14.4 Tangent planes & linear approximations

1. tangent plane & linear approximation
2. differentiable
3. total differential

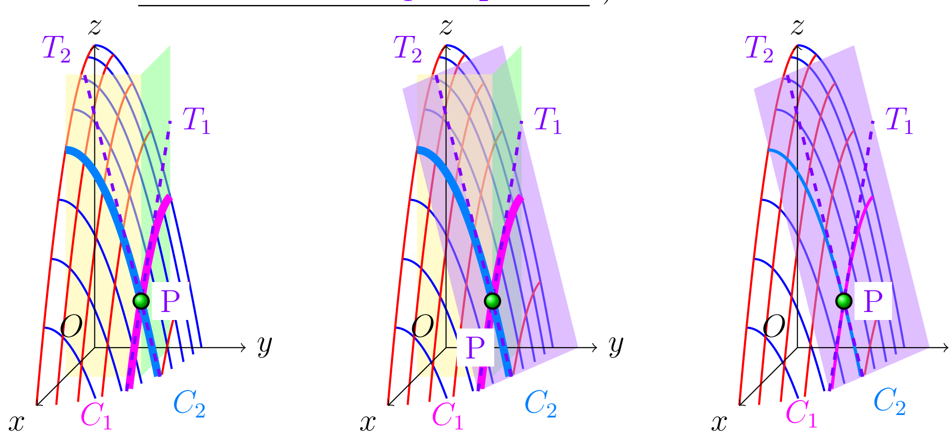
0.1 Tangent plane & linear approximation

Story: Suppose a surface S has equation $z = f(x, y)$, where f has **continuous first partial derivatives**, and let $P(x_0, y_0, z_0)$ be a point on S.
(S 是一個有連續偏導數的函數 f 的曲面, P 是 S 上的一點 ($z_0 = f(x_0, y_0)$)。)

Let curves C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. P lies on both C_1 and C_2.
(C_1 & C_2 是平面 $y = y_0$ & $x = x_0$ 跟 S 交集的曲線。P 在 C_1 & C_2 上。)

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at P.
(T_1 & T_2 是 C_1 & C_2 在 P 的切線。)

Then the ***tangent plane*** 切平面 to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.
(對 S 在 P 的切平面定義爲包含 T_1 & T_2 的平面。)



Assume the equation of the plane is $A(x-x_0)+B(y-y_0)+C(z-z_0) = 0$.
(scalar equation, normal vector $\mathbf{n} = \langle A, B, C\rangle$, assume $C \neq 0$.)
Let $a = -A/C$ and $b = -B/C$, then $z - z_0 = a(x - x_0) + b(y - y_0)$.
When $y = y_0$, $z - z_0 = a(x - x_0)$, $a = f_x(x_0, y_0)$ (T_1 的斜率)。
When $x = x_0$, $z - z_0 = b(y - y_0)$, $b = f_y(x_0, y_0)$ (T_2 的斜率)。 ∎

Recall: The ***tangent line*** 切線 to the curve $y = f(x)$ at the point $P_0(x_0, y_0)$ is
$$y - y_0 = f'(x_0)(x - x_0).$$

Define: ♠ Suppose f has **continuous (first) partial derivatives 連續的 (一階) 偏導數**. An equation of the ***tangent plane*** 切平面 to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$ is

$$\boxed{\boldsymbol{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}}.$$

Example 0.1 *Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $(1, 1, 3)$.*

Let $f(x, y) = 2x^2 + y^2$.
(偏微代入) Then $f_x(x, y) = 4x$, $f_x(1, 1) = 4$; $f_y(x, y) = 2y$, $f_y(1, 1) = 2$.
(放進公式) $z - 3 = 4(x - 1) + 2(y - 1)$ *or* $4x + 2y - z - 3 = 0$. ■

Recall: The ***linear approximation*** 線性逼近 or ***tangent line approximation*** 切線逼近 of f at a is

$$f(x) \approx f(a) + f'(a)(x - a) := L(x),$$

where $L(x)$ is the ***linearization*** 線性化 of f at a.

