Advanced Data Structure and Algorithm

Asymptotic Analysis

The plan

- Sorting Algorithms
  - InsertionSort: does it work and is it fast?
  - MergeSort: does it work and is it fast?
  - Skills:
    - Analyzing correctness of iterative and recursive algorithms.
    - Analyzing running time of recursive algorithms

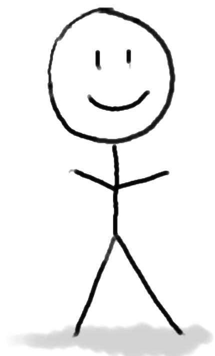
- How do we measure the runtime of an algorithm?
  - Worst-case analysis
  - Asymptotic Analysis

Worst-case analysis

Sorting a sorted list should be fast!!

The "running time" for an algorithm is its running time on the worst possible input.

Here is my algorithm!



Algorithm:
  Do the thing
  Do the stuff
  Return the answer

Algorithm designer

Here is an input! (Which I designed to be terrible for your algorithm!)

Big-O notation

- What do we mean when we measure runtime?

  - We probably care about wall time: how long does it take to solve the problem, in seconds or minutes or hours?
- This is heavily dependent on the programming language, architecture, etc.
- These things are very important, but are not the point of this class.
- We want a way to talk about the running time of an algorithm, independent of these considerations.

Main idea:

Focus on how the runtime **scales** with n (the input size).

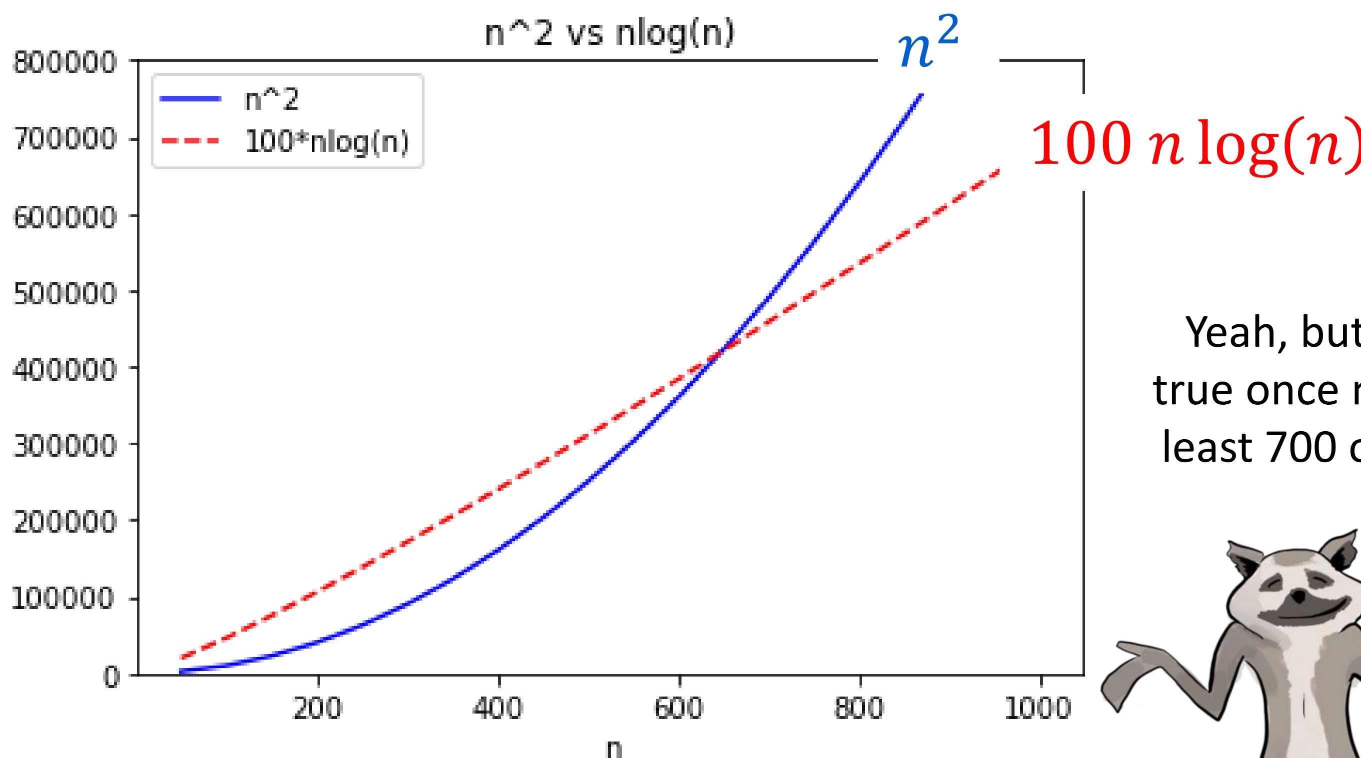
Informally….

