$Taylor\ \ Series\ (Part\ 2)$

In last lecture we discovered that if a function $f(x)$ is represented by a power series ,

then the series must be a Maclaurin or Taylor series.

What conditions on a function guarantee that a power representation exists?

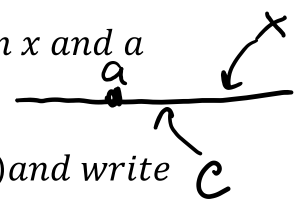
Taylor's Theorem:

If $f(x)$ has derivatives of all orders in an open interval I containing a
then for each positive integer n and for each x in I,

$$\boldsymbol{f(x) = P_n(x) + R_n(x,a)}$$

where $\boldsymbol{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$ *is the $n-th$ order Taylor polynomial of $f(x)$*

and $\boldsymbol{R_n(x,a) = \frac{f^{(n+1)}(C)}{(n+1)!}(x-a)^{n+1}}$*, for some* $\boldsymbol{C}$ *between x and a*



If $\boldsymbol{\lim_{n\to\infty} R_n(x) = 0}$ *for all x in I we say*

that the ***Taylor series*** *for $f(x)$at a* ***converges to $f(x)$****and write*

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Then, the magnitude of the error

$$\boldsymbol{|E_n| = |f(x) - P_n(x)| = |R_n(x,a)| \le \frac{M}{(n+1)!}|x-a|^{n+1}}$$

where $\left|f^{(n+1)}(t)\right| \le M$ for all t betweenx and a

Example#1

Show that the Taylor series of $f(x) = e^x$about $a = 0$ converges to $f(x)$

for every real value of x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$R_n(x,a) = \frac{f^{(n+1)}(C)}{(n+1)!}(x-a)^{n+1} = \frac{e^C}{(n+1)!}(x-a)^{n+1}$$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{e^C}{(n+1)!}(x-a)^{n+1} = 0$$

Example#2

a. Find the approximation to the function $f(x) = e^x$ *by the fourth Maclaurin polynomial* $\longrightarrow$ $a=0$

$$\cancel{e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!} + \cdots}$$

$$f(x) \sim P_4(x) = f(0) + f'(0)\cdot x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$$

a)

$$f(x) = e^x \Big|_{x=0} = 1$$

$$\vdots$$

$$f^{(4)} = e^x \Big|_{x=0} = 1$$

$$f(x) \sim P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$f^{(4+1)} = f^{(5)}(x) = e^x \Big|_{x=c} = e^c$$

b. Determine the maximum error in using this approximation for $-4 \le x \le 4$

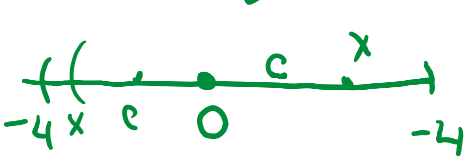
$$|E_n| = |f(x) - P_n(x)| \le \frac{f^{(n+1)}(c)}{(n+1)!}|x-a|^{n+1}$$

$\underline{n=4}$

$$\boxed{|E_5| \le \frac{f^{(5)}(c)}{5!}|x|^5 \le \boxed{\frac{e^4 \cdot 4^5}{5!}}}$$

$$-4 \le x \le 4 \Rightarrow |x| \le 4 \Rightarrow \boxed{|x|^5 \le 4^5}$$

$\downarrow$



$$\Rightarrow -4 \le c \le 4 \Rightarrow \boxed{e^c \le e^4}$$

Example#3

a. Find the Taylor Polynomial $P_3(x)$ for the function $f(x) = e^{x^2}$ about $a = 0$

$$f(x) = e^{x^2}\Big|_{x=0} = 1 \qquad f(x) \sim \underbrace{P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3}$$

a

$$f'(x) = e^{x^2}\cdot 2x = 0$$

$$f''(x) = e^{x^2}\cdot 2x\cdot 2x + e^{x^2}\cdot 2 = e^{x^2}\cdot(4x^2+2) = 2$$

$$f'''(x) = e^{x^2}\cdot 2x(4x^2+2) + e^{x^2}\cdot(8x) = e^{x^2}\cdot(8x^3+12x) = 0$$

