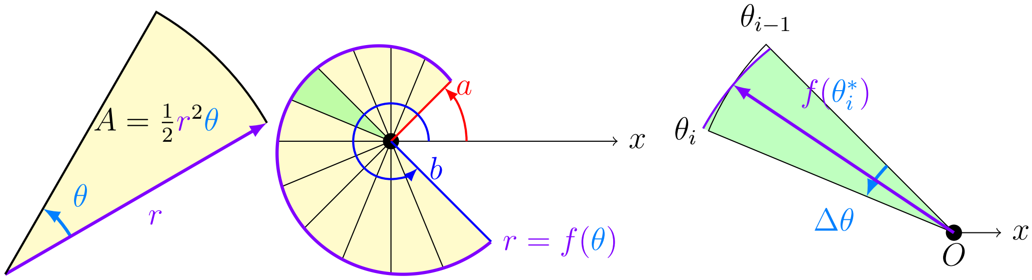
10.4 Areas and lengths in polar coordinates

1. area 面積 $A = \int \frac{1}{2} r^2 \, d\theta$

2. arc length 弧長 $L = \int \sqrt{r^2 + (\frac{dr}{d\theta})^2} \, d\theta$

0.1 Area



The area A of the region bounded by a polar curve $r = f(\theta)$ and two rays $\theta = a$ and $\theta = b$, where $f(\theta) \geq 0$ for $a \leq \theta \leq b$ and $0 < b - a \leq 2\pi$, is

$$\boxed{\boldsymbol{A = \int_a^b \frac{1}{2} r^2 \, d\theta}}$$

Proof. 分成 n 等角, 再用扇形面積去逼近:
$\Delta\theta = \dfrac{b-a}{n}$ and $\theta_i = a + i\Delta\theta$, $\theta_i^* \in [\theta_{i-1}, \theta_i]$ (sample point).

$$\begin{aligned}
A &\approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta, \\
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta \\
&= \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta \\
&= \int_a^b \frac{1}{2} r^2 \, d\theta.
\end{aligned}$$

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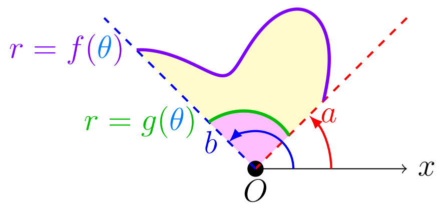
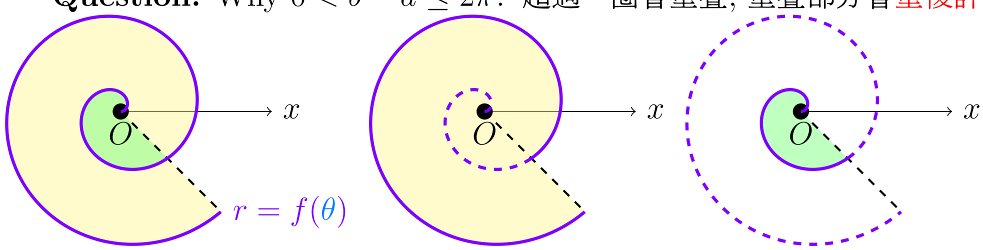
The area A of the region bounded by polar curves $r = f(\theta)$ and $r = g(\theta)$ and two rays $\theta = a$ and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$, is

$$\boxed{A = \int_a^b \frac{1}{2}[f(\theta)]^2\ d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2\ d\theta}$$



Question: Why $0 < b - a \leq 2\pi$? 超過一圈會重疊, 重疊部分會重複計算!



