Final exam M01 - Linear algebra and probability

Master AI - Idiap and Unidistance

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Read all instructions and WAIT for the signal before starting the exam.

For ALL the questions, choose ALL the correct statements. Indicate your answer by shading the appropriate box with your pencil.

Selecting a correct statement gives you points proportionally to the number of correct statements.

Selecting an incorrect statement cancels all the points for the question. Select your answers carefully! However, the minimum number of points you can get for a question is 0.

Do not stay to long on one question. Move to the next and try it later!

There is a total of 200 points: 125 for Linear Algebra and 75 for Probability

You can report (on the sheets) a LA/Proba related error if you think you saw one.

You have 2 hours to complete the test.

Do not forget to write your full name below! Good luck!

Your full name: ____

1 Linear Algebra (25 questions - 125 pts out of 200)

1. (5 points) Let $\boldsymbol{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix}3\\6\\9\end{bmatrix}$, and $\boldsymbol{w} = \begin{bmatrix}1\\0\\3\end{bmatrix}$. Which of the following statements are TRUE? We consider here the geometrical description of all the linear combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.
   - ◯ Their linear combinations fill a line in $\mathbb{R}^3$
   - ◯ Their linear combinations fill a plane in $\mathbb{R}^3$
   - ◯ Their linear combinations fill all $\mathbb{R}^3$
   - ◯ Their linear combinations fill all $\mathbb{R}^2$
   - ◯ Their linear combinations fill a plane in $\mathbb{R}^2$

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2. (5 points) Let $A = \begin{bmatrix}5 & 0 & 5\\0 & 5 & 0\\-5 & 0 & -5\end{bmatrix}$, $B = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$, $C = \begin{bmatrix}1 & 0\\0 & 1\\1 & 0\end{bmatrix}$ Which of the following statements are CORRECT?
   - ◯ $ABC = \begin{bmatrix}10 & 0\\0 & 5\\-10 & 0\end{bmatrix}$
   - ◯ $(BC)^{-1} = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$
   - ◯ $A^{-1} = \frac{1}{100}\begin{bmatrix}5 & 0 & 5\\0 & 20 & 0\\-5 & 0 & -5\end{bmatrix}$
   - ◯ $BCA^T = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$
   - ◯ All the above statements are incorrect

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3. (5 points) Let A, B, C be square matrices. If $AB = CA$ and $det(A) \neq 0$, does this ALWAYS imply that
   - ◯ $B = C$
   - ◯ $BA^{-1} = A^{-1}C$
   - ◯ ABA^{-1} is non-singular
   - ◯ $A = C^{-1}AB$

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4. (5 points) If $\boldsymbol{u}$ is perpendicular (orthogonal) to $\boldsymbol{v}$ and $\boldsymbol{w}$, which of the following statements are TRUE knowing that $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ are unit vectors?
   - ◯ $\boldsymbol{v}$ is parallel to $\boldsymbol{w}$
   - ◯ $\boldsymbol{u}$ is always perpendicular to any linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$
   - ◯ $\boldsymbol{u}$ is perpendicular to any linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$ only if $\boldsymbol{v}$ and $\boldsymbol{w}$ are independent.
   - ◯ The length of $\boldsymbol{u} - \boldsymbol{v}$ is $\sqrt{2}$
   - ◯ The length of $\boldsymbol{w} - \boldsymbol{v}$ is $\sqrt{2}$
   - ◯ The length of $\boldsymbol{w} - \boldsymbol{u}$ is $\sqrt{2}$

5. (5 points) For which of the following values of α, the elimination will fail to produce three pivots for the matrix $A = \begin{bmatrix} \alpha & 2 & 3 \\ \alpha & \alpha & 4 \\ \alpha & \alpha & \alpha \end{bmatrix}$

   ○ $\alpha = -1$

   ○ $\alpha = 0$

   ○ $\alpha = 1$

   ○ $\alpha = 2$

   ○ $\alpha = 3$

   ○ $\alpha = 4$

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6. (5 points) Suppose A is a 3×3 matrix such that $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Which of the following statements are TRUE?

