$\boldsymbol{Series\ (Part6)} - Alternating\ Series\ Estimation$

$$Let \sum_{n=1}^{\infty} (-1)^{n+1} a_n \ \textbf{satisfies conditions Leibniz's Theorem}$$

$$\boldsymbol{Then, S \approx S_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n}$$

and the error of this approximation is less or equal to $\boldsymbol{a_{n+1}}$

that is ,
$$\boldsymbol{|S - S_n| \le a_{n+1}}$$

The remainder , $S - S_n$, *has the same sign as the first unused term*

Example#1

Estimate the magnitude of the eror involved in using the sum of the $\underbrace{\overbrace{\textit{first three terms}}^{S_3}}$ *to approximate the sum of the entire series. Compare it with the actual error.*

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2}{3}\right)^n = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \underbrace{\left(\frac{2}{3}\right)^4} + \dots$$

a) $$S \approx S_3 = \frac{2}{3} - \frac{4}{9} + \frac{8}{27}$$

$$a_{n+1} = a_4 = \left(\frac{2}{3}\right)^4$$

The error $$|E_3| = |S - S_3| \le |a_4| = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

$$E_3 \le \frac{16}{81} \left(\approx 0{,}198\right)$$

b) For actual error we will find S and S_3

$$S = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots = \frac{2}{3}\Bigg(\underbrace{1 + \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)^2 + \dots}_{|r| = \left|-\frac{2}{3}\right| < 1 \Rightarrow \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{3}{5}}\Bigg) = \boxed{\frac{2}{5}}$$

$$S_3 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} + \frac{8}{27} = \frac{18-12+8}{27} = \frac{14}{27}$$

$$|S - S_3| = \left|\frac{2}{5} - \frac{14}{27}\right| = \left|-\frac{16}{5 \cdot 27}\right| = \underline{0{,}118}$$

***Example*#2**

Approximate the following sum with an error less than 0.001

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+n} = S \qquad 1)\ \underline{S \text{ converges}}\ \underline{\text{absolutely}}$$

$S \approx S_N$, $N = ?$ such that $|S - S_N| < \frac{1}{1000}$

$$|S - S_N| \le |a_{N+1}| < \frac{1}{1000}$$

$$a_n = \frac{(-1)^{n+1}}{n^2+n} \Rightarrow |a_{N+1}| = \left|\frac{(-1)^{N+2}}{(N+1)^2 + (N+1)}\right| < \frac{1}{1000}$$

$$(N+1)^2 + N + 1 > 1000$$

$$N^2 + 3N + 2 > 1000$$

Try $N = 30$ $\quad 900 + 90 + 2 = 992 < 1000$

almost...

$N = 31$ will work $\Rightarrow \underline{S \approx S_{31}}$

Example#3

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{10^n \cdot n!}$$

a. *Tell if the series converges absolutely or conditionally*

$\sum \frac{(-1)^{n+1}}{10^n n!}$ converges absolutely: ① $\sum \frac{1}{10^n n!}$

$\frac{1}{10^n n!} < \frac{1}{10^n} = \left(\frac{1}{10}\right)^n$

so $\sum \frac{1}{10^n n!}$ conv. by DCT with $\sum \left(\frac{1}{10}\right)^n$ – geometric

$r = \frac{1}{10} < 1$

conv.

b. *Find the reasonable partial sum that is within* $\frac{1}{20}$ *of the sum of the entire series.*

b) again, we are looking for N:

$S \approx S_N$, and $|S - S_N| < \frac{1}{20}$

$$|a_{N+1}| < \frac{1}{20}$$

$$\left| \frac{(-1)^{N+2}}{10^{N+1} (N+1)!} \right| < \frac{1}{20}$$

$$10^{N+1} \cdot (N+1)! > 20$$

$\underline{\underline{N=1}}$ $\quad 10^2 \cdot 2! = 100 \cdot 2 > 20$ ✓

$$S \approx S_1$$

$$S \approx S_1 = \frac{(-1)^2}{10 \cdot 1!} = \frac{1}{10}$$

Example#4

$\boldsymbol{a}.\ \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{3n^2+1}$

Tell if the series converges absolutely or conditionally

① $\sum \frac{n}{3n^2+1}$ $\qquad \frac{n}{3n^2+1} \sim \frac{n}{3n^2} = \frac{1}{3n}$

$$\text{LCT: } \lim_{n\to\infty} \frac{\frac{n}{3n^2+1}}{\frac{1}{n}} = \lim \frac{n^2}{3n^2+1} = \lim \frac{1}{3+\frac{1}{n^2}} = 3, \quad 0<3<\infty$$

So $\sum \frac{n}{3n^2+1}$ div. by LCT with $\sum \frac{1}{n}$, $p=1$ $\Rightarrow$ No abs convergen.

