Closure properties of CFLs

- There are some differences when compared with regular languages.
- We deviate from both text books (to simplify the things).
- We want to skip homomorphism.

CFLs are --

- Closed under
  - Union
  - Concatenation
  - Kleene star
  - Reversal
  - Intersection with regular languages
  -
  -

- Not closed under
  - Intersection
  - Difference
  - Complement
  - Repetition
  -
  -

CFLs are closed under Union

- L_1 and L_2 are CFLs $\Rightarrow L_1 \cup L_2$ is a CFL.

CFLs are closed under Union

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- Proof: Let the CFG for these are $G_1 = (V_1, T_1, P_1, S_1)$ and $G_2 = (V_2, T_2, P_2, S_2)$.
- We rename variables so that V_1 and V_2 such that they are disjoint and does not have a variable with name S. Note T_1 and T_2 need not be disjoint. Perhaps they are same.
- Now the grammar $(V_1 \cup V_2, T_1 \cup T_2, P_1 \cup P_2 \cup \{S \rightarrow S_1 | S_2\}, S)$ will generate the union.

Closed under concatenation

- Add the production $S \rightarrow S_1 S_2$

Closed under Kleene star

- Add productions $S \rightarrow S_1 S | \epsilon$

Closed under reversal

Theorem 7.25: If L is a CFL, then so is L^R.

PROOF: Let $L = L(G)$ for some CFL $G = (V, T, P, S)$. Construct $G^R = (V, T, P^R, S)$, where P^R is the "reverse" of each production in P. That is, if $A \to \alpha$ is a production of G, then $A \to \alpha^R$ is a production of G^R. It is an easy induction on the lengths of derivations in G and G^R to show that $L(G^R) = L^R$. Essentially, all the sentential forms of G^R are reverses of sentential forms of G, and vice-versa. We leave the formal proof as an exercise. $\square$

Not closed under intersection☹

Not closed under intersection☹

- Proof [by counter example]:

$L = \{0^n 1^n 2^n \mid n \geq 1\}$ is not a context-free language.

However, the following two languages *are* context-free:

$$L_1 = \{0^n 1^n 2^i \mid n \geq 1, i \geq 1\}$$
$$L_2 = \{0^i 1^n 2^n \mid n \geq 1, i \geq 1\}$$

- And, $L = L_1 \cap L_2$

A grammar for L_1 is:

$$\begin{aligned} &S \rightarrow AB \\ &A \rightarrow 0A1 \mid 01 \\ &B \rightarrow 2B \mid 2 \end{aligned}$$

In this grammar, A generates all strings of the form $0^n 1^n$, and B generates all strings of 2's.

A grammar for L_2 is:

$$\begin{aligned} &S \rightarrow AB \\ &A \rightarrow 0A \mid 0 \\ &B \rightarrow 1B2 \mid 12 \end{aligned}$$

CFLs are closed when intersected with regular languages

Theorem 7.27: If L is a CFL and R is a regular language, then $L \cap R$ is a CFL.

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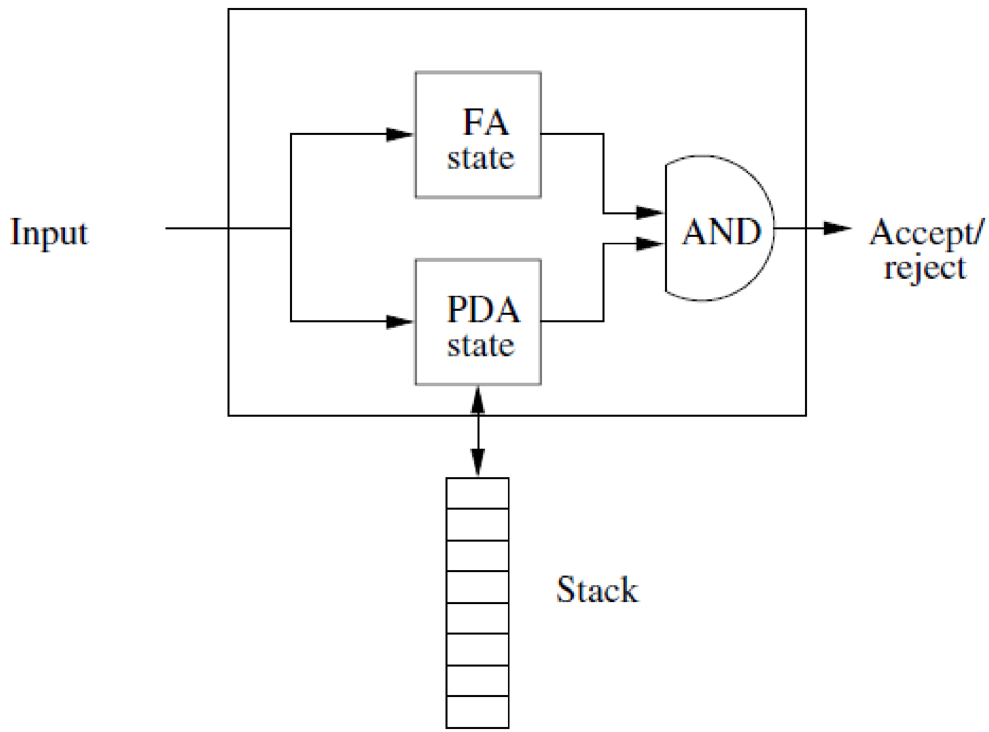

Figure 7.9: A PDA and a FA can run in parallel to create a new PDA

- Dyck set (strings of balanced parentheses) is a CFL
- $(^*)^*$ is a regular language
- Intersection of above two is a CFL .

CFLs are not closed under complementation

- We know $L = \{ww | w \in \{0,1\}^* \}$ is not a CFL.

CFLs are not closed under complementation

- We know $L = \{ww | w \in \{0,1\}^* \}$ is not a CFL.
- But its complement is a CFL !!
- This is a proof by counter example.

CFG for the $\bar{L}$

- $S \to S_o | S_e$
- $S_o \to 0R|1R|0|1,\ R \to 0S_o|1S_o$
- $S_e \to XY|YX,\ X \to ZXZ\ |0$,
  $Y \to ZYZ\ |1$, $Z \to 0|1$

- S_o generates odd length strings, whereas S_e generates even length strings.

Other proof

- Other proof is through contradiction.
- If it is closed under complementation then it has to be closed under intersection.
- Refer the Ullman book.

Not closed under set difference

- Proof by contradiction.
- If closed under set difference, then it has to be closed under complementation.

Not closed under set difference

- Proof by contradiction.
- If closed under set difference, then it has to be closed under complementation.
- Σ^* is a CFL
- Let L be any CFL
- If $\Sigma^* - L$ is a CFL, then $\overline{L} = \Sigma^* - L$ must be a CFL.

Set difference with regular is okay☺

- If L is a CFL and R is a regular language, then $L - R$ is a CFL.
- $L - R = L \cap \bar{R}$
- If R is regular then $\bar{R}$ is regular.

- Show that the language L which consists of strings of equal number of 0s, 1s, and 2s is not a CFL.

- Show that the language L which consists of strings of equal number of 0s, 1s, and 2s is not a CFL.
- Proof by contradiction.
- Assume L is a CFL.
- We know $0^*1^*2^*$ is regular. Let us call this R.
- Then, $L \cap R = \{0^n 1^n 2^n | n \geq 0\}$ which must be a CFL. But is NOT. So the contradiction.