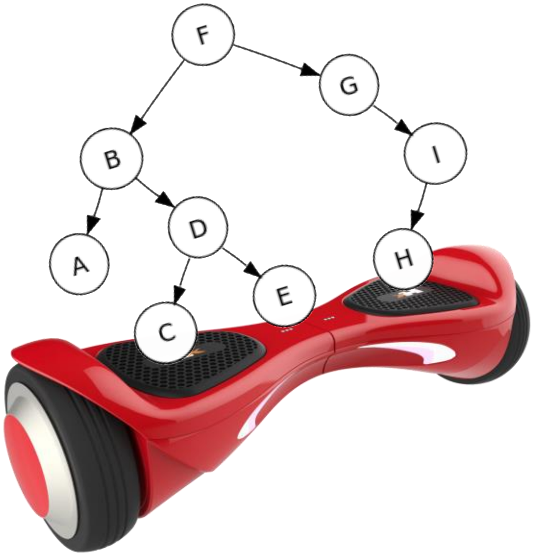
Advanced Data Structure and Algorithm

Red-Black Trees

Today

- Self-Balancing Binary Search Trees
  - **Red-Black** trees.

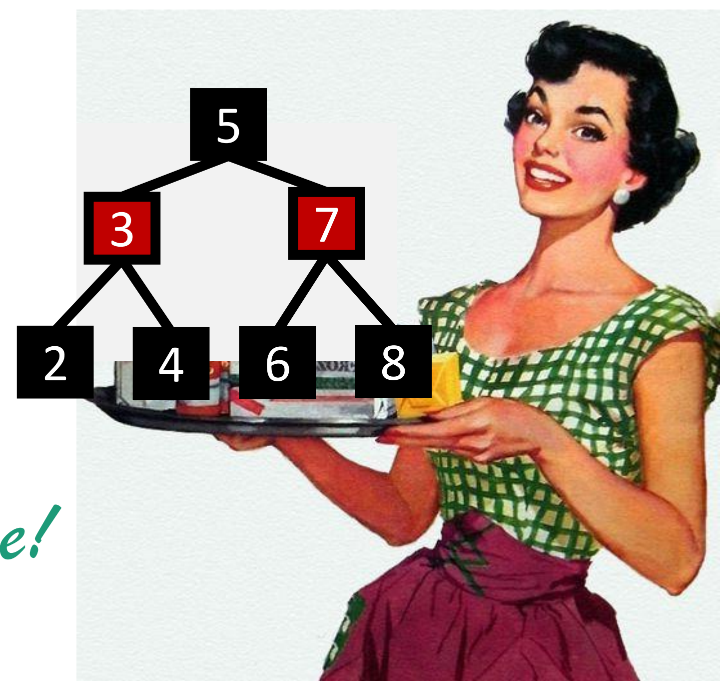


Red-Black Trees

- A Binary Search Tree that balances itself!
- No more time-consuming by-hand balancing!
- Be the envy of your friends and neighbors with the time-saving…

Red-Black tree!

Maintain balance by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes**.

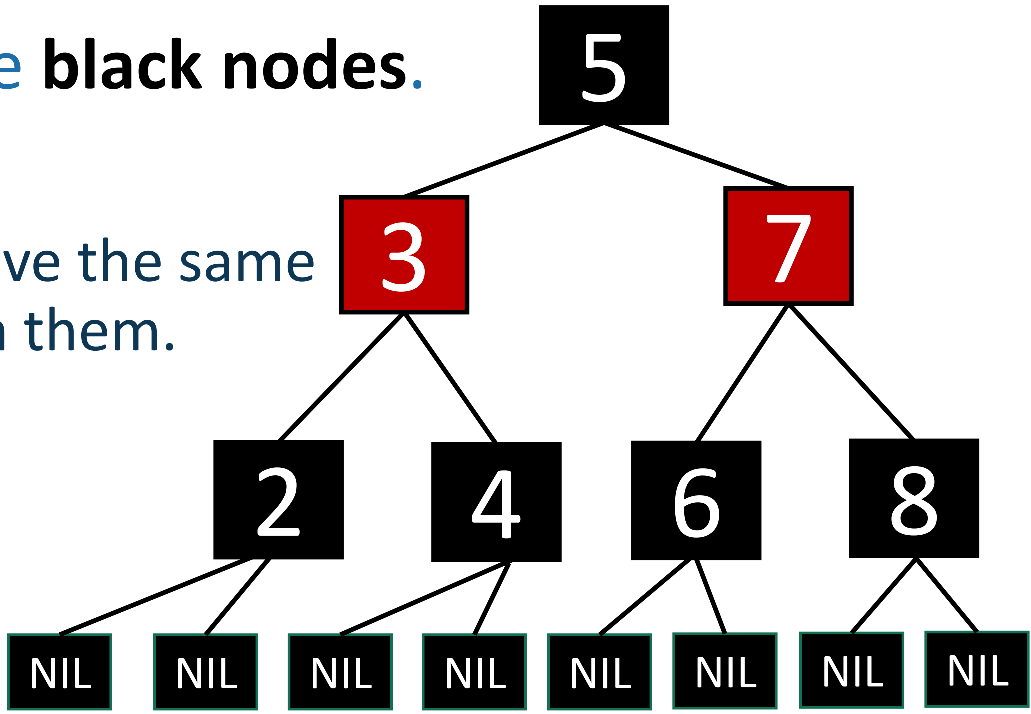


It's just good sense!

Red-Black Trees

obey the following rules (which are a proxy for balance)

- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NIL's have the same number of **black nodes** on them.



I'm not going to draw the NIL children in the future, but they are treated as black nodes.

Examples(?)

- Every node is colored **red** or **black.**
- The root node is a **black node**.
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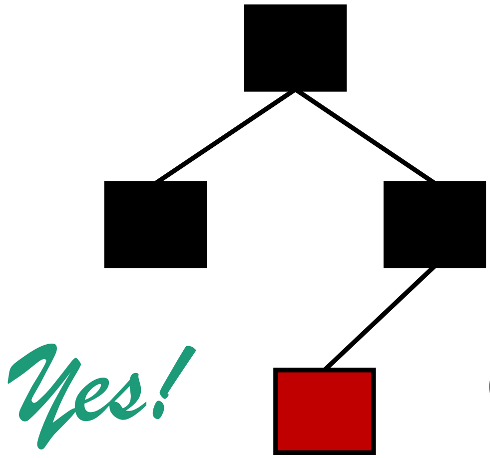
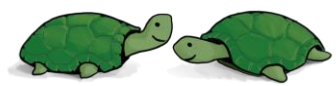
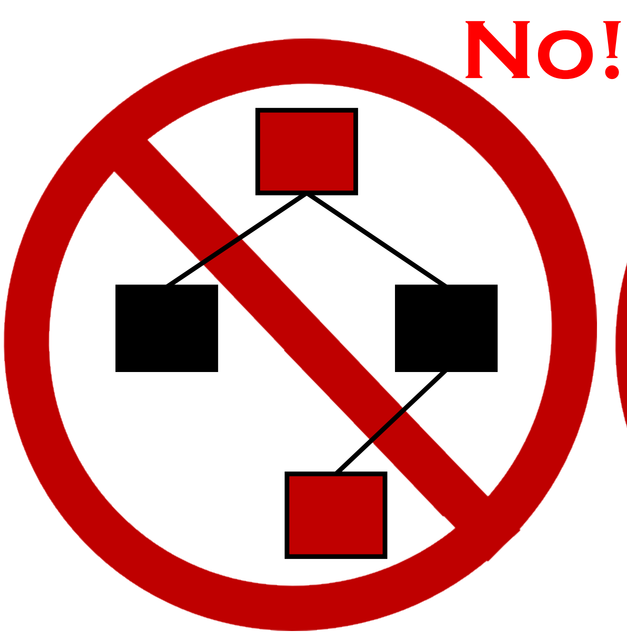
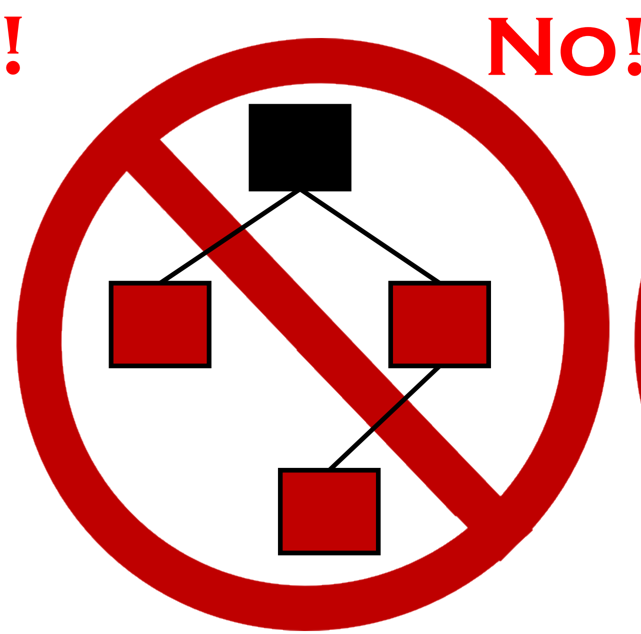
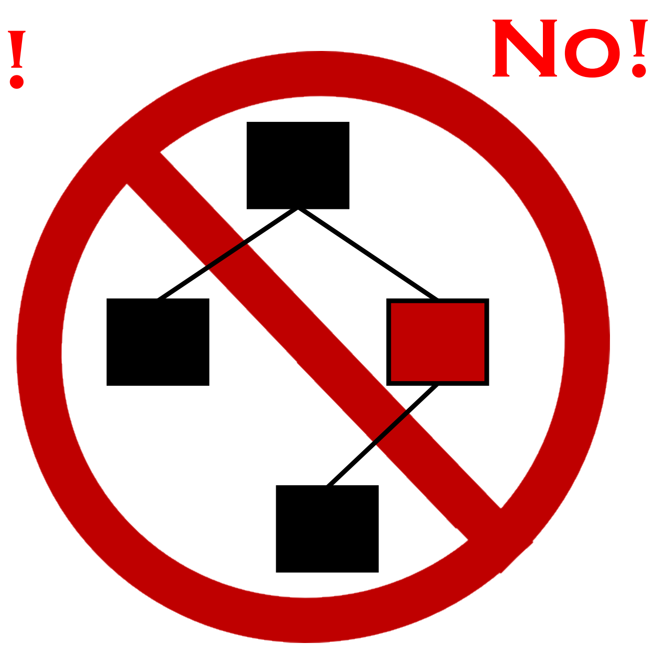


Which of these are red-black trees? (NIL nodes not drawn)



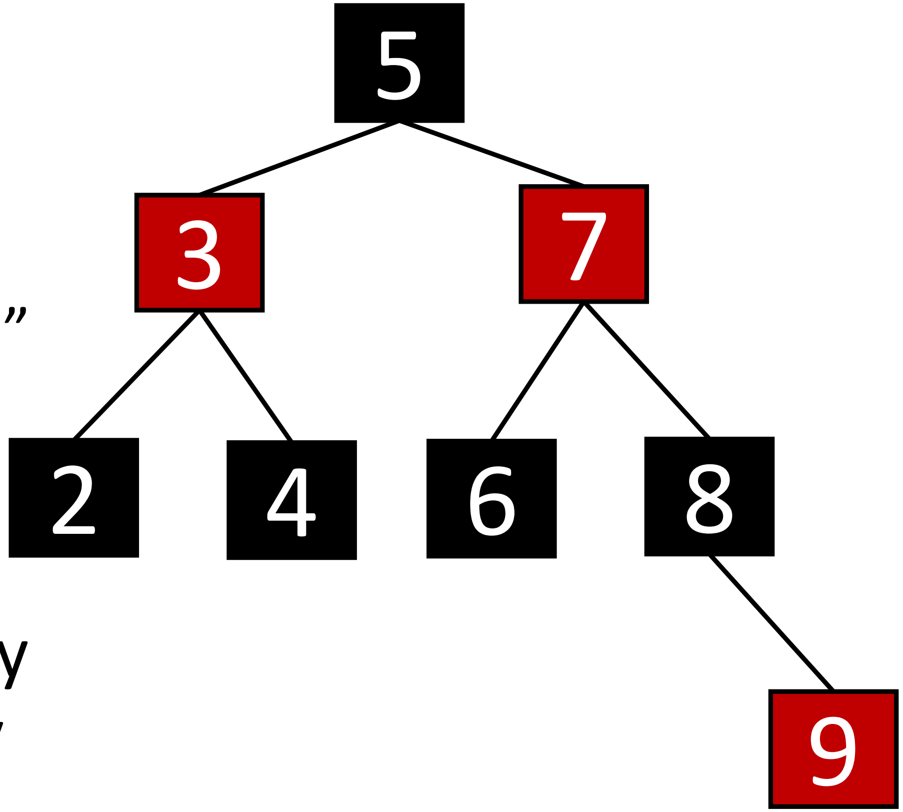






Why **these** rules???????

