4.9 Antiderivatives

1. anti-derivative[ˌæntaɪ-dəˈrɪvətɪv] 反導函數

Recall: §2.8 導函數:

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ is the derivative of f at where the limit exists.

Define: A function F is called an ***antiderivative*** 反導(函)數 of f on an interval I if $F'(x) = f(x)$ for all x on I.

$$\begin{tikzcd} F \arrow[r, bend right, "\text{導函數}"'] & f \arrow[l, bend right, "\text{反導數}"'] \end{tikzcd}$$

你(f')是我(f)的導函數, 我是你的反導數。

Ex: Let $f(x) = x^2$. If $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$.

$\frac{1}{3}x^3,\ \frac{1}{3}x^3 + 1,\ \frac{1}{3}x^3 + 2,\ \ldots$ are antiderivatives of x^2. 反導數不止一個。

Theorem 1 *If F is an antiderivative of f on I, then* ***the most general antiderivative*** 最一般反導 *(*函*)* 數 *of f on I is*

$$\boxed{\boldsymbol{F} + \boldsymbol{C}}$$

where C is an ***arbitrary constant*** 任意常數.

凡走過必留下痕跡, 最一般反導數必加個大C。

Ex: $(\because \frac{d}{dx}C = 0)$ $\frac{1}{3}x^3 + C$ is the most general antiderivative of x^2.

Note: 找 f 的反導數沒指定範圍 (I), 就要看 f 的定義域 (domain)。

Example 0.1 *Find the most general antiderivative.*
(a) $f(x) = \sin x$, *(b)* $f(x) = \frac{1}{x}$, *(c)* $f(x) = x^n$, $n \neq -1$.

(a) $(\cos x)' = -\sin x$, $(-\cos x)' = \sin x$. $F(x) = -\cos x + C$.

(b) $(\ln x)' = \frac{1}{x}$ *on* $(0, \infty)$, *(*但是 *domain* 不一樣*)*
but also $(\ln|x|)' = \frac{1}{x}$ *on* $(-\infty, 0) \cup (0, \infty)$. $F(x) = \ln|x| + C$.

(c) $(\frac{1}{n+1}x^{n+1})' = \frac{n+1}{n+1}x^n = x^n$ $(n \neq -1)$. $F(x) = \frac{1}{n+1}x^{n+1} + C$. ■

Table 1: Table of derivatives and antiderivatives formulas: $(F' = f, G' = g)$

Derivative	Function	Antiderivative
cf'	cf	cF
$f' \pm g'$	$f \pm g$	$F \pm G$
nx^{n-1}	$x^n, n \neq -1$	$\frac{1}{n+1}x^{n+1}$
$-\frac{1}{x^2}$	$\frac{1}{x}$	$\ln \lvert x \rvert$
$\frac{1}{x}$	$\ln x$	$x \ln x - x$
e^x	e^x	e^x
$\cos x$	$\sin x$	$-\cos x$
$-\sin x$	$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$	$\ln \lvert \sec x \rvert$
$\sec x \tan x$	$\sec x$	$\ln \lvert \sec x + \tan x \rvert$
$2\sec^2 x \tan x$	$\sec^2 x$	$\tan x$
$2\sec^3 x - \sec x$	$\sec x \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$x \sin^{-1} x + \sqrt{1-x^2}$
$\frac{1}{1+x^2}$	$\tan^{-1} x$	$x \tan^{-1} x - \ln \sqrt{1+x^2}$
$\frac{x}{\sqrt{1-x^2}^3}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{-2x}{(1+x^2)^2}$	$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\cosh x$	$\sinh x$	$\cosh x$
$\sinh x$	$\cosh x$	$\sinh x$

黑色必背, 藍色多背, 綠色免背, 紅色別背。

Example 0.2 *Find g such that* $g' = 4\sin x + \dfrac{2x^5 - \sqrt{x}}{x}$.

$g' = 4\sin x + 2x^4 - x^{-1/2}$; $(-\cos x)' = \sin x$, $(\frac{1}{5}x^5)' = x^4$, $(2x^{1/2})' = x^{-1/2}$.
$g(x) = 4(-\cos x) + 2(\frac{1}{5}x^5) - (2x^{1/2}) + C = -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C$. ■

Example 0.3 *Find f such that $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$.*

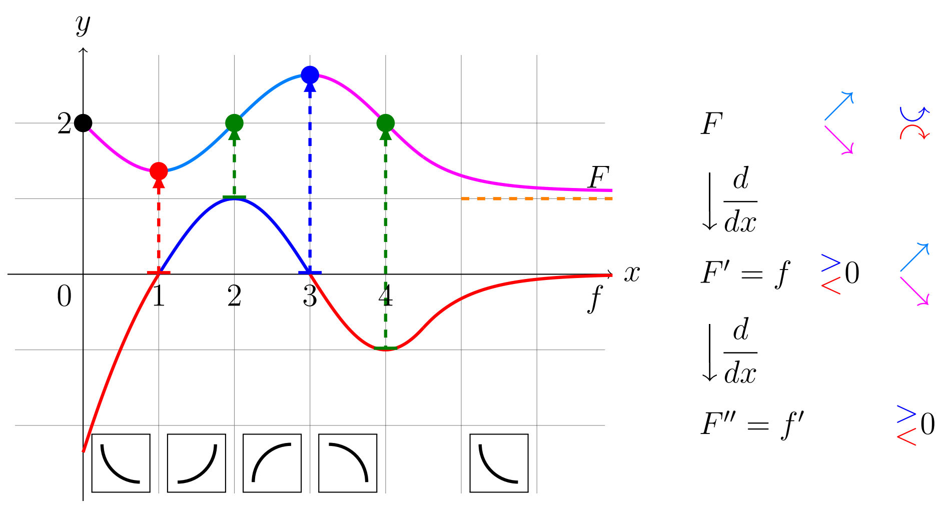
$(e^x)' = e^x$, $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$, $f(x) = (e^x) + 20(\tan^{-1} x) + C$.
$f(0) = 1 + 20 \cdot 0 + C = -2$, $C = -3$. $f(x) = e^x + 20\tan^{-1} x - 3$. ■