$[0, b(> 2\pi)]$ $[b - 2\pi, b]$ $[0, b - 2\pi]$

怎麼算? 減掉重複的部分 : $\displaystyle\int_a^b \frac{1}{2} r^2\ d\theta - \int_a^{b-2\pi} \frac{1}{2} r^2\ d\theta = \int_{b-2\pi}^b \frac{1}{2} r^2\ d\theta.$

Question: Why $f(\theta) \geq 0$? 想想看: $f(\theta) < 0$ 時的意義, 它的黎曼和是什麼, 要怎麼算。

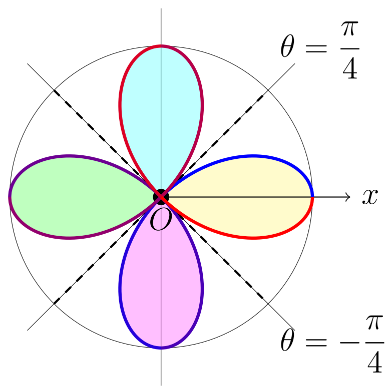
Skill: 1. 常用到半角公式:

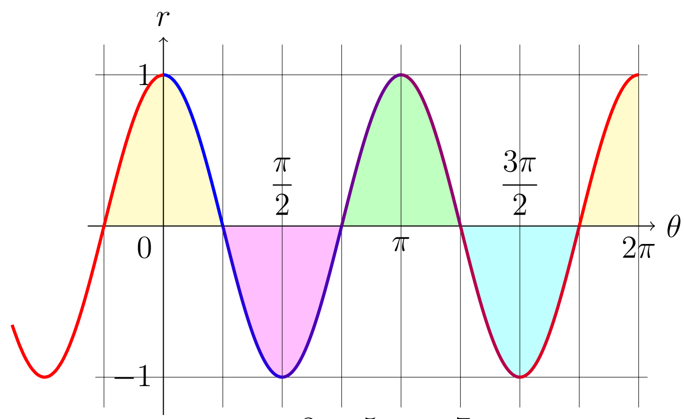
$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta), \quad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta).$$

$$\int \sin^2\theta\ d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C, \quad \int \cos^2\theta\ d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C.$$

2. 畫圖可以幫助認清邊界。
3. 善用對稱性可以簡化計算。
4. 要注意誰大誰小。

Example 0.1 *Find the area enclosed by one loop of the four-leaved rose* $r = \cos 2\theta$.





(找出積分上下限) $r = \cos 2\theta = 0 \iff \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ *or* $\frac{7\pi}{4} = -\frac{\pi}{4}$.

Choose $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

$$
\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta \\
&= \int_{0}^{\pi/4} \cos^2 2\theta \, d\theta \qquad \text{(對稱性)} \\
&= \int_{0}^{\pi/4} \frac{1}{2}(1 + \cos 4\theta) \, d\theta \qquad \text{(半角)} \\
&= \left[\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right]_0^{\pi/4} = \frac{\pi}{8}.
\end{aligned}
$$

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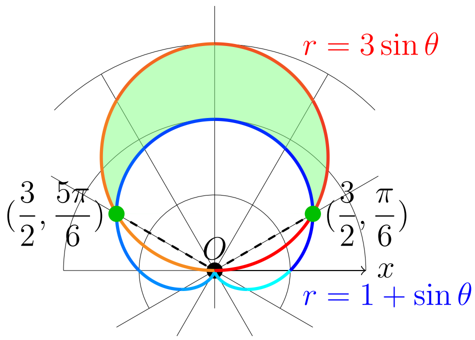
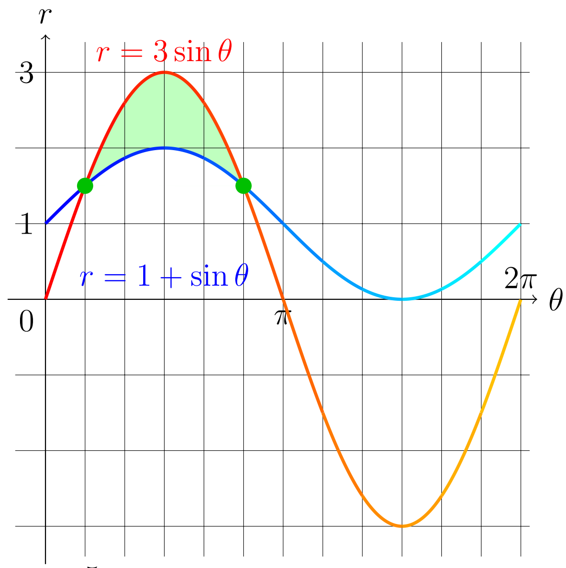
Example 0.2 *Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.*





$$r = 3\sin\theta = 1 + \sin\theta \iff \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}.$$

