ศศิกานต์ อังคณากร
61070507233

EXERCISE

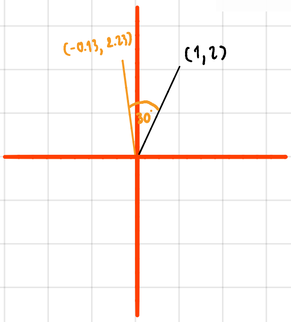
GEOMETRIC TRANSFORMATION

3.1 a) Find the matrix that represents the rotation of an object by 30° about origin.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\theta = 30^\circ;\ R_{30^\circ} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

b) Calculate the coordinates of the point (1, 2) after the rotation.

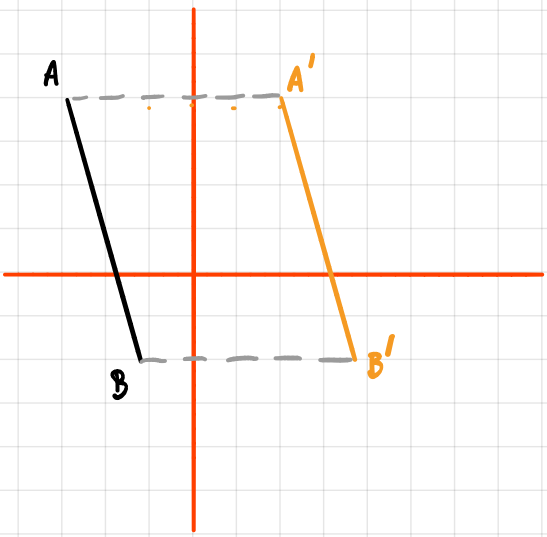


$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$x' = (1)\cos 30^\circ - (2)\sin 30^\circ = \frac{\sqrt{3}}{2} - 2\left(\frac{1}{2}\right) = \frac{\sqrt{3}-2}{2} = -0.13$$

$$y' = (1)\sin 30^\circ + 2\cos 30^\circ = \frac{1}{2} + 2\frac{\sqrt{3}}{2} = 2.23$$

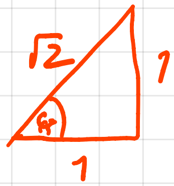
3.2 Show that the translated straight line of AB is the straight line connecting the translated points A′ and B′

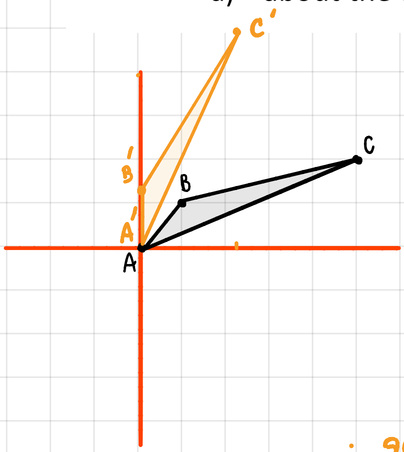


∴ จะเห็นได้ว่าเปลี่ยนแปลงไปตามแกน x บวก 5 หน่วย

3.3 Perform a 45° rotation of the triangle ABC with A(0, 0) B(1, 1) and C(5, 2)

a) about the origin





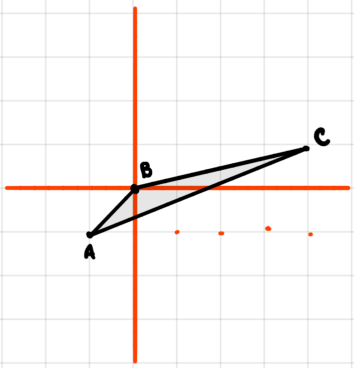
$$x' = x\cos\theta - y\sin\theta \qquad \theta = 45^\circ$$
$$y' = x\sin\theta + y\cos\theta$$

