$Taylor\ \ Series\ (Part\ 3)$

$1.$ *Binomial Series*

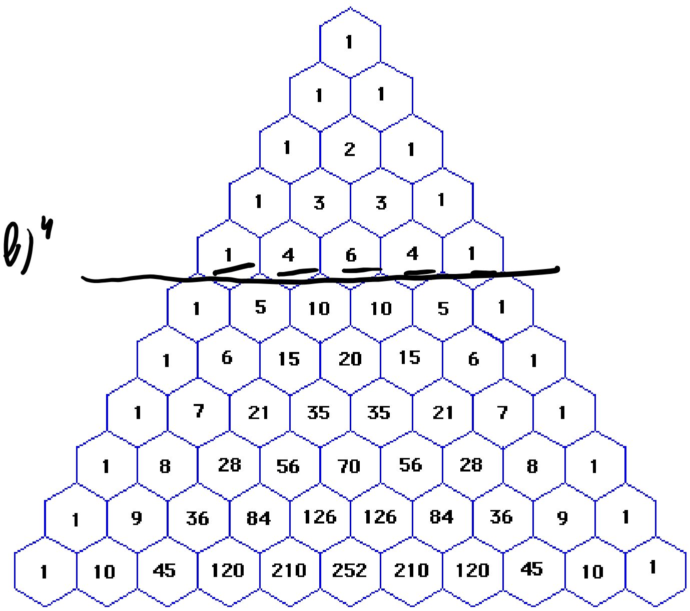
We know $(a+b)^1 = a+b$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

1 *question*: $\boxed{(a+b)^k = ?\ \ for\ any\ positive\ integer\ k}$

$(a+b)^4$



Pascal's Triangle finds coefficients for

$(a+b)^k, for\ k > 0\ integer$

$$\boxed{(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n}$$

$$\boxed{\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}}$$

or $\boxed{\binom{k}{n} = \frac{k!}{(k-n)!\,n!} \qquad n \le k}$ ✓

$n =$

$$\boxed{\binom{k}{0} = 1,\quad \binom{k}{1} = k,\quad \binom{k}{2} = \frac{k!}{(k-2)!\,2!} = \frac{(k-1)k}{2},\quad \dots \quad \binom{k}{k} = 1,}$$

$$(1+x)^{k} = \sum_{n=0}^{\cancel{\infty}\,k} \binom{k}{n} x^n = 1 + \sum_{n=1}^{\cancel{\infty}\,k} \binom{k}{n} x^n$$

$\binom{k}{n} = \frac{k!}{(k-n)!\,n!}$, $n \le k$: $\binom{k}{0} = 1$, $\binom{k}{1} = k$, $\binom{k}{2} = \frac{k!}{(k-2)!\,2!} = \frac{(k-1)k}{2}$, ... $\binom{k}{k} = 1$,

Example #1

(a)

$$(1+x)^4 = 1 + \sum_{n=1}^{4} \binom{4}{n} x^n$$

$\binom{4}{1} = \boxed{4}$

$\binom{4}{2} = \frac{4!}{(4-2)!\cdot 2!} = \frac{24}{2\cdot 2} = \boxed{6}$

$\binom{4}{3} = \frac{4!}{(4-3)!\,3!} = \frac{24}{1!\cdot 6} = \boxed{4}$

$\binom{4}{4} = \frac{4!}{0!\cdot 4!} = \boxed{1}$

$$(1+x^4) = 1 + 4x + 6x^2 + 4x^3 + x^4$$

$(1+X)^6$

(b)

$$(1-5x)^6 = \Big(1 + \underbrace{(-5x)}_{X}\Big)^6 = 1 + \sum_{n=1}^{6} \binom{6}{n} X^n = 1 + \sum_{n=1}^{6} \binom{6}{n} (-5x)^n =$$

$$= 1 + \sum_{n=1}^{6} \binom{6}{n} (-1)^n 5^n x^n = 1 - 6\cdot 5x + 15\cdot 5^2 x^2 - 20\cdot 5^3 x^3 + 15\cdot 5^4 x^4 - 6\cdot 5^5 x^5 + 5^6 x^6$$