$3\sin\theta \geq 1 + \sin\theta \geq 0$ *for* $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$.

$$\begin{aligned}
A &= \int_{\pi/6}^{5\pi/6} \frac{1}{2}(3\sin\theta)^2\, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2\, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta)\, d\theta \qquad \text{(範圍一樣)} \\
&= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\, d\theta \qquad \text{(對稱性)} \\
&= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \qquad \text{(半角)} \\
&= \Big[3\theta - 2\sin 2\theta + 2\cos\theta\Big]_{\pi/6}^{\pi/2} \\
&= \Big[3(\frac{\pi}{2}) - 2\cancelto{0}{\sin\pi} + 2\cancelto{0}{\cos\frac{\pi}{2}}\Big] - \Big[3(\frac{\pi}{6}) - 2\sin\frac{\pi}{3} + 2\cos\frac{\pi}{6}\Big] \\
&= \frac{3\pi}{2} - \frac{\pi}{2} + \cancel{\sqrt{3}} - \cancel{\sqrt{3}} \;=\; \pi.
\end{aligned}$$

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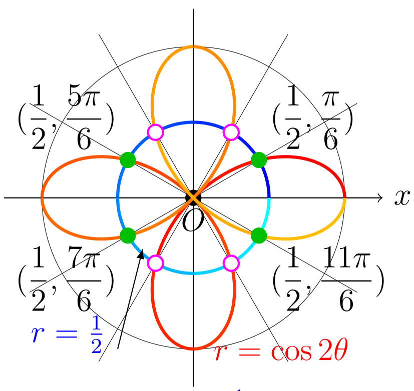
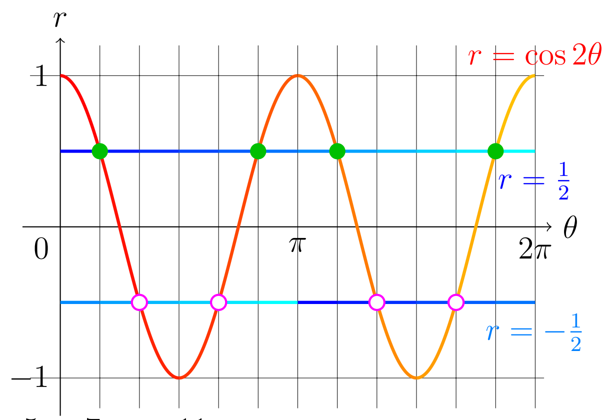
Example 0.3 *Find all points of intersection of the curves* $r = \cos 2\theta$ *and* $r = \dfrac{1}{2}$.



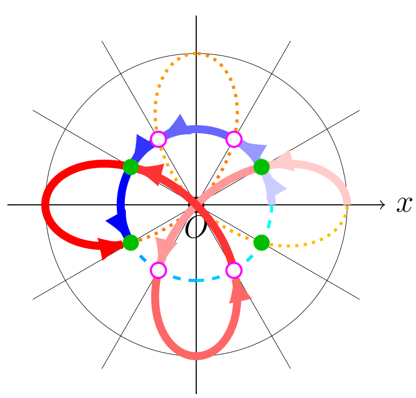


$$r = \cos 2\theta = \frac{1}{2} \iff \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}.$$

Four points: $(\frac{1}{2}, \frac{\pi}{6})$, $(\frac{1}{2}, \frac{5\pi}{6})$, $(\frac{1}{2}, \frac{7\pi}{6})$ *and* $(\frac{1}{2}, \frac{11\pi}{6})$.

