GRADUATION PROJECT REPORT

GOES TO GENERAL RECORD DATA

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1. Introduction

The ordered random variables play important roles in the theory and practice of statistics. They possess significant statistical properties. Over the last few decades, many articles on various topics of ordered statistical data have appeared. It was a special interest to coordinate and edit an interesting research problem based on material contributed by several important researchers from all over the world. In this study the ordering of random k^{th} record values with hazard function and Burr XII distribution function well be discussed. In beginning it will explain the stochastic ordering of random k^{th} record data, hazard function and burr xii distribution function.

Order statistics and record values appear in many statistical applications and are widely used in statistical modeling and inference. A form of the joint distribution of n ordered random variables is presented that enables a unified approach to a variety of models of ordered random variables, e.g. order statistics and record values. In particular, sequential order statistics are introduced as a modification of order statistics. Generalized order statistics, provide a suitable approach to explain similarities and analogies in the two models and to generalize related results. The definition of *random k^{th} record* can be shown as below.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with continuous distribution function $F(x) = P(X_1 \leq x)$. Denote by $X_{1,n} \leq \ldots \leq X_{n,n}$ the order statistics of $X_1, \ldots, X_n$. For a fixed integer $k \geq 1$, we define the corresponding k^{th} *record times*, $\{L(n, k), n \geq 1\}$, and k^{th} *record values* , $\{X(n, k), n \geq 1\}$, by setting

$L(1, k) = k, \quad L(n + 1, k) = \min\{j > L(n, k) : X_j > X_{j-k,j-1}\}$ for $n \geq 1$, and

$X(n, k) = X_{L(n,k)-k+1,L(n,k)}$ for $n \geq 1$.

Let N be a positive integer-valued random variable which is independent of the X_i. The random variables $X(N, k)$ are called the *random kth record*.

Now we will discuss with the hazard function. Hazard function is generally using when calculating the age of an electronic device or any material. The failure rate of a system usually depends on time, with the rate varying over the life cycle of the system. For example, an automobile's failure rate in its fifth year of service may be many times greater than its failure rate during its first year of service.

The hazard function can be defined as

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{R(t)}.$$

Many probability distributions can be used to model the failure distribution.

Burr distribution was first discussed by Burr (1942) as a two-parameter family. An additional scale parameter was introduced by Tadikamalla (1980). The Burr distribution can fit a wide range of empirical data. Different values of its parameters cover a broad set of skewness and kurtosis. Hence, it is used in various fields such as finance, hydrology, and reliability to model a variety of data types. Examples of data modeled by the Burr distribution are household income, crop prices, insurance risk, travel time, flood levels, and failure data.

Hazard functions of Burr type XII distribution is,

$$h(x|\alpha, c, k) = \frac{\frac{kc}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}}{1+\left(\frac{x}{\alpha}\right)^{c}}.$$

Kamps (1995) introduced the concept of generalized order statistics (gos) as a unified approach to order statistics, record values, and sequential order statistics. The gos are defined using quintile transformation based on the distribution function F.

Let X(1, n, m,k),X(r, n,m,k), k > 1, m is a real number, be gos based on absolutely continuous distribution function F with density function f. The joint density function of the above quantities is given by

$$\begin{aligned} &f^{X(1,n,m,k),\ldots,X(r,n,m,k)}(x_1,\ldots,x_r) \\ &= C_{r-1}\left(\prod_{i=1}^{r-1}[1-F(x_i)]^m f(x_i)\right)[1-F(x_r)]^{\gamma_r-1}f(x_r), \\ &F^{-1}(0+) < x_1 \le \cdots \le x_r < F^{-1}(1), \end{aligned} \tag{1.3}$$

where

$$\left.\begin{aligned} &\gamma_r = k + (n-r)(m+1) > 0, \\ &C_{r-1} = \prod_{j=1}^{r}\gamma_j, \quad r = 1,2,\ldots,n, \ \ \gamma_n = k. \end{aligned}\right\} \tag{1.4}$$

with $n \in N$, $k > 0$ and $m \in R$. For more details of gos, see Kamps (1995). In the case $m = 0$ and $k = 1$, $X(r, n, m, k)$ reduces to the ordinary r-th order statistics and (1.3) is the joint pdf of r ordinary order statistics, $X1{:}n \le X2{:}n \le \cdots \le Xr{:}n$. For various distributional properties of ordinary order statistics, see David (1981) and Arnold et al. (1992). If $m = -1$ and $k = 1$, then (1.3) is the joint pdf of r upper record values. For some distributional properties of record values, see Ahsanullah (1995)

and Arnold et al. (1998).