Define: ♣ The ***linear approximation*** or ***tangent plane approximation*** 切平面逼近 of f at (a, b) is

$$\boxed{\boldsymbol{f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) := L(x, y)}},$$

where $L(x, y)$ is the ***linearization*** of f at (a, b).

Note: 比較 切平面方程式 與 切平面逼近: $(x_0, y_0, z_0) = (a, b, f(a, b))$

$$\begin{array}{ccccccccc}
z & = & z_0 & + & f_x(x_0, y_0) & (x - x_0) & + & f_y(x_0, y_0) & (y - y_0) \\
\updownarrow & & \updownarrow & & \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\
f(x, y) & \approx & f(a, b) & + & f_x(a, b) & (x - a) & + & f_y(a, b) & (y - b) \; := L(x, y)
\end{array}$$

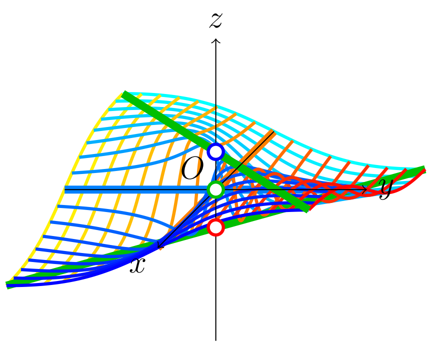
Note: 有偏導數 (f_x, f_y) 就有線性化 $(L(x, y))$, 但是<u>不一定</u>有切平面!

Question: 一定要有連續的 (一階) 偏導函數? Yes, 否則可能沒有切平面。

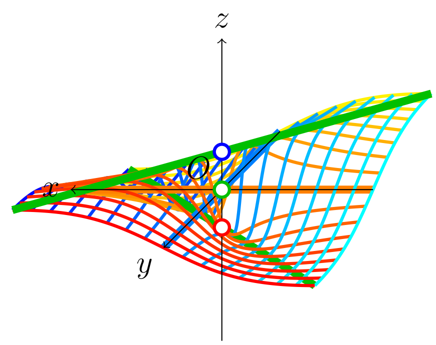
Example 0.2 (extra) *(Exercise 14.4.46, 有偏導數無切平面)*

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\[2ex] 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

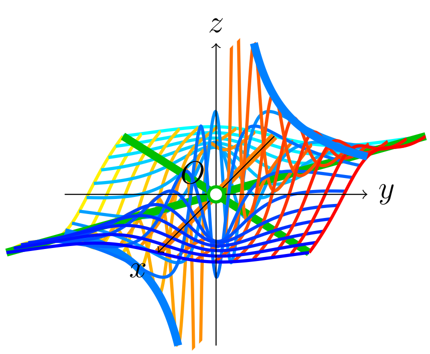
$f_x(0,0)=0=f_y(0,0)$, *then* $L(x,y): z=0+0(x-0)+0(y-0)=0$.
but $f(x,y)\to\frac{1}{2}$ *along* $x=y$ *and* $f(x,y)\to-\frac{1}{2}$ *along* $x=-y$,
so $f(x,y)\not\approx L(x,y)$.
(在 $(0,0)$ 沒有切平面, 線性化不逼近; f_x 與 f_y 在 $(0,0)$ 不連續。) ■



