2. Prologue

2.1 Introduction

2.2 Computational complexity

2.3 Time complexity of common functions

2.4 Recurrence relation

2.5 Fibonacci

2.4 Recurrence relation

- Why recurrence relation?

- What is the time complexity of this function?

```
int Fib ( int n )
{
    if ( n == 0 || n == 1 )
        return n;

    return Fib (n-1) + Fib (n-2);
}
```

$$T(n) = T(n-1) + T(n-2) + k$$

2.4 Recurrence relation

- Recurrence relation?

$$t_n = at_{n-1} + b$$

$$T(n) = aT(n/b) + n^d$$

- An equation in which each term of the sequence is defined as a function of the preceding terms
- In many cases, the time complexity is represented by the recurrence relations
- Solution

  (1) Characteristic equation

  (2) Repeated substitution or telescoping

  (3) Master theorem

2.4 Recurrence relation

(1) Characteristic equation

- An equation created from a recurrence relation by substituting t_n by x^n
- Homogeneous recurrence relation

$$\boxed{a_0 t_n + a_1 t_{n-1} + \ldots + a_k t_{n-k} = 0}$$

- Inhomogeneous recurrence relation

$$\boxed{a_0 t_n + a_1 t_{n-1} + \ldots + a_k t_{n-k} = b^n p(n)}$$

2.4 Recurrence relation

(1) Characteristic equation

- Homogeneous recurrence relations

$$\boxed{a_0 t_n + a_1 t_{n-1} + \ldots + a_k t_{n-k} = 0}$$

- Strategy
  1. Build a characteristic equation
     - Set $t_n = x^n$. $\boxed{a_0 x^n + a_1 x^{n-1} + \ldots + a_k x^{n-k} = 0}$
  2. Solve characteristic equation

$$\boxed{a_0 x^k + a_1 x^{k-1} + \ldots + a_k = 0}$$

     - Solutions: $r_1, r_2, \ldots, r_k$.
  3. Solution to the recurrence relation

$$\boxed{t_n = \sum_{i=1}^{k} c_i r_i^n}$$

2.4 Recurrence relation

(1) Characteristic equation

- Homogeneous recurrence relations
  - Example
    - Example 1:

$$\boxed{t_n - 3t_{n-1} - 4t_{n-2} = 0, \quad n \geq 2, t_0 = 0, t_1 = 1.}$$

    - Example 2:

$$\boxed{t_n = t_{n-1} + t_{n-2}, \quad n \geq 2, t_0 = 0, t_1 = 1.}$$

    - Example 3:

$$\boxed{t_n = 5t_{n-1} - 8t_{n-2} + 4t_{n-3}, \quad n \geq 3, t_0 = 0, t_1 = 1, t_2 = 2.}$$

2.4 Recurrence relation

(1) Characteristic equation

- Inhomogeneous recurrence relations

$$a_0 t_n + a_1 t_{n-1} + \ldots + a_k t_{n-k} = b^n p(n)$$

- Case 1: p(n) = 1

$$t_n - 2t_{n-1} = 3^n.$$

- Case 2: p(n): a polynomial in n of degree d

$$t_n - 2t_{n-1} = (n+5)3^n.$$

2.4 Recurrence relation

(1) Characteristic equation

- Inhomogeneous recurrence relations
  - Case 1: p(n) = 1

$$t_n - 2t_{n-1} = 3^n.$$

- Convert to homogeneous recurrence relation and solve it.

$$t_{n+1} - 2t_n = 3^{n+1} = 3 \cdot 3^n$$
$$3t_n - 6t_{n-1} = 3 \cdot 3^n$$

$$t_{n+1} - 5t_n + 6t_{n-1} = 0$$

2.4 Recurrence relation

(1) Characteristic equation

- Inhomogeneous recurrence relations
  - Case 2: p(n): a polynomial in n of degree d

$$\boxed{t_n - 2t_{n-1} = (n+5)3^n.}$$

- Convert to homogeneous recurrence relation and solve it.

$$\boxed{\begin{aligned} &9t_n - 18t_{n-1} = (n+5)3^{n+2} \\ &t_{n+2} - 2t_{n+1} = (n+7)3^{n+2} \\ &-6t_{n+1} + 12t_n = -6(n+6)3^{n+1} \end{aligned}}$$

$$\boxed{t_{n+2} - 8t_{n+1} + 21t_n - 18t_{n-1} = 0}$$

2.4 Recurrence relation

(2) Repeated substitution

- Continually substitute the recurrence relation on the right hand side
- Substitute a value into the original equation and then derive a previous version of the equation

- Examples

  ① $T(n) = T(n-1) + n$

  $T(1) = 1$

  ② $T(n) = 2T(n/2) + n$

  $T(1) = 1$

2.4 Recurrence relation

(2) Repeated substitution

① $T(n) = T(n-1) + n$

$T(1) = 1$

$$T(n) = T(n-1) + n$$
$$T(n-1) = T(n-2) + (n-1)$$
$$T(n-2) = T(n-3) + (n-2)$$
$$\dots$$
$$T(2) = T(1) + 2$$

Sum them all

$$T(n) = T(1) + 2 + 3 + \dots + n = n(n+1)/2 = O(n^2)$$

2.4 Recurrence relation

(2) Repeated substitution

② $T(n) = 2T(n/2) + n$

$T(1) = 1$

$$T(n) = 2T(n/2) + n$$
$$T(n/2) = 2T(n/4) + n/2$$
$$T(n/4) = 2T(n/8) + n/4$$
$$\ldots$$
$$T(2) = 2T(1) + 2$$

$$T(n) = 2T(n/2) + n$$
$$2T(n/2) = 4T(n/4) + n$$
$$4T(n/4) = 8T(n/8) + n$$
$$\ldots$$
$$2^{k-1}\,T(2) = 2^k\,T(1) + n$$
$$(n = 2^k)$$

$$T(n) = n + n + \ldots + n$$
$$= (k+1)\,n = n \log n + n$$
$$= O(n \log n)$$

2.4 Recurrence relation

(3) Master theorem

– If $T(n) = aT\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$ for a > 0, b > 1, and d≥0, then

$$T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b a \\ O(n^d \log n), & \text{if } d = \log_b a \\ O(n^{\log_b a}), & \text{if } d < \log_b a \end{cases}$$

– Example: Merge sort

$$T(n) = 2T(n/2) + n$$

Quiz6

Solve T(n) for the following questions.

- $T(n) = 2\,T(n/3) + c$
- $T(n) = 3\,T(n/2) + c$
- $T(n) = 2\,T(n/4) + n$
- $T(n) = T(n/2) + c$