Deterministic Finite Automaton and Non-deterministic Finite Automaton

DFA and NFA

(Finite State Machines)

Notation and Definitions

- Alphabet
- String
- Language
- Operations on languages

Strings

- An alphabet is any finite set of distinct symbols
  - $\{0, 1\}$, $\{0,1,2,\ldots,9\}$, $\{a,b,c\}$
  - We denote a generic alphabet by Σ
- A string is any finite-length sequence of elements of Σ.
- e.g., if $\Sigma = \{a, b\}$ then *a*, *aba*, *aaaa*, ..., *abababbaab* are some strings over the alphabet Σ

Strings

- The length of a string ω is the number of symbols in ω. We denote it by $|\omega|$. $|aba| = 3$.
- The symbol ϵ denotes a special string called the empty string
  - ϵ has length 0
- String concatenation
  - If $\omega = a_1, \ldots, a_n$ and $\nu = b_1, \ldots, b_m$ then $\omega \cdot \nu$ (or $\omega\nu$) $= a_1, \ldots, a_n b_1, \ldots, b_m$
  - Concatenation is associative with ϵ as the identity element.
- If $a \in \Sigma$, we use a^n to denote a string of n a's concatenated
  - $\Sigma = \{0, 1\}, 0^5 = 00000$
  - $a^0 =_{def} \epsilon$
  - $a^{n+1} =_{def} a^n a$

Strings

- The reverse of a string ω is denoted by ω^R.
  - $\omega^R = a_n, \ldots, a_1$
- A substring y of a string ω is a string such that $\omega = xyz$ with $|x|, |y|, |z| \geq 0$ and $|x| + |y| + |z| = |\omega|$
- If $\omega = xy$ with $|x|, |y| \geq 0$ and $|x| + |y| = |\omega|$, then x is prefix of ω and y is a suffix of ω.
  - For $\omega = abaab$,
    - ϵ, a, aba, and $abaab$ are some prefixes
    - ϵ, $abaab$, aab, and $baab$ are some suffixes.

Strings

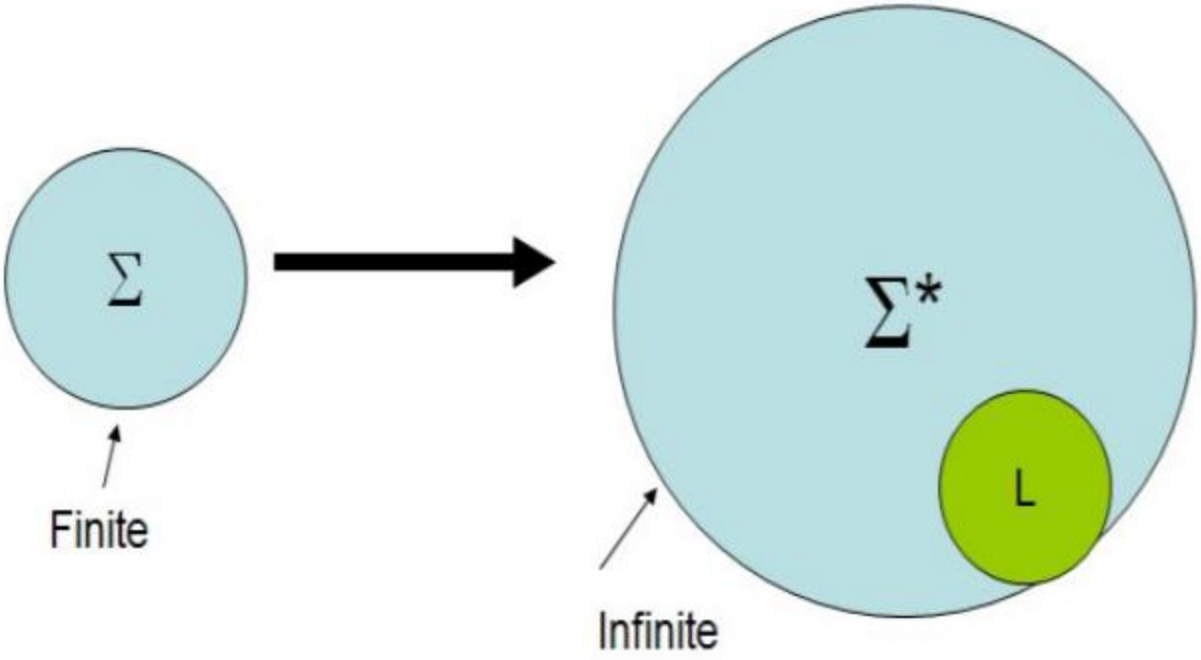
- The set of all possible strings over Σ is denoted by Σ^*.
- We define $\Sigma^0 = \{\epsilon\}$ and $\Sigma^n = \Sigma^{n-1} \cdot \Sigma$
  - with some abuse of the concatenation notation applying to sets of strings now
- So $\Sigma^n = \{\omega | \omega = xy \text{ and } x \in \Sigma^{n-1} \text{ and } y \in \Sigma\}$
- $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots \Sigma^n \cup \cdots = \bigcup_0^\infty \Sigma^i$
  - Alternatively, $\Sigma^* = \{x_1 x_2 \ldots x_n | n \geq 0 \text{ and } x_i \in \Sigma \text{ for all } i\}$
- Φ denotes the empty set of strings $\Phi = \{\}$,
  - but $\Phi^* = \{\epsilon\}$

Strings

- Σ^* is a countably infinite set of finite length strings
- If x is a string, we write x^n for the string obtained by concatenating n copies of x.
  - $(aab)^3 = aabaabaab$
  - $(aab)^0 = \epsilon$

Languages

- A language *L* over Σ is any subset of Σ^*



- *L* can be finite or (countably) infinite

Some Languages

- $L = \Sigma^*$ – The mother of all languages!
- $L = \{a, ab, aab\}$ – A fine finite language.
  - Description by enumeration
- $L = \{a^n b^n : n \geq 0\} = \{\epsilon,\ \text{ab},\ \text{aabb},\ \text{aaabbb}, \ldots\}$
- $L = \{\omega | n_a(\omega) \text{ is even}\}$
  - $n_x(\omega)$ denotes the number of occurrences of x in ω
  - all strings with even number of a's.
- $L = \{\omega | \omega = \omega^R\}$
  - All strings which are the same as their reverses – palindromes.
- $L = \{\omega | \omega = xx\}$
  - All strings formed by duplicating some string once.
- $L = \{\omega | \omega$ is a syntactically correct Java program $\}$

Languages

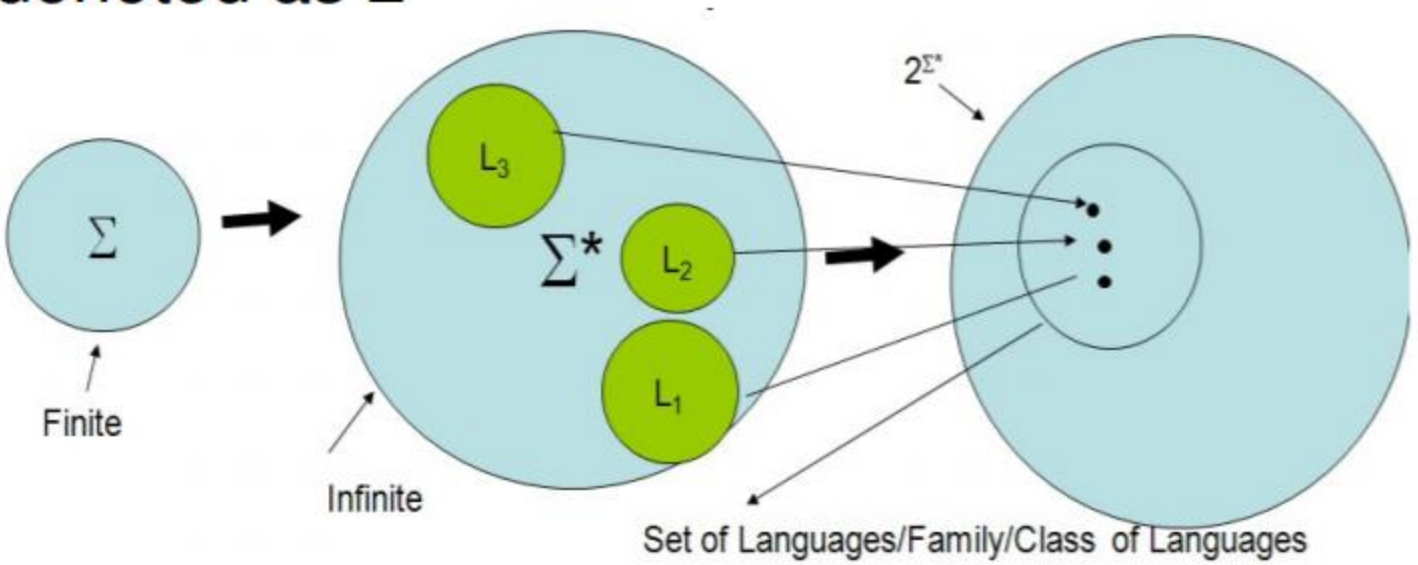
- Since languages are sets, all usual set operations such as intersection and union, etc. are defined.
- Complementation is defined with respect to the universe Σ^* : $\overline{L} = \Sigma^* - L$