(Only pay attention to the largest function of n that appears.)

Number of operations	Asymptotic Running Time
$\frac{1}{10} \cdot n^2 + 100$	$O(n^2)$
$0.063 \cdot n^2 - .5\,n + 12.7$	$O(n^2)$
$100 \cdot n^{1.5} - 10^{10000}\sqrt{n}$	$O(n^{1.5})$
$11 \cdot n\log(n) + 1$	$O(n\log(n))$

We say this algorithm is "asymptotically faster" than the others.

So $100\, n \log(n)$ operations is "better" than n^2 operations?



But when n=200, that's not true at all!

Yeah, but it's true once n is at least 700 or so.

Asymptotic Analysis

One algorithm is "faster" than another if its runtime scales better with the size of the input.

Pros:

- Abstracts away from hardware- and language-specific issues.
- Makes algorithm analysis much more tractable.

Cons:

- Only makes sense if n is large (compared to the constant factors).

1000000000 n
is "better" than n^2 ?!?!

pronounced “big-oh of …” or sometimes “oh of …”

O(…) means an upper bound

- Let $T(n)$, $g(n)$ be functions of positive integers.
  - Think of $T(n)$ as a runtime: positive and increasing in n.
- We say “$T(n)$ is $O\left(g(n)\right)$” if $T(n)$ grows no faster than $g(n)$ as n gets large.
- Formally,

$$T(n) = O\left(g(n)\right)$$

$$\Leftrightarrow$$

$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$

$$0 \leq T(n) \leq c \cdot g(n)$$

Example

$2n^2 + 10 = O(n^2)$

$$T(n) = O\big(g(n)\big)$$
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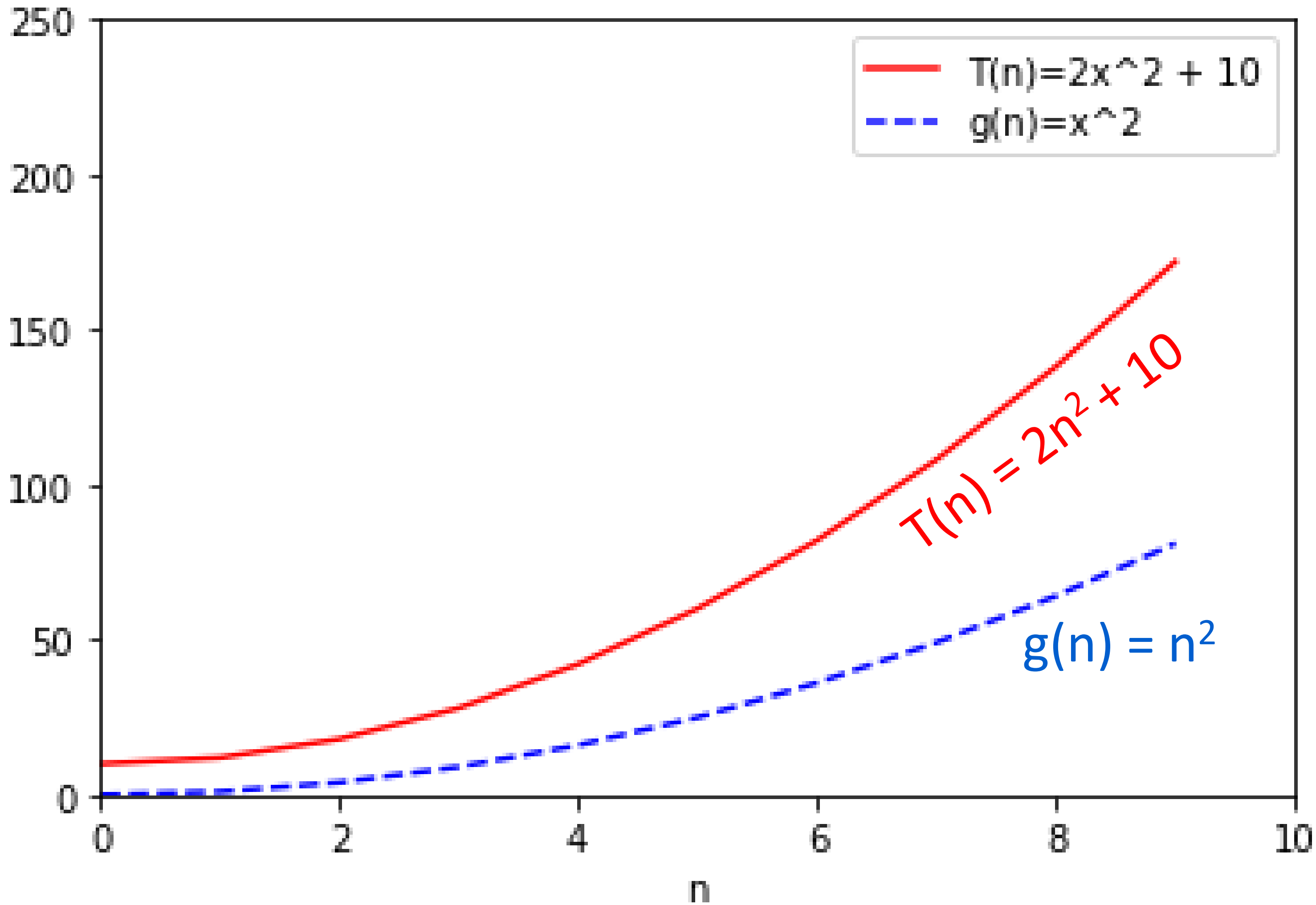