- This is pretty balanced.
  - The **black nodes** are balanced
  - The **red nodes** are "spread out" so they don't mess things up too much.
- We can maintain this property as we insert/delete nodes, by using rotations.



This is the really clever idea!

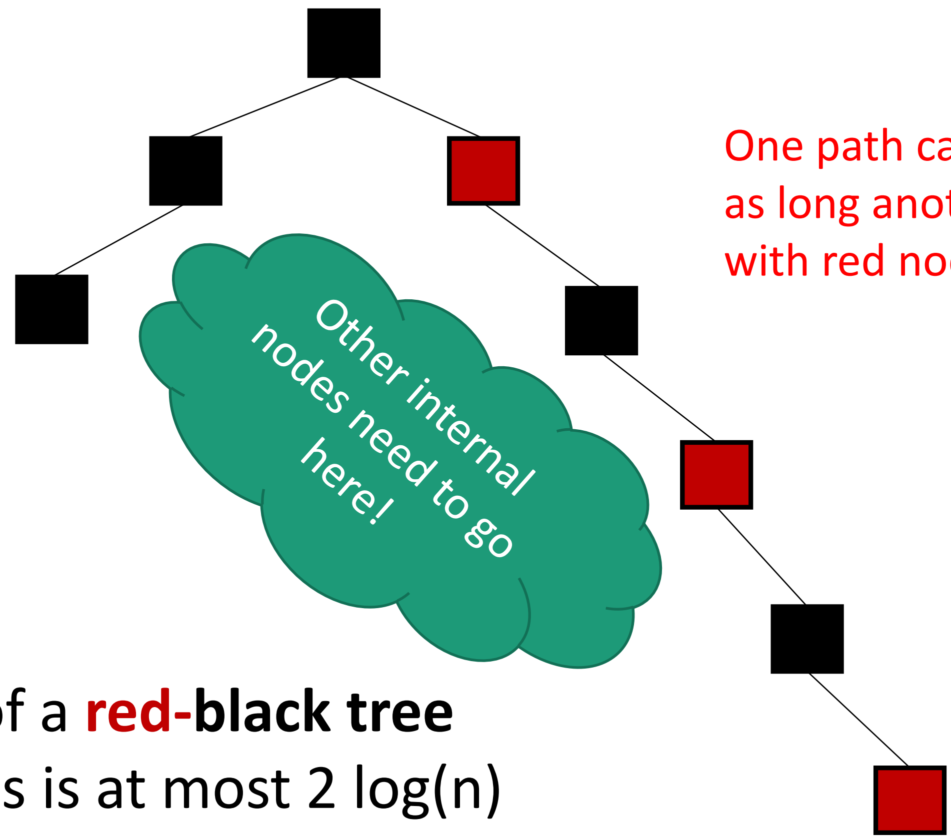
This **Red-Black** structure is a proxy for balance.

It's just weaker than perfect balance, but we can actually maintain it!

Let’s build some intuition!

This is “pretty balanced”

- To see why, intuitively, let’s try to build a Red-Black Tree that’s unbalanced.

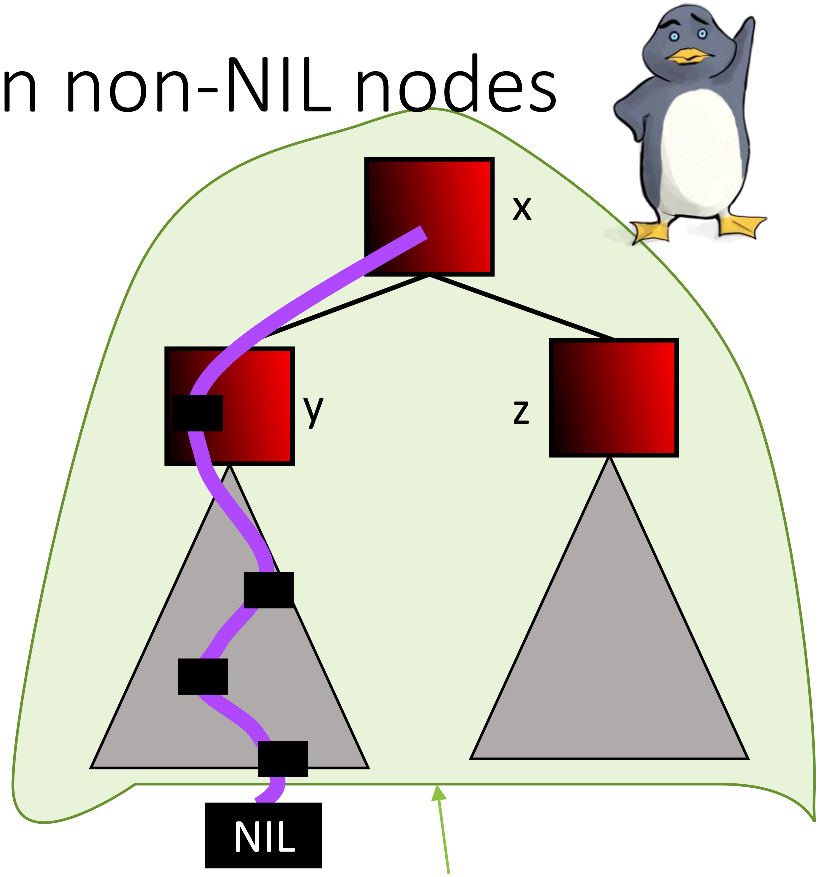


One path can be at most twice as long another if we pad it with red nodes.

Conjecture:
the height of a **red-black tree** with n nodes is at most 2 log(n)

The height of a RB-tree with n non-NIL nodes is at most $2\log(n+1)$

- Define b(x) to be the number of black nodes in any path from x to NIL.
  - (excluding x, including NIL).
- Claim:
  - There are at least $2^{b(x)} - 1$ non-NIL nodes in the subtree underneath x. (Including x).
- [Proof by induction]



Claim: at least $2^{b(x)} - 1$ nodes in this WHOLE subtree (of any color).

Then:

$$n \geq 2^{b(root)} - 1$$ using the Claim

$$\geq 2^{height/2} - 1$$ b(root) >= height/2 because of RBTree rules.

Rearranging:

$$n + 1 \geq 2^{height/2} \Rightarrow height \leq 2\log(n+1)$$

This is great!

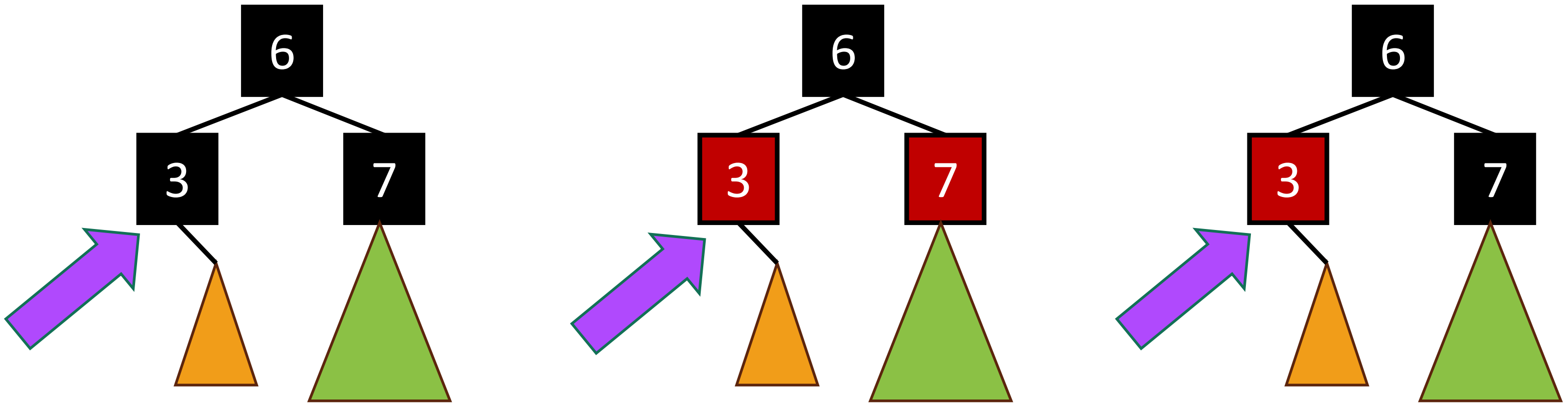
- SEARCH in an RBTree is immediately O(log(n)), since the depth of an RBTree is O(log(n)).

- What about INSERT/DELETE?
  - Turns out, you can INSERT and DELETE items from an RBTree in time O(log(n)), while maintaining the RBTree property.

INSERT/DELETE

- INSERT/DELETE for RBTrees
  - You should know what the “proxy for balance” property is and why it ensures approximate balance.

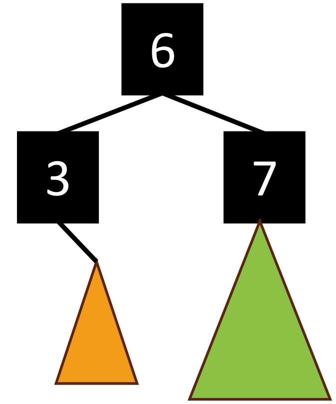
INSERT: Many cases



- Suppose we want to insert 0 **here**.
- There are 3 "important" cases for different colorings of the existing tree, and there are 9 more cases for all of the various symmetries of these 3 cases.

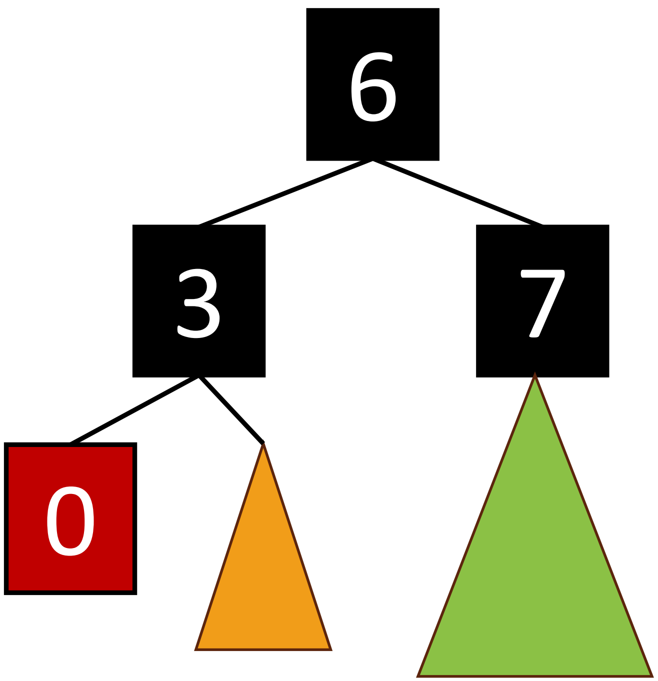
INSERT: Case 1

- Make a new **red node**.
- Insert it as you would normally.

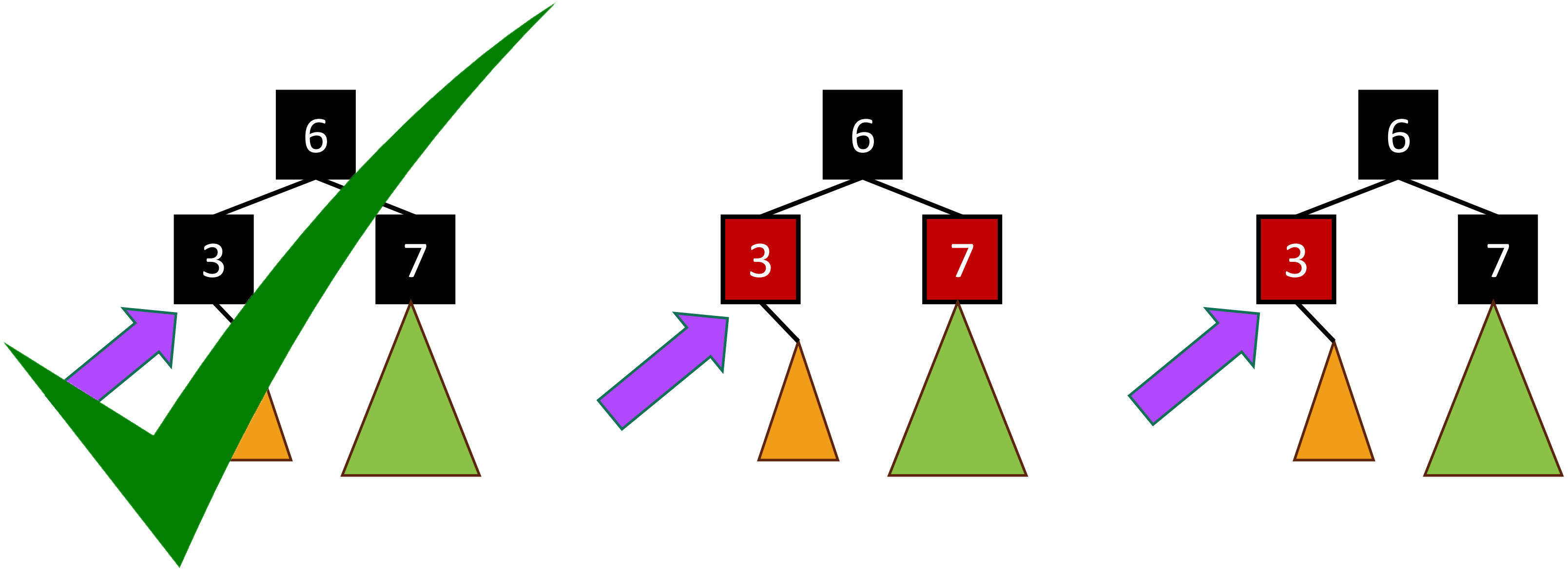


What if it looks like this?

