Rotating neutron stars

Balázs Pál

Eötvös Loránd University

The structure of compact stars
June 13, 2022

Overview

Motivation

- Rotating objects are always problematic
- Rotation in relativistic conditions are even more problematic
- We still want to determine interesting attributes of compact rotating objects (stars), e.g. mass, angular momentum etc.
- We want to unveil the effects of rotation on the star's composition and inner structure

How to approach the problem?

1. Describe the frame-dragging effect
2. Derivate the Kepler-frequency
3. Discuss the consequences arising from the nature of different descriptions

Frame-dragging

Effect of rotation on metric tensor

- The metric tensor in case of a static, stationary star is described in the Schwarzschild metric with $G = c = 1$ contains only diagonal elements

$$g = d\tau^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 sin^2(\theta)\, d\varphi^2 \qquad (1)$$

- Rotation however introduces off-diagonal elements too (where $\omega \neq 0$ is the angular velocity of local interial frames)

$$\begin{aligned} g = d\tau^2 = {} & e^{2\nu(r,\theta)} dt^2 - e^{2\lambda(r,\theta)} dr^2 \\ & - e^{2\mu(r,\theta)} \left[r^2 d\theta^2 + r^2 \sin^2(\theta) \cdot (d\varphi - \omega(r,\theta)\, dt)^2 \right] \end{aligned} \qquad (2)$$

- Due to rotation the star deforms around the equator (centrifugal flattening) and loses spherical symmetry

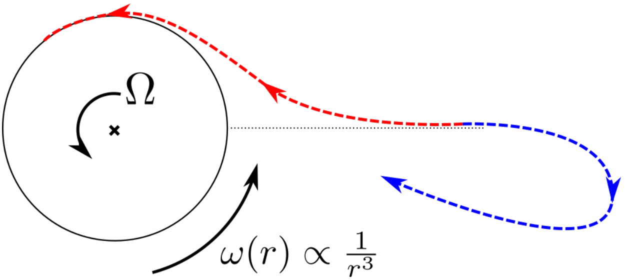


Figure 1: Frame-dragging effect over a massive rotating object on a free-falling body (red) and on a body that tries to move away from it (blue). Frame-dragging induces a force similar to the Coriolis–effect by „dragging" the compass of intertia in the direction of the star's rotation. Bodies falling towards the star's center are deflected in the direction of the rotation, while bodies moving away from it are deflected in the opposite direction.

Dimensional analysis of $\omega(r)$

- Since frame-dragging is caused by the rotation of the star, it will be proportional to the angular momentum. Since we're only interested in dimensions, we can write the classical formula for this:

$$J = I \cdot \Omega \propto MR^2\Omega \tag{3}$$

- Using CGS units to make dimensional analysis possible (since GR units won't make us any good obviously...), also using CGS, since astronomers likes it much better than SI for some reason

$$\begin{aligned}
[G] &= cm^3 \cdot g^{-1} \cdot s^{-2} = L^3 \cdot M^{-1} \cdot T^{-2} \\
[J] &= g \cdot cm^2 \cdot s^{-1} = M \cdot L^2 \cdot T^{-1} \\
[\omega] &= s^{-1} = T^{-1}
\end{aligned}$$

Dimensional analysis of $\omega(r)$

- Now substituting into (3) and expanding $\omega(r)$ we get the following:

$$\omega(r) \propto \frac{GJ^l}{c^m r^n} \propto L^{3+2l-m-n} T^{-2-l+m} M^{-1+l} \tag{4}$$

- Solving the system of equations for the exponents gives us the solution $l = 1$, $m = 2$ and $n = 3$:

$$\omega(r) \propto \frac{GJ}{c^2 r^3} \xRightarrow{G=1,\ c=1} \frac{J}{r^3} = \frac{I\Omega}{r^3}, \quad r \geq R \tag{5}$$

Solutions for all r values

Interior solution for J

- The centrifugal force acting on a fluid element depends on the coupled frequencies of rotating inertial frames and the rotation of the star:

$$\bar{\omega}(r) = \Omega - \omega(r) \tag{6}$$

- J. B. Hartle obtained a formula from the Einstein equations

$$\frac{1}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r} + \frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\omega} = 0\right) \tag{7}$$

- ...that can be used to derive the formula for the angular momentum:

$$J = \frac{8\pi}{3}\int_0^R \mathrm{d}r\, r^4 \frac{\epsilon(r) + p(r)}{\sqrt{1 - 2M(r)/r}}\left[\Omega - \omega(r)\right] e^{-\nu(r)} \tag{8}$$

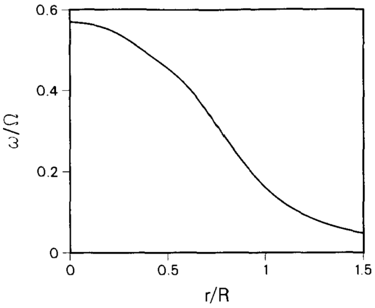
Solutions for all r values

Final form og $\omega(r)$

- From this, dimensional analysis can show the true relationship between frame-dragging and rotation of the star:

$$\omega(r) = \frac{2J}{r^3} = \frac{2I\Omega}{r^3}, \quad r \geq R \tag{9}$$

- From (8) and (9), $\omega(r)$ can be finally described for every r values.



Problems with previous solutions

- The equation (8) is only valid for static or slowly rotating objects.
- Rotation introduces θ dependence of ω.
- Different attributes of the system (change in angular momentum, change in centrifugal forces, changes in the shape of the star etc.) are all intertwined.

How to approach these problems?

All the phenomenons and quantities mentioned above are connected by I, the moment of inertia of the star, hence we're interested in the description of this quantity.

- The z-component (the axis of rotation) of the angular momentum can be computed as

$$J = I\Omega = \int \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\sqrt{-\det g_{\alpha\beta}}\,T_{\varphi}^{t} \tag{10}$$

- Our goal is to determine the metric tensor g and the mass-energy tensor T to be able to calculate the J angular momentum and the I moment of inertia. After some algebra we get the final form for J:

$$J = I\Omega = -\int \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\frac{(p+\epsilon)(\Omega-\omega)\,r^4\sin^3(\theta)\,e^{\nu+\lambda+2\mu}}{e^{2(\nu-\mu)} - r^2\sin^2(\theta)(\Omega-\omega)^2} \tag{11}$$

Kepler frequency in GR

- The Kepler angular velocity or Kepler frequency for short is the limit for a star to remain stable. A rotational velocity above the Kepler frequency causes the star to start shredding along its equator due to centrifugal forces.
- The Newtonian equilibrium of gravitational and centrifugal force is

$$\frac{Mm}{R^2} = m\Omega^2 R \tag{12}$$

- A self-consistency condition for a fluid element moving along the equator of the star with V velocity (relative to an observer with zero angular velocity along the φ direction):

$$\Omega_K = e^{\nu-\mu} \frac{V}{R + \omega(R)} \tag{13}$$