Series (Part5) – Alternating Series

DEF: *Alternating Series is a series in which the terms are alternately positive and negative*:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n + \cdots$$

or

$a_n > 0$

$$\sum_{n=1}^{\infty} (-1)^{n} a_n = -a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^{n} a_n + \cdots$$

Alternating Series Test – Leibniz's Test:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges if } \begin{cases} 1.\ \text{The } a_n \text{ are all positive} \\ 2.\ a_n \geq a_{n+1} \text{ for every } n \\ 3.\ \lim_{n\to\infty} a_n = 0 \end{cases}$$

to check $a_n \searrow$

Reminder:

DEF: *A series* $\sum a_n$ ***converges absolutely*** *if* $\sum |a_n|$ *converges*

DEF: *A series* $\sum a_n$ ***converges conditionally*** *if* $\sum a_n$ ***converges*** $\sum |a_n|$ *diverges*

1. $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}}$ $= +1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$

① Check abs. convergency: $\sum \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ –

$p=1$, "harmonic" diverges

series does not conv. abs

② $\sum \frac{(-1)^{n+1}}{n}$ should be check with AST

1 – 2 – 3

$$\left\{\begin{array}{ll} 1) & \frac{1}{n} > 0 \\ 2) & \frac{1}{n} \downarrow \qquad \frac{1}{n} = \frac{1}{\text{incr}} = \text{decr} \\ 3) & \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 \end{array}\right\} \text{ AST cond } \underline{\text{satisfied}}$$

$$\Rightarrow \sum \frac{(-1)^{n+1}}{n} \text{ conv. conditionally}$$

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1) Abs. conv
2) Cond conv
3) Diverg.

$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ → try $\sum_{n=1}^{\infty}|(-1)^{n+1}a_n|$ →

1. If $\sum_{n=1}^{\infty}|(-1)^{n+1}a_n|$ *converges* than $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ ***converges absolutely***

2. If $\sum_{n=1}^{\infty}|(-1)^{n+1}a_n|$ *diverges*

↓

If the reason is $\lim (a_n) \neq 0$ => $\lim (-1)^n(a_n) \neq 0$ => ***divergent***

If the reason is Root or Ratio test => ***divergent***

↓

Try Leibniz's Test → If conditions are satisfied $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ ***converges conditionally***

↓

$\boldsymbol{Examples}$: *Check if the following series are* $\boldsymbol{absolutely}$ *convergent,* $\boldsymbol{conditionally}$ *convegent, or divergent:*

1. $\sum_{n=2}^{\infty} (-1)^n n^2 e^{-n}$

1) For abs conv. $\sum_{n=2} n^2 e^{-n} = \sum_{n=2} \frac{n^2}{e^n}$

Root: $\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{e^n}} = \lim_{n\to\infty} \frac{(\overset{\to 1}{\sqrt[n]{n}})^2}{e} = \frac{1}{e} < 1$, $\rho < 1 \Rightarrow$ conv!

$$\boxed{\sum_{n=2}^{\infty} n^2 e^{-n} \text{ converges by Root Test} \Rightarrow \sum_{n=2}^{\infty} (-1)^n n^2 e^{-n} \textbf{ converges absolutely}}$$

2. $\sum_{n=2}^{\infty} \left(\frac{(-4)^{n+1}}{3^n}\right)$

1) abs $\sum \left|\frac{(-1)^{n+1} 4^{n+1}}{3^n}\right| = \sum \frac{4^{n+1}}{3^n} = 4\sum \left(\frac{4}{3}\right)^n$

$a_n = 4\left(\underbrace{\frac{4}{3}}_{\frac{4}{3} > 1}\right)^n \not\to 0,\ n \to \infty$

$\lim_{n\to\infty} a_n = \infty \neq 0 \Rightarrow$ Div. by n-th term test

$\lim\left(\frac{(-4)^{n+1}}{3^n}\right) = (\pm\infty) = DNE \neq 0 \Rightarrow$ Div. by Divergency Test

$$\sum_{n=2}^{\infty} \frac{(4)^{n+1}}{3^n} \textit{ diverges by } n-\textit{term test} \Rightarrow \sum_{n=2}^{\infty} \frac{(-4)^{n+1}}{3^n} \textbf{ diverges by } \mathbf{n-th\ term\ test}$$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$

