UFR de mathématique et d'informatique
Université de Strasbourg

Inria

Reduced order models for transport

Internship presentation

Master 1 Calcul Scientifique et Mathématiques de l'Information

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Presented by
Claire Schnoebelen

Supervised by
Emmanuel Franck
Emmanuel Opshtein
Laurent Navoret

Objectives

1) Test reduced order model for Burgers' equation.
2) Understand reduced order models for linear Hamiltonian systems.
3) Test them on piano string model.
4) Think about how to improve reduction in the non-linear case.

Reduced order models

We consider the problem

$$\begin{cases} \partial_t u = \mathcal{F}_g(u, \partial_z u, \partial_{zz} u, \dots) & \forall z \in \Omega \quad \forall t \in I, \\ u(z,0) = u_0(z) & \forall x \in \Omega. \end{cases} \tag{1}$$

Hypothesis : It exists a differential manifold M and a *decoding* operator $D : \mathbb{R}^k \to M$, such that $u(z,t,g) = D(\tilde{u}(z,t,g))$, where $\tilde{u}(z,t,g) \in \mathbb{R}^k$ and $u(\cdot,\cdot,g) \in M$ the solution of (1).

The chain rule gives

$$\dot{u}(z,t,g) = \mathcal{J}_D(\tilde{u}(z,t,g))\dot{\tilde{u}}(z,t,g).$$

Using Galerkin projection, we get

$$\dot{\tilde{u}}(z,t,g) = \mathcal{J}_D^+(\tilde{u}(z,t,g))D(\tilde{u}(z,t,g)),$$

with $\mathcal{J}_D^+$ the Moore-Penrose inverse of the jacobian of D $\mathcal{J}_D$.

Symplectic structures [Arnold, 1989]

Definition (Alternate form)

A *k-form* ω on $\mathbb{R}^n$ is a k-linear skew-symmetric application of $\mathbb{R}^n$ in $\mathbb{R}$.

Definition (Differential form)

Let M be a differential manifold. A *k-differential form* ω is a collection of k-forms $\omega_x : (T_x M)^k \to \mathbb{R}$ defined at each $x \in M$ and varying differentiably with x.

Definition (Symplectic form)

We call a *symplectic form* on M a differential 2-form which is closed and non-degenerate.
The manifold M endowed with this form is called a *symplectic manifold*.

Symplectic matrices and symplectic inverse [Afkham and Hesthaven, 2017]

Definition (Symplectic map)

Let (M, ω) and (N, η) be two symplectic manifolds. A differentiable map $f : M \to N$ is said to be *symplectic* if $f^*\eta = \omega$.

Take $M = \mathbb{R}^{2n}$ and note $x \in \mathbb{R}^{2n}$ as (p, q) with $p, q \in \mathbb{R}^n$. Consider the symplectic form $\omega_{2n}^2 = \sum_{i=1}^n p_i dq_i$ and the usual euclidian structure on $\mathbb{R}^{2n}$.

We have

$$\omega_{2n}^2(\cdot,\cdot) = \langle \cdot, \mathbb{J}_{2n} \cdot \rangle, \qquad \mathbb{J}_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

so

$$\omega_{2n}^2(A\cdot, A\cdot) = \omega_{2k}^2(\cdot,\cdot) \quad \iff \quad A^T \mathbb{J}_{2n} A = \mathbb{J}_{2k}. \tag{2}$$

Matrices verifying (2) are said to be *symplectic*.

We call $A^+ := \mathbb{J}_{2k} A \mathbb{J}_{2n}$ the *symplectic inverse* of A.

Hamiltonian systems [Arnold, 1989]

Let (M, ω) be a symplectic manifold of even dimension $2n$.

Consider a differentiable function $H : M \to \mathbb{R}$.

$$\text{Non-degenerecy of } \omega \quad \Longrightarrow \quad \exists X_H \in \Gamma(M) \quad | \quad \omega(X_H, \cdot) = dH.$$

Definition

A *Hamiltonian equation* is a differential equation of the form

$$\dot{z}(t) = X_H(z(t)), \quad \forall t \in I. \tag{3}$$

Proposition

1) Solutions of (3) conserve H,
2) ω and the phase space volume are preserved by the phase flow of (3).

Reduced order models for Hamiltonian systems [Afkham and Hesthaven, 2017]

Goal: for $M = \mathbb{R}^{2n}$, build linear decoder A and encoder B such that :

- A preserves ω^2,
- $\mathcal{L}(A) = \|Z - BAZ\|$ has a small value.