Languages

- If L, L_1 and L_2 are languages:
  - $L_1 \cdot L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
  - $L^0 = \{\epsilon\}$ and $L^n = L^{n-1} \cdot L$
  - $L^* = \bigcup_0^\infty L^i$
  - $L^+ = \bigcup_1^\infty L^i$

Sets of Languages

- The power set of Σ^*, the set of all its subsets, is denoted as 2^{Σ^*}



AUTOMATA

- The control unit has some finite memory and it keeps track of what step to execute next.
- Additional memory (if any) is infinite - we never run out of memory!
  - Infinite but like a stack - only the top item is accessible at a given time.
  - Infinite but like a tape, any cell is (sequentially) accessible.

Finite State Automata

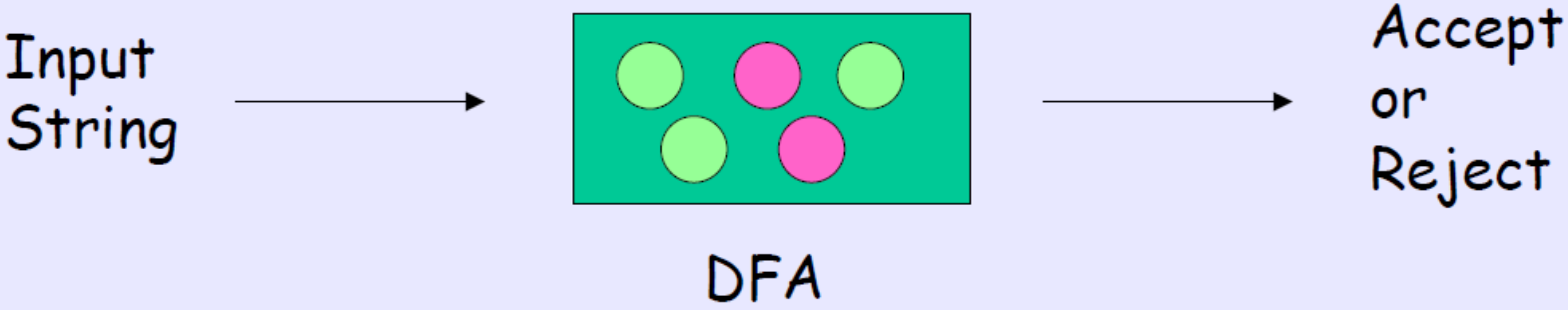
- Finite State Automata (FSA) are the simplest automata.
- Only the finite memory in the control unit is available.
- The memory can be in one of finite states at a given time – hence the name.
  - One can remember only a (fixed) finite number of properties of the past input.
  - Since input strings can be of arbitrary length, it is not possible to remember unbounded portions of the input string.
- It comes in Deterministic and Nondeterministic flavors.

Deterministic Finite State Automata (DFA)

- A DFA starts in a start state and is presented with an input string.
- It moves from state to state, reading the input string one symbol at a time.
- What state the DFA moves next depends on
  - the current state,
  - current input symbol
- When the last input symbol is read, the DFA decides whether it should accept the input string

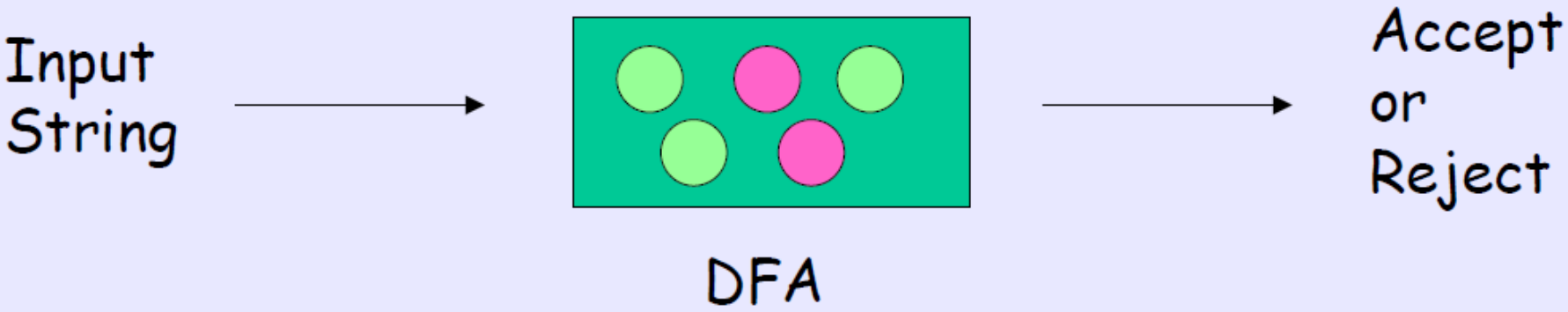
Finite State Machines

DFA



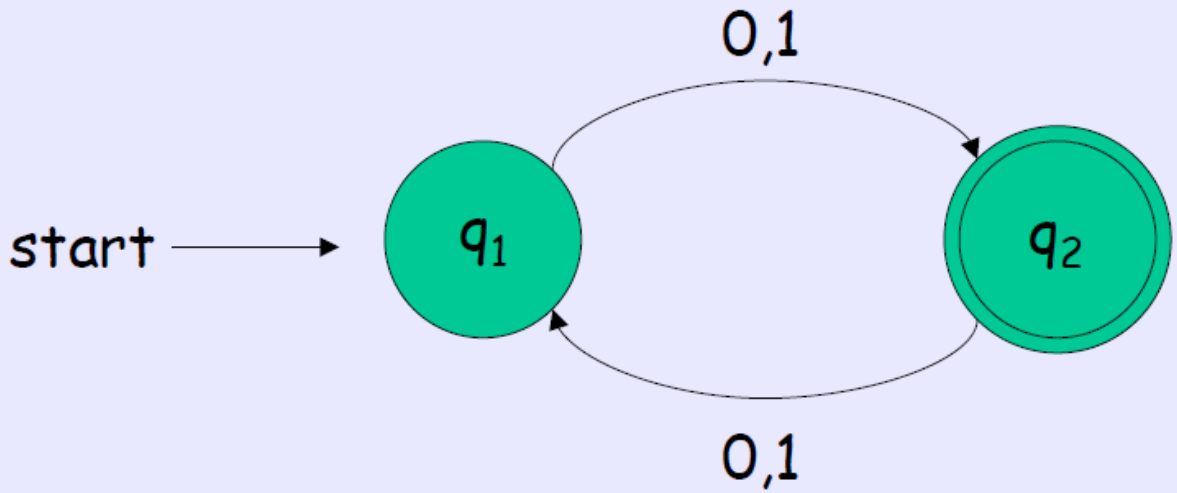
- A machine with finite number of states, some states are accepting states, others are rejecting states
- At any time, it is in one of the states
- It reads an input string, one character at a time

DFA



- After reading each character, it moves to another state depending on what is read and what is the current state
- If reading all characters, the DFA is in an accepting state, the input string is accepted.
- Otherwise, the input string is rejected.

Example of DFA



- The circles indicates the states
- If accepting state is marked with double circle
- The arrows pointing from a state q indicates how to move on reading a character when current state is q

DFA – Formal Definition

- A Deterministic Finite State Acceptor (DFA) is defined as the 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$ where
  - Q is a finite set of states
  - Σ is a finite set of symbols – the alphabet
  - $\delta : Q \times \Sigma \to Q$ is the next-state function
  - $q_0 \in Q$ is the (label of the) start state
  - $F \subseteq Q$ is the set of final (accepting) states

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Note, there must be exactly one start state.
Final states can be many or even empty !

