Statistical Physics (MSc)
Homework 3.

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QUESTION

(On the next page there is a table.) Next to your name you can find five numbers under the column $\mathcal{G}(X, X')$. Give the contribution of those graphs to $\mathcal{G}(X, X')$ in coordinate space. Choose appropriate coordinates at the vertices and write it on your figures in the solutions. Next to your name you can find another five numbers under the column $\mathcal{G}(\boldsymbol{k}, i\omega_n)$. Give the contribution of those graphs to $\mathcal{G}(\boldsymbol{k}, i\omega_n)$. Once again, use clear notations for your conventions together with a picture showing the newly introduced momenta and frequencies. Classify your graphs if they are reducible or irreducible.

The row of the table with my name:

No.	Name	$G(X, X')$					$G(\boldsymbol{k}, i\omega_n)$				
⋮	⋮			⋮					⋮		
17	Pál Balázs	8	5	6	10	3	7	2	4	1	9
⋮	⋮			⋮					⋮		

REDUCIBILITY

Def. An **internal vertex** is a node which is either part of a propagator loop, or have more than one connections. Internal vertices are marked with filled dots ($\bullet$) on the figures. Similarly, external vertices could be defined as the opposite of internal vertices, which are marked as empty dots ($\circ$) on the diagrams.
Def. An **internal edge/line** is an edge which connects two internal vertices. In contrast, external edges/lines are always connected to at least one external vertex.
**Def.** We call a diagram **reducible** which fall into two disjunct pieces if we cut one internal propagator line.
Def. Similarly, we call a diagram **irreducible** which does not fall into two disjunct pieces if we cut one internal propagator line.

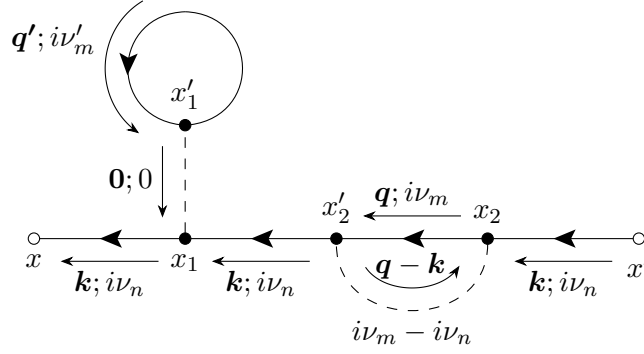Using these definitions, we can easily identify the reducible and irreducible graphs:

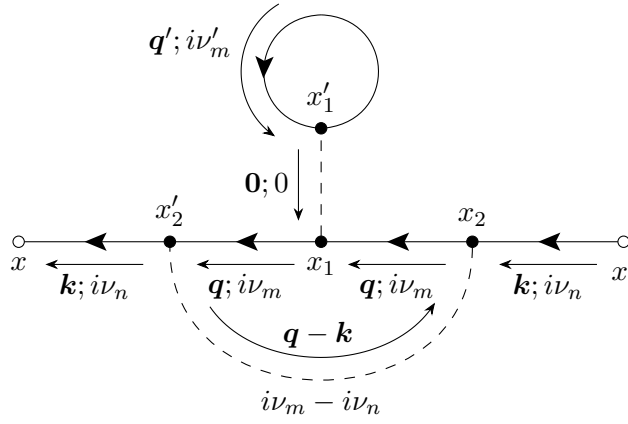| Reducible graphs				Irreducible graphs					
1	3	7	8	2	4	5	6	9	10

