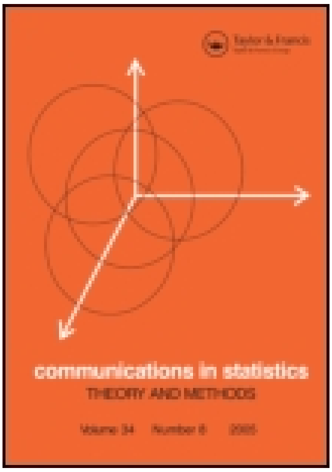

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Estimation Based on Generalized Order Statistics from the Burr Model

ZEINHUM F. JAHEEN [a]
[a] Department of Mathematics , Assiut University , Egypt
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ZEINHUM F. JAHEEN

Department of Mathematics, Assiut University, Egypt

The concept of generalized order statistics was introduced by Kamps (1995) to unify several concepts that have been used in statistics such as order statistics, record values, and sequential order statistics. Estimation of the parameters of the Burr type XII distribution are obtained based on generalized order statistics. The maximum likelihood and Bayes methods of estimation are used for this purposes. The Bayes estimates are derived by using the approximation form of Lindley (1980). Estimation based on upper records from the Burr model is obtained and compared by using Monte Carlo simulation study. Our results are specialized to the results of AL-Hussaini and Jaheen (1992) which are based on ordinary order statistics.

Keywords Bayes estimation; Burr model; Generalized order statistics; Monte Carlo simulation; Record values.

Mathematics Subject Classification Primary 62F10; Secondary 62F15.

1. Introduction

As a member of the Burr (1942) family of distributions, which includes 12 types of cumulative distribution functions with a variety of density shapes, the two-parameter Burr type XII (denoted by BurrXII(a, b)) distribution has a probability density function (pdf) of the form

$$f(x) = abx^{a-1}(1 + x^a)^{-(b+1)}, \quad x > 0, \ (a > 0, b > 0), \tag{1.1}$$

and a cumulative distribution function (cdf)

$$F(x) = 1 - (1 + x^a)^{-b}, \quad x > 0. \tag{1.2}$$

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Address correspondence to Zeinhum F. Jaheen, Department of Mathematics, Assiut University, Egypt; E-mail: zjaheen@hotmail.com

The Burr type XII distribution has been proposed as a lifetime model, and its properties have been studied by Burr and Cislak (1968), Rodriguez (1977), Tadikamalla (1980), and Lewis (1981), among others.

Bayesian inferences based on the Burr type XII distribution and some of its testing measures have been discussed by several authors. Papadopoulos (1978) obtained Bayes estimation for the parameter b and the reliability function when a is known based on Type II censored samples. Evans and Ragab (1983) considered Bayesian estimation based on a discrete prior for the two unknown parameters a and b. AL-Hussaini and Jaheen (1992, 1994) considered estimation of the parameters, reliability, and failure rate functions of the model based on Type-II censored samples from a Bayesian approach. Shah and Gokhale (1993) used the maximum likelihood and maximum product of spacings methods to estimate the parameters of the BurrXII(a, b) distribution and compared the estimates for small and large samples. Ali Mousa (1995) considered empirical Bayes estimation for one of the two shape parameters and the reliability function of the Burr type XII distribution based on Type-II censored data obtained from an accelerated life test. Ali Mousa and Jaheen (2002) considered Bayesian estimation of the parameters of the Burr distribution based on progressively censored samples.

Kamps (1995) introduced the concept of generalized order statistics (gos) as a unified approach to order statistics, record values, and sequential order statistics. The gos are defined using quintile transformation based on the distribution function F.