② AST: 1 - 2 - 3

1) $a_n = \frac{n}{3n^2+1} > 0$

2) $a_n = \frac{1 \text{ degree}}{2 \text{ degree}} \searrow$

3) $\lim \frac{n}{3n^2+1} = \lim \frac{\frac{1}{n}}{3+\frac{1}{n^2}} = 0$

$\sum \frac{(-1)^n n}{3n^2+1}$ conver. conditionally.

$\boldsymbol{b}.$ *Find the smallest positive integer k so the error* $|E_k| < \frac{1}{10}$

$$\sum_{n=1} \frac{(-1)^n n}{3n^2+1} = \underbrace{-\overset{a_1}{\frac{1}{4}} + \overset{a_2}{\frac{2}{13}} - \overset{a_3}{\frac{3}{28}}}_{S_3} + \underbrace{\overset{a_4}{\frac{4}{49}}}_{<\frac{1}{10}} - \dots \quad \Rightarrow \underline{\underline{k=3}}$$

or same way as we did before

$$|a_{N+1}| = \left| \frac{(-1)^{N+1}(N+1)}{3(N+1)^2+1} \right| = \frac{N+1}{3(N+1)^2+1} < \frac{1}{10}$$

$$\frac{3(N+1)^2+1}{N+1} > 10$$

$$3(N+1)+\frac{1}{N+1} > 10$$

Try $N=1$ $\quad 6+\frac{1}{2} < 10$

$N=2$ $\quad 3\cdot 3+\frac{1}{3} < 10$

$\underline{\underline{N=3}}$ $\quad 3\cdot 4+\frac{1}{4} > 10$

$$S \approx S_{\underline{\underline{3}}}$$

***Example*#5**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot lnn}$$

a. *Tell if the series converges absolutely or conditionally*

① $\sum_{n=2} \frac{1}{n \cdot \ln n}$, $\quad f(x)=\frac{1}{x \ln x}$ $\begin{cases} a) \ f(x)>0 \ \checkmark \\ b) \ f(x) \text{ cont } \checkmark \\ c) \ f(x) \searrow \ \checkmark \end{cases}$

$\sum_{n=2} \frac{1}{n \ln n}$ behaves as $\int_2^{\infty} \frac{1}{x \ln x} dx$ = diverges. $\Big\}$ no $\underline{\text{abs}}$ convergency

② AST: 1-2-3 $\begin{cases} 1) >0 \\ 2) \searrow \\ 3) \lim \frac{1}{n \ln n} = \left(\frac{1}{\infty}\right) = 0 \end{cases}$

$\sum_{n=2} \frac{(-1)^n}{n \ln n}$ converges conditionally

$\boldsymbol{b}$. *What is the errors upper bound in using* $\underline{10 - th\ partial\ sum}$ *approximation for the series.*

$$S \approx S_{10}$$

(I found a small mistake here, we used $\sum_{n=1}$)

$$\sum_{n=2} \frac{(-1)^n}{n \ln n} = \underbrace{\frac{1}{2\ln 2}}_{a_1} - \underbrace{\frac{1}{3\ln 3}}_{a_2} + \underbrace{\frac{1}{4\ln 4}}_{a_4} - \cdots \underbrace{\frac{1}{11\ln 11}}_{a_{10}}$$

$$\text{so } |a_{n+1}| = |a_{10+1}| = |a_{11}| = \left|\frac{1}{12\ln 12}\right|$$

$$|E_{10}| \leq \frac{1}{12\ln 12}$$

Strategy for Testing Series

It is not wise to apply a list of the tests in a specific order until one finally works.

That would be a waste of time and effort.

Instead, as with integration, the main strategy is to classify the series according to its form.

So, the following general questions are useful:

1. ***Does the nth term approach zero as n approaches infinity?***

(If not, the Divergence Test implies the series diverges)

2. ***Is the series one of the special types – geometric, $p-$series or alternating series?***

3. ***If the series has a form that is similar to a $p-$series or a geometric series,*** *then one of the comparison tests should be considered.*

(In particular, if a_n is a rational function or algebraic function of n (involving roots of polynomials), then the series should be compared with a $p-$series)

4. ***Can the Ratio Test be applied?***

(Series that involve factorials or other products (including a constant raised to the th power) are often conveniently tested using the Ratio Test)

5. ***Can the Root Test be applied?***

6. ***Can the Integral Test be applied?***

(If $a_n = f(n)$, where $f(n)$ can be easily integrated, then the Integral Test is effective. But check the conditions!)