Example: insert 0



INSERT: Many cases



- Suppose we want to insert 0 **here**.
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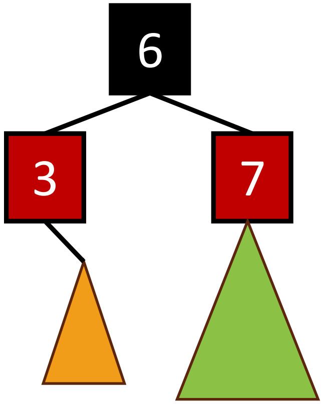
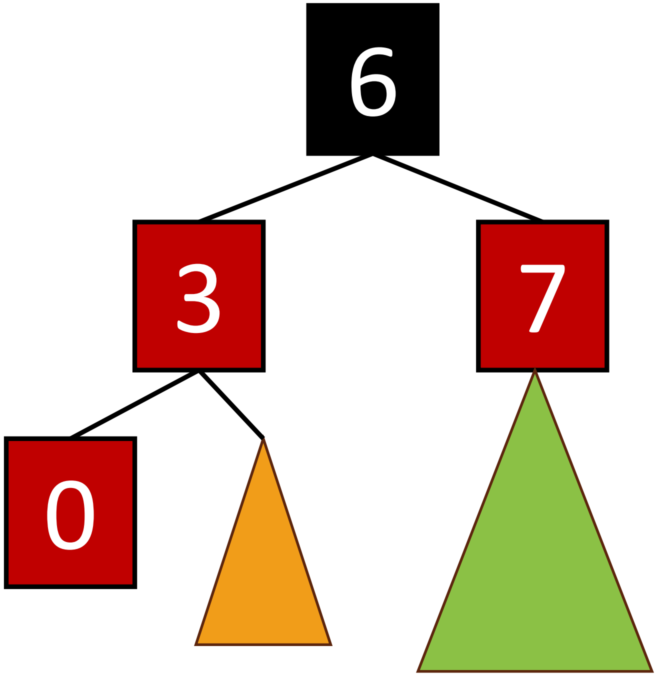
INSERT: Case 2

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.

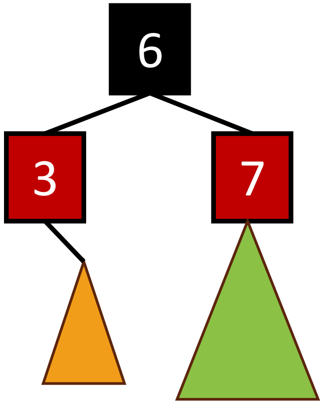


What if it looks like this?

Example: insert 0



INSERT: Case 2



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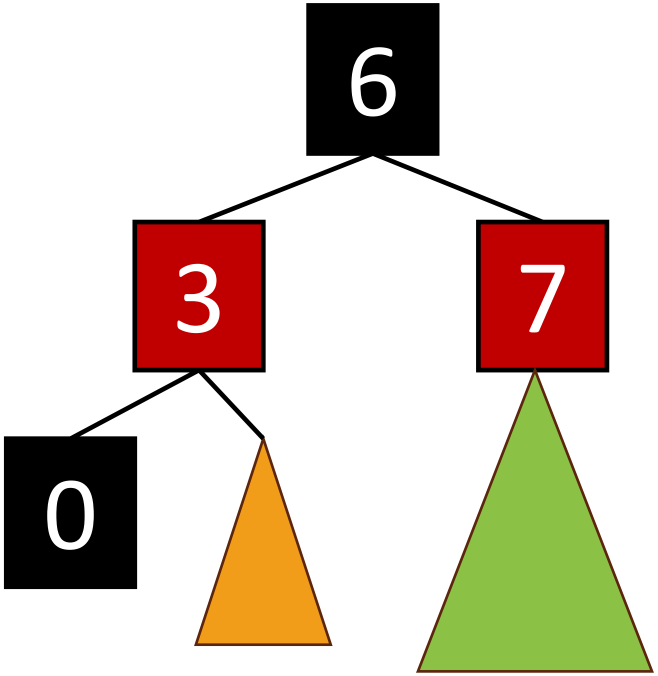
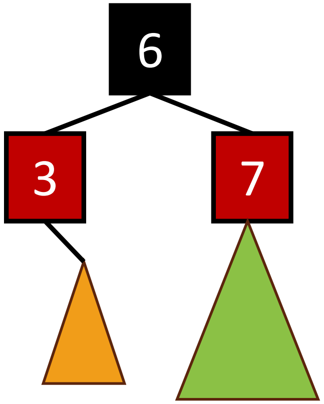


Example: insert 0

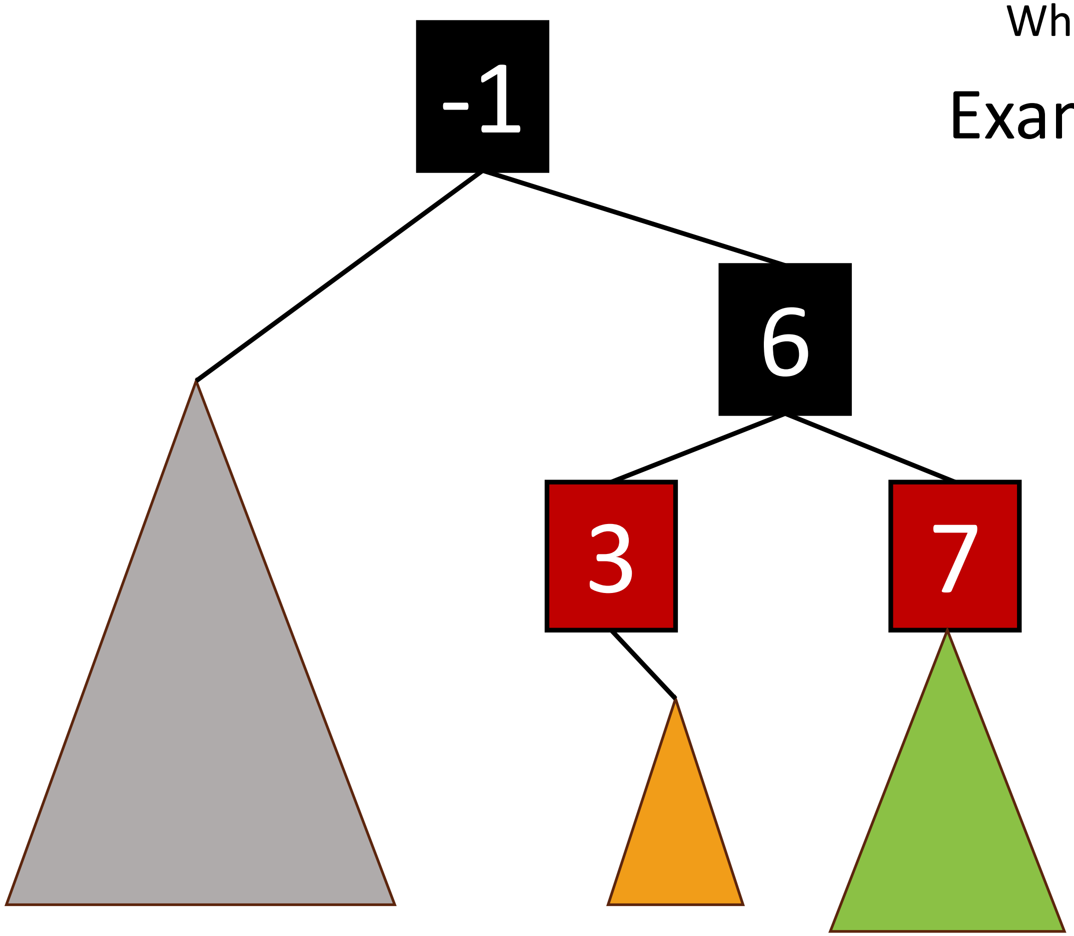
Can’t we just insert 0 as a **black node?**

We need a bit more context



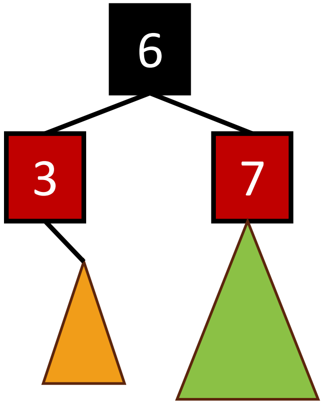
What if it looks like this?

Example: insert 0



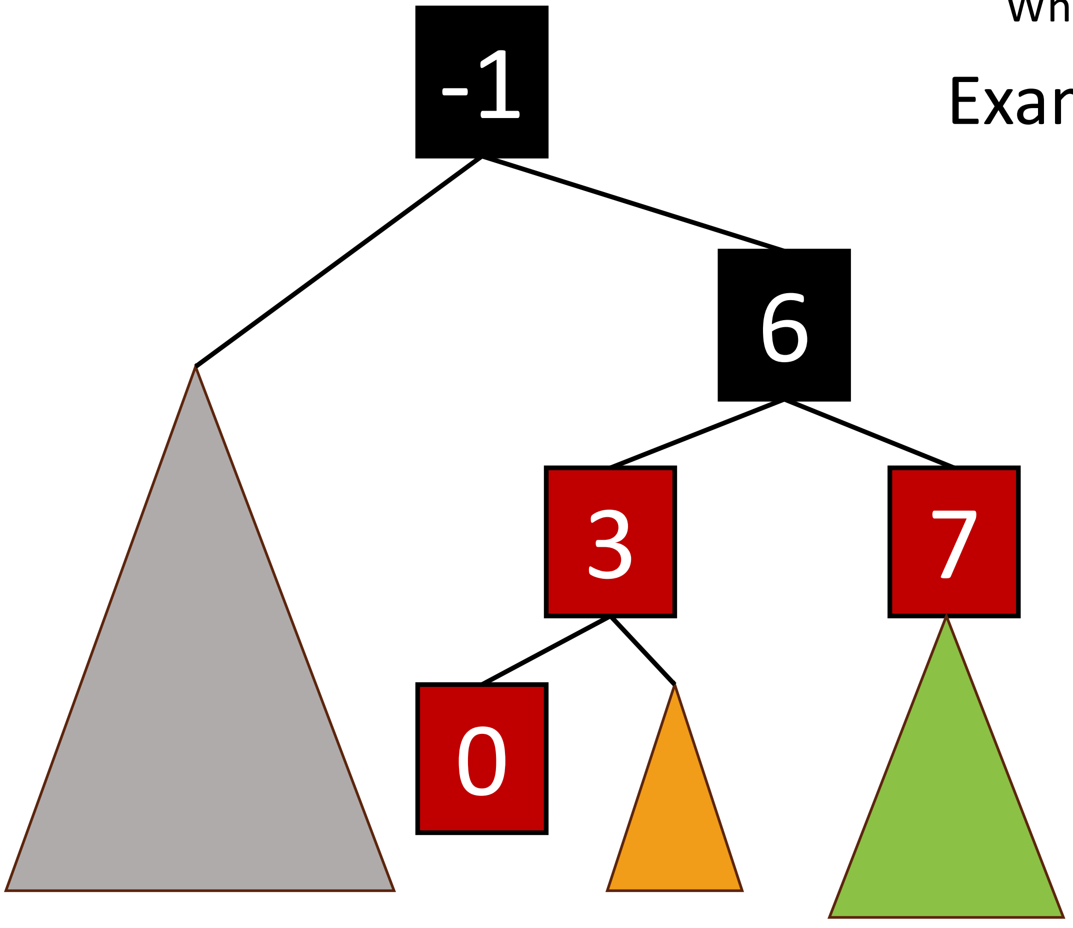
We need a bit more context

- Add 0 as a red node.

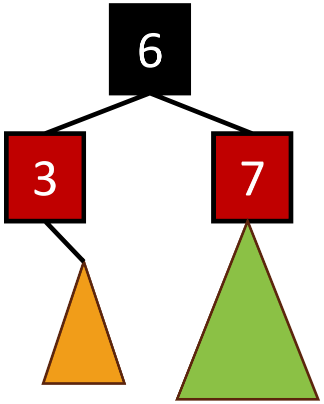


What if it looks like this?