1) $\displaystyle\sum_{n=2} \frac{\ln n}{n}$, DCT: $\frac{\ln n}{n} > \frac{1}{n}$

no abs. conv

$\sum \frac{\ln n}{n}$ diverges by DCT $\sum \frac{1}{n}$, $p=1$ dive

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2) AST. 1-2-3

✓ 1) $a_n = \frac{\ln n}{n} > 0$

✓ 2) $a_n = \frac{\ln n}{n} \searrow$ $= \left(f' = \left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \right.$

$= \frac{1-\ln x}{x^2} < 0$

✓ 3) $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty} = \lim \frac{(\ln x)'}{(x)'} = \lim \frac{\frac{1}{x}}{1} = 0$

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n} \textit{ converges conditionally by AST}$$

3. $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(n)\left(\frac{3}{4}\right)^{n}$

1) $\sum_{n=1}^{\infty} \ln n \cdot \left(\frac{3}{4}\right)^{n}$

Ratio: $\lim_{n\to\infty} \frac{\ln(n+1)\cdot\left(\frac{3}{4}\right)^{\cancel{n}+1}}{\ln(n)\cdot\left(\frac{3}{4}\right)^{\cancel{n}}} = \lim_{n\to\infty} \overbrace{\frac{\ln(n+1)}{\ln n}}^{1}\cdot\left(\frac{3}{4}\right) = \frac{3}{4}$

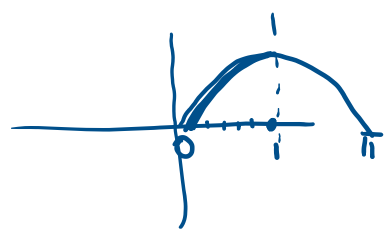
$\overset{\frac{\infty}{\infty}}{=} \lim \frac{(\ln(x+1))'}{(\ln x)'} = \lim \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim \frac{x}{x+1} = \lim \frac{1}{1+\frac{1}{x}} = 1$

$\rho = \frac{3}{4} < 1 \Rightarrow$ abs conv!

$\sum_{n=1}^{\infty}\ln(n)\left(\frac{3}{4}\right)^{n}$ *converges by Ratio Test* $=>$ $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(n)\left(\frac{3}{4}\right)^{n}$ ***converges absolutely***

4. $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$

1) $\sum_{n=1} \sin\left(\frac{1}{n}\right)$



$\sin\frac{1}{n} \sim \frac{1}{n}$

LCT: $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{\frac{1}{n}=t\to 0} \frac{\sin t}{t} = 1$, $0 < 1 < \infty$

$\sum \sin\frac{1}{n}$ div. by LCT with $\sum\frac{1}{n}$ (harmonic, $p=1$) div

no abs. conv.

2) AST   1 - 2 - 3

✓ 1) $\sin\frac{1}{n} > 0$

✓ 2) $\sin\frac{1}{n} \searrow$

✓ 3) $\lim_{n\to\infty} \sin\frac{1}{n} = \sin(0) = 0$

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{1}{n}\right) \quad \textit{converges conditionally by AST}}$$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{1+e^{\frac{1}{n}}}$

1) $\displaystyle\sum_{n=2} \left(\frac{1}{1+e^{1/n}}\right)$ $a_n = \dfrac{1}{1+e^{1/n}} \longrightarrow \dfrac{1}{1+1} = \dfrac{1}{2} \neq 0$

$\Rightarrow \displaystyle\lim_{n\to\infty} \underline{\underline{(-1)^n}} \cdot \frac{1}{1+e^{\frac{1}{n}}} = \pm\frac{1}{2} = DNE \neq 0$

