$\boldsymbol{Series \quad (Part3)}$

$\boldsymbol{What\ we\ know\ so\ far}$:

part 1

$\mathbf{1.}$ $\boldsymbol{Method\ of\ Partial\ Sums}$: $Works\ for\ very\ specific\ series,$

$called\ telescopic\ series.$

$With\ this\ method\ we\ can$ $check\ the\ convergency\ and\ find\ the\ Sum.$

$\mathbf{2.}$ $\boldsymbol{Geometric\ Series}$

$\boldsymbol{Geometric\ Series\ of\ the\ form}$
$\displaystyle\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots = \begin{cases} \text{converges to } \dfrac{1}{1-r}, & \text{if } \lvert r\rvert < 1 \\ \text{diverges}, & \text{if } \lvert r\rvert \geq 1 \end{cases}$

$With\ this\ method\ we\ can$ $check\ the\ convergency\ and\ find\ the\ Sum.$

$\mathbf{3.}$ $\boldsymbol{Divergency\ Test\ or\ N-th\ term\ test}$:

$If\ \lim\limits_{k\to\infty} a_k \neq 0\ (\ or\ \lim\limits_{k\to\infty} a_k = DNE) \quad \rightarrow \quad \displaystyle\sum_{k=1}^{\infty} a_k\ is\ divergent$

$With\ this\ method\ we\ can\ prove$ $divergency\ only$

$\boldsymbol{Positive - Term\ Tests}$: $For \sum a_n \quad such\ that\ a_n > 0\ for\ every\ n$

Part 2

$\mathbf{4.}$ $\boldsymbol{Integral\ Test}$:

$$Let \sum a_n\ be\ a\ series\ such\ that\ a_n = f(n)$$

$$and \begin{cases} f(x) > 0 \\ Continuous \\ Decreasing\ for\ any\ x \geq k \end{cases}$$

$$Then\ \textbf{the series} \sum_{n=k}^{\infty} \boldsymbol{a_n}\ \textbf{and the integral} \int_{k}^{\infty} \boldsymbol{f(x)}\ \textbf{converge or diverge together}$$

$\mathbf{5.}$ $\boldsymbol{DCT}$

$$Let \sum a_n\ and \sum b_n\ be\ a\ positive\ term\ series\ , a_n \leq b_n\ for\ every\ n,$$

1. $If \sum b_n\ converges, then \sum a_n\ converges$
2. $If \sum a_n\ diverges, then \sum b_n\ diverges$

$$\textbf{Positive} - \textbf{Term Series}: \sum a_n \quad \textit{such that } a_n > 0 \textit{ for every } n$$

$\textbf{Limit Comparison Test}$: $\textbf{LCT}$

$$\textit{Let} \sum a_n \textit{ and} \sum b_n \textit{ be a positive term series}$$

1. $\textit{If} \lim_{n\to\infty} \frac{a_n}{b_n} = l \ , 0 < l < \infty \quad \sum b_n \textit{ and} \sum a_n \textit{ converge or diverge together}$
2. $\textit{If} \lim_{n\to\infty} \frac{a_n}{b_n} = 0 \textit{ and} \sum b_n \textit{ converges} \to \sum a_n \textit{ converges too}$
3. $\textit{If} \lim_{n\to\infty} \frac{a_n}{b_n} = \infty \textit{ and} \sum b_n \textit{ diverges} \to \sum a_n \textit{ diverges}$

$\textbf{Example\#1}$

1. $\displaystyle \sum_{n=1}^{\infty} \boldsymbol{sin}\left(\frac{\mathbf{1}}{\boldsymbol{n}}\right)$

$\textit{Almost everything we learned so far will fail}$:

- Not telescopic
- Not geometric
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} sin\frac{1}{n} = \sin(0) = 0$ $\quad$ $, \textit{hence Not a Divergency Test}$ $\quad a_n \xrightarrow{?} 0$
- Integral Test (?) $\quad f(x) = \sin\frac{1}{x}$

a-b-c?