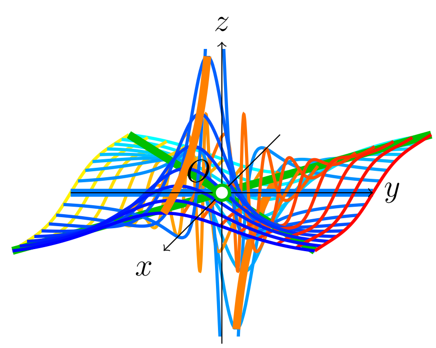
$z=\dfrac{xy}{x^2+y^2}$



$z=\dfrac{xy}{x^2+y^2}$



$z=f_x$

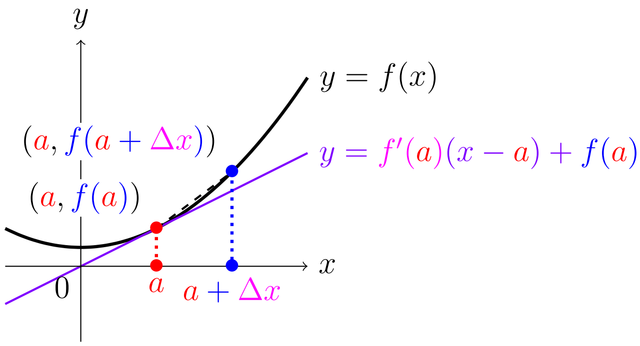


$z=f_y$

★ 差異之三: 有切$\dfrac{\text{線}}{\text{平面}}$的條件: 有(一階) 導數 v.s. 有連續的 (一階) 偏導數。

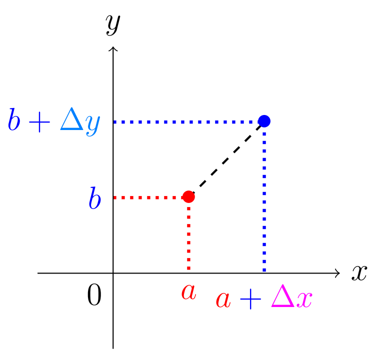
0.2 Differentiable

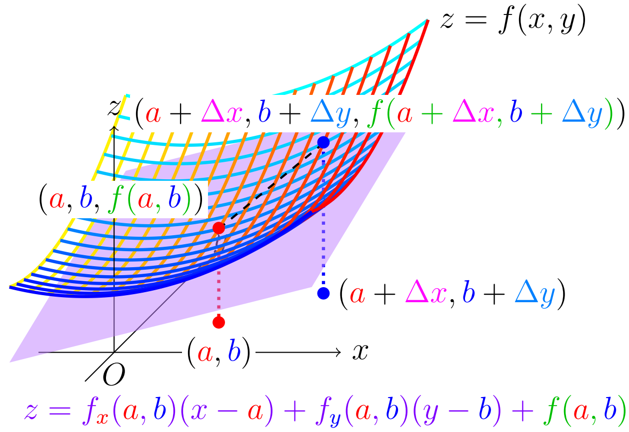
Recall: $y = f(x)$, when x increases from a to $a + \Delta x$, the ***increment*** 增量 of y is $\Delta y = f(a + \Delta x) - f(a)$. If f is differentiable at a, then $\Delta y = f'(a)\Delta x + \varepsilon \Delta x$, where $\varepsilon \to 0$ as $\Delta x \to 0$.



$$\lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a} - f'(a)\right] = \lim_{\Delta x \to 0} \frac{f(x) - L(x)}{\Delta(x)} = 0.$$

Now: $z = f(x, y)$, when x increases from a to $a + \Delta x$ and y increases from b to $b + \Delta y$, the increment of z is $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$.





Define: If $z = f(x, y)$, then f is ***differentiable*** 可微分 at (a, b) if

$$\boxed{\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. (越近越靠近切平面。)

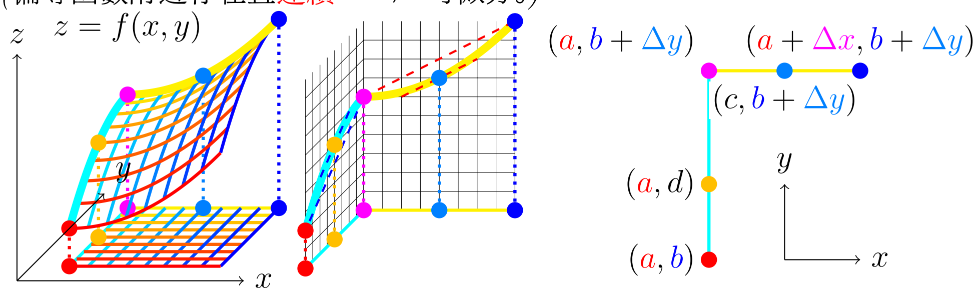
★ 差異之四: 可微分定義: 導數存在 v.s. 線性逼近逼得好。

♦: 其他書上可微分 (等價) 定義:

$$\lim_{(x,y) \to (a,b)} \frac{\Delta z - f_x(a, b)\Delta x - f_y(a, b)\Delta y}{\sqrt{(x - a)^2 + (y - b)^2}} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(x, y) - L(x, y)}{\sqrt{\Delta x^2 + \Delta y^2}} = 0.$$

