Sampling Distributions 2

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t-distribution

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DATE

t - distribution

Let $Z \sim N(0,1)$, $U \sim \chi^2(\nu)$ and they are independent. Then the distribution of the random variable of the form $\frac{Z}{\sqrt{U/\nu}}$ is known to have t distribution with ν degrees of freedom

pdf: $$f(t) = \frac{1}{\nu^{1/2} B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

$-\infty < t < \infty$.

Ref: Fundamentals of Statistics, Vol 1 by Gun, Gupta, Dasgupta

Notation

mean = 0.

$t_{\alpha,\nu}$: The value of t with df $= \nu$

s.t. $P(t > t_{\alpha,\nu}) = \alpha$.

$t_{1-\alpha,\nu} = -t_{\alpha,\nu}$. [Symmetry]

Ref: Fundamentals of Statistics, Vol 1 by Gun, Gupta, Dasgupta

Graph

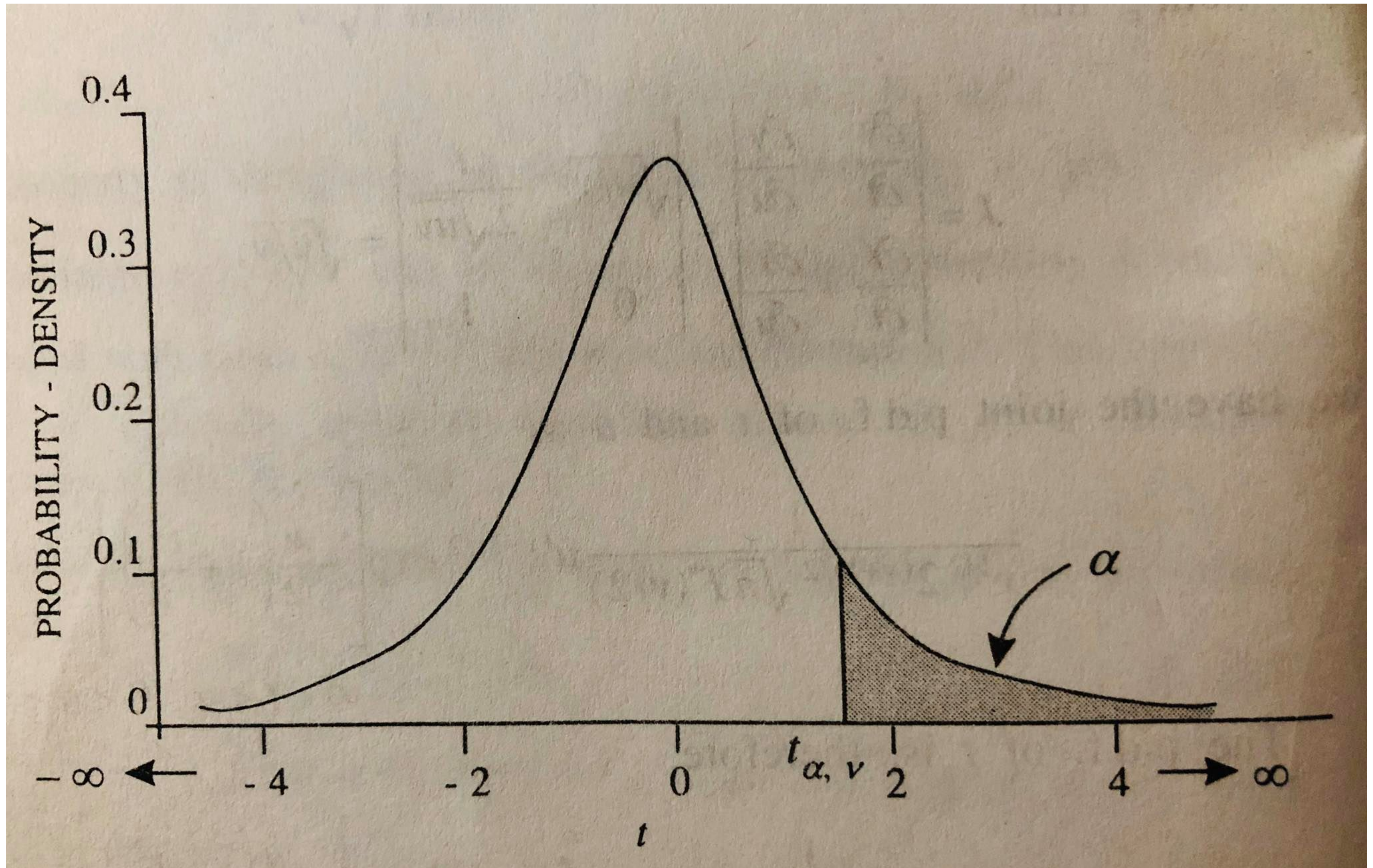
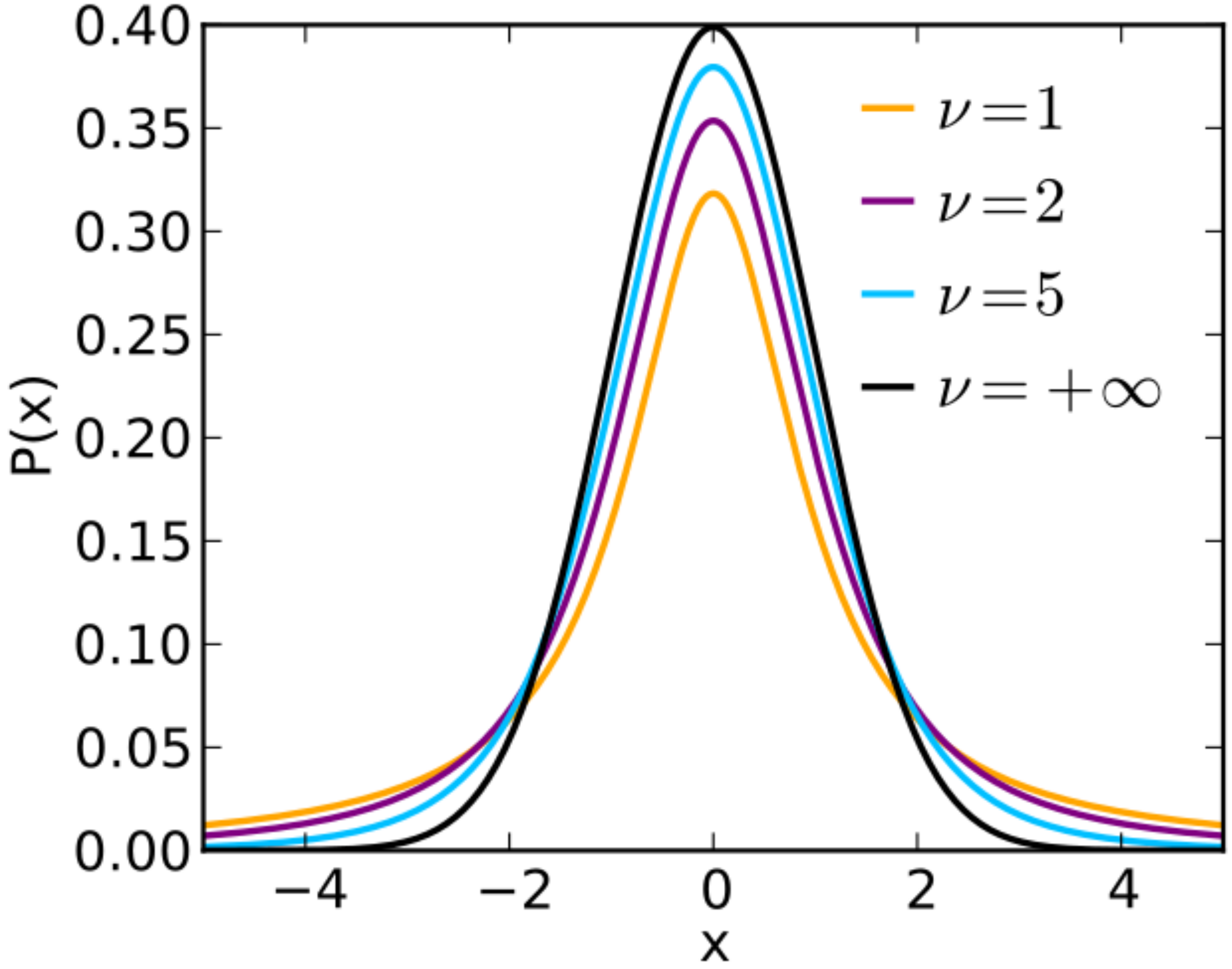