a) $\sin\left(\frac{1}{x}\right) > 0$ ✓

$\downarrow$ 0 $\quad x > 0$

b) $f(x) = \sin\frac{1}{x}$ is cont, $x \neq 0$ ✓

c) $f(x)$ is decreasing ✓

$\int_1^{\infty} f(x)\,dx$ and $\sum_1^{\infty} \sin\frac{1}{n}$ behave the same

$\int_1^\infty \sin\frac{1}{x}$

DCT for integrals

$\sin\frac{1}{x} < \frac{1}{x}$ does not work

LCT for integral

$\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{t=\frac{1}{x}\to 0} \frac{\sin t}{t} = 1 \Rightarrow \int_1^\infty \sin\frac{1}{x}$ div

by LCT test wit $\int \frac{1}{x}$ div

$\sum \sin\frac{1}{n}$ div $\Leftarrow$

- DCT ?

Will not work

$\sin\frac{1}{n} < \frac{1}{n}$

correct, but useless for divergency, and we know that $\sum \frac{1}{n}$, $p=1$ diverg

LCT: $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{\frac{1}{n}=t\to 0} \frac{\sin t}{t} = 1$

$\boxed{\sum_{n=1}^{\infty} \sin\frac{1}{n} \text{ diverges by LCT with series } \sum_{n=2}^{\infty} \frac{1}{n} \quad (p = 1 \text{ diverges})}$

This is the shortest way to solve it.

2. $\sum_{n=1}^{\infty}\left(1-2^{-\frac{1}{n}}\right)$

Let's try n-th term:

$a_n = \left(1-\underbrace{2^{-\frac{1}{n}}}_{\downarrow\, 1}\right) \longrightarrow 1-1=0$ did not work!

$1-2^{-\frac{1}{n}} \sim \frac{1}{n}$

LCT: $\lim_{n\to\infty} \dfrac{1-2^{-\frac{1}{n}}}{\frac{1}{n}} = \left(\frac{0}{0}\right) = \lim_{x\to\infty} \dfrac{1-2^{-\frac{1}{x}}}{\frac{1}{x}} = \lim \dfrac{\left(2^{-\frac{1}{x}}\right)\cdot \ln 2 \left(\cancel{\frac{1}{x}}\right)'}{\cancel{\left(\frac{1}{x}\right)'}} = \ln 2$

$0 < \ln 2 < \infty$

$\boxed{\sum_{n=1}^{\infty}\left(1-2^{-\frac{1}{n}}\right) \textit{ diverges by LCT with series } \sum_{n=2}^{\infty}\frac{1}{n} \quad (p=1 \textit{ diverges})}$

3. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{3^n}\right)$ a_n

n-th term test?

$a_n = \ln\left(1+\underset{\downarrow\, 0}{\frac{1}{3^n}}\right) \longrightarrow \ln 1 = 0$

did not work.

LCT:

$\lim_{n\to\infty} \dfrac{\ln\left(1+\frac{1}{3^n}\right)}{\frac{1}{3^n}} = \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{n\to\infty} \dfrac{\ln\left(1+\frac{1}{3^x}\right)}{\frac{1}{3^x}} = \lim \dfrac{\frac{1}{1+\frac{1}{3^x}}\cdot\cancel{\left(\frac{1}{3^x}\right)'}}{\cancel{\left(\frac{1}{3^x}\right)'}}$

$= \lim_{x\to\infty} \dfrac{1}{1+\underset{\downarrow\, 0}{\frac{1}{3^x}}} = 1$

$0 < 1 < \infty$

$$\boxed{\sum_{n=1}^{\infty} ln\left(1+\frac{1}{3^n}\right) \text{ converges by LCT with convergent geometric series } \sum_{n=1}^{\infty}\frac{1}{3^n}, \text{where } r=\frac{1}{3}<1}$$