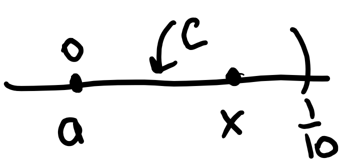
$$f^{(4)}(x) = e^{x^2}\cdot 2x(8x^3+12x) + e^{x^2}(24x^2+12) = e^{x^2}(16x^4+48x^2+12)$$

$$f(x) \approx P_3(x) = 1 + \frac{2x^2}{2!} = 1 + x^2.$$

b. Find the maximum error of replacing $f(x)$ by $P_3(x)$ on $\left[0, \frac{1}{10}\right]$

$$|E_3| \le \frac{f^{(4)}(c)}{4!}|x|^4 \le \frac{14}{4!}\cdot\frac{1}{10^4} = \boxed{\frac{14}{4!\,10^4}}$$

$$0 \le x \le \frac{1}{10} \Rightarrow |x| \le \frac{1}{10} \Rightarrow |x|^4 \le \boxed{\frac{1}{10^4}}$$



$$\Rightarrow 0 \le c \le \frac{1}{10} \Rightarrow$$

$$f^{(4)}(c) = e^{c^2}(16c^4+48c^2+12), \quad c \le \frac{1}{10}$$

$$\boxed{e^{\frac{1}{100}}\left(\frac{16}{10^4} + \frac{48}{10^2} + 12\right)} \le \boxed{14}$$

$$e^{\frac{1}{100}} \le 1, \quad \underbrace{\frac{16}{10^4} \le 1, \quad \frac{48}{10^2} \le 1, \quad 12}_{\le 14}$$

Example#4

a. Find the Taylor Polynomial $P_3(x)$ *centered at* $x=\frac{\pi}{4}$ *to approximate* $f(x)=\sin x$

$$f(x) \sim P_3(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3$$

a)

$$f(x) = \sin x \Big|_{\pi/4} = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x \Big|_{\pi/4} = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x \Big|_{\pi/4} = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = -\cos x \Big|_{\pi/4} = -\frac{\sqrt{2}}{2}$$

$$P_3(x) = \frac{\sqrt{2}}{2}\left(1 + \left(x-\frac{\pi}{4}\right) - \frac{1}{2}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x-\frac{\pi}{4}\right)^3\right)$$

$$f^{(4)}(x) = \sin x \Big|_c$$

b. Suppose we used $P_3(x)$ *to estimate* $f\left(\frac{\pi}{12}\right)$*. Then, by Taylor's formula, the best error bound for using this approximation is (circle one). Show your work justifying your answer.*

1. $|E_3| \le \dfrac{\sin\left(\frac{\pi}{12}\right)(\pi)^4}{24 \cdot 6^4}$

2. $|E_3| \le \dfrac{\sqrt{2}(\pi)^4}{48 \cdot 6^4}$

3. $\cancel{|E_3| \le \dfrac{\sin\left(\frac{\pi}{12}\right)(\pi)^3}{6^4}}$

4. $\cancel{|E_3| \le \dfrac{\sqrt{2}(\pi)^3}{2 \cdot 6^4}}$

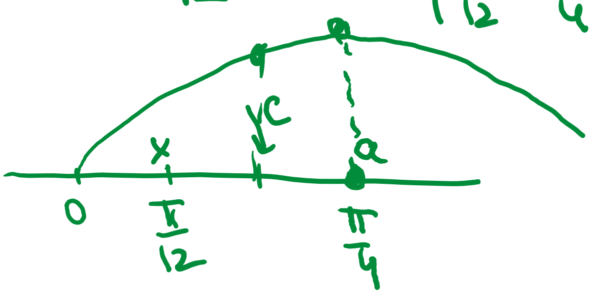
5. $|E_3| \le \dfrac{(\pi)^4}{4! \cdot (12)^4}$

$$|E_3| \le \frac{|f^{(4)}(c)|}{4!}\left|x-\frac{\pi}{4}\right|^4$$

$$f(x) \sim P_3(x)$$

$$f\left(\frac{\pi}{12}\right) \sim P_3\left(\frac{\pi}{12}\right)$$

$$x = \frac{\pi}{12} \Rightarrow \left|\frac{\pi}{12}-\frac{\pi}{4}\right|^4 = \left|\frac{\pi-3\pi}{12}\right|^4 = \left|-\frac{\pi}{6}\right|^4 = \frac{\pi^4}{6^4}$$



$$\frac{\pi}{12} < c < \frac{\pi}{4}$$

$$\sin c < \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$|E_3| \le \frac{\frac{\sqrt{2}}{2}\cdot\frac{\pi^4}{6^4}}{4!} = \frac{\sqrt{2}\cdot\pi^4}{48\cdot 6^4}$$