$\binom{k}{n} = \frac{k!}{(k-n)!\,n!}$, $n \le k$: $\binom{k}{0} = 1$, $\binom{k}{1} = k$, $\binom{k}{2} = \frac{k!}{(k-2)!\,2!} = \frac{(k-1)k}{2}$, ... $\binom{k}{k} = 1$,

$n=1$ $\binom{6}{1} = 6$

$n=2$ $\binom{6}{2} = \frac{6!}{4!\cdot 2!} = \frac{(6-1)\cdot 6}{2} = 15.$

$n=3$ $\binom{6}{3} = \frac{6!}{3!\cdot 3!} = \frac{\cancel{3!}\cdot 4\cdot 5\cdot \cancel{6}}{\cancel{3!}\cdot \cancel{6}} = 20$

$n=4$ $\binom{6}{4} = \frac{6!}{2!\cdot 4!} = \frac{\cancel{4!}\cdot 5\cdot 6}{2\cdot \cancel{4!}} = 15$

$n=5$ $\binom{6}{5} = \frac{6!}{1!\cdot 5!} = 6$

$n=6$ $\binom{6}{6} = 1$

(c)

$$\left(2+\frac{x}{3}\right)^{4} = \left(2\left(1+\frac{x}{6}\right)\right)^{4} = 2^4\left(1+\underbrace{\frac{x}{6}}_{X}\right)^{4} = 2^4\left(1+\sum_{n=1}^{\cancel{\infty}\,4}\binom{4}{n}X^n\right) = 2^4\left(1+\sum_{n=1}^{\cancel{\infty}\,4}\binom{4}{n}\left(\frac{x}{6}\right)^n\right)$$

$$(\quad)^4 = 1 \qquad \underset{\binom{4}{1}}{\underline{4}} \qquad \underset{\binom{4}{2}}{\underline{6}} \qquad \underset{\binom{4}{3}}{\underline{4}} \qquad \underset{\binom{4}{4}}{\underline{}}\,1$$

The question: $\boxed{(a+b)^k = ?\ \ for\ any\ k}$

Newton expanded the binomial theorem so that k did not have to be a positive integer:

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1+\sum_{n=1}^{\infty}\binom{k}{n}x^n}$$

converges for $|x|<1$, *for any real number k*: $\boxed{\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}}$

$$\boxed{\binom{k}{0} = 1,\quad \binom{k}{1} = k,\quad \binom{k}{2} = \frac{k(k-1)}{2}}$$

!!The expansion for positive integer powers terminates, i.e. it has only a finite number of terms. However, for powers that are not positive integers the series is an infinite series that goes on forever.

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-(n-1))}{n!}: \quad \binom{k}{0} = 1, \quad \binom{k}{1} = k, \quad \binom{k}{2} = \frac{k(k-1)}{2}$$

Example #2

(a)

$k = -2$

$$(1-x)^{-2} = 1 + \sum_{n=1}^{\infty} \binom{-2}{n} x^n$$

$n=1 \quad \binom{-2}{1} = -2$

$n=2 \quad \binom{-2}{2} = \frac{(-2)(-2-1)}{2!} = \frac{(-2)(-3)}{2!} = \frac{(-1)^2 \cdot 2 \cdot 3}{2!}$

$n=2$

$n-1 = \underline{\underline{1}}$

$n=3 \quad \binom{-2}{3} = \frac{(-2)(-2-1)(-2-2)}{3!} = \frac{(-2)(-3)(-4)}{3!} = \frac{(-1)^3 \cdot 2 \cdot 3 \cdot 4}{3!}$

$n=3$

$n-1=2$

$\vdots$

$$\binom{-2}{n} = \frac{(-1)^n (n+1)!}{n!} = (-1)^n (n+1)$$

$$(1-x)^{-2} = (1+(-x))^{-2} = 1 + \sum \underbrace{(-1)^n (n+1)}_{\binom{k}{n}} \cdot (-x)^n = \boxed{1 + \sum_{n=1} (n+1) x^n}$$

(b)

Expand $\sqrt{\mathbf{1+2x}}$ *and state for what values of x the series is valid*

