Using uWUE to remove dependence of ET on GPP

This document is to analytically caluclate ET and $\frac{\partial ET}{\partial VPD}$.
Penman montieth is given by:

$$\lambda E = \frac{\Delta R + g_a \rho_a c_p D_s}{\Delta + \gamma(1 + \frac{g_a}{g_s})} \tag{1}$$

where from Medlyn we have:

$$g_s = \text{LAI } 1.6 \left(1 + \frac{g_1}{\sqrt{D_s}}\right) \frac{A}{c_s} \tag{2}$$

Here, g_{1M} has units of kPa $^{0.5}$, A is the net CO $_2$ assimilation rate in μ mol m $^{-2}$ s $^{-1}$, c_s is the CO $_2$ concentration in ppm (supposed to be at leaf surface). The problem is the A term that we need to get rid of. Also g_w as above has units of mol (air?) m $^{-2}$ s $^{-1}$.

we can use:

$$\text{uWUE} = \frac{\text{GPP} \cdot \sqrt{\text{VPD}}}{ET} \tag{3}$$

As long as we put uWUE into μmol , we can use the uWUE provided by Zhou et. al. To do this just multiply be a factor $\frac{1molC}{12.011gC} \frac{1.e6\mu mol}{1mol}$. Then make sure that units of VPD match. If using Franks et al. 2017 and Zhou et al. then VPD within the medlyn model should be in kPa, and VPD cancelling uWUE should be in hPa.

Then we can show that if we multiply g_s by $\frac{uWUE\ ET}{A\sqrt{D_s}}$ (which is equivalent to 1 with unit conversion above), we can rearrange Penman Moneith to get:

$$\lambda E = \frac{\Delta R + g_a \left(\rho_a c_p D_s - \frac{\gamma c_s \sqrt{D_s}}{\text{LAI 1.6 uWUE } (1 + \frac{g_1}{\sqrt{D_s}})}\right)}{\Delta + \gamma} \tag{4}$$

If we want to solve for LAI as a function of all other knowns, we have:

$$LAI = \frac{g_a \gamma c_s \sqrt{D_s}}{(\text{ ET } (\Delta + \gamma) - \Delta R - g_a \rho_a c_p D_s)\, 1.6 \text{ uWUE } (1 + \frac{g_1}{\sqrt{D_s}})} \tag{5}$$

Now taking the derivative w.r.t. D_s we get:

$$\frac{\partial\ \lambda E}{\partial\ D_s} = \frac{g_a}{\Delta + \gamma} \left(\rho_a c_p - \frac{\gamma c_s}{\text{LAI 1.6 uWUE}} \left(\frac{2 g_1 + \sqrt{D_s}}{2(g_1 + \sqrt{D_s})^2}\right)\right) \tag{6}$$