

Using uWUE to analytically evaluate VPD effect on WUE, GPP, and ET

This document is to analytically calculate ET and $\frac{\partial ET}{\partial VPD}$.
Penman montieith is given by:

$$ET = \frac{\Delta R + g_a \rho_a c_p D_s}{\Delta + \gamma(1 + \frac{g_a}{g_s})} \quad (1)$$

where from Medlyn we have:

$$g_s = LAI \frac{RT}{P} 1.6 \left(1 + \frac{g_1}{\sqrt{D_s}}\right) \frac{A}{c_s} \quad (2)$$

Here, g_{1M} has units of $\text{kPa}^{0.5}$, A is the net CO_2 assimilation rate in $\mu \text{ mol m}^{-2} \text{ s}^{-1}$, c_s is the CO_2 concentration in ppm (supposed to be at leaf surface). The problem is the A term that we need to get rid of. Also g_w as above has units of $\text{mol (air?) m}^{-2} \text{ s}^{-1}$.

we can use:

$$\text{uWUE} = \frac{\text{GPP} \cdot \sqrt{\text{VPD}}}{ET} \quad (3)$$

As long as we put uWUE into μmol , we can use the uWUE provided by Zhou et. al. To do this just multiply by a factor $\frac{1 \text{ mol C}}{12.011 \text{ g C}} \frac{1.66 \mu \text{ mol}}{1 \text{ mol}}$. Then make sure that units of VPD match. Note the table in Franks et al. 2017 and Zhou et al. have VPD in kPa and hPa, respectively.

Then we can show that if we multiply g_s by $\frac{\text{uWUE} ET}{A \sqrt{D_s} L_v}$ (which is equivalent to 1 with unit conversion above), we can rearrange Penman Moneith to get:

$$ET = \frac{\Delta R + g_a \left(\rho_a c_p D_s - \frac{\gamma c_s \sqrt{D_s} L_v P}{LAI R T 1.6 \text{ uWUE} (1 + \frac{g_1}{\sqrt{D_s}})} \right)}{\Delta + \gamma} \quad (4)$$

If we want to solve for a “pseudo-LAI” (in practice this will also be a function of noise and model error in addition to leaf area) as a function of all other knowns, we have:

$$LAI = - \frac{g_a \gamma c_s \sqrt{D_s} L_v P}{(ET (\Delta + \gamma) - \Delta R - g_a \rho_a c_p D_s) 1.6 R T \text{ uWUE} (1 + \frac{g_1}{\sqrt{D_s}})} \quad (5)$$

Now also taking the derivative of ET w.r.t. D_s we get:

$$\frac{\partial ET}{\partial D_s} = \frac{g_a}{\Delta + \gamma} \left(\rho_a c_p - \frac{\gamma c_s L_v P}{LAI 1.6 R T \text{ uWUE}} \left(\frac{2g_1 + \sqrt{D_s}}{2(g_1 + \sqrt{D_s})^2} \right) \right) \quad (6)$$

Now that we have ET, we can also use uWUE to get GPP:

$$\frac{ET}{L_v} = \frac{\text{GPP} \cdot \sqrt{D_s}}{\text{uWUE}} \quad (7)$$

Plugging this into Equation (4) gives:

$$\text{GPP} = \frac{\text{uWUE} \Delta R + g_a \left(\text{uWUE} \rho_a c_p D_s - \frac{\gamma c_s \sqrt{D_s} L_v P}{LAI R T 1.6 (1 + \frac{g_1}{\sqrt{D_s}})} \right)}{L_v \sqrt{D_s} (\Delta + \gamma)} \quad (8)$$

Taking the derivative w.r.t D_s gives:

$$\frac{\partial \text{GPP}}{\partial D_s} = \frac{1}{2L_v D_s^{3/2} (\Delta + \gamma)} \left(-\text{uWUE} \Delta R + g_a \text{uWUE} \rho_a c_p D_s - \frac{g_a g_1 \gamma c_s L_v P}{LAI R T 1.6 (1 + \frac{g_1}{\sqrt{D_s}})^2} \right) \quad (9)$$

We can also solve for WUE, again using uWUE (just make sure units are correct: mol C/mol H₂O).

$$\frac{GPP}{ET} = \frac{\text{uWUE}}{\sqrt{D_s}} \quad (10)$$

And the derivative:

$$\frac{\partial \text{WUE}}{\partial D_s} = -\frac{\text{uWUE}}{2D_s^{3/2}} \quad (11)$$