   ○ At least one column of A is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ but we do not know which one

   ○ The first column of A is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

   ○ The first row of A is $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

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7. (5 points) Which of the following statements are TRUE?

   ○ The set of all vectors $\begin{bmatrix} a \\ 0 \\ a \\ a+1 \end{bmatrix}$ is a subspace of $\mathbb{R}^4$

   ○ The set of all vectors $\begin{bmatrix} a \\ 1 \end{bmatrix}$ is a subspace of $\mathbb{R}^2$

   ○ Any set of k column vectors of the inverse of the $(n \times n)$ identity matrix is a subspace of $\mathbb{R}^n$ $(k \leq n)$

   ○ The set of all vectors $\begin{bmatrix} a \\ 0 \\ b \\ c \end{bmatrix}$ is a subspace of $\mathbb{R}^4$

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8. (5 points) Let A be an $M \times N$-dimensional matrix with $M < N$. Consider the system of equations $A\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{b}$ being an $M \times 1$-dimensional non-zero vector. Which of the following statements are **always** TRUE?

   ○ The system of equations has no solution

   ○ The system of equations has a unique solution

   ○ The system of equations has at least one solution

   ○ The system of equations has a solution if and only if it has infinitely many solutions

9. (5 points) Consider the equation $A\boldsymbol{x} = \boldsymbol{b}$, with A a $(M \times N)$-dimensional matrix. When does this equation **always** have a unique exact solution?

   - ◯ The dimension of the nullspace of A is 0 and $M = N$
   - ◯ The rank of A is zero
   - ◯ $\boldsymbol{b}$ is in the row space of A
   - ◯ The dimension of the left-nullspace of A is 0
   - ◯ The rank of A is N

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10. (5 points) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be linearly independent vectors. Which of the following statements are **always** TRUE?

   - ◯ $\alpha_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} = \mathbf{0}$ for some non-zero scalars α_1, α_2, and α_3
   - ◯ The rank of the matrix created by stacking $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its rows is exactly 3
   - ◯ The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ are also linearly independent
   - ◯ The rank of the matrix created by stacking $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its columns is at most 3
   - ◯ $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in the same hyperplane
   - ◯ The determinant of the matrix created by stacking $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its columns is exactly 3
   - ◯ The determinant of the matrix created by stacking $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its columns is at most 3

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11. (5 points) Which of the following statements are TRUE regarding the equation $A\boldsymbol{x} = \boldsymbol{b}$?

   - ◯ Any vector in the nullspace of A is orthogonal to the columns of A
   - ◯ Any vector in the nullspace of A is orthogonal to the rows of A
   - ◯ Any vector in the left nullspace of A is orthogonal to the columns of A
   - ◯ Any vector in the left nullspace of A is orthogonal to the rows of A.
   - ◯ If the solution $\boldsymbol{x_s}$ of the equation has a nullspace component, this solution is unique
   - ◯ You can always find the exact solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ using the least squares method.

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12. (5 points) Consider the following two statements:
(S_1) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are orthonormal vectors, then the projection of $\mathbf{v}_3$ onto the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ is the zero vector.
(S_2) The Gram-Schmidt process applied to the orthonormal vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ produces a different set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Which of the following statements are TRUE?

   - ◯ Both statements (S_1) and (S_2) are correct
   - ◯ Neither statement (S_1) nor statement (S_2) is correct
   - ◯ Only statement (S_1) is correct
   - ◯ Only statement (S_2) is correct

13. (5 points) Consider the $(M \times N)$-dimensional matrix A with $M > N$ and $\text{rank}(A) = N$. Which of the following statements are TRUE?

- ◯ The matrix AA^T is singular
- ◯ The matrix AA^T is non-singular
- ◯ The matrix A^TA is singular
- ◯ The matrix AA^T cannot be defined because the dimensions do not fit for the matrix multiplication
- ◯ The matrix A^TA is non-singular

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14. (5 points) Complete the following sentence by using one or several of the answers below. Note that we are not considering the generalized least squares method here.
"If $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, then the least squares method gives an approximation of $\boldsymbol{x}$ by solving $A\boldsymbol{x} = \boldsymbol{p}$, where $\boldsymbol{p}$ is ________."