Example: insert 0



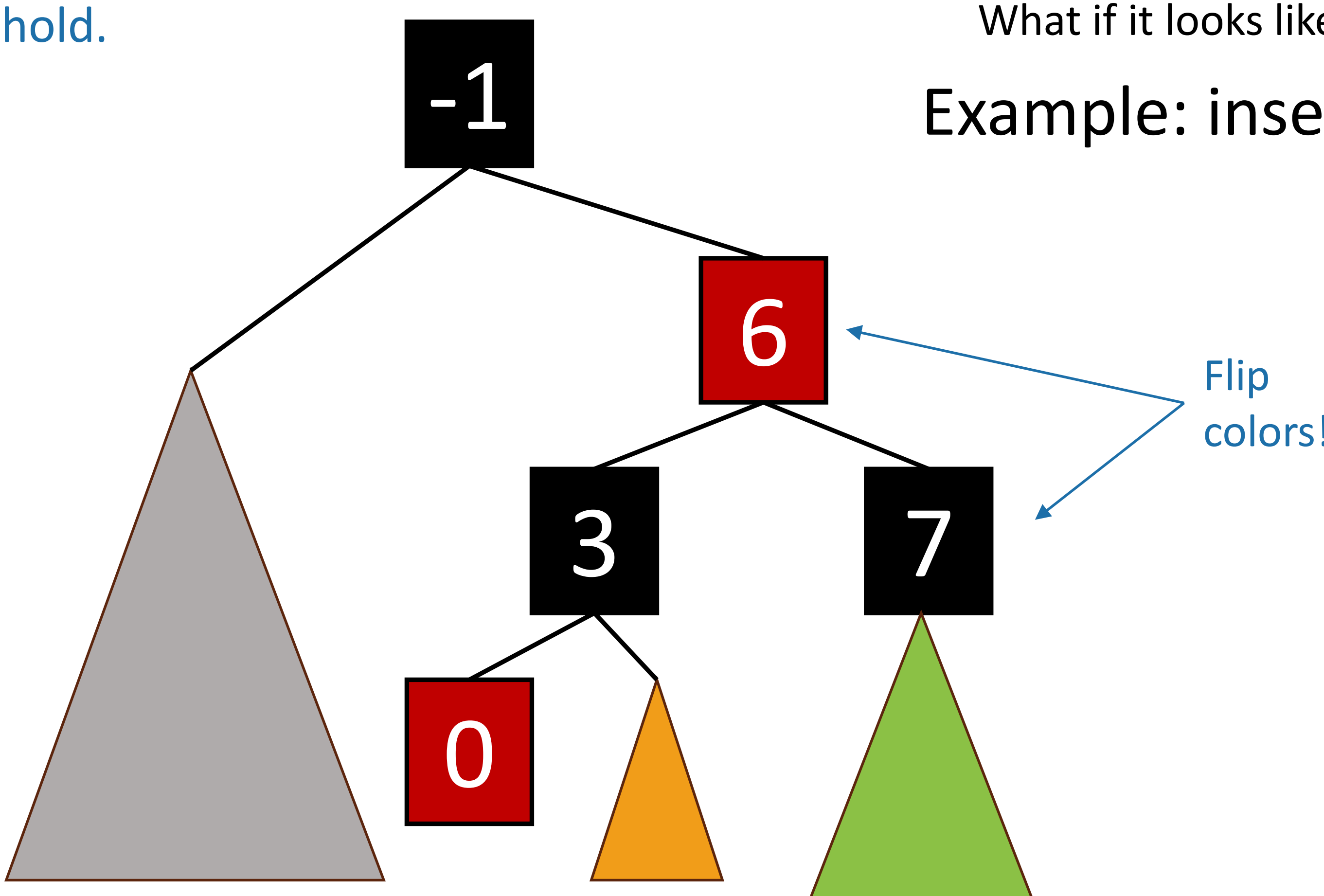
We need a bit more context



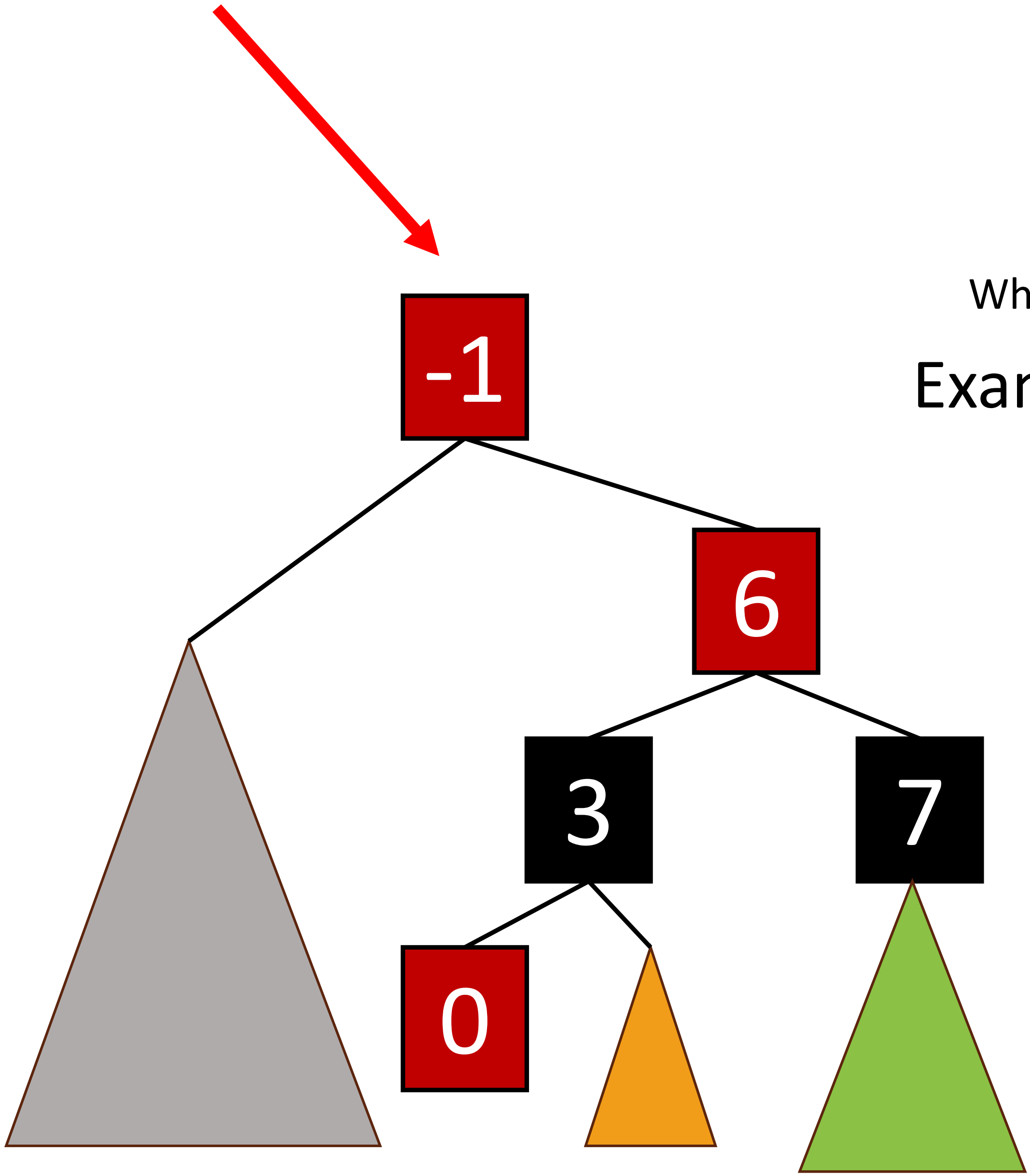
What if it looks like this?

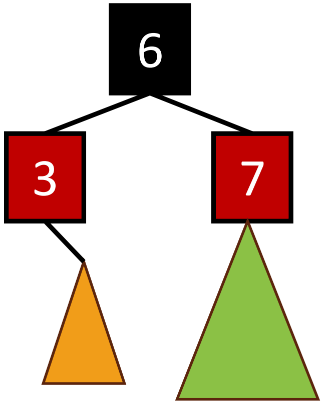
Example: insert 0

- Add 0 as a red node.
- **Claim:** RB-Tree properties still hold.



But what if that was red?

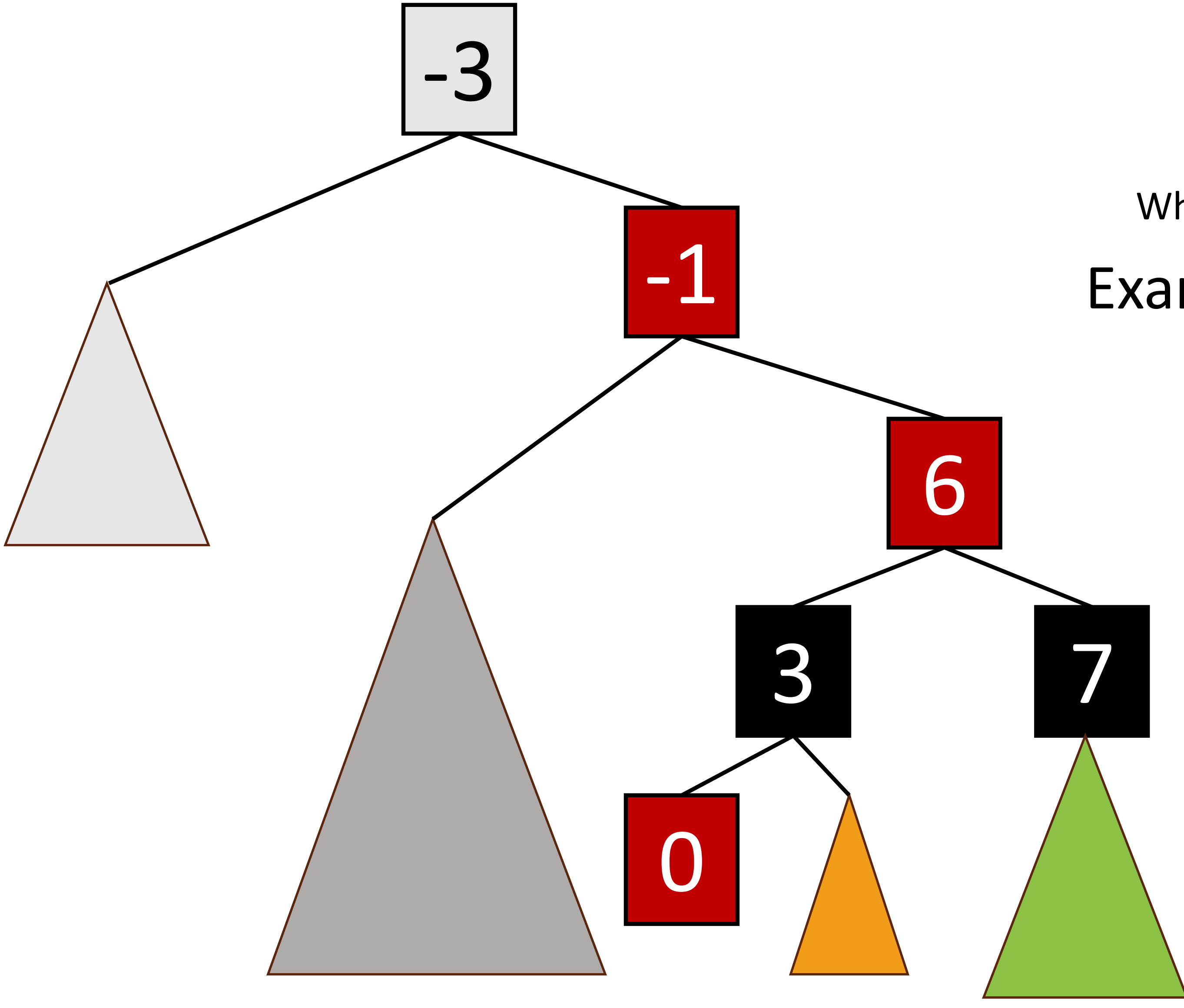


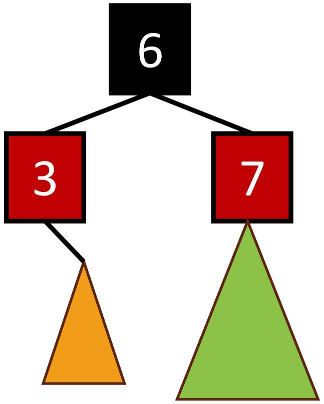


What if it looks like this?