Let $X(1, n, m, k), \ldots, X(r, n, m, k)$, $k \geq 1$, m is a real number, be gos based on absolutely continuous distribution function F with density function f. The joint density function of the above quantities is given by

$$\begin{aligned} &f^{X(1,n,m,k),\ldots,X(r,n,m,k)}(x_1, \ldots, x_r) \\ &\quad = C_{r-1}\left(\prod_{i=1}^{r-1}[1 - F(x_i)]^m f(x_i)\right)[1 - F(x_r)]^{\gamma_r - 1} f(x_r), \\ &\quad F^{-1}(0+) < x_1 \leq \cdots \leq x_r < F^{-1}(1), \end{aligned} \tag{1.3}$$

where

$$\left.\begin{aligned} &\gamma_r = k + (n - r)(m + 1) > 0, \\ &C_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad r = 1, 2, \ldots, n, \ \gamma_n = k. \end{aligned}\right\} \tag{1.4}$$

with $n \in N$, $k > 0$ and $m \in R$. For more details of gos, see Kamps (1995). In the case $m = 0$ and $k = 1$, $X(r, n, m, k)$ reduces to the ordinary r-th order statistics and (1.3) is the joint pdf of r ordinary order statistics, $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$. For various distributional properties of ordinary order statistics, see David (1981) and Arnold et al. (1992). If $m = -1$ and $k = 1$, then (1.3) is the joint pdf of r upper record values. For some distributional properties of record values, see Ahsanullah (1995) and Arnold et al. (1998).

Distribution properties of gos from a uniform distribution are given by Ahsanullah (1996). He obtained the minimum variance linear unbiased estimators of the parameters of the two parameters of uniform distribution based on the first m gos Kamps (1996) characterized the uniform distribution based on distribution properties of subranges of gos. Kamps and Gather (1997) characterized the exponential distributions by distributional properties of gos Cramer and Kamps (1996, 1998, 2001) studied some estimation problems with different sequential k-out-of-n systems. Ahsanullah (2000) gave some distributional properties of the gos from the two parameter exponential distribution. He also obtained the minimum variance linear unbiased estimators of the two parameters and characterized the exponential distribution based on gos. Cramer and Kamps (2000) derived relations for expectations of functions of gos from a class of distributions which includes the exponential, uniform, Pareto, Lomax and, Pearson I. Habibullah and Ahsanullah (2000) obtained estimates for the parameters of the Pareto distribution based on gos. Kamps and Cramer (2001) studied some distribution properties of the gos from the Pareto, power and Weibull distributions. Jaheen (2002) considered the prediction of future gos from a general class of distributions which includes the Weibull, compound Weibull, Burr type XII, Pareto, beta, and Gompertz by using Bayesian two-sample prediction technique.

2. Maximum Likelihood Estimation

Suppose that $X(1, n, m, k), \ldots, X(r, n, m, k)$, $k > 0$ and $m \in R$ be a generalized ordered random sample of size r drawn from the BurrXII(a, b) population whose pdf is given by (1.1). The likelihood function (LF) may be obtained from (1.1), (1.2), and (1.3), and written as

$$L(a, b; \underline{x}) = C_{r-1} a^r b^r v(a; \underline{x}) \exp[-bH(a; \underline{x})], \tag{2.1}$$

where $\underline{x} = (x_1, \ldots, x_r)$,

$$\left.\begin{aligned} v(a; \underline{x}) &= \prod_{i=1}^{r} \left(\frac{x_i^{a-1}}{1 + x_i^a} \right), \\ H(a; \underline{x}) &= (m+1) \sum_{i=1}^{r-1} \ln(1 + x_i^a) + \gamma_r \ln(1 + x_r^a), \end{aligned}\right\} \tag{2.2}$$

and γ_r is as given by (1.4).

Assuming that the parameter a is known, the Maximum likelihood (ML) estimate of the parameter b can be shown to be of the form

$$\hat{b}_{ML} = \frac{r}{H(a; \underline{x})}, \tag{2.3}$$

where $H(a; \underline{x})$ is as given in (2.2). In this case, the Burr type XII distribution is a particular case of the set-up considered in Cramer and Kamps (1996).

When the two parameters a and b are unknown, the likelihood equation for a can be written as

$$\frac{r}{a} + \sum_{i=1}^{r} \omega_i - b[(m+1) \sum_{i=1}^{r-1} x_i^a \omega_i + \gamma_r x_r^a \omega_r] = 0, \tag{2.4}$$

where, for $i = 1, 2, \ldots, r$,

$$\omega_i \equiv \omega(a; x_i) = \frac{\ln x_i}{1 + x_i^a}. \tag{2.5}$$

Substituting the value of b given by (2.3) in (2.4), yields a nonlinear equation in a, by solving it numerically we obtain the ML estimate of the parameter a. Then, substitute the ML estimate of a in (2.3), we obtain the ML estimate of b.

