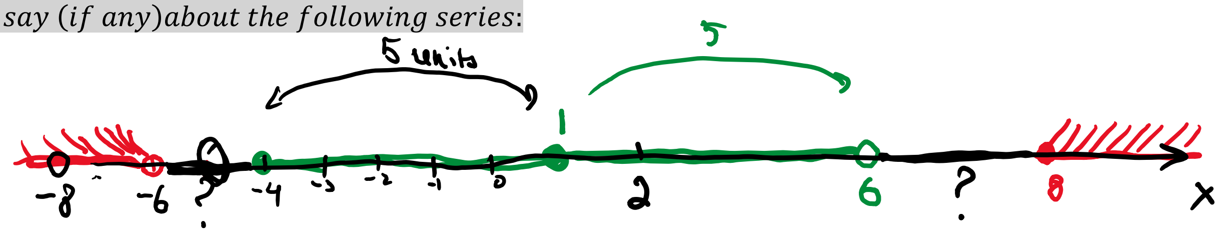
$Power\ \textbf{Series}\ (\textbf{part2})$

Example from Part1

$Suppose\ \sum_{n=1}^{\infty} c_n(x-1)^n\ converges\ for\ x=-4\ and\ diverges\ for\ x=8$

$What\ can\ you\ say\ (if\ any) about\ the\ following\ series:$



$$\sum_{n=1}^{\infty} c_n = \sum_{n=1} c_n(2-1)^n = \sum_{n\geq 1}^{\infty} c_n 1^n = \sum c_n = \text{conver. absolutely}$$

$x=2$

$$\sum_{n=1}^{\infty} c_n(-1)^n 9^n = \sum c_n(-9)^n = \sum c_n(-8-1)^n = \text{diverging.}$$

$x=-8$

$$\sum_{n=1}^{\infty} c_n(-6)^n = \sum c_n(-5-1)^n =$$

it can be conv \ diverging
we can not say.

$x=-5$

A power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ ***determines a function*** $f(x)$ ***whose domain is the interval of convergence of the series.***

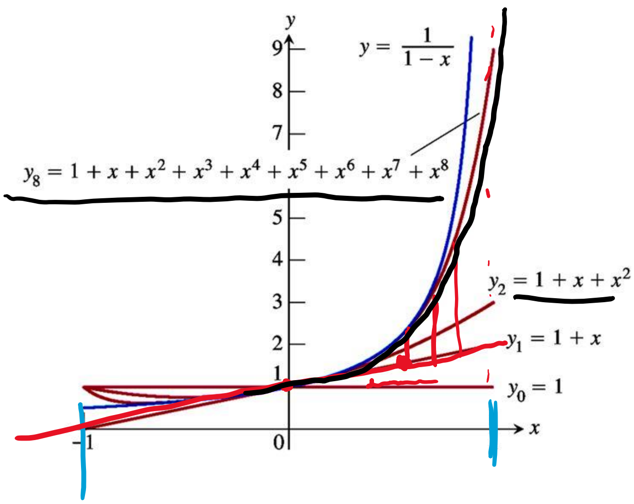
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad when\ |r| < 1$$

geom. series

$$\underbrace{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}, \quad when\ |x| < 1 \quad or \quad \boxed{f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, when\ |x| < 1}$$

$$\frac{1}{1-x} = \underbrace{1 + x + x^2 + x^3 + \ldots}_{\text{polynomial}}$$

The graph of f(x) = 1/(1-*x*) and four of its polynomial approximations

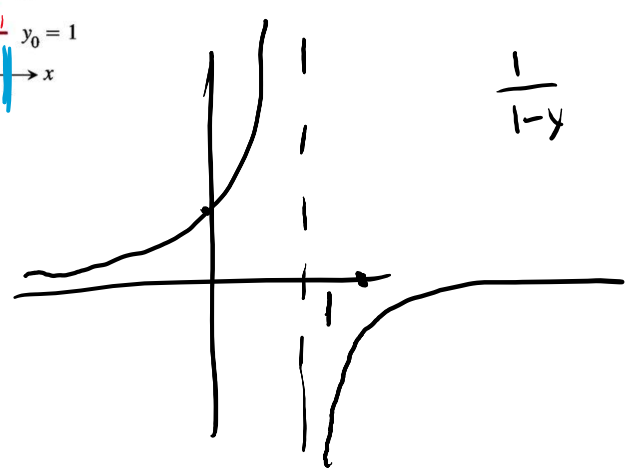


$N=0$

$$\frac{1}{1-x} \sim \sum_{n=0}^{0} = 1, \quad |x|<1$$

$$\frac{1}{1-x} \sim \sum_{n=0}^{1} = 1 + x$$

$$\frac{1}{1-x} = \sum_{n=0}^{2} = 1 + x + x^2$$



$\frac{1}{1-y}$

$$\boxed{\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}, \quad |t|<1}$$

***Example* #1**

Use a power series representation for $\frac{1}{1-x}$ *to obtain a power series representation for:*

