Estimation of Burr XII distribution based on generalized order statistics

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Abstract

The ordered random variables play important roles in the theory and practice of statistics. They possess significant statistical properties. Over the last few decades, many articles on various topics of ordered statistical data have appeared. It was a special interest to coordinate and edit an interesting research problem based on material contributed by several important researchers from all over the world. In this study the estimation of parameters with different methods using the kth record values from Burr XII distribution will be discussed and based on different estimation hazard function will be obtained. At the beginning we will give brief definition of kth record data, hazard function and Burr XII distribution function.

1 Order Statistics and Record Values

Order statistics and record values appear in many statistical applications and are widely used in statistical modeling and inference. A form of the joint distribution of n ordered random variables is presented that enables a unified approach to a variety of models of ordered random variables, e.g. order statistics and record values. Generalized order statistics, provide a suitable approach to explain similarities and analogies in the two models and to generalize related results.

The definition of random

The log likelihood function $l(\alpha,\beta|\mathbf{x}) = \log L(\alpha,\beta|\mathbf{x})$, dropping terms that do not involve α and β, is

$$l(\alpha,\beta|\mathbf{X}=\mathbf{x}) = nln\alpha + nln\beta + (\alpha-1)\sum_{i=1}^{n-1} lnx_i - (m\beta+\beta+1)\sum_{i=1}^{n-1} ln(1+x_i^{\alpha})$$
$$+(\alpha-1)lnx_n - (k\beta+1)ln(1+x_n^{\alpha}). \tag{1.1}$$

We assume that the parameters α and β are unknown. To obtain the normal equations for the unknown parameters, we differentiate (22) partially with respect to α and β and equate to zero, the resulting equations are

$$0=\frac{\partial l(\alpha,\beta|\mathbf{x})}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}lnx_i-\sum_{i=1}^{n}x_i^{\alpha}\upsilon_i-\beta\left[(1+m)\sum_{i=1}^{n-1}x_i^{\alpha}\upsilon_i+kx_n^{\alpha}\upsilon_n\right], \tag{1.2}$$

and

$$0=\frac{\partial l(\alpha,\beta|\mathbf{x})}{\partial\beta}=\frac{n}{\beta}+(m+1)\sum_{i=1}^{n-1}lnx_i^{\alpha}\delta_i+k\ lnx_n^{\alpha}\delta_n, \tag{1.3}$$

where $\delta_i=\frac{x_i^{\alpha}}{1+x_i^{\alpha}}$ and $\upsilon_i=\frac{lnx_i}{1+x_i^{\alpha}}$.
The solutions of the above equations are the maximum likelihood estimators of the Burr XII (α,β) parameters α and β, denoted $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$, respectively. As the equations expressed in (23) and (24) cannot be solved analytically, one must use a numerical procedure to solve them.

2 Bayesian estimation

In this section we consider Bayesian estimation of the unknown parameters of the Burr XII (α,β) under squared error loss function (SEL). It is assumed that α and β has the independent gamma prior distributions with probability density functions

$$h(\alpha)\propto\alpha^{a-1}e^{-b\alpha},\quad \alpha>0 \tag{2.1}$$

and

$$h(\beta)\propto\beta^{c-1}e^{-d\beta},\quad \beta>0. \tag{2.2}$$

The hyper-parameters a, b, c, and d are known and non-negative. If both the parameters α and β are unknown, joint conjugate priors do not exist. It is not unreasonable to assume independent gamma priors on the shape and scale parameters for a two-parameter Burr XII (α,β), because gamma distributions are very flexible, and the Jeffrey's (non-informative) prior, introduced by Jeffrey (1946) is a special case of this. The joint prior distribution in this case is

$$h(\alpha,\beta)\propto\alpha^{a-1}e^{-b\alpha}\beta^{c-1}e^{-d\beta},\quad \alpha,\ \beta>0. \tag{2.3}$$

Combining (27) with (21) and using Bayes theorem, the joint posterior distribution is derived as

$$\pi(\alpha,\beta|\mathbf{x})\propto\alpha^{n+a-1}\beta^{n+c-1}e^{-b\alpha-d\beta}\left(\prod_{i=1}^{n-1}\frac{x_i^{\alpha-1}}{(1+x_i^{\alpha})^{m\beta+\beta+1}}\right)\frac{x_n^{\alpha-1}}{(1+x_n^{\alpha})^{k\beta+1}}. \tag{2.4}$$

Bayes estimator of any function of α and β, say $g(\alpha,\beta)$ under the SE loss function is its posterior mean. Therefore, the Bayes estimator of $g(\alpha,\beta)$ under the SE loss function is

$$\widehat{g}_{BS}=E_{\alpha,\beta|\mathbf{r},\mathbf{k}}(g(\alpha,\beta))=\frac{\int_0^{\infty}\int_0^{\infty}g(\alpha,\beta)L(\alpha,\beta;\mathbf{r},\mathbf{k})\pi(\alpha,\beta)d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty}L(\alpha,\beta;\mathbf{r},\mathbf{k})\pi(\alpha,\beta)d\alpha d\beta}. \tag{2.5}$$

It is not possible to compute Equation (2.5) analytically. Two approaches are suggested here to approximate Equation (2.5), namely (i) Lindley's approximation and (ii) MCMC method.