Problem: the set of symplectic matrices is not bounded.

Greedy algorithm [Afkham and Hesthaven, 2017]

After iteration k, we have $A_k \in \mathcal{M}_{2n,2k}$. At iteration $k+1$:

1) Choice of the parameter g_{k+1} maximising

$$\Delta H_k(t) := |H(z(t)) - H_(A_k y_k(t))|.$$

2) Computation of the solution for this parameter: $S := \{z(t_i, g_{k+1})\}_{i=1,\dots,m}$.

3) Choice of the sample with the "worst" projection:

$$s_{k+1} = \mathrm{argmax}_{s \in S} \|s - A_k A_k^+ s\|_2.$$

4) Obtain $\tilde{v}$ with Gramm-Schmidt symplectic orthogonalization procedure.

5) Set $A_{k+1} = \Big(A_k[:, : k] \quad | \quad \tilde{v} \quad | \quad A_k[:, : k] \quad | \quad \mathbb{J}_{2n}\tilde{v} \Big)$.

Application to piano vibrating strings [Chabassier and Joly, 2010]

General model:

$$\partial_{tt}^2 U(x,t) = \partial_x \Big[\nabla V(\partial_x U(x,t)) \Big], \qquad \forall (x,t) \in [0,1] \times \mathbb{R}_+,$$

$U(x,t) = (v(x,t), u(x,t))$: variations of x on the oscillation plane

V : real differentiable function of $\mathbb{R}^{2n}$.

with Hamiltonian formulation with:

$$H : (p,q,t) \mapsto \int_{[0,1]} \frac{1}{2}|p|^2 + V(\partial_x q) dx.$$

Linear model: $V(u,v) = \frac{(1-\alpha)}{2} u^2 + \frac{1}{2} v^2$, $\alpha \in [0,1]$.

Non-linear model: $V(u,v) = \frac{1-\alpha}{2} u^2 + \frac{1}{2} v^2 + \frac{\alpha}{2}(u^2 v + \frac{1}{4} u^4)$, $\alpha \in [0,1]$.

Application to piano vibrating strings : linear model

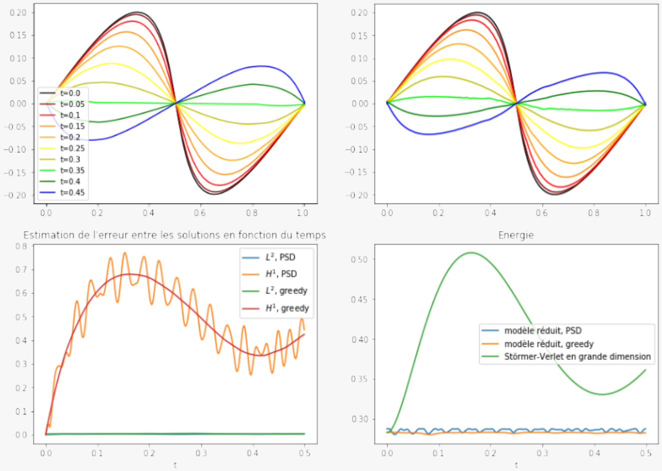

Figure 1: Numerical solution of piano string equation with $V = \frac{1}{2}((1-g)u^2 + v^2)$, where $g = 0.537$. The reductions were made using PSD with $k = 5$.

Application to piano vibrating strings : non-linear model

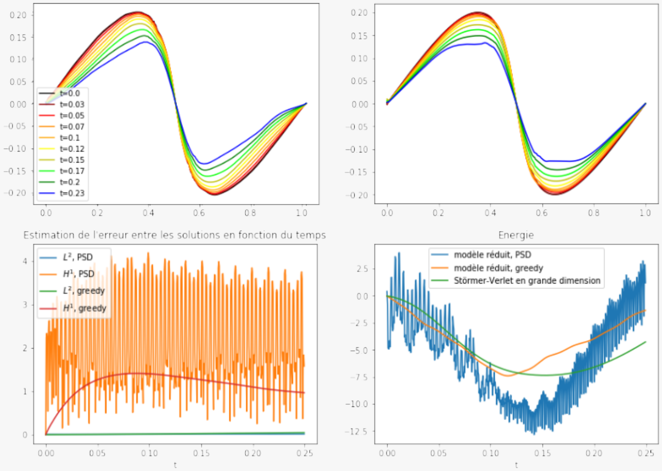

Figure 2: Numerical solution of piano string equation with non-linear model, where $g = 0.8$. The reductions were made using PSD with $k = 5$.

