Statistical Physics (MSc)
Homework 3.

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QUESTION

(On the next page there is a table.) Next to your name you can find five numbers under the column $\mathcal{G}(X, X')$. Give the contribution of those graphs to $\mathcal{G}(X, X')$ in coordinate space. Choose appropriate coordinates at the vertices and write it on your figures in the solutions. Next to your name you can find another five numbers under the column $\mathcal{G}(\boldsymbol{k}, i\omega_n)$. Give the contribution of those graphs to $\mathcal{G}(\boldsymbol{k}, i\omega_n)$. Once again, use clear notations for your conventions together with a picture showing the newly introduced momenta and frequencies. Classify your graphs if they are reducible or irreducible.

The row of the table with my name:

No.	Name	$G(X, X')$					$G(\boldsymbol{k}, i\omega_n)$				
⋮	⋮	⋮					⋮				
17	Pál Balázs	8	5	6	10	3	7	2	4	1	9
⋮	⋮	⋮					⋮				

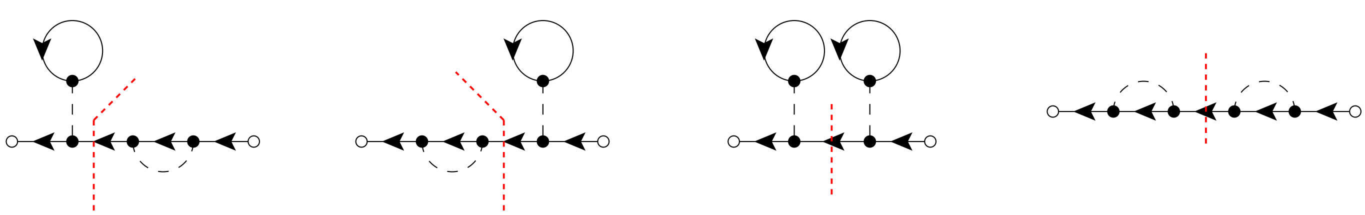
REDUCIBILITY

Def. An **internal vertex** is a node which is either part of a propagator loop, or have more than one connections. Internal vertices are marked with filled dots (•) on the figures. Similarly, external vertices could be defined as the opposite of internal vertices, which are marked as empty dots (∘) on the diagrams.
Def. An **internal edge/line** is an edge which connects two internal vertices. In contrast, external edges/lines are always connected to at least one external vertex.
Def. We call a diagram **reducible** which fall into two disjunct pieces if we cut one internal propagator line.
Def. Similarly, we call a diagram **irreducible** which does not fall into two disjunct pieces if we cut one internal propagator line.

Using these definitions, we can easily identify the reducible and irreducible graphs:

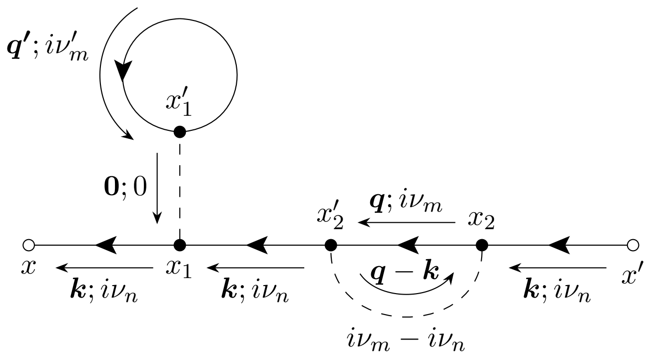
Reducible graphs				Irreducible graphs					
1	3	7	8	2	4	5	6	9	10

REDUCIBLE GRAPHS



EXPRESSING THE GRAPHS

GRAPH 1. — MOMENTUM REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(x' \to x_2)$, $(x_2 \to x'_2)$, $(x'_2 \to x_1)$, $(x_1 \to x)$ and $(x'_1 \to x'_1)$, where the last one is a fermion loop. Their contributions are:

$$- \mathcal{G}_0 \left(\boldsymbol{k}; i\nu_n\right) = -\frac{1}{i\nu_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}} - \mu\right)} \tag{1}$$

Using this relation, the contribution of fermion propagator lines in this graph are the following:

$$\begin{aligned} \mathcal{G}_F &= \left[-\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)\right] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{q}; i\nu_m\right)\right] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)\right] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)\right] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{q'}; i\nu'_m\right)\right] = \\ &= -\left[\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)\right]^3 \cdot \mathcal{G}_0\left(\boldsymbol{q}; i\nu_m\right) \cdot \mathcal{G}_0\left(\boldsymbol{q'}; i\nu'_m\right) \end{aligned} \tag{2}$$

Which could be expressed as follows according to Eq. (1):

$$\mathcal{G}_F = -\left[\frac{1}{i\nu_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}} - \mu\right)}\right]^3 \cdot \frac{1}{i\nu_m - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{q}} - \mu\right)} \cdot \frac{1}{i\nu'_m - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{q'}} - \mu\right)} \tag{3}$$

