Control Barrier Functions

Internship M1

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Subject

- *High Relative Degree Control Barrier Functions Under Input Constraints* from Joseph Breeden and Dimitra Panagou (2021)
- Control Barrier Functions (CBF) to avoid collisions
- Understand the article, the case of simply avoiding the obstacle
- New objective : the case of getting closer to a target while avoiding the obstacle
- No time for 2 satellites

Some notation

If $h : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$:

$$\mathcal{L}_f h(x) = \langle \nabla h(x), f(x) \rangle$$

where $\nabla h(x)$ is the gradient of h at the point x.
If $h : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, we reformulate this derivative as

$$\mathcal{L}_g h(x) = \begin{bmatrix} \langle \nabla h(x), g_1(x) \rangle \\ \vdots \\ \langle \nabla h(x), g_m(x) \rangle \end{bmatrix}$$

where $g(x) = (g_1(x), ..., g_m(x))$ and $g_i : \mathbb{R}^n \to \mathbb{R}^n$.

Formulation of the problem

$$\begin{cases} \dot{y}(t) = \begin{bmatrix} \dot{r}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} v(t) \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} v(t) \\ 0_3 \end{bmatrix}}_{f(y(t))} + \underbrace{\begin{bmatrix} 0_{3\times 3} \\ I_{3\times 3} \end{bmatrix}}_{g(y(t))} u(t) \\ y(0) = x \end{cases}$$

$$\iff$$

$$\begin{cases} \dot{y}(t) = f\left(y(t)\right) + g\left(y(t)\right) u\left(y(t)\right) = F\left(t, y(t)\right) \\ y(0) = x \end{cases}$$

where
$$\begin{cases} x \in \mathbb{R}^n, \\ u \in U \ \text{ compact}, \\ f : \mathbb{R}^n \to \mathbb{R}^n \in \mathcal{C}^r, \\ g : \mathbb{R}^n \to \mathbb{R}^{n\times m} \in \mathcal{C}^r. \end{cases}$$

Definitions & Properties

Relative degree 2 functions

$h : \mathbb{R}^n \to \mathbb{R}$ is said to be of relative degree 2 with respect to the previous dynamics if

1. $h \in \mathcal{C}^2$ on an open,
2. $\mathcal{L}_g h(x) = 0 \;\; \forall x \in \mathbb{R}^n$
3. $\exists C \subseteq \mathbb{R}^n$ such that $\mathcal{L}_g \mathcal{L}_f h(x) \neq 0 \;\; \forall x \in C$.

Let $\mathcal{G}^2$ be the space of functions of relative degree 2 satisfying the given system.

Let $h \in \mathcal{G}^2$, we define the safe set S such that

$$S = \{x \in \mathbb{R}^n \mid h(x) \leq 0\}.$$

Formulation of the problem

Let

$$H(x) = \sup_{t \geq 0} h\left(\varphi_u\left(t, x\right)\right)$$

where $t \mapsto \varphi_u(t, x)$ is the solution of the previous ODE with $\varphi_u(0, x) = x$

and

$$S_H = \{x \in \mathbb{R}^n \mid H(x) \leq 0\}.$$

Now we set a control u^* that we have previously defined.

Definitions & Properties

ZCBF

For our dynamics, a function of class $\mathcal{C}^1$ $h : \mathbb{R}^n \to \mathbb{R}$ is a zeroing control barrier function (ZCBF) on S if

$$\inf_{u \in U} \left[\dot{h}(x,u) + h(x)\right] \leq 0 \ \forall x \in S$$

Lemma 1

If h is a ZBCF on S under our dynamics then for any Lipschitzian controller u such that

$$\dot{h}(x,u) \leq -h(x) \ \forall x \in S$$

will make the set S forward invariant.