線性逼近可以很好的逼近 ($\exists\ \varepsilon_1, \varepsilon_2$) 的叫作可微分, 但是定義很難用。

Theorem 1 ♢ *If the* **partial derivatives** f_x *and* f_y **exist** *near* (a, b) *and are* **continuous** *at* (a, b)*, then* f *is differentiable at* (a, b)*.*
(偏導函數附近存在且連續 $\Longrightarrow$ 可微分。)



Proof. $\because$ f_x and f_y exist near (a, b), by Mean Value Theorem, $\exists\ c$ between a and $a + \Delta x$, $\exists\ d$ between b and $b + \Delta y$, such that

$$\begin{aligned}
\Delta z &= f(a + \Delta x, b + \Delta y) - f(a, b) \\
&= f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) + f(a, b + \Delta y) - f(a, b) \\
&= f_x(c, b + \Delta y)\Delta x + f_y(a, d)\Delta y \\
&= f_x(a, b)\Delta x + f_y(a, b)\Delta y \\
&\quad + [f_x(c, b + \Delta y) - f_x(a, b)]\Delta x + [f_y(a, d) - f_y(a, b)]\Delta y \\
&= f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,
\end{aligned}$$

where $\varepsilon_1 = f_x(c, b + \Delta y) - f_x(a, b)$ and $\varepsilon_2 = f_y(a, d) - f_y(a, b)$.

When $(\Delta x, \Delta y) \to (0, 0)$, $(c, b + \Delta y) \to (a, b)$ and $(a, d) \to (a, b)$, and since f_x and f_y are continuous, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. ■

Note: 只有偏導數不一定可微分。
Recall: 有連續偏導函數 $\Longrightarrow$ 有切平面 & 可微分。

Example 0.3 *Show that* $f(x, y) = xe^{xy}$ *is differentiable at* $(1, 0)$ *and find its linearization to approximate* $f(1.1, -0.1)$*.*

$f_x(x, y) = e^{xy} + xye^{xy}$, $f_x(1, 0) = 1$; $f_y(x, y) = x^2 e^{xy}$, $f_y(1, 0) = 1$.
$\because$ f_x *and* f_y *are continuous at* $(1, 0)$*, by the Theorem,* f *is differentiable.*
$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$
$\quad = 1 + 1(x - 1) + 1(y - 0) = x + y.$
$f(1.1, -0.1) \approx L(1.1, -0.1) = 1.1 + (-0.1) = 1.$ ■
(Compare $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$*.)*

0.3 Differentials & total differential

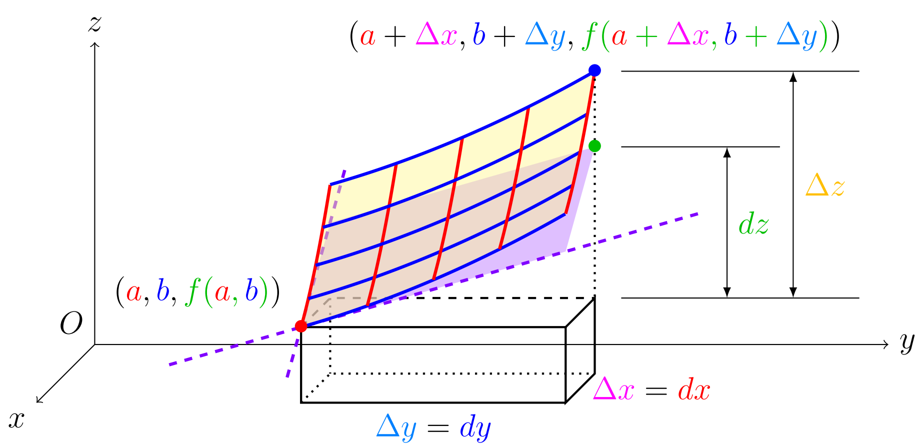
Recall: For a differentiable function $y = f(x)$, we define differential dx to be an independent variable, then the differential $dy = f'(x)\ dx$ and $\Delta y \approx dy$ when $\Delta x = dx$.