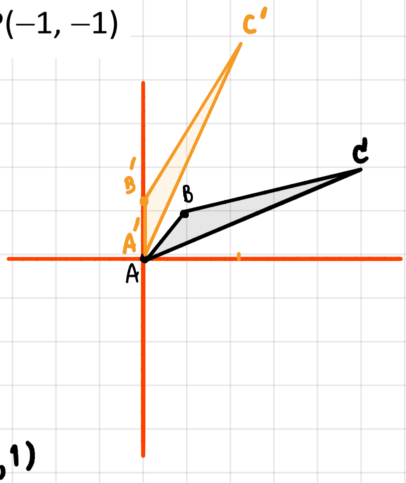
ณ จุด B ; $x' = \cos 45^\circ - \sin 45^\circ$, $x' = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$ | $y' = \sin 45^\circ + \cos 45^\circ$, $y' = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \approx 1.41$ → $(0, 1.41)$

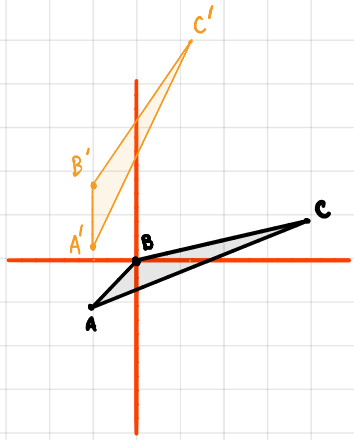
ณ จุด C ; $x' = 5\cos 45^\circ - 2\sin 45^\circ$, $x' = 5\left(\frac{1}{\sqrt{2}}\right) - 2\left(\frac{1}{\sqrt{2}}\right) = 2.12$ | $y' = 5\sin 45^\circ + 2\cos 45^\circ$, $= 5\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) = 4.95$ → $(2.12, 4.95)$

∴ จุดที่หมุนได้ คือ A(0,0), B(0, 1.41), C(2.12, 4.95)

b) about P(−1, −1)







A = (-1,-1) , B = (0,0) , C = (4,1)

การหมุนรอบจุดใดๆ

(A) $P_1(x', y') = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_1(1-\cos\theta) + y\sin\theta \\ \sin\theta & \cos\theta & y_1(1-\cos\theta) - x_1\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & -(1-\cos 45^\circ) - \sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ & -(1-\cos 45^\circ) + \sin 45^\circ \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1+\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

A(-1, 0.41)

$$(A',B',C') = P(x',y')\cdot(A,B,C) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1+\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & \frac{3\sqrt{2}}{2}-1 \\ -1+\sqrt{2} & 2\sqrt{2}-1 & \frac{9\sqrt{2}}{2}-1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\therefore \quad A'(-1,-1+\sqrt{2}),\ B'(-1,2\sqrt{2}-1),\ C'\left(\frac{3\sqrt{2}}{2}-1,\ \frac{9\sqrt{2}}{2}-1\right)$

3.4 For the same triangle ABC in Exercise 3.3, find the transformed triangle A′ B′ C′ by magnifying it to twice its size while keeping C(5, 2) fixed Hint. Use eq. 3.8 to obtain

$$R = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Then A′(–5, –2), B′(–3, 0) and C′(5, 2)

$S_x = 2$

$S_y = 2$

$$P(x', y') = T(x, y) \cdot S(S_x, S_y) \cdot T(-x, -y)$$

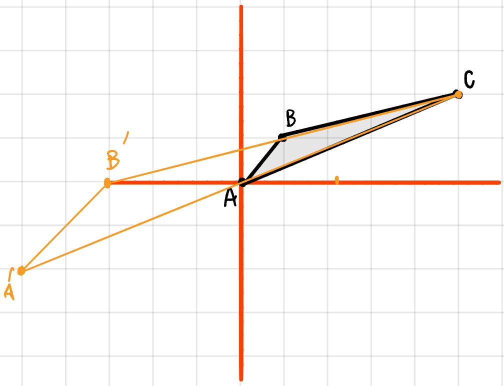
$$= \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x & 0 & x \\ 0 & S_y & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & x(1-S_x) \\ 0 & S_y & y(1-S_y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 5(1-2) \\ 0 & 2 & 2(1-2) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A', B', C') = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -3 & 5 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$