Some Terminology

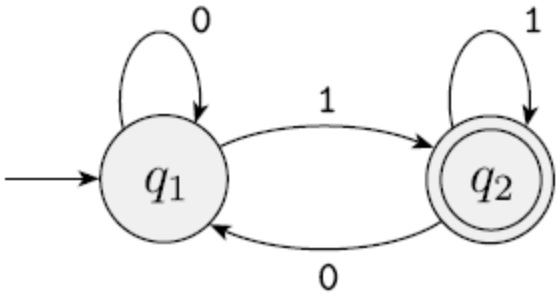
Let M be a DFA

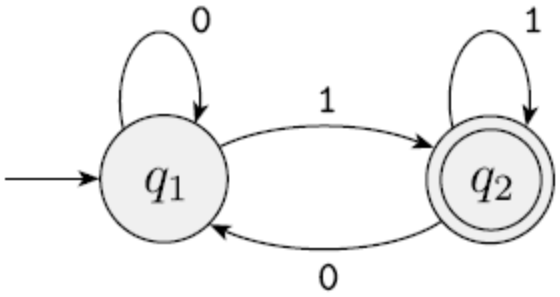
- Among all possible strings, M will accept some of them, and M will reject the remaining
- The set of strings which M accepts is called the language recognized by M
- That is, M recognizes A if

$$A = \{ w \mid M \text{ accepts } w \}$$

L(M)

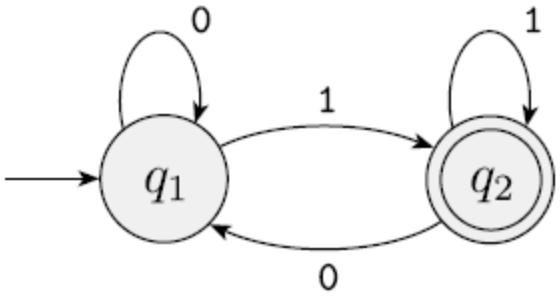
If A is the set of all strings that machine M accepts, we say that A is the ***language of machine M*** and write $L(M) = A$. We say that ***M recognizes A*** or that ***M accepts A***.


0
1
1
q_1
q_2
0



In the formal description, M_2 is $(\{q_1, q_2\}, \{0,1\}, \delta, q_1, \{q_2\})$. The transition function δ is

	0	1
q_1	q_1	q_2
q_2	q_1	q_2.



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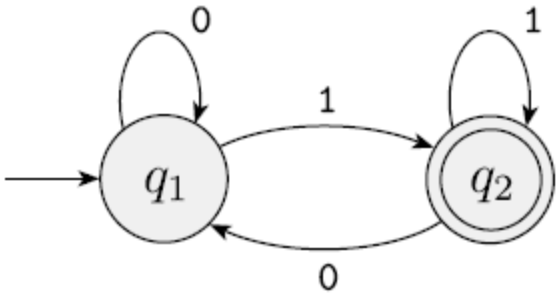
	0	1
q_1	q_1	q_2
q_2	q_1	q_2.

Remember that the state diagram of M_2 and the formal description of M_2 contain the same information, only in different forms. You can always go from one to the other if necessary.



M_2

$$L(M_2) = \{w | \ w \text{ ends in a } 1\}.$$

Consider the finite automaton M_3.

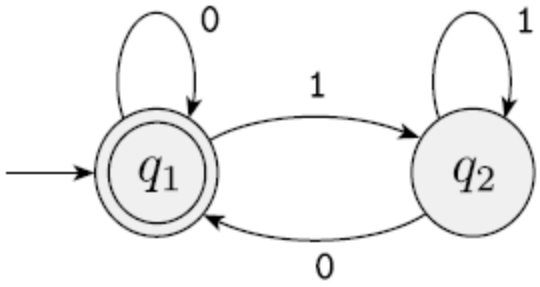


FIGURE **1.10**
State diagram of the two-state finite automaton M_3

Can you describe this in the 5 tuple form?
In particular, can you write down the transition table?

Consider the finite automaton M_3.

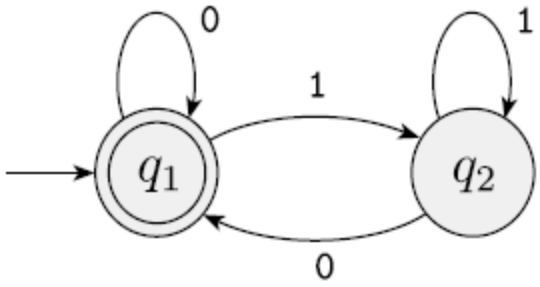


FIGURE 1.10
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What language M_3 recognizes?

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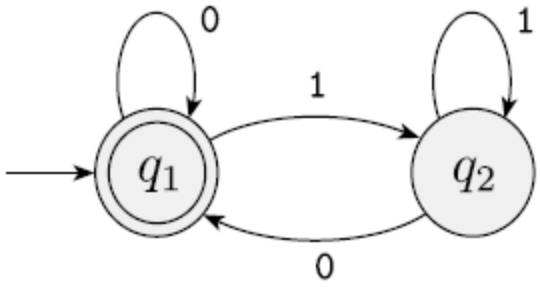


FIGURE 1.10
State diagram of the two-state finite automaton M_3

What language M_3 recognizes?

$L(M_3) = \{w | \ w \text{ is the empty string } \varepsilon \text{ or ends in a } 0\}.$

EXAMPLE 1.11

The following figure shows a five-state machine M_4.

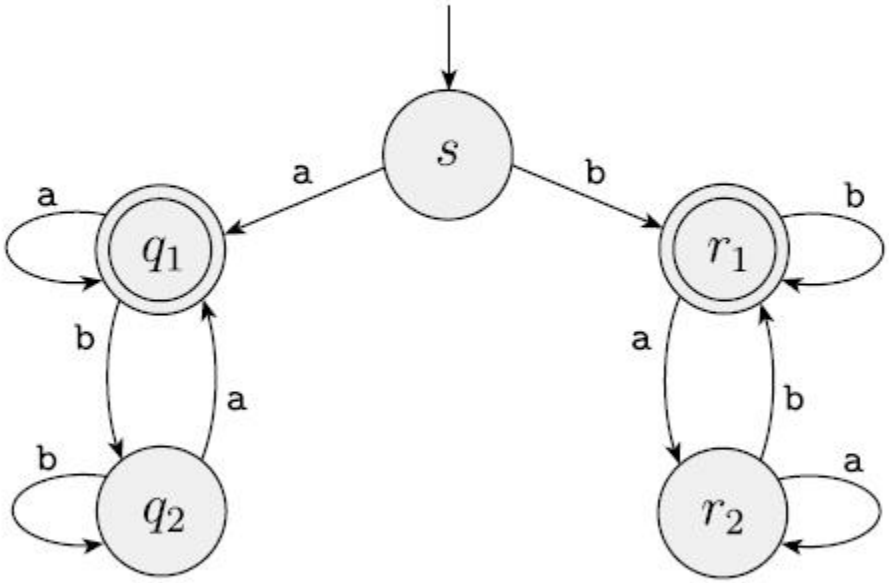


FIGURE 1.12
Finite automaton M_4

EXAMPLE 1.11

The following figure shows a five-state machine M_4.

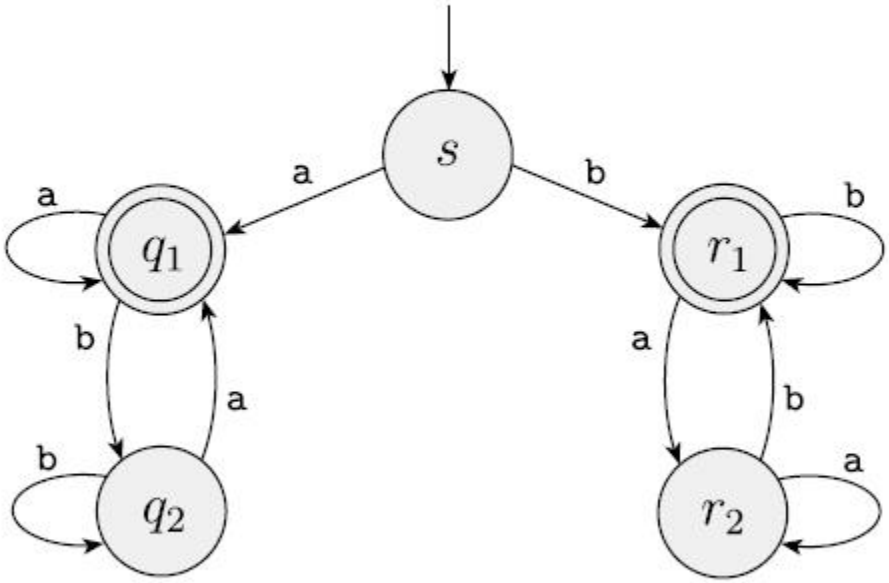


FIGURE 1.12
Finite automaton M_4

$L(M_4) =$ all strings that begin and end with the same character.