All contribution from $\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)$ propagators should be multiplied by the factor $e^{i\nu_n \eta}$, if the propagator line is a closed loop itself, or whether its endpoints are connected with an interaction line. Here both $\mathcal{G}_0\left(\boldsymbol{q}; i\nu_m\right)$ and $\mathcal{G}_0\left(\boldsymbol{q'}; i\nu'_m\right)$ are subjects to this condition, and should be multiplied by the previously mentioned factor. Thus the contribution from the propagator lines are changing as the following:

$$\mathcal{G}_F = -\left[\frac{1}{i\nu_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}} - \mu\right)}\right]^3 \cdot \frac{e^{i\nu_m \eta}}{i\nu_m - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{q}} - \mu\right)} \cdot \frac{e^{i\nu'_m \eta}}{i\nu'_m - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{q'}} - \mu\right)} \tag{4}$$

Interaction lines running between the points (x_1, x'_1) and (x_2, x'_2). The contribution from an $(\boldsymbol{k}; i\nu_n)$ interaction line is frequency-independent and is the following:

$$\mathcal{G}_I\left(\boldsymbol{k}; i\nu_n\right) = -\frac{1}{\hbar} v\left(\boldsymbol{k}\right) \tag{5}$$

Using this relation the total contribution of interaction lines in this graph is

$$\mathcal{G}_I = \left[-\frac{1}{\hbar} v\left(\boldsymbol{0}\right)\right] \cdot \left[-\frac{1}{\hbar} v\left(\boldsymbol{q} - \boldsymbol{k}\right)\right] = \frac{1}{\hbar^2} v\left(\boldsymbol{0}\right) v\left(\boldsymbol{q} - \boldsymbol{k}\right) \tag{6}$$

To put everything together, we need to sum over all independent frequencies and momentums:

$$\mathcal{G}\left(\boldsymbol{k}; i\nu_n\right) = \frac{1}{\beta\hbar}\sum_{i\nu_n}\frac{1}{\beta\hbar}\sum_{i\nu_m}\frac{1}{\beta\hbar}\sum_{i\nu'_m}\frac{1}{V}\sum_{\boldsymbol{k}}\frac{1}{V}\sum_{\boldsymbol{q}}\frac{1}{V}\sum_{\boldsymbol{q'}}\mathcal{G}_F \cdot \mathcal{G}_I = \frac{1}{\beta^3\hbar^3}\frac{1}{V^3}\sum_{i\nu_n}\sum_{i\nu_m}\sum_{i\nu'_m}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{q'}}\mathcal{G}_F \cdot \mathcal{G}_I \tag{7}$$

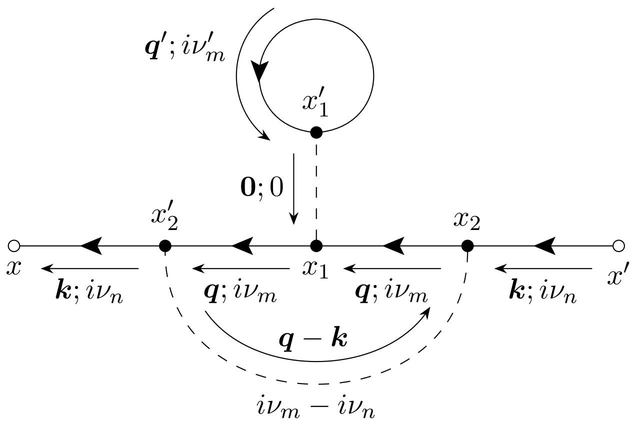
Finally this contribution should be multiplied by another factor, which is the contribution of the propagator loops itself:

$$\mathcal{G}_L = \left[\pm\left(2s+1\right)\right]^L \tag{8}$$

Where L is the number of propagator loops. Since there are only one propagator loop, $L = 1$. Thus the final Green's function is the following:

$$\mathcal{G}\left(\boldsymbol{k}; i\nu_n\right) = \left[\pm\left(2s+1\right)\right]^1 \cdot \frac{1}{\beta^3\hbar^3}\frac{1}{V^3}\sum_{i\nu_n}\sum_{i\nu_m}\sum_{i\nu'_m}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{q}'}\mathcal{G}_F \cdot \mathcal{G}_I \tag{9}$$

GRAPH 2. — MOMENTUM REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(x' \to x_2)$, $(x_2 \to x_1)$, $(x_1 \to x'_2)$, $(x'_2 \to x)$ and $(x'_1 \to x'_1)$, where the last one is a fermion loop. Their contributions are:

$$-\mathcal{G}_0(\boldsymbol{k}; i\nu_n) = -\frac{1}{i\nu_n - \frac{1}{\hbar}(\varepsilon_{\boldsymbol{k}} - \mu)} \tag{10}$$

Using this relation, the contribution of fermion propagator lines in this graph are the following:

$$\begin{aligned}\mathcal{G}_F &= [-\mathcal{G}_0(\boldsymbol{k}; i\nu_n)] \cdot [-\mathcal{G}_0(\boldsymbol{q}; i\nu_m)] \cdot [-\mathcal{G}_0(\boldsymbol{q}; i\nu_m)] \cdot [-\mathcal{G}_0(\boldsymbol{k}; i\nu_n)] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{q}'; i\nu'_m\right)\right] = \\ &= -\left[\mathcal{G}_0(\boldsymbol{k}; i\nu_n)\right]^2 \cdot \left[\mathcal{G}_0(\boldsymbol{q}; i\nu_m)\right]^2 \cdot \mathcal{G}_0\left(\boldsymbol{q}'; i\nu'_m\right)\end{aligned} \tag{11}$$