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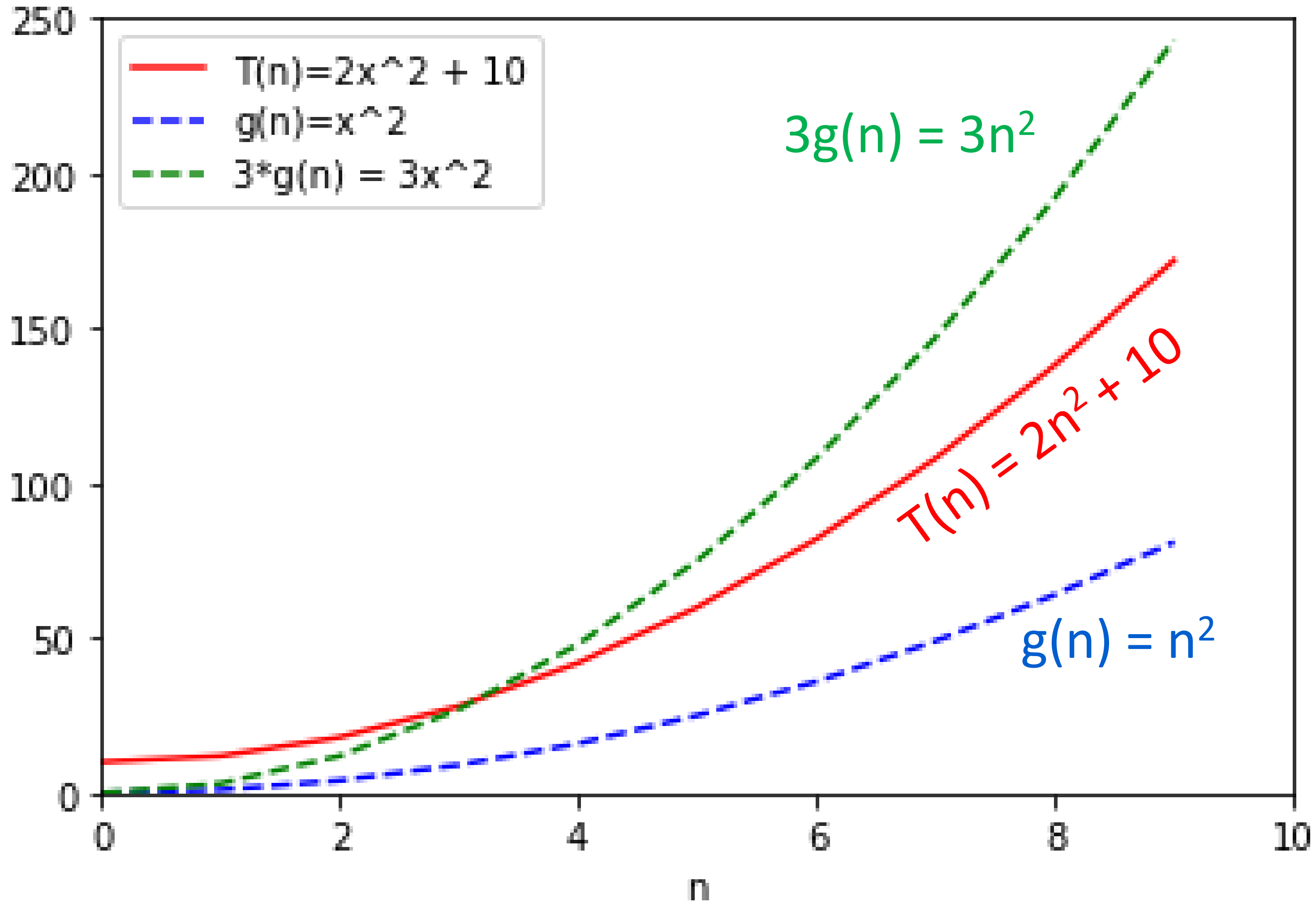

