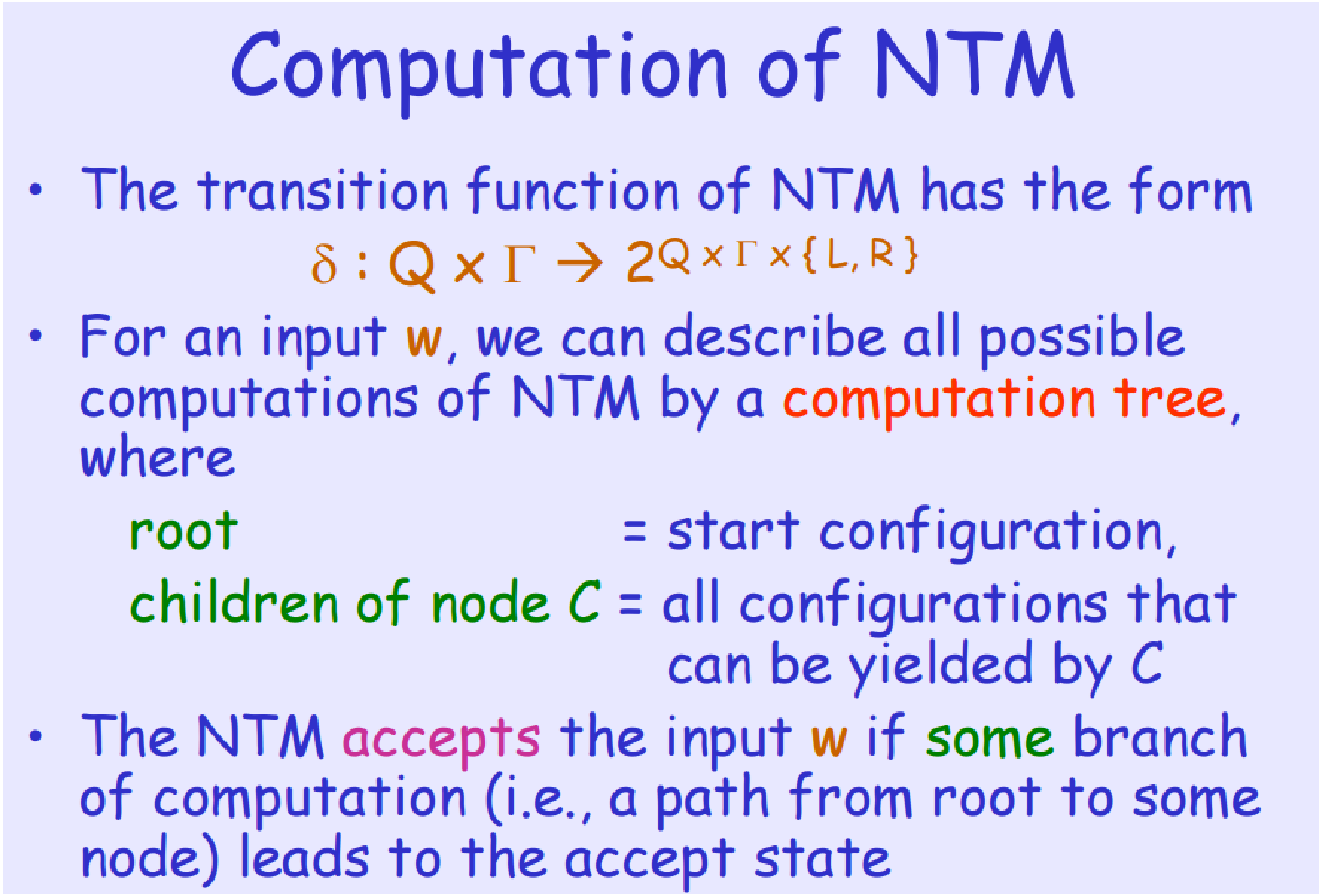
# Time Complexity of NTM

Relationship with DTM

# Recap of Converting a NTM to DTM



## Computation of NTM

- The transition function of NTM has the form

  $$\delta : Q \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L, R\}}$$

- For an input $w$, we can describe all possible computations of NTM by a computation tree, where

  root = start configuration,

  children of node $C$ = all configurations that can be yielded by $C$

- The NTM accepts the input $w$ if some branch of computation (i.e., a path from root to some node) leads to the accept state