- ◯ the projection of $\boldsymbol{b}$ on $C(A)$
- ◯ the projection of $\boldsymbol{b}$ on $C(A^T)$
- ◯ the projection of $\boldsymbol{b}$ on $N(A^T)$
- ◯ the projection of $\boldsymbol{b}$ on $N(A)$
- ◯ the pseudo-inverse of A

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15. (5 points) Suppose the equation $A\boldsymbol{x} = \boldsymbol{b}$ has no solution. The standard (not generalized) least squares method associated to the projection matrix $A(A^TA)^{-1}A^T$ can be used to find an approximate solution if and only if:

- ◯ $\boldsymbol{b}$ has a component in the left-nullspace of A
- ◯ A^TA is invertible
- ◯ The left-nullspace of A contains only the zero vector
- ◯ The nullspace of A contains only the zero vector

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16. (5 points) Which of the following statements are TRUE about the determinant of the square matrix A?

- ◯ $det(A) =$ sum of the eigenvalues of A
- ◯ $det(A) =$ product of the eigenvalues of A
- ◯ $det(A) =$ sum of the pivots of A
- ◯ $det(A) =$ product of the pivots of A
- ◯ A is invertible if and only if $det(A) > 0$

17. (5 points) Which of the following statements are TRUE?
   - ◯ If A is a (3×3) matrix and $\det(A) = 3$, A is diagonalizable using the eigendecomposition
   - ◯ If A is an invertible matrix, A does not have eigenvalues equal to 0
   - ◯ If the matrix B has only strictly positive eigenvalues, it is diagonalizable
   - ◯ If the matrix B has only strictly positive eigenvalues, B is symmetric
   - ◯ If a matrix has one or more eigenvalue(s) equal to zero, it is not diagonalizable
   - ◯ If all the N eigenvalues of a (N x N) matrix are different, then the matrix is invertible

18. (5 points) Consider the 3×3-dimensional matrix A with $|A - \lambda I| = \lambda(1-\lambda)(\lambda - 1)$. Which of the following statements are TRUE?
   - ◯ $\mathrm{Tr}(A) = 0$
   - ◯ One cannot compute the eigenvalues of A using only the provided information
   - ◯ The eigenvalues of A are 0, 1, and 0
   - ◯ $|A| = 0$

19. (5 points) Let $A = X\Lambda X^{-1}$ be the eigendecomposition of A with Λ the diagonal matrix containing the positive eigenvalues with no repetition and X the corresponding invertible matrix containing the eigenvectors of A as its columns. Which of the following statements are TRUE?
   - ◯ The matrix of eigenvectors of A^k is X^k
   - ◯ The matrix of eigenvalues of A^k is Λ^k
   - ◯ The matrix of eigenvalues of A^{-1} is Λ^{-1}
   - ◯ We cannot say what the matrix of eigenvalues of A^{-1} is
   - ◯ We cannot say what the matrix of eigenvectors of A^k is
   - ◯ We cannot say what the matrix of eigenvalues of A^k is

20. (5 points) Let $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = X\Lambda X^{-1}$. Which of the following statements are TRUE?
   - ◯ The eigenvalues of A are 5, 2, and 3
   - ◯ The eigenvectors of A are the columns of the identity matrix
   - ◯ X is invertible
   - ◯ $XX^T = I$, where I is the identity matrix
   - ◯ We cannot say if A is diagonalizable, but we can say that it is invertible

21. (5 points) You do not know the (3×3)-dimensional symmetric matrix A but you know that its eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 0$. Furthermore, you know the eigenvectors corresponding to λ_1 and λ_2, but you do not know the eigenvector corresponding to λ_3. Which of the following statements are TRUE?
   - ◯ The singular vectors of the SVD of A are the same as the eigenvectors of A
   - ◯ Using the known eigenvalues and eigenvectors, one can perfectly reconstruct the matrix A
   - ◯ The matrix A cannot be perfectly reconstructed unless one knows also the third eigenvector corresponding to $\lambda_3 = 0$
   - ◯ The singular values of A^TA are given by $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, and $\sigma_3 = \sqrt{\lambda_3}$
   - ◯ The singular values of A^TA are given by $\sigma_1 = \lambda_1^2$, $\sigma_2 = \lambda_2^2$, and $\sigma_3 = \lambda_3^2$