Example 0.4 *Find f such that $f'' = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$.*

$f'(x) = 4x^3 + 3x^2 - 4x + C$, $f(x) = x^4 + x^3 - 2x^2 + Cx + D$,
$f(0) = D = 4$, $f(1) = C + D = 1$, $C = -3$.
$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$. ■

Example 0.5 *Sketch an antiderivative F of a given f with $F(0) = 2$.*

$f(1) = f(3) = 0$ and f change sign $- \to +$ at 1 and $+ \to -$ at 3
$\Longrightarrow$ F has local min at 1 and local max at 3.
$f'(2) = f'(4) = 0$ and f' change sign $+ \to -$ at 2 and $- \to +$ at 4
$\Longrightarrow$ F has IP at 2 and 4.
When $x \to \infty$, $f \to 0 \Longrightarrow F \to c$ constant $\Longrightarrow F$ has a H.A. $y = c$. ■



Additional: $f \to c \Longrightarrow F \to cx + d \Longrightarrow F$ has a S.A. $y = cx + d$.

Application: Rectilinear[ˌrɛktɪˈlɪnɪɚ] motion 直線運動

$s(t)$ position 位置 function,
$v(t) = s'(t)$ velocity 速率 function,
$a(t) = v'(t) = s''(t)$ acceleration[ækˌsɛləˈreʃən] 加速度 function.
$s(t)$ is an antiderivative of $v(t)$, $v(t)$ is an antiderivative of $a(t)$.

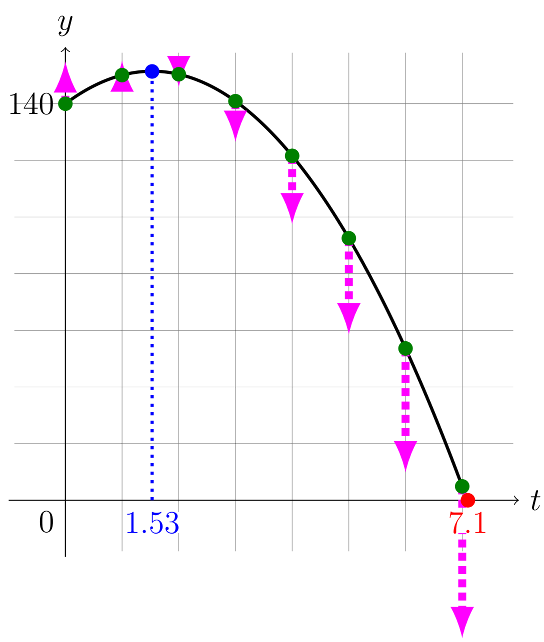
Example 0.6 *$a(t) = 6t + 4$, $v(0) = -6$ cm/s, $s(0) = 9$ cm. Find $s(t)$.*

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C,\ v(0) = C = -6. \Longrightarrow v(t) = 3t^2 + 4t - 6.$$

$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$, $s(0) = D = 9$.
$\Longrightarrow s(t) = t^3 + 2t^2 - 6t + 9$. ■

Note: Gravity 重力加速度: -9.8 m/s^2 (往下).

Example 0.7 *A ball is thrown upward with a speed of 15 m/s from the edge of cliff 140 m above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?*
於 ($s(0) =$)140 *m* 崖向上以 ($v(0) =$)15 *m/s* 丟球. $s(t) =$? *max at* $t =$? $s(t) = 0$, $t =$?



$a(t) = -9.8$, $v(t) = -9.8t + C$,
$v(0) = C = 15$,
$\Longrightarrow v(t) = -9.8t + 15$ *m/s*.
$s(t) = -4.9t^2 + 15t + D$,
$s(0) = D = 140$,
$\Longrightarrow s(t) = -4.9t^2 + 15t + 140$ *m*.

$s'(t) = v(t) = 0$
when $t = \frac{15}{9.8} \approx 1.53$ *s*.

$s(t) = -4.9t^2 + 15t + 140 = 0$
when $t = \frac{15 \pm \sqrt{2969}}{9.8} \approx 7.1$ *s*.
(負不合, negative time)

Ans: $s(t) = -4.9t^2 + 15t + 140$ *m*, $\frac{15}{9.8}$ *s*, $\frac{15 + \sqrt{2969}}{9.8}$ *s*. ■