Define: For a differentiable function of two variables $z = f(x, y)$, we define ***differentials*** 微分 dx and dy to be independent variables, then the ***(total) differential*** 全微分

$$\boxed{dz = f_x(x,y)\ dx + f_y(x,y)\ dy = \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy}$$

When $\Delta x = dx = x - a$ and $\Delta y = dy = y - b$,

$$\overset{\text{(實際改變)}}{f(x,y) - f(a,b)} = \Delta z \approx dz = \overset{\text{(從切平面算的差距)}}{f_x(a,b)(x-a) + f_y(a,b)(y-b)}\ .$$



Note: 比較 切平面方程式 與 全微分:

$$\begin{array}{ccccccc}
z - f(a,b) & = & f_x(a,b) & (x-a) & + & f_y(a,b) & (y-b) \\
\shortparallel & & \shortparallel & \shortparallel & & \shortparallel & \shortparallel \\
\Delta z & = & f_x(a,b) & \Delta x & + & f_y(a,b) & \Delta y \\
\updownarrow & & & \updownarrow & & & \updownarrow \\
dz & = & f_x(a,b) & dx & + & f_y(a,b) & dy
\end{array}$$

Example 0.4 *(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find dz.*
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare Δz and dz.

(a) $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = (2x + 3y)dx + (3x - 2y)dy.$

(b) $\Delta z = f(2.05, 2.96) - f(2, 3) = 0.6449.$
$dz = [2(2) + 3(3)](2.05 - 2) + [3(2) - 2(3)](2.96 - 3) = 0.65.$ *($\Delta z \approx dz$.)* ■

Example 0.5 *A circular cone of base radius $r = 10$ cm and height $h = 25$ cm, with error ≤ 0.1 cm. Use differentials to estimate the maximum error of the volume of the cone.*

Volume $V = \frac{\pi}{3} r^2 h$, $|\Delta r| \leq 0.1 = dr$, $|\Delta h| \leq 0.1 = dh$.
(做偏微分找全微分) $\Delta V \approx dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = \frac{2\pi}{3} rh\ dr + \frac{\pi}{3} r^2\ dh$
(代入對應值) $= \frac{2\pi}{3}(10)(25)(0.1) + \frac{\pi}{3}(10)^2(0.1) = 20\pi \text{cm}^3 \approx 63\text{cm}^3.$ ■

Functions of three variables

If $w = f(x, y, z)$ is differentiable, then

$$\begin{aligned}
\Delta w &= f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z), \\
dw &= \frac{\partial w}{\partial x}\ dx + \frac{\partial w}{\partial y}\ dy + \frac{\partial w}{\partial z}\ dz, \\
\text{and } \Delta w &\approx dw.
\end{aligned}$$

Example 0.6 *A box of dimensions 75 cm, 60 cm, 40 cm, with error 0.2 cm. Use differentials to estimate the maximum error of the volume of the box.*

Volume $V = xyz$, $|\Delta x| \leq 0.2 = dx$, $|\Delta y| \leq 0.2 = dy$, $|\Delta z| \leq 0.2 = dz$.
$\Delta V \approx dV = \frac{\partial V}{\partial x}\ dx + \frac{\partial V}{\partial y}\ dy + \frac{\partial V}{\partial z}\ dz = yz\ dx + xz\ dy + xy\ dz$
$= (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980.$ ■
($V = 75 \times 60 \times 40 = 180000$, $\Delta V = 75.2 \times 60.2 \times 40.2 - 180000 = 1987.008$,
$\frac{\Delta V}{V} \approx \frac{dV}{V} = 0.011$ *(relative error 相對誤差)*
$= 1.1\%$ *(percentage error 百分誤差).*)