1. $f(x) = \dfrac{\mathbf{1}}{\mathbf{1}+x}$

$$\frac{1}{1+x} = \frac{1}{1-\underbrace{(-x)}_{t}} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$, |t| = \underbrace{|-x|<1} \longrightarrow |x|<1$$

$$\underline{\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x|<1}$$

2. $f(x) = \dfrac{\mathbf{4}}{\mathbf{1}+x^2}$

$$\frac{4}{1+x^2} = 4\cdot\left(\frac{1}{1-\underbrace{(-x^2)}}\right) = 4\sum_{n=0}^{\infty} t^n = 4\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} 4(-1)^n x^{2n}$$

$$|t| = |-x^2|<1 \longrightarrow |x^2| = x^2 < 1$$

$$-1 < x < 1$$

$$\sum_{n=0}^{\infty} (-1)^n 4x^{2n} = \frac{4}{1+x^2}, \quad -1<x<1$$

$$\boxed{\frac{1}{1-t} = \sum_{n=0} t^n, \quad |t|<1}$$

3. $f(x) = \dfrac{2}{3-x}$

$$\frac{2}{3-x} = 2\left(\frac{1}{3-x}\right) = 2\left(\frac{1}{3\left(1-\frac{x}{3}\right)}\right) = \frac{2}{3}\left(\frac{1}{1-\underbrace{\frac{x}{3}}_{t}}\right) = \frac{2}{3}\sum_{n=0}^{\infty} t^n =$$

$$|t| = \left|\frac{x}{3}\right| < 1$$

$$= \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{2}{3^{n+1}}x^n$$

$$\boxed{\frac{2}{3-x} = \sum \frac{2x^n}{3^{n+1}},} \quad \left|\frac{x}{3}\right|<1, \Rightarrow \quad |x|<3$$

or

$$\boxed{-3<x<3}$$

4. $f(x) = \dfrac{x}{1+2x^2}$

$$\frac{x}{1+2x^2} = x\left(\frac{1}{1-\underbrace{(-2x^2)}_{t}}\right) = x\sum_{n=0} t^n = x\sum_{n=0}^{\infty}(-2x^2)^n = \sum(-2)^n x^{2n+1}$$

$$|t| = |-2x^2| < 1 \longrightarrow \quad 2x^2<1$$

$$x^2 < \frac{1}{2}$$

$$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

$$\boxed{\sum(-2)^n x^{2n+1} = \frac{x}{1+2x^2}}$$

$$\sum_{n=0} t^n = \frac{1}{1-t} \quad , \quad |t|<1$$

5. *Find the interval of convergence for* $\sum_{n=0}^{\infty} \boldsymbol{7^n x^n}$ *and the sum of the series as a function* $f(x)$

$$\sum_{n=0} 7^n x^n = \sum_{n=0} (\underbrace{7x}_{t})^n = \frac{1}{1-7x} \, ,$$

$|7x|<1$

$|x|<\frac{1}{7}$

$\left(-\frac{1}{7}, \frac{1}{7}\right)$

6. *Find the interval of convergence for* $\sum_{n=0}^{\infty} (\boldsymbol{\ln x})^n$ *and the sum of the series as a function* $f(x)$

$$\boxed{\sum_{n=0} (\underbrace{\ln x}_{t})^n = \frac{1}{1-\ln x} \qquad \left(\frac{1}{e}, e\right)} \, ,$$

$|\ln x|<1$

$-1<\ln x<1$

$\frac{1}{e}<x<e$

$$\text{Suppose that a series } \sum c_n x^n \text{ converges on } (-r, r), \text{ and}$$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ for every } x \text{ in the interval of convergence}$$

$(c_n x^n)'_x = c_n \cdot n\, x^{n-1}$

$$\text{Then, } \quad f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}, \quad \text{for } -r < x < r$$

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} + C, \quad \text{for } -r < x < r$$