Which could be expressed as follows according to Eq. (10):

$$\mathcal{G}_F = -\left[\frac{1}{i\nu_n - \frac{1}{\hbar}(\varepsilon_{\boldsymbol{k}} - \mu)}\right]^2 \cdot \left[\frac{1}{i\nu_m - \frac{1}{\hbar}(\varepsilon_{\boldsymbol{q}} - \mu)}\right]^2 \cdot \frac{1}{i\nu'_m - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{q}'} - \mu\right)} \tag{12}$$

All contribution from $\mathcal{G}_0(\boldsymbol{k}; i\nu_n)$ propagators should be multiplied by the factor $e^{i\nu_n \eta}$, if the propagator line is a closed loop itself, or whether its endpoints are connected with an interaction line. Here only $\mathcal{G}_0(\boldsymbol{q}'; i\nu'_m)$ is subject to this condition, and should be multiplied by the previously mentioned factor. Thus the contribution from the propagator lines are changing as the following:

$$\mathcal{G}_F = -\left[\frac{1}{i\nu_n - \frac{1}{\hbar}(\varepsilon_{\boldsymbol{k}} - \mu)}\right]^2 \cdot \left[\frac{1}{i\nu_m - \frac{1}{\hbar}(\varepsilon_{\boldsymbol{q}} - \mu)}\right]^2 \cdot \frac{e^{i\nu'_m \eta}}{i\nu'_m - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{q}'} - \mu\right)} \tag{13}$$

Interaction lines running between the points (x_1, x'_1) and (x_2, x'_2). The contribution from an $(\boldsymbol{k}; i\nu_n)$ interaction line is frequency-independent and is the following:

$$\mathcal{G}_I(\boldsymbol{k}; i\nu_n) = -\frac{1}{\hbar} v(\boldsymbol{k}) \tag{14}$$

Using this relation the total contribution of interaction lines in this graph is

$$\mathcal{G}_I = \left[-\frac{1}{\hbar} v(\boldsymbol{0})\right] \cdot \left[-\frac{1}{\hbar} v(\boldsymbol{q}-\boldsymbol{k})\right] = \frac{1}{\hbar^2} v(\boldsymbol{0})\, v(\boldsymbol{q}-\boldsymbol{k}) \tag{15}$$

To put everything together, we need to sum over all independent frequencies and momentums:

$$\mathcal{G}(\boldsymbol{k}; i\nu_n) = \frac{1}{\beta\hbar}\sum_{i\nu_n} \frac{1}{\beta\hbar}\sum_{i\nu_m} \frac{1}{\beta\hbar}\sum_{i\nu'_m} \frac{1}{V}\sum_{\boldsymbol{k}} \frac{1}{V}\sum_{\boldsymbol{q}} \frac{1}{V}\sum_{\boldsymbol{q}'} \mathcal{G}_F \cdot \mathcal{G}_I = \frac{1}{\beta^3\hbar^3}\frac{1}{V^3}\sum_{i\nu_n}\sum_{i\nu_m}\sum_{i\nu'_m}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{q}'} \mathcal{G}_F \cdot \mathcal{G}_I \tag{16}$$

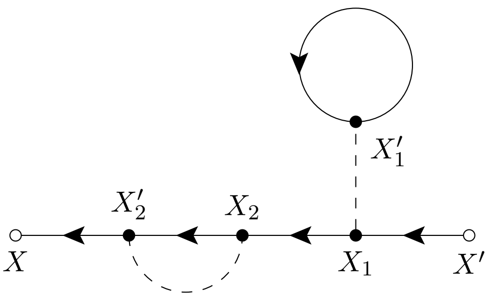
Finally this contribution should be multiplied by another factor, which is the contribution of the propagator loops itself:

$$\mathcal{G}_L = \left[\pm (2s+1)\right]^L \tag{17}$$

Where L is the number of propagator loops. Since there are only one propagator loop, $L = 1$. Thus the final Green's function is the following:

$$\mathcal{G}\left(\boldsymbol{k}; i\nu_n\right) = \left[\pm (2s+1)\right]^1 \cdot \frac{1}{\beta^3 \hbar^3} \frac{1}{V^3} \sum_{i\nu_n} \sum_{i\nu_m} \sum_{i\nu'_m} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{q}} \sum_{\boldsymbol{q}'} \mathcal{G}_F \cdot \mathcal{G}_I \tag{18}$$

GRAPH 3. — COORDINATE REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points ($X' \to X_1$), ($X_2' \to X$), ($X_1 \to X_2$), ($X_2 \to X_2'$) and ($X_1' \to X_1'$), where the last one is a fermion loop. Their contributions are:

$$\begin{aligned} -\mathcal{G}_0\left(X_i, X_j\right) \to \left[-\mathcal{G}_0\left(X_2, X_1\right)\right] \cdot \left[-\mathcal{G}_0\left(X_2', X_2\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1', X_1'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1, X'\right)\right] \cdot \left[-\mathcal{G}_0\left(X, X_2'\right)\right] = \\ = -\mathcal{G}_0\left(X_2, X_1\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1'\right) \cdot \mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0\left(X, X_2'\right) \end{aligned} \tag{19}$$