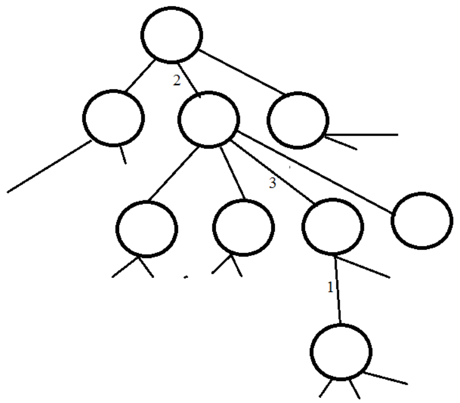
# NTM = TM

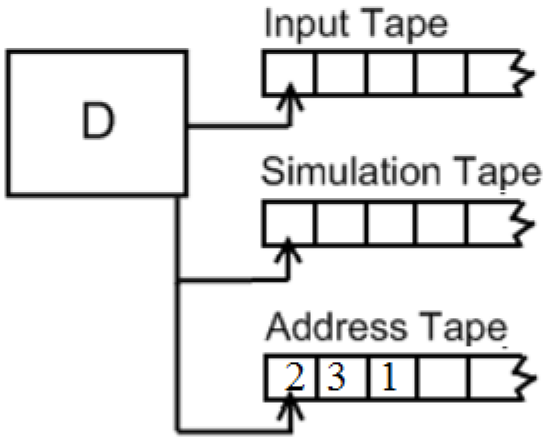
Theorem: Given an NTM that recognizes a language L, we can find a TM that recognizes the same language L.

Proof: Let N be the NTM. We show how to convert N into some TM D. The idea is to simulate N by trying all possible branches of N's computation. If one branch leads to an accept state, D accepts. Otherwise, D's simulation will not terminate.

# NTM = TM (Proof)

- To simulate the search, we use a 3-tape TM for D
  - first tape stores the input string
  - second tape is a working memory, and
  - third tape "encodes" which branch to search
- What is the meaning of "encode"?





## NTM = TM (Proof)

- Let $b = |Q \times \Gamma \times \{ L, R \}|$, which is the maximum number of children of a node in N's computation tree.
- We encode a branch in the tree by a string over the alphabet {1,2,…,b}.
  - E.g., 231 represents the branch:

    root r $\rightarrow$ r's 2nd child c $\rightarrow$
    c's 3rd child d $\rightarrow$ d's 1st child

## NTM = TM (Proof)

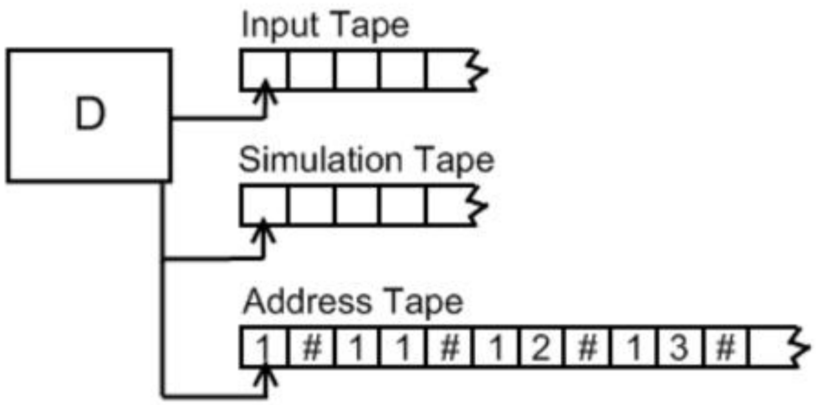
```
On input string w,
  Step 1.  D stores w in Tape 1 and □ in Tape 3
  Step 2.  Repeat
     2a.  Copy Tape 1 to Tape 2
     2b.  Simulate N using Tape 2, with the branch
          of computation specified in Tape 3.
          Precisely, in each step, D checks the
          next symbol in Tape 3 to decide which
          choice to make.   (Special case ...)
```

# NTM = TM (Proof)

2b [Special Case].

1. If this branch of N enters accept state, accepts w
2. If no more chars in Tape 3, or a choice is invalid, or if this branch of N enters reject state, D aborts this branch

2c. Copy Tape 1 to Tape 2, and update Tape 3 to store the next branch (in Breadth-First Search order)



## NTM = TM (Proof)

- In the simulation, D will first examine the branch ε (i.e., root only), then the branch 1 (i.e., root and 1st child only), then the branch 2, and then 3, 4, …, b, then the branches 11, 12, 13, …, 1b, then 21, 22, 23, …, 2b, and so on, until the examined branch of N enters an accept state (what if N enters a reject state?)
- If N does not accept w, the simulation of D will run forever
- Note that we cannot use DFS (depth-first search) instead of BFS (why?)

## TIME COMPLEXITY RELATION BETWEEN DTM & NTM

## NTM decider

An NTM is a decider if all its computation branches halt on all inputs.

