Taylor Series (Part 1)

In the last section, we could only write functions into a power series if we could get $f(x)$ *into the form* $\dfrac{1}{1-x}$ *by differentiating or integrating.*

A natural question is, "Is there a formula for c_n *based on* $f(x)$*?"*

Suppose f is a function which can be differentiated over and over again at $\boldsymbol{x = 0}$.

Then if f has a power series of the form $f(x) = \sum_{n=0}^{\infty} c_n x^n$,

$(x-a)^n$

$a=0$

then the coefficients c_n ***must satisfy*** $c_n = \dfrac{f^{(n)}(0)}{n!}$ ***for all n***

In other words, the only power series of the form $\sum_{n=0}^{\infty} c_n x^n$ ***which can represent f is the series***

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

This series is called the **Taylor series (centered at 0)** *of* $f(x)$ *or the* **Maclaurin series** *of* $f(x)$

If f has a power series of the form $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *then the coefficients* c_n *must satisfy* $c_n = \dfrac{f^{(n)}(a)}{n!}$ *and the series is called the* ***Taylor series centered at a***

The ***nth − order Taylor Polynomial*** $P_n(x)$ *is*

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\underbrace{f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n}, \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

$$\downarrow$$

$$x=a$$

$$\boxed{f(a) = c_0}$$

$$f'(x) = 0 + c_1 + c_2 \cdot 2(x-a) + c_3 \cdot 3(x-a)^2 + c_4 \cdot 4(x-a)^3 + \dots$$

$$\downarrow$$

$$x=a$$

$$\boxed{f'(a) = c_1}$$

$$\downarrow$$

$$f''(x) = 0 + 0 + 2c_2 + 3 \cdot 2 \cdot c_3(x-a) + 4 \cdot 3\, c_4(x-a)^2 + \dots$$

$$\downarrow$$

$$x=a$$

$$\boxed{f''(a) = 2c_2}$$

$$f'''(x) = 0 + 0 + 0 + 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2\, c_4(x-a) + \dots$$

$$\downarrow$$

$$x=a$$

$$\boxed{f'''(a) = 2 \cdot 3 \cdot c_3}$$

$$\vdots$$

$$f^{(n)}(a) = 1 \cdot 2 \cdot 3 \cdots n\, c_n \Rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Example#1

Find a ***Maclaurin Series for*** $\boldsymbol{f(x)}$:

1. $f(x) = \dfrac{1}{1-x}$

For $|x| < 1$, $f(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$

$$f(x) = \frac{1}{1-x}\Big|_{x=0} = 1$$

$$f'(x) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}\Big|_{x=0} = \frac{1!}{1}$$

$$f''(x) = \frac{-2}{(1-x)^3}\cdot(-1) = \frac{2}{(1-x)^3}\Big|_{x=0} = \frac{2!}{1}$$

$$f^{(3)}(x) = \frac{2\cdot(-3)}{(1-x)^4}(-1) = \frac{2\cdot 3}{(1-x)^4}\Big|_{x=0} = \frac{3!}{1}$$

$$\vdots$$

$$\boxed{f^{(n)}(x) = \frac{2\cdot 3\cdots n}{(1-x)^{n+1}} = \frac{n!}{(1-x)^{n+1}}\Big|_{x=0} = \frac{n!}{1}}$$

$$C_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1$$

$$f(x) = \sum C_n x^n = \sum_{n=0}^{\infty} x^n$$

2. $f(x) = e^x$

Macl. $\Rightarrow$ $x = 0$ $\Rightarrow$ $e^x = \sum_{n=0}^{\infty} c_n x^n$

$f(x) = e^x \Big|_{x=0} = 1$

$f'(x) = e^x \Big|_{x=0} = 1$

$\vdots$

$f^{(n)}(x) = e^x$ $\Rightarrow$ $f^{(n)}(0) = 1$

$c_n = \frac{f^{n}(0)}{n!} = \frac{1}{n!}$

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \overbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

Ratio $\lim \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim \frac{|x|^{\cancel{n+1}} \cdot n!}{|x|^{\cancel{n}} (n+1)!} = |x| \underbrace{\lim \frac{1}{n+1}}_{0} = 0 < 1$ alw

conv. for any x, $-\infty < x < \infty$, $(-\infty, \infty)$

3. $f(x) = \cos x$

$a = 0$ $\Rightarrow$ $\cos x = \sum_{n=0}^{\infty} c_n x^n$ $\Rightarrow$ $c_n = ?$