DFA for complement of a language

- Flip final and non-final states.

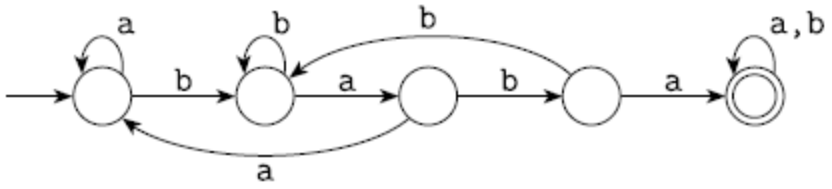
1.5 Each of the following languages is the complement of a simpler language. In each part, construct a DFA for the simpler language, then use it to give the state diagram of a DFA for the language given. In all parts, $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

[A] **a.** $\{w | \ w$ does not contain the substring $\mathtt{ab}\}$

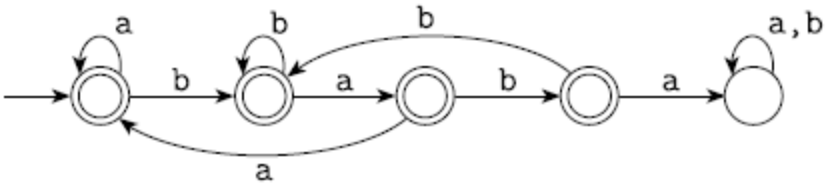
[A] **b.** $\{w | \ w$ does not contain the substring $\mathtt{baba}\}$

1.5 (a) The left-hand DFA recognizes $\{w|\ w$ contains ab$\}$. The right-hand DFA recognizes its complement, $\{w|\ w$ doesn't contain ab$\}$.

(b) This DFA recognizes $\{w|\ w$ contains baba$\}$.



This DFA recognizes $\{w|\ w$ does not contain baba$\}$.



Designing a DFA (Quick Quiz)

- How to design a DFA that accepts all binary strings representing a multiple of 5? (E.g., 101, 1111, 11001, ...)

Formally

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton and let $w = w_1 w_2 \cdots w_n$ be a string where each w_i is a member of the alphabet Σ. Then M ***accepts*** w if a sequence of states $r_0, r_1, \ldots, r_n$ in Q exists with three conditions:

1. $r_0 = q_0$,
2. $\delta(r_i, w_{i+1}) = r_{i+1}$, for $i = 0, \ldots, n-1$, and
3. $r_n \in F$.

Regular language [Ref: Sipser Book]

DEFINITION 1.16

A language is called a ***regular language*** if some finite automaton recognizes it.

The regular operations

DEFINITION 1.23

Let A and B be languages. We define the regular operations ***union***, ***concatenation***, and ***star*** as follows:

- **Union**: $A \cup B = \{x | \, x \in A \text{ or } x \in B\}$.
- **Concatenation**: $A \circ B = \{xy | \, x \in A \text{ and } y \in B\}$.
- **Star**: $A^* = \{x_1 x_2 \ldots x_k | \, k \geq 0 \text{ and each } x_i \in A\}$.

- These are similar to arithmetic operations.
- Note, * is a unary operator.

THEOREM 1.25

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

- The proof is by construction.
- We build a DFA for the union from the individual DFAs.

- The idea is simple: While reading the input simultaneously follow both machines.
  - Put a finger on current state. You need two fingers. You can move these two fingers as per the respective transition function.

PROOF

Let M_1 recognize A_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, and
M_2 recognize A_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct M to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q_0, F)$.

1. $Q = \{(r_1, r_2) | \, r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$.
This set is the ***Cartesian product*** of sets Q_1 and Q_2 and is written $Q_1 \times Q_2$.
It is the set of all pairs of states, the first from Q_1 and the second from Q_2.

2. Σ, the alphabet, is the same as in M_1 and M_2. In this theorem and in all subsequent similar theorems, we assume for simplicity that both M_1 and M_2 have the same input alphabet Σ. The theorem remains true if they have different alphabets, Σ_1 and Σ_2. We would then modify the proof to let $\Sigma = \Sigma_1 \cup \Sigma_2$.

3. δ, the transition function, is defined as follows. For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let

$$\delta\big((r_1, r_2), a\big) = \big(\delta_1(r_1, a), \delta_2(r_2, a)\big).$$

Hence δ gets a state of M (which actually is a pair of states from M_1 and M_2), together with an input symbol, and returns M's next state.

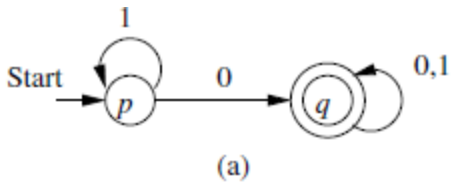
4. q_0 is the pair (q_1, q_2).

5. F is the set of pairs in which either member is an accept state of M_1 or M_2. We can write it as

$$F = \{(r_1, r_2) |\ r_1 \in F_1 \text{ or } r_2 \in F_2\}.$$

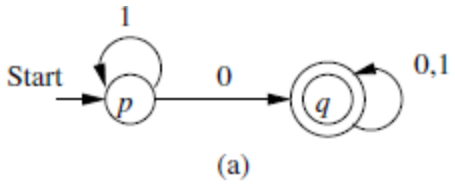
This expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$. (Note that it is *not* the same as $F = F_1 \times F_2$. What would that give us instead?[3])

Union Example



What is the language recognized by this DFA?

Union Example



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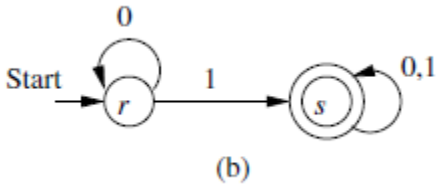
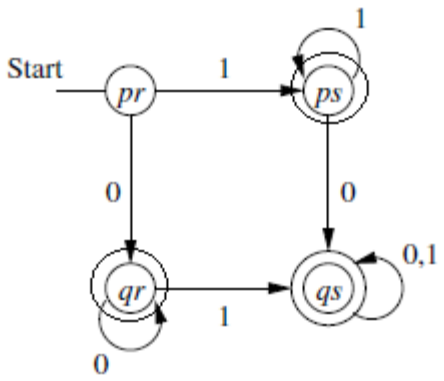


What is the language recognized by this DFA?

Find DFA for the union

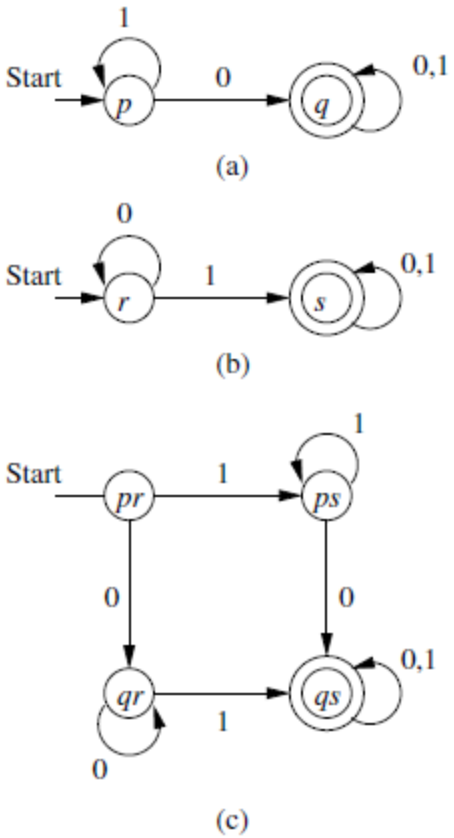
Find DFA for the union



What about intersection?