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{1+e^{\frac{1}{n}}} \; \textit{diverges by n-term test} \Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{1+e^{\frac{1}{n}}} \; \textbf{diverges by n-th term test}}$$

6. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \cos(n\pi)}{\sqrt{n^3}}$

1) $\displaystyle\sum \frac{|\cos \pi n|}{\sqrt{n^3}}$ $\underline{\underline{DCT}}$ $\dfrac{|\cos \pi n|}{\sqrt{n^3}} \le \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$

$\displaystyle\sum_{n=2} \frac{|\cos \pi n|}{\sqrt{n^3}}$ conv

by DCT with $\displaystyle\sum \frac{1}{n^{3/2}}$, $p = \frac{3}{2} > 1$, conv

$\Rightarrow$ abs. conv

$$\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{\sqrt{n^3}} \text{ converges by DCT} \Rightarrow \sum_{n=2}^{\infty}\frac{(-1)^n\cos(n\pi)}{\sqrt{n^3}} \text{ converges absolutely}$$

6. $\sum_{n=2}^{\infty}\frac{(-1)^n n}{\sqrt{n^3+3}}$

1) $\sum \frac{n}{\sqrt{n^3+3}}$ $\quad \frac{n}{\sqrt{n^3+3}} \sim \frac{n}{\sqrt{n^3}} = \frac{n}{n^{3/2}} = \frac{1}{\sqrt{n}}$

LCT: $\lim \frac{\frac{n}{\sqrt{n^3+3}}}{\frac{1}{\sqrt{n}}} = \lim \frac{n\sqrt{n}}{\sqrt{n^3+3}} = \lim \sqrt{\frac{n^3}{n^3+3}} = 1$

$0 < 1 < \infty \Rightarrow$

$\sum \frac{n}{\sqrt{n^3+3}}$ div by LCT with $\sum \frac{1}{\sqrt{n}}$, $p = \frac{1}{2} < 1$ divergent

no abs. convergency

AST: 1 - 2 - 3

1) $\frac{n}{\sqrt{n^2+3}} > 0$ ✓

2) $a_n = \frac{n}{(n^3+3)^{1/2}} =$ (increases faster than n) ↘ ✓

3) $\lim \frac{n}{\sqrt{n^3+3}} = \lim \sqrt{\frac{n^2}{n^3+3}} = \lim \sqrt{\frac{1/n}{1+\frac{3}{n^3}}} = 0$ ✓

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+3}} \text{ converges conditionally by AST}}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n \cos^2 n}{n^2}$$

1) $\displaystyle\sum_{n=2} \frac{(\cos n)^2}{n^2}$ $\qquad \underbrace{\frac{(\cos n)^2}{n^2} \leq \frac{1}{n^2}}_{DCT}$

$\displaystyle\sum \frac{(\cos n)^2}{n^2}$ conv by DCT with $\displaystyle\sum \frac{1}{n^2}$, $p = 2 > 1$ conv

we have abs. conv

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n \cos^2 n}{n^2} \text{ converges by DCT} \Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n \cos^2 n}{n^2} \textbf{ converges absolutely}}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$$

1) $\displaystyle\sum \frac{\sqrt{n}}{n+1}$ $\qquad \displaystyle\frac{\sqrt{n}}{n+1} \sim \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

DCT: $\displaystyle\frac{\sqrt{n}}{n+1} > \frac{\sqrt{n}}{n+n} = \frac{\sqrt{n}}{2n} = \frac{1}{2} \cdot \frac{1}{\sqrt{n}}$

$$\underbrace{\sum \frac{\sqrt{n}}{n+1} \text{ diverges by DCT wit } \frac{1}{2}\sum \frac{1}{\sqrt{n}},\ p = \tfrac{1}{2} < 1 \text{ div}}_{\text{no abs. conv}}$$

2) AST

1) $\frac{\sqrt{n}}{n+1} > 0$ ✓

2) $\frac{\sqrt{n}}{n+1} \searrow$ ✓

3) $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim \frac{\frac{\sqrt{n}}{n}}{\frac{n}{n}+\frac{1}{n}} = \lim \frac{\left(\frac{1}{\sqrt{n}}\right)^{\to 0}}{1+\underbrace{\frac{1}{n}}_{0}} = \frac{0}{1} = 0$ ✓

