Advanced statistics and modelling

2020. február 12.

Technical information
Contacts of the teachers

- **Gergely Palla** / pallag@hal.elte.hu
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- **Zoltán Kaufmann** / kaufmann@complex.elte.hu
  Dept. of Physics of Complex Systems, Room 5.53
- **Illés J. Farkas** / fij@hal.elte.hu
  Quant AVP at Citi MQA,
  Dept. of Biological Physics, Room 3.90

Grading:

- 2 written tests,
- activity during the lectures affects the mark positively.

Course material:

- all slides, jupyter notebooks, etc. are going to be uploaded to the ELTE moodle site of the course.

Schedule

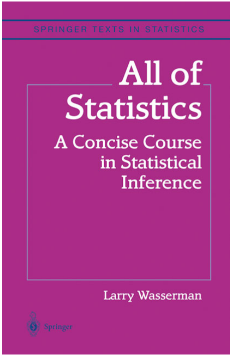
- **Sample space, Probability, Variable and Distribution**
- **Expectation, Inequalities, Convergence**
- **Models, Inference, Learning**
- **Empirical PDF, CDF, Smoothing, Binning**
- **Bootstrap, Maximum Likelihood, Hypothesis testing**
- **Regression, Inference About Independence**
- **Extreme Statistics, Posthoc Analysis**
- **Hierarchical Bayesian models**
- **Classical chaos, Ljapunov exponents, Frobenius Perron operators**
- **Random Matrices and Eigenvalue Statistics**
- **Statistics in fincance**
- **Stochastic Processes with applications in finance**

Schedule

Advanced statistics and modelling 2020 spring		
1. week	II.. 11. kedd Gergely Palla	Probability, variable, distribution
2. week	II.. 18. kedd Gergely Palla	Expectation, inequalities, convergence
3. week	II.. 25. kedd Gergely Palla	Statistical inference, estimating the CDF, PDF, smoothing
4. week	III.. 3. kedd Gergely Palla	Bootstrap, Maximum Likelihood, Hypothesis testing
5. week	III.. 10. kedd Péter Pollner	Regression, inference about independence
6. week	III.. 17. kedd Péter Pollner	Extreme statistics, posthoc analysis
7. week	III.. 24. kedd	!! MIDTERM TEST !!
8. week	III.. 31. kedd Illés Farkas	Statistics in finance
9. week	IV.. 7. kedd Illés Farkas	Stochastic processes with applications in finance
10. week	IV.. 14. kedd Holiday	
11. week	IV.. 21. kedd Péter Pollner	Bayesian models
12. week	IV.. 28. kedd Zoltán Kaufmann	Classical chaos, Ljapunov exponent, Frobenius-Perron
13. week	V.. 5. kedd László Oroszlány	Random matrices and eigenvalue statistics
14. week	V.. 12. kedd	!! FINAL TEST !!

Recommended literature

Larry Wasserman: **All of Statistics**
A Concise Course in Statistical Inference



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SAMPLE SPACE, PROBABILITY, VARIABLE AND DISTRIBUTION

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SAMPLE SPACE

Sample space and events

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Sample space

$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ is the set of possible outcomes of an experiment. The points ω_i in Ω are called **sample outcomes** or **realisations**.

Events

A subset $A \subset \Omega$ is called an **event**.

- True event: $A = \Omega$, (always true).
- Null event: $A = \emptyset$, (always false).

Examples

- Coin tossing: $\Omega = \{\omega = (\omega_1, \omega_2, \omega_3, \ldots,), \ \ \omega_i \in \{H, T\}\}$
  Event A can be e.g., that the first head appears on the third toss:
  $A = \{(\omega_1, \omega_2, \omega_3, \ldots,) : \omega_1 = T, \omega_2 = T, \omega_3 = H, \omega_i \in \{H, T\}\}$.
- Let ω be the outcome of a measurement of some physical quantity, e.g., temperature. Then $\Omega = [-\infty, \infty]$.
  Event A can be e.g., that he measurement is between 10 and 20: $A = [10, 20]$.