Test the series for convergency/divergency:

1. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

div by n-th term

$a_n = \frac{n^2-1}{n^2+n} \longrightarrow 1 \neq 0$

2. $\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

$\frac{n-1}{n^2+n} \sim \frac{1}{n}$

DCT or LCT

diverges by comparison with $\sum \frac{1}{n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$\frac{1}{n^2+n} \sim \frac{1}{n^2}$

Show DCT or LCT with $\sum \frac{1}{n^2}$

converges

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$

converges condit. (see #1) by AST (check 1-2-3)

5. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

6. $\sum_{n=1}^{\infty} \frac{n \ln n}{(n+1)^3}$

7. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

8. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}}$

#5) ① $\sum_{n=2} \frac{1}{n \ln n}$ (Show divergence by Integral Test)

② AST 1-2-3 $\Rightarrow$ conv. conditionally

#8) ① $\sum \frac{\ln n}{\sqrt{n}}$, $\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}}$,

so diverges by DCT with $\sum \frac{1}{\sqrt{n}}$, $p = \frac{1}{2} < 1$ div

$\Rightarrow$ no abs conv

#6 1) $\sum \frac{n \ln n}{(n+1)^3}$: $\frac{n \ln n}{(n+1)^3} \sim \frac{n \ln n}{n^3} = \frac{\ln n}{n^2}$

so, first, by LCT $\lim_{n\to\infty} \frac{\frac{n \cancel{\ln n}}{(n+1)^3}}{\frac{\cancel{\ln n}}{n^2}} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 1$

we can say, $\sum_{n=1} \frac{n \ln n}{(n+1)^3}$ behaves as $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

How $\sum \frac{\ln n}{n^2}$ behaves?

Use Integral Test: $f(x) = \frac{\ln x}{x^2}$ $\left(\begin{matrix}\text{Check} \\ a-b-c \checkmark\end{matrix}\right)$

$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{R\to\infty} \int_1^R \frac{\ln x}{x^2} dx = ?$

$\int \frac{\ln x}{x^2} dx = \left(\begin{matrix} u = \ln x \to du = \frac{1}{x} dx \\ dv = \frac{1}{x^2} dx \to v = -\frac{1}{x}\end{matrix}\right) = -\frac{1}{x}\ln x + \int \frac{1}{x^2} dx =$

$= -\frac{1}{x} \ln x - \frac{1}{x} + C$

$= \lim -\left(\left(\underset{\downarrow 0}{\frac{\ln R}{R}} + \underset{\downarrow 0}{\frac{1}{R}}\right) - \left(0 + \frac{1}{1}\right)\right) = 1$ (conv)!

so, $\sum \frac{\ln n}{n^2}$ conv $\Rightarrow$ $\sum \frac{n \ln n}{(n+1)^3}$ conv. too.

#7 we can show $(\ln n)^{\ln n} > n^2$ for n big enough

$$\ln\left(\ln n^{\ln n}\right) > \ln(n^2)$$

$$\cancel{\ln n} \cdot \ln(\ln n) > 2 \cancel{\ln n}$$

$$\ln(\ln n) > 2$$

Then, $\dfrac{1}{\ln n^{\ln n}} < \dfrac{1}{n^2}$

and $\sum \frac{1}{\ln n^{\ln n}}$ conv by DCT with $\sum \frac{1}{n^2}$, $p=2>1$ conv.

$$9. \sum_{n=2}^{\infty} \frac{1}{n + n\cos^2 n} \qquad 10. \sum_{n=2}^{\infty} \frac{\cos\left(\frac{n}{2}\right)}{n^2+4n} \qquad 11. \sum_{n=2}^{\infty} \frac{\sin\frac{1}{n}}{\sqrt{n}} \qquad 12. \sum_{n=2}^{\infty} \frac{\arctan n}{n\sqrt{n}}$$

$$13. \sum_{n=2}^{\infty} \frac{3^n n^2}{n!} \qquad 14 \sum_{n=2}^{\infty} \frac{2^n n!}{(n+2)!} \qquad 15. \sum_{n=2}^{\infty} \frac{n^2+1}{5^n} \qquad 16. \sum_{n=2}^{\infty} \frac{n!}{e^{n^2}}$$

$$17. \sum_{n=2}^{\infty} (\sqrt[n]{2} - 1)$$

From Series_Part4

4. $\sum_{n=1}^{\infty} \frac{n^2}{(n+2)(n-2)}$

5. $\sum_{n=1}^{\infty} \frac{4^n}{3(n+1)^{n-1}}$

6. $\sum_{n=1}^{\infty} \frac{\cos^2 n + 1}{n^2 + 1}$

7. $\sum_{n=1}^{\infty} \frac{n + 2^{-n}}{n2^n - 1}$

8. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{\ln n}}$

9. $\sum_{n=1}^{\infty} \frac{1}{\ln (n^2)}$

10. $\sum_{n=1}^{\infty} \frac{n}{e^n}$

11. $\sum_{n=1}^{\infty} \frac{1}{e^n}$

12. $\sum_{n=1}^{\infty} \frac{e^n}{n}$

13. $\sum_{n=1}^{\infty} \frac{\sin(n) + 2^n}{n + 5^n}$

14. $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n}\right)^n$