The Discovery of an Algebraic structure

ASSIGBE Komi . RAHOUTI Chahid .

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1 Introduction

1.1 Definition

What is an Algebraic structure? An algebraic structure consists of a nonempty set A (called the underlying set, carrier set or domain), a collection of operations on A (typically binary operations such as addition and multiplication), and a finite set of identities, known as axioms, that these operations must satisfy.

Among the multiples algebraic structures, we can name:

- Group
- Ring
- Field
- Vector space
- .....

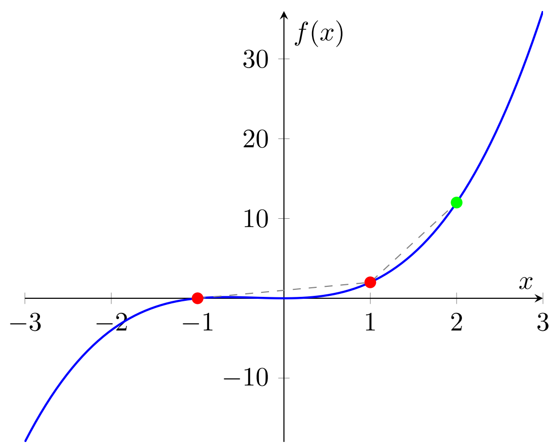
Example 1. *A simple example of a group for addition is the additive group of integers* $(\mathbb{Z}, +)$ *satisfies the following properties for any* $a, b, c \in \mathbb{Z}$*:*

- *Closure under the addition operation:* $a + b$ *is an integer.*
- *Associativity:* $(a + b) + c = a + (b + c)$*.*
- *Existence of the identity element: There exists an element* $0 \in \mathbb{Z}$ *such that* $a + 0 = a$ *for every* $a \in \mathbb{Z}$*.*
- *Existence of inverses: For each element* $a \in \mathbb{Z}$*, there exists an element* $-a \in \mathbb{Z}$ *such that* $a + (-a) = 0$*.*
- *Commutativity:* $a + b = b + a$ *for every* $a, b \in \mathbb{Z}$*.*

These properties make $(\mathbb{Z}, +)$ *a fundamental example of an additive group.*

2 Objective

Our problem is a mathematical problem known as data detection on structured surfaces or varieties. When presented with a dataset V defined on such structured surfaces, the challenge arises in determining whether there exists a discernible algebraic pattern within the data. Essentially, it's about investigating whether there are underlying mathematical relationships or structures governing the given dataset. This problem is crucial in various fields such as algebraic geometry and data analysis, where understanding these structures aids in making predictions or drawing meaningful conclusions from the data. Thus, the goal is to identify and characterize the algebraic properties inherent in the dataset for further analysis and interpretation.



We have three main sub-objectives to address:

1. Selecting a set of points in a space or variety and defining an algebraic structure on them.
2. Defining a vector space over this set of points using a technique based on a one-to-one function.
3. Implementing these two approaches in Python while minimizing the losses for each axiom.

3 First Objective

Let's take an example of a group structure. let consider a surface $M \in \mathbb{R}^{n\times n}$

Like, M $= \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$ with $x_i \in \mathbb{R}^d$

The question is if we consider M as a group structure, it is possible to find a binary operation ($\circ$) on M such that M is a group.

1. $\exists e \in M, \forall x \in M, e \circ x = x \circ e = x$
2. $\forall x, y \in M, x \circ y = y \circ x$
3. $\forall x, y, z \in M, (x \circ y) \circ z = x \circ (y \circ z)$
4. $\forall x \in M, \exists - x \in M, x \circ -x = -x \circ x = e$

4 Second Objective

let consider a set of points V and let $f : R \to V$ a one-to-one function from R into a codomain V. We define the vector addition by

$$x \oplus y = f(f^{-1}(x) + f^{-1}(y))$$

and the scalar multiplication by

$$\alpha \odot x = f(\alpha . f^{-1}(x))$$

The question is if we consider V as a group structure, it is possible to find a binary operation $(\oplus)$ and $(\otimes)$ on V such that this relations is satisfying.

Example 2 (Trivial Example)**.** *Let f be a function from R to $Vect(e_1)$ such that $f(x) = x.e_1$. we have $f^{-1}(x) = \lambda$. we take x and y in $vect(e_1)$, we have*
$x \oplus y = f(f^{-1}(x) + f^{-1}(y)) = x + y$
$\alpha \odot x = f(f^{-1}(x) * \alpha) = \alpha x$

Example 3. *Let β be any positive real number and let $f : \mathbb{R} \to \mathbb{R}_+^*$ be defined by $f(x) = (1/\beta)e^x$. Then f is a one-to-one function from $\mathbb{R}$ onto the set of positive real numbers, and $f^{-1}(x) = \ln(\beta x)$ for $x > 0$. we would define vector addition and scalar multiplication by*

$$x \oplus y = \frac{1}{\beta} e^{\ln(\beta x) + \ln(\beta y)} = \beta x y$$

$$\alpha \odot x = \frac{1}{\beta} e^{\alpha \ln(\beta x)} = \beta^{\alpha - 1} x^{\alpha}.$$

