\frac{1}{\ln(n+2)}}_{\downarrow 0}\right) = -\frac{1}{ln2},$

and the series converges to $\boxed{\frac{-1}{ln2}} = S$

4. $(extra - practice)$ $\quad \sum_{k=1}^{\infty} \arccos\left(\frac{\sqrt{3}}{n+1}\right) - \arccos\left(\frac{\sqrt{3}}{n+2}\right)$ $\quad a_n$

$S_n = ?$

$$S_1 = \underbrace{arccos\left(\frac{\sqrt{3}}{1+1}\right) - arccos\left(\frac{\sqrt{3}}{1+2}\right)}_{a_1} = \boxed{arccos\left(\frac{\sqrt{3}}{2}\right) - arccos\left(\frac{\sqrt{3}}{3}\right)}$$

$$S_2 = \left(\underbrace{arccos\left(\frac{\sqrt{3}}{1+1}\right) - arccos\left(\frac{\sqrt{3}}{1+2}\right)}_{a_1}\right) + \left(\underbrace{arccos\left(\frac{\sqrt{3}}{2+1}\right) - arccos\left(\frac{\sqrt{3}}{2+2}\right)}_{a_2}\right) =$$

$$= \boxed{arccos\left(\frac{\sqrt{3}}{2}\right) - arccos\left(\frac{\sqrt{3}}{4}\right)}$$

$$S_3 = \left(\underbrace{arccos\left(\frac{\sqrt{3}}{1+1}\right) - \cancel{arccos\left(\frac{\sqrt{3}}{1+2}\right)}}_{a_1}\right) + \left(\underbrace{\cancel{arccos\left(\frac{\sqrt{3}}{2+1}\right)} - \cancel{arccos\left(\frac{\sqrt{3}}{2+2}\right)}}_{a_2}\right) +$$

$$+ \left(\underbrace{\cancel{arccos\left(\frac{\sqrt{3}}{3+1}\right)} - arccos\left(\frac{\sqrt{3}}{3+2}\right)}_{a_3}\right) = \boxed{arccos\left(\frac{\sqrt{3}}{2}\right) - arccos\left(\frac{\sqrt{3}}{5}\right)}$$

It looks like we have $S_n = \boxed{arccos\left(\frac{\sqrt{3}}{2}\right) - arccos\left(\frac{\sqrt{3}}{n+2}\right)}$

$$S_n$$

$$So,\ \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(arccos\left(\frac{\sqrt{3}}{2}\right) - arccos\left(\frac{\sqrt{3}}{n+2}\right)\right) = arccos\left(\frac{\sqrt{3}}{2}\right) - \arccos(0) = \frac{\pi}{6} - \frac{\pi}{2} = -\frac{\pi}{3} = S$$

and the series converges to $-\frac{\pi}{3}$ $\qquad S_n \to S$

Geometric Series of the form
$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots = \begin{cases} \text{converges to } \dfrac{1}{1-r}, & \text{if } \lvert r\rvert < 1 \\ \text{diverges}, & \text{if } \lvert r\rvert \geq 1 \end{cases}$$

Example#2

Check if the following series converges or diverges. If it converges, find its sum.

1. $$\sum_{k=1}^{\infty} 2^{k-1}\cdot 3^{2-k} = \sum_{k=1} 2^k\cdot 2^{-1}\cdot 3^2\cdot 3^{-k} = \sum_{k=1} 2^k\cdot\left(\frac{1}{3}\right)^k\cdot\frac{1}{2}\cdot 9 =$$

$$= \sum_{k=1}\left(\frac{2}{3}\right)^k\cdot\frac{9}{2} = \frac{9}{2}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k, \ |r| = \left|\frac{2}{3}\right| < 1 \Rightarrow \text{conv. series}$$

$$= \frac{9}{2}\left(\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots\right) = \frac{9}{2}\cdot\frac{2}{3}\underbrace{\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots\right)}_{r = 2/3} = \frac{9}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\cancel{3} = \boxed{9}$$

$$\frac{1}{1-r} = \frac{1}{1-2/3} = \frac{1}{\frac{1}{3}} = 3$$

$S = 9$

$$\boxed{\sum_{n=1}(a_n) \text{ conv as geom. with } r = \tfrac{2}{3} < 1, \text{ and } S = 9}$$

2. $\sum_{n=1}^{\infty} 3^{2n-1} \cdot (-5)^{2-n} = \sum_{n=1} 3^{2n} \cdot 3^{-1} \cdot (-5)^2 \cdot (-5)^{-n} =$