Additional: (♡ 考觀念)

Theorem 2 ♡ *If f is differentiable then f is continuous.* (可微分就連續。)

Proof. (Exercise 14.4.45) Let $x = a + \Delta x$ and $y = b + \Delta y$.
As $(x, y) \to (a, b) \iff (\Delta x, \Delta y) \to (0, 0) \implies \varepsilon_1, \varepsilon_2 \to 0$,

$$\begin{aligned} f(x,y) &= f(a+\Delta x, b+\Delta y) - f(a,b) + f(a,b) \\ &\overset{\text{可微分}}{=} f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y + f(a,b) \\ &\to f_x(a,b)\cdot 0 + f_y(a,b)\cdot 0 + 0\cdot 0 + 0\cdot 0 + f(a,b) \\ &= f(a,b). \end{aligned}$$

$$\implies \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b). \qquad \blacksquare$$

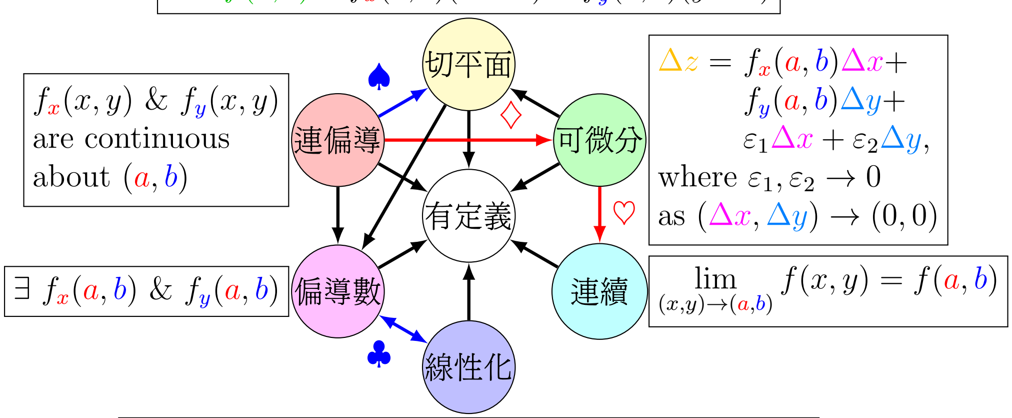
★ 相同之一: 可微分 $\implies$ 連續 (而且反向不對)。

★ 相同之二: 用微分估計誤差 ($dz \approx \Delta z$)。

Remark:

$z = f(x, y)$, $\Delta x = x - a$, $\Delta y = y - b$, $\Delta z = f(x, y) - f(a, b)$.

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

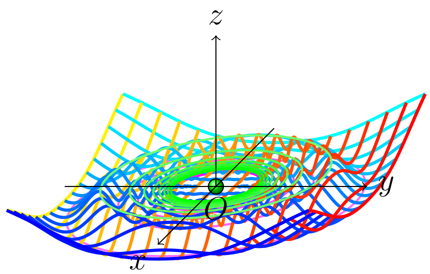


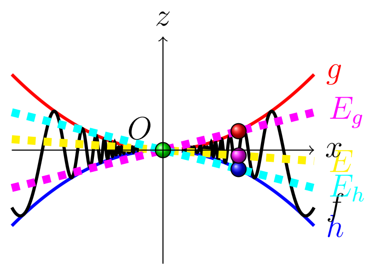
$$\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

(能找到單向箭頭的逆向反例?
Ex: $f(x, y) = |x| + |y|$ 連續, 但在 $(0, 0)$ 不可微, 沒有偏導數, 也沒有切平面。)