Sample space and events

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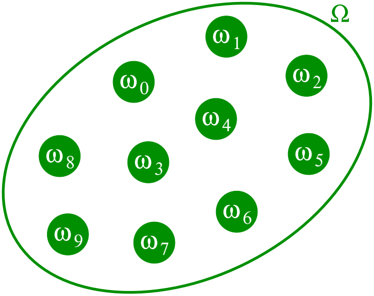
Operations between events

- Union, $A \cup B$: at least one of them is true (OR): $A + B$.
- Intersection, $A \cap B$: both of them are true (AND): AB.
- Complement, A^c: it is false (NEGATION): $\overline{A}$.
- Inclusion: $A \subset C$.
- Set difference, $A \setminus B$: A is true and B is false: $A \setminus B = A \cap \overline{B}$.



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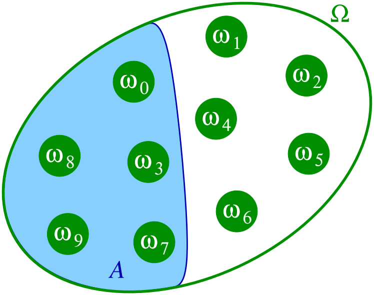

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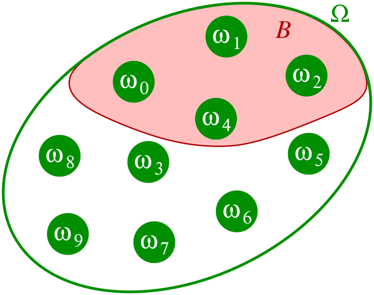

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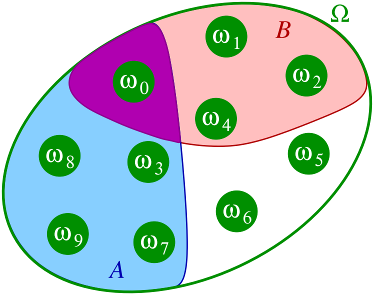

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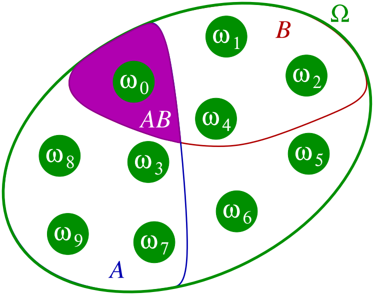

Operations between events

Sample space, Probability, Variable and Distribution

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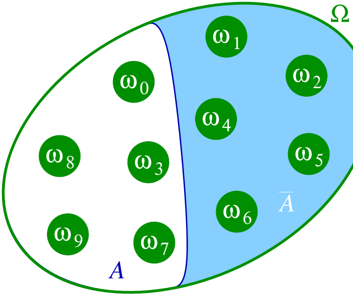

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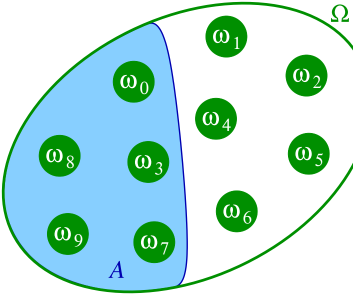

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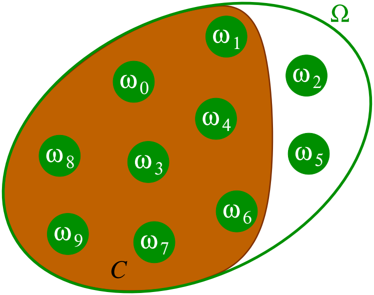

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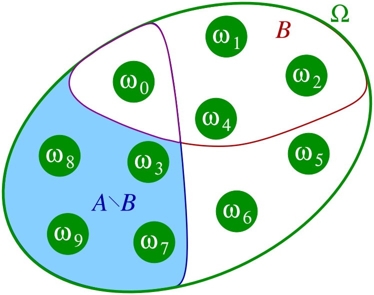

