$Series\ (Part2)$

What we know so far:

To answer the question about convergency or divergency of $\sum_{n=k}^{\infty} a_n$ ***we have 3 tools:***

1. *Method of Partial Sums* **:** *Works for very specific series, called telescopic series.*

With this method we can check the convergency and find the Sum.

2. *Geometric Series*: *again,* *Works for very specific series*

Geometric Series of the form

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots = \begin{cases} \text{converges to } \dfrac{1}{1-r}, & \text{if } |r| < 1 \\ \text{diverges}, & \text{if } |r| \geq 1 \end{cases}$$

With this method we can check the convergency and find the Sum.

3. *Divergency Test or N − th term test*:

$$\boxed{If\ \lim_{k\to\infty} a_k \neq 0\ (\ or\ \lim_{k\to\infty} a_k = DNE) \ \rightarrow\ \sum_{k=1}^{\infty} a_k \text{ is divergent}}$$

With this method we can prove divergency only

Positive – Term Series : $\sum a_n$ such that $a_n > 0$ for every n

Integral Test:

Let $\sum_{n=k} a_n$ be a series such that $a_n = f(n)$

and
a $f(x) > 0$
b Continuous
c Decreasing for any $x \geq k$

a, b, c

Then **the series** $\sum_{n=k}^{\infty} a_n$ **and the integral** $\int_k^{\infty} f(x)$ **converge or diverge together**

The function does not necessarily have to be decreasing for all $x \geq k$ as long as function is decreasing eventually, starting some N

Example#1

1. Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ **diverges**!

$a_n = \frac{1}{n} \longrightarrow f(n) = \frac{1}{n} \longrightarrow \boxed{f(x) = \frac{1}{x}}$

a - b - c

a) $f(x) = \frac{1}{x} > 0$

b) $f(x) = \frac{1}{x} = \frac{1}{polynomial}$ continuous for $x \neq 0$ ✓

c) $f(x) = \frac{1}{x}$ decreasing

$f(x) = \frac{1}{\text{increasing}} = \text{decreasing}$

or, we can do it more accurate and check $f'(x) = \frac{1}{-x^2} < 0 \Rightarrow f(x) \searrow$ is decreasing

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges by integral Test with } \int_1^{\infty} \frac{1}{x} dx \quad \text{is } p = 1 \text{ divergent integral}}$$

2. ***Show The P − Series*** $\displaystyle \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$

$a_n = \frac{1}{n^p} \longrightarrow a_n = f(n) \longrightarrow f(x) = \frac{1}{x^p}$

First, check a-b-c:

$(x > 0 \leftarrow n = 1, 2 \ldots)$ $n > 0$

a) $f(x) = \frac{1}{x^p} > 0$ ✓

b) $f(x) = \frac{1}{x^p} = \frac{1}{polynomial}$ continuous for $x \neq 0$ ✓

c) $f(x) = \frac{1}{x^p}$ decreasing $\Rightarrow \frac{1}{\text{increasing}}$, $p > 0$. ✓

Checked!

Can use Integral Test

$\int_1^{\infty} \frac{1}{x^p} dx$ and series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ behave the same:

$\int_1^{\infty} \frac{1}{x^p} dx$ conv, when $p > 1 \longrightarrow$ so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv. when $p > 1$

$\int_1^{\infty} \frac{1}{x^p} dx$ div, when $p \leq 1 \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ div. when $p \leq 1$

3. $\sum_{n=1}^{\infty} n^{-e} = \qquad \sum_{n=1}^{\infty} \frac{1}{n^{e}}$ *is p − series ,* $\quad p = e > 1 \quad$ *converges*

$p = e$

Best way to solve it is recognize in this series $p = \frac{1}{2} < 1$ divergent series.

