Objectives

- More discussion on the class NP
- Cook-Levin Theorem

The Class NP revisited

- Recall that NP is the class of language that has a polynomial time verifier
- If a language L is in NP, there is a polynomial time TM V (verifier) such that
  - For each w in L, there is some string c such that ⟨w,c⟩ is accepted by V, and
  - For every w not in L, ⟨w,c⟩ is rejected by V for all c

The Class NP revisited (2)

The reason why V is called a verifier for L:

- if a string w is in L, it has at least one certificate c to present to V, allowing V to verify that it is in L
- if a string w is not in L, it will not have any certificate c to present to V, so that V will never make a mistake to verify that it is in L

V is actually like a judge in the old days: It tends to say you are guilty, unless you can prove yourself to be innocent!

Examples (HAMPATH)

- A trivial certificate for a string $\langle G \rangle$ in HAMPATH (i.e., for a graph G to be Hamiltonian) is the order of vertices visited in a Hamiltonian path in G
  - We can find a verifier V that uses this trivial certificate to prove $\langle G \rangle$ is in HAMPATH in polynomial time (in terms of the length of $\langle G \rangle$)
- Also, using this V, a non-Hamiltonian graph can never find a certificate to "prove" that it is Hamiltonian

Examples (COMPOSITE)

- The certificate for a string $\langle x \rangle$ in COMPOSITE (I.e., the number x is a composite number) is a factor of x between 2 to x-1
  - The corresponding verifier can use the certificate to prove $\langle x \rangle$ is in COMPOSITE in polynomial time (in terms of $\log x$ --- the length of $\langle x \rangle$)
- Also, when a number not composite, no certificate can fool this verifier

Properties of NP

Theorem: A language is in NP if and only if it is decided by a NTM that runs in polynomial time.

Prove idea: We show how to convert a verifier into a NTM and vice versa…

Properties of NP (2)

Proof: (➔) Let A be a language in NP, so that it can be verified by some polynomial time verifier V. Let n^k be the running time of V. We create a polynomial time NTM N that decides A as follows:

N = "On input w,

1. Select a string c of length at most n^k
2. Run V on $\langle w,c \rangle$
3. If V accepts, accept. Else, reject."

Properties of NP (3)

Proof: (←) Let A be a language decided by some polynomial time NTM N. We construct a polynomial time verifier V as follows:

V = "On input $\langle w,c \rangle$,

1. Simulate N on w, treating c as the description of the non-deterministic choice of N at each step
2. If this branch of computation in N accepts, *accept*. Else, *reject*."

Properties of NP (4)

Definition: NTIME($t(n)$) = the set of languages that can be decided by an NTM that runs in $O(t(n))$ time

Based on the above definition and the previous theorem, we have:

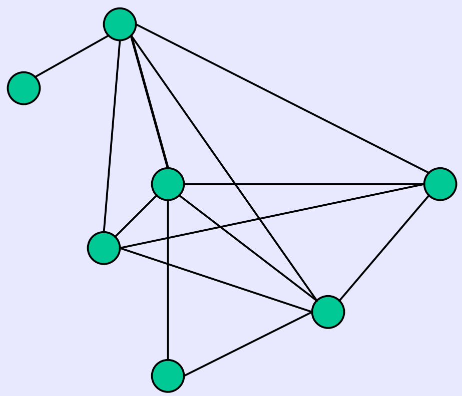
Corollary: $NP = \bigcup_k NTIME(n^k)$

More Examples of NP

Definition: A clique is a subgraph G' in an undirected graph G such that every two nodes in G' are connected.

Definition: A k-clique is a clique that contains k nodes.

E.g.,



Can you find a 5-clique here?

More Examples in NP (2)

Let CLIQUE be the language

{ $\langle G,k \rangle$ | G is a graph with a k-clique }

Theorem: CLIQUE is in NP.

How to prove??

CLIQUE is in NP

Proof 1 (using DTM verifier): What should be the certificate that a graph can prove itself is a k-clique??

The following is the verifier for CLIQUE:

V = "On input $\langle G,k,c \rangle$

1. Test if c is a set of k nodes in G
2. Test whether every two nodes in c are connected
3. If both pass, accept. Else, reject."