$\sum_{n=1} \underline{(9)^n} \cdot \frac{1}{3} \cdot 25 \cdot \underline{\left(-\frac{1}{5}\right)^n} =$

$= \frac{25}{3} \sum_{n=1} \left(-\frac{9}{5}\right)^n, \ |r| = \left|-\frac{9}{5}\right| > 1$

Series <u>diverges</u> as <u>geom. ser. with $r = \left(-\frac{9}{5}\right)$</u>

3. $\sum_{n=0}^{\infty} \frac{5^n + 3^n}{4^n}$

$$\sum_{n=0}^{\infty} \left(\frac{5^n}{4^n} + \frac{3^n}{4^n}\right) =$$

$$\sum_{n=0}^{\infty} \left(\left(\frac{5}{4}\right)^n + \left(\frac{3}{4}\right)^n\right) = \underbrace{\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n}_{r=\frac{5}{4}>1 \text{ div}} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n}_{r=\frac{3}{4}<1 \text{ con.}} = \underbrace{\text{div series}}_{\text{the sum is infinite}} + \underbrace{\text{conv series}}_{\text{there is a finite sum}} = \underline{\underline{\text{div}}}$$

4. $\displaystyle\sum_{n=4}^{\infty} \frac{7 \cdot 2^n - 3^n}{5^n}$

$r = \frac{2}{5} < 1 \to$ conv $\to S = ?$

$r = \frac{3}{5} < 1 \to$ conv $\to S = ?$

$$\sum_{n=4}^{\infty} \frac{7 \cdot 2^n}{5^n} - \frac{3^n}{5^n} = 7\sum_{n=4}^{\infty}\left(\frac{2}{5}\right)^n - \sum_{n=4}^{\infty}\left(\frac{3}{5}\right)^n = S + S$$

$$7 \cdot \left(\left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \left(\frac{2}{5}\right)^6 + \cdots\right) - \left(\left(\frac{3}{5}\right)^4 + \left(\frac{3}{5}\right)^5 + \left(\frac{3}{5}\right)^6 + \cdots\right)$$

factoring factoring

$$7 \cdot \left(\frac{2}{5}\right)^4 \cdot \underbrace{\left(1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \cdots\right)}_{r = \frac{2}{5},\ \frac{1}{1-r} = \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{3}{5}} = \frac{5}{3}} - \left(\frac{3}{5}\right)^4 \cdot \underbrace{\left(1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right)}_{r = \frac{3}{5},\ \frac{1}{1-r} = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}} =$$

$$= 7 \cdot \left(\frac{2}{5}\right)^4 \cdot \frac{5}{3} - \left(\frac{3}{5}\right)^4 \cdot \frac{5}{2} =$$

$$\frac{7\cdot 16\cdot 5}{5^4\cdot 3}-\frac{3^4\cdot 5}{5^4\cdot 2}=\frac{112}{5^3\cdot 3}-\frac{81}{5^3\cdot 2}=\frac{112\cdot 2-81\cdot 3}{5^3\cdot 6}=\frac{224-243}{5^3\cdot 6}=\boxed{\frac{-19}{750}}$$

Series conv to

5. $\displaystyle\sum_{n=1}^{\infty}\left(e^n+e^{-n}\right)$

$$\sum_{n=1}^{\infty}\left(e^n+\left(\frac{1}{e}\right)^n\right)=\underbrace{\sum_{n=1}^{\infty}e^n}_{\substack{r=e>1\\ \text{div}}}+\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n}_{\substack{r=\frac{1}{e}<1\\ \text{conv}}}=\text{div series}+\text{conv series}\Rightarrow \text{divergees}$$

6. $\displaystyle\frac{1}{e^2}+\frac{2\pi}{e^5}+\frac{4\pi^2}{e^8}+\frac{8\pi^3}{e^{11}}+\cdots\;=$

$r=\frac{2\pi}{e^3}<1\rightarrow$ conv

$$\frac{1}{e^2}\cdot\left(1+\frac{2\pi}{e^3}+\frac{4\pi^2}{e^6}+\frac{8\pi^3}{e^9}+\cdots\right)=\frac{1}{e^2}\cdot\left(\frac{1}{1-\frac{2\pi}{e^3}}\right)=\frac{1}{e^2}\cdot\frac{e^3}{e^3-2\pi}=\boxed{\frac{e}{e^3-2\pi}}$$