Operational identities

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Operational identities

„AND"

- $A \cap B = B \cap A$
- $A \cap A = A$
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cap \overline{A} = \varnothing$
- $A \cap \varnothing = \varnothing$
- $A \cap \Omega = A$

„OR"

- $A \cup B = B \cup A$
- $A \cup A = A$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cup \overline{A} = \Omega$
- $A \cup \varnothing = A$
- $A \cup \Omega = \Omega$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (A \cap B) = A$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Partition

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Mutually exclusive events

A and B are mutually exclusive events if $A \cap B = \emptyset$

Partition

$A_1, A_2, \dots, A_n$ is a partition of Ω, if for all $k = 1, \dots, n$

a) $A_k \neq \emptyset$,

b) $A_j \cap A_k = \emptyset$ if $j \neq k$,

c) $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$.

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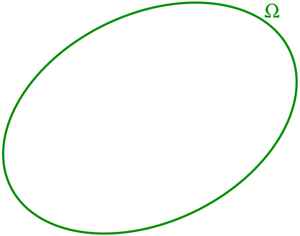

Partition

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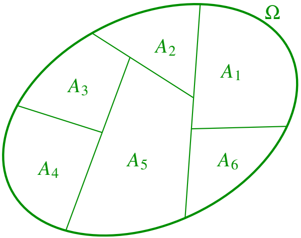

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PROBABILITY

Probability

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Definition of probability (Kolmogorov)

Given a sample space Ω, a function $P(A)$ over the subsets of Ω is a **probability distribution** or a **probability measure** if

K1 $0 \leq P(A) \leq 1 \; \forall A \subset \Omega$,

K2 $P(\Omega) = 1$,

K3 If $A_1, A_2, \ldots$ are mutually exclusive, then

$$P\left(\bigcup_{i=1}^{\infty} A_k\right) = \sum_{i=1}^{\infty} P(A_i).$$

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The probability of the false event $\varnothing$ is 0.

$$\begin{aligned} A \cup \varnothing = A, &\qquad A \cap \varnothing = \varnothing \\ P(A) = P(A \cup \varnothing) &\underset{K3}{=} P(A) + P(\varnothing) \\ P(\varnothing) &= 0 \end{aligned}$$

For a partition $A_1, A_2, \ldots, A_n$ the sum of the probabilities is $\sum_i P(A_i) = 1$.

- $A_i \cap A_j = \varnothing$ and $\bigcup_{i=1}^{n} A_i = \Omega$.
- Since $P(\Omega) = 1$, according to (K3) we obtain $\sum_i P(A_i) = 1$.

The probability of the complement $\overline{A}$ is $P(\overline{A}) = 1 - P(A)$.

(A and $\overline{A}$ together form a partition.)

Probability identities

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Probability identities

Probability of a subset

If A is included in B, i.e., $A \subset B$, then $P(B \setminus A) = P(B) - P(A)$.

- Since $A \subset B$, we can write B as $B = A \cup (B \setminus A)$.
- Also $A(B \setminus A) = \varnothing$, thus according to (K3) $P(B) = P(A) + P(B \setminus A)$.

Probability of the union

For any events A and B the probability of their union is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

- Since $A \cup B = A \cup (B \setminus A \cap B)$ and $A(B \setminus A \cap B) = \varnothing$, according to (K3) $P(A \cup B) = P(A) + P(B \setminus A \cap B)$.
- Since $A \cap B \subset B$, according to the previous result $P(B \setminus A \cap B) = P(B) - P(A \cap B)$, which substituted back yields the end result.

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- Also $A(B \setminus A) = \varnothing$, thus according to (K3) $P(B) = P(A) + P(B \setminus A)$.

Probability of the union

For any events A and B the probability of their union is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

- Since $A \cup B = A \cup (B \setminus A \cap B)$ and $A(B \setminus A \cap B) = \varnothing$, according to (K3) $P(A \cup B) = P(A) + P(B \setminus A \cap B)$.
- Since $A \cap B \subset B$, according to the previous result $P(B \setminus A \cap B) = P(B) - P(A \cap B)$, which substituted back yields the end result.