4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Or, to practice Integral Test:

$a_n = \frac{1}{\sqrt{n}} \longrightarrow f(x) = \frac{1}{\sqrt{x}} \longrightarrow$ Check a-b-c:

Conditions for I.Test

$\frac{1}{\sqrt{x}} > 0$ ✓ clear

$\frac{1}{\sqrt{x}}$ *is continuous* ✓ $\frac{1}{\text{continuous}, \neq 0} = \text{cont}$

$\frac{1}{\sqrt{x}}$ *is decreasing* ✓ $\frac{1}{\text{increasing}} = \text{decreasing}$

a-b-c checked ⟶ can use I.T:

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *behaves as* $\int_{1}^{\infty} \frac{1}{\sqrt{x}} dx \; p = \frac{1}{2} < 1$ *divergent integral*

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges by integral test with } \int_{1}^{\infty} \frac{1}{\sqrt{x}} dx}$$

5. $\sum_{n=1}^{\infty} ne^{-n^2}$ $\quad f(x) = xe^{-x^2}$

a - b - c :

a) $f(x) = \underbrace{x}_{>0}\underbrace{e^{-x^2}}_{>0} > 0$

b) $f(x) = \underbrace{x}\cdot\underbrace{e^{-x^2}}$ *continuous for all x,* as a product of two cont. functions

c) $f(x) = xe^{-x^2}$ *is decreasing ....*

easy $f(x) = \frac{x}{e^{x^2}} \rightarrow \frac{\text{polynomial}}{\text{exponent.}} \text{ - } \frac{}{\text{faster}} \searrow$

or $f'(x) = ?$

$$f'(x) = \underbrace{\left(xe^{-x^2}\right)'}_{\text{derivative of product}} = 1\cdot e^{-x^2} + xe^{-x^2}\cdot(-2x) = \underbrace{e^{-x^2}}_{>0}\cdot(\underbrace{1-2x^2}) \underbrace{< 0}$$

$f(x) \searrow$ $\quad$ a - b - c ✓

$\sum_{n=1}^{\infty} ne^{-n^2}$ *and* $\int_1^{\infty} xe^{-x^2}\,dx$ *behave the same*

How $\int_1^{\infty} xe^{-x^2}\,dx$ *behaves?*

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{R\to\infty}\int_1^{R} xe^{-x^2}\,dx = \lim_{R\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)_{x=1}^{x=R} = \lim_{R\to\infty} -\frac{1}{2}\left(e^{-R^2}-e^{-1}\right)$$

$= finite \qquad Integral\ converges!$

Solution for the integral:

$$\int xe^{-x^2}\,dx = \left(\begin{matrix} u=-x^2 \\ du=-2xdx \end{matrix}\right) = \int -\frac{1}{2}e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

$$\boxed{Hence, \sum_{n=1}^{\infty} ne^{-n^2}\ converges\ by\ Integral\ Test\ with\ \int_1^{\infty} xe^{-x^2}\,dx}$$

6. $$\sum_{n=1}^{\infty} \frac{\boldsymbol{n}}{\boldsymbol{(1+n^2)^2}}$$

$$f(x) = \frac{x}{(1+x^2)^2}$$

$Check\ a-b-c$:

✓ a) $f(x) > 0$

✓ b) $f(x)$ is cont, $\frac{\text{polyn}}{\text{polyn}}$, for any x

✓ c) $f(x)$ decreases

$f(x) = \frac{\text{polyn of degree 1}}{\text{polyn. of degree 4}} \overline{=}$ faster

$$\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^2}\ and\ \underbrace{\int_1^{\infty} \frac{x}{(1+x^2)^2}\,dx}_{\text{how it behaves?}}\ behave\ the\ same$$

$x \to \infty$

$$\int_1^{\infty} \frac{x}{(1+x^2)^2}\,dx$$

$$\frac{x}{(1+x^2)^2} \sim \frac{x}{x^4} = \frac{1}{x^3}$$

(we "think": $p = 3 > 1$ .... "conv")

DCT

$$\frac{x}{(1+x^2)^2} \leq \frac{x}{(x^2)^2} = \frac{1}{x^3}$$

$\int_1^{\infty} \frac{x}{(1+x^2)^2}\,dx$ ***converges by DCT with*** $\int_1^{\infty} \frac{1}{x^3}\,dx$ $p = 3 > 1$ ***convergent*** ✓

$$\boxed{\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^2} \quad \textit{converges by Integral Test with} \int_1^{\infty} \frac{x}{(1+x^2)^2}\,dx}$$