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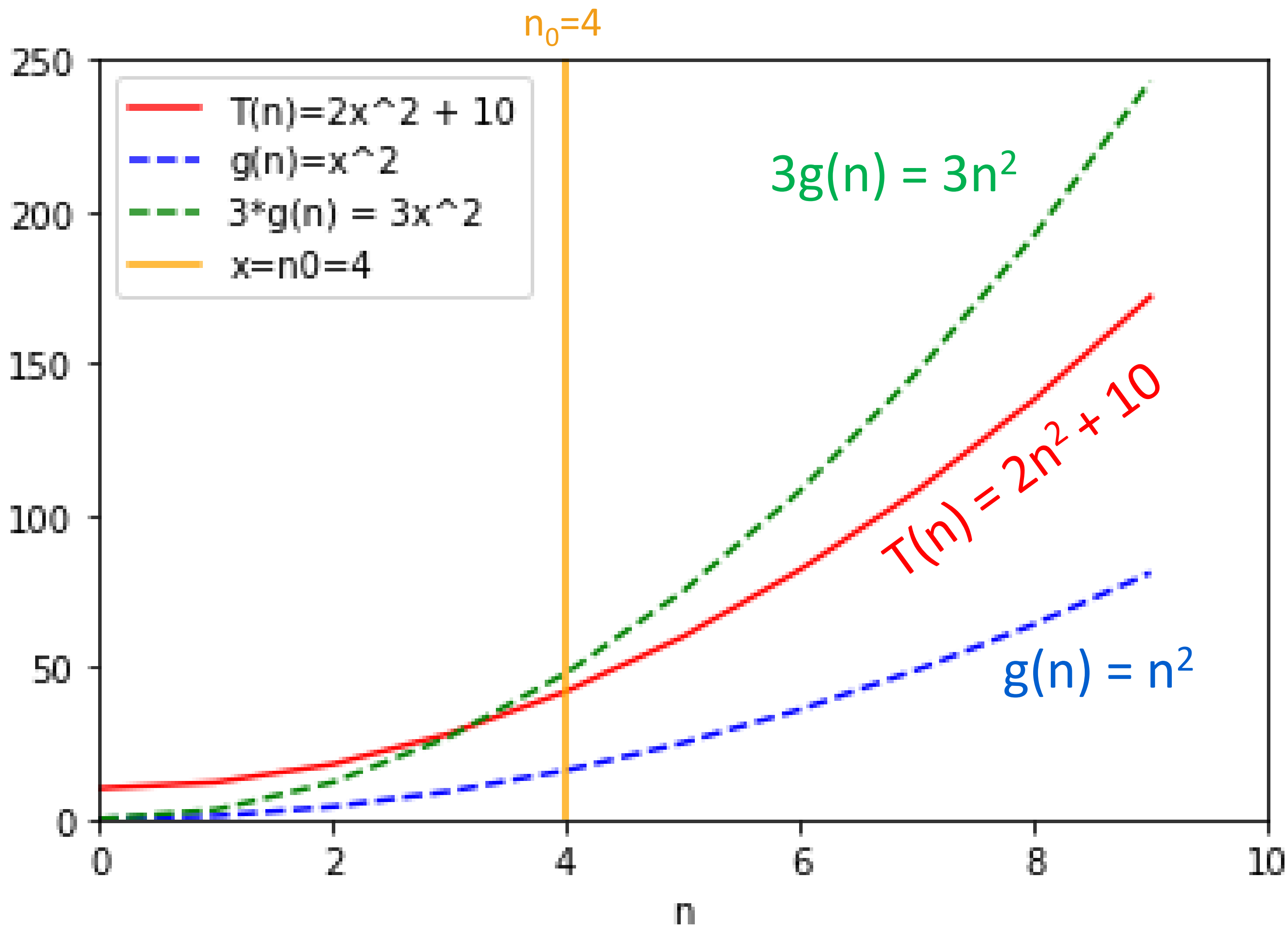