Probability identities

Probability of a subset

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Limit theorems

L I. If $A_1, A_2, \ldots$ is an infinite series of events where $A_n \supset A_{n+1}$ for all n, and $\bigcap_{i=1}^{\infty} A_i = A$, then $\lim_{n\to\infty} P(A_n) = P(A)$.

L II. If $A_1, A_2, \ldots$ is an infinite series of events where $A_n \subset A_{n+1}$ for all n, and $\bigcup_{i=1}^{\infty} A_i = A$, then $\lim_{n\to\infty} P(A_n) = P(A)$.

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Conditional probability

The conditional probability of event A given event B is defined as

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)},$$

(where we assumed that $P(B) > 0$).

Properties:

- $0 \leq P(A \mid B) \leq 1$,
- $P(B \mid B) = 1$,
- If $A_1, A_2, \ldots,$ are pairwise disjoint, then $P(\bigcup_i A_i \mid B) = \sum_i P(A_i \mid B)$.
- The $P(A \mid B)$ function over the subsets A in Ω is also a probability distribution.
- $P(A \mid B)P(B) = P(A \cap B) = P(B \mid A)P(A)$.

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Multiplication rule

Multiplication rule

For any events $A_1, A_2, \ldots, A_n$

$$\begin{aligned} P(A_1 \cap A_2 \cap \cdots \cap A_n) &= P(A_1 \mid A_2 \cap \cdots \cap A_n) P(A_2 \mid A_3 \cap \cdots \cap A_n) \cdots \\ & \cdots P(A_{n-1} \mid A_n) P(A_n) = \\ & P(A_n) \prod_{i=1}^{n-1} P(A_i \mid A_{i+1} \cap \cdots \cap A_n) \end{aligned}$$

Proof: by applying the multiplication rule for two events recursively, e.g.,

$$P(A_1 \cap A_2 \cap A_3) = P(A_1 \mid A_2 \cap A_3) P(A_2 \cap A_3) = P(A_1 \mid A_2 \cap A_3) P(A_2 \mid A_3) P(A_3),$$

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Law of total probability

Law of total probability

If $A_1, A_2, \ldots, A_n$ provide a partition and $P(A_i) > 0\ \forall i$, then for any event B

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i) P(A_i).$$

- Since the A_i are pairwise disjoint, so are $B \cap A_i$.
- Based on $\bigcup_i A_i = \Omega$, the union of the events $B \cap A_i$ is

$$(B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n) = B \cap (A_1 \cup A_2 \cup \cdots \cup A_n) = B \cap \Omega = B.$$

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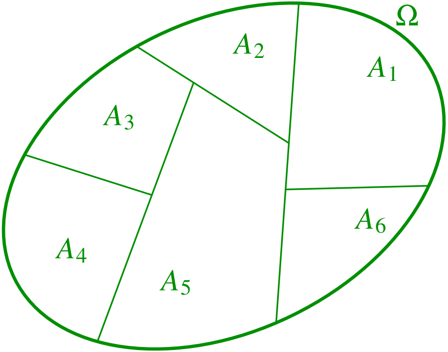
Illustration:



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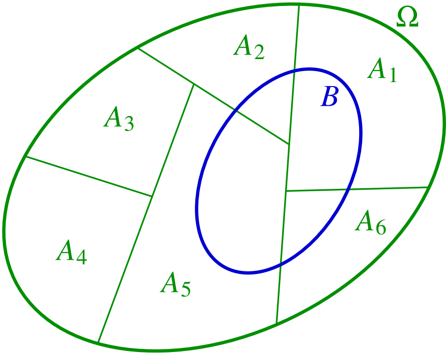

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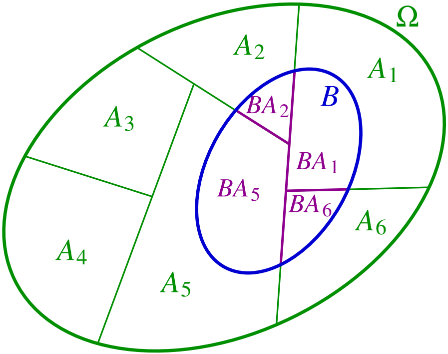