4. $\sum_{n=1}^{\infty}\left(1-cos\frac{1}{n}\right)$

Lets review, what we know for sin(x), when x is small

$\sin x \sim x \Rightarrow$

$\sin\frac{1}{n} \sim \frac{1}{n}$

$$cosA - cosB = -2sin\frac{A+B}{2}sin\frac{A-B}{2}$$

$A=0$, $\cos 0 = 1$, $B=\frac{1}{n}$

odd

$$\underline{1-\cos\frac{1}{n}} = -2\sin\left(\frac{0+\frac{1}{n}}{2}\right)\sin\left(\frac{0-\frac{1}{n}}{2}\right) = -2\sin\left(\frac{1}{2n}\right)(-1)\sin\left(\frac{1}{2n}\right) = 2\sin\left(\frac{1}{2n}\right)\sin\left(\frac{1}{2n}\right)$$

$$1-\cos\frac{1}{n} = 2\sin\left(\frac{1}{2n}\right)\cdot\sin\left(\frac{1}{2n}\right) \sim 2\left(\frac{1}{2n}\right)\cdot\left(\frac{1}{2n}\right) = \frac{1}{2}\cdot\frac{1}{n^2}$$

$$\left(1-\cos\frac{1}{n}\right) \sim \frac{1}{2}\cdot\frac{1}{n^2}$$

$$\underline{LCT} \quad \lim_{n\to\infty}\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}} = \frac{0}{0} = \lim_{x=\frac{1}{n}\to 0}\frac{1-\cos x}{x^2} \overset{LH}{=} \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}$$

$$0 < \frac{1}{2} < \infty$$

$$\boxed{\sum_{n=1}^{\infty}\left(1-cos\frac{1}{n}\right) \text{ converges by LCT with series } \sum_{n=1}^{\infty}\frac{1}{n^2} \quad (p=2>1 \text{ converges})}$$

5. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$

$$\frac{n}{\sqrt{n^3+1}} \sim \frac{n}{\sqrt{n^3}} = \frac{1}{\sqrt{n}}$$

LCT: $$\lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n\sqrt{n}}{\sqrt{n^3+1}} = \lim \sqrt{\frac{n^3}{n^3+1}} = \lim \sqrt{\frac{1}{1+\underbrace{\frac{1}{n^3}}_{0}}} = 1$$

$0 < 1 < \infty$ It worked!

$$\boxed{\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}} \text{ diverges by LCT with series } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \left(p = \frac{1}{2} < 1 \text{ diverges}\right)}$$

If using DCT: $$\frac{n}{\sqrt{n^3+1}} > \frac{n}{\sqrt{2n^3}} = \frac{1}{\sqrt{n}\sqrt{2}}$$, so our series will diverge by DCT with $\frac{1}{\sqrt{2}}\sum \frac{1}{\sqrt{n}}$ $p = \frac{1}{2} < 1$ divergent.

$$\sqrt{n^3+1} < \sqrt{n^3+n^3}$$

6. $\sum_{n=1}^{\infty} \frac{1}{5^n - 1}$

$$\frac{1}{5^n-1} \sim \frac{1}{5^n}$$

"think" $\left(\frac{1}{5^n}\right) = \left(\frac{1}{5}\right)^n$ geom?, $r = \frac{1}{5} < 1$ conv?

LCT: $$\lim_{n\to\infty} \frac{\left(\frac{1}{5^n-1}\right)}{\left(\frac{1}{5^n}\right)} = \lim_{n\to\infty} \frac{5^n}{5^n-1} = \lim_{n\to\infty} \frac{1}{1-\underbrace{\frac{1}{5^n}}_{0}} = 1$$

$0 < 1 < \infty$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{5^n-1} \text{ converges by LCT with convergent geometric series } \sum_{n=1}^{\infty} \frac{1}{5^n}, \text{ where } r = \frac{1}{5} < 1}$$

For DCT: way #1

$5^n - 1 > e^n$ ✓

probably need to show...

$$\frac{1}{5^n-1} < \frac{1}{e^n} = \left(\frac{1}{e}\right)^n$$

$r = \frac{1}{e} < 1 \ldots$

way #2: $5^n - 1 > 5^n - \frac{5^n}{2} = \frac{5^n}{2}$ $\quad \frac{1}{5^n-1} < \frac{2}{5^n} = 2\left(\frac{1}{5}\right)^n$!