Example#5

Find N such that Maclaurin polynomial for $f(x) = e^x$ has $f(1)$ approximated with an error $|E_N| \le 0.000005$

$$f(1) \sim P_N(1) : \quad |E_N| \le \frac{5}{10^6} \Rightarrow N = ?$$

$$|E_N| \le \frac{f^{N+1}(c)}{(N+1)!}\, x^{N+1} = \frac{f^{N+1}(c)}{(N+1)!} < \frac{5}{10^6}$$

$x = 1$

$$f^{(N+1)}(x) = e^x\Big|_{x=c} = e^c < e$$

$a=0,\ x=1 \Rightarrow 0 < c < 1$

$$\frac{e}{(N+1)!} < \frac{5}{10^6}$$

$$(N+1)! > 10^6$$

$9! = 362880$

$10! = 3628800$

$$f(1) \sim P_9(1)$$

$$E_9 < \frac{5}{10^6}$$

Example#6

a. Find the Taylor Polynomial $\underline{P_2(x)}$ *centered at* $\underline{x=1}$ *to approximate* $f(x) = xe^x$

$$\boxed{P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2}$$

a

$$f(x) = x\cdot e^x \Big|_{x=1} = e$$

$$f'(x) = 1\cdot e^x + x\cdot e^x = (1+x)\cdot e^x \Big|_{x=1} = 2e$$

$$f''(x) = (1)e^x + (1+x)e^x = (2+x)e^x \Big|_{x=1} = 3e$$

$$f'''(x) = (3+x)e^x$$

$$\boxed{P_2(x) = e\left(1 + 2(x-1) + \frac{3}{2}(x-1)^2\right)}$$

$x=1$ $\quad f(1) = 1\cdot e$

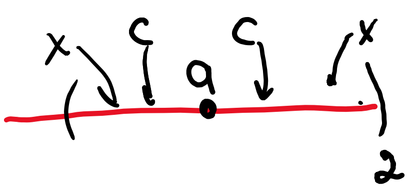
$\underline{e} \sim \underline{e}(1)$

b. Supposed $f(x)$ *is replaced by the Taylor polynomial from part a. Estimate the error when* $0 \le x \le 2$

$$|E_2| \le \frac{|f^{(3)}(c)|}{3!}|x-1|^3$$

$$0 \le x \le 2 \implies |x| \le 2$$

$$\left.\begin{aligned} 0-1 &\le x-1 \le 2-1 \\ -1 &\le x-1 \le 1 \end{aligned}\right\} \implies |x-1| \le 1 \implies |x-1|^3 \le 1$$



$$\implies 0 \le c \le 2$$

$$|f'''(c)| = |(3+c)e^c|, \quad 3+c \le 5, \quad e^c \le e^2$$

$$|E_2| \le \frac{5e^2}{3!}\cdot 1 = \boxed{\frac{5e^2}{6}}$$

Extra

$\boldsymbol{Example}: The\ function\ f\ has\ derivatives\ of\ all\ orders\ for\ all\ real\ numbers\ x.$

$Assume\ f(2) = -3, f'(2) = 5\ f''(2) = 3\ and\ f'''(2) = -8$

$(a) Write\ the\ third\ degree\ Taylor\ polynomial\ for\ f\ about\ x = 2\ and\ use\ it\ to$

$approximate\ f(1.5).$

$$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3$$

$$f(x) \approx P_3(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$$

$x = 1.5 = 3/2$

$$f\left(\tfrac{3}{2}\right) \approx \underbrace{-3 + 5\left(-\tfrac{1}{2}\right) + \tfrac{3}{2}\left(-\tfrac{1}{2}\right)^2 - \tfrac{4}{3}\left(-\tfrac{1}{2}\right)^3} = \frac{-119}{24}$$

?

$(b) Assume\ that\ the\ fourth\ derivative\ of\ f\ satisfies\ \left|f^{(4)}(x)\right| \le 3 for\ all\ x\ in\ the$

$interval\ [1.5,2]. Use\ the\ Taylor\ error\ bound\ on\ the\ approximation\ to\ f(1.5) to\ explain$

$why\ f(1.5) \ne -5$