$f(x) = \cos x \Big|_{x=0} = 1$

1) $f'(x) = -\sin x \Big|_{x=0} = 0$

$f''(x) = -\cos x \Big|_{x=0} = -1$

3) $f'''(x) = \sin x \Big|_{x=0} = 0$

$f^{(4)} = \cos x \Big|_{x=0} = 1$

$\vdots$

$f^{(k)}(0)$ $\nearrow$ $k = \text{odd} \Rightarrow f^{(k)}(0) = 0$

$\searrow$ $k = \text{even}$, $k = 2n$ $\quad f^{(2n)}(0) = \pm 1 = (-1)^n$

c_k $\nearrow$ $0 = \frac{f^{\text{odd}}(0)}{k!} = 0$

$\searrow$ $c_{2n} = \frac{f^{(2n)}(0)}{(2n)!} = \frac{(-1)^n}{(2n)!}$

$$\sum_{n=0} c_{2n} x^{2n}$$

$$\underbrace{c_1}_{=0} x_1 \quad \overset{0}{\overset{\shortparallel}{c_3}} x^3 \quad \overset{0}{\overset{\shortparallel}{c_5}} x^5$$

$$\boxed{cosx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots (-1)^n \frac{x^{2n}}{(2n)!} + \cdots}$$

4. $f(x) = sinx$

$\underline{a=0} \Rightarrow sin\,x = \sum_{n=0}^{\infty} C_n x^n$

$\underline{n=0}$ $\quad f(x) = sin\,x \Big|_{x=0} = 0$

$f'(x) = cos\,x \Big|_{x=0} = 1$

$\underline{n=2}$ $\quad f''(x) = -sin\,x \Big|_{x=0} = 0$

$f^3(x) = -cos\,x \Big|_{x=0} = -1$

$\underline{f^{(4)}(x) = sin\,x} \Big|_{x=} = 0$

$\vdots$

$f^k(0)$

$k = 2n \Rightarrow f^{(2n)}(0) = 0$

$k = 2n+1 \Rightarrow f^{(2n+1)}(0) = (\pm 1)$

or $f^{(2n+1)}(0) = (-1)^{n+1}$

$$C_n = \frac{(-1)^n}{(2n+1)!}$$

$$\boxed{sinx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Example #2

$\mathbf{1(}\boldsymbol{a}\mathbf{)}$*Find the full Taylor Series representation for* $\boldsymbol{f(x) = e^{x-5}}$ ***centered at*** $\boldsymbol{x = 5}$.

$$\cancel{f(x) = e^{x-5} = \sum c_n (x-5)^n}$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!} + \cdots$$

$t = x - 5$

$$e^{x-5} = e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(x-5)^n}{n!}} = 1 + (x-5) + \frac{(x-5)^2}{2!} + \ldots$$

We can check it : $e^{x-5} = \sum_{n=0}^{\infty} c_n (x-5)^n$, *where* $\boldsymbol{c_n} = \dfrac{\boldsymbol{f^{(n)}(5)}}{\boldsymbol{n!}}$

$f^{n}(5) = 1$

$f(x) = e^{x-5}|_{x=5} = 1$

$f'(x) = e^{x-5}|_{x=5} = 1$

$f''(x) = e^{x-5}|_{x=5} = 1$ => $\boldsymbol{c_n} = \dfrac{\boldsymbol{f^{(n)}(5)}}{\boldsymbol{n!}} = \dfrac{1}{n!}$ *and* $e^{x-5} = \boxed{\sum_{n=0}^{\infty} \frac{1}{n!}(x-5)^n}$

⋮

$f^{(n)}(x) = e^{x-5}|_{x=5} = 1$

$\mathbf{1(}\boldsymbol{b}\mathbf{)}$*Find the full Taylor Series representation for* $\boldsymbol{f(x) = e^{x}}$ ***centered at*** $\boldsymbol{x = 5}$