$1 < \frac{5^n}{2}$ $\qquad r = \frac{1}{5} < 1$

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7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{4n^3-5n}}$

$$\frac{1}{\sqrt{4n^3-5n}} \sim \frac{1}{\sqrt{4n^3}} = \frac{1}{2n^{3/2}}$$

? $p = \frac{3}{2} > 1$

LCT

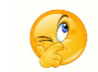
$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{4n^3-5n}}}{\frac{1}{\sqrt{4n^3}}} = \lim_{n\to\infty} \sqrt{\frac{4n^3}{4n^3-5n}} = \lim_{n\to\infty} \sqrt{\frac{1}{1-\underbrace{\frac{5}{4n^2}}_{0}}} = 1$$

$0 < 1 < \infty$!

$$\boxed{\sum_{n=2}^{\infty} \frac{1}{\sqrt{4n^3-5n}} \textit{ converges by LCT with series } \sum_{n=2}^{\infty} \frac{1}{n^{3/2}} \quad \left(p = \frac{3}{2} > 1 \textit{ converges}\right)}$$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^{7/3}-4}$

$$\frac{n^2}{n^{7/3}-4} \sim \frac{n^2}{n^{7/3}} = \frac{1}{n^{1/3}}$$



$\frac{1}{3} < 1$ ... div

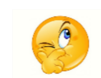
$$\text{LCT: } \lim_{n\to\infty} \frac{\left(\frac{n^2}{n^{7/3}-4}\right)}{\frac{1}{n^{1/3}}} = \lim_{n\to\infty} \frac{n^2 n^{1/3}}{n^{7/3}-4} = \lim \frac{1}{\frac{n^{7/3}}{n^{7/3}} - \frac{4}{n^{7/3}}} = \lim \frac{1}{1-\frac{1}{n^{7/3}}} = 1$$

$0 < 1 < \infty$

$$\boxed{\sum_{n=2}^{\infty}\frac{n^2}{n^{7/3}-4}\ \textit{diverges by LCT with series}\ \sum_{n=2}^{\infty}\frac{1}{n^{1/3}}\quad \left(p=\frac{1}{3}<1\,\textit{diverges}\right)}$$

9. $$\sum_{n=1}^{\infty}\frac{\sin(n)+2^n}{n+5^n}$$

$$\frac{\sin(n)+2^n}{n+5^n}\sim\frac{2^n}{5^n}=\left(\frac{2}{5}\right)^n$$



$r=\frac{2}{5}<1$

LCT:

$$\lim_{n\to\infty}\frac{\dfrac{\sin n+2^n}{n+5^n}}{\left(\dfrac{2^n}{5^n}\right)}=\lim_{n\to\infty}\frac{5^n\sin n+10^n}{2^n n+10^n}=\lim\frac{\underbrace{\dfrac{\sin n}{2^n}}_{\text{sandwich}\to 0}+1}{\underbrace{\dfrac{n}{5^n}}_{\text{L'H}\to 0}+1}=1$$

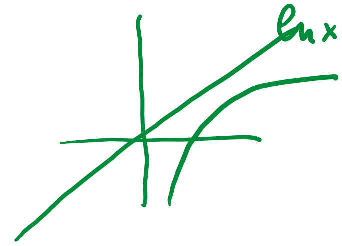
$0<1<\infty$

$$\boxed{\sum_{n=1}^{\infty}\frac{\sin(n)+2^n}{n+5^n}\ \textit{converges by LCT with convergent geometric series}\ \sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n,\textit{where}\ r=\frac{2}{5}<1}$$

DCT

$$\frac{\sin n+2^n}{n+5^n}\le\frac{2^n+2^n}{n+5^n}\le\frac{2\cdot 2^n}{5^n}=2\cdot\left(\frac{2}{5}\right)^n$$

$\ln x < x$



$\ln n < n$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \ln n}$

$$\frac{1}{\sqrt{n} + \ln n} \sim \frac{1}{n}$$

<u>LCT</u> $$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}+\ln n} = \lim \frac{1}{\underset{\downarrow 0}{\frac{1}{\sqrt{n}}} + \underset{\downarrow 0}{\frac{\ln n}{n}}} = \infty$$

$\sum \frac{1}{n}$ diverges => „∞" will work!

$$\lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty} = L'H_{rule} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

(for „fun" check if you can compare it with $\sum \frac{1}{\sqrt{n}}$!)