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Proof:

- According to the definition of the conditional probability $P(A_k \mid B)P(B) = P(B \mid A_k)P(A_k)$.
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$$P(A_k \mid B) = \frac{P(B \mid A_k)P(A_k)}{P(B)} = \frac{P(B \mid A_k)P(A_k)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)}.$$

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Independent events

Events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

- According to the definition the conditional probability of A given B in case of independence is

$$P(A \mid B) = P(A \cap B)/P(B) = \frac{P(A)P(B)}{P(B)} = P(A).$$

→ This can also be the alternative definition of independence.

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Random variable

Random variable

A **random variable** is usually a mapping $X : \Omega \longrightarrow \mathbb{R}$ assigning a real number $X(\omega)$ to each event. However in general a random variable can map also as

$$X(\omega) : \Omega \longrightarrow \begin{cases} \mathbb{N} \\ \mathbb{R} \\ \mathbb{C} \\ \mathbb{R}^n \end{cases}$$

Cumulative distribution function

Cumulative distribution function (CDF)

The **cumulative distribution function** of a random variable X denoted by $F_X(x)$ is defined as

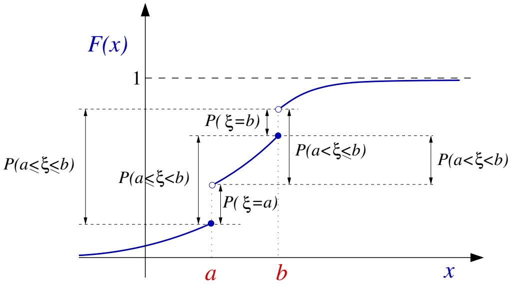
$$F_X(x) := P(X < x) = P\left(\{\omega \in \Omega\} : X(\omega) < x\right)$$

Properties:

- If $x_1 < x_2$ then $F(x_1) \leq F(x_2)$,
  (because event $X < x_1$ includes $X < x_2$).
- $\lim\limits_{x \to -\infty} F(x) = 0$ and $\lim\limits_{x \to \infty} F(x) = 1$.
  (These are the consequences of L I. and L II. limit theorems).
- $F(x)$ is continuous from the left:
  ha $x_1 < x_2 \cdots < x_i < \cdots$ and $\lim\limits_{n \to \infty} x_n = x$ then $\lim\limits_{n \to \infty} F(x_n) = F(x)$.
  (This comes from the limit theorem L II.)

Cumulative distribution function

Illustration:



Point mass function

Point mass function

- A random variable is **discrete** if its values $X(\omega) = x$ can take up finite or countable many different values.

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Point mass function

Point mass function

- A random variable is **discrete** if its values $X(\omega) = x$ can take up finite or countable many different values.
- Discrete random variables can be also characterised by their **point mass function** $f_X(x)$. By denoting the i-th discrete value $X(\omega)$ can take as x_i, the $f_X(x)$ can be defined as

$$f_X(x_i) := P(X(\omega) = x_i) = P(\{\omega_j\} : X(\omega_i) = x_i).$$

Point mass function

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- Discrete random variables can be also characterised by their **point mass function** $f_X(x)$. By denoting the i-th discrete value $X(\omega)$ can take as x_i, the $f_X(x)$ can be defined as

$$f_X(x_i) := P(X(\omega) = x_i) = P(\{\omega_j\} : X(\omega_i) = x_i).$$

- The relation between the CDF and the point mass function can be written as

$$F(x) = \sum_{i:\, x_i < x} f_X(x_i) = \sum_{i:\, x_i < x} P(\{\omega_j\} : X(\omega_i) = x_i).$$

Probability density function

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Probability density function

- A random variable is **continuous**, if there is a $\rho(x) \geq 0$ function fulfilling

$$F(b) - F(a) = P(a \leq X < b) = P(a < X < b) = \int_a^b \rho(x) dx.$$

Probability density function

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$$F(b) - F(a) = P(a \leq X < b) = P(a < X < b) = \int_a^b \rho(x)dx.$$

- Using $F(-\infty) = 0$ we can express the CDF as

$$F(x) = \int_{-\infty}^{x} \rho(x')dx'.$$

Probability density function

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- The function $\rho(x)$ is called the **probability density function**.