7. $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ $f(x)=\frac{1}{1+x^2}$

Check a − b − c:

a. $f(x)=\frac{1}{1+x^2}>0$ ✓

b. $f(x)=\frac{1}{1+x^2}$ *continuous* ✓

c. $f(x)=\frac{1}{1+x^2}=\frac{1}{\text{increasing function}}=\text{decreasing}$ ✓

So, $\underbrace{\text{the series } \sum_{n=1}^{\infty}\frac{1}{1+n^2}}$ *and the* $\underbrace{\text{integral } \int_1^{\infty}\frac{1}{1+x^2}dx \text{ behave the same}}$

$x \sim \infty$

How does $\int_1^{\infty}\frac{1}{1+x^2}dx$ **behave?** $\frac{1}{1+x^2} \sim \frac{1}{x^2}$ ("think" $p=2>1$ conv)

Let's use DCT again: $\frac{1}{1+x^2} \leq \frac{1}{x^2}$

$\underbrace{\int_1^{\infty}\frac{1}{1+x^2}}dx$ **converges by DCT with** $\underbrace{\int_1^{\infty}}\frac{1}{x^2}dx$ $p=2>1$ **convergen**

Conclusion for integral

$$\boxed{\sum_{n=1}^{\infty}\frac{n}{1+n^2} \quad \textbf{converges by Integral Test with} \int_1^{\infty}\frac{1}{1+x^2}dx}$$

← Conclusion for series

$f(x) = \frac{\ln x}{x}$

8. $\sum_{n=1}^{\infty} \frac{lnn}{n}$

$Check\ a - b - c$

a) $f(x) = \frac{\ln x}{x} > 0$

b) $f(x)$ - cont - $\frac{\text{cont}}{\text{cont} \neq 0} = \text{cont}$

c) $f(x)$ decreasing ? $\rightarrow \frac{\ln x}{x}$ "slow" / polyn

$$f' = \left(\frac{lnx}{x}\right)' = \frac{\frac{1}{x} \cdot x - lnx \cdot 1}{x^2} = \frac{1 - lnx}{x^2} < 0, for\ x\ bif\ enough$$

(or show that $f'(x) < 0 \rightarrow f \downarrow$)

a-b-c ✓

?

$The\ series\ \sum_{n=1}^{\infty} \frac{lnn}{n}\ and\ the\ integral\ \int_1^{\infty} \frac{lnx}{x} dx\ \ behave\ the\ same$

$$To\ check\ how\ does\ \int_1^{\infty} \frac{lnx}{x} dx\ \ behave, let's\ use\ DCT\ again$$

$$\frac{lnx}{x} > \frac{1}{x}, \quad for\ x\ big\ enough$$

div.

$$\int_1^{\infty} \frac{\mathbf{lnx}}{\mathbf{x}} \mathbf{dx}\ \ \textbf{diverges by DCT with} \int_1^{\infty} \frac{\mathbf{1}}{\mathbf{x}} \mathbf{dx}\ \ \mathbf{p = 1}\ \textbf{divergent}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{\mathbf{lnn}}{\mathbf{n}}\ \ \textbf{diverges by Integral Test with} \int_1^{\infty} \frac{\mathbf{lnx}}{\mathbf{x}} \mathbf{dx}}$$

9. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$f(x) = \frac{1}{x(\ln x)^2}$ $\qquad$ *Check a − b − c:*

a. $f(x) = \frac{1}{x(\ln x)^2} > 0$ ✓

b. $f(x) = \frac{1}{x(\ln x)^2}$ *continuous* ✓ – cont, ≠0 $\quad x \geq 2$,

c. $f(x) = \frac{1}{x(\ln x)^2} = \frac{1}{\text{increasing function}} = \text{decreasing}$ ✓

⇒ we allowed to use I.Test

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \int_2^{R} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty} \left(-\frac{1}{\ln x}\right)\Bigg|_{x=2}^{x=R} = \lim_{R\to\infty} \left(-\frac{1}{\ln R} + \frac{1}{\ln 2}\right)$$

$= \frac{1}{\ln 2}$ $\qquad$ *finite limit* => *integral converges*

Here is the solution for the integral:

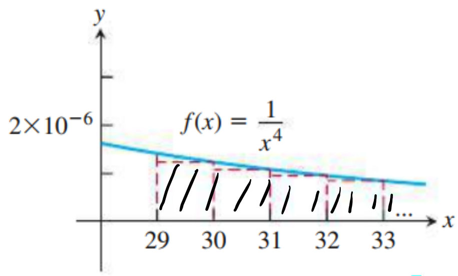
$$\int \frac{1}{x(\ln x)^2}\,dx = \left(\begin{matrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{matrix}\right) = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

$$\boxed{\sum_{n=2}^{\infty} \frac{\mathbf{1}}{\boldsymbol{n(\ln n)^2}} \quad \textbf{converges by Integral Test with} \int_2^{\infty} \frac{\mathbf{1}}{\boldsymbol{x(\ln x)^2}}\,\boldsymbol{dx}}$$

Example* #54 *from the textbook

4. $\sum_{n=1}^{\infty}(1/n^4)$ **converges**

a. Use the accompanying graph to find an upper bound for the error if $s_{30} = \sum_{n=1}^{30}(1/n^4)$ is used to estimate the value of $\sum_{n=1}^{\infty}(1/n^4)$.



b. Find n so that the partial sum $s_n = \sum_{i=1}^{n}(1/i^4)$ estimates the value of $\sum_{n=1}^{\infty}(1/n^4)$ with an error of at most 0.000001.

$$\sum_{n=1}^{\infty}\frac{1}{n^4}, \quad p=4>1$$

convergent

$$\underbrace{S}_{\text{finite}} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$S \sim \underbrace{S_n}_{\text{partial sum}} = a_1 + a_2 + \dots + a_n$$

$$S \sim S_{30} = a_1 + a_2 + \dots a_{30}$$

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le \underbrace{s_n + \int_{n}^{\infty} f(x)\,dx}_{\text{upper bound}}$$

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \ge n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx. \qquad (1)$$

$$f(x) = \frac{1}{x^4} \Leftarrow a_n = \frac{1}{n^4}$$

$n=30$,

$$S \le S_{30} + \int_{30}^{\infty}\frac{1}{x^4}\,dx \quad \Big| -S_{30}$$

$$\underbrace{S - S_{30}}_{\text{Error}} \le \int_{30}^{\infty}\frac{1}{x^4}\,dx$$

number

$$\int_{30}^{\infty}\frac{1}{x^4}\,dx = \lim_{n\to\infty}\int_{30}^{R}\frac{1}{x^4}\,dx = \lim_{R\to\infty}\left(\frac{x^{-3}}{-3}\right)_{30}^{R} = \lim_{R\to\infty} -\frac{1}{3}\left(\frac{1}{R^3} - \frac{1}{(30)^3}\right) = \frac{1}{81}\cdot 10^{-3}$$

Upper bound for the error is $\frac{1}{81}\cdot 10^{-3}$

$N = ?$ $S \approx S_N$, such that an error

$$E = S - S_N \leq 10^{-6}$$

$$E \leq \int_N^\infty \frac{1}{x^4}\,dx \leq 10^{-6}$$

$$\int_N^\infty \frac{1}{x^4}\,dx = \lim_{n\to\infty}\int_N^R \frac{1}{x^4}\,dx = \lim_{R\to\infty}\left(\frac{x^{-3}}{-3}\right)_N^R = \lim_{R\to\infty} -\frac{1}{3}\left(\frac{1}{R^3} - \frac{1}{(N)^3}\right) = \frac{1}{3(N)^3} < \frac{1}{10^6}$$

$$\frac{1}{3N^3} < \frac{1}{10^6}$$ } flip

$$3N^3 > 10^6$$

$$N^3 > \frac{10^6}{3} \Rightarrow N > \sqrt[3]{\frac{10^6}{3}}$$

$$N > 70$$

Comparison Test: DCT

> Let $\sum a_n$ and $\sum b_n$ be a positive term series, $a_n \le b_n$ for every n,
> 1. If $\sum b_n$ converges, then $\sum a_n$ converges
> 2. If $\sum a_n$ diverges, then $\sum b_n$ diverges

Example#3

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+5^n}$

$$\frac{1}{2+5^n} \sim \frac{1}{5^n} = \left(\frac{1}{5}\right)^n$$

„think" conv.
$\underbrace{\sum r^n,\ r = \frac{1}{5} < 1}_{\text{looks like}}$ — geometric...