3. Bayes Estimation

In the following, Bayesian estimation for the parameters of the Burr type XII distribution is considered for two cases, the first is when the parameter a is known and the second is when the two parameters a and b are assumed to be unknown.

3.1. *One Parameter Case* (*a is Known*)

When the parameter a is assumed to be known, we use the gamma conjugate prior density for the parameter b, that was first used by Papadopoulos (1978) and AL-Hussaini et al. (1992), when a was assumed known, in the following form

$$g(b) = \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} b^{\alpha} e^{-b\beta}, \quad b > 0, \ (\alpha > 0, \beta > 0). \tag{3.1}$$

It follows, from (2.1) and (3.1), that the posterior density of b is then

$$q_1(b \mid \underline{x}, a) = \frac{[\beta + H(a; \underline{x})]^{r+\alpha+1}}{\Gamma(r+\alpha+1)} b^{r+\alpha} \exp[-b(\beta + H(a; \underline{x})]. \tag{3.2}$$

Under a squared error loss function, the Bayes estimate of the parameter b is the posterior mean in the form

$$\hat{b}_B = \frac{r+\alpha+1}{\beta + H(a; \underline{x})}. \tag{3.3}$$

3.2. *Two Parameter Case* (*a and b are Unknown*)

When both of the two parameters a and b are unknown, AL-Hussaini and Jaheen (1992) suggested a bivariate prior density function, given by

$$g(a, b) = g_1(b \mid a) g_2(b), \tag{3.4}$$

where

$$g_1(b \mid a) = \frac{a^{\alpha+1}}{\Gamma(\alpha+1)\beta^{\alpha+1}} b^{\alpha} e^{-ab/\beta}, \quad b > 0, \ (\alpha > -1, \beta > 0), \tag{3.5}$$

is the gamma conjugate prior, that was first used by Papadopoulos (1978), when a was assumed known, and

$$g_2(a) = \frac{1}{\Gamma(\delta)\gamma^{\delta}} a^{\delta-1} e^{-a/\gamma}, \quad a > 0, \ (\gamma > 0, \delta > 0), \tag{3.6}$$

is the gamma density function.

Multiplying $g_1(b \mid a)$ by $g_2(a)$, we obtain the bivariate density of a and b, given from (3.4), by

$$g(a, b) = A_1 a^{\alpha+\delta} b^{\alpha} \exp\left[-a\left\{\frac{1}{\gamma} + \frac{b}{\beta}\right\}\right], \quad a > 0, \ b > 0, \tag{3.7}$$

where $\alpha > -1$, β, γ, and δ are positive real numbers and $A_1^{-1} = \Gamma(\delta)\Gamma(\alpha+1)\gamma^{\delta}\beta^{\alpha+1}$.

The four-parameter gamma-gamma prior (3.7) is so chosen that it would be rich enough to cover the prior belief of the experimenter (AL-Hussaini and Jaheen, 1992). It follows, from (2.1) and (3.7), that the joint posterior density function of a and b given the data is thus

$$\begin{aligned} q_2(a, b \mid \underline{x}) = {} & A_2 a^{r+\alpha+\delta} b^{r+\alpha} e^{-a/\gamma} v(a; \underline{x}) \\ & \times \exp\left[-b\left(H(a; \underline{x}) + \frac{a}{\beta}\right)\right], \quad a > 0, \ b > 0, \end{aligned} \tag{3.8}$$

where

$$A_2^{-1} = \Gamma(r + \alpha + 1) I_0(\underline{x}), \tag{3.9}$$

and

$$I_0(\underline{x}) = \int_0^{\infty} a^{r+\alpha+\delta} e^{-a/\gamma} v(a; \underline{x}) \left(H(a; \underline{x}) + \frac{a}{\beta}\right)^{-(r+\alpha+1)} da. \tag{3.10}$$

