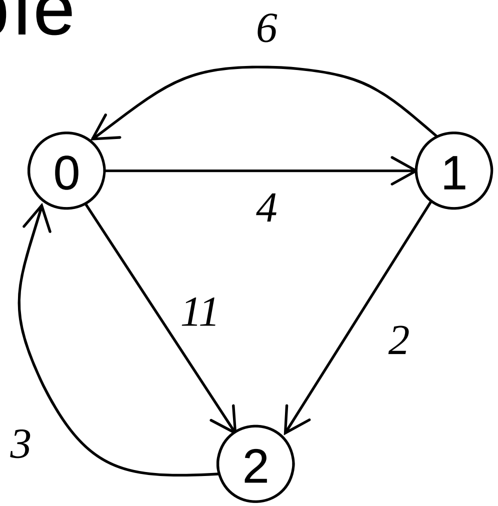
Contents

6.4 All Pairs Shortest path

- Problem: All-pairs shortest path (feat. 4.9)
  - The problem of finding shortest paths for every two vertices *v* to *u*.
  - Solving single-source shortest path for all vertices in G

- Example



	0	1	2
0			
1			
2			

6.4 All Pairs Shortest path

- Algorithm: Floyd's algorithm
  - Finding the all-pair's shortest path.
    - Input: adjacency matrix of a graph.
    - Output: minimum cost distance + path.
  - The weight of a path between two vertices is the sum of the weights of the edges along that path.
    - Negative weight is allowed.
    - Negative cycle is not allowed.

$A^k[i][j] = \min \{ A^{k-1}[i][j], A^{k-1}[i][k] + A^{k-1}[k][j] \}$

$A^{-1}[i][j] = w[i][j]$

6.4 All Pairs Shortest path

- Strategy
  - Suppose we wish to find a shortest path from vertex i to vertex j.
  - Let A_i be the vertices adjacent to i.
  - Which of the vertices in A_i should be the second vertex?

6.4 All Pairs Shortest path

- Strategy
  - Principle of optimality

Whatever the initial state and decision are, the remaining decisions must constitute an optimal decision sequence

- Let $A_1 = \{i_x\}$.
- The optimal path will be $i \rightarrow i_x \rightarrow \dots \rightarrow j$.
- Whatever i_x will be, $i_x \rightarrow j$ must be optimal.

$$Cost(i, i_1, i_2, \ldots, j) = \min(Cost(i, i_x) + Cost(i_x, \ldots, j))$$

6.4 All Pairs Shortest path

- Strategy
  - Let k be an intermediate vertex on a shortest path from i to j such that {i → ... → k → ... → j}.
    - k is max (i_1, i_2, ..., k, p_1, p_2, ...)
  - Path {i → i_1 → i_2 → ... → k} is shortest from i to k
  - Path {k → p_1 → p_2 → ... → j} is shortest from k to j

$$
\begin{aligned}
&Cost(i, i_1, i_2, ..., k, p_1, p_2, ..., j) \\
&= Cost(i, i_1, i_2, ..., k,) + Cost(k, p_1, p_2, ..., j)
\end{aligned}
$$

6.4 All Pairs Shortest path

- Algorithm: Floyd's algorithm
  - $A^k[i][j]$:
    - The cost of the shortest path from vertex i to j, using only those intermediate vertices with an index $\leq$ k.
  - $A^{-1}[i][j]$: the weight of an edge connecting vertex i and vertex j

Minimum cost with those vertices less than k

Minimum cost from i to k those vertices less than k

Minimum cost from k to j those vertices less than k

$$A^k[i][j] = \min \{ A^{k-1}[i][j], A^{k-1}[i][k] + A^{k-1}[k][j] \}$$

$$A^{-1}[i][j] = w[i][j]$$

6.4 All Pairs Shortest path

- Algorithm: Floyd's algorithm
  - Answer: $A^n[i][j]$:
    - The cost of the shortest path from vertex i to j, using only those intermediate vertices with an index $\leq$ n.
    - n: number of vertices

6.4 All Pairs Shortest path

- Algorithm: Floyd's algorithm
  - Answer: $A^n[i][j]$:
    - The cost of the shortest path from vertex i to j, using only those intermediate vertices with an index $\leq$ n.
    - n: number of vertices

$$A^n[i][j] = \min \{ A^{n-1}[i][j], A^{n-1}[i][n] + A^{n-1}[n][j] \}$$

$$A^{n-1}[i][j] = \min \{ A^{n-2}[i][j], A^{n-2}[i][n-1] + A^{n-2}[n-1][j] \}$$

...

$$A^1[i][j] = \min \{ A^0[i][j], A^0[i][1] + A^0[1][j] \}$$

$$A^0[i][j] = \min \{ A^{-1}[i][j], A^{-1}[i][0] + A^{-1}[0][j] \}$$

$$A^{-1}[i][j] = w[i][j]$$

6.4 All Pairs Shortest path

- Algorithm: Floyd's algorithm

```
void Floyd ( int cost[][], int dist[][], int n )
{
    int i, j, k;
    for ( i = 0; i < n; i++ )
        for ( j = 0; j < n; j++ )
            dist[i][j] = cost[i][j];

    for ( k = 0; k < n; k++ ) {
         for ( i = 0; i < n; i++ )
             for ( j = 0; j < n; j++ )
                 dist[i][j] = min(dist[i][j], dist[i][k] + dist[k][j];
    }
}
```