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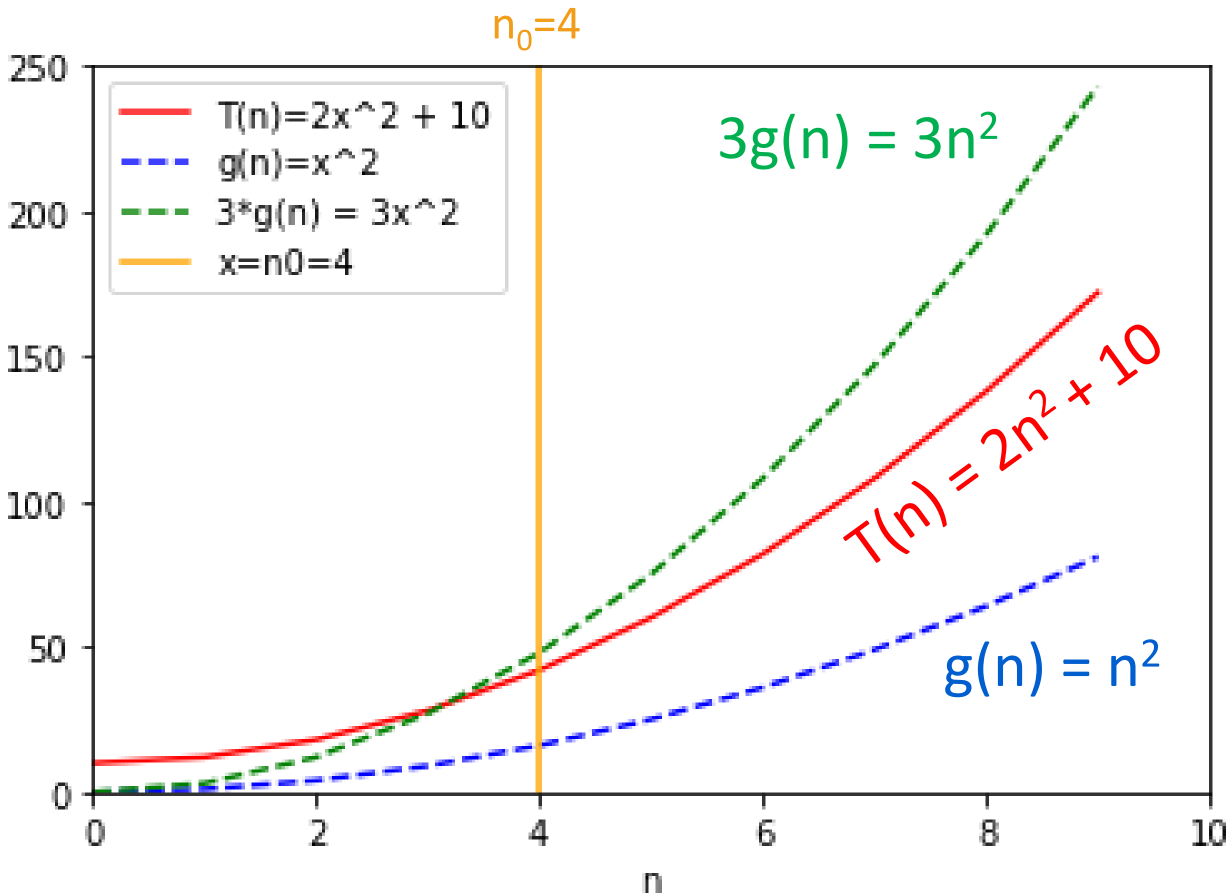


Formally:

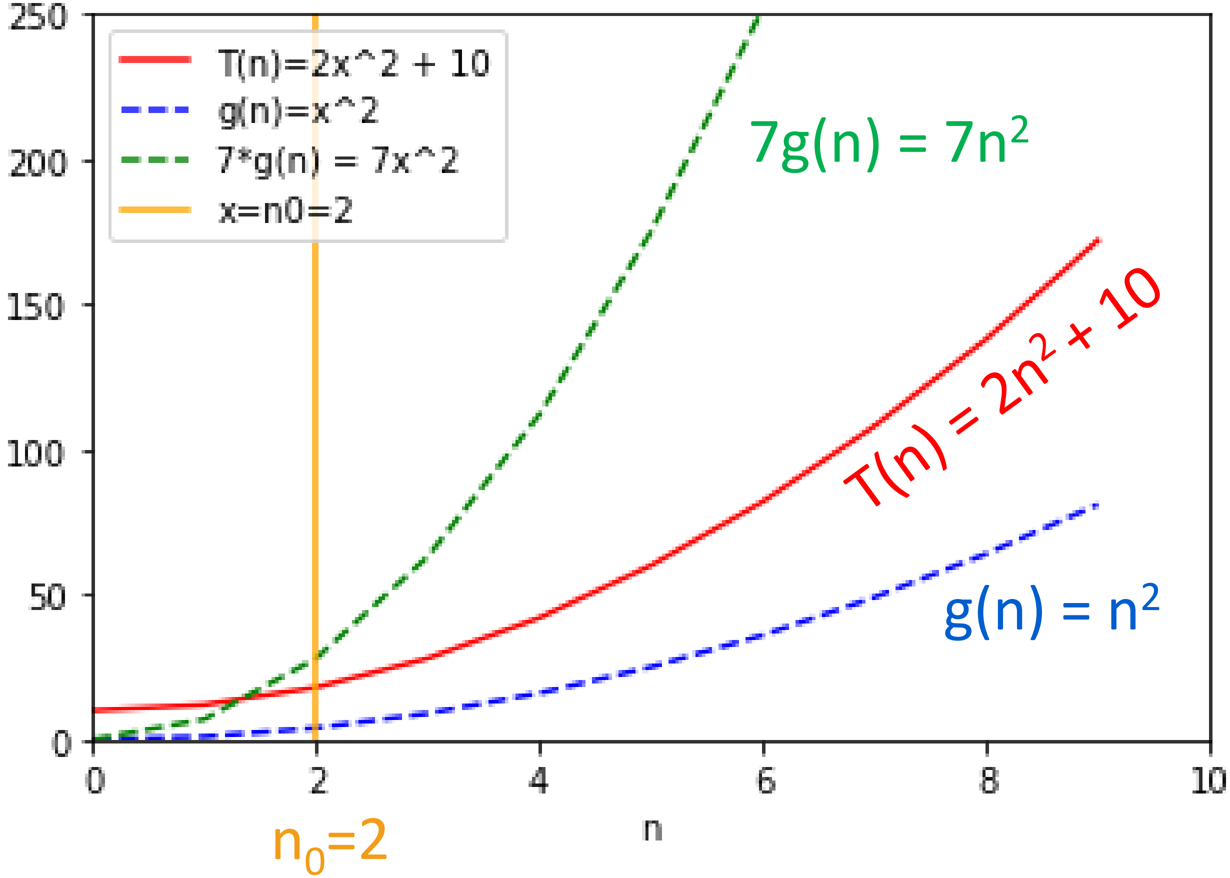
- Choose c = 3
- Choose n_0 = 4
- Then:

$$\forall n \geq 4,$$
$$0 \leq 2n^2 + 10 \leq 3 \cdot n^2$$

Same example

$$2n^2 + 10 = O(n^2)$$

$$T(n) = O(g(n))$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot g(n)$$



Formally:

- Choose c = 7
- Choose $n_0 = 2$
- Then:

$$\forall n \geq 2,$$
$$0 \leq 2n^2 + 10 \leq 7 \cdot n^2$$

There is not a "correct" choice of c and n_0

$\Omega(\ldots)$ means a lower bound

- We say "$T(n)$ is $\Omega\big(g(n)\big)$" if $T(n)$ grows at least as fast as $g(n)$ as n gets large.

- Formally,

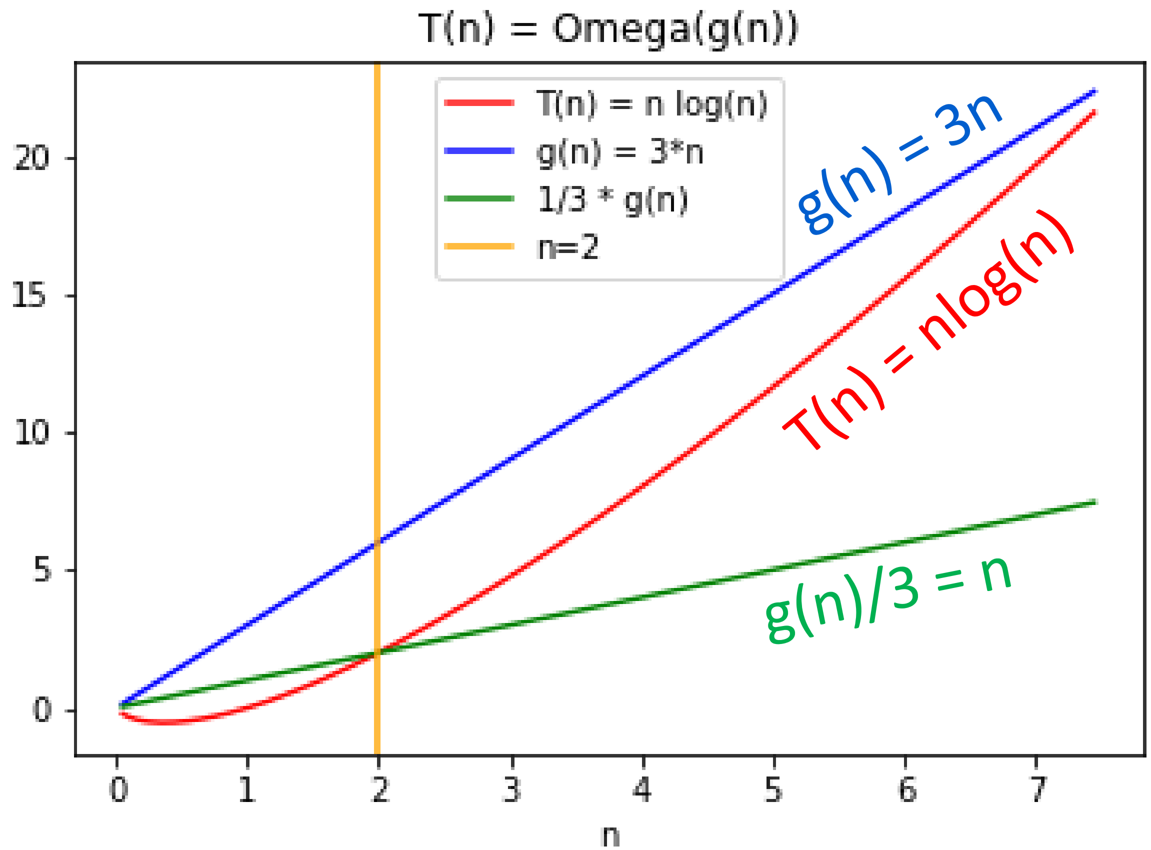
$$T(n) = \Omega\big(g(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq c \cdot g(n) \leq T(n)$$