Fig. 13.3 t distribution with 5 degrees of freedom ($\nu = 5$).



Ref: Fundamentals of Statistics, Vol 1 by Gun, Gupta, Dasgupta
wikipedia

Example

Suppose we have a Normal population with mean μ and s.d. σ.

$$X \sim N(\mu, \sigma^2).$$

Take n i.i.d samples $X_1, X_2, \ldots, X_n$ from the population.

$$\bar{X} \sim N(\mu, \sigma^2/n).$$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim ?$$

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$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Example

The problem is that we might not know σ always. So what could be the best guess of σ given a sample?

'S' is ~~the~~ an estimator of σ.

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim ?$$

Example

$$T = \frac{\bar{X}-\mu}{s/\sqrt{n}} = \frac{\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right) \longrightarrow Z}{S/\sigma \longrightarrow \sqrt{\frac{U}{n-1}}}.$$

$$U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

numerator and denominator are independent.

$$\frac{\bar{x}-\mu}{s/\sqrt{n}} \sim t \text{ with } (n-1) \text{ df.}$$

F-distribution

F-distribution

Let Y_1 and Y_2 be independently distributed as χ^2 with ν_1 and ν_2 d.f. respectively. The random variable

$$\frac{Y_1/\nu_1}{Y_2/\nu_2}$$ is said to have

F distribution with (ν_1, ν_2) degrees of freedom.

Ref: Fundamentals of Statistics, Vol 1 by Gun, Gupta, Dasgupta

Properties

pdf: $$f(F) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B(\nu_1/2, \nu_2/2)} \cdot F^{(\nu_1-2)/2} \cdot \left(1 + \frac{\nu_1}{\nu_2} \cdot F\right)^{-(\nu_1+\nu_2)/2}.$$

$0 < F < \infty$.

Mean $= \dfrac{\nu_2}{\nu_2 - 2}$, f distn +vely skewed.

$X \sim t(n)$ then, $X^2 \sim F(1, n)$.

Ref: Fundamentals of Statistics, Vol 1 by Gun, Gupta, Dasgupta

Notation

$F_{\alpha;\nu_1,\nu_2}$: the upper α-point of the F distn with df $= (\nu_1, \nu_2)$

$$P(F > F_{\alpha;\nu_1,\nu_2}) = \alpha.$$

$F_{1-\alpha;\nu_1,\nu_2}$: the lower α-point.

$$P(F < F_{1-\alpha;\nu_1,\nu_2}) = \alpha.$$

$$P\left(\frac{1}{F} > \frac{1}{F_{1-\alpha;\nu_1,\nu_2}}\right) = \alpha.$$

$\frac{1}{F}$ is of the form $\frac{Y_2/\nu_2}{Y_1/\nu_1}$ is distributed as F with df $= (\nu_2, \nu_1)$.

$$\frac{1}{F_{1-\alpha;\nu_1,\nu_2}} = F_{\alpha;\nu_2,\nu_1}$$

or, $F_{1-\alpha;\nu_1,\nu_2} = \frac{1}{F_{\alpha;\nu_2,\nu_1}}$.