Distribution properties of gos from a uniform distribution are given by Ahsanullah (1996). He obtained the minimum variance linear unbiased estimators of the parameters of the two parameters of uniform distribution based on the first m gos Kamps (1996) characterized the uniform distribution based on distribution properties of subranges of gos. Kamps and Gather (1997) characterized the exponential distributions by distributional properties of gos Cramer and Kamps (1996, 1998, 2001) studied some estimation problems with different sequential k-out-of-n systems. Ahsanullah (2000) gave some distributional properties of the gos from the two parameter exponential distribution. He also obtained the minimum variance linear unbiased estimators of the two parameters and characterized the exponential distribution based on gos. Cramer and Kamps (2000) derived relations for expectations of functions of gos from a class of distributions which includes the exponential, uniform, Pareto, Lomax and, Pearson I. Habibullah and Ahsanullah (2000) obtained estimates for the parameters of the Pareto distribution based on gos. Kamps and Cramer (2001) studied some distribution properties of the gos from the Pareto, power and Weibull distributions. Jaheen (2002) considered the prediction of future gos from a general class of distributions which includes the Weibull, compound Weibull, Burr type XII, Pareto, beta, and Gompertz by using Bayesian two-sample prediction technique.

2. Maximum Likelihood Estimation

Suppose that $X(1, n, m, k), \ldots, X(r, n, m, k)$, $k > 0$ and m e R be a generalized ordered random sample of size r drawn from the BurrXII(a, b) population whose pdf is given by (1.1). The likelihood function (LF) may be obtained from (1.1), (1.2), and (1.3), and written as

$$L(a, b; \underline{x}) = C_{r-1} a^r b^r v(a; \underline{x}) \exp[-bH(a; \underline{x})], \tag{2.1}$$

where $\underline{x} = (x_1, \ldots, x_r)$,

$$\left.\begin{aligned} v(a; \underline{x}) &= \prod_{i=1}^{r} \left(\frac{x_i^{a-1}}{1 + x_i^a} \right), \\ H(a; \underline{x}) &= (m+1) \sum_{i=1}^{r-1} \ln(1 + x_i^a) + \gamma_r \ln(1 + x_r^a), \end{aligned}\right\} \tag{2.2}$$

and γ_r is as given by (1.4).

Assuming that the parameter a is known, the Maximum likelihood (ML) estimate of the parameter b can be shown to be of the form

$$\hat{b}_{ML} = \frac{r}{H(a; \underline{x})}, \tag{2.3}$$

where $H(a; \mathrm{x})$ is as given in (2.2). In this case, the Burr type XII distribution is a particular case of the set-up considered in Cramer and Kamps (1996).

When the two parameters a and b are unknown, the likelihood equation for a can be written as

$$\frac{r}{a} + \sum_{i=1}^{r} \omega_i - b[(m+1) \sum_{i=1}^{r-1} x_i^a \omega_i + \gamma_r x_r^a \omega_r] = 0, \tag{2.4}$$

where, for $i = 1, 2, \ldots, r$,

$$\omega_i \equiv \omega(a; x_i) = \frac{\ln x_i}{1 + x_i^a}. \tag{2.5}$$

Substituting the value of b given by (2.3) in (2.4), yields a nonlinear equation in a, by solving it numerically we obtain the ML estimate of the parameter a. Then, substitute the ML estimate of a in (2.3), we obtain the ML estimate of b.

3. Bayes Estimation

In the following, Bayesian estimation for the parameters of the Burr type XII distribution is considered for two cases, the first is when the parameter a is known and the second is when the two parameters a and b are assumed to be unknown.