- Intersection of two regular languages is also regular.
- Proof: by construction. Similar. Only final states will change.

Intersection



What else we can do with product principle?

- Set difference.
  - How?

1.4 Each of the following languages is the intersection of two simpler languages. In each part, construct DFAs for the simpler languages, then combine them using the construction discussed in footnote 3 (page 46) to give the state diagram of a DFA for the language given. In all parts, $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

a. $\{w \mid w$ has at least three a's and at least two b's$\}$

[A] **b.** $\{w \mid w$ has exactly two a's and at least two b's$\}$

c. $\{w \mid w$ has an even number of a's and one or two b's$\}$

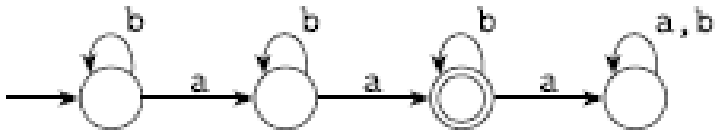
[A] **d.** $\{w \mid w$ has an even number of a's and each a is followed by at least one b$\}$

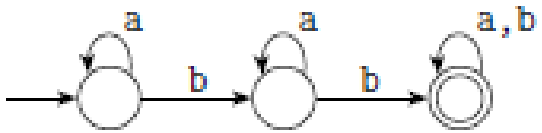
e. $\{w \mid w$ starts with an a and has at most one b$\}$

f. $\{w \mid w$ has an odd number of a's and ends with a b$\}$

g. $\{w \mid w$ has even length and an odd number of a's$\}$

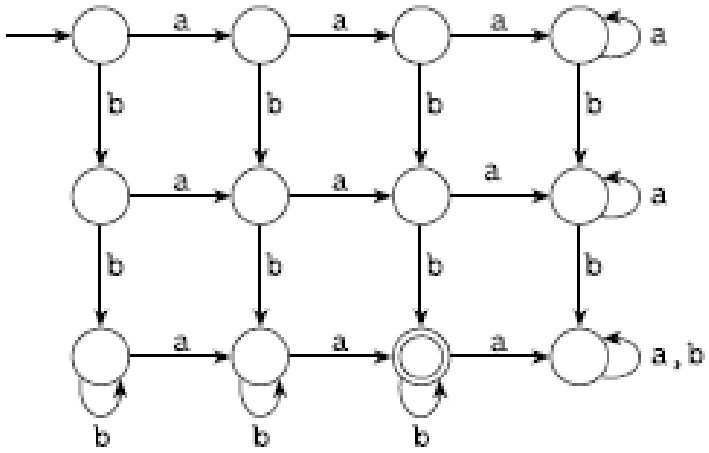
1.4 (b) The following are DFAs for the two languages $\{w|\ w$ has exactly two a's$\}$ and $\{w|\ w$ has at least two b's$\}$.





- Now find product machine.

Combining them using the intersection construction gives the following DFA.



- This can be minimized. {Some states are redundant}.

NONDETERMINISM

- Useful concept, has great impact on ToC/algorithms.
- DFA is deterministic: every step of a computation follows in a unique way from the preceding step.
  - When the machine is in a given state, and upon reading the next input symbol, we know deterministically what would be the next state.
  - Only one next state.
  - No choice !!

NONDETERMINISM

- In a nondeterministic machine, several choices may exist for the next state at any point.
- Nondeterminism is a generalization of determinism.

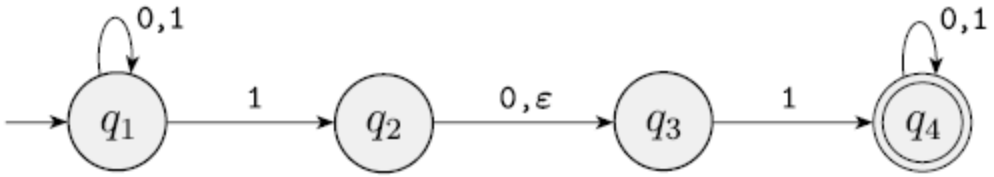


FIGURE **1.27**
The nondeterministic finite automaton N_1

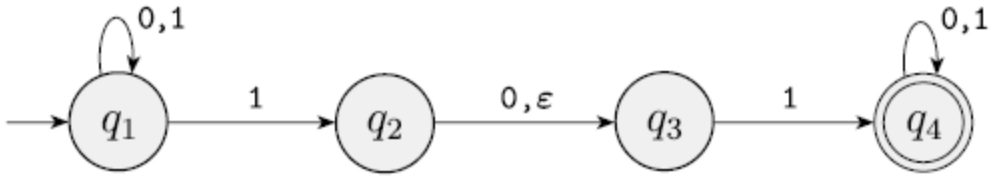


FIGURE **1.27**
The nondeterministic finite automaton N_1

- More than one arrow from from q_1 on symbol 1.
- No arrow at all from q_3 on 0.
- There is Ɛ over an arrow !

How does an NFA compute?

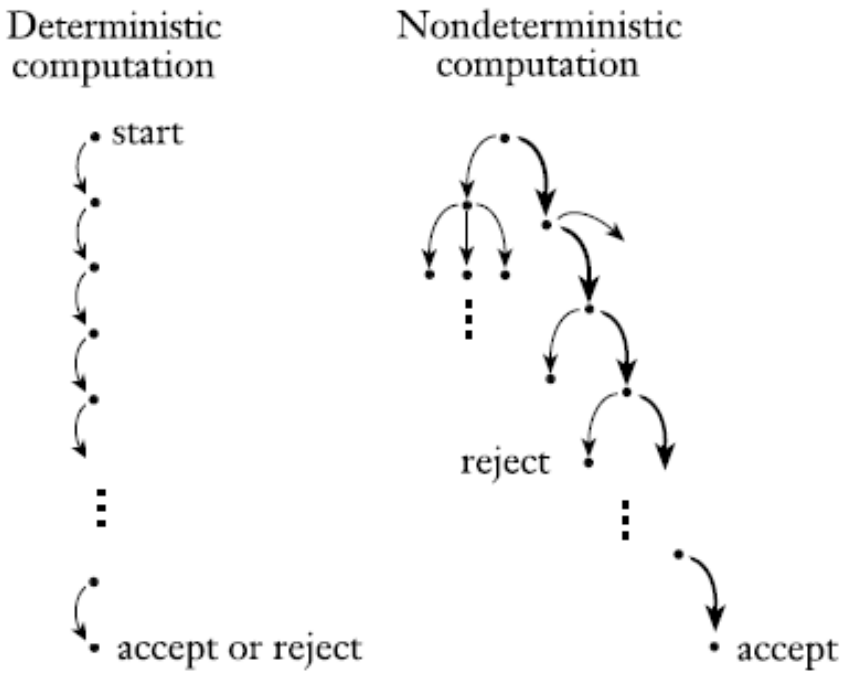


FIGURE **1.28**
Deterministic and nondeterministic computations with an accepting branch

