Adaptive Implicit Schemes for Hyperbolic Equations

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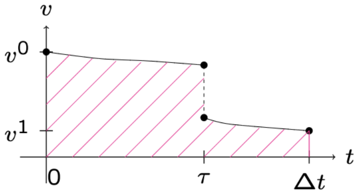
Introduction

We want to study Hyperbolic PDEs such as the linear Advection Equation:

$$\partial_t u + a\partial_x u = 0$$

But with a function u carrying a discontinuity (= a shock) in space! (and smooth everywhere else.)



Such continuity can be used to modelize a boundary between two materials.

Theta Schemes

Usually, representing a discontinuity in space is achievable without issues, but implementing a solver in time is another problem.
A fixed scheme that works well for a certain step may not for the next ones...

A Theta Scheme for this equation can be written like this:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\theta\partial_t u^{n+1} + a(1-\theta)\partial_x u^n = 0$$

The idea is to use θ as a parameter of our model responsible of handling the shock in time, in order to have a better scheme.

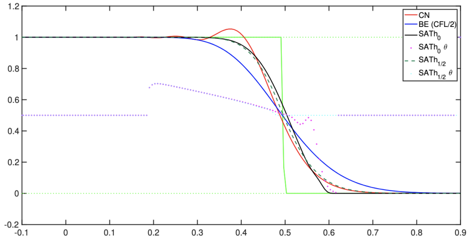
Self-Adaptive Theta Scheme

We want a small model that varies θ at each time step. We can define these bounds and conditions:

$$\theta_i^{n+1} = \begin{cases} max(\theta_{min}, h(u_i^n, u_i^{n+1})) & \text{if } |u_i^{n+1} - u_i^n| > \epsilon \\ \theta^* & \text{else} \end{cases}$$

We can visualize this type of scheme resolution with various values of θ_{min} and θ^*:

$u_t + u_x = 0$ for $0 < x < 1$ at $t = 0.5$



CFL 5, $\Delta x = 1/160$, $\Delta t = 5\Delta x$

We can already see some overlearning!

Goals

- Study the variations of the basic Theta scheme in function of θ.
- Find an optimal θ for the initial PDE through heuristic methods.
- Implement a Self-Adaptive Theta Scheme. Experiment with θ variations.
- Study a scheme based on the flux formula in space, with the Crank-Nicholson adaptive scheme in time.
- Expand the method to more complex Hyperbolic systems, non-linear Transport for example.

Bibliography

[1] Todd Arbogast and Chieh-Sen Huang. “Self-Adaptive Theta Schemes for solving Hyperbolic Conservation Laws”. In: *IMA Journal of Numerical Analysis* 42.4 (2022), pp. 3430–3463.

[2] Siddhartha Mishra, Ulrik Skre Fjordholm, and Rémi Abgrall. “Numerical methods for conservation laws and related equations”. In: ().