3.1. *One Parameter Case (a is Known)*

When the parameter a is assumed to be known, we use the gamma conjugate prior density for the parameter b, that was first used by Papadopoulos (1978) and AL-Hussaini et al. (1992), when a was assumed known, in the following form

$$g(b) = \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} b^{\alpha} e^{-b\beta}, \quad b > 0, \ (\alpha > 0, \beta > 0). \tag{3.1}$$

It follows, from (2.1) and (3.1), that the posterior density of b is then

$$q_1(b \mid \underline{x}, a) = \frac{[\beta + H(a; \underline{x})]^{r+\alpha+1}}{\Gamma(r+\alpha+1)} b^{r+\alpha} \exp[-b(\beta + H(a; \underline{x})]. \tag{3.2}$$

Under a squared error loss function, the Bayes estimate of the parameter b is the posterior mean in the form

$$\hat{b}_B = \frac{r+\alpha+1}{\beta + H(a; \underline{x})}. \tag{3.3}$$

3.2. *Two Parameter Case (a and b are Unknown)*

When both of the two parameters a and b are unknown, AL-Hussaini and Jaheen (-992) suggested a bivariate prior density function, given by

$$g(a, b) = g_1(b \mid a) g_2(b), \tag{3.4}$$

where

$$g_1(b \mid a) = \frac{a^{\alpha+1}}{\Gamma(\alpha+1)\beta^{\alpha+1}} b^{\alpha} e^{-ab/\beta}, \quad b > 0, \ (\alpha > -1, \beta > 0), \tag{3.5}$$

is the gamma conjugate prior, that was first used by Papadopoulos (1978), when a was assumed known, and

$$g_2(a) = \frac{1}{\Gamma(\delta)\gamma^{\delta}} a^{\delta-1} e^{-a/\gamma}, \quad a > 0, \ (\gamma > 0, \delta > 0), \tag{3.6}$$

is the gamma density function.

Multiplying *g*$_1$*(b | a)* by *g*$_2$*(a)*, we obtain the bivariate density of a and *b*, given from (3.4), by

$$g(a, b) = A_1 a^{\alpha+\delta} b^{\alpha} \exp\left[-a \left\{ \frac{1}{\gamma} + \frac{b}{\beta} \right\} \right], \quad a > 0, \ b > 0, \tag{3.7}$$

where a > -1, *f*, y, and *5* are positive real numbers and A^{-1} = r(£)T(a + *1)y*^{5}*f*^{a} *+*^{1}.

The four-parameter gamma-gamma prior (3.7) is so chosen that it would be rich enough to cover the prior belief of the experimenter (AL-Hussaini and Jaheen, 1992). It follows, from (2.1) and (3.7), that the joint posterior density function of *a* and b given the data is thus

$$q_2(a, b \mid \underline{x}) = A_2 a^{r+\alpha+\delta} b^{r+\alpha} e^{-a/\gamma} \upsilon(a; \underline{x}) \times \exp\left[-b \left(H(a; \underline{x}) + \frac{a}{\beta} \right) \right], \quad a > 0, \ b > 0, \tag{3.8}$$

where

$$A_2^{-1} = \Gamma(r + \alpha + 1) I_0(\underline{x}), \tag{3.9}$$

and

$$I_0(\underline{x}) = \int_0^{\infty} a^{r+\alpha+\delta} e^{-a/\gamma} \upsilon(a; \underline{x}) \left(H(a; \underline{x}) + \frac{a}{\beta} \right)^{-(r+\alpha+1)} da. \tag{3.10}$$

It is well known that, under squared error loss function, the Bayes estimator of a function, say *U(a, b)*, is the posterior mean of the function and is given by a ratio of two integrals which may be written as

$$E[U(a,b)\mid \underline{x}] = \int_0^\infty \int_0^\infty U(a,b)q_2(a,b\mid \underline{x})\,da\,db$$
$$= \frac{\int_0^\infty \int_0^\infty U(a,b)L(a,b;\underline{x})g(a,b)\,da\,db}{\int_0^\infty \int_0^\infty L(a,b;\underline{x})g(a,b)\,da\,db}. \tag{3.11}$$

Generally, the ratio of two integrals (3.11) cannot be obtained in a simple closed form. Numerical methods of integration may be used in this case, which can be computationally intensive, especially in high dimensional parameter space. Instead, we can use the approximation form due to Lindley (1980). In the following, a review of Lindley's approximation form is given.