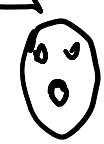
DCT: $\displaystyle \frac{1}{2+5^n} < \frac{1}{5^n} = \left(\frac{1}{5}\right)^n$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{2+5^n} \text{ converges by DCT with convergent geometric series } \sum_{n=1}^{\infty} \frac{1}{5^n}, \quad \text{where } r = \frac{1}{5} < 1}$$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\sqrt{n}-1}$

$$\frac{3}{\sqrt{n}-1} \sim \frac{3}{\sqrt{n}} = 3 \cdot \frac{1}{n^{1/2}}$$

„think"
$p = \frac{1}{2} < 1$ ....
div...



DCT: $\displaystyle \frac{3}{\sqrt{n}-1} > \frac{3}{\sqrt{n}} > \frac{1}{\sqrt{n}}$

$\sqrt{n}-1 < \sqrt{n}$

$$\boxed{\sum_{n=2}^{\infty} \frac{3}{\sqrt{n}-1} \text{ diverges by DCT with series } \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \quad \left(p = \frac{1}{2} < 1 \text{ diverges}\right)}$$

3. $\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{n}}{n-8}$

$$a_n=\frac{\sqrt{n}}{n-8}\sim\frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}\quad n\sim\infty$$

„think"…. $p=\frac{1}{2}<1$ …. div ….

$$\frac{\sqrt{n}}{n-8}\underset{n-8<n}{>}\frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}$$

$$\boxed{\sum_{n=2}^{\infty}\frac{\sqrt{n}}{n-8}\ \textit{diverges by DCT with series}\ \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\quad \left(p=\frac{1}{2}<1\ \textit{diverges}\right)}$$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3+1}}$

this is not a proof, not DCT, we only „think" here…

$$a_n=\frac{n}{\sqrt{n^3+1}}\sim\frac{n}{n^{3/2}}=\frac{1}{\sqrt{n}}$$

$p=\frac{1}{2}<1$ „think" div

DCT:

$$\frac{n}{\sqrt{n^3+1}}\underset{n^3+1<n^3+n^3}{>}\frac{n}{\sqrt{n^3+n^3}}=\frac{n}{\sqrt{2n^3}}=\frac{n}{n^{3/2}\cdot\sqrt{2}}=\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{2}}$$

! It's easy to get here $\dfrac{n}{\sqrt{n^3+1}}<\dfrac{n}{\sqrt{n^3}}=\dfrac{1}{\sqrt{n}}$

wrong sign!.

Correct, but useless!

But! this part makes us „think" $p=\frac{1}{2}<1$ divergent

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}} \text{ diverges by DCT with series } \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \quad \left(p = \frac{1}{2} < 1 \text{ diverges}\right)$$

The end of class-notes!

Extra-examples:

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}\cdot 3^n}$

$$\frac{1}{\sqrt{n+1}\cdot 3^n} \sim \frac{1}{\sqrt{n}\cdot 3^n} \sim \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

(looks like geometric ... $r = \frac{1}{3} < 1$ .... „think" conv)

$$\text{DCT: } \frac{1}{\sqrt{n+1}\cdot 3^n} < \frac{1}{\sqrt{n}\cdot 3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

$n \geq 1$

$\sqrt{n} \geq 1$

$\sqrt{n}\cdot 3^n \geq 1 \cdot 3^n$

$\sqrt{n}\cdot 3^n > 3^n$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}\cdot 3^n} \text{ converges by DCT with convergent geometric series } \sum_{n=1}^{\infty} \frac{1}{3^n}, \quad \text{where } r = \frac{1}{5} < 1$$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos(n)}{n^2}$