It is well known that, under squared error loss function, the Bayes estimator of a function, say $U(a, b)$, is the posterior mean of the function and is given by a ratio of two integrals which may be written as

$$\begin{aligned} E[U(a, b) \mid \underline{x}] &= \int_0^{\infty} \int_0^{\infty} U(a, b) q_2(a, b \mid \underline{x}) da db \\ &= \frac{\int_0^{\infty} \int_0^{\infty} U(a, b) L(a, b; \underline{x}) g(a, b) da db}{\int_0^{\infty} \int_0^{\infty} L(a, b; \underline{x}) g(a, b) da db}. \end{aligned} \tag{3.11}$$

Generally, the ratio of two integrals (3.11) cannot be obtained in a simple closed form. Numerical methods of integration may be used in this case, which can be computationally intensive, especially in high dimensional parameter space. Instead, we can use the approximation form due to Lindley (1980). In the following, a review of Lindley's approximation form is given.

3.2.1. The Approximation form of Lindley. Lindley (1980) developed approximate procedures for the evaluation of the ratio of two integrals in the form

$$\int_{\Theta} \omega(\lambda) e^{L(\lambda)} d\lambda \Big/ \int_{\Theta} g(\lambda) e^{L(\lambda)} d\lambda, \tag{3.12}$$

where $\lambda \equiv (\lambda_1, \ldots, \lambda_N)$, $L(\lambda)$ is the logarithm of the likelihood function, $g(\lambda)$ and $\omega(\lambda) = U(\lambda) g(\lambda)$ are arbitrary functions of λ. From Eq. (3.12), the posterior expectation of the function $U(\lambda)$, for given $\underset{\sim}{x}$, is

$$E[U(\lambda) \mid \underset{\sim}{x}] = \frac{\int U(\lambda) e^{Q(\lambda)} d\lambda}{\int e^{Q(\lambda)} d\lambda}, \tag{3.13}$$

where $Q(\lambda) = L(\lambda) + \rho(\lambda)$ is the logarithm of the posterior distribution of λ except for the normalizing constant and $\rho(\lambda) = \ln g(\lambda)$. Expanding $Q(\lambda)$ in (3.13) into a Taylor series expansion about the posterior mode of λ, Lindley obtained the required expression for $E[U(\lambda) \mid \underset{\sim}{x}]$. For more details, see Lindley (1980).

In this article, we consider Lindley's approximation form expanding about the posterior mode. For the two parameter case $\lambda = (\lambda_1, \lambda_2)$, Lindley's approximation leads to

$$\widehat{U}_B = U(\lambda) + \frac{1}{2}\left[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}\right], \tag{3.14}$$

where $B = \sum_{i=1}^{2}\sum_{j=1}^{2} U_{ij}\tau_{ij}$, $Q_{\eta\xi} = \frac{\partial^{\eta+\xi}Q}{\partial^{\eta}\lambda_1 \partial^{\xi}\lambda_2}$, $\eta, \xi = 0, 1, 2, 3$, $\eta + \xi = 3$, for $i, j = 1, 2$, $U_i = \frac{\partial U}{\partial \lambda i}$, $U_{ij} = \frac{\partial^2 U}{\partial \lambda i \partial \lambda j}$, and for $i \neq j$,

$$B_{ij} = (U_i\tau_{ii} + U_j\tau_{ij})\tau_{ii}, \quad C_{ij} = 3U_i\tau_{ii}\tau_{ij} + U_j(\tau_{ii}\tau_{jj} + 2\tau_{ij}^2),$$

τ_{ij} is the $(i, j)th$ element in the inverse of the matrix $Q^* = (-Q_{ij}^*)$, $i, j = 1, 2$ such that $Q_{ij}^* = \frac{\partial^2 Q}{\partial \lambda_i \partial \lambda_j}$. Expansion (3.14) is to be evaluated at $(\hat{\lambda}_1, \hat{\lambda}_2)$, the mode of the posterior density.

In our case, $(\lambda_1, \lambda_2) \equiv (a, b)$ and Q is then given by

$$Q = \ln q_2 \propto (r + \alpha + \delta)\ln a + (r + \alpha)\ln b - \frac{a}{\gamma} + \ln v(a; \underset{\sim}{x}) - b\left[H + \frac{a}{\beta}\right], \tag{3.15}$$

where $H \equiv H(a; \underset{\sim}{x})$ is as given in (2.2).