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-(n-1))}{n!}: \ \binom{k}{0} = 1, \quad \binom{k}{1} = k, \quad \binom{k}{2} = \frac{k(k-1)}{2}$$

$$\sqrt{(1+2x)} = (1+\underbrace{2x}_{X})^{\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (X)^n = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} 2^n \cdot x^n =$$

$$\approx 1 + \frac{1}{2}\cdot 2\cdot x - \frac{1}{8}\cdot 2^2 x^2 + \frac{1}{16}\cdot 2^3 x^3 - \frac{5}{16\cdot 8}\cdot 2^4 x^4 + \dots$$

$n=1 \quad \binom{\frac{1}{2}}{1} = \frac{1}{2}$ ✓

$n=2 \quad \binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} = \frac{\frac{1}{2}(-\frac{1}{2})}{2} = -\frac{1}{8}$

$n=3 \quad \binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} = \frac{1}{16}$

$n=4 \quad \binom{\frac{1}{2}}{4} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!} = \frac{-5}{16\cdot 8}$

$\vdots$

Extra Examples:

1.

Expand $\left(1-\frac{x}{2}\right)^{-5}$. For what values of x is the expansion valid?

Solution.

$$\begin{aligned}\left(1-\frac{x}{2}\right)^{-5} &= 1+(-5)\left(-\frac{x}{2}\right)+\frac{(-5)(-6)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-5)(-6)(-7)}{3!}\left(-\frac{x}{2}\right)^3+\cdots \\ &= 1+\frac{5}{2}x+\frac{15}{4}x^2+\frac{35}{8}x^3+\cdots\end{aligned}$$

This is valid when $-1 < -\frac{x}{2} < 1$, i.e. when $-2 < x < 2$.

2.

Expand $(3+x)^{-\frac{1}{2}}$.

Solution. Remember that when the power is not a positive integer your expression has to be of the form $(1+\text{something})^{\text{power}}$. Deal with this as follows:

$$(3+x)^{-\frac{1}{2}} = \left(3\left(1+\frac{x}{3}\right)\right)^{-\frac{1}{2}} = 3^{-\frac{1}{2}}\underbrace{\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}}_{\text{expand this}}$$

$$\begin{aligned}&= 3^{-\frac{1}{2}}\left(1+(-\frac{1}{2})(\frac{x}{3})+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(\frac{x}{3})^2+\cdots\right) \\ &= \frac{1}{\sqrt{3}}\left(1-\frac{x}{6}+\frac{x^2}{24}+\cdots\right)\end{aligned}$$

This is valid when $-1 < x/3 < 1$, i.e. when $-3 < x < 3$.

3.

Find expansions for $\left(1+\frac{1}{x}\right)^{1/2}$ for the cases (i) $|x|>1$ and (ii) $0<x<1$.

Solution. the following calculation produces an expansion which will be valid when $1/|x|<1$, i.e. $|x|>1$:

$$\begin{aligned}\left(1+\frac{1}{x}\right)^{1/2} &= 1+\frac{1}{2}\left(\frac{1}{x}\right)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(\frac{1}{x}\right)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\left(\frac{1}{x}\right)^3+\cdots \\ &= 1+\frac{1}{2x}-\frac{1}{8x^2}+\frac{1}{16x^3}+\cdots\end{aligned}$$

valid for $|x|>1$.

The above expansion is no good if $|x|<1$. For this case the following trick produces a valid expansion:

$$\begin{aligned}\left(1+\frac{1}{x}\right)^{1/2} &= \left(\frac{x+1}{x}\right)^{1/2}=\frac{1}{x^{1/2}}\underbrace{(1+x)^{1/2}}_{\text{expand this}} \\ &= \frac{1}{x^{1/2}}\left(1+\frac{1}{2}x+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3+\cdots\right) \\ &= \frac{1}{x^{1/2}}\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots\right) \\ &= \frac{1}{x^{1/2}}+\frac{1}{2}x^{1/2}-\frac{1}{8}x^{3/2}+\frac{1}{16}x^{5/2}+\cdots\end{aligned}$$

Note that this is actually defined only for $0<x<1$.