Application to piano vibrating strings : non-linear model

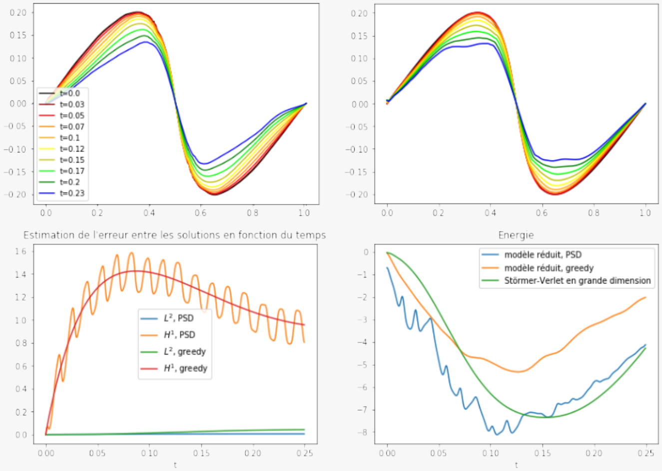

Figure 3: Numerical solution of piano string equation with non-linear model, where $g = 0.8$. The reductions were made using PSD with $k = 20$.

Hyper-reduction

Goal: faster the computation of the reduced model

Kernel regression: build $\bar{H}$ of the form

$$\bar{H}(x) = \sum_{i=1}^{m} \theta_i K(x^i, x),$$

with K a kernel function.

Loss:

$$\mathcal{L}_{\alpha,\beta}(\theta) = \alpha \sum_{i=1}^{m} \| \frac{A^+ z^{i+1} - A^+ z^i}{\Delta t} - \mathbb{J}_{2k} \nabla \bar{H}(x^i) \|^2 + \beta \sum_{i=1}^{m} |\bar{H}(x^i) - H(Ax^i)|^2.$$

Hyper-reduction : results

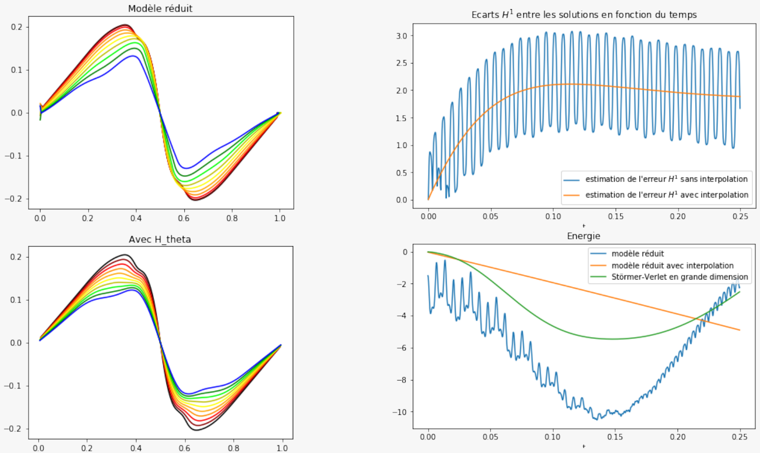

Figure 4: Numerical solution of piano string equation with non-linear model, where $g = 0.783$. The first column represents the piano string position in the oscillation plane for different times computed in low dimension with $H \circ A$ (top) and H_θ (bottom). The second column represents an estimation of the errors made on the string position through time on H^1 norm (top) and the variation of energy durng time for the three different solutions (bottom). The reduction was made using PSD with $k = 5$.

Conclusion

- Symplectic structure induce particluar constraint on the reduced model,
- useful to find good reduced model
- PSD and greedy algorithm good for linear systems...
- ...but fail for non-linear ones.
- Hyper-reduction gives bad results
- and shows that we have to take into account ∇H (take $\alpha \neq 0$).
- Ideas to look into with generating functions but theorical difficulties...

Afkham, B. and Hesthaven, J. (2017).
Structure preserving model reduction of parametric hamiltonian systems.
Preprint at https://arxiv.org/abs/1703.08345.

Arnold, V. (1989).
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Springer.

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Computer Methods in Applied Mechanics and Engineering, 199(45-48):2779–2795.