Example: insert 0

More context…

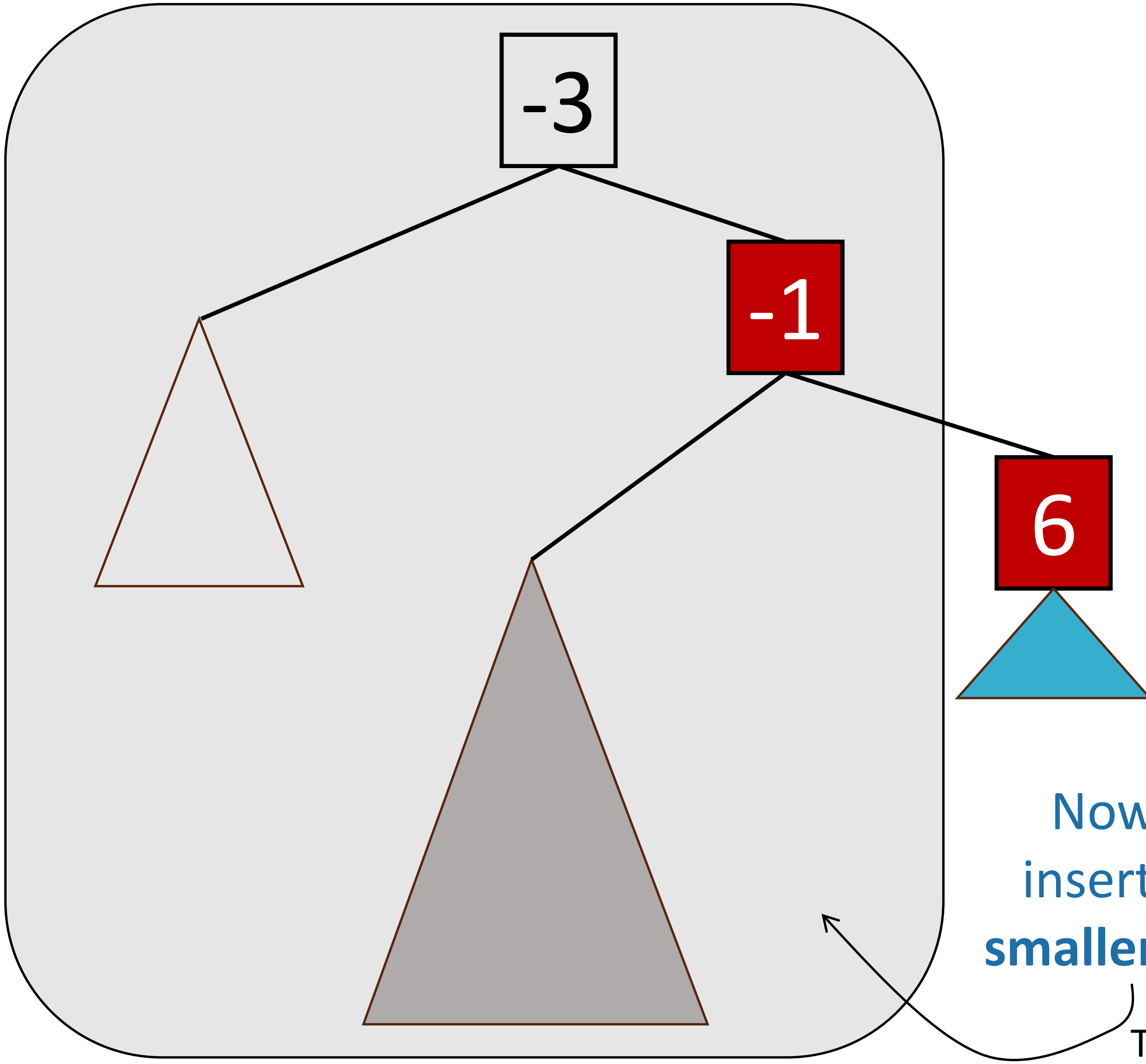


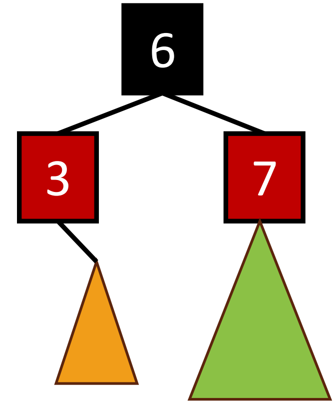


What if it looks like this?

Example: insert 0

More context…





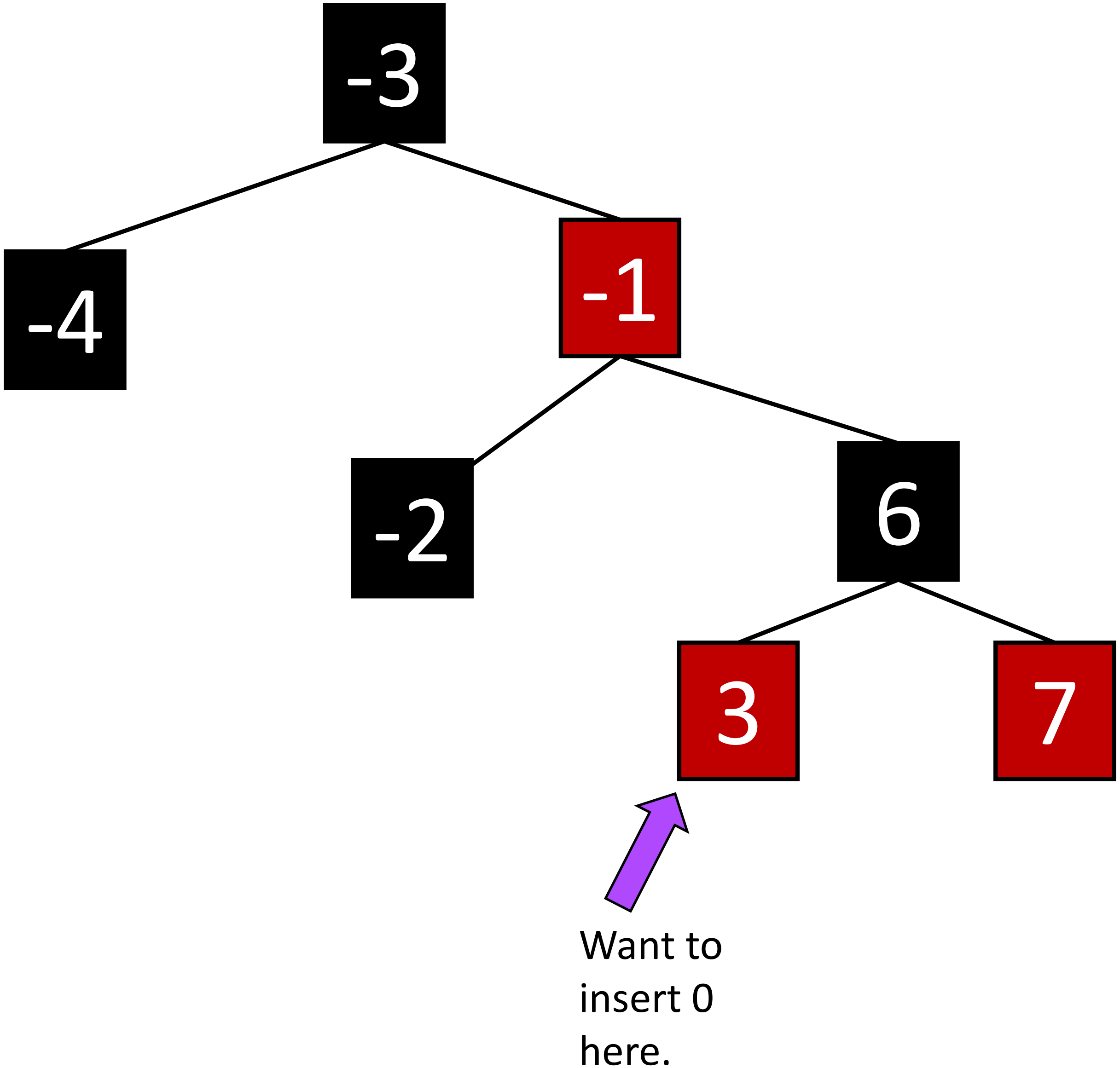
What if it looks like this?

Example: insert 0

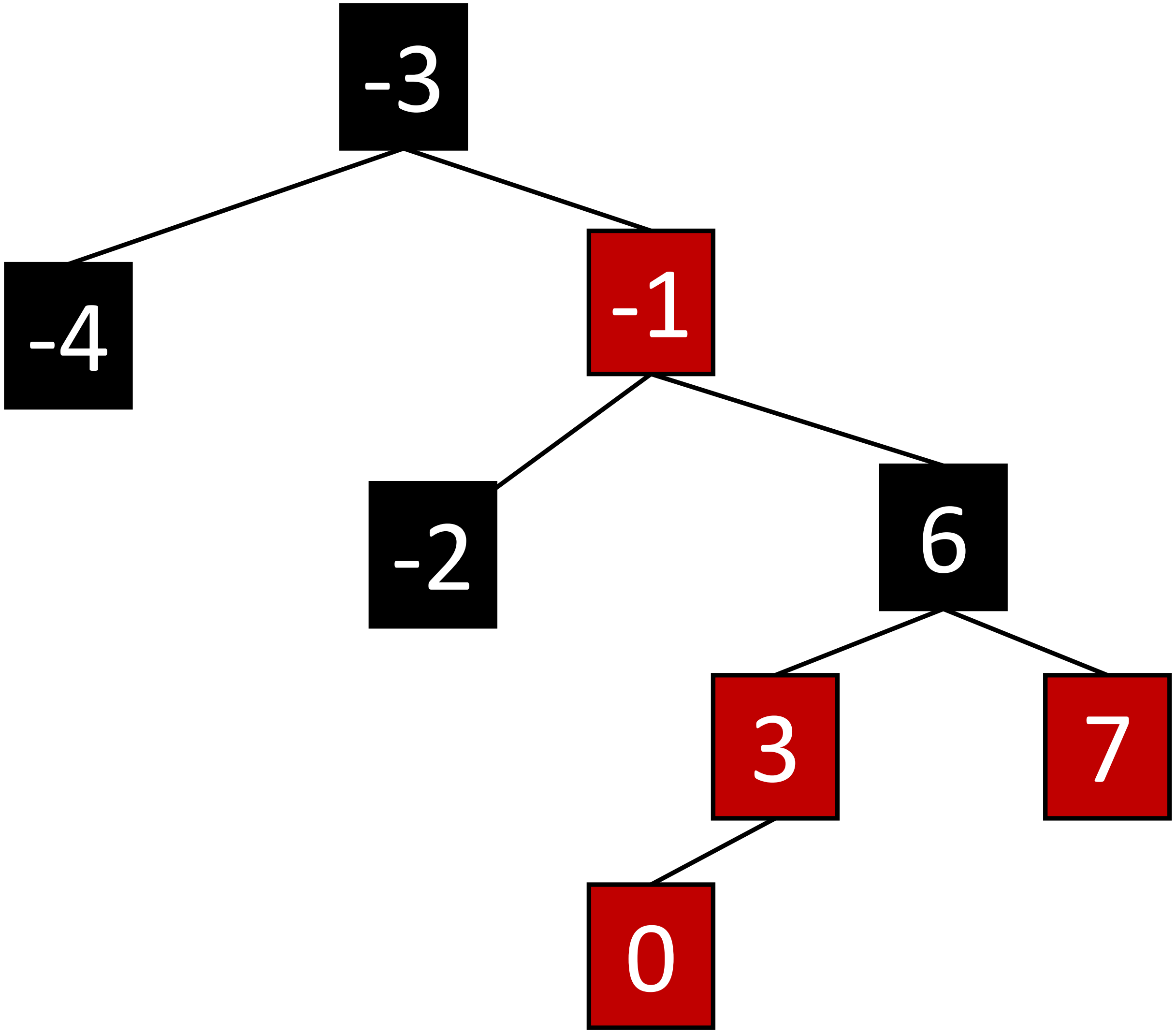
Now we're basically inserting 6 into some **smaller tree**. Recurse!

This one!