(i) Lindley's approximation Lindley (1980) proposed a method to approximate the ratio of integrals such as Equation (2.5). For the two parameter case (α, β), the Lindley's approximation can be written as

$$\widehat{g}_{Lind}(\alpha, \beta) = g(\widetilde{\alpha}, \widetilde{\beta}) + \frac{1}{2}\left[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}\right], \tag{2.6}$$

where $B = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\tau_{ij}$, $Q_{ij} = \partial^{i+j}Q/\partial^{i}\alpha\partial^{j}\beta$, for $i,j = 0,1,2,3$ and $i+j = 3$, $g_1 = \partial g/\partial\alpha$, $g_2 = \partial g/\partial\beta$, $g_{ij} = \partial^2 g/\partial\alpha^i\partial\beta^j$ for $i,j = 1,2$ and $B_{ij} = (g_i\tau_{ii} + g_j\tau_{ij})\tau_{ii}$ and $C_{ij} = 3g_i\tau_{ii}\tau_{ij} + g_j(\tau_{ii}\tau_{ij} + 2\tau_{ij}^2)$ for $i \neq j$, where τ_{ij} is the (i,j)th element in the inverse of the matrix $Q^* = (-Q^*_{ij})$, $i,j = 1,2$ such that $Q^*_{ij} = \partial^2 Q/\partial\alpha^i\partial\beta^j$, Q is the logarithm of the posterior density function, dropping terms that do not involve α and β, $(\widetilde{\alpha}, \widetilde{\beta})$ is the joint posterior mode of Q and $\widehat{g}_{Lind}(\alpha, \beta)$ is evaluated at $(\widetilde{\alpha}, \widetilde{\beta})$.

For our case, we have from Equation (28)

$$\begin{aligned} Q =& (n+a-1)\ln\alpha + (n+c-1)\ln\beta - (\alpha-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\ln(1+x_i) \\ & -\beta\big[d + (m+1)\sum_{i=1}^{n-1}\ln(1+x_i)^{\alpha} + k\ln(1+x_n^{\alpha})\big] - b\alpha \end{aligned} \tag{2.7}$$

The joint posterior mode is obtained from system equations $\partial Q/\partial\alpha = 0$ and $\partial Q/\partial\beta = 0$. Therefore,

$$\widetilde{\beta} = \frac{n+c-1}{d + \big[(m+1)\sum\limits_{i=1}^{n-1}\ln(1+x_i^{\widetilde{\alpha}}) + k\ln(1+x_n^{\widetilde{\alpha}})\big]}, \tag{2.8}$$

and $\widetilde{\alpha}$ is the solution of the following nonlinear equation

$$\frac{n+a-1}{\alpha} - \sum_{i=1}^{n}\frac{\ln(x_i)}{1+x_i^{\alpha}} - \beta\big[(m+1)\sum_{i=1}^{n-1}\frac{x_i^{\alpha}\ln(x_i)}{1+x_i^{\alpha}} + \frac{kx_n^{\alpha}\ln(x_n)}{1+x_n^{\alpha}}\big] - b = 0 \tag{2.9}$$

It can be solved by using the same procedure in Equations (1.2) and (1.3). The elements of the Q^* are given by

$$Q^*_{11} = -\frac{(n+a-1)}{\alpha^2} - \sum_{i=1}^{n}x_i^{\alpha}\big(\frac{\ln(x_i)}{1+x_i^{\alpha}}\big)^2 - \beta(m+1)\sum_{i=1}^{n-1}x_i^{\alpha}\big(\frac{\ln(x_i)}{1+x_i^{\alpha}}\big)^2 - \beta kx_n^{\alpha}\big(\frac{\ln(x_n)}{1+x_n^{\alpha}}\big)^2 \tag{2.10}$$

$$Q^*_{12} = Q^*_{21} = -(m+1)\sum_{i=1}^{n-1}x_i^{\alpha}\frac{\ln(x_i)}{1+x_i^{\alpha}} - kx_n^{\alpha}\frac{\ln(x_n)}{1+x_n^{\alpha}} \tag{2.11}$$

$$Q_{22}^{*}=-\frac{(n+c-1)}{\beta^{2}} \tag{2.12}$$

and $\tau_{ij},\ i,j=1,2$ are obtained by using Equation (2.10)-(2.12). Moreover, we have

$$Q_{12}=0,\ Q_{21}=-(m+1)\sum_{i=1}^{n-1}x_i^{\alpha}(\frac{\ln(x_i)}{1+x_i^{\alpha}})^2-kx_n^{\alpha}(\frac{\ln(x_n)}{1+x_n^{\alpha}})^2,\ Q_{03}=\frac{2\,(n+c-1))}{\beta^{3}},$$

$$Q_{30}=\frac{2\,(n+a-1)}{\alpha^{3}}-\sum_{i=1}^{n}x_i^{\alpha}(1-x_i^{\alpha})\left(\frac{\ln x_i}{1+x_i^{\alpha}}\right)^3-\beta(m+1)\sum_{i=1}^{n-1}x_i^{\alpha}(1-x_i^{\alpha})\left(\frac{\ln x_i}{1+x_i^{\alpha}}\right)^3$$
$$-\beta kX_n^{\alpha}(1-x_n^{\alpha})\left(\frac{\ln x_n}{1+x_n^{\alpha}}\right)^3$$

Therefore, the approximate Bayes estimators of α and β under the SE loss function is obtained as

$$\widehat{\alpha}_{Lind}=\widetilde{\alpha}+\frac{1}{2}\left[Q_{30}\tau_{11}^{2}+3Q_{21}\tau_{11}\tau_{12}+Q_{03}\tau_{21}\tau_{22}\right], \tag{2.13}$$

$$\widehat{\beta}_{Lind}=\widetilde{\beta}+\frac{1}{2}\left[Q_{30}\tau_{11}\tau_{12}+Q_{21}(\tau_{11}\tau_{22}+2\tau_{12}^{2})+Q_{03}\tau_{22}^{2}\right]. \tag{2.14}$$

Notice that all approximate Bayes estimators are evaluated at $(\widetilde{\alpha},\widetilde{\beta})$.