Fermion loops also contribute to the Green's function. Since there are only one of them its contribution is

$$(-1)^F \to (-1)^1 = -1 \tag{20}$$

Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$-\frac{1}{\hbar} v\left(X_i X_i'\right) \to \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \tag{21}$$

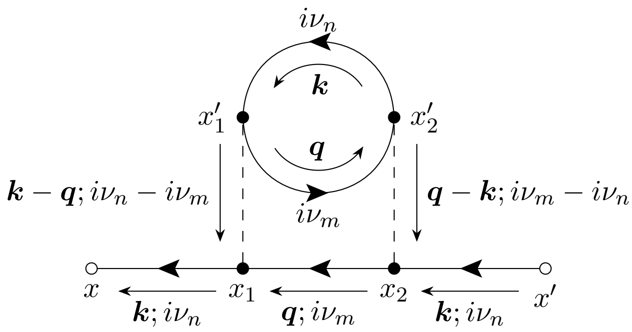
Putting them all together, we need to integrate over all the internal X_i points to get the Green's function:

$$\begin{aligned} \mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot (-1) \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times \\ \times \left[-\mathcal{G}_0\left(X_2, X_1\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1'\right) \cdot \mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0\left(X, X_2'\right)\right] \end{aligned} \tag{22}$$

Where the (-1) term cancels out the minus sign at the $\mathcal{G}_0$ contributions. Thus the Green's function is the following:

$$\begin{aligned} \mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times \\ \times \mathcal{G}_0\left(X_2, X_1\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1'\right) \cdot \mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0\left(X, X_2'\right) \end{aligned} \tag{23}$$

GRAPH 4. — MOMENTUM REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(x' \to x_2)$, $(x_2 \to x_1)$, $(x_1 \to x)$, $(x_1 \to x_1')$, $(x_1' \to x_2')$ and $(x_2' \to x_1')$, where the last two forms a fermion loop. Their contributions are:

$$- \mathcal{G}_0 (\boldsymbol{k}; i\nu_n) = - \frac{1}{i\nu_n - \frac{1}{\hbar} (\varepsilon_{\boldsymbol{k}} - \mu)} \tag{24}$$

Using this relation, the contribution of fermion propagator lines in this graph are the following:

$$\begin{aligned} \mathcal{G}_F &= [-\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)] \cdot [-\mathcal{G}_0 (\boldsymbol{q}; i\nu_m)] \cdot [-\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)] \cdot [-\mathcal{G}_0 (\boldsymbol{q}; i\nu_m)] \cdot [-\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)] = \\ &= - [\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)]^3 \cdot [\mathcal{G}_0 (\boldsymbol{q}; i\nu_m)]^2 \end{aligned} \tag{25}$$

Which could be expressed as follows according to Eq. (24):

$$\mathcal{G}_F = - \left[\frac{1}{i\nu_n - \frac{1}{\hbar} (\varepsilon_{\boldsymbol{k}} - \mu)} \right]^3 \cdot \left[\frac{1}{i\nu_m - \frac{1}{\hbar} (\varepsilon_{\boldsymbol{q}} - \mu)} \right]^2 \tag{26}$$

All contribution from $\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)$ propagators should be multiplied by the factor $e^{i\nu_n \eta}$, if the propagator line is a closed loop itself, or whether its endpoints are connected with an interaction line. Here none of them nodes are subjects to this condition, so we should leave this contribution as it is.

Interaction lines running between the points (x_1, x_1') and (x_2, x_2'). The contribution from an $(\boldsymbol{k}; i\nu_n)$ interaction line is frequency-independent and is the following:

$$\mathcal{G}_I (\boldsymbol{k}; i\nu_n) = - \frac{1}{\hbar} v (\boldsymbol{k}) \tag{27}$$

Using this relation the total contribution of interaction lines in this graph is

$$\mathcal{G}_I = \left[- \frac{1}{\hbar} v (\boldsymbol{q} - \boldsymbol{k}) \right] \cdot \left[- \frac{1}{\hbar} v (\boldsymbol{k} - \boldsymbol{q}) \right] = \frac{1}{\hbar^2} v (\boldsymbol{q} - \boldsymbol{k}) v (\boldsymbol{k} - \boldsymbol{q}) \tag{28}$$

To put everything together, we need to sum over all independent frequencies and momentums:

$$\mathcal{G} (\boldsymbol{k}; i\nu_n) = \frac{1}{\beta\hbar} \sum_{i\nu_n} \frac{1}{\beta\hbar} \sum_{i\nu_m} \frac{1}{V} \sum_{\boldsymbol{k}} \frac{1}{V} \sum_{\boldsymbol{q}} \mathcal{G}_F \cdot \mathcal{G}_I = \frac{1}{\beta^2\hbar^2} \frac{1}{V^2} \sum_{i\nu_n} \sum_{i\nu_m} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{q}} \mathcal{G}_F \cdot \mathcal{G}_I \tag{29}$$