Ref: Fundamentals of Statistics, Vol 1 by Gun, Gupta, Dasgupta

Graph

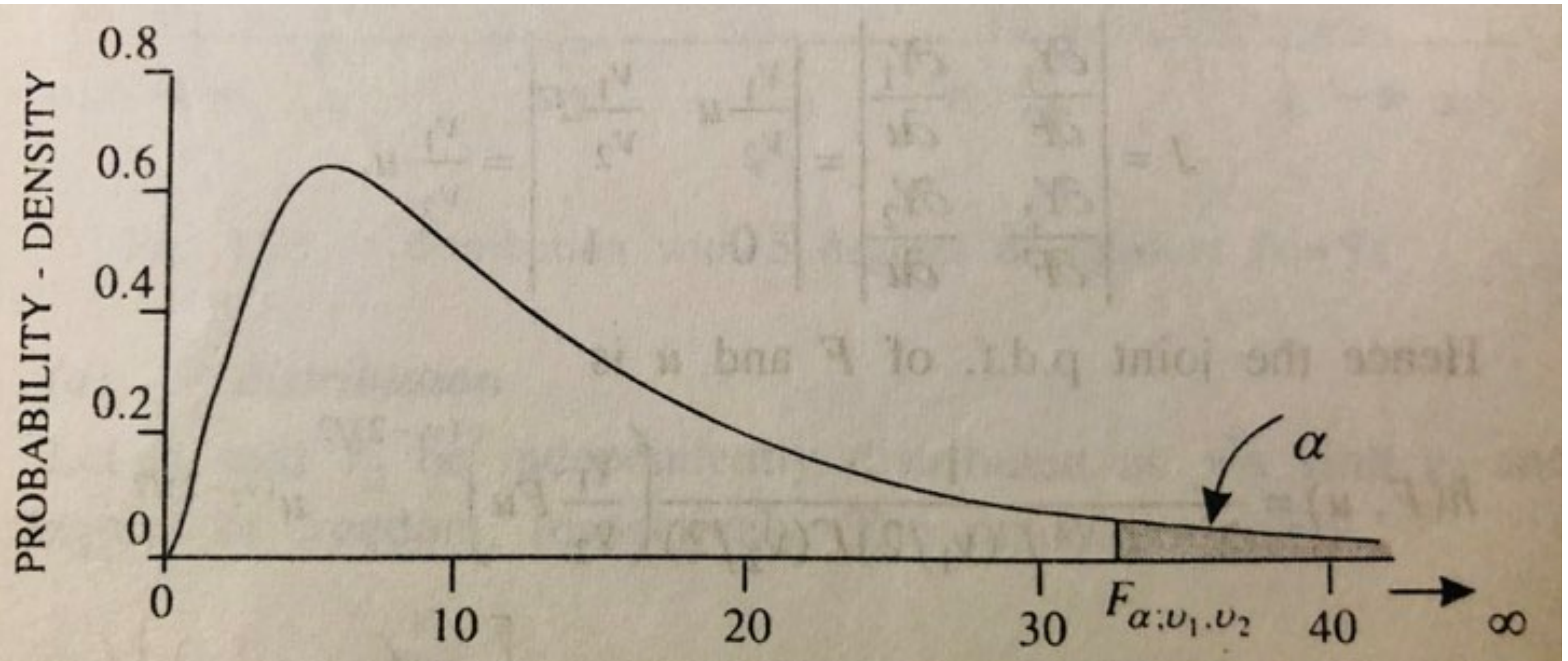
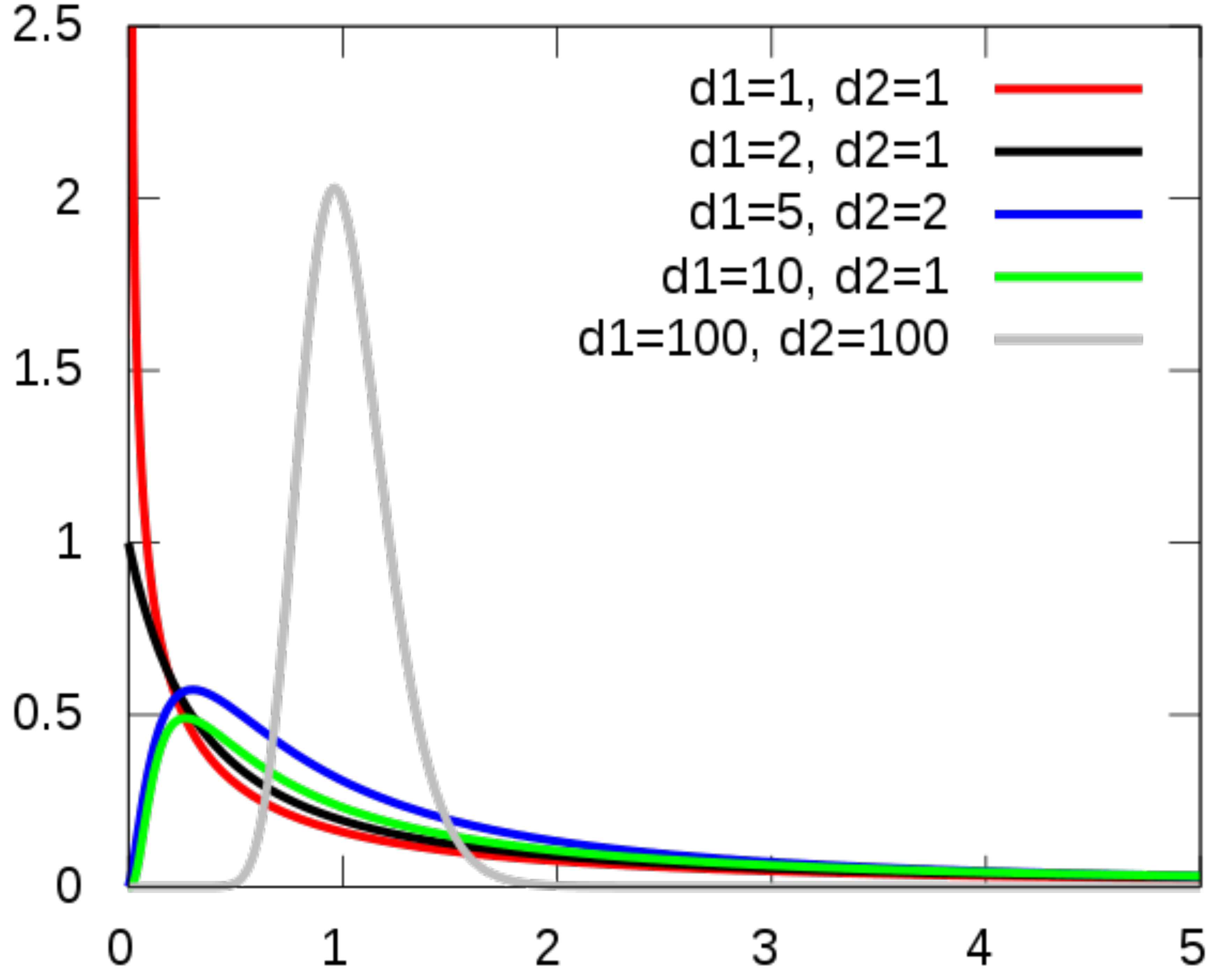


Fig. 13.4 F distribution with (10, 4) degrees of freedom ($\nu_1 = 10$, $\nu_2 = 4$)



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wikipedia

Example

F-distribution often arises when the ratios of variances are involved.

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} = \frac{\chi^2_{(n_1-1)}/(n_1-1)}{\chi^2_{(n_2-1)}/(n_2-1)} \sim F_{n_1-1,\,n_2-1}$$

Ref: nptel: https://drive.google.com/file/d/0BwSBts69cKElekk3dWFRTUFkbTg/view

Question

Let $X_1, X_2, \ldots, X_{2n}$ are i.i.d random samples from a Normal population with mean μ and variance σ^2. Find the distribution of the following statistic:

$$[x_1 + x_2 + \ldots + x_n - x_{n+1} - x_{n+2} - \ldots . - x_{2n}]/\sqrt{2n}$$