H is a ZCBF

It can be shown that H is a ZCBF. Thus we have the following implications :

$$\begin{aligned} H \text{ ZCBF} &\Rightarrow \exists u \in U \;\; \dot{H}(x,u) + H(x) \leq 0 \\ &\Rightarrow \text{We can make } S_H \text{ forward invariant.} \end{aligned}$$

Illustration

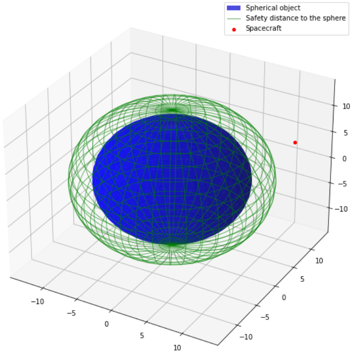


Figure 1: Illustration of the problem

- Spherical object : radius ρ_a and center r_a
- Minimum distance from the sphere : ρ_s
- Distance between the satellite and the safety bubble : h

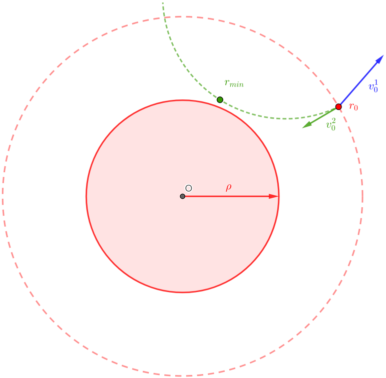
$$h(y(t)) = \rho - \|r_a - r(t)\|_2$$

where $\rho = \rho_a + \rho_s$ and we take $u^* = u_{ball}$.

Sketch of $H(x) = \sup h\left(\varphi_{u_{ball}}(t, x)\right)$

$$H(r_0, v_0^1) = r_0$$

$$H(r_0, v_0^2) = r_{min}$$



Control Lyapunov Function

Control Lyapunov Function (CLF)

$$V(y,t) = \frac{1}{2}\|r - r_p(t)\|_2^2 + \frac{1}{2}k_2\|v - k_1(r - r_p(t))\|_2^2$$

with k_1 and k_2 as constants.

Optimisation problem

The control function we are looking for here is defined by the following expression:

$$\begin{aligned}\tilde{u}(x,t) &= \underset{(u,\delta)\in C}{\operatorname{argmin}}\, u^T u + k\delta^2 \\ &= \underset{(u,\delta)\in C}{\operatorname{argmin}}\, \|(u, \sqrt{k}\delta) - (0_3, 0)\|_2^2\end{aligned}$$

$$C = \left\{ (u,\delta) \in \mathbb{R}^3 \times \mathbb{R} \;\middle|\; \begin{array}{c} \mathcal{L}_f H(x) + \mathcal{L}_g H(x) u + H(x) \leq 0 \\ \mathcal{L}_f V(x,t) + \mathcal{L}_g V(x,t) u + \delta k_3 V(x,t) \leq 0 \\ \|u\|_2 - u_{max} \leq 0 \end{array} \right\}.$$

Set of constraints

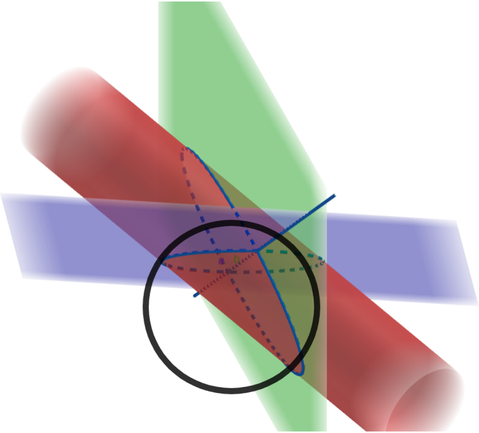


Figure 3: Illustration of constraints with Geogebra 3D

Resolution

We want to solve the following differential equation

$$\begin{cases} \dot{y}(t) = \begin{bmatrix} \dot{r}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} v(t) \\ \tilde{u}(y(t), t) \end{bmatrix} \\ y(0) = x \end{cases}$$

where $\tilde{u}$ is the solution of the following problem

$$\tilde{u}(x,t) = \underset{(u,\delta) \in C}{\operatorname{argmin}}\, u^T u + k\delta^2$$