$\because$ 極座標表示法不唯一, $r = \frac{1}{2}$ 與 $r = -\frac{1}{2}$ 同一條, 要解 $r = \cos 2\theta = -\frac{1}{2}$.

Another four: $(\frac{1}{2}, \frac{\pi}{3})$, $(\frac{1}{2}, \frac{2\pi}{3})$, $(\frac{1}{2}, \frac{4\pi}{3})$ *and* $(\frac{1}{2}, \frac{5\pi}{3})$. ■



Skill: 只解 $r = f(\theta) = g(\theta)$ 是解出 (同時) ***collision*** 碰撞;
要一起解 $f(\theta) = -g(\theta)$ 才會解出所有 (可以不同時) ***intersection*** 交點。

0.2 Arc length

The arc length of the curve of polar equation $r = f(\theta)$, $a \le \theta \le b$, where $f(\theta)$ is smooth ($f'(\theta)$ is continuous) on $[a, b]$, is

$$\boxed{\boldsymbol{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}}$$

Proof. Parametric equations:

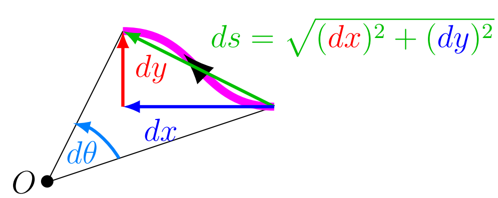
$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Use Product Rule (注意, 這裡 $r = f(\theta)$, $\dfrac{dr}{d\theta} = f'(\theta)$.)

$$\begin{aligned}
\frac{dx}{d\theta} &= \frac{dr}{d\theta}\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta, \\
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 \\
&= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - \cancel{2r\frac{dr}{d\theta}\cos\theta\sin\theta} + r^2\sin^2\theta \\
&+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + \cancel{2r\frac{dr}{d\theta}\sin\theta\cos\theta} + r^2\cos^2\theta \\
&= \left(\frac{dr}{d\theta}\right)^2\left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\sin^2\theta + \cos^2\theta\right) \\
&= \left(\frac{dr}{d\theta}\right)^2 + r^2.
\end{aligned}$$

Arc length formula:

$$\begin{aligned}
L &= \int ds \\
&= \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \blacksquare
\end{aligned}$$

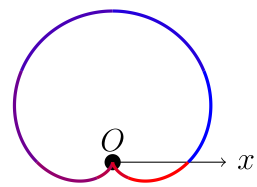


Attention: 弧長公式算出來的 L 是里程數, 是實際走的距離。如果有重複繞, 可以算完再除以繞的圈數, 或是找到繞一圈的範圍積分。

Note: $\sqrt{[f(x)]^2} = |f(x)|$, 積分時從 $f(x) = 0$ 的地方分開積分。

Example 0.4 *Find the length of the cardioid $r = 1+\sin\theta$.*

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2+2\sin\theta}\, d\theta \end{aligned}$$



$r = 1+\sin\theta$

(Sol 1) $\sqrt{2+2\sin\theta} = \sqrt{2+2\sin\theta}\cdot\dfrac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} = \dfrac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}$

$= \dfrac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}} = \dfrac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}$, $\cos\theta \le 0$ *when* $\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$.