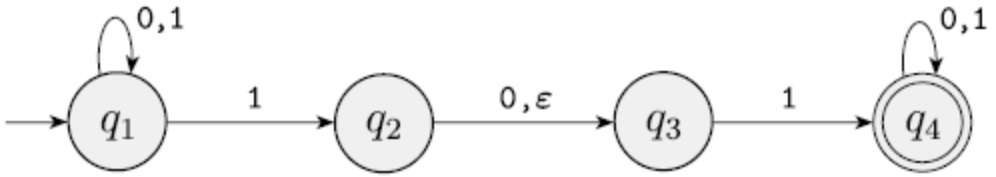


FIGURE **1.27**
The nondeterministic finite automaton N_1

On input **010110**

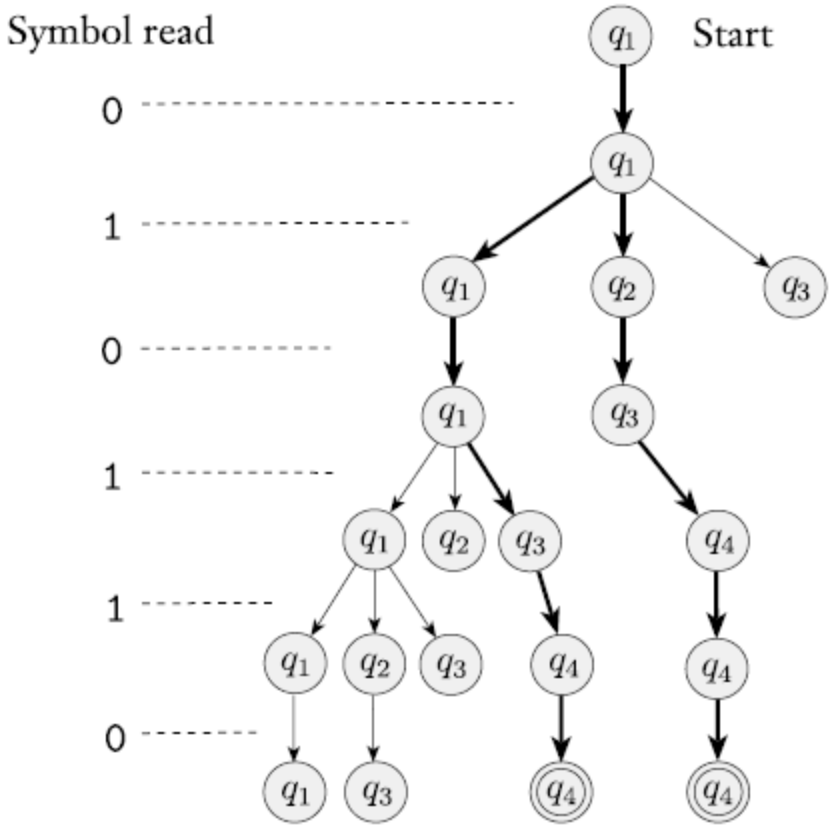


FIGURE **1.29**
The computation of N_1 on input 010110

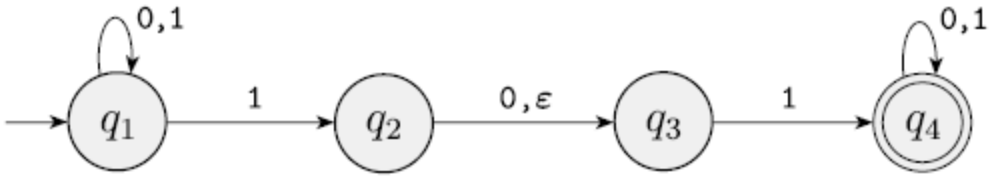


FIGURE **1.27**
The nondeterministic finite automaton N_1

- What is the language accepted by this NFA?

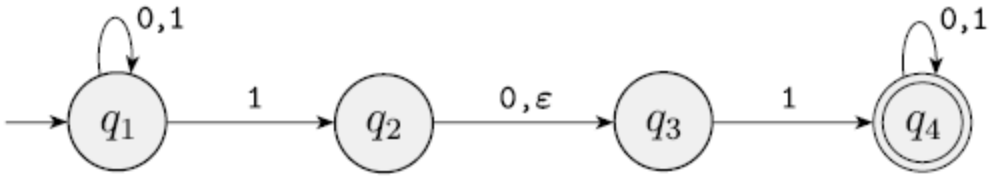


FIGURE **1.27**
The nondeterministic finite automaton N_1

- It accepts all strings that contain either 101 or 11 as a substring.

- Constructing NFAs is sometimes easier than constructing DFAs.
  - Later we see that every NFA can be converted into an equivalent DFA.

EXAMPLE 1.30

Let A be the language consisting of all strings over $\{0,1\}$ containing a 1 in the third position from the end (e.g., 000100 is in A but 0011 is not). The following four-state NFA N_2 recognizes A.

- Building DFA for this is possible, but difficult.
- Try this.

But NFA is easy to build.

EXAMPLE 1.30

Let A be the language consisting of all strings over {0,1} containing a 1 in the third position from the end (e.g., 000100 is in A but 0011 is not). The following four-state NFA N_2 recognizes A.

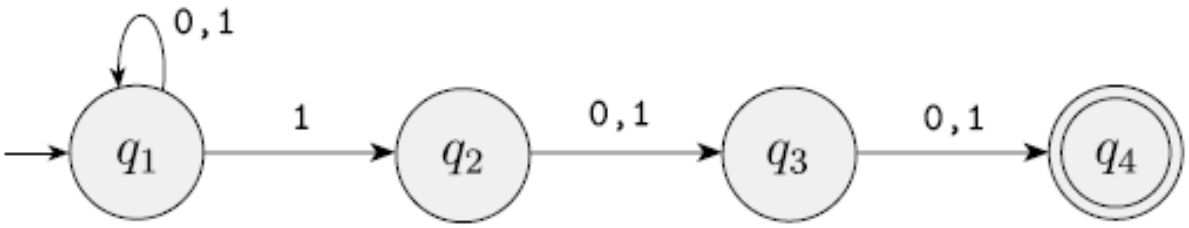


FIGURE 1.31
The NFA N_2 recognizing A

DFA for A

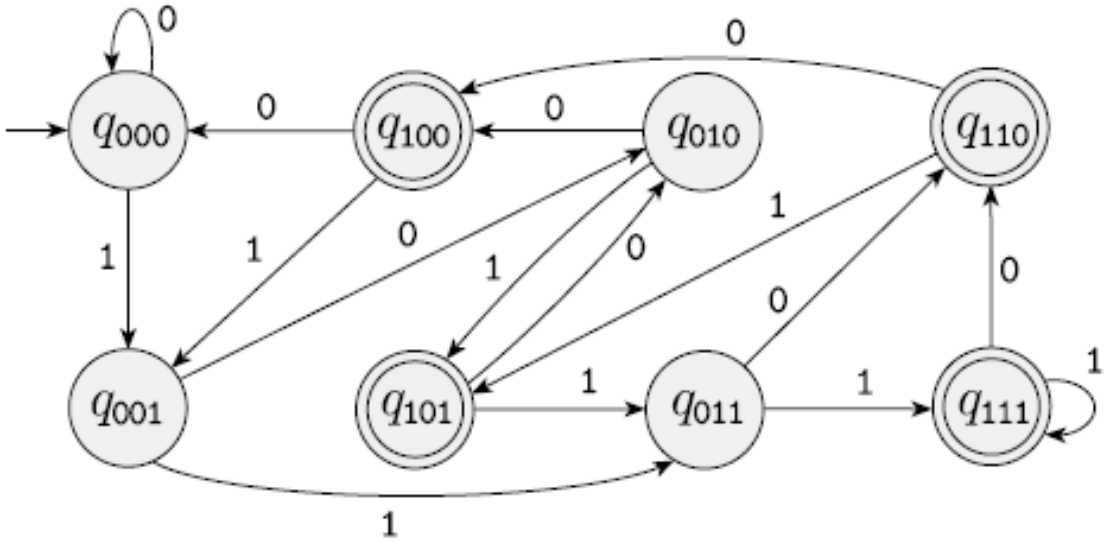


FIGURE **1.32**
A DFA recognizing A

- See number of states and complexity !

Formal definition of NFA

We use $\sum_{\varepsilon}$ to mean $\sum \cup \{\varepsilon\}$

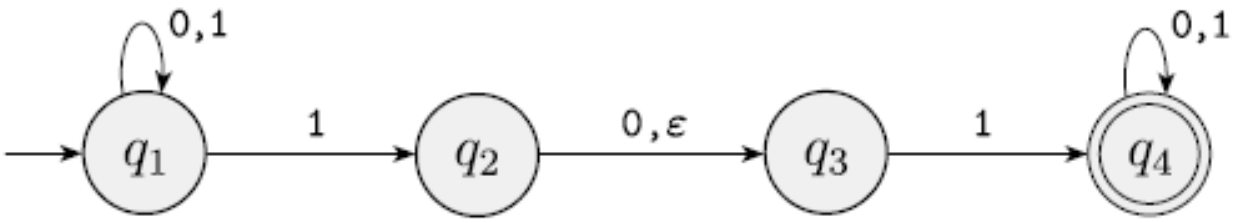
DEFINITION 1.37

A *nondeterministic finite automaton* is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta: Q \times \Sigma_{\varepsilon} \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

EXAMPLE 1.38

Recall the NFA N_1:



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,

4. q_1 is the start state, and
5. $F = \{q_4\}$.

The formal definition of computation for an NFA is similar to that for a DFA. Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and w a string over the alphabet Σ. Then we say that N ***accepts*** w if we can write w as $w = y_1 y_2 \cdots y_m$, where each y_i is a member of Σ_ε and a sequence of states $r_0, r_1, \ldots, r_m$ exists in Q with three conditions:

1. $r_0 = q_0$,
2. $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0, \ldots, m-1$, and
3. $r_m \in F$.