7. *Change* $0.\overline{13}$ *into a fraction*

$0.\overline{13}=0.1313131313=$

$$0,\overline{13} = 0.13\,13\,13\,13$$

$$\left.\begin{aligned} & 0.13 \\ + \; & 0.0013 \\ + \; & 0.000013 \\ + \; & 0.00000013 \end{aligned}\right\} \quad \frac{13}{100}+\frac{13}{100^2}+\frac{13}{100^3}+\dots =$$

$$r=\frac{1}{100}<1$$

$$13\cdot\frac{1}{100}\underbrace{\left(1+\frac{1}{100}+\frac{1}{100^2}+\dots\right)}_{r=\frac{1}{100}} = \heartsuit$$

$$\frac{1}{1-r}=\frac{1}{1-\frac{1}{100}}=\frac{100}{99}$$

$$\heartsuit = \frac{13}{100}\cdot\frac{100}{99}=\boxed{\frac{13}{99}}$$

n-th term test / divergency test

$$If \sum_{k=1}^{\infty} a_k \text{ is convergent } \rightarrow \lim_{k\to\infty} a_k = 0$$

$$If \lim_{k\to\infty} a_k \neq 0 \ (or \ \lim_{k\to\infty} a_k = DNE) \rightarrow \sum_{k=1}^{\infty} a_k \text{ is divergent}$$

$$\boldsymbol{If \lim_{k\to\infty} a_k = 0 \rightarrow \ then \ldots WE\ KNOW\ NOTHING!}$$

Remember, 1) $\sum \underbrace{\ln(n+1) - \ln(n+2)}$

$$a_n = \ln\left(\frac{n+1}{n+2}\right) \to \ln(1) = \underline{\underline{0}}$$

2) $\sum \frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}$

$$a_n = \underset{\downarrow \atop 0}{\frac{1}{\ln(n+2)}} - \underset{\downarrow \atop 0}{\frac{1}{\ln(n+1)}} \to 0$$

In this case the series $\sum_{k=1}^{\infty} a_k$ *has* a chance *to be convergent*

$a_n = \frac{1}{n} \to 0$, and still $\underline{\sum \frac{1}{n} \text{ div}}$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{n} \quad \textit{DIVERGES!} \qquad (\textit{Trust me and we will prove it later})}$$

$$\sum_{k=1}^{\infty} \frac{3n^2}{n(n+3)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^{\cancel{2}}}{\cancel{n}(n+3)} = \lim_{n\to\infty} \frac{3n}{n+3} = \lim_{n\to\infty} \frac{3}{1+\underbrace{\frac{3}{n}}_{\to 0}} = 3$$

$$a_n \to 3 \neq 0$$

$\sum \frac{3n^2}{n(n+3)}$ div by n-th term test.

$$\underbrace{\sum_{k=1}^{\infty} \overset{a_n}{\frac{lnn}{ln(lnn)}}}_{}$$

diverges by n-th term test.

$$a_n = \frac{\ln n}{\ln(\ln n)}$$

$$\lim_{n\to\infty} \frac{lnn}{\ln(lnn)} = \frac{\infty}{\infty} = L'H_{rule}$$

$(\ln x)'$

$$= \lim_{x\to\infty} \frac{lnx}{\ln(lnx)} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{lnx}\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{\cancel{\frac{1}{x}}}{\frac{1}{lnx}\cdot\cancel{\frac{1}{x}}} = \lim_{x\to\infty} \frac{1}{\frac{1}{lnx}} =$$

$$\lim_{x\to\infty} lnx = \boxed{\infty}$$

$$a_n \to \infty \neq 0$$

$$\sum_{k=1}^{\infty} n-\sqrt{n^2-n} \qquad a_n$$

$$\lim_{n\to\infty} a_n = ?$$

$$\lim_{n\to\infty}\left(n-\sqrt{n^2-n}\right) = \lim_{n\to\infty}\frac{\left(n-\sqrt{n^2-n}\right)\left(n+\sqrt{n^2-n}\right)}{n+\sqrt{n^2-n}} =$$

$$\lim_{n\to\infty}\frac{(\cancel{n^2}-\cancel{n^2}+n)}{n+\sqrt{n^2-n}} = \lim_{n\to\infty}\frac{n}{n+\sqrt{n^2-n}} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty}\frac{n}{n+n\sqrt{1-\frac{1}{n}}}$$

$$\lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}\left(1+\sqrt{1-\frac{1}{n}}\right)} = \lim_{n\to\infty}\frac{1}{\left(1+\sqrt{1-\underbrace{\frac{1}{n}}_{0}}\right)} = \frac{1}{2} \neq 0$$

$$\frac{1}{1+\sqrt{1+0}}$$

$$a_n \to \frac{1}{2} \neq 0$$

So , the series $\sum_{k=1}^{\infty} n-\sqrt{n^2-n}$ $\boxed{\textit{diverges by } n-\textit{th term test}}$

the end!