Tutorial 2

Mainak Thakur

10.09.2020

Problem 1

Ex: The lifetime of a brand tubelight bulbs is modeled by an exponential distribution with (unknown parameter λ. A sample of 5 bulbs were found to have lifetimes of 2, 3, 1, 3, and 4 years respectively. What is the MLE of λ?

Solution

Sol^n: $X_i \leftarrow i$th random sample

$x_i \leftarrow$ value that X_i takes

pdf of $X_i \rightarrow f_{X_i}(x_i) = \lambda e^{-\lambda x_i}$.

i.i.d samples (assumption)

$$L(\lambda) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2} \cdot \lambda e^{-\lambda x_3} \cdot \lambda e^{-\lambda x_4} \cdot \lambda e^{-\lambda x_5}.$$

$$= \lambda^5 \left[e^{-\lambda(x_1 + x_2 + \dots + x_5)} \right]$$

$$= \lambda^5 e^{-13\lambda} \quad [x_1 = 2, x_2 = 3, x_3 = 1, x_4 = 3, x_5 = 4].$$

$$\frac{d}{d\lambda}(\text{log likelihood}) = \frac{5}{\lambda} - 13 = 0$$

$$\hat{\lambda}_{MLE} = \frac{5}{13}.$$

Note: $\bar{x} = \frac{13}{5}$.

Ref: https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading10b.pdf

Problem 2

Ex. Let the population distribution is continuous Uniform $U(a, b)$ i.i.d samples $x_1, x_2, \ldots, x_n$ drawn from the population. Find MLE for a and b.

Solution

Sol^n: $U(a, b)$ density is $\frac{1}{b-a}$ on $[a, b]$

$$f(x_1, x_2, \dots, x_n \mid a, b) = \left(\frac{1}{b-a}\right)^n, \quad x_i \in [a, b]$$

No need of calculus here.

The above mentioned function is maximized by making $b-a$ as small as possible. The only condition is that $[a, b]$ must include all the data.

$$\hat{a}_{MLE} = \min(x_1, x_2, \dots, x_n), \quad \hat{b}_{MLE} = \max(x_1, x_2, \dots, x_n)$$

Ref: https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading10b.pdf

Problem 3

Ex. Population random variable X has pdf: $f(x) = \frac{2}{\theta^2} \cdot (\theta - x)$, $0 < x < \theta$.

A random sample of two observations of X yields values 0.5 and 0.9.

Determine $\hat{\theta}$, the method of moment estimator of θ.

Solution

Sol^n. $E(X) = \frac{2}{\theta^2}\int_0^\theta x(\theta - x)\,dx = \frac{2}{\theta^2}\left(\theta\int_0^\theta x\,dx - \int_0^\theta x^2\,dx\right)$

$= \frac{2}{\theta^2}\left(\frac{\theta^3}{2} - \frac{\theta^3}{3}\right) = \frac{\theta}{3}.$

Sample first moment is

$\frac{0.5 + 0.9}{2} = 0.7.$

Equating, $\frac{\theta}{3} = 0.7 \Rightarrow \hat{\theta}_{MME} = 2.1.$

Ref: http://home.cc.umanitoba.ca/~farhadi/ASPER/Method%20of%20Moments.pdf

Problem 4

Ex. Four samples are observed from a Gamma distribution. The observed values are 200, 300, 350 and 450. Find the method of moment estimate for shape parameter α.

Solution

Sol^n. Gamma has two parameters, α and θ.

$$E(X) = \alpha\theta$$

$$E(X^2) = \alpha(\alpha+1)\theta^2.$$

1st sample moment $= \dfrac{200+300+350+450}{4}$

$= 325.$

2nd " " $= \dfrac{200^2+300^2+350^2+450^2}{4}$

$= 113750.$

Equating, $\alpha\theta = 325.$ —— (1)

$\alpha(\alpha+1)\theta^2 = 113750.$ —— (2)

(2) $\div$ (1)^2 $\Rightarrow \dfrac{\alpha+1}{\alpha} = \dfrac{113750}{325^2}$

$\Rightarrow \dfrac{\alpha+1}{\alpha} = 1.0769$

$\Rightarrow \alpha = 13.$

Ref: http://home.cc.umanitoba.ca/~farhadi/ASPER/Method%20of%20Moments.pdf