Purushottam

Two Wheeled Balancing bot (Inverted Pendulm Cart-model)

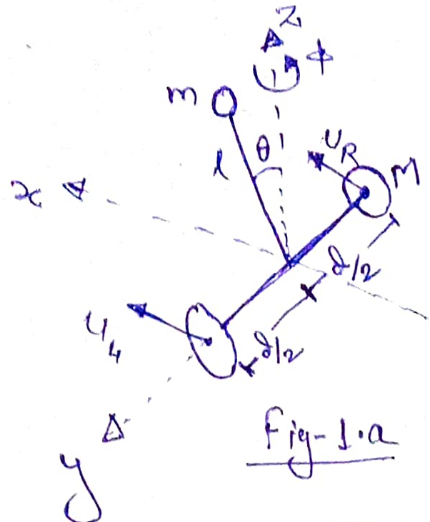

fig-1.a

or

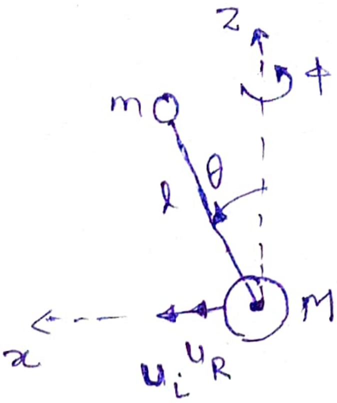

fig-1.b

m = mass of body,

l = height of the body from axle

M = mass of both the wheels ($M/2$ for each)

d = width between wheels

U_L, U_R = Forces on the respective wheels

θ = Pitch.

$\dot{\theta}$ = Pitch velocity

ϕ = Yaw

$\dot{\phi}$ = Yaw velocity

x = x-position

$\dot{x}$ = x velocity

} State variables

Now

Let (x_m, y_m) Position of mass 'm' (body)

$$y_m = l\cos\theta \quad , \quad \dot{y}_m = -l\sin\theta\,\dot{\theta}$$

$$x_m = x + l\sin\theta \quad , \quad \dot{x}_m = \dot{x} + l\cos\theta\,\dot{\theta}$$

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For $\ddot{\phi}$ (yaw)

$$\tau = I\alpha \Rightarrow (U_R - U_L)\cdot \frac{d}{2} = \left(M\left(\frac{d}{2}\right)^2 + m(l \sin\theta)^2\right)\ddot{\phi}$$

$$\Rightarrow \boxed{\ddot{\phi} = \frac{(U_R - U_L)\, d/2}{\frac{Md^2}{4} + ml^2\sin^2\theta}} \quad — (1)$$

for, $\ddot{x}$ and $\ddot{\theta}$ we use, Lagrangian approach

Kinetic energy

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right) + \frac{1}{2}I\dot{\phi}^2$$

$$\left[K = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\left(l^2\dot{\theta}^2 + 2\dot{x}l\cos\theta\,\dot{\theta}\right) + \frac{1}{2}\left(\frac{Md^2}{4} + ml^2\sin^2\theta\right)\dot{\phi}^2\right]$$

Potential Energy

$$[V = mgl\cos\theta]$$

$$[L = K - V], \quad \text{and} \quad \boxed{\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = U_i} \quad — (L)$$

for x-direction

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\cos\theta\cdot\dot{\theta},$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2$$

So, Puting in eqn (L)

$$\left[(M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = F\right] \quad — (i)$$

Here, $\{F = U_R + U_L\}$

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for θ dirⁿ

$$\frac{\partial L}{\partial \theta} = -ml\sin\theta\,\dot\theta\dot x + ml^2\sin\theta\cos\theta\,\dot\phi^2 + mgl\sin\theta$$

$$= -ml\sin\theta\,\dot x\dot\theta + m(g + l\cos\theta\,\dot\phi^2)\,l\sin\theta$$

{Note, Net vertical accⁿ changes from (g) to $(g + l\cos\theta\,\dot\phi^2)$}

$$\frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta + ml\dot x\cos\theta$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2\ddot\theta + ml\cos\theta\,\ddot x - ml\sin\theta\,\dot x\dot\theta$$

from eqⁿ (L)

$$ml^2\ddot\theta + ml\cos\theta\,\ddot x - \cancel{ml\sin\theta\,\dot x\dot\theta} + \cancel{ml\sin\theta\,\dot x\dot\theta} - m(g + l\cos\theta\,\dot\phi^2)l\sin\theta = 0$$

$$\Rightarrow \quad ml^2\ddot\theta + ml\cos\theta\,\ddot x = m(g + l\cos\theta\,\dot\phi^2)\,l\sin\theta$$

$$\left[\,l\ddot\theta + \cos\theta\,\ddot x = (g + l\cos\theta\,\dot\phi^2)\sin\theta\,\right] \quad -\text{(ii)}$$

Solving eqⁿ (i) and (ii), (i) − $m\cos\theta$ (ii)

$$\boxed{\ddot x = \frac{k(U_L+U_R) + ml\sin\theta\,\dot\theta^2 - m(g + l\cos\theta\,\dot\phi^2)\sin\theta\cos\theta}{M + m\sin^2\theta}} \quad -(2)$$

and

$$\boxed{\ddot\theta = \frac{(g + l\cos\theta\,\dot\phi^2)\sin\theta}{l} - \frac{\cos\theta}{l}\left(\frac{(U_L+U_R) + ml\sin\theta\,\dot\theta^2 - m(g + l\cos\theta\,\dot\phi^2)\sin\theta\cos\theta}{M + m\sin^2\theta}\right)}$$

$-(3)$

for State Space eqn

$$y = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} \qquad \dot{y} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} \qquad U = \begin{bmatrix} U_R \\ U_L \end{bmatrix}$$

$$\dot{y} = Ay + BU$$

equilibrium ⇒
$\theta = 0$
$\dot{\theta} = 0$
$\dot{\phi} = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & g/l + \frac{gm}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1/M & 1/M \\ 0 & 0 \\ -1/Ml & -1/Ml \\ 0 & 0 \\ \frac{2}{Md} & \frac{-2}{Md} \end{bmatrix}$$