CLIQUE is in NP (2)

Proof 2 (using NTM decider): What should be the "guess" made by the NTM in order to check if G contains a k-clique??

The NTM below is one that decides CLIQUE:

N = "On input $\langle G,k \rangle$

1. Non-deterministically select a subset c containing k nodes of G
2. Check if every two nodes in c are connected
3. If yes, *accept*. Else, *reject*."

Another Examples in NP

Let SUBSET-SUM be the language

$\{ \langle S, t \rangle \mid S$ is a set of integers such that

a subset of S adds up to $t \}$

Theorem: SUBSET-SUM is in NP.

How to prove??

SUBSET-SUM is in NP

Proof 1 (using DTM verifier): What should be the certificate that S can prove itself has a subset that adds up to t??

The verifier below is one for SUBSET-SUM:

V = "On input $\langle S,t,c \rangle$

1. Test if c is a set of numbers in S
2. Test if the numbers in c adds up to t
3. If both pass, accept. Else, reject."

SUBSET-SUM is in NP (2)

Proof 2 (using NTM decider): What should be the non-deterministic guess made by the NTM in order to check if S contains a subset that adds up to t??

The NTM below is one that decides SUBSET-SUM:

N = "On input ⟨S,t⟩

1. Non-deterministically find a subset c of S
2. Check if the numbers in c adds up to t
3. If yes, *accept*. Else, *reject*."

P versus NP

Roughly speaking:

P = the class of language that can be
decided "quickly"
NP = the class of language that can be
verified "quickly"

- The power of the polynomial time NTM decider seems to be much greater than the polynomial time DTM decider…

P versus NP (2)

- … Unfortunately, so far, nobody can tell whether $P = NP$, or $P \neq NP$, is true
- A general belief (which may not be true) is $P \neq NP$ because people has input a lot of effort to find polynomial time algorithms for certain problems in NP, but fail
- What we can conclude safely so far is:

$$P \subseteq NP \subseteq \bigcup_k TIME(2^{n^k}) = EXPTIME$$

NP-Completeness

In the early 1970s, Stephen Cook and Leonid Levin (separately) discovered that:

Some languages in NP, if any of them are decidable by a DTM in polynomial time, will imply ALL problems in NP can be decided by a DTM in polynomial time

That is, they discovered some language L, such that if L is in P, then P = NP

NP-Completeness (2)

These problems are called NP-complete problems, and they form a class NP-C (We shall give formal definition later)

The first NP-complete problem we present is called the satisfiability problem

Satisfiability Problem

Definition: A variable v is called a Boolean variable if it has a value either 1 (TRUE) or 0 (FALSE)

Definition: The Boolean operations $\wedge$, $\vee$, $\neg$, are defined as follows:

$$0 \wedge 0 = 0, \quad 0 \vee 0 = 0, \quad \neg 0 = 1$$
$$0 \wedge 1 = 0, \quad 0 \vee 1 = 1, \quad \neg 1 = 0$$
$$1 \wedge 0 = 0, \quad 1 \vee 0 = 1,$$
$$1 \wedge 1 = 1, \quad 1 \vee 1 = 1$$

Satisfiability Problem (2)

Definition: *A* **Boolean formula** is an expression involving Boolean variables and operations

E.g., The following is a Boolean formula:

$$F = (\neg x \land y) \lor (x \land \neg z)$$

Satisfiability Problem (2)

Definition: A Boolean formula is satisfiable if some assignments of 0 and 1 to the variables makes the value of the formula equal to 1

E.g., The previous Boolean formula

$$F = (\neg x \wedge y) \vee (x \wedge \neg z)$$

is satisfiable because if we set $x = 0$, $y = 1$, and $z = 1$, the value of F becomes 1

Cook-Levin Theorem

Let SAT be the language

$\{ \langle F \rangle \mid F$ is a satisfiable Boolean formula $\}$

Theorem: SAT is P if and only if P = NP

Next Time

- Polynomial Time Reducibility
- Prove Cook-Levin Theorem
- Proving other problems to be NP-Complete