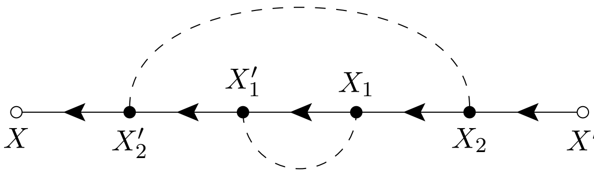
Finally this contribution should be multiplied by another factor, which is the contribution of the propagator loops itself:

$$\mathcal{G}_L = [\pm (2s + 1)]^L \tag{30}$$

Where L is the number of propagator loops. Since there are only one propagator loop $(x_1' \to x_2' \to x_1')$, $L = 1$. Thus the final Green's function is the following:

$$\mathcal{G}\left(\boldsymbol{k};i\nu_n\right) = \left[\pm\left(2s+1\right)\right]^1 \cdot \frac{1}{\beta^2\hbar^2}\frac{1}{V^2}\sum_{i\nu_n}\sum_{i\nu_m}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}}\mathcal{G}_F\cdot\mathcal{G}_I \tag{31}$$

GRAPH 5. — COORDINATE REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(X' \to X_2)$, $(X_2 \to X_1)$, $(X_1 \to X_1')$, $(X_1' \to X_2')$ and $(X_2' \to X)$. Their contributions are:

$$\begin{aligned}-\mathcal{G}_0\left(X_i, X_j\right) \to \left[-\mathcal{G}_0\left(X_2, X'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1, X_2\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1', X_1\right)\right] \cdot \left[-\mathcal{G}_0\left(X_2', X_1'\right)\right] \cdot \left[-\mathcal{G}_0\left(X, X_2'\right)\right] = \\ = -\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0\left(X_1, X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1\right) \cdot \mathcal{G}_0\left(X_2', X_1'\right) \cdot \mathcal{G}_0\left(X, X_2'\right) \qquad (32)\end{aligned}$$

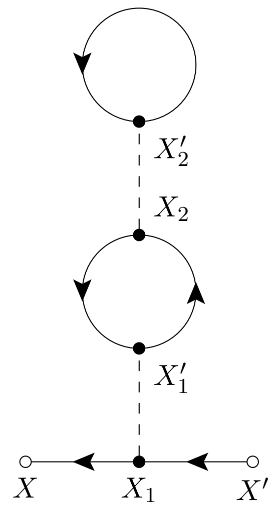
Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$-\frac{1}{\hbar} v\left(X_i X_i'\right) \to \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \qquad (33)$$

Putting them all together, we need to integrate over all the internal X_i points to get the final form for the integral of the Green's function:

$$\begin{aligned}\mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times \\ \times \left[-\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0\left(X_1, X_2\right) \cdot \mathcal{G}_0\left(X_1', X_1\right) \cdot \mathcal{G}_0\left(X_2', X_1'\right) \cdot \mathcal{G}_0\left(X, X_2'\right)\right] \qquad (34)\end{aligned}$$

GRAPH 6. — COORDINATE REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(X' \to X_1)$, $(X_1 \to X)$, $(X_1' \to X_2)$, $(X_2 \to X_1')$ and $(X_2' \to X_2')$, where the last three are forming two separate fermion loops. Their contributions are:

$$-\mathcal{G}_0(X_i, X_j) \to \left[-\mathcal{G}_0\left(X_1, X'\right)\right] \cdot \left[-\mathcal{G}_0(X, X_1)\right] \cdot \left[-\mathcal{G}_0\left(X_2, X_1'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1', X_2\right)\right] \cdot \left[-\mathcal{G}_0\left(X_2', X_2'\right)\right] = \\ = -\mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0(X, X_1) \cdot \mathcal{G}_0\left(X_2, X_1'\right) \cdot \mathcal{G}_0\left(X_1', X_2\right) \cdot \mathcal{G}_0\left(X_2', X_2'\right) \tag{35}$$

Fermion loops also contribute to the Green's function. Since there are two of them their contribution is

$$(-1)^F \to (-1)^2 = 1 \tag{36}$$

Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

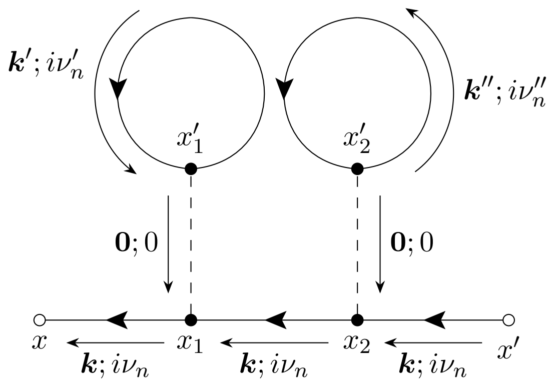
$$-\frac{1}{\hbar} v\left(X_i X_i'\right) \to \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \tag{37}$$

Putting them all together, we need to integrate over all the internal X_i points to get the Green's function:

$$\mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot (1) \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times \\ \times \left[-\mathcal{G}_0\left(X_1, X'\right) \cdot \mathcal{G}_0(X, X_1) \cdot \mathcal{G}_0\left(X_2, X_1'\right) \cdot \mathcal{G}_0\left(X_1', X_2\right) \cdot \mathcal{G}_0\left(X_2', X_2'\right)\right] \tag{38}$$

Where the $\cdot\,(1)$ term stand for the contribution from the fermion loops.