The joint posterior mode, denoted by $(\tilde{a}, \tilde{b})$, is obtained from (3.15) and is given by

$$\tilde{b} = \frac{\beta(r + \alpha)}{a + \beta H}, \tag{3.16}$$

where $\tilde{a}$ is the solution of the following nonlinear equation

$$\frac{r + \alpha + \delta}{a} - \frac{1}{\gamma} + \sum_{i=1}^{r}\omega_i - \frac{\beta(r + \alpha)B_1}{a + \beta H} = 0, \tag{3.17}$$

where

$$B_1 \equiv B_1(a; \underset{\sim}{x}) = \frac{1}{\beta} + (m + 1)\sum_{i=1}^{r-1} x_i^a \omega_i + \gamma_r x_r^a \omega_r,$$

and ω_i is as given by (2.5).

The τ_{ij} elements of the inverse of the matrix $Q^* = (-Q^*_{ij})$, $i, j = 1, 2$ are given by

$$\left.\begin{aligned} \tau_{11} &= \frac{a^2(r+\alpha)}{D}, \\ \tau_{12} &= \tau_{21} = -\frac{a^2 b^2 B_1}{D}, \\ \tau_{22} &= \frac{b^2[r+\alpha+\delta+a^2B_2]}{D}, \end{aligned}\right\} \tag{3.18}$$

where

$$B_2 \equiv B_2(a; \underline{x}) = \sum_{i=1}^{r} x_i^a \omega_i^2 + bB_3,$$

$$B_3 \equiv B_3(a; \underline{x}) = (m+1)\sum_{i=1}^{r-1} x_i^a \omega_i^2 + \gamma_r x_r^a \omega_r^2,$$

$$D = (r+\alpha)[r+\alpha+\delta+a^2B_2] - [abB_1]^2.$$

Furthermore,

$$Q_{12} = 0, \quad Q_{21} = -B_3, \quad Q_{30} = \frac{2(r+\alpha+\delta)}{a^3} - B_4, \quad Q_{03} = \frac{2(r+\alpha)}{b^3},$$

where

$$\begin{aligned} B_4 \equiv B_4(a; \underline{x}) &= \sum_{i=1}^{r} x_i^a (1 - x_i^a)\omega_i^3 \\ &\quad + (m+1)\sum_{i=1}^{r-1} x_i^a (1 - x_i^a)\omega_i^3 + \gamma_r x_r^a (1 - x_r^a)\omega_r^3. \end{aligned}$$

Substituting the above values in (3.14) yields the Bayes estimate of a function $U \equiv U(a, b)$, of the unknown parameters a and b, given by:

$$\widehat{U}_B = E[U(a, b) \mid \underline{x}] = U + \frac{W}{2D} + \frac{1}{2D^2}\big[aW_1U_1 + bW_2U_2\big], \tag{3.19}$$

where

$$\left.\begin{aligned} W &= a^2[(r+\alpha)U_{11} - b^2B_1(U_{21}+U_{12})] \\ &\quad + b^2(r+\alpha+\delta+a^2B_2)U_{22}, \\ W_1 &= (r+\alpha)\big[(r+\alpha)\big\{2(r+\alpha+\delta-a^3B_4)\big\} \\ &\quad + 3a^3b^2B_1B_3 - 2abB_1(r+\alpha+\delta+a^2B_2)\big], \\ W_2 &= \big[2(r+\alpha)\big\{(r+\alpha+\delta+a^2B_2)^2 \\ &\quad - abB_1(r+\alpha+\delta-a^3B_4)\big\} + a^2bB_3 \\ &\quad \times \big\{(r+\alpha)(r+\alpha+\delta+a^2B_2) + 2(abB_1)^2\big\}\big]. \end{aligned}\right\} \tag{3.20}$$

All functions in the right-hand side of (3.19) are to be evaluated at the posterior mode $(\tilde{a}, \tilde{b})$. Now, the Bayes estimates of a and b are computed as follows:

(i) If $U(a, b) = a$, we have from (3.19),

$$\hat{a}_{BG} = a\left[1 + \frac{W_1}{2D^2}\right], \tag{3.21}$$

evaluated at the posterior mode $(\tilde{a}, \tilde{b})$.