Probability density function

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Properties of the PDF:

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Properties of the PDF:

- It is normalised:

$$\int_{-\infty}^{\infty} \rho(x)dx = F(\infty) - F(-\infty) = 1.$$

Probability density function

Properties of the PDF:

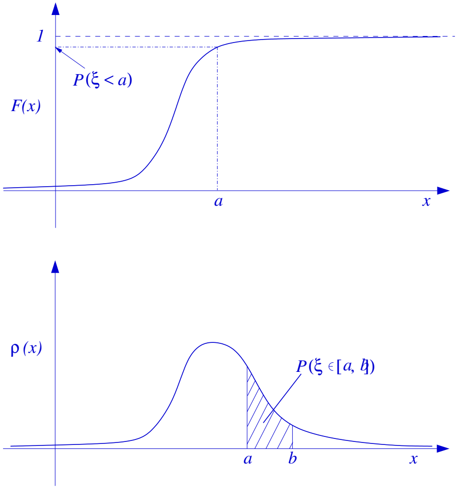
- It is normalised:

$$\int_{-\infty}^{\infty} \rho(x)dx = F(\infty) - F(-\infty) = 1.$$

- The probability that $X \in [a, b]$ can be expressed as

$$P(X \in [a, b]) = \int_{a}^{b} \rho(x)dx.$$

F(x) and $\rho(x)$
Illustration



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Discrete uniform distribution

X has a uniform distribution on $\{x_1, x_2, \ldots, x_n\}$ if its point mass function is given by

$$f_X(x_i) = \frac{1}{n}.$$

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Bernoulli distribution

Let X represent an experiment with possibly two different outcomes (e.g., coin flip), where $P(X = 1) = p$ and $P(X = 0) = 1 - p$. Then X has a **Bernoulli distribution**, with a point mass function for $x \in 0, 1$ written as

$$f_X(x) = p^x(1-p)^{1-x} = \begin{cases} p, & \text{if } x = 1, \\ 1-p, & \text{if } x = 0. \end{cases}$$

This is usually denoted as $X \sim$Bernoulli(p).

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Binomial and Geometric distributions

- Assume a coin flip with probability p for heads and probability $q = 1 - p$ for tails, repeated N times independently.

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Binomial and Geometric distributions

- Assume a coin flip with probability p for heads and probability $q = 1 - p$ for tails, repeated N times independently.
- Let X count to the number of heads. X has a **Binomial distribution** with a point mass function

$$f_X(x = k) = P(X = k) = \binom{N}{k} p^k (1-p)^{N-k} = \binom{N}{k} p^k q^{N-k}.$$

This is usually denoted as $X \sim \text{Binomial}(N, p)$.

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  This is usually denoted as $X \sim \text{Binomial}(N, p)$.
- Let Y count the number of flips needed until the first heads. Y has a **Geometrical distribution** with a point mass function

$$f_Y(y = k) = P(Y = k) = (1-p)^{k-1} p.$$

  This is usually denoted as $Y \sim \text{Geom}(p)$.

Important distributions

Discrete variables

Poisson distribution

X taking up non-negative integer values has a Poisson distribution if its point mass function can be written as

$$f_X(x=k) = P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}.$$

This is usually denoted as $X \sim$Poisson(λ).