2. *Important Maclaurin Series to Memorize*!

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)} + \cdots = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}, \quad |x| \leq 1$$

Example #2.1

Consider the function $f(x) = xe^{-x^2}$

a. Find the Maclaurin Series for $f(x)$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$$

$$t = -x^2$$

$$f(x) = x \cdot e^{-x^2} = x \cdot \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}} =$$

$$= \underbrace{x - x^3} + \frac{x^5}{2!} - \frac{x^7}{3!} + \ldots$$

b. Find $P_3(x)$

$$P_3(x) = x - x^3$$

(If asked here about $P_4(x) = x - x^3$)

c. Find a Maclaurin series for $\int xe^{-x^2}\,dx$

$$\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\int x^{2n+1}\,dx\right) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+2}}{2n+2} + C}$$

$$= \overbrace{\frac{1}{2}}^{n=0} x^2 - \overbrace{\frac{1}{4} x^4}^{n=1} + \overbrace{\frac{1}{2!} \cdot \frac{x^6}{6}}^{n=2} - \frac{1}{3!}\frac{x^8}{8} + \frac{1}{4!}\frac{x^{10}}{10} - \ldots \quad C$$

d. Estimate the integral $\int_0^1 xe^{-x^2}\,dx$ *with an error of magnitude less than* 0.01 $\left(\frac{1}{100}\right)$ *by using as few terms as possible*

alt

$$|\text{Error}| = |S - P_n| \le |a_{n+1}|$$

$$\int_0^1 xe^{-x^2}\,dx \approx \boxed{\frac{1}{2} - \frac{1}{4} + \frac{1}{12} - \frac{1}{48}} + \frac{1}{240} - \ldots$$

$$\frac{1}{240} \le \frac{1}{100}$$

$$\int_0^1 xe^{-x^2}dx \approx \frac{\overset{12}{\overbrace{24-12}}+4-1}{48} = \frac{15}{48} \approx 0.3$$

$e.\ Now\ integrate \int_0^1 xe^{-x^2}dx\ \ directly$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}\,du = x\,dx$

$$\int_0^1 xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}\Big|_0^1 =$$

$$= -\frac{1}{2}\left(e^{-1}-1\right) = \underbrace{-\frac{1}{2}\left(\frac{1}{e}-1\right)}_{0.31} > 0$$

Example **#2.2**

$Consider\ the\ function\ f(x) = xe^{-x^5}$

$a.\ Find\ the\ Maclaurin\ Series\ for\ f(x)$

$b.\ Use\ part(a) to\ find\ the\ Series\ for \int_0^{0.1} xe^{-x^5}dx$

$c.\ Approximate\ the\ value\ of\ the\ integral\ from\ part(b) with\ an\ error\ of\ absolute\ value\ less\ than\ 10^{-8}$

Example#2.3

Consider the function $f(x) = x^2 \sin(x^2)$

a. Find the Maclaurin Series for $f(x)$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots = \sum_{n=0} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$x^2 \sin(x^2) = x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{\overbrace{2n+1}^{4n+2}}}{(2n+1)!} = \boxed{\sum_{n=0} \frac{(-1)^n x^{4n+4}}{(2n+1)!}} =$$

$$= x^4 - \frac{x^8}{3!} + \frac{x^{12}}{5!} - \frac{x^{16}}{7!} + \dots$$

b. Use part(a) to find the Series for $\int_0^{0.1} x^2 \sin(x^2)dx$

$$\int_0^{\frac{1}{10}} x^2 \sin x^2 dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\int_0^{\frac{1}{10}} x^{4n+4} dx \right) =$$

$$= \sum \frac{(-1)^n}{(2n+1)!} \frac{x^{4n+5}}{(4n+5)} \Bigg|_0^{\frac{1}{10}} = \sum_{n=0} \frac{(-1)^n}{(2n+1)!(4n+5)10^{4n+5}}$$

c. Suppose that first two terms in the series in part (b) are used to approximate the value of the integral. Bound the magnitude of the error of this aproximation.