$e^x = \sum c_n (x-5)^n \Rightarrow c_n = ?$

$t = x - 5 \Rightarrow x = t + 5$

$$e^x = e^{t+5} = e^5 e^t = e^5 \sum \frac{t^n}{n!} = e^5 \sum_{n=0}^{\infty} \frac{(x-5)^n}{n!} = \boxed{\sum \frac{e^5 (x-5)^n}{n!}}$$

$Check\ again:\quad e^x = \sum_{n=0}^{\infty} c_n(x-5)^n, where\ \boldsymbol{c_n} = \dfrac{\boldsymbol{f^{(n)}(5)}}{\boldsymbol{n!}}$

$f(x) = e^x|_{x=5} = e^5$

$f'(x) = e^x|_{x=5} = e^5$ $\qquad f^{(n)}(x) = e^5 \Rightarrow \boldsymbol{c_n} = \dfrac{\boldsymbol{f^{(n)}(5)}}{\boldsymbol{n!}} = \dfrac{e^5}{n!}$ and $e^x = \boxed{\sum_{n=0}^{\infty} \frac{e^5}{n!}(x-5)^n}$

$f''(x) = e^x|_{x=5} = e^5$

$\vdots$

$f^{(n)}(x) = e^x|_{x=5} = e^5$

$\mathbf{1}(\boldsymbol{c})$ *Find the full Taylor Series representation for* $\boldsymbol{f(x) = e^{-\frac{x}{2}}}$ ***centered at*** $\boldsymbol{x = 1}$

$$f(x) = e^{-\frac{x}{2}} = \sum_{n=0} c_n (\overbrace{x-1}^{t})^n$$

$$t = x-1 \Rightarrow x = t+1 \qquad u = -\frac{t}{2},\quad e^u = \sum \frac{u^n}{n!}$$

$$e^{-\frac{x}{2}} = e^{-\frac{1}{2}(t+1)} = e^{-\frac{1}{2}} e^{\left(-\frac{t}{2}\right)} =$$

$$= e^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\left(-\frac{t}{2}\right)^n}{n!} = e^{-\frac{1}{2}} \sum \frac{(-1)^n t^n}{2^n \cdot n!} = \boxed{\sum_{n=0} \frac{(-1)^n (x-1)^n}{\sqrt{e}\ 2^n n!}}$$

$Check:\quad e^{-\frac{x}{2}} = \sum_{n=0}^{\infty} c_n(x-1)^n, where\ \boldsymbol{c_n} = \dfrac{\boldsymbol{f^{(n)}(1)}}{\boldsymbol{n!}}$

$f(x) = e^{-\frac{x}{2}}\Big|_{x=1} = e^{-\frac{1}{2}}$

$f(x) = e^{-\frac{x}{2}} \cdot \left(-\frac{1}{2}\right)\Big|_{x=1} = e^{-\frac{1}{2}}\left(-\frac{1}{2}\right)$

$f(x) = e^{-\frac{x}{2}} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right)\Big|_{x=1} = e^{-\frac{1}{2}}\left(-\frac{1}{2}\right)^2$

$\vdots$

$f^{(n)}(1) = e^{-\frac{1}{2}}\left(-\frac{1}{2}\right)^n \Rightarrow \boldsymbol{c_n} = \dfrac{\boldsymbol{f^{(n)}(1)}}{\boldsymbol{n!}} = \dfrac{e^{-\frac{1}{2}}\left(-\frac{1}{2}\right)^n}{n!}$ and $e^{-\frac{x}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-\frac{1}{2}}}{2^n n!}(x-1)^n$

$2.\ a.$ *Find the Taylor Polynomial* P_3 ***for*** $\boldsymbol{f(x) = \ln(x-2)}$ ***centered at*** $\boldsymbol{x = 3}$

$$\boxed{\sum \frac{f^{k}(3)}{n!}(x-3)^k}$$

$$\boxed{P_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)(x-3)^2}{2!} + \frac{f^{(3)}(3)(x-3)^3}{3!}}$$

$$f(x) = \ln(x-2)\Big|_{x=3} = 0$$

$$f'(x) = \frac{1}{x-2}\Big|_{x=3} = 1$$

$$f''(x) = \frac{-1}{(x-2)^2}\Big|_{x=1} = -1$$

$$f^{(3)}(x) = \frac{(-1)(-2)}{(x-2)^3}\Big|_{x=1} = 2$$

$$P_3(x) = 0 + 1(x-3) - \frac{1}{2!}(x-3)^2 + \frac{2}{3!}(x-3)^3 \Rightarrow$$

$$\boxed{P_3(x) = (x-3) - \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3}$$

$$f^{(4)}(x) = \frac{(-1)(-2)(-3)}{(x-2)^4}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(x-2)^n}$$