How to solve integration

$$\dot{y}(t) = \begin{bmatrix} v(t) \\ \tilde{u}(y(t), t) \end{bmatrix} \implies \text{RK4 } (k_1, k_2, k_3, k_4)$$

$$\Downarrow$$

Calculate $\tilde{u}$ for each k

$$\Downarrow$$

Solve optimisation problem

$$\Downarrow$$

Determine C

$$\Downarrow$$

Another integration to find the sup $\Longleftarrow$ Calculate H and $\dot{H}$

Another integration to find the $\dot{H}$

Target point and conflict path

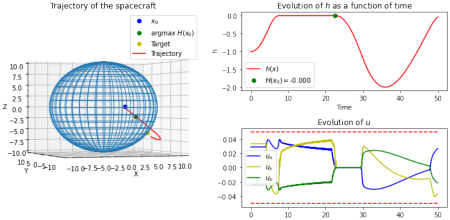


Figure 4: Target point

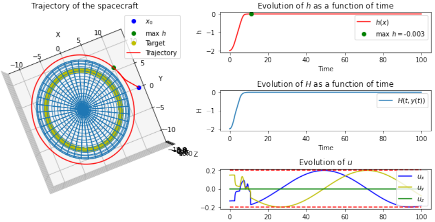


Figure 5: Conflict path

Target path

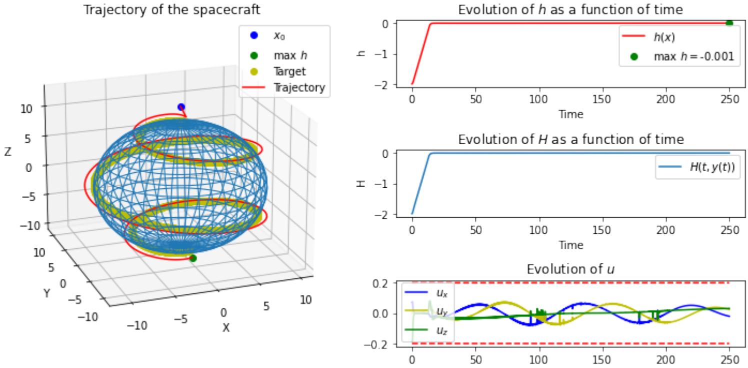


Figure 6: Beautiful trajectory

Gravitation

$$f(y) = \begin{bmatrix} v \\ f_\mu(r) \end{bmatrix} \quad \text{with} \quad f_\mu(r) = \mu \frac{r_a - r}{\|r_a - r\|^3}$$

where $\mu = \mathcal{G}M$ with G the gravitational constant and M the mass of the object.

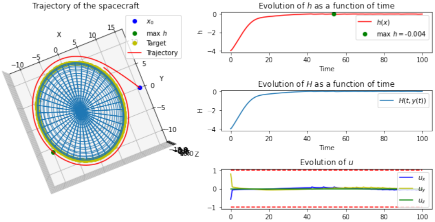


Figure 7: $\mu = 0$

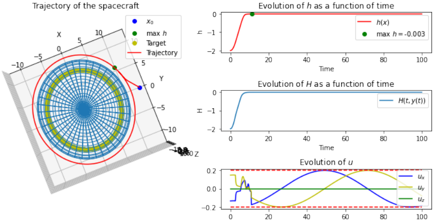
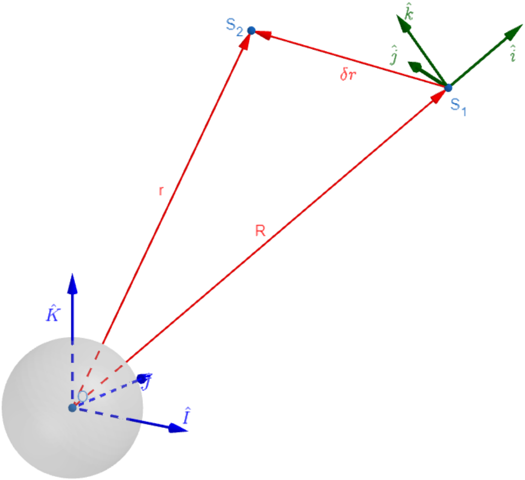