$\sum_{n=0}^{\infty} c_n (x-1)^n$

$\sum c_n n\, 3^{n-1}$

$x=4$

$\sum c_n (4-1)^n = \sum_{n=0}^{\infty} c_n \cdot 3^n$ abs. conv $\Rightarrow$

$\left(\sum c_n (x-1)^n\right)'$ also conv. absolutely for $x=4$

$\sum c_n (n) (x-1)^{n-1} \Big|_{x=4} = \sum c_n n\, 3^{n-1}$ conv. abs

$$\sum t^n = \frac{1}{1-t}, \quad |t|<1$$

Example #2

Find a power series representation for the function $f(x)$:

1. $f(x) = \frac{4}{(1+x)^2}$

almost

$$\frac{4}{(1+x)^2} = 4\left(\frac{1}{(1+x)^2}\right) = 4\left(\frac{1}{1+x}\right)'(-1) = -4\left(\frac{1}{1-(-x)}\right)' =$$

$$= -4\left(\sum_{n=0}^{\infty} (-x)^n\right)' = -4\left(\sum\left((-1)^n x^n\right)'\right) = -4\sum_{n=0}^{\infty} (-1)^n n\, x^{n-1}$$

$|-x|<1$

$-1<x<1$

$$= \sum_{n=1} (-4)(-1)^n n x^{n-1} = \sum_{n=1} (-1)^{n+1} 4n\, x^{n-1} \qquad -1<x<1$$

2. $f(x) = \ln(1+x)$

how we can get $\ln(1+x)$

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx + C =$$

$|x|<1$ (see Ex#1)

$$= \int\left(\sum_{n=0}^{\infty} (-x)^n\right)dx + C = \sum_{n=0}^{\infty}\left(\int (-1)^n x^n\,dx\right) + C =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C,$$

$x=0$

$\ln(1+0) = \sum (-1)^n \frac{0}{n+1} + C$

$0 = 0 + C$

$\boxed{C=0}$

$$\boxed{\ln(1+x) = \sum (-1)^n \frac{x^{n+1}}{n+1} + \cancel{C}}$$

$-1<x<1$

3. $f(x) = \arctan(x)$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$ see before

$$\left(\frac{1}{1+x^2} = \sum (-x^2)^n = \sum (-1)^n x^{2n},\ -1<x<1 \right)$$

$$\arctan x = \int \frac{1}{1+x^2}\,dx + C =$$

$$= \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx + C =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$-1<x<1$, $x=0$

$$\arctan 0 = \sum(0) + C$$

$$0 = 0 + C \Rightarrow C = 0$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} \qquad -1<x<1}$$

4. *Use the power series to find the sum* $\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n}$

$$x = \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n} \stackrel{?}{=} f(x)$$

$$\left(\sum_{n=1} \frac{x^n}{n} \right)' \stackrel{?}{=} (f(x))'$$

$$\sum_{n=1} \left(\frac{x^n}{n} \right)' \stackrel{?}{=} f'(x)$$

$$\sum_{n=1} \frac{1}{\cancel{n}} \cdot \cancel{n}\, x^{n-1} \stackrel{?}{=} f'(x)$$

$$\sum_{n=1} x^{n-1} \stackrel{?}{=} (f'(x))'$$

$m = n-1$

$n = m+1$

$n=1 \Rightarrow m=0$

$$\underbrace{\sum_{m=0}^{\infty} x^m}_{} \stackrel{?}{=} f'(x)$$

$-1 < x < 1$

$x = \frac{1}{2}$

$$\frac{1}{1-x} \stackrel{?}{=} f'(x)$$

$$f(x) = \int \frac{1}{1-x} dx = -\ln(1-x) + C$$

$$f(x) = -\ln(1-x) + C$$

$f(x) = \sum \frac{1}{n} x^n$

$f(0) = 0$

$0 = -\underbrace{\ln(1-0)}_{0} + C$

$C = 0$

$$\underline{f(x) = -\ln(1-x)}$$

$$f(\tfrac{1}{2}) = -\ln(1-\tfrac{1}{2}) = -\ln(\tfrac{1}{2})$$

$$f(\tfrac{1}{2}) = -(\underbrace{\ln 1}_{0} - \ln 2) = \ln 2$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(\frac{1}{2})^n}{n} = \ln 2}$$

$\cancel{f(-5) = \sum \frac{(-5)^n}{n} = -\ln(1+5)}$

$-5 \notin (-1, 1)$