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n}}{n+1} \text{ converges conditionally by AST}}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$$

1) $\sum \frac{1}{n\sqrt{\ln n}}$

$f(x) = \frac{1}{x\sqrt{\ln x}}$

a) $f(x) > 0$ ✓

b) $f(x) = \frac{1}{\text{increas}}$ = decreasing ✓

c) $f(x)$ is cont = as a comp of cont. funct ✓

$\Rightarrow \sum_{n=2} \frac{1}{n\sqrt{\ln n}}$ and $\int_2^\infty \frac{dx}{x\sqrt{\ln x}}$ will behave the same

$$\int_2^\infty \frac{dx}{x\sqrt{\ln x}} = \lim_{R\to\infty} \int_2^R \frac{dx}{x\sqrt{\ln x}} = \lim 2\left(\underbrace{\sqrt{\ln R}}_{\infty} - \sqrt{\ln 2}\right) = \infty$$

$$\left(\int \frac{dx}{x\sqrt{\ln x}} = \left(\begin{matrix} u = \ln x \\ du = \frac{1}{x}dx \end{matrix} \right) = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{\ln x} + C \right.$$

$\Rightarrow \int_2^\infty \frac{dy}{x\sqrt{\ln x}}$ diverges $\Rightarrow \sum \frac{1}{n\sqrt{\ln n}}$ diverges too $\Rightarrow$

no abs. conv

2) AST: 1) $\frac{1}{n\sqrt{\ln n}} > 0$ (see (a)) ✓

2) $\frac{1}{n\sqrt{\ln n}} \searrow$ (see (c)) ✓

3) $\lim_{n\to\infty} \frac{1}{n\sqrt{\ln n}} = \left(\frac{1}{\infty}\right) = 0$ ✓

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}} \text{ converges conditionally by AST}}$$

Example .

1. *Suppose that the series* $\sum_{1}^{\infty} (-1)^n f(n)$ *is shown to be convergent by ratio test.*

Is the series $\sum_{1}^{\infty} f(n)$ *convergent , divergent , or is it unable to determine?*

$\sum \underbrace{(-1)^n f(n)}_{a_n} \Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{f(n+1)}{f(n)}\right| = \rho < 1$

$\sum f(n) \Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{f(n+1)}{f(n)}\right| = \rho < 1$ ✓

2. For each of the following, circle ALL VALUES of k for which the following converge.

$$a. \sum_{n=1}^{\infty} \frac{(-1)^n}{(k+2)n^k}$$

$k = 3$

$k = 1$

$k = \frac{1}{2}$

$k = 0$

$k = -1$

none of these

$k=3 \quad \sum \frac{(-1)^n}{5n^3}$ (abs. conv)

$k=1 \quad \sum \frac{(-1)^n}{3n}$ (cond. conv by AST)

$k=\frac{1}{2} \quad \sum \frac{(-1)^n}{(2+\frac{1}{2})\sqrt{n}}$ (cond. conv by AST)

$k=0 \quad \sum \frac{(-1)^n}{2n^0} = \sum \frac{(-1)^n}{2} = \frac{1}{2}(-1+1-1+1$ $a_n \not\to 0)$

$\underline{\text{div}}$

$k=-1 \quad \sum \frac{(-1)^n}{1n^{-1}} = \sum((-1)^n \cdot n)$ - div $\quad \lim a_n = \pm\infty \neq 0$

$$b. \sum_{n=1}^{\infty} \frac{k}{n^k}$$

$k = 3$

$k = 1$

$k = \frac{1}{2}$

$k = 0$

$k = -1$

none of these

$c.\ the\ sequence\ \left\{\frac{1}{n^k}\right\}_{n=1}^{\infty}$

$k = 3$

$k = 1$

$k = \frac{1}{2}$

$k = 0$

$k = -1$

$none\ of\ these$