Equivalence of NFAs and DFAs

- We say two machines are equivalent if they recognize the same language.

THEOREM 1.39

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

Proof

- Proof by construction.
  - We build a equal DFA for the given NFA

Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A.

We construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing A.

- First, for understanding purpose, we assume that there are no edges with Ɛ transitions.

Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A.

We construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing A.

1. $Q' = \mathcal{P}(Q)$.
   Every state of M is a set of states of N. Recall that $\mathcal{P}(Q)$ is the set of subsets of Q.

2. For $R \in Q'$ and $a \in \Sigma$, let $\delta'(R, a) = \{q \in Q | \, q \in \delta(r, a) \text{ for some } r \in R\}$.

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a).$$

3. $q_0' = \{q_0\}$.
   M starts in the state corresponding to the collection containing just the start state of N.

4. $F' = \{R \in Q' | \, R \text{ contains an accept state of } N\}$.
   The machine M accepts if one of the possible states that N could be in at this point is an accept state.

Can you convert the following

FIGURE **1.31**
The NFA N_2

- What is the language accepted by this?

Now, considering Ɛ arrows

- For this purpose, we define Ɛ-CLOSURE of a set of states *R*.

Formally, for $R \subseteq Q$ let

$E(R) = \{q|\ q$ can be reached from R by traveling along 0 or more ε arrows$\}$.

- *E(R)* is Ɛ-CLOSURE of *R*.

- Then the transition is defined as,

$$\delta'(R, a) = \{q \in Q |\ q \in E(\delta(r, a)) \text{ for some } r \in R\}.$$

- Now the start state of the DFA should be

$$q_0{}' = E(\{q_0\})$$

Example

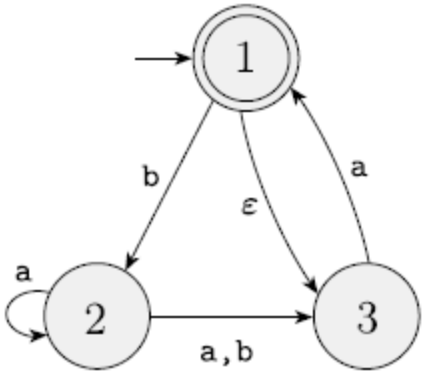
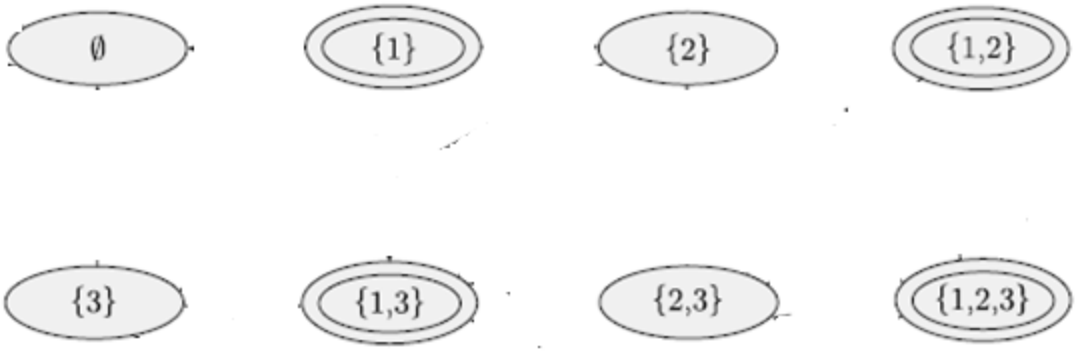
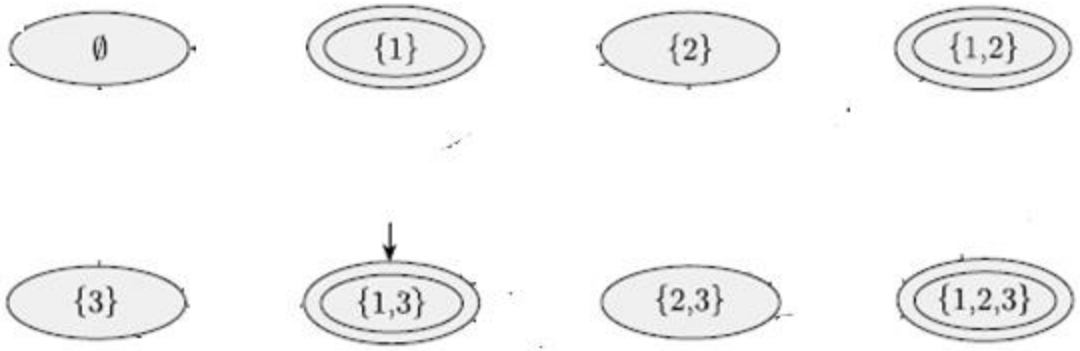
- 

FIGURE **1.42**
The NFA N_4

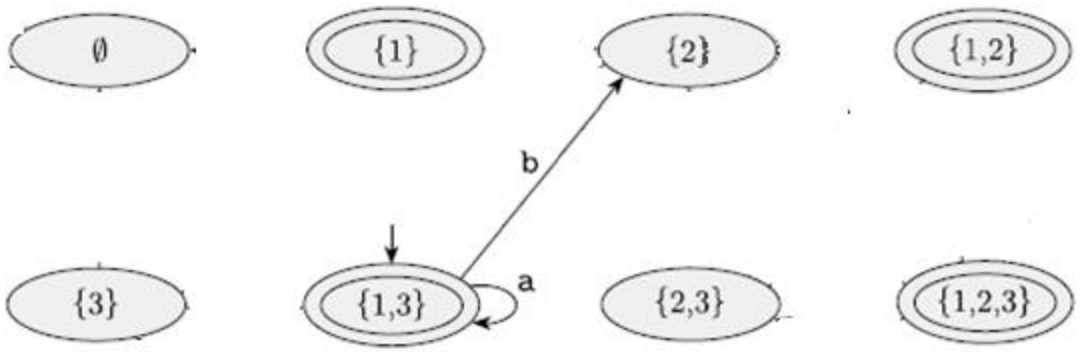


All possible states of the DFA.
(to be constructed; Final states are shown)

- Now we need to add edges, and
- identify the initial state.



- Identify the initial state.
  - Note, it is not {1}



- Adding edges, …

After all edges …

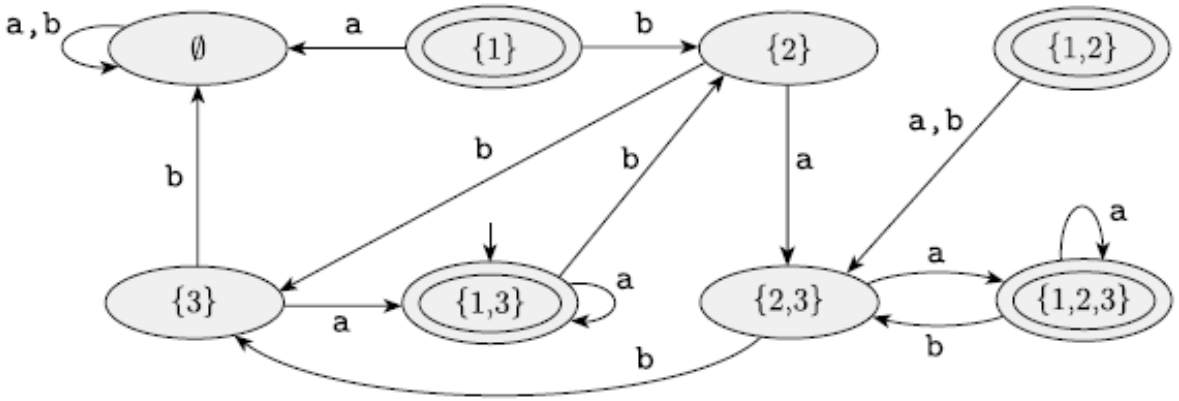


FIGURE **1.43**
A DFA D that is equivalent to the NFA N_4

- But, some states are not reachable !
- Simplification can remove this.