5 losses and Implementing

In implementing algebraic structures, the primary objective is to minimize the losses associated with each axiom governing these structures. Here are the four losses for the first approach

1. Existence of the identity element:

$$L_1(\theta) = \sum_{v \in V} (e_{\text{obs}} - e_{\text{pred}})^2$$

2. Commutativity:

$$L_2(\theta) = \sum_{(x,y)\in V\times V} (x \oplus y - y \oplus x)^2$$

3. Associativity:

$$L_3(\theta) = \sum_{(x,y,z)\in V\times V\times V} ((x \oplus y) \oplus z - x \oplus (y \oplus z))^2$$

4. Existence of inverses:

$$L_4(\theta) = \sum_{x\in V} (-x_{\text{obs}} + (-x_{\text{pred}}))^2$$

Here, θ represents the parameters of our model. These functions $L_i(\theta)$ measure the discrepancies between the observed and predicted values for each group axiom, where $i = 1, 2, 3, 4$. By minimizing these functions $L_i(\theta)$, we aim to adjust our model to be as close as possible to the real data, ensuring that our algebraic structure accurately satisfies the group axioms.

$$L(\theta) = \min_{\theta} L_1(\theta) + L_2(\theta) + L_3(\theta) + L_4(\theta)$$

This function $L(\theta)$ represents the sum of losses associated with each group axiom. By minimizing this function $L(\theta)$, we aim to adjust our model so that it optimally satisfies the four group axioms, ensuring the accuracy and consistency of our algebraic structure with respect to the provided data.
Here are the two losses for the second approach

1. Loss for vector addition:

$$L_1(\theta) = \sum_{(x,y)\in V\times V} \left\| x \oplus y - \left(f(f^{-1}(x)) + f(f^{-1}(y)) \right) \right\|^2$$

2. Loss for scalar multiplication:

$$L_2(\theta) = \sum_{(\alpha,x)\in \mathbb{R}\times V} \left\| \alpha \odot x - f(\alpha \cdot f^{-1}(x)) \right\|^2$$

Function $L(\theta)$, the sum of these two functions:

$$L(\theta) = L_1(\theta) + L_2(\theta)$$

Here, θ represents the parameters of our model. These functions $L_1(\theta)$ and $L_2(\theta)$ measure the discrepancies between the observed and predicted values for vector addition and scalar multiplication operations, respectively. By minimizing these functions, we aim to adjust our model to accurately represent the algebraic structure present in the dataset.

We implement all of these functions in Python using the PyTorch library to obtain the optimal parameters θ that minimize the losses and calculate the accuracy of our model. To effectively tackle this challenge, we leverage the Python programming language to develop and test these algebraic structures across diverse datasets

6 Application

Many differential equations encountered in solving have parametric solutions. Thus, to find the solution for each parameter, this often requires numerous calculations. To optimize computation and storage times, we define an algebraic structure whereby, if we compute the solution for parameters λ_1 and λ_2, we can deduce λ_3 ...

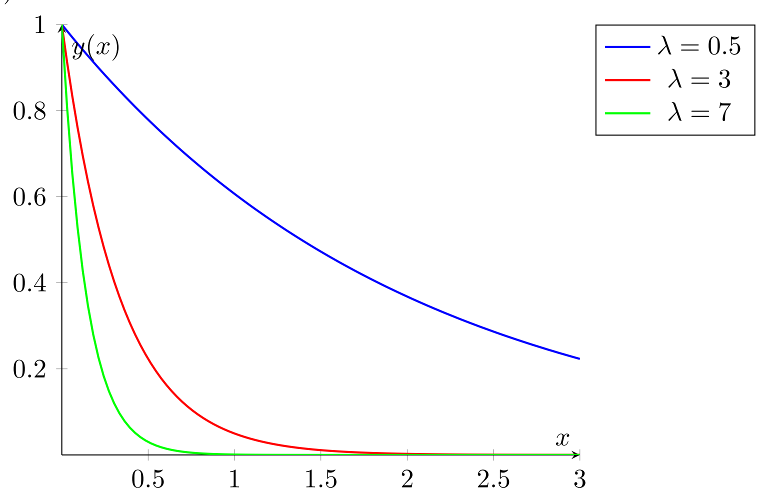
Example 4. *Consider the general form of a first-order linear ODE with a parameter λ:*

$$\frac{dy}{dx} + \lambda y = 0$$

The solution to this differential equation depends on the parameter λ, the solution to this ODE is given by:

$$y(x) = Ce^{-\lambda x}$$

where C is the constant of integration. The solution on $y(0) = 1$ is given by: $y(x) = e^{-\lambda x}$



In this example, the parameter λ affects the behavior of the solution function $y(x)$. Different values of λ lead to different solutions, each with its own characteristic behavior. Thus, the solution is a function with a parameter λ.

7 Conclusion

this project provides a comprehensive exploration of algebraic structures and their relevance in data analysis, offering insights into how these structures can be effectively applied to detect patterns and derive meaningful conclusions from datasets. Through Python implementation and testing, the project demonstrates practical approaches to optimize algebraic structures for improved accuracy and performance in various applications