$$\begin{aligned} L &= \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\, d\theta \\ &= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta \\ & \quad (\textit{Let } u = 2-2\sin\theta,\ du = -2\cos\theta\ d\theta,\ \theta = 0, \tfrac{\pi}{2}, \tfrac{3\pi}{2}, 2\pi,\ u = 2, 0, 4, 2.) \\ &= \int_2^0 \frac{-1}{\sqrt{u}}\, du + \int_0^4 \frac{1}{\sqrt{u}}\, du + \int_4^2 \frac{-1}{\sqrt{u}}\, du \\ &= \int_0^2 \frac{1}{\sqrt{u}}\, du + \int_0^4 \frac{1}{\sqrt{u}}\, du + \int_2^4 \frac{1}{\sqrt{u}}\, du \;=\; 2\int_0^4 \frac{1}{\sqrt{u}}\, du \quad (\textit{improper}) \\ &= \lim_{t\to 0^+} 2\int_t^4 \frac{1}{\sqrt{u}}\, du \;=\; \lim_{t\to 0^+} 2\Big[2\sqrt{u}\Big]_t^4 \;=\; 8 - \lim_{t\to 0^+} 4\sqrt{t} \;=\; 8; \\ & \quad (\textit{Or let } v = \sqrt{2-2\sin\theta},\ dv = \frac{-\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta,\ v = \sqrt{2}, 0, 2, \sqrt{2}.) \\ &= \int_{\sqrt{2}}^0 -2\ dv + \int_0^2 2\ dv + \int_2^{\sqrt{2}} -2\ dv \\ &= 2\int_0^{\sqrt{2}} dv + 2\int_0^2 dv + 2\int_{\sqrt{2}}^2 dv \;=\; 4\int_0^2 dv \;=\; 8. \end{aligned}$$

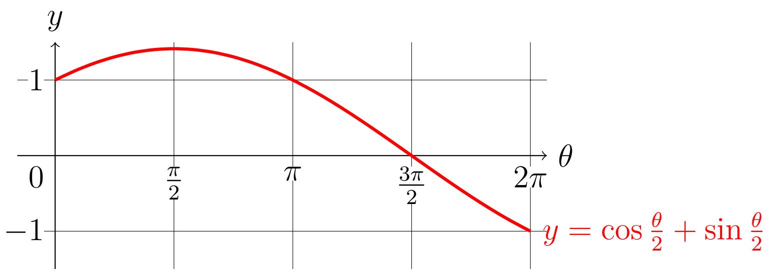
Note: $2-2\sin\theta = 0$ *when* $\theta = \dfrac{\pi}{2}$, 可以乘 $\dfrac{0}{0}$*? No!* 會變成瑕積分。

(Sol 2) [變型成真積分]

$\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$,

$$\begin{aligned}
\sqrt{2+2\sin\theta} &= \sqrt{2}\sqrt{(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2})} \\
&= \sqrt{2}\sqrt{(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})^2} \\
&= \sqrt{2}\left|\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right|,
\end{aligned}$$

$\cos\frac{\theta}{2} + \sin\frac{\theta}{2} \le 0$ *when* $\frac{3\pi}{2} \le \theta \le 2\pi$.



$$\begin{aligned}
L &= \sqrt{2}\int_0^{2\pi} |\cos\frac{\theta}{2} + \sin\frac{\theta}{2}|\ d\theta \\
&= \sqrt{2}\int_0^{3\pi/2} (\cos\frac{\theta}{2} + \sin\frac{\theta}{2})\ d\theta + \sqrt{2}\int_{3\pi/2}^{2\pi} -(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})\ d\theta \\
&= 2\sqrt{2}\Big[\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\Big]_0^{3\pi/2} - 2\sqrt{2}\Big[\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\Big]_{3\pi/2}^{2\pi} \\
&= 2\sqrt{2}\Big[(\frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}}) - (0-1)\Big] - 2\sqrt{2}\Big[(0-(-1)) - (\frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}})\Big] \\
&= 4 + 2\sqrt{2} - 2\sqrt{2} + 4 \\
&= 8.
\end{aligned}$$

(Try yourself: integration for θ *from* $-\frac{\pi}{2}$ *to* $\frac{3\pi}{2}$*:* $L = \int_{-\pi/2}^{3\pi/2} \cdots\ d\theta$*,*

or from $-\frac{\pi}{2}$ *to* $\frac{\pi}{2}$ *then double it by symmetry:* $L = 2\int_{-\pi/2}^{\pi/2} \cdots\ d\theta$*.)* ■