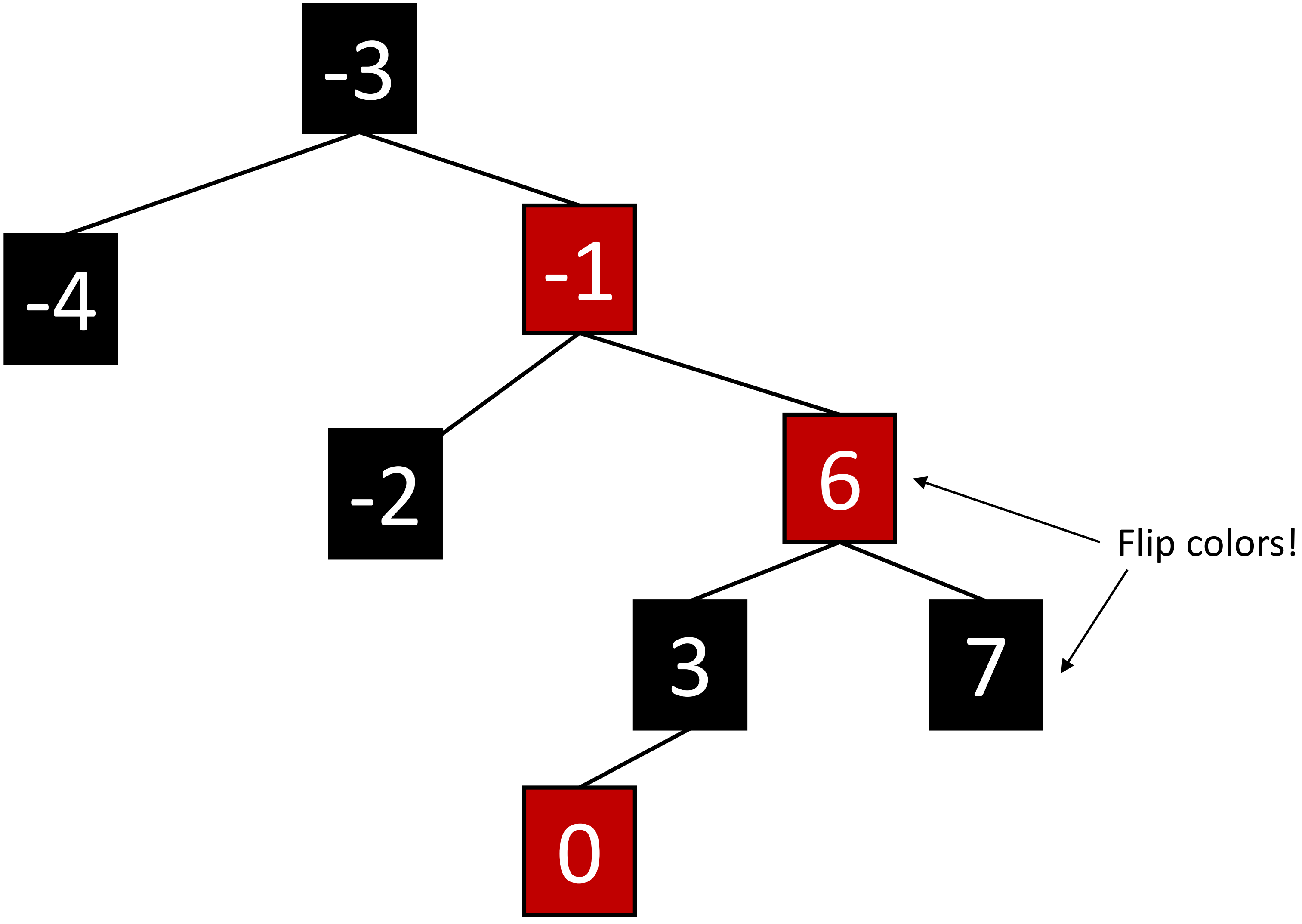
Example, part I



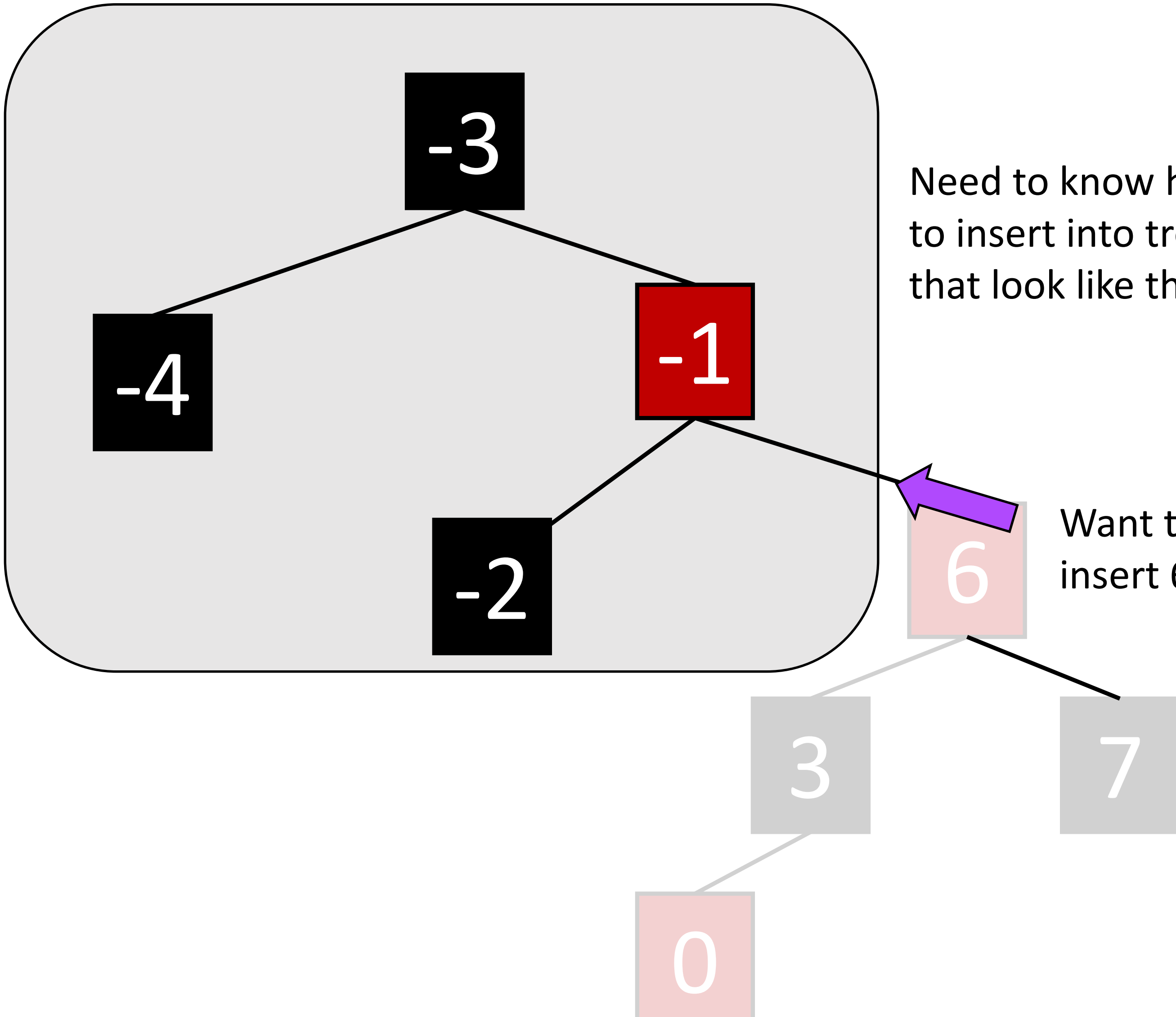
Example, part I



Example, part I



Example, part I

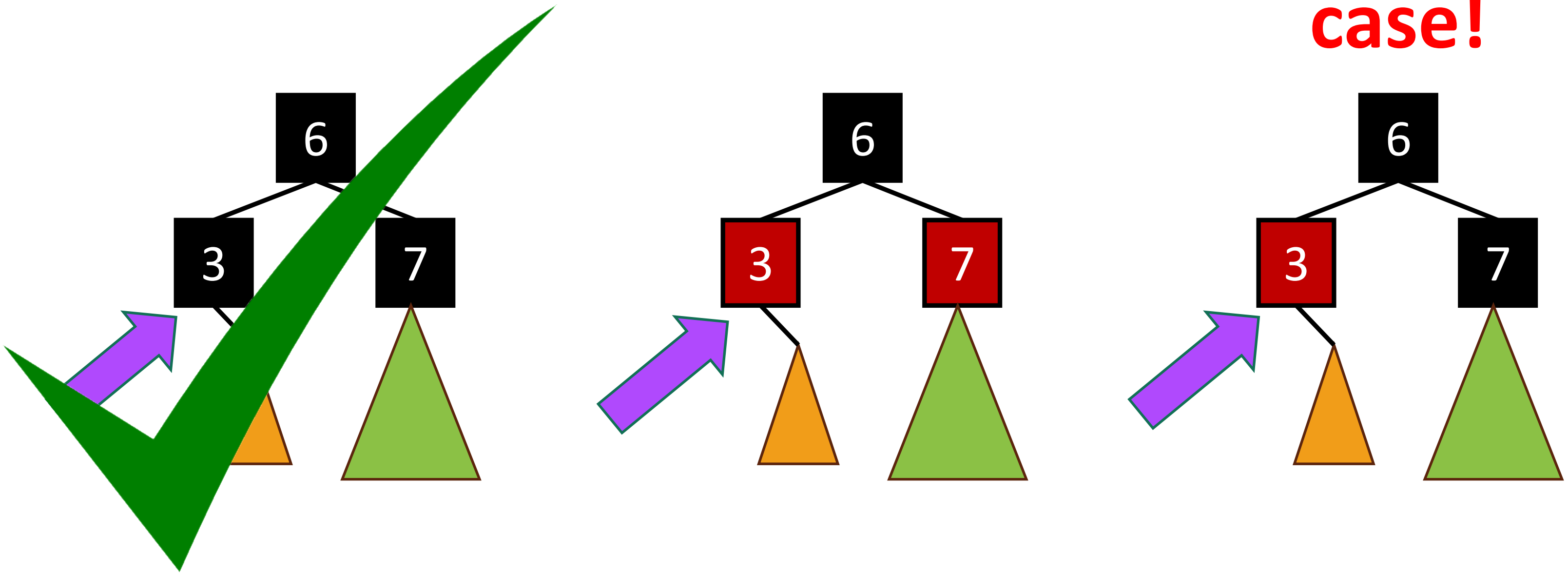


Need to know how to insert into trees that look like this…

Want to insert 6 here.

INSERT: Many cases

That’s this case!



- Suppose we want to insert 0 **here**.
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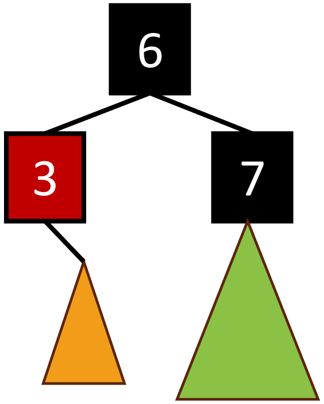
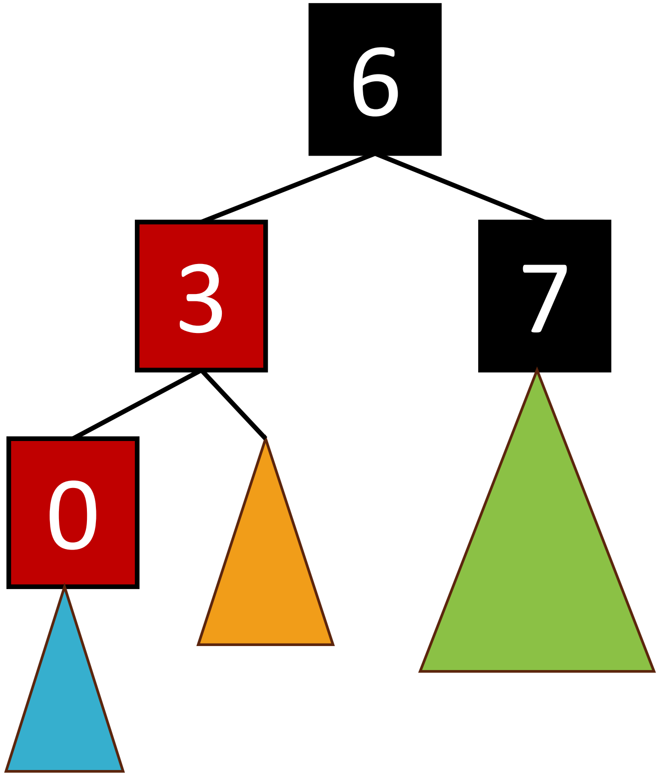
INSERT: Case 3



What if it looks like this?

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.