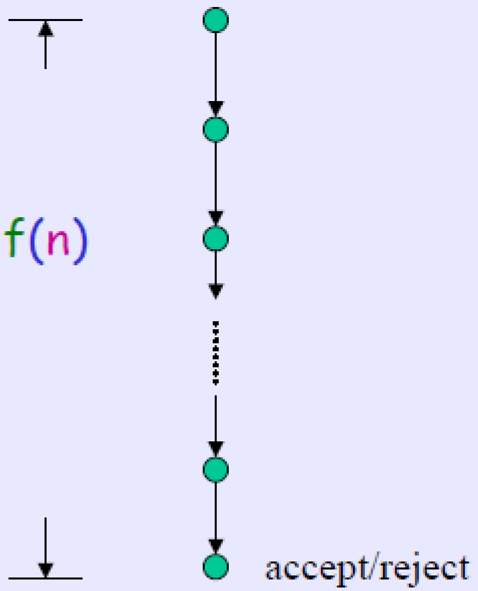
Definition: Let M be an NTM decider. The running time of M is the function $f:N\rightarrow N$, where $f(n)$ is the maximum number of steps that M uses on any branch of its computation on any input of length n

## DTM versus NTM decider

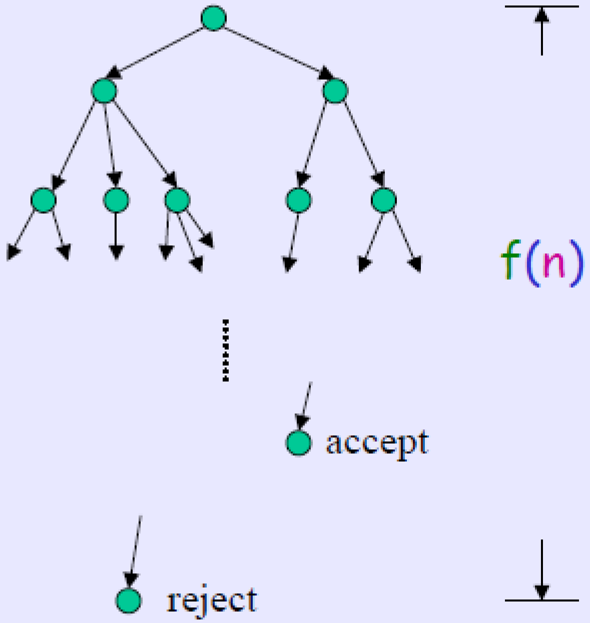
Theorem: Let $t(n)$ be a function, $t(n) \geq n$.
Then every $t(n)$-time single-tape NTM decider has an equivalent $2^{O(t(n))}$-time single-tape DTM

Proof: Let $M$ be a NTM that runs in $t(n)$ time. We construct a DTM $D$ that simulates $M$ by searching $M$'s computation tree. We now analyze $D$'s simulation.

# Comparison of Running Times



Deterministic time



Non-deterministic time

## DTM versus NTM decider (2)

- On an input of length n, every branch of computation of M has at most t(n) steps
- Every node in the computation tree has at most b children, where b is the maximum number of choices in M's transition ➔ number of leaves is at most $O(b^{t(n)})$
- Also, total number of nodes + leaves is at most $O(t(n)\ b^{t(n)})$ (why??)

## DTM versus NTM decider (3)

- The simulation proceeds by visiting the nodes (including leaves) in BFS order. Here, when we visit a node $v$, we always travel starting from the root

  ➔ time to visit $v$ is $O(t(n))$

Thus, the total time for $D$ to simulate $M$ is
$O(t(n)^2 \, b^{t(n)}) = 2^{O(t(n))}$ (why??)

## Time Complexity Class

Definition: Let $t: N \rightarrow R^+$ be a function. We define the time complexity class, TIME(t(n)), to be the collection of all languages that are decidable by an O(t(n))-time Turing machine

the language $A = \{0^k1^k \mid k \geq 0\}$ is in TIME($n^2$)

## The Class P

Definition: **P** is the class of languages that are decidable in polynomial time on a single-tape DTM. In other words,

$$\bigcup_{k=1} \mathrm{TIME}(n^k)$$

- **P** is invariant for all computation models that are polynomially equivalent to the single-tape DTM, and
- **P** roughly corresponds to the class of problems that are realistically solvable

## Further points to notice

- When we describe an algorithm, we usually describe it with stages, just like a step in the TM, except that each stage may actually consist of many TM steps
- Such a description allows an easier (and clearer) way to analyze the running time of the algorithm

## Further points to notice (2)

- So, when we analyze an algorithm to show that it runs in poly-time, we usually do:
  1. Give a polynomial upper bound on the number of stages that the algorithm uses when its input is of length $n$
  2. Ensure that each stage can be implemented in polynomial time on a reasonable deterministic model
- When the two tasks are done, we can say the algorithm runs in poly-time (why??)

