Kalman Filter for Solving Parameterized Partial Differential Equations (PDEs)

Abdoul Aziz Sarr

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Presentation Outline

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Introduction

- Parameterized PDEs model many physical phenomena in mechanics, thermics, engineering.
- Parameters include physical constants, domain shapes, boundary conditions.
- The Kalman filter estimates system state optimally in noisy linear settings.
- It fuses numerical model predictions and noisy observational data.
- Application: Assimilation of measurements into PDE simulations to improve solution accuracy and robustness.

Problem Description: Heat Equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) = f \quad \text{in } \Omega$$

- u: temperature,
- k: thermal diffusivity,
- f: source term,
- Ω: spatial domain.

Numerical solution uses `Feel++`, a finite element method library.

Numerical Tool: Feel++ Library

- C++/Python library for PDE resolution (Galerkin method).
- Mesh generation (structured/unstructured).
- Optimized linear/non-linear solvers for parallel architectures

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Kalman Filter Overview

- Recursive algorithm for optimal state estimation under uncertainty.
- Combines model prediction and noisy measurements.
- Provides correction to predictions to improve accuracy.
- Especially powerful for dynamic systems and noisy data.

Kalman Filter: Prediction Step

$$X_{k|k-1} = A_k X_{k-1|k-1} + B_k u_k$$
$$P_{k|k-1} = A_k P_{k-1|k-1} A_k^\top + Q_k$$

- $X_{k|k-1}$: predicted state vector,
- $P_{k|k-1}$: predicted covariance,
- A_k: model matrix,
- Q_k: process noise covariance.

Kalman Filter: Observation Step

$$Z_k = H_k X_{k|k-1} + v_k$$

- Z_k: measurement vector,
- H_k: observation matrix (selects measured nodes),
- v_k: measurement noise with covariance R_k.

Kalman Filter: Update Step

$$K_k = P_{k|k-1} H_k^\top (H_k P_{k|k-1} H_k^\top + R_k)^{-1}$$
$$X_{k|k} = X_{k|k-1} + K_k (Z_k - H_k X_{k|k-1})$$
$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$

- K_k: Kalman gain matrix,
- $X_{k|k}$: updated state,
- $P_{k|k}$: updated covariance.

Summary of Kalman Filter Variables

- X_k: state vector (temperature values at nodes).
- P_k: error covariance matrix.
- Q_k: process noise covariance.
- H_k: observation matrix.
- R_k: measurement noise covariance.
- K_k: Kalman gain matrix.

Algorithm Workflow

1. Initialize X_0 and P_0.
2. For each time step k:
    - Predict: $X_{k|k-1}, P_{k|k-1}$.
    - Observe noisy measurements Z_k.
    - Compute gain K_k and update $X_{k|k}$.
    - Update error covariance $P_{k|k}$.

Naturally integrated with time-stepping PDE solutions.

Importance and Robustness of Kalman Filter

- Adapts correction of predictions under model and measurement uncertainties.
- Provides consistent solutions even with approximate models and noisy or partial data.
- Reduces overall error and enhances numerical stability.
- Produces solutions less sensitive to noise and closer to reality.

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Setting up the PDE Problem with Feel++

- Define exact solution and gradient for comparison.
- Specify source term and thermal diffusivity.
- Set initial conditions and boundary conditions.
- Choose temporal discretization parameters (time step, final time).
- Configure noise levels for measurements and process.

Integration of Kalman Filter parameters for assimilation.

Mesh Sizes and Numerical Runs

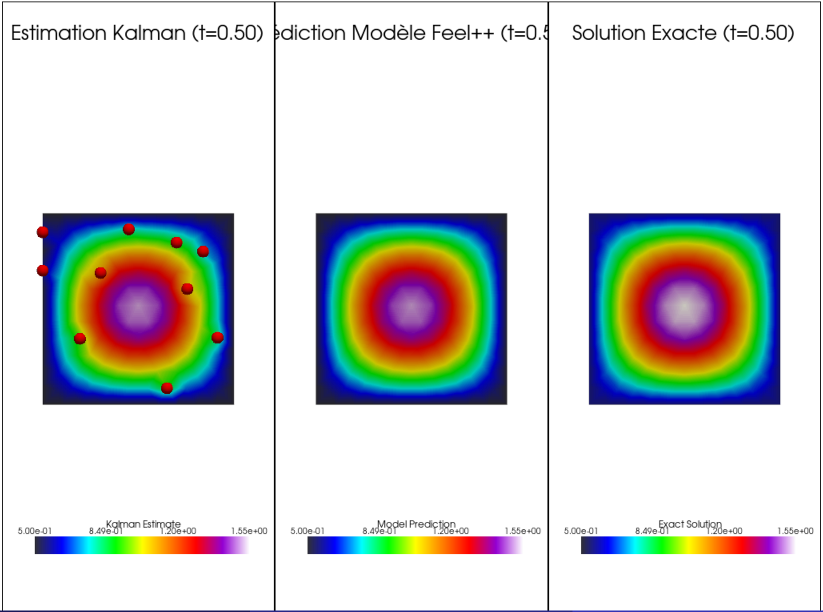
- Simulations are ran for different mesh sizes: coarse to fine.
- For each mesh:
  - Run numerical PDE resolution.
  - Apply Kalman filter assimilation with measurements.
  - Store and analyze estimation results.

Allows study of convergence and filter effect.