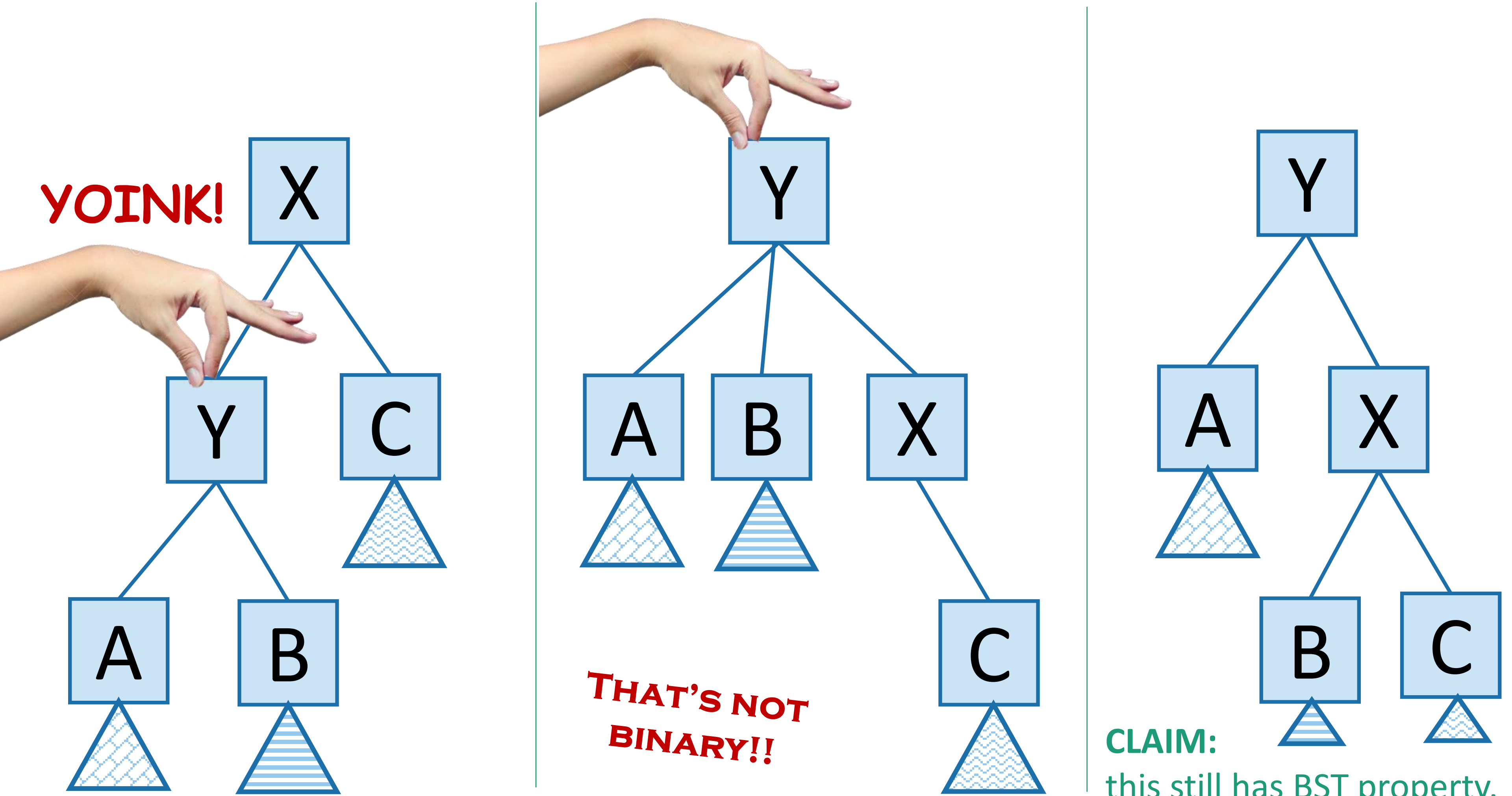


Example: Insert 0.

- Maybe with a subtree below it.

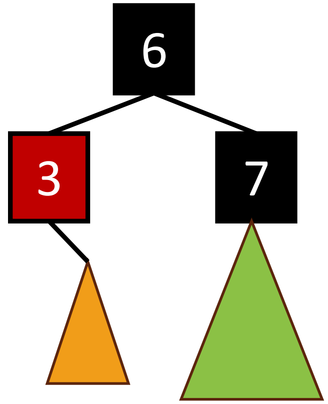
Recall Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.



CLAIM:
this still has BST property.

Inserting into a Red-Black Tree



What if it looks like this?

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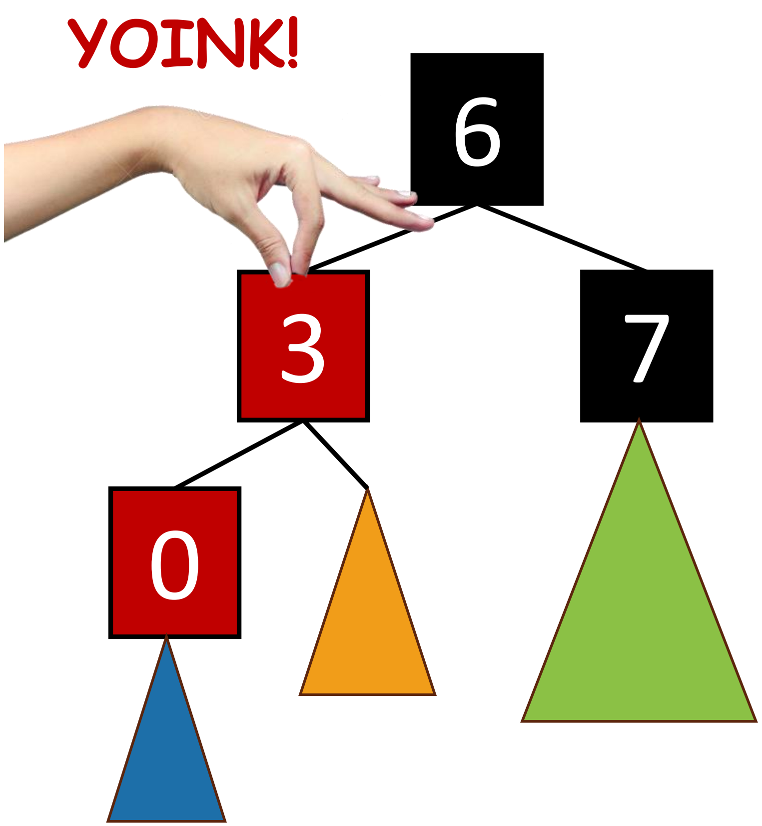
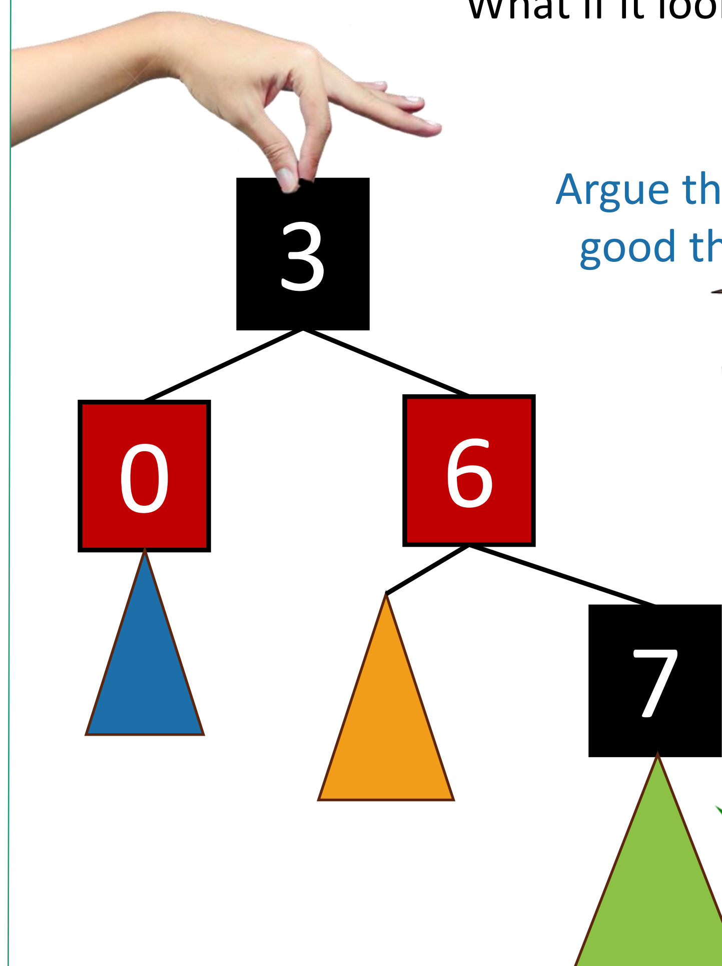
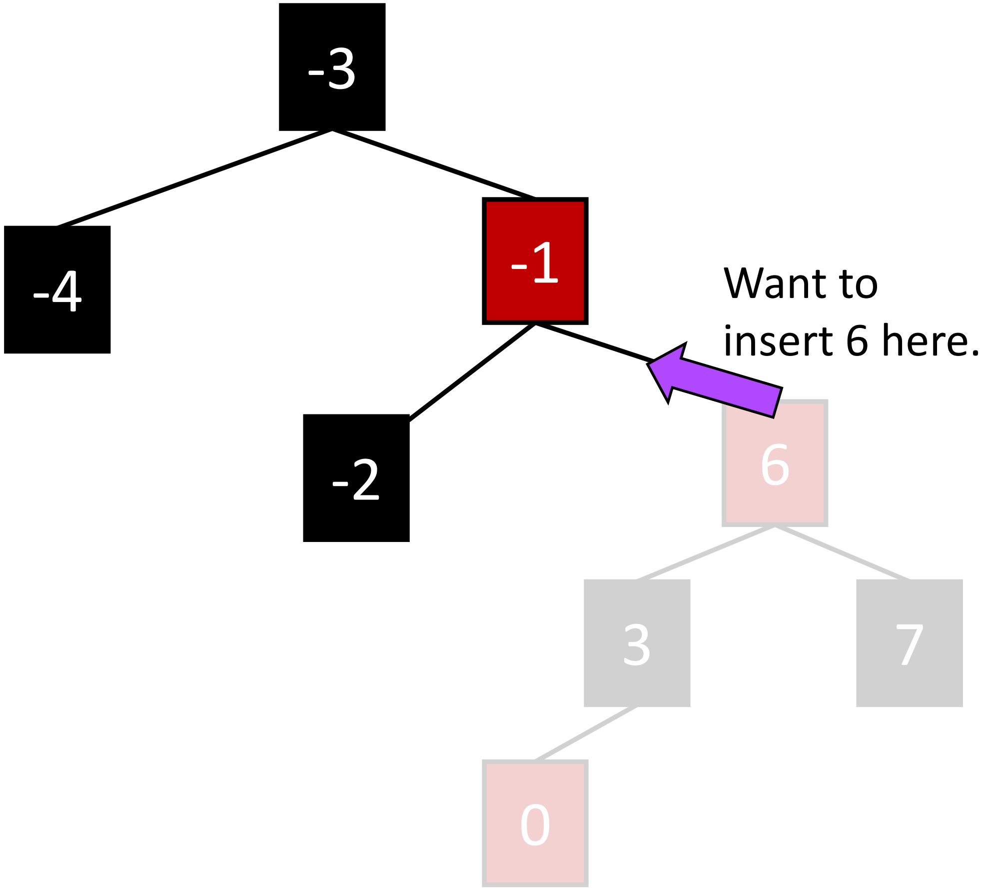


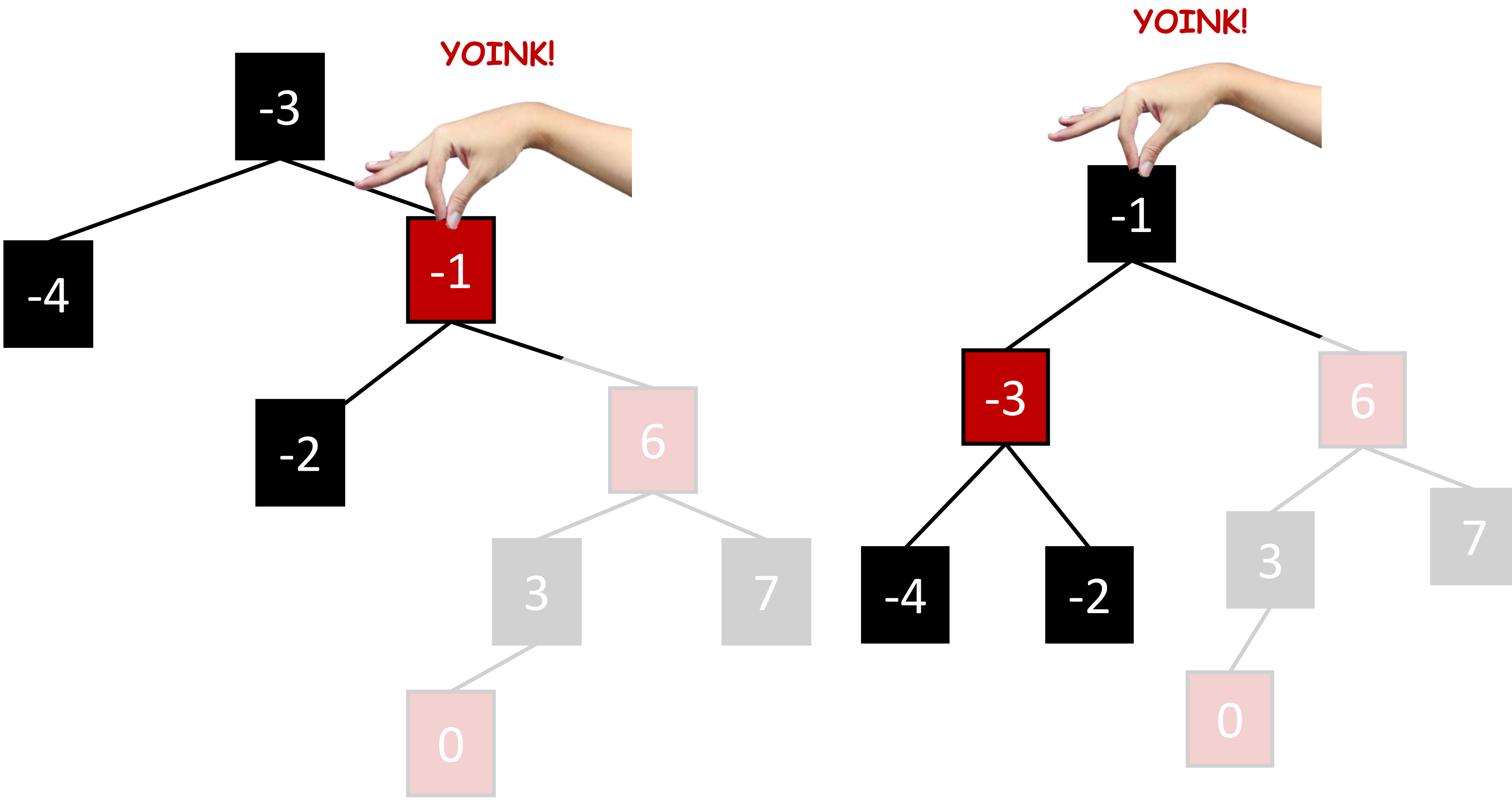


Argue that this is a good thing to do!