3.2.1. The Approximation form of Lindley. Lindley (1980) developed approximate procedures for the evaluation of the ratio of two integrals in the form

$$\int_\Theta \omega(\lambda)e^{L(\lambda)}d\lambda \Big/ \int_\Theta g(\lambda)e^{L(\lambda)}d\lambda, \tag{3.12}$$

where $k = (k_1,...,k_N)$, $L(k)$ is the logarithm of the likelihood function, g(A) and $m(k) = U(k)g(k)$ are arbitrary functions of k. From Eq. (3.12), the posterior expectation of the function $U(k)$, for given x, is

$$E[U(\lambda)\mid \underline{x}] = \frac{\int U(\lambda)e^{Q(\lambda)}d\lambda}{\int e^{Q(\lambda)}d\lambda}, \tag{3.13}$$

where $Q(k) = L(k) + p(k)$ is the logarithm of the posterior distribution of k except for the normalizing constant and $p(k) = \ln g(k)$. Expanding $Q(k)$ in (3.13) into a Taylor series expansion about the posterior mode of k, Lindley obtained the required expression for $E[U(k) \mid x]$. For more details, see Lindley (1980).

In this article, we consider Lindley's approximation form expanding about the posterior mode. For the two parameter case $k = (k_1, k_2)$, Lindley's approximation leads to

$$\widehat{U}_B = U(\lambda) + \frac{1}{2}\left[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}\right], \tag{3.14}$$

where $B = \sum_{i=1}^{2}\sum_{j=1}^{2} U_{ij}\tau_{ij}$, $Q_{\eta\xi} = \frac{\partial^{\eta+\xi}Q}{\partial^{\eta}\lambda_1\partial^{\xi}\lambda_2}$, $\eta, \xi = 0, 1, 2, 3$, $\eta + \xi = 3$, for $i, j = 1, 2$, $U_i = \frac{\partial U}{\partial \lambda i}$, $U_{ij} = \frac{\partial^2 U}{\partial \lambda i \partial \lambda j}$, and for $i \neq j$,

$$B_{ij} = (U_i\tau_{ii} + U_j\tau_{ij})\tau_{ii}, \quad C_{ij} = 3U_i\tau_{ii}\tau_{ij} + U_j(\tau_{ii}\tau_{jj} + 2\tau_{ij}^2),$$

τ_{ij} is the $(i, j)th$ element in the inverse of the matrix $Q^* = (-Q^*_{ij})$, $i, j = 1, 2$ such that $Q^*_{ij} = \frac{\partial^2 Q}{\partial\lambda_i\partial\lambda_j}$. Expansion (3.14) is to be evaluated at $(\hat{\lambda}_1, \hat{\lambda}_2)$, the mode of the posterior density.

In our case, $(\lambda_1, \lambda_2) \equiv (a, b)$ and Q is then given by

$$Q = \ln q_2 \propto (r + \alpha + \delta)\ln a + (r + \alpha)\ln b - \frac{a}{\gamma} + \ln v(a; \underset{\sim}{x}) - b\left[H + \frac{a}{\beta}\right], \tag{3.15}$$

where $H \equiv H(a; \underset{\sim}{x})$ is as given in (2.2).

The joint posterior mode, denoted by (a,b), is obtained from (3.15) and is given by

$$\tilde{b} = \frac{\beta(r + \alpha)}{a + \beta H}, \tag{3.16}$$

where a is the solution of the following nonlinear equation

$$\frac{r + \alpha + \delta}{a} - \frac{1}{\gamma} + \sum_{i=1}^{r}\omega_i - \frac{\beta(r + \alpha)B_1}{a + \beta H} = 0, \tag{3.17}$$

where

$$B_1 \equiv B_1(a; \underset{\sim}{x}) = \frac{1}{\beta} + (m + 1)\sum_{i=1}^{r-1} x_i^a\omega_i + \gamma_r x_r^a\omega_r,$$

and ω_i is as given by (2.5).