REDUCIBLE GRAPHS
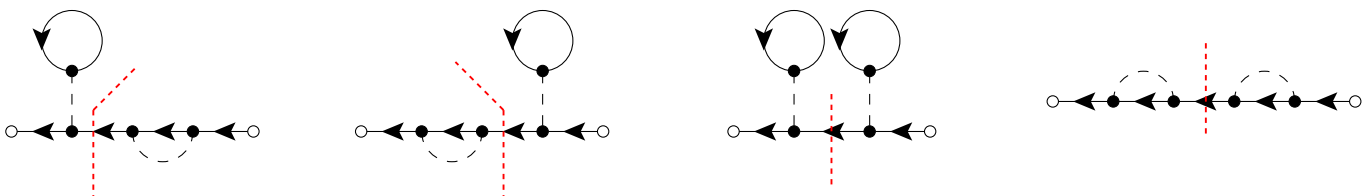
EXPRESSING THE GRAPHS

GRAPH 1. — MOMENTUM REPRESENTATION
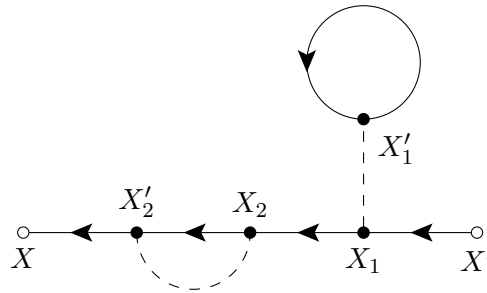

SOLUTION

GRAPH 2. — MOMENTUM REPRESENTATION

SOLUTION

GRAPH 3. — COORDINATE REPRESENTATION

SOLUTION

Fermion propagator lines are building connections between the points $(X' \to X_1)$, $(X'_2 \to X)$, $(X_1 \to X_2)$, $(X_2 \to X'_2)$ and $(X'_1 \to X'_1)$, where the last one is a fermion loop. Their contributions are:

$$-\mathcal{G}_0(X_i, X_j) \to [-\mathcal{G}_0(X_2, X_1)] \cdot [-\mathcal{G}_0(X'_2, X_2)] \cdot [-\mathcal{G}_0(X'_1, X'_1)] \cdot [-\mathcal{G}_0(X_1, X')] \cdot [-\mathcal{G}_0(X, X'_2)] =$$
$$= -\mathcal{G}_0(X_2, X_1) \cdot \mathcal{G}_0(X'_2, X_2) \cdot \mathcal{G}_0(X'_1, X'_1) \cdot \mathcal{G}_0(X_1, X') \cdot \mathcal{G}_0(X, X'_2) \tag{1}$$

Fermion loops also contribute to the Green's function. Since there are only one of them its contribution is

$$(-1)^F \rightarrow (-1)^1 = -1 \tag{2}$$

Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$-\frac{1}{\hbar} v\left(X_i X_i'\right) \rightarrow \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \tag{3}$$

Putting them all together, we need to integrate over all the internal X_i points to get the Green's function:

$$\mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot (-1) \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times$$
$$\times \left[-\mathcal{G}_0\left(X_2, X_1\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1'\right) \cdot \mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0\left(X, X_2'\right)\right] \tag{4}$$

Where the (-1) term cancels out the minus sign at the $\mathcal{G}_0$ contributions. Thus the Green's function is the following:
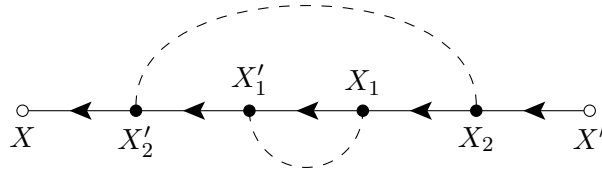
$$\mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times$$
$$\times \mathcal{G}_0\left(X_2, X_1\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1'\right) \cdot \mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0\left(X, X_2'\right) \tag{5}$$

GRAPH 4. — MOMENTUM REPRESENTATION
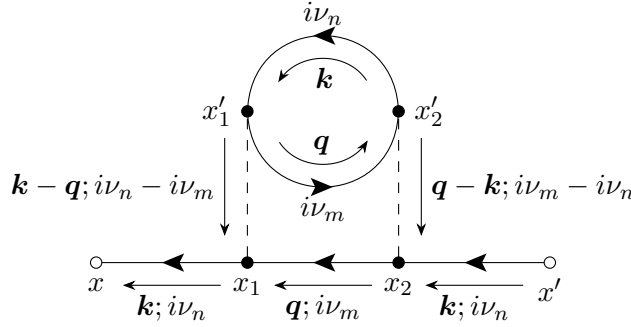
SOLUTION