$$\int_0^{\frac{1}{10}} x^2 \sin x^2 dx = \underbrace{\frac{1}{5 \cdot 10^5} - \frac{1}{6 \cdot 9\, 10^9}}_{P_N} + \frac{1}{5!\, 13\, 10^{13}}$$

$$\boxed{|E| \leq \frac{1}{5!(13)10^{13}}}$$

***Example* #2.4**

Consider the function $f(x) = x\cos(x)$

a. Find the Maclaurin Series for $f(x)$

$$x\cdot\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!} =$$

$$= x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\cdots\right) = x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\ldots$$

b. Use part(a) to find the first two nonzero terms in Maclaurin Series for

$g(x) = x\cos(x) - \sin(x)$

$\sin x$

$$g(x) = \left(\cancel{x}-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\ldots\right)-\left(\cancel{x}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\right)$$

$$x^3\left(-\frac{1}{2!}+\frac{1}{3!}\right)+x^5\left(\frac{1}{4!}-\frac{1}{5!}\right)+\ldots =$$

$$= -\frac{1}{3}x^3+\left(\frac{1}{24}-\frac{1}{120}\right)x^5+\ldots = \boxed{-\frac{1}{3}x^3+\left(\frac{1}{30}x^5\right.}+\ldots$$

c. Use the result from part b to evaluate the $\lim_{x\to 0}\frac{x\cos(x)-\sin(x)}{x^3}$

$$\lim_{x\to 0}\frac{\left(-\frac{1}{3}x^3+\frac{1}{30}x^5\right)+(\ )x^7+\ldots}{x^3} = \lim\left(-\frac{1}{3}+\frac{1}{30}x^2\right)+(\ )x^4+\ldots$$

$0 \quad 0$

$$= \boxed{-\frac{1}{3}}$$

Check

$$\left(\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} = \frac{0}{0} = \lim \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{3x^2} = \right.$$

$$\left. = \lim_{x\to 0} \frac{-x\sin x}{3x^2} = \lim_{x\to 0} -\frac{\sin x}{3\,x} = \boxed{-\frac{1}{3}} \right)$$

$$\frac{\sin x}{x} \downarrow 1$$

Example **#2.5**

Consider the function $f(x) = \ln(x)$

a. Find the Taylor Series for $f(x)$ *around* $a = 1$

$$\ln(1+t) = \sum_{n=0} (-1)^{n-1} \frac{t^n}{n}$$

$$\ln x = \ln(x+1-1) = \ln(1 + \underbrace{(x-1)}_{t}) = \boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n} =}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$

b. Use part (a) to find the first two nonzero terms in Taylor Series for

$$g(x) = \frac{\ln(x)}{x-1} = \frac{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots}{(x-1)} = \underbrace{\boxed{1 - \frac{1}{2}(x-1)} + \frac{1}{3}(x-1)^2 + \dots}$$

c. Use the result from part b to evaluate the $\lim_{x\to 1} \frac{\ln(x)}{x-1}$ $\left(= \frac{0}{0} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \boxed{1} \right)$

$$\lim_{x\to 1} 1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 + \dots = \boxed{1}$$

$$\frac{1}{2}(x-1) \downarrow 0 \qquad \frac{1}{3}(x-1)^2 \downarrow 0 \qquad \dots \downarrow 0$$

$\boldsymbol{Example}$ **#2.6**

$Find\ the\ Taylor\ expansion\ about\ x = 3\ and\ its\ radius\ and\ interval\ of\ convergence\ for$

$$f(x) = \frac{2x - 6}{x}$$

Example **#2.7**

$Consider\ f(x) = \dfrac{3}{1 - x^2} \quad Use\ Maclaurin\ Series\ to\ find\ f^{(13)}(0)\ and\ f^{(16)}(0)$

***Example* #2.8**

Consider the function $f(x) = sinx - tanx$

$a.$ *Find the Maclaurin Series for* $f(x)$

$b.$ *Use the result from part* (a) *to evaluate the* $\lim_{x\to 0}\frac{\sin(x) - tanx}{x^3}$

***Example*#2.9**

Use Maclaurin Series to evaluate

$$\lim_{x\to 0}\frac{1}{sinx} - \frac{1}{x}$$