GRAPH 7. — MOMENTUM REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(x' \to x_2)$, $(x_2 \to x_1)$, $(x_1 \to x)$, $(x'_1 \to x'_1)$ and $(x'_2 \to x'_2)$, where the last two are fermion loops. Their contributions are:

$$- \mathcal{G}_0 \left(\boldsymbol{k}; i\nu_n\right) = - \frac{1}{i\nu_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}} - \mu\right)} \tag{39}$$

Using this relation, the contribution of fermion propagator lines in this graph are the following:

$$\begin{aligned} \mathcal{G}_F &= \left[-\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)\right] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)\right] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)\right] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{k}'; i\nu'_n\right)\right] \cdot \left[-\mathcal{G}_0\left(\boldsymbol{k}''; i\nu''_n\right)\right] = \\ &= - \left[\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)\right]^3 \cdot \mathcal{G}_0\left(\boldsymbol{k}'; i\nu'_n\right) \cdot \mathcal{G}_0\left(\boldsymbol{k}''; i\nu''_n\right) \end{aligned} \tag{40}$$

Which could be expressed as follows according to Eq. (39):

$$\mathcal{G}_F = - \left[\frac{1}{i\nu_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}} - \mu\right)}\right]^3 \cdot \frac{1}{i\nu'_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}'} - \mu\right)} \cdot \frac{1}{i\nu''_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}''} - \mu\right)} \tag{41}$$

All contribution from $\mathcal{G}_0\left(\boldsymbol{k}; i\nu_n\right)$ propagators should be multiplied by the factor $e^{i\nu_n\eta}$, if the propagator line is a closed loop itself, or whether its endpoints are connected with an interaction line. Here both $\mathcal{G}_0\left(\boldsymbol{k}'; i\nu'_n\right)$ and $\mathcal{G}_0\left(\boldsymbol{k}''; i\nu''_n\right)$ are subjects to this condition, and should be multiplied by the previously mentioned factor. Thus the contribution from the propagator lines are changing as the following:

$$\mathcal{G}_F = - \left[\frac{1}{i\nu_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}} - \mu\right)}\right]^3 \cdot \frac{e^{i\nu'_n}}{i\nu'_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}'} - \mu\right)} \cdot \frac{e^{i\nu''_n}}{i\nu''_n - \frac{1}{\hbar}\left(\varepsilon_{\boldsymbol{k}''} - \mu\right)} \tag{42}$$

Interaction lines running between the points (x_1, x'_1) and (x_2, x'_2). The contribution from an $(\boldsymbol{k}; i\nu_n)$ interaction line is frequency-independent and is the following:

$$\mathcal{G}_I\left(\boldsymbol{k}; i\nu_n\right) = -\frac{1}{\hbar} v\left(\boldsymbol{k}\right) \tag{43}$$

Using this relation the total contribution of interaction lines in this graph is

$$\mathcal{G}_I = \left[-\frac{1}{\hbar} v\left(\boldsymbol{0}\right)\right] \cdot \left[-\frac{1}{\hbar} v\left(\boldsymbol{0}\right)\right] = \left[\frac{1}{\hbar} v\left(\boldsymbol{0}\right)\right]^2 \tag{44}$$

To put everything together, we need to sum over all independent frequencies and momentums:

$$\mathcal{G}\left(\boldsymbol{k}; i\nu_n\right) = \frac{1}{\beta\hbar}\sum_{i\nu_n}\frac{1}{\beta\hbar}\sum_{i\nu'_n}\frac{1}{\beta\hbar}\sum_{i\nu''_n}\frac{1}{V}\sum_{\boldsymbol{k}}\frac{1}{V}\sum_{\boldsymbol{k}'}\frac{1}{V}\sum_{\boldsymbol{k}''}\mathcal{G}_F \cdot \mathcal{G}_I = \frac{1}{\beta^3\hbar^3}\frac{1}{V^3}\sum_{i\nu_n}\sum_{i\nu'_n}\sum_{i\nu''_n}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\sum_{\boldsymbol{k}''}\mathcal{G}_F \cdot \mathcal{G}_I \tag{45}$$

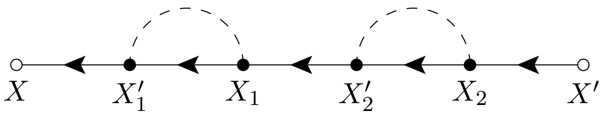
Finally this contribution should be multiplied by another factor, which is the contribution of the propagator loops itself:

$$\mathcal{G}_L = \left[\pm\left(2s+1\right)\right]^L \tag{46}$$

Where L is the number of propagator loops. Since there are two propagator loops, $L = 2$. Thus the final Green's function is the following:

$$\mathcal{G}\left(\boldsymbol{k}; i\nu_n\right) = \left[\pm\left(2s+1\right)\right]^2 \cdot \frac{1}{\beta^3\hbar^3}\frac{1}{V^3}\sum_{i\nu_n}\sum_{i\nu_n'}\sum_{i\nu_n''}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\sum_{\boldsymbol{k}''}\mathcal{G}_F \cdot \mathcal{G}_I \tag{47}$$