GRAPH 5. — COORDINATE REPRESENTATION

SOLUTION

Fermion propagator lines are building connections between the points $(X' \rightarrow X_2)$, $(X_2 \rightarrow X_1)$, $(X_1 \rightarrow X_1')$, $(X_1' \rightarrow X_2')$ and $(X_2' \rightarrow X)$. Their contributions are:

$$-\mathcal{G}_0\left(X_i, X_j\right) \rightarrow \left[-\mathcal{G}_0\left(X_2, X'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1, X_2\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1', X_1\right)\right] \cdot \left[-\mathcal{G}_0\left(X_2', X_1'\right)\right] \cdot \left[-\mathcal{G}_0\left(X, X_2'\right)\right] =$$
$$= -\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0\left(X_1, X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1\right) \cdot \mathcal{G}_0\left(X_2', X_1'\right) \cdot \mathcal{G}_0\left(X, X_2'\right) \tag{6}$$
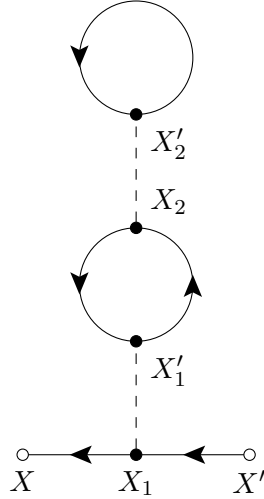
Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$-\frac{1}{\hbar} v\left(X_i X_i'\right) \to \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \tag{7}$$

Putting them all together, we need to integrate over all the internal X_i points to get the final form for the integral of the Green's function:

$$\mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times$$
$$\times \left[-\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0\left(X_1, X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1\right) \cdot \mathcal{G}_0\left(X_2', X_1'\right) \cdot \mathcal{G}_0\left(X, X_2'\right)\right] \tag{8}$$

GRAPH 6. — COORDINATE REPRESENTATION

SOLUTION

Fermion propagator lines are building connections between the points $(X' \to X_1)$, $(X_1 \to X)$, $(X_1' \to X_2)$, $(X_2 \to X_1')$ and $(X_2' \to X_2')$, where the last three are forming two separate fermion loops. Their contributions are:

$$-\mathcal{G}_0\left(X_i, X_j\right) \to \left[-\mathcal{G}_0\left(X_1, X'\right)\right] \cdot \left[-\mathcal{G}_0\left(X, X_1\right)\right] \cdot \left[-\mathcal{G}_0\left(X_2, X_1'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1', X_2\right)\right] \cdot \left[-\mathcal{G}_0\left(X_2', X_2'\right)\right] =$$
$$= -\mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0\left(X, X_1\right) \cdot \mathcal{G}_0\left(X_2, X_1'\right) \cdot \mathcal{G}_0\left(X_1', X_2\right) \cdot \mathcal{G}_0\left(X_2', X_2'\right) \tag{9}$$

Fermion loops also contribute to the Green's function. Since there are two of them their contribution is

$$(-1)^F \to (-1)^2 = 1 \tag{10}$$

Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$-\frac{1}{\hbar} v\left(X_i X_i'\right) \to \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \tag{11}$$

Putting them all together, we need to integrate over all the internal X_i points to get the Green's function:

$$\mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot (1) \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times$$
$$\times \left[-\mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0\left(X, X_1\right) \cdot \mathcal{G}_0\left(X_2, X_1'\right) \cdot \mathcal{G}_0\left(X_1', X_2\right) \cdot \mathcal{G}_0\left(X_2', X_2'\right)\right] \tag{12}$$
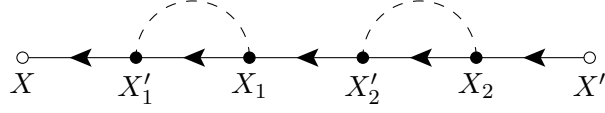
Where the $\cdot (1)$ term stand for the contribution from the fermion loops.

GRAPH 7. — MOMENTUM REPRESENTATION
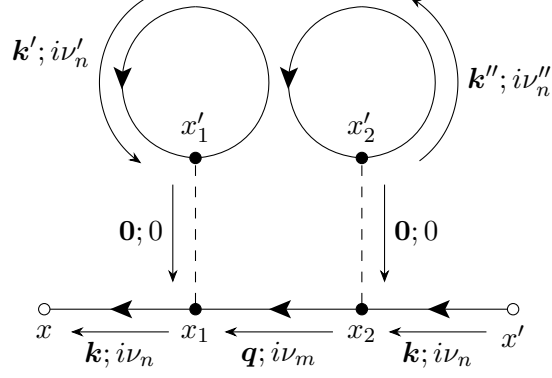
SOLUTION