Simeon Poisson

If we take a binomial distribution $X \sim \mathrm{Binom}(N,p)$ in the following limit:

$$\lim N = \infty, \quad \lim p = 0 \quad \lim_{\substack{N\to\infty \\ p\to 0}} pN = \lambda,$$

then its point mass function is converging to a Poisson distribution

$$f_X(x=k) = P(X=k) = \binom{N}{k} p^k (1-p)^{N-k} \xrightarrow{N\to\infty} \frac{(Np)^k}{k!}e^{-Np} = \frac{\lambda^k}{k!}e^{-\lambda}.$$

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Continuous uniform distribution

The continuous random variable $X(\omega) \in [x_1, x_2]$ has a uniform distribution if

$$\rho_X(x) = \begin{cases} \frac{1}{x_2 - x_1} & \text{if } x_1 \leq x \leq x_2 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < x_1 \\ \frac{x - x_1}{x_2 - x_1} & \text{if } x_1 \leq x \leq x_2 \\ 1 & \text{if } x > x_2 \end{cases}$$

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Exponential distribution

A continuous random variable X over the non-negative real numbers has an **Exponential distribution** if

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x}{\lambda}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \rho_X(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

This is usually denoted as $X \sim \text{Exp}(\lambda)$.

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Gamma distribution

- A continuous random variable X over the non-negative real numbers has a **Gamma distribution**, usually denoted as $X \sim$Gamma(q, λ) if

$$\rho_X(x) = \frac{1}{\lambda^q \Gamma(q)} x^{q-1} e^{-\frac{x}{\lambda}},$$

where the Gamma function $\Gamma(z)$ is defined as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

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- Connections with the exponential distribution:
  - The exponential distribution corresponds to the special case of Gamma$(q = 1, \lambda)$.

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- Connections with the exponential distribution:
  - The exponential distribution corresponds to the special case of Gamma$(q = 1, \lambda)$.
  - The sum of n independent random variables $X_i \sim$Exp(λ) has a Gamma distribution Gamma(n, λ).

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- Connections with the exponential distribution:
  - The exponential distribution corresponds to the special case of Gamma$(q = 1, \lambda)$.
  - The sum of n independent random variables $X_i\sim$Exp(λ) has a Gamma distribution Gamma(n, λ).

  → Thus, also the sum of independent $X_i\sim$Gamma(q_i, λ) has a distribution $\sum_{i=1}^n X_i\sim$Gamma$(\sum_{i=1}^n q_i, \lambda)$.

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Normal distribution

A continuous random variable X has a **Normal** (Gaussian) distribution (usually denoted by $X \sim \mathrm{N}(\mu, \sigma)$) if

$$\rho_X(x) = \mathcal{N}(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad F_X(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right],$$

A standard normal distribution is corresponding to $\mathrm{N}(\mu = 0, \sigma = 1)$.

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A standard normal distribution is corresponding to $\mathrm{N}(\mu = 0, \sigma = 1)$.

χ^2 distribution

- A continuous random variable X over the non-negative real numbers has a χ^2 distribution with n degrees of freedom (usually denoted by $X \sim \chi^2_n$) if

$$\rho_X(x) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$$

- Connection with the Normal distribution:
  If $X_1, X_2, \ldots, X_n$ are independent standard Normal random variables, then $\sum_{i=1}^{n} Z_i^2 \sim \chi^2_n$.

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t distribution and Cauchy distribution

- A continuous random variable X over the non-negative real numbers has a t **distribution** (also called as Student's t distribution) with n degrees of freedom (usually denoted by $X \sim t_n$) if

$$\rho_X(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}.$$

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- Connection with the Normal distribution:
  If $X_1, X_2, \ldots, X_n$ and Y are independent standard Normal random variables, then the variable

$$Z := \frac{\sqrt{n}Y}{\sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}} \quad \text{has a } t \text{ ditribution, } Z \sim t_n$$

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- Connection with the Normal distribution:
  If $X_1, X_2, \ldots, X_n$ and Y are independent standard Normal random variables, then the variable

$$Z := \frac{\sqrt{n}Y}{\sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}} \quad \text{has a } t \text{ ditribution, } Z \sim t_n$$

- The $n = 1$ special case of the t distribution corresponds to the **Cauchy distribution**, where

$$\rho_X(x) = \frac{1}{\pi} \frac{1}{1 + x^2}.$$