GRAPH 8. — COORDINATE REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(X' \to X_2)$, $(X_2 \to X_2')$, $(X_2' \to X_1)$, $(X_1 \to X_1')$ and $(X_1' \to X)$. Their contributions are:

$$
\begin{aligned}
-\mathcal{G}_0\left(X_i, X_j\right) \to &\left[-\mathcal{G}_0\left(X_2, X'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_2', X_2\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1, X_2'\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1', X_1\right)\right] \cdot \left[-\mathcal{G}_0\left(X, X_1'\right)\right] = \\
&= -\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1, X_2'\right) \cdot \mathcal{G}_0\left(X_1', X_1\right) \cdot \mathcal{G}_0\left(X, X_1'\right)
\end{aligned} \tag{48}
$$

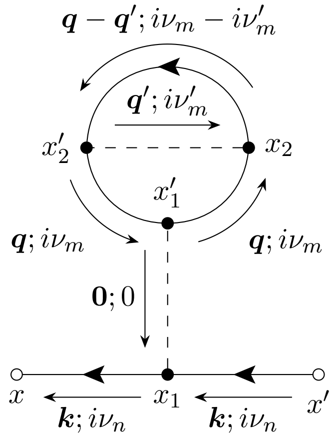
Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$
-\frac{1}{\hbar} v\left(X_i X_i'\right) \to \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \tag{49}
$$

Putting them all together, we need to integrate over all the internal X_i points to get the final form for the integral of the Green's function:

$$
\begin{aligned}
\mathcal{G}\left(X, X'\right) = &\int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times \\
&\times \left[-\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0\left(X_2', X_2\right) \cdot \mathcal{G}_0\left(X_1, X_2'\right) \cdot \mathcal{G}_0\left(X_1', X_1\right) \cdot \mathcal{G}_0\left(X, X_1'\right)\right]
\end{aligned} \tag{50}
$$

GRAPH 9. — MOMENTUM REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(x' \to x_1)$, $(x_1 \to x)$, $(x'_1 \to x_2)$, $(x_2 \to x'_2)$ and $(x'_2 \to x'_1)$. Their contributions are:

$$- \mathcal{G}_0 (\boldsymbol{k}; i\nu_n) = - \frac{1}{i\nu_n - \frac{1}{\hbar} (\varepsilon_{\boldsymbol{k}} - \mu)} \tag{51}$$

Using this relation, the contribution of fermion propagator lines in this graph are the following:

$$\begin{aligned} \mathcal{G}_F &= [-\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)] \cdot [-\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)] \cdot [-\mathcal{G}_0 (\boldsymbol{q}; i\nu_m)] \cdot \left[-\mathcal{G}_0 \left(\boldsymbol{q}-\boldsymbol{q}'; i\nu_m - i\nu'_m\right)\right] \cdot [-\mathcal{G}_0 (\boldsymbol{q}; i\nu_m)] = \\ &= - [\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)]^2 \cdot [\mathcal{G}_0 (\boldsymbol{q}; i\nu_m)]^2 \cdot \mathcal{G}_0 \left(\boldsymbol{q}-\boldsymbol{q}'; i\nu_m - i\nu'_m\right) \end{aligned} \tag{52}$$

Which could be expressed as follows according to Eq. (51):

$$\mathcal{G}_F = - \left[\frac{1}{i\nu_n - \frac{1}{\hbar} (\varepsilon_{\boldsymbol{k}} - \mu)}\right]^2 \cdot \left[\frac{1}{i\nu_m - \frac{1}{\hbar} (\varepsilon_{\boldsymbol{q}} - \mu)}\right]^2 \cdot \frac{1}{(i\nu_m - i\nu'_m) - \frac{1}{\hbar} \left(\varepsilon_{\boldsymbol{q}-\boldsymbol{q}'} - \mu\right)} \tag{53}$$

All contribution from $\mathcal{G}_0 (\boldsymbol{k}; i\nu_n)$ propagators should be multiplied by the factor $e^{i\nu_n \eta}$, if the propagator line is a closed loop itself, or whether its endpoints are connected with an interaction line. Here only $\mathcal{G}_0 (\boldsymbol{q}-\boldsymbol{q}'; i\nu_m - i\nu'_m)$ is subject to this condition, and should be multiplied by the previously mentioned factor. Thus the contribution from the propagator lines are changing as the following:

$$\mathcal{G}_F = - \left[\frac{1}{i\nu_n - \frac{1}{\hbar} (\varepsilon_{\boldsymbol{k}} - \mu)}\right]^2 \cdot \left[\frac{1}{i\nu_m - \frac{1}{\hbar} (\varepsilon_{\boldsymbol{q}} - \mu)}\right]^2 \cdot \frac{e^{(i\nu_m - i\nu'_m)\eta}}{(i\nu_m - i\nu'_m) - \frac{1}{\hbar} \left(\varepsilon_{\boldsymbol{q}-\boldsymbol{q}'} - \mu\right)} \tag{54}$$