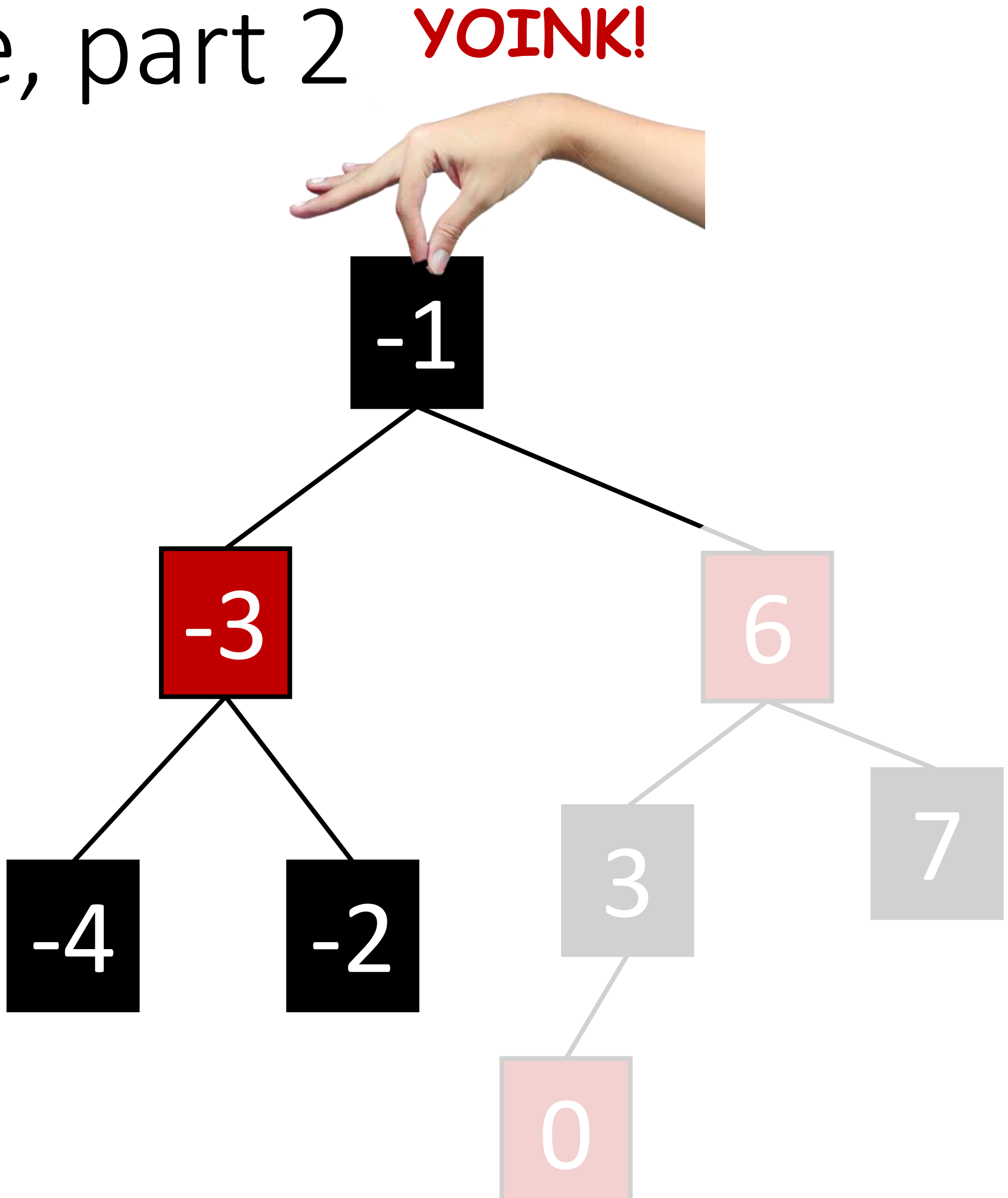
Example, part 2



Example, part 2

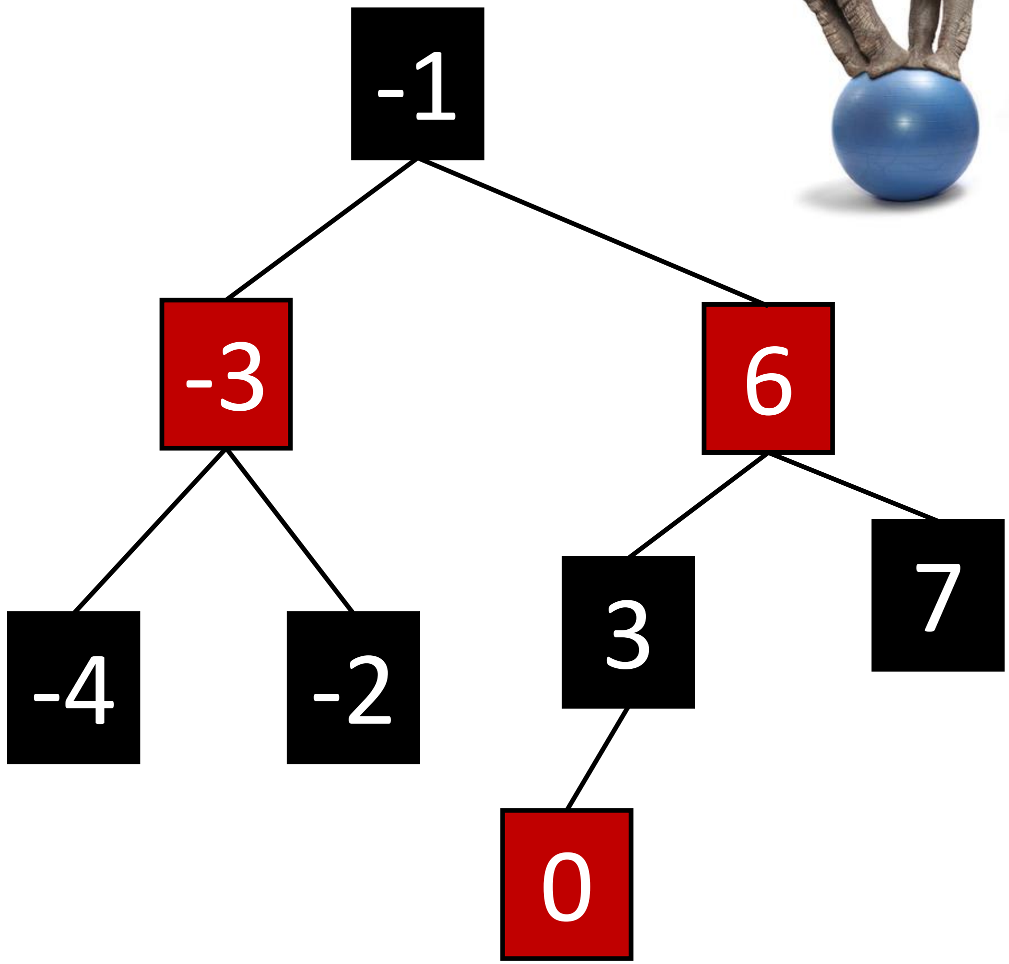


Example, part 2

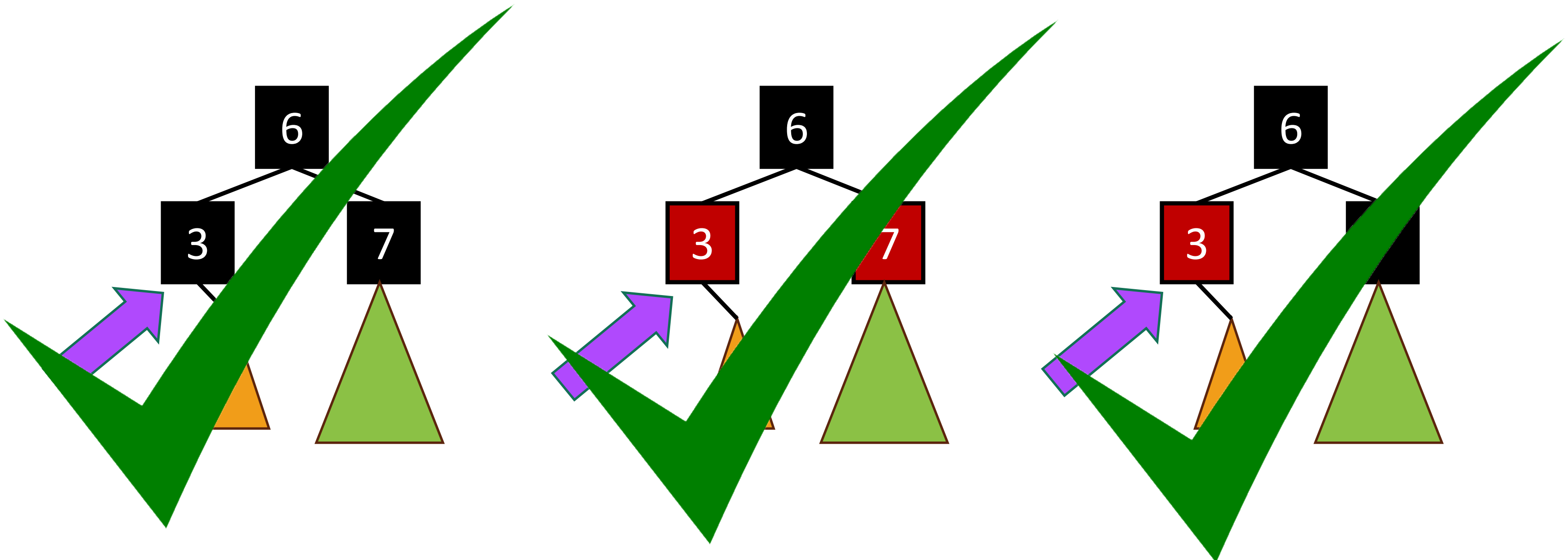


Example, part 2

TA-DA!



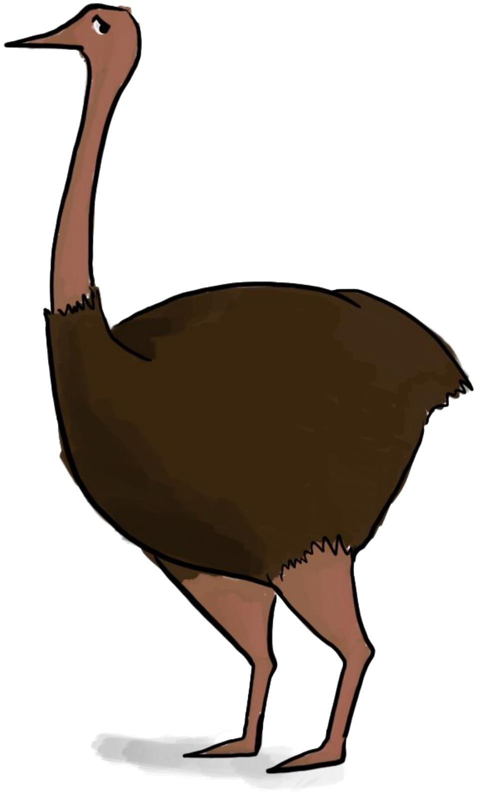
Many cases



- Suppose we want to insert 0 **here**.
- There are 3 "important" cases for different colorings of the existing tree, and there are 9 more cases for all of the various symmetries of these 3 cases.

Deleting from a Red-Black tree

Fun exercise!



That’s a lot of cases!

What have we learned?

- Red-Black Trees always have height at most 2log(n+1).
- As with general Binary Search Trees, all operations are O(height)
- So all operations with RBTrees are O(log(n)).

Conclusion: The best of both worlds

	Sorted Arrays	Linked Lists	Red Black Trees
Search	O(log(n))	O(n)	O(log(n))
Delete	O(n)	O(n)	O(log(n))
Insert	O(n)	O(1)	O(log(n))

Recap

- Balanced binary trees are the best of both worlds!
- But we need to keep them balanced.
- **Red-Black Trees** do that for us.
    - We get O(log(n))-time INSERT/DELETE/SEARCH
    - Clever idea: have a proxy for balance