The τ_{ij} elements of the inverse of the matrix $Q^* = (-Q^*_{ij}),\ i, j = 1, 2$ are given by

$$\left.\begin{aligned} \tau_{11} &= \frac{a^2(r+\alpha)}{D}, \\ \tau_{12} = \tau_{21} &= -\frac{a^2 b^2 B_1}{D}, \\ \tau_{22} &= \frac{b^2[r+\alpha+\delta+a^2 B_2]}{D}, \end{aligned}\right\} \tag{3.18}$$

where

$$B_2 \equiv B_2(a; \underset{\sim}{x}) = \sum_{i=1}^{r} x_i^a \omega_i^2 + bB_3,$$

$$B_3 \equiv B_3(a; \underset{\sim}{x}) = (m+1)\sum_{i=1}^{r-1} x_i^a \omega_i^2 + \gamma_r x_r^a \omega_r^2,$$

$$D = (r+\alpha)[r+\alpha+\delta+a^2 B_2] - [abB_1]^2.$$

Furthermore,

$$Q_{12} = 0, \quad Q_{21} = -B_3, \quad Q_{30} = \frac{2(r+\alpha+\delta)}{a^3} - B_4, \quad Q_{03} = \frac{2(r+\alpha)}{b^3},$$

where

$$B_4 \equiv B_4(a; \underset{\sim}{x}) = \sum_{i=1}^{r} x_i^a(1-x_i^a)\omega_i^3 + (m+1)\sum_{i=1}^{r-1} x_i^a(1-x_i^a)\omega_i^3 + \gamma_r x_r^a(1-x_r^a)\omega_r^3.$$

Substituting the above values in (3.14) yields the Bayes estimate of a function U = U(a, b), of the unknown parameters a and b, given by:

$$\widehat{U}_B = E[U(a,b) \mid \underset{\sim}{x}] = U + \frac{W}{2D} + \frac{1}{2D^2}[aW_1 U_1 + bW_2 U_2], \tag{3.19}$$

where

$$\left.\begin{aligned}
W &= a^2[(r+\alpha)U_{11} - b^2B_1(U_{21}+U_{12})] \\
&\quad + b^2(r+\alpha+\delta+a^2B_2)U_{22}, \\
W_1 &= (r+\alpha)\left[(r+\alpha)\left\{2(r+\alpha+\delta-a^3B_4)\right\}\right. \\
&\quad \left. + 3a^3b^2B_1B_3 - 2abB_1(r+\alpha+\delta+a^2B_2)\right], \\
W_2 &= \left[2(r+\alpha)\left\{(r+\alpha+\delta+a^2B_2)^2\right.\right. \\
&\quad \left. - abB_1(r+\alpha+\delta-a^3B_4)\right\} + a^2bB_3 \\
&\quad \left. \times \left\{(r+\alpha)(r+\alpha+\delta+a^2B_2) + 2(abB_1)^2\right\}\right].
\end{aligned}\right\} \tag{3.20}$$

All functions in the right-hand side of (3.19) are to be evaluated at the posterior mode *(a*, b). Now, the Bayes estimates of a and *b* are computed as follows:

(i) If *U(a*, b) = *a*, we have from (3.19),

$$\hat{a}_{BG} = a\left[1 + \frac{W_1}{2D^2}\right], \tag{3.21}$$

evaluated at the posterior mode (**a, b**).

(ii) If U(a, b) = b, we have from (3.19),

$$\hat{b}_{BG} = b\left[1 + \frac{W_2}{2D^2}\right], \tag{3.22}$$

evaluated at the posterior mode $(\tilde{a}, \tilde{b})$.

Resources

http://www.sciencedirect.com/science/article/pii/037837589400147N

http://www.itl.nist.gov/div898/handbook/apr/section1/apr123.htm

https://en.wikipedia.org/wiki/Stochastic_ordering

http://www.mathworks.com/help/stats/burr-type-xii-distribution.html?requestedDomain=www.mathworks.com

https://en.wikipedia.org/wiki/Burr_distribution https://en.wikipedia.org/wiki/Failure_rate#hazard_function

https://books.google.com.tr/books?id=3c7IZjbGVdIC&pg=PA88&lpg=PA88&dq=ordering+of+random+k+th+record+value&source=bl&ots=nTxf_RR4Ax&sig=nlLbL7SMQlbChe081E_oKKClhWE&hl=tr&sa=X&ved=0ahUKEwiBucbN_ZjMAhVhIJoKHSC0BPAQ6AEIOTAD#v=onepage&q=ordering%20of%20random%20k%20th%20record%20value&f=false