Interaction lines running between the points (x_1, x'_1) and (x_2, x'_2). The contribution from an $(\boldsymbol{k}; i\nu_n)$ interaction line is frequency-independent and is the following:

$$\mathcal{G}_I (\boldsymbol{k}; i\nu_n) = - \frac{1}{\hbar} v (\boldsymbol{k}) \tag{55}$$

Using this relation the total contribution of interaction lines in this graph is

$$\mathcal{G}_I = \left[- \frac{1}{\hbar} v (\boldsymbol{0})\right] \cdot \left[- \frac{1}{\hbar} v \left(\boldsymbol{q}'\right)\right] = \frac{1}{\hbar^2} v (\boldsymbol{0})\, v \left(\boldsymbol{q}'\right) \tag{56}$$

To put everything together, we need to sum over all independent frequencies and momentums:

$$
\mathcal{G}(\boldsymbol{k}; i\nu_n) = \frac{1}{\beta\hbar}\sum_{i\nu_n}\frac{1}{\beta\hbar}\sum_{i\nu_m}\frac{1}{\beta\hbar}\sum_{i\nu'_m}\frac{1}{V}\sum_{\boldsymbol{k}}\frac{1}{V}\sum_{\boldsymbol{q}}\frac{1}{V}\sum_{\boldsymbol{q}'}\mathcal{G}_F\cdot\mathcal{G}_I = \frac{1}{\beta^3\hbar^3}\frac{1}{V^3}\sum_{i\nu_n}\sum_{i\nu_m}\sum_{i\nu'_m}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{q}'}\mathcal{G}_F\cdot\mathcal{G}_I \tag{57}
$$

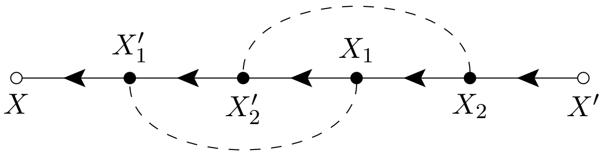
Finally this contribution should be multiplied by another factor, which is the contribution of the propagator loops itself:

$$
\mathcal{G}_L = [\pm(2s+1)]^L \tag{58}
$$

Where L is the number of propagator loops. Since there are only one propagator loop ($x'_1 \to x_2 \to x'_2$), $L = 1$. Thus the final Green's function is the following:

$$
\mathcal{G}(\boldsymbol{k}; i\nu_n) = [\pm(2s+1)]^1 \cdot \frac{1}{\beta^3\hbar^3}\frac{1}{V^3}\sum_{i\nu_n}\sum_{i\nu_m}\sum_{i\nu'_m}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{q}'}\mathcal{G}_F\cdot\mathcal{G}_I \tag{59}
$$

GRAPH 10. — COORDINATE REPRESENTATION



SOLUTION

Fermion propagator lines are building connections between the points $(X' \to X_2)$, $(X_2 \to X_1)$, $(X_1 \to X_2')$, $(X_2' \to X_1')$ and $(X_1' \to X)$. Their contributions are:

$$-\mathcal{G}_0(X_i, X_j) \to \left[-\mathcal{G}_0\left(X_2, X'\right)\right] \cdot \left[-\mathcal{G}_0(X_1, X_2)\right] \cdot \left[-\mathcal{G}_0\left(X_2', X_1\right)\right] \cdot \left[-\mathcal{G}_0\left(X_1', X_2'\right)\right] \cdot \left[-\mathcal{G}_0\left(X, X_1'\right)\right] = \\ = -\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0(X_1, X_2) \cdot \mathcal{G}_0\left(X_2', X_1\right) \cdot \mathcal{G}_0\left(X_1', X_2'\right) \cdot \mathcal{G}_0\left(X, X_1'\right) \tag{60}$$

Interaction happens between (X_1, X_1') and (X_2, X_2'). Their contributions are

$$-\frac{1}{\hbar} v\left(X_i X_i'\right) \to \left(-\frac{1}{\hbar}\right) v\left(X_1 X_1'\right) \cdot \left(-\frac{1}{\hbar}\right) v\left(X_2 X_2'\right) = \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \tag{61}$$

Putting them all together, we need to integrate over all the internal X_i points to get the final form for the integral of the Green's function:

$$\mathcal{G}\left(X, X'\right) = \int \mathrm{d}X_1 \int \mathrm{d}X_2 \int \mathrm{d}X_1' \int \mathrm{d}X_2' \cdot \frac{1}{\hbar^2} v\left(X_1 X_1'\right) v\left(X_2 X_2'\right) \times \\ \times \left[-\mathcal{G}_0\left(X_2, X'\right) \cdot \mathcal{G}_0(X_1, X_2) \cdot \mathcal{G}_0\left(X_2', X_1\right) \cdot \mathcal{G}_0\left(X_1', X_2'\right) \cdot \mathcal{G}_0\left(X, X_1'\right)\right] \tag{62}$$