Figure 8: $\mu = 50$

Illustration of the problem



Linearization of the equations of relative motion in orbit

$$\begin{cases} R, V : & \text{The position and speed of } S_1 \\ \delta r, \delta v : & \text{The position and speed of } S_2 \text{ in the motion frame} \\ \Omega = \frac{h}{\|R\|} : & \text{The angular acceleration of } S_1 \end{cases}$$

$$\begin{cases} \dot{R} = V \\ \dot{V} = -\mu \frac{R}{\|R\|^3} \\ \delta \dot{r} = \delta v \\ \delta \ddot{x} = \left(\frac{2\mu}{\|R\|^3} + \frac{\|h\|^2}{\|R\|^4} \right) \delta x - 2\|h\| \frac{\langle V \mid R \rangle}{\|R\|^4} \delta y + 2 \frac{\|h\|}{\|R\|^2} \delta \dot{y} \\ \delta \ddot{y} = -\left(\frac{\mu}{\|R\|^3} - \frac{\|h\|^2}{\|R\|^4} \right) \delta y + 2\|h\| \frac{\langle V \mid R \rangle}{\|R\|^4} \delta x - 2 \frac{\|h\|}{\|R\|^2} \delta \dot{x} \\ \delta \ddot{z} = -\frac{\mu}{\|R\|^3} \delta z \end{cases}$$

Trajectories

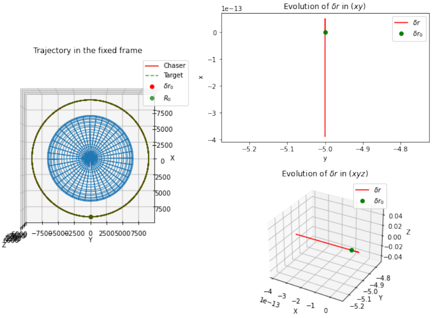


Figure 10: Circular orbit and $\delta r_0 = (0, -5, 0)$

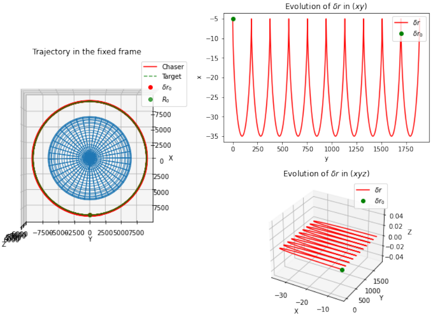


Figure 11: Circular orbit and $\delta r_0 = (-5, 0, 0)$

Trajectories

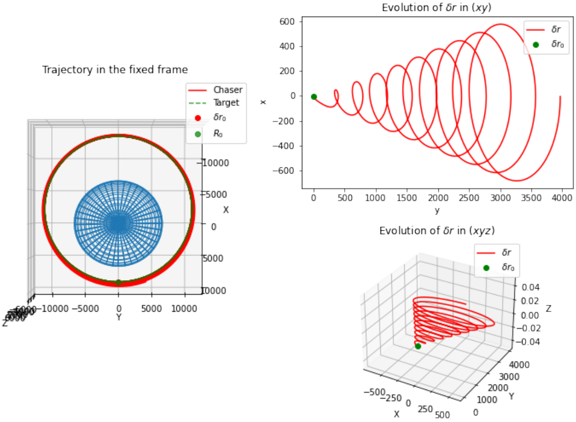


Figure 12: Elliptical orbit and $\delta r_0 = (-5, 0, 0)$

Hill-Clohessy–Wiltshire equations

$$\begin{cases} \langle V \mid R \rangle = 0, \\ \|h\| = \sqrt{\mu \|R\|} \\ n = \frac{\|V\|}{\|R\|} = \sqrt{\frac{\mu}{\|R\|^3}} \end{cases}$$

$$\delta r(t) = \Phi_{rr}(t)\delta r_0 + \Phi_{rv}(t)\delta v_0$$

and

$$\delta v(t) = \Phi_{vr}(t)\delta r_0 + \Phi_{vv}(t)\delta v_0$$

where

$$\Phi_{rr}(t) = \begin{bmatrix} 4 - 3\cos(nt) & 0 & 0 \\ 6\left(\sin(nt) - nt\right) & 1 & 0 \\ 0 & 0 & \cos(nt) \end{bmatrix}$$