## Further points to notice (3)

- Since time is measured in terms of $n$, we have to be careful how to encode a string
- We continue to use the notation $\langle\ \rangle$ to indicate a reasonable encoding
- E.g., the graph encoding in $(V,E)$, DFA encoding in $(Q,\Sigma,\delta,q_0,F)$, are reasonable
- E.g., to encode a number in unary, such as using 11111111111111111 to represent 17, is not reasonable since it is exponentially larger than any base-$k$ encoding with $k > 1$

## Examples of Languages in P

Let PATH be the language

{ ⟨G,s,t⟩ | G is a graph with path from s to t}

Theorem: PATH is in P.

How to prove??

… Find a decider for PATH that runs in polynomial time

## PATH is in P

Proof: A polynomial time decider $M$ for PATH operates as follows:

```
M = "On input ⟨G,s,t⟩,
    1. Mark node s
    2. Repeat until no new nodes are marked
       i. Scan all edges of G to find an edge
          that has exactly one marked node.
          Mark the other node
    3. If t is marked, accept. Else, reject."
```

## PATH is in P (2)

What is the running time for M?

- Let m be the number of nodes in G
- Stages 1 and 3 each involves O(1) scan of the input
- Stage 2 has at most m runs, each run checks at most all the edges of G. Thus, each run involves at most $O(m^2)$ scans of the input ➔ Stage 2 involves $O(m^3)$ scans
- Since m = O(n), where n = input length, the total time is polynomial in n

## Every CFL is in P

Theorem: Every CFL is in P

How to prove??

… Let's recall an old idea for deciding a particular CFL …

## Every CFL is in P (2)

Proof(?): Let C be the CFL and G be the CFG in Chomsky Normal form that generates C. Define M as follows:

```
M = "On input $w = w_1 w_2 \ldots w_n$,
  1. Construct all possible derivations in G
     with 2n-1 steps
  2. If any derivation generates w, accept.
     Else, reject."
```

Quick Quiz: Does M run in polynomial time?

But, CYK algorithm runs in $O(n^3)$ time.

## The Class NP

Definition: A verifier for a language $A$ is an algorithm $V$, where

$A = \{ w \mid V \text{ accepts } \langle w,c \rangle \text{ for some string } c\}$

A polynomial-time verifier is a verifier that runs in time polynomial in the length of the input $w$.

## The Class NP

A language A is polynomially verifiable if it has a polynomial time verifier.

Definition: NP is the class of language that is polynomially verifiable.

## Examples of Languages in NP

Let HAMILTON be the language

$\{ \langle G \rangle \mid G$ is a Hamiltonian graph $\}$

Theorem: HAMILTON is in NP.

How to prove?? … Define a polynomial time verifier $V$, and for each $\langle G \rangle$ in HAMILTON, define a string $c$, and show $\{ \langle G \rangle \mid V$ accepts $\langle G,c \rangle \}$ = HAMILTON

## HAMILTON is in NP

Proof: Define a TM V as follows:

```
V = "On input ⟨G,c⟩ ,
    1. If c is a cycle in G that visits each
       vertex once, accept
    2. Else, reject."
```

- Note: V runs in time polynomial in length of ⟨G⟩ (why?)
- To show HAMILTON is in NP, it remains to show V is a verifier for HAMILTON

## HAMILTON is in NP (2)

To show V is a verifier, we let H = { ⟨G⟩ | V accepts ⟨G,c⟩ }, and show H = HAMILTON

For every ⟨G⟩ in H, there is some c that V accepts ⟨G,c⟩. This implies ⟨G⟩ is a Hamiltonian graph, and H $\subseteq$ HAMILTON

For every ⟨G⟩ in HAMILTON, let c be one of the hamilton cycle in the graph. Then, V accepts ⟨G,c⟩, and so HAMILTON $\subseteq$ H

## Examples of Languages in NP (2)

Let COMPOSITE be the language

$\{ x \mid x$ is a composite number $\}$

Theorem: COMPOSITE is in NP.

How to prove?? ... Define a polynomial time verifier $V$, and for each $x$ in COMPOSITE, define a string $c$, and show that $\{ x \mid V$ accepts $\langle x,c \rangle \}$ = COMPOSITE

## COMPOSITE is in NP

Proof: Define a TM V as follows:

```
V = "On input ⟨x,c⟩ ,
    1. If c is not 1 or x, and c divides x,
       accept
    2. Else, reject."
```

- Note: V runs in time polynomial in length of ⟨x⟩ (why?)
- To show COMPOSITE is in NP, it remains to show V is a verifier for COMPOSITE

## COMPOSITE is in NP (2)

To show V is a verifier, we let $C = \{ x \mid V$ accepts $\langle x,c \rangle \}$, and show $C$ = COMPOSITE

For every x in $C$, there is some c that V accepts $\langle G,c \rangle$. This implies x is a composite number, and $C \subseteq$ COMPOSITE

For every x in COMPOSITE, let c be one of the divisor of x with $1 < c < x$. Then, V accepts $\langle x,c \rangle$, and so COMPOSITE $\subseteq C$