Remeber !!!! 3. ***If*** $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ ***and*** $\sum b_n$ ***diverges*** $\rightarrow \sum a_n$ ***diverges***

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \ln n}$ *diverges by LCT with series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ $(p = 1$ *diverges*$)$

Try DCT : $\ln n < n$ => $\frac{1}{\sqrt{n}+\ln n} > \frac{1}{\sqrt{n}+n} > \frac{1}{n+n} = \frac{1}{2n}$

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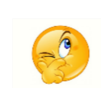
11. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$

$$\frac{3n+5}{n(n-1)(n-2)} \sim \frac{3n}{n(n)(n)} = \frac{3}{n^2}$$



$$\lim_{n\to\infty} \frac{\frac{3n+5}{n(n-1)(n-2)}}{\frac{3}{n^2}} = \lim_{n\to\infty} \frac{(3n+5)n^2}{n(n-1)(n-2)3} = \lim \frac{3n^3+5n^2}{n(n-1)(n-2)} = \lim \frac{3+\overset{\nearrow 0}{\frac{5}{n}}}{\underset{1}{\frac{n}{n}} \cdot \underset{\downarrow 1}{\left(\frac{n-1}{n}\right)} \cdot \underset{\downarrow 1}{\left(\frac{n-2}{n}\right)}} = 3$$

$0 < 3 < \infty$

$\boxed{\sum_{n=1}^{\infty} \frac{\mathbf{3n+5}}{\mathbf{n(n-1)(n-2)}} \text{ converges by LCT with series } \sum_{n=2}^{\infty} \frac{1}{n^2} \quad (p = 2 > 1 \text{ converges})}$

Mixed Problems

1. $\sum_{n=1}^{\infty} sin\frac{1}{n^2}$

DCT $\quad \sin\frac{1}{n^2} \le \frac{1}{n^2}$

$\left.\begin{array}{l} \sin\frac{1}{n^2} \le \frac{1}{n^2} \\ \sum \frac{1}{n^2} \quad p=2>1 \text{ conv} \end{array}\right\} \to$ Convergent by DCT

2. $\sum_{n=1}^{\infty} \frac{sin^2(n)}{1+2^n}$

DCT: $\quad \frac{\sin^2(n)}{1+2^n} \le \frac{1}{1+2^n} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

$\sum$ is convergent by DCT with $\sum\left(\frac{1}{2}\right)^n$; geom. conver. $r=\frac{1}{2}<1$

3. $\sum_{n=1}^{\infty} \frac{e^n}{1+3e^n}$

n-th term test:

$a_n = \frac{e^n}{1+3e^n} = \frac{1}{\frac{1}{e^n}+3} \xrightarrow{n\to\infty} \frac{1}{3} \neq 0$

$\frac{1}{e^n} \to 0$

$a_n \not\to 0 \Rightarrow$ Series diverges by n-th term <u>test</u>

5. $(fall\ 2016, C\#3)$ $\sum_{n=1}^{\infty} \frac{n^2}{(n+2)(n-2)}$

6. $(fall\ 2016, C\#3)$ $\sum_{n=1}^{\infty} \frac{2n^2}{\sqrt[3]{n^7+n}}$

7. $(fall\ 2016, C\#3)$ $\sum_{n=1}^{\infty} \frac{n}{e^{-n}+n^3}$

8. $(\ spring\ 2016, C\#3)$ $\sum_{n=1}^{\infty} \frac{\sqrt{n}-1}{\sqrt{n}+n}$

9. $(\ spring\ 2016, C\#3)$ $\sum_{n=1}^{\infty} \frac{2+cos^2 n}{n^2+n}$

10. $(\ spring\ 2016, C\#3)$ $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{lnn}}$