3.5 Prove that a 2D rotation and scaling commute if and only if $s_x = s_y$.

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S \cdot R = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x\cos\theta & -S_x\sin\theta & 0 \\ S_y\sin\theta & S_y\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

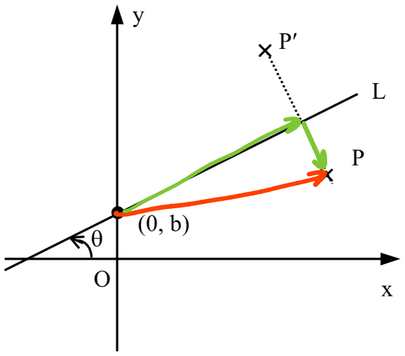
if $S_x = S_y$; $S \cdot R = \begin{bmatrix} S_x\cos\theta & -S_x\sin\theta & 0 \\ S_x\sin\theta & S_x\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$R \cdot S = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x\cos\theta & S_y\sin\theta & 0 \\ S_x\sin\theta & S_y\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

if $S_x = S_y$; $R \cdot S = \begin{bmatrix} S_x\cos\theta & S_x\sin\theta & 0 \\ S_x\sin\theta & S_y\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ $S \cdot R = R \cdot S$ เมื่อ $S_x = S_y$

3.6 Describe the transformation which reflects an object about the line L of equation $y = mx + b$.



$y = mx$

$m = \tan\theta$

วิธีการทำ

① ทำการ Translate เส้นตรง ด้วย $(0,-b)$ ดังนั้น สมการ $y = mx+b$ เป็น $y = mx$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

② Reflect ผ่าน เส้น $y = mx$

$L = \begin{bmatrix} 1 \\ m \end{bmatrix}$

สูตรการหาเวกเตอร์ภาพฉาย

$\text{proj.}\ \vec{v} = \left(\frac{\vec{v}\cdot L}{|L|^2}\right) L$

$\text{proj.}\ \vec{P} = \frac{\vec{P}\cdot L}{|L|^2} L$

$= \left(\frac{\begin{bmatrix} x \\ y \end{bmatrix}\cdot\begin{bmatrix} 1 \\ m \end{bmatrix}}{1+m^2}\right)\begin{bmatrix} 1 \\ m \end{bmatrix}$

$$M = 2\vec{p} - p$$

$$= 2\left(\frac{\begin{bmatrix} x \\ y \end{bmatrix}\cdot\begin{bmatrix} 1 \\ m \end{bmatrix}}{1+m^2}\right)\begin{bmatrix} 1 \\ m \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 2\left(\frac{x+ym}{1+m^2}\right)\begin{bmatrix} 1 \\ m \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2x+ym}{1+m^2} \\ \frac{2mx+2m^2y}{1+m^2} \end{bmatrix} - \begin{bmatrix} \frac{(1+m^2)x}{1+m^2} \\ \frac{(1+m^2)y}{1+m^2} \end{bmatrix}$$

$$= \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m & 0 \\ 2m & m^2-1 & 0 \\ 0 & 0 & 1+m^2 \end{bmatrix}$$

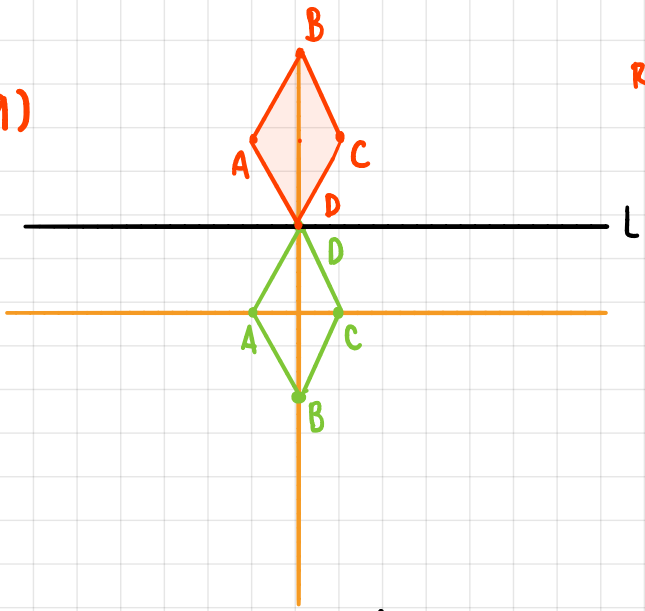
③ Translate กลับ ด้วย $(0, b)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