Additional: 有切平面, 可微分, 但是偏導(函)數不連續的函數

$$\text{Let } f(x,y)=\begin{cases}(x^2+y^2)\sin(\dfrac{1}{x^2+y^2}) & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$





[f 在 $(0,0)$ 有切平面]

考慮 $g(x,y)=x^2+y^2$ 與 $h(x,y)=-(x^2+y^2)$, 在 $(0,0)$ 都可微分, 有切平面 $z=0$。通過 $f(0,0)$, $f(x_0,y_0)$, $f(x_1,y_1)$ 的割平面 (如右上圖中 E), 會被通過 $g(0,0)$, $g(x_0,y_0)$, $g(x_1,y_1)$ 的割平面 (如右上圖中 E_g), 與通過 $h(0,0)$, $h(x_0,y_0)$, $h(x_1,y_1)$ 的割平面 (如右上圖中 E_h) 夾住, 所以當 (x_0,y_0) 與 (x_1,y_1) 靠近 $(0,0)$, g,h 在 $(0,0)$ 的切平面是 $z=0$。根據夾擠定理, f 在 $(0,0)$ 的切平面也就得是 $z=0$, 所以有切平面。($\Longrightarrow$ $f_x(0,0)=f_y(0,0)=0$.)

[f 在 $(0,0)$ 可微分 $\Longrightarrow$ 連續]

$$(x^2+y^2)\sin(\frac{1}{x^2+y^2})=\underbrace{f(x,y)-f(0,0)}_{\Delta z}$$

$$=\underbrace{f_x(0,0)}_{=0}x+\underbrace{f_y(0,0)}_{=0}y+\underbrace{x\sin(\frac{1}{x^2+y^2})}_{\varepsilon_1}\cdot x+\underbrace{y\sin(\frac{1}{x^2+y^2})}_{\varepsilon_2}\cdot y.$$

當 $(x,y)\to(0,0)$, $-|x|\le\varepsilon_1\le|x|$, $-|y|\le\varepsilon_2\le|y|$.
By 夾擠定理, $\varepsilon_1,\varepsilon_2\to 0$. By 定義, f 可微分, 所以也連續。

[f_x, f_y 在 $(0,0)$ 不連續]

$$f_x=2x\sin(\frac{1}{x^2+y^2})-\underline{\frac{2x}{x^2+y^2}\cos(\frac{1}{x^2+y^2})},$$

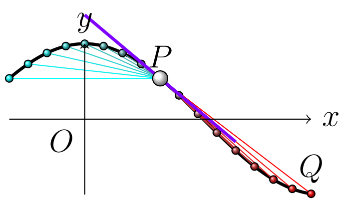
當 $(x,y)\to(0,0)$, 前項 (無底線處) 極限為零, 但後項 (有底線處) 極限不存在. 因此 f_x 在 $(0,0)$ 極限不存在, 因而不連續。f_y 的情況類似, 在 $(0,0)$ 極限不存在也不連續。

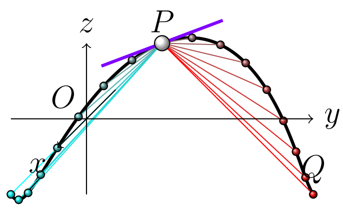
♦ Additional: Define tangent line/plane in geometry 從幾何觀點定義切線/面

曲線 C 上一點 P, 考慮線上另一點 Q, 會決定唯一一條 (割) 直線通過 P & Q, 平行 $\mathbf{n}_P(Q)(:= \overline{PQ})$。定義對 C 在 P 的切線 (***tangent line*** to C at P) 是通過 P 平行 $\lim\limits_{Q\to P} \mathbf{n}_P(Q)$ 的直線, 若且爲若極限存在。(lim 割線 = 切線。)

函數曲線 $y = f(x)$, f 在 a 可微分, $\implies L : y = f(a) + f'(a)(x - a)$.

參數曲線 $\mathbf{r}(t)$, 在 a 可微分, $\implies L : \mathbf{L}(t) = \mathbf{r}(a) + \mathbf{r}'(a)(t - a)$.