GRAPH 8. — COORDINATE REPRESENTATION

SOLUTION

Fermion propagator lines are building connections between the points $(X' \to X_2)$, $(X_2 \to X_2')$, $(X_2' \to X_1)$, $(X_1 \to X_1')$ and $(X_1' \to X)$. Their contributions are:

$$-\mathcal{G}_0\left(X_i, X_j\right) \to \left[-\mathcal{G}_0\left(X_2, X'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_2', X_2\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1, X_2'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1', X_1\right)\right] \cdot \left[-\mathcal{G}_0\left(X, X_1'\right)\right] =$$
$$= -\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1, X_2'\right) \cdot \mathcal{G}_0\left(X_1', X_1\right) \cdot \mathcal{G}_0\left(X, X_1'\right) \tag{13}$$
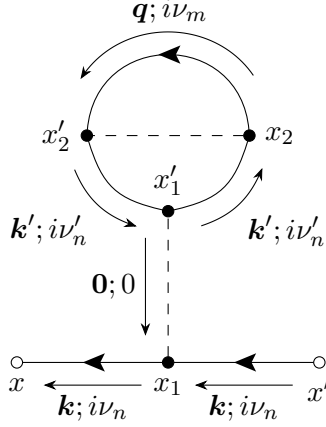
Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$-\frac{1}{\hbar}v\left(X_i X_i'\right) \to \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2}v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \tag{14}$$

Putting them all together, we need to integrate over all the internal X_i points to get the final form for the integral of the Green's function:
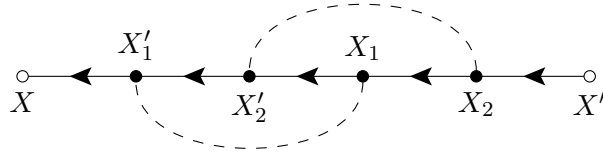
$$\mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot \frac{1}{\hbar^2}v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times$$
$$\times \left[-\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1, X_2'\right) \cdot \mathcal{G}_0\left(X_1', X_1\right) \cdot \mathcal{G}_0\left(X, X_1'\right)\right] \tag{15}$$

GRAPH 9. — MOMENTUM REPRESENTATION

SOLUTION

GRAPH 10. — COORDINATE REPRESENTATION

SOLUTION

Fermion propagator lines are building connections between the points $(X' \to X_2)$, $(X_2 \to X_1)$, $(X_1 \to X_2')$, $(X_2' \to X_1')$ and $(X_1' \to X)$. Their contributions are:

$$-\mathcal{G}_0 (X_i, X_j) \to \left[-\mathcal{G}_0 (X_2, X')\right] \cdot \left[-\mathcal{G}_0 (X_1, X_2)\right] \cdot \left[-\mathcal{G}_0 (X_2', X_1)\right] \cdot \left[-\mathcal{G}_0 (X_1', X_2')\right] \cdot \left[-\mathcal{G}_0 (X, X_1')\right] =$$
$$= -\mathcal{G}_0 (X_2, X') \cdot \mathcal{G}_0 (X_1, X_2) \cdot \mathcal{G}_0 (X_2', X_1) \cdot \mathcal{G}_0 (X_1', X_2') \cdot \mathcal{G}_0 (X, X_1') \tag{16}$$

Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$-\frac{1}{\hbar} v (X_i X_i') \to \left(-\frac{1}{\hbar}\right) v (X_1 X_1') \cdot \left(-\frac{1}{\hbar}\right) v (X_2 X_2') = \frac{1}{\hbar^2} v (X_1 X_1') v (X_2 X_2') \tag{17}$$

Putting them all together, we need to integrate over all the internal X_i points to get the final form for the integral of the Green's function:

$$\mathcal{G} (X, X') = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot \frac{1}{\hbar^2} v (X_1 X_1') v (X_2 X_2') \times$$
$$\times \left[-\mathcal{G}_0 (X_2, X') \cdot \mathcal{G}_0 (X_1, X_2) \cdot \mathcal{G}_0 (X_2', X_1) \cdot \mathcal{G}_0 (X_1', X_2') \cdot \mathcal{G}_0 (X, X_1')\right] \tag{18}$$