3.7 Reflect the quadrangle ABCD with A(–1, 0), B(0, –2), C(1, 0) and D(0, 2) about

a) the horizontal line $y = 2$,
b) the vertical line $x = 2$,
c) the line $y = x + 2$.

a)



Reflect through L

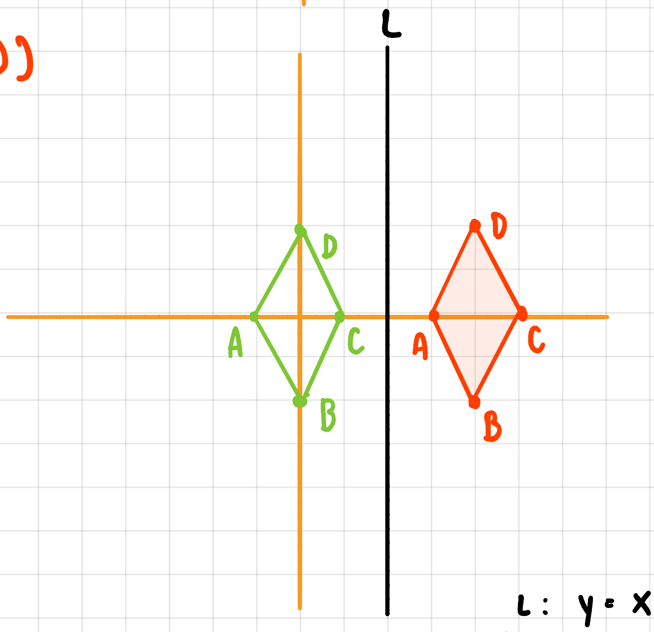
$A'(-1,2)$

$B'(0,4)$

$C'(1,2)$

$D'(0,0)$

b)



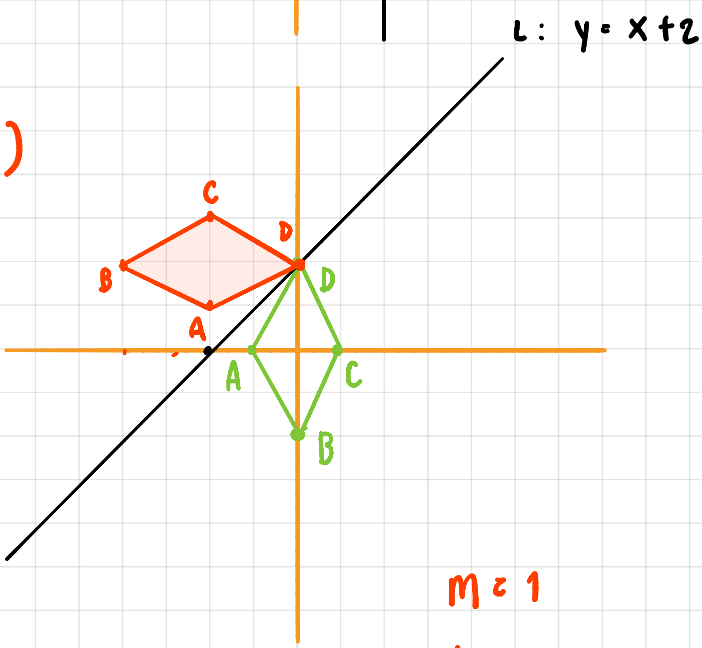
$A'(1,0)$

$B'(-2,2)$

$C'(3,0)$

$D'(2,2)$

c)



$A'(-2,1)$

$B'(-4,2)$

$C'(-2,3)$

$D'(0,2)$

$m = 1$

$b = 2$