Rotating neutron stars

A presentation based on Glendenning's Compact Stars

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The structure of compact stars
June 13, 2022

Overview

Motivation

- Rotating objects are always problematic
- Rotation in relativistic conditions are even more problematic
- We still want to determine interesting attributes of compact rotating objects (stars), e.g. mass, angular momentum etc.
- We want to unveil the effects of rotation on the star's composition and inner structure

How to approach the problem?

1. Describe the frame-dragging effect
2. Determine the J function for a star with a specific EoS with the frame-dragging effect included
3. Construct a real EoS

Frame-dragging

Effect of rotation on moving bodies

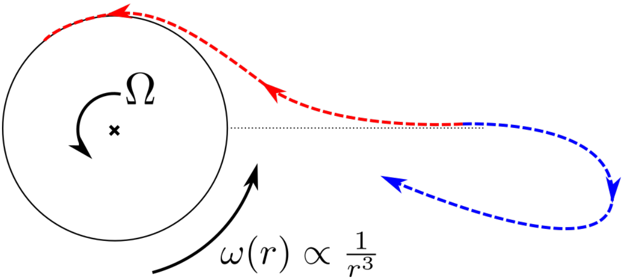


Figure 1: Frame-dragging effect over a massive rotating object on a free-falling body (red) and on a body that tries to move away from it (blue). Frame-dragging induces a force similar to the Coriolis–effect by „dragging" the compass of intertia in the direction of the star's rotation. Bodies falling towards the star's center are deflected in the direction of the rotation, while bodies moving away from it are deflected in the opposite direction.

Frame-dragging

Effect of rotation on metric tensor

- The metric tensor in case of a static, stationary star is described in the Schwarzschield metric with $G = c = 1$ contains only diagonal elements

$$g = d\tau^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 sin^2(\theta)\, d\varphi^2 \tag{1}$$

- Rotation however introduces off-diagonal elements too (where $\omega \neq 0$ is the angular velocity of local interial frames)

$$\begin{aligned} g = d\tau^2 = {} & e^{2\nu(r,\theta)} dt^2 - e^{2\lambda(r,\theta)} dr^2 \\ & - e^{2\mu(r,\theta)} \left[r^2 d\theta^2 + r^2 \sin^2(\theta) \cdot (d\varphi - \omega(r,\theta)\, dt)^2 \right] \end{aligned} \tag{2}$$

- Due to rotation the star deforms around the equator (centrifugal flattening) and loses spherical symmetry

Dimensional analysis of $\omega(r)$

- Since frame-dragging is caused by the rotation of the star, it will be proportional to the angular momentum. Since we're only interested in dimensions, we can write the classical formula for this:

$$J = I \cdot \Omega \propto MR^2\Omega \tag{3}$$

- Using CGS units to make dimensional analysis possible (since GR units won't make us any good obviously...), also using CGS, since astronomers likes it much better than SI for some reason

$$[G] = cm^3 \cdot g^{-1} \cdot s^{-2} = L^3 \cdot M^{-1} \cdot T^{-2}$$
$$[J] = g \cdot cm^2 \cdot s^{-1} = M \cdot L^2 \cdot T^{-1}$$
$$[\omega] = s^{-1} = T^{-1}$$

Dimensional analysis of $\omega(r)$

- Now substituting into (3) and expanding $\omega(r)$ we get the following:

$$\omega(r) \propto \frac{GJ^l}{c^m r^n} \propto L^{3+2l-m-n} T^{-2-l+m} M^{-1+l} \tag{4}$$

- Solving the system of equations for the exponents gives us the solution $l = 1$, $m = 2$ and $n = 3$:

$$\omega(r) \propto \frac{GJ}{c^2 r^3} \xRightarrow{G=1,\ c=1} \frac{J}{r^3} = \frac{I\Omega}{r^3}, \quad r \geq R \tag{5}$$

Solutions for all r values

Interior solution for J

- The centrifugal force acting on a fluid element depends on the coupled frequencies of rotating inertial frames and the rotation of the star:

$$\bar{\omega}(r) = \Omega - \omega(r) \tag{6}$$

- J. B. Hartle obtained a formula from the Einstein's equations

$$\frac{1}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r} + \frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\omega} = 0\right) \tag{7}$$

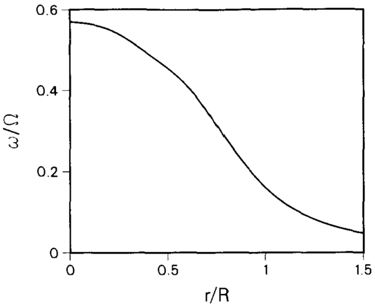
- ...that can be used to derive the formula for the angular momentum:

$$J = \frac{8\pi}{3}\int_0^R \mathrm{d}r\, r^4 \frac{\epsilon(r) + p(r)}{\sqrt{1 - 2M(r)/r}}\left[\Omega - \omega(r)\right] e^{-\nu(r)} \tag{8}$$

- From this, dimensional analysis can show the true relationship between frame-dragging and rotation of the star:

$$\omega(r) = \frac{2J}{r^3} = \frac{2I\Omega}{r^3}, \quad r \geq R \tag{9}$$

- From (7) and (9), $\omega(r)$ can be finally described for every r values.



Problems with previous solutions

- The equation (8) is only valid for static or slowly rotating objects.
- Rotation introduces θ dependence of ω.
- Different attributes of the system (change in angular momentum, change in centrifugal forces, changes in the shape of the star etc.) are all intertwined.

How to approach these problems?

All the phenomenons and quantities mentioned above are connected by I, the moment of inertia of the star, hence we're interested in the description of this quantity.

Solutions for relativistic stars

Derivation

- The z-component (the axis of rotation) of the angular momentum can be computed as

$$J = I\Omega = \int \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\sqrt{-\det g_{\alpha\beta}}\,T^{t}_{\varphi} \tag{10}$$

- Our goal is to determine the determinant g of the metric tensor $g_{\alpha\beta}$ and the mass-energy tensor T to be able to calculate the J angular momentum and the I moment of inertia. After some algebra we get the final form for J and subsequently for I:

$$J = I\Omega = -\int \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\frac{(p+\epsilon)\,(\Omega-\omega)\,r^{4}\sin^{3}(\theta)\,e^{\nu+\lambda+2\mu}}{e^{2(\nu-\mu)} - r^{2}\sin^{2}(\theta)\,(\Omega-\omega)^{2}} \tag{11}$$

Realistic nuclear matter equation of state (EoS)

- The goal of constructing an EoS is to describe the thermodynamical properties (mainly the energy density ϵ and the pressure p) of the nuclear matter that the NS consists of.
- The crust and core of the NS usually described using different models.
- There is a real abundance of models trying to describe NS in any way. In the modern NS literature, models based on relativistic mean-field theory became the most widely used ones.
- In Glendenning's Compact stars there are 17 different EoS mentioned and compared with each other (pp. 296–297, Table 6.3 and 6.4).

Main question

What are the real constraints on the EoS?