(ii) If $U(a, b) = b$, we have from (3.19),

$$\hat{b}_{BG} = b\left[1 + \frac{W_2}{2D^2}\right], \tag{3.22}$$

evaluated at the posterior mode $(\tilde{a}, \tilde{b})$.

4. Numerical Computations Based on Upper Records

The upper record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(r)}$ of size r can be obtained from the gos scheme as a special case by taking $m = -1$ and $k = 1$. In this case, estimation for the parameters of the BurrXII(a, b) distribution based on upper records can be obtained from the above results by taking $m = -1$, $k = 1$.

The ML and Bayes estimates of the two unknown parameters a and b are computed and compared based on a Monte Carlo simulation study according to the following steps:

1. For a given vector of prior parameters $(\alpha, \beta, \gamma, \delta)$, we generate a and b from the joint prior density (3.7). The IMSL (1984) is used in the generation of the gamma random variates.
2. For given a and b obtained in Step 1, we generate $n = (5, 8, 12)$ upper record values from the BurrXII(a, b) with pdf (1.1).
3. The ML estimate of a is computed by solving the nonlinear Eq. (2.4), with $m = -1$, $k = 1$, and $\gamma_r = 1$, by using ZSPOW routine from the IMSL (1984) library. Substituting the ML estimate of a we compute the ML estimate of b from $\hat{b}_{MLR} = \frac{r}{\ln(1+x_r^a)}$.
4. The Bayes estimates of a and b are computed from (3.21) and (3.22) with $m = -1$, $k = 1$ and $\gamma_r = 1$.
5. The squared deviations $(\psi^* - \psi)^2$ are computed for different sizes n where $(*)$ stands for an estimate (ML or Bayes) and ψ stands for the parameter (a or b).

Table 1
Estimated Risk (ER) for the ML and Bayes estimates of a and b when $(\alpha = 2.5, \beta = 1.4, \gamma = 3.4, \delta = 2.0)$ for different values of n and 1000 repetitions

n	$ER(\hat{a}_{MLR})$	$ER(\hat{a}_{BR})$	$ER(\hat{b}_{MLR})$	$ER(\hat{b}_{BR})$
5	0.1473	0.1435	0.2765	0.2637
8	0.1395	0.1318	0.2551	0.2449
12	0.1364	0.1287	0.2482	0.2378

Table 2
Estimated risk (ER) for the ML and Bayes estimates of a and b when ($\alpha = 2.0$, $\beta = 2.5$, $\gamma = 2.3$, $\delta = 3.5$) for different values of n and 1000 repetitions

n	$ER(\hat{a}_{MLR})$	$ER(\hat{a}_{BR})$	$ER(\hat{b}_{MLR})$	$ER(\hat{b}_{BR})$
5	0.2136	0.2033	0.3254	0.3164
8	0.2087	0.1947	0.3146	0.3055
12	0.1935	0.1752	0.3027	0.2866

6. The above steps are repeated 1000 times and the estimated risk (ER) is computed by averaging the squared deviations over the 1000 repetitions. The computational results are displayed in Tables 1 and 2.

5. Concluding Remarks

1. In this article, the ML and Bayes methods of estimation are used for estimation of the parameters of the Burr type XII distribution based on generalized order statistics. Estimation based on upper records from the Burr model are obtained as a special case. It has been noticed, from Tables 1 and 2, that the estimated risks of the estimates decrease as n increases and the Bayes estimates have the smallest estimated risks as compared with their corresponding ML estimates.
2. It has not been possible to provide necessary and sufficient conditions for the existence and uniqueness of the ML estimators for the two unknown parameters of the Burr type XII distribution. A graphical investigation may be used for studied the existence of the global maximum or the local maximum of the likelihood function (2.1) in the case of upper record values.
3. If $m = 0$ and $k = 1$ (ordinary order statistics), then $\gamma_r = n - r + 1$ and the Bayes estimators (3.21) and (3.22) reduce to those obtained by AL-Hussaini and Jaheen (1992).
4. If the prior parameters are unknown, the empirical Bayes approach may be used to estimate such parameters, (see, for example, Maritz and Lwin, 1989).

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