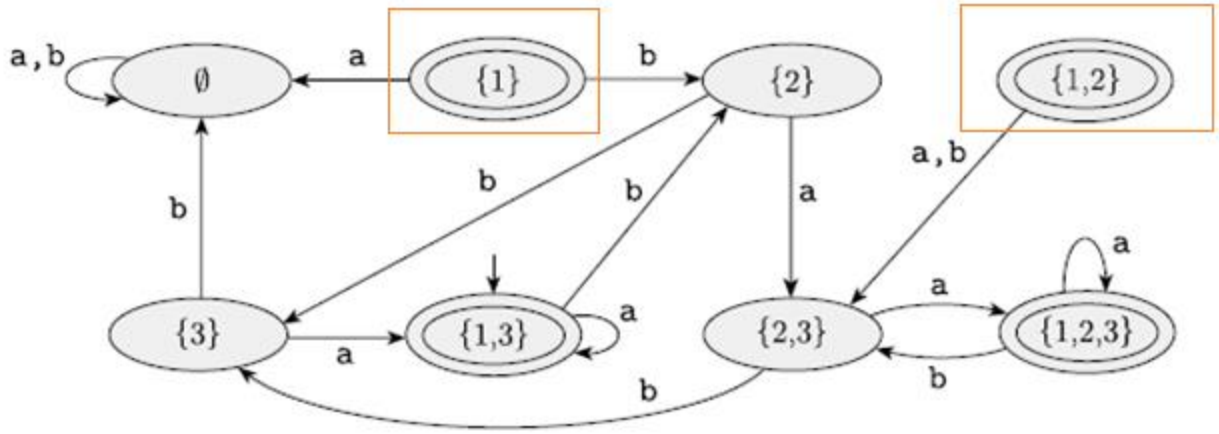
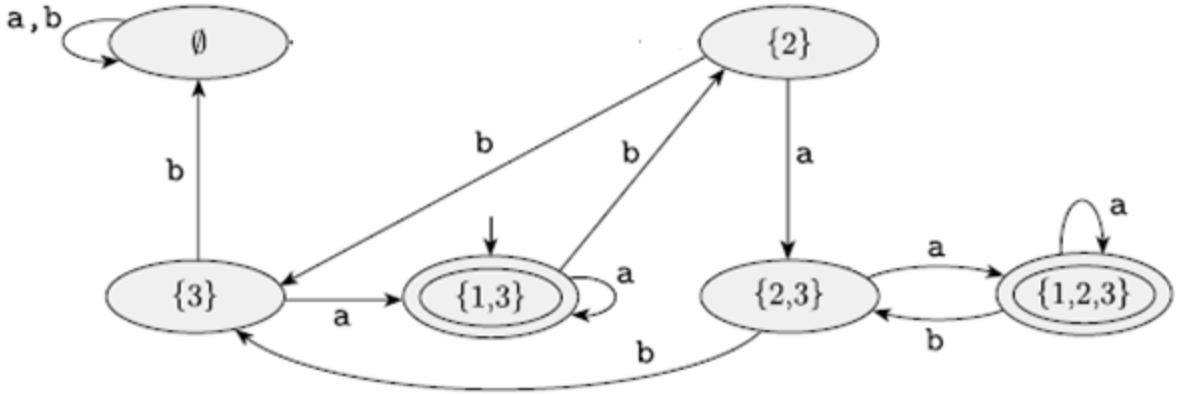
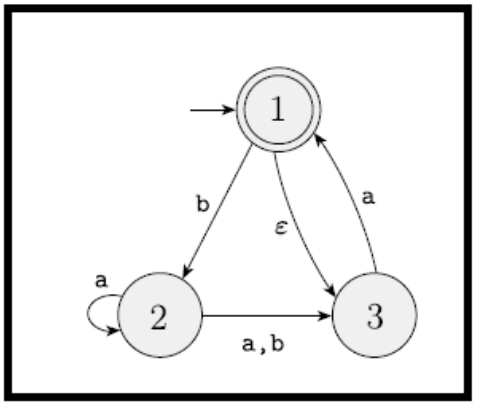


FIGURE 1.43
A DFA D that is equivalent to the NFA N_4

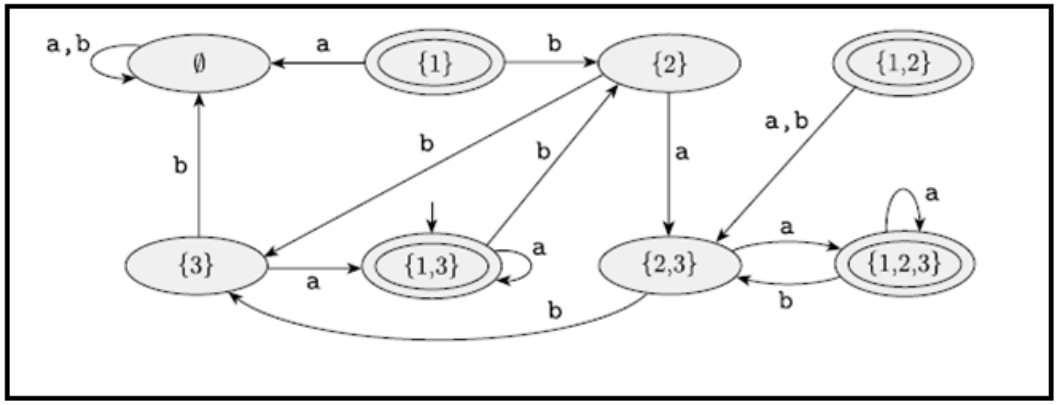


DFA D' which is equivalent to D.

Note: D' and D are different machines; but, they are equivalent.



N_4

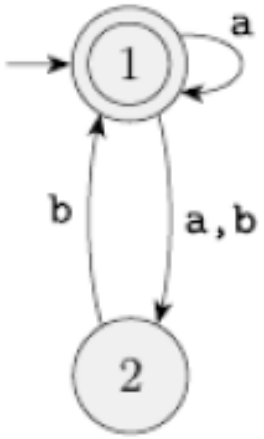


A DFA D that is equivalent to the NFA N_4

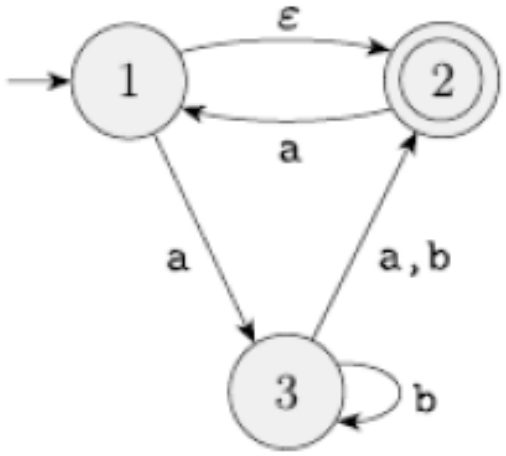
- Being in state 1 of N_4 upon reading input a the machine N_4 can be in state 1.
- Convince yourself that in the DFA D there are no mistakes.
  - From state $\{1\}$ with input a the DFA D goes to state ϕ.

Exercise

Convert the following NFAs to equivalent DFAs.



(a)



(b)

(Problem Source: Sipser's book exercise problem 1.16)

Exercise

- **1.7** Give state diagrams of NFAs with the specified number of states recognizing each of the following languages. In all parts, the alphabet is $\{0,1\}$.

  [A]a. The language $\{w \mid w$ ends with $00\}$ with three states

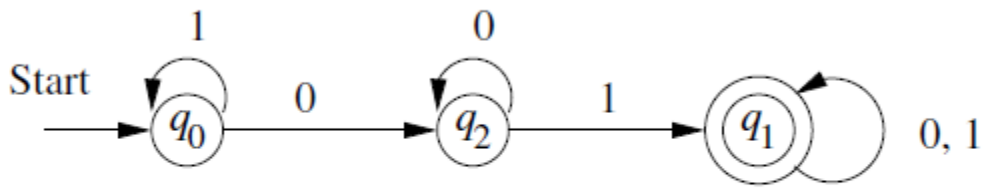
  d. The language $\{0\}$ with two states

  g. The language $\{\varepsilon\}$ with one state

  h. The language 0^* with one state

- Can you convert each of above NFAs into a corresponding DFA.

Some Notation adapted



The transition diagram for the DFA accepting all strings with a substring 01

	0	1
$\rightarrow q_0$	q_2	q_0
$*q_1$	q_1	q_1
q_2	q_2	q_1

This also has all 5 components. This table is complete description of the DFA as the diagram.