$$\ldots$$

Trajectories

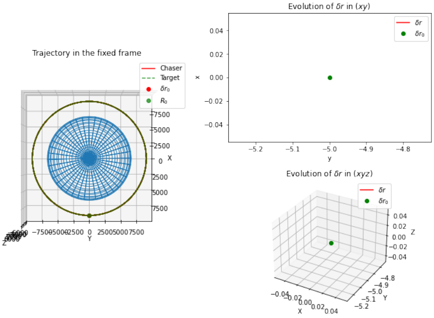


Figure 13: $\delta r_0 = (0, -5, 0)$

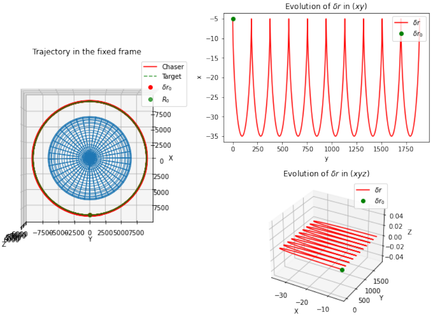


Figure 14: $\delta r_0 = (-5, 0, 0)$

With control

$$f_n\left(\delta r, \delta v\right) = \begin{bmatrix} 3n^2\delta x + 2n\delta\dot{y} \\ -2n\delta\dot{x} \\ -n^2\delta z \end{bmatrix}.$$

$$f\left(\delta r, \delta v\right) = \begin{bmatrix} \delta v \\ f_n(\delta r, \delta v) \end{bmatrix}$$

$$h(y) = \rho - \|\delta r\|$$

$$u^* = u_{ball}.$$

Trajectories

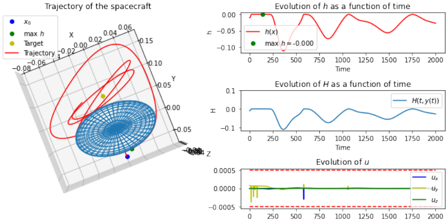


Figure 15: With H

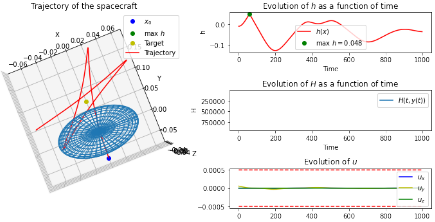


Figure 16: Without H

New optimisation problem

$$\tilde{u}(x,t) = \underset{\|u\| \leq u_{max}}{\operatorname{argmin}} \ \mathcal{L}_f V + \mathcal{L}_g V u$$

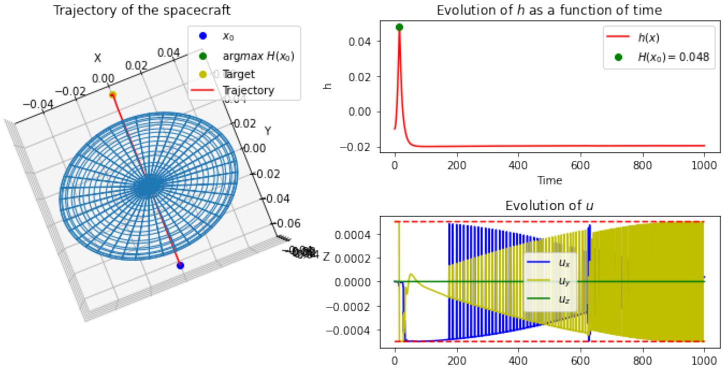


Figure 17: New optimisation problem for the attraction

New strategy

$$\tilde{u}(x,t) = \begin{cases} \text{argmin}_{\|u\| \leq u_{max}} & \mathcal{L}_f V + \mathcal{L}_g V u \quad \text{if } H(x) \leq -m \\ \text{argmin}_{(u,\delta) \in C} & u^T u + k\delta^2 \quad \text{else} \end{cases}$$

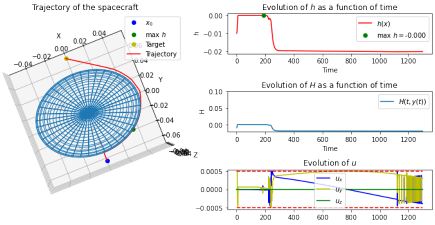


Figure 18: $m = 0.002$

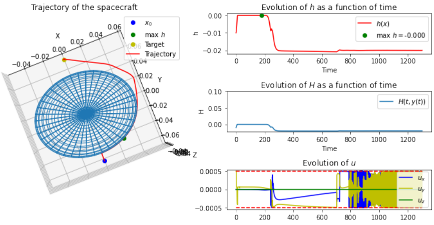


Figure 19: $m = 0.008$

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