$\cos n \leq 1$

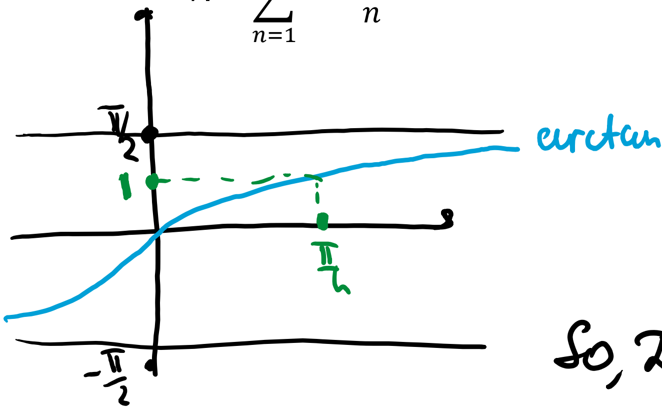
$$\text{DCT: } \frac{2+\cos n}{n^2} \leq \frac{2+1}{n^2} \leq \frac{3}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{2+\cos(n)}{n^2} \text{ converges by DCT with series } \sum_{n=2}^{\infty} \frac{1}{n^2} \quad (p = 2 > 1 \text{ converges})$$

when n is large, $\arctan n \sim \frac{\pi}{2}$

7. $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n}$

$$\frac{\arctan n}{n} \sim \frac{\frac{\pi}{2}}{n}$$ (so... $\frac{1}{n}$ ... $p=1$ ... „think" div)



(It's easy to say $\frac{\arctan(n)}{n} < \frac{\pi/2}{n}$

But, again, it's a wrong sign (if we want to show divergency)

So, DCT: $\arctan n > 1$ (for $n > \frac{\pi}{4}$ see graph!)

$$\frac{\arctan n}{n} > \frac{1}{n}$$

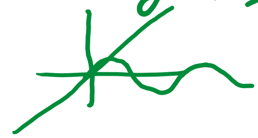
$$\boxed{\sum_{n=1}^{\infty} \frac{\arctan(n)}{n} \text{ diverges by DCT with series } \sum_{n=2}^{\infty} \frac{1}{n} \quad (p = 1 \text{ diverges})}$$

8. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

DCT: $\frac{\ln n}{n} > \frac{1}{n}$ (for $n \geq 3$)

$$\boxed{\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \text{ diverges by DCT with series } \sum_{n=2}^{\infty} \frac{1}{n} \quad (p = 1 \text{ diverges})}$$

We saw it in integrals: $\sin x \le x$ for any $x \ge 0$



9. $\displaystyle\sum_{n=1}^{\infty} sin\left(\frac{1}{n^2}\right)$

DCT: $\sin\left(\frac{1}{n^2}\right) \le \frac{1}{n^2}$

$$\boxed{\sum_{n=1}^{\infty} sin\left(\frac{1}{n^2}\right) \text{ converges by DCT with series } \sum_{n=2}^{\infty}\frac{1}{n^2} \quad (p = 2 > 1 \text{ converges})}$$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2^n}{n+3^n}$

DCT: $\displaystyle\frac{n^2+2^n}{n+3^n} \le \frac{2^n+2^n}{3^n} = \frac{2\cdot 2^n}{3^n} = 2\underbrace{\left(\frac{2}{3}\right)^n}_{r=\frac{2}{3}<1}$

$n^2+2^n \le 2^n+2^n$ for $n \ge 5$
(you can say n big enough

$n+3^n > 3^n$

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2+2^n}{n+3^n} \text{ converges by DCT with convergent geometric series } 2\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n, \text{where } r = \frac{2}{3} < 1}$$

11. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

$n^n > 2^n, \quad n \geq 3$

DCT: $\frac{1}{n^n} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^n} \text{ converges by DCT with convergent geometric series } \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n, \text{where } r = \frac{2}{3} < 1}$$

12. $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)n} < \frac{1}{(n-1)n} = \frac{1}{n^2 - n} < \frac{1}{n^2 - \frac{n^2}{2}} = \frac{2}{n^2}$$

$n < \frac{n^2}{2}$ for $n \geq 2$

$-n > -\frac{n^2}{2}$

$n^2 - n > n^2 - \frac{n^2}{2}$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n!} \text{ converges by DCT with series } 2\sum_{n=2}^{\infty} \frac{1}{n^2} \quad (p = 2 > 1 \text{ converges})}$$