22. (5 points) The eigenvalues of the (4×4)-dimensional matrix A^TA are $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 9$, $\lambda_4 = 0$. Which of the following statements are TRUE?
   - ◯ The rank of the matrix A^TA is 3 whereas the rank of the matrix A is 4
   - ◯ The singular values of A cannot be computed without knowing the matrix A
   - ◯ The singular values of A are $\sigma_1 = 4^2$, $\sigma_2 = 2^2$, $\sigma_3 = 9^2$, and $\sigma_4 = 0^2$
   - ◯ The singular values of A are $\sigma_1 = -\sqrt{4}$, $\sigma_2 = -\sqrt{2}$, $\sigma_3 = -\sqrt{9}$, and $\sigma_4 = 0$
   - ◯ The rank of the matrix A^TA is 3 and the rank of the matrix AA^T is 3

23. (5 points) Which of the following statements are **always** TRUE about the equation $A\boldsymbol{x} = \boldsymbol{b}$?
   - ◯ If $\boldsymbol{b}$ is not in the column space of A, then $A^+\boldsymbol{b}$ gives an exact solution of the equation
   - ◯ $A^+\boldsymbol{b}$ is the unique solution to the equation
   - ◯ $A^+\boldsymbol{b}$ is always in the rowspace of A
   - ◯ $AA^+\boldsymbol{b}$ is always in the columnspace of A
   - ◯ $AA^+\boldsymbol{b}$ is always equal to $\boldsymbol{b}$

24. (5 points) Let $[\mathbf{u}]_E = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be the coordinates of $\mathbf{u}$ in the standard basis (i.e. basis vectors = columns of the identity matrix). Let B be a basis defined by the set of vectors $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$. What are the coordinates of $[\mathbf{u}]_B$?
   - ◯ $[\mathbf{u}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
   - ◯ $[\mathbf{u}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
   - ◯ $[\mathbf{u}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
   - ◯ There is an infinite number of possible coordinates in the B basis
   - ◯ $[\mathbf{u}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

25. (5 points) Consider the (100×30)-dimensional matrix A and the 100-dimensional vector $\mathbf{b}$. We are interested in finding a solution to $A\mathbf{x} = \mathbf{b}$. However, we are told that the data in the last 20 rows of A is unreliable. Which of the following statements are TRUE?

- ◯ Both weighted least squares and regularized least squares can help in such scenarios
- ◯ To put less weight on the least reliable data, we could use weighted least squares with $\mathbf{W} = \text{diag}\{\mathbf{w}\}$ and $\mathbf{w}$ being a 100-dimensional vector where the first 20 entries are $\frac{1}{2}$ and the remaining entries are 1, i.e.,

$$\mathbf{w} = [\underbrace{\frac{1}{2}\ \frac{1}{2}\ \dots\ \frac{1}{2}}_{20\ \text{elements}}\ 1\ 1\ \dots\ 1]^T$$

- ◯ To put less weight on the least reliable data, we could use weighted least squares with $W = \text{diag}\{\mathbf{w}\}$ and $\mathbf{w}$ being a 20-dimensional vector
- ◯ To put less weight on the least reliable data, we could use weighted least squares with $\mathbf{W} = \text{diag}\{\mathbf{w}\}$ and $\mathbf{w}$ being a 100-dimensional vector where the first 80 entries are 1 and the remaining entries are $\frac{1}{2}$, i.e.,

$$\mathbf{w} = [\underbrace{1\ 1\ \dots\ 1}_{80\ \text{elements}}\ \frac{1}{2}\ \frac{1}{2}\ \dots\ \frac{1}{2}]^T$$