Results Visualization Overview

- Comparison among:
  - Kalman filter estimation,
  - Feel++ model prediction,
  - Exact analytical solution.
- Time snapshot: $t = 0.50$.
- Mesh sizes analyzed: $h = 0.1$, $h = 0.05$, $h = 0.025$.

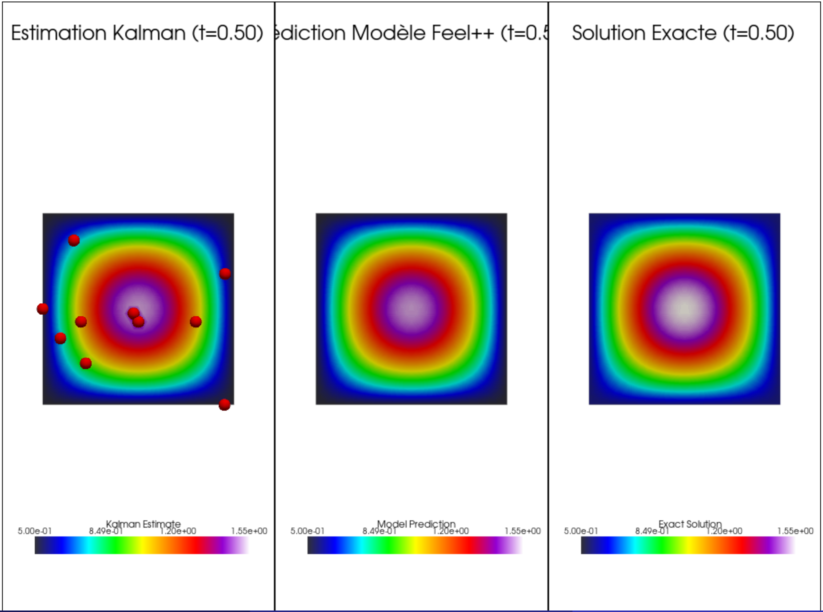
Results for Mesh Size $h = 0.1$



Results for Mesh Size $h = 0.1$

- Kalman estimate close to exact solution despite coarse mesh.
- Measurement points visibly improve prediction locally.
- Model prediction already good; Kalman filter refines around observations.
- Improves solution robustness against model and mesh uncertainties.

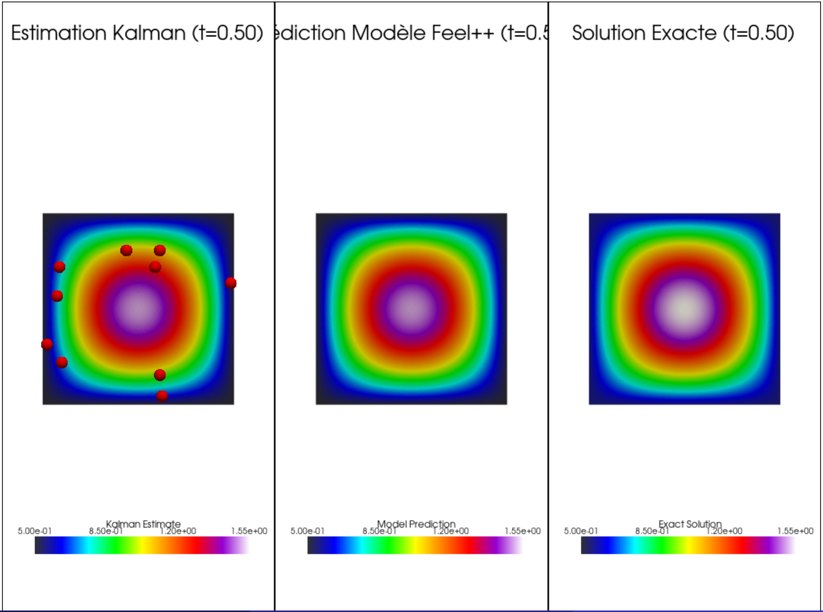
Results for Mesh Size $h = 0.05$



Results for Mesh Size $h = 0.05$

- Model prediction nearly identical to the exact solution.
- Kalman filter fine-tunes residual noise and errors.
- Influence of measurements less visually obvious due to model accuracy.
- Demonstrates rapid convergence with mesh refinement and filter.

Results for Mesh Size $h = 0.025$



Results for Mesh Size $h = 0.025$

- Kalman estimate, model prediction, and exact solution visually indistinguishable.
- Excellent convergence of numerical scheme.
- Filter ensures robustness in presence of measurement and process noise.
- Provides subtle but essential corrections to maintain solution optimality.

Smoothing Effect of the Kalman Filter

- Produces smooth, coherent field estimates without measurement artifacts.
- Fusion of model and observations attenuates noise impact.
- Visual result: smooth color transitions, well-defined contours.
- Key to interpreting noisy measurement data reliably.

Robustness Across Mesh Refinement

- Effective estimate maintained even on coarse mesh ($h = 0.1$).
- With mesh refinement, prediction improves, but filter assimilates residual uncertainties.
- Guarantees solution reliability close to physical reality.
- Acts as continuous regularization and correction tool.

Practical Applications and Impact

- Real-time assimilation of experimental data into simulations.
- Improved solution quality in engineering and physics problems.
- Risk assessment and control relying on accurate state estimates.
- Robustness against sensor noise and incomplete data.

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Conclusion

- The Kalman filter successfully blends numerical models with noisy observations.
- Enhances solution accuracy, stability, and robustness for parameterized PDEs.
- Effective even with coarse discretizations; essential for realistic simulations.
- Future work may include extension to nonlinear models and adaptive filtering.