Mathematical formulation

Fama-French 3-factor model

To reduce estimation error, noise, and overfitting risk, expected returns are obtained from an FF3 model rather than raw sample means.

For each asset $i = 1, \ldots, N$, a time-series regression of excess returns is estimated on the three Fama-French factors:

$$R_{i,t} - R_{f,t} = \alpha_i + \beta_{i,M}(R_{M,t} - R_{f,t}) + \beta_{i,\text{SMB}}\,\text{SMB}_t + \beta_{i,\text{HML}}\,\text{HML}_t + \varepsilon_{i,t}$$

The expected daily return for each asset is then estimated as:

$$\mu_i = r_f + \hat{\alpha}_i + \hat{\beta}_{i,M}\,\overline{(R_M - R_f)} + \hat{\beta}_{i,\text{SMB}}\,\overline{\text{SMB}} + \hat{\beta}_{i,\text{HML}}\,\overline{\text{HML}}$$

where bars denote sample means and r_f is the sample average of $R_{f,t}$. In the implementation, $\hat{\alpha}_i$ is excluded by default (a flag `include_alpha` allows otherwise).

Collecting all assets, the expected return vector $\mu \in \mathbb{R}^{N\times 1}$ is obtained.

Minimum-variance frontier

Let Y_i be the random daily return of asset i. For any portfolio $x \in \mathbb{R}^{N\times 1}$:

$$\sigma_x^2 = x^\top V x \qquad \text{where} \qquad V = \begin{bmatrix} \text{Var}(Y_1) & \text{Cov}(Y_1, Y_2) & \cdots & \text{Cov}(Y_1, Y_N) \\ \text{Cov}(Y_2, Y_1) & \text{Var}(Y_2) & \cdots & \text{Cov}(Y_2, Y_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(Y_N, Y_1) & \text{Cov}(Y_N, Y_2) & \cdots & \text{Var}(Y_N) \end{bmatrix}_{N\times N}$$

The covariance matrix V is symmetric and (assumed) positive definite, which ensures invertibility and that all portfolio variances are nonnegative.

Let $e = [1\ \ldots\ 1]^\top \in \mathbb{R}^{N\times 1}$. Having chosen an arbitrary target return $m \in \mathbb{R}$, the minimum-variance portfolio $x^*(m) \in \mathbb{R}^{N\times 1}$ solves:

$$x^*(m) = \arg\min_{x \in \mathbb{R}^{N\times 1}} \; x^\top V x \quad \text{subject to} \quad \begin{cases} \mu^\top x = m \\ e^\top x = 1 \end{cases}$$

Define $A = [\mu \;\; e] \in \mathbb{R}^{N\times 2}$ and compute:

$$B = A^\top V^{-1} A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

The elements a, b, and c yield the variance of $x^*(m)$:

$$\sigma^2_{x^*(m)} = \frac{cm^2 - 2bm + a}{ac - b^2} \in \mathbb{R}_+$$

In the implementation, μ and V are replaced with their estimates, and the above formula is used to trace the minimum-variance frontier by letting m vary over a grid of target returns.

GMVP and efficient frontier

Among all efficient portfolios x^*, the one with the lowest variance is the GMVP (Global Minimum Variance Portfolio), denoted as $\tilde{x}$. Its weight vector, expected return, and variance are:

$$\tilde{x} = \frac{1}{c} V^{-1} e \in \mathbb{R}^{N\times 1} \qquad m_{\tilde{x}} = \mu^\top \tilde{x} = \frac{b}{c} \qquad \sigma^2_{\tilde{x}} = \tilde{x}^\top V \tilde{x} = \frac{1}{c}$$

The efficient frontier is the portion of the minimum-variance frontier with expected return at least $m_{\tilde{x}}$.

CML and tangency portfolio

When a risk-free asset with return r_f is introduced, the new efficient frontier is the line:

$$\sigma = \bar{A}\,(m - r_f) \qquad \text{where} \qquad \bar{A} = \frac{1}{\sqrt{a + c r_f^2 - 2 r_f b}} \in \mathbb{R}$$

This is the capital market line (CML). It meets the original minimum variance frontier in exactly one point, corresponding to the portfolio with the highest Sharpe ratio (known as the tangency portfolio, x_t).

The tangency portfolio's weight vector, expected return, and variance are:

$$x_t = \frac{V^{-1}(\mu - r_f e)}{e^\top V^{-1}(\mu - r_f e)} \in \mathbb{R}^{N\times 1} \qquad m_t = \mu^\top x_t \qquad \sigma^2_{x_t} = x_t^\top V x_t$$

Using the tangency portfolio's volatility and expected return, it is possible to rewrite the CML equation:

$$m = r_f + \frac{m_t - r_f}{\sigma_{x_t}}\,\sigma$$

where $\frac{m_t - r_f}{\sigma_{x_t}}$ is the Sharpe ratio of x_t.

With a risk-free asset, any efficient portfolio invests a weight $(1-\lambda)$ in the risk-free asset and λ in the tangency portfolio. If leverage at r_f is allowed, $\lambda \in \mathbb{R}$; otherwise $\lambda \in [0, 1]$. The risky-asset weights are $x = \lambda\, x_t$ (so $e^\top x = \lambda$), and the expected return of the efficient portfolio is:

$$m_x = (1 - \lambda)\, r_f + \lambda\, m_t$$

In this way, the entire set of efficient portfolios with a risk-free asset lies on the CML.