$1.\ b.$ *Find the full Taylor Series representation for* $f(x) = \ln(x-2)$ *centered at* $x = 3$

$$f^{n}(3) = (-1)^{n-1} \Rightarrow C_n = \frac{(-1)^{n-1}(n-1)!}{n!}$$

$$f(x) = \ln(x-2) = \sum_{n=0} \frac{(-1)^{n-1}(n-1)!}{n!}(x-3)^n = \left(\begin{matrix} n=0 \\ (-1)! \\ \text{not defined} \end{matrix} \right)$$

$n=1$

$$= \sum_{n=1} \frac{(-1)^{n-1}(n-1)!}{n!}(x-3)^n = \boxed{\sum_{n=1} \frac{(-1)^{n-1}}{n}(x-3)^n}$$

a

$$\ln(1+t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots (-1)^n \frac{t^n}{n} + \cdots$$ ✓

we want it around $x = 3 \Rightarrow t = x - 3$

$$\underbrace{\ln(1+t)} = \ln(1+(x-3)) = \underbrace{\ln(x-2)},$$

$$\ln(x-2) = \ln(1+t) = \sum_{n=1} (-1)^{n-1} \frac{t^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-3)^n}{n}$$

$$\ln(x-2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-3)^n}{n} = \underbrace{\boxed{(x-3) - \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3}}}_{P_3(t)} - \frac{(x-3)^4}{4} + \cdots (-1)^n \frac{(x-3)^n}{n} + \cdots$$

3. *Find the full Taylor Series representation for* $f(x) = \ln(x) - 1$ *centered at* $x = e$

(Here, the „long way", with derivatives, is easier, than the „short cut". The last one is too tricky, I think

(Here is the idea for „short cut", anyway:

$$f(x) = \ln x - \underbrace{\ln e}_{=1} = \ln\left(\frac{x}{e}\right) = \ln\left(1 + \frac{x}{e} - 1\right) =$$

$$= \ln\Big(1 + \underbrace{\left(\frac{x-e}{e}\right)}_{t}\Big) = \ln(1+t) = \ldots \text{ use the formula.}$$)

☺

4. *Find the first three non − zero terms of the* **Maclaurin series for the** $\boldsymbol{f(x) = x^2 sin(2x)}$

$$sint = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \cdots$$

$t = 2x$

$$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}$$

$$f(x) = x^2 \cdot \sin 2x : \quad x^2\left(2x - \frac{8x^3}{3!} + \frac{32x^5}{5!}\right) = 2x^3 - \frac{8}{3!}x^5 + \frac{32}{5!}x^7$$

5. *Find the first three non − zero terms of the* **Maclaurin series for the** $\boldsymbol{f(x) = 2xe^x}$

6. *Find the first three non − zero terms of the* **Taylor series for the $f(x) = \frac{1}{x}$ about $a = 1$**

$$f(x) = \sum C_n (x-1)^n \qquad \left(\frac{1}{1-t}\right)$$

short cut

$$\frac{1}{x} = \frac{1}{(x-1)+1} = \frac{1}{1+(x-1)} = \frac{1}{1-\underbrace{(-(x-1))}_{t}} = \sum_{n=0}^{\infty} \left(-(x-1)\right)^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n = \boxed{1 - (x-1) + (x-1)^2} - (x-1)^3$$

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other way

$$f(x) = \frac{1}{x} \Big|_{x=1} = 1 \qquad C_0 = \frac{f(1)}{0!} = 1$$

$$f'(x) = -\frac{1}{x^2} \Big|_{x=1} = -1 \qquad C_1 = \frac{f'(1)}{1!} = (-1)$$

$$f''(x) = \frac{-1(-2)}{x^3} \Big|_{x=1} = 2 \qquad C_2 = \frac{f''(1)}{2!} = \frac{2}{2} = 1$$

three term $\boxed{1 + (-1)(x-1) + 1(x-1)^2}$