Switched these!!

Example

$n \log_2(n) = \Omega(3n)$

$$T(n) = \Omega(g(n))$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq c \cdot g(n) \leq T(n)$$



- Choose c = 1/3
- Choose $n_0 = 2$
- Then

$$\forall n \geq 2,$$

$$0 \leq \frac{3n}{3} \leq n \log_2(n)$$

$\Theta(\ldots)$ means both!

- We say "$T(n)$ is $\Theta(g(n))$" iff both:

$$T(n) = O\big(g(n)\big)$$

and

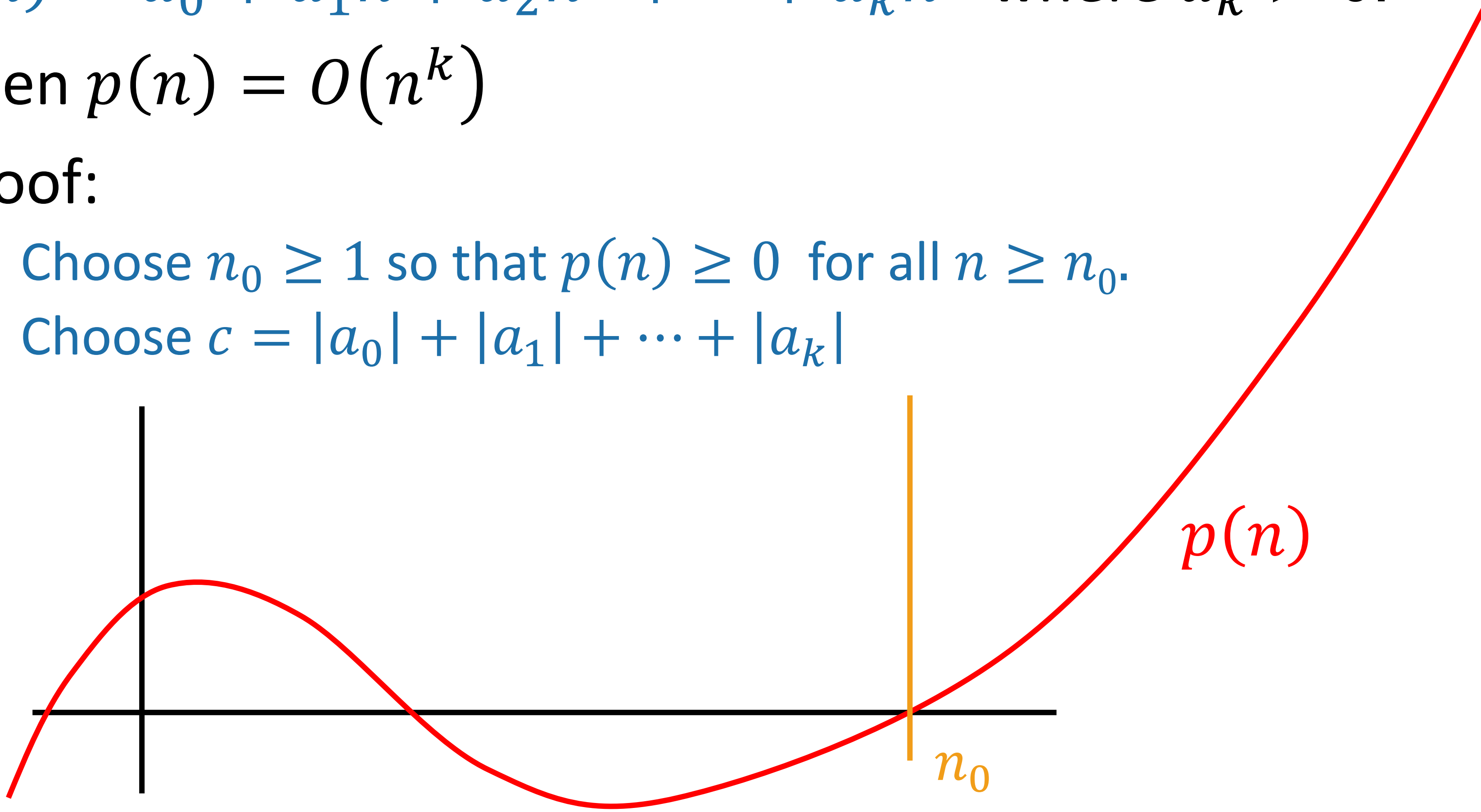
$$T(n) = \Omega\big(g(n)\big)$$

Example: polynomials

- Suppose the p(n) is a polynomial of degree k:

$p(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k$ where $a_k > 0$.

- Then $p(n) = O\left(n^k\right)$
- Proof:
  - Choose $n_0 \geq 1$ so that $p(n) \geq 0$ for all $n \geq n_0$.
  - Choose $c = |a_0| + |a_1| + \cdots + |a_k|$



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- Proof:
  - Choose $n_0 \geq 1$ so that $p(n) \geq 0$ for all $n \geq n_0$.
  - Choose $c = |a_0| + |a_1| + \cdots + |a_k|$
  - Then for all $n \geq n_0$:
  - $0 \leq p(n) = |p(n)| \leq |a_0| + |a_1|n + \cdots + |a_k|n^k$
  - $\leq |a_0|n^k + |a_1|n^k + \cdots + |a_k|n^k$
  - $= c \cdot n^k$

Because $n \leq n^k$ for $n \geq n_0 \geq 1$.

Definition of c

Example: more polynomials

- For any $k \geq 1, n^k$ is NOT $O\left(n^{k-1}\right)$.
- Proof:
  - Suppose that it were. Then there is some c, n_0 so that
    $$n^k \leq c \cdot n^{k-1} \text{ for all } n \geq n_0$$
  - Aka, $n \leq c$ for all $n \geq n_0$
  - But that's not true! What about $n = n_0 + c + 1$?
  - We have a contradiction! It *can't* be that $n^k = O\left(n^{k-1}\right)$.

Take-away from examples

- To prove $T(n) = O(g(n))$, you have to come up with c and n_0 so that the definition is satisfied.

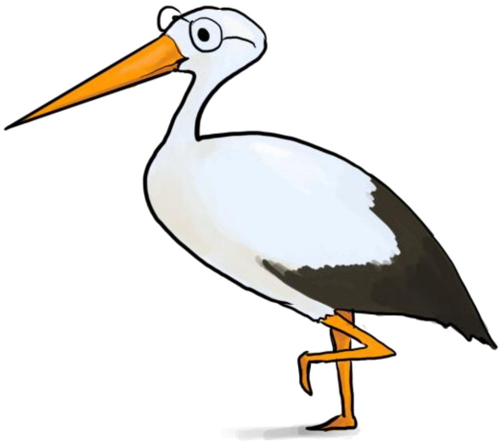
- To prove $T(n)$ is NOT $O(g(n))$, one way is **proof by contradiction**:
  - Suppose (to get a contradiction) that someone gives you a c and an n_0 so that the definition ***is*** satisfied.
  - Show that this someone must by lying to you by deriving a contradiction.

Yet more examples

- $n^3 + 3n = O(n^3 - n^2)$
- $n^3 + 3n = \Omega(n^3 - n^2)$
- $n^3 + 3n = \Theta(n^3 - n^2)$

- 3^n is **NOT** $O(2^n)$
- $\log(n) = \Omega(\ln(n))$
- $\log(n) = \Theta(\ 2^{\log\log(n)}\ )$

Work through these on your own!



remember that $\log = \log_2$ in this class.

Some brainteasers

- Are there functions f, g so that NEITHER $f = O(g)$ nor $f = \Omega(g)$?
- Are there non-decreasing functions f, g so that the above is true?
- Define the n'th fibonacci number by $F(0) = 1$, $F(1) = 1$, $F(n) = F(n-1) + F(n-2)$ for $n > 2$.
  - 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

  True or false:
  - $F(n) = O(2^n)$
  - $F(n) = \Omega(2^n)$

Recap

- InsertionSort runs in time $O(n^2)$
- MergeSort is a divide-and-conquer algorithm that runs in time $O(n \log(n))$

- How do we show an algorithm is correct?
  - Today, we did it by induction
- How do we measure the runtime of an algorithm?
  - Worst-case analysis
  - Asymptotic analysis

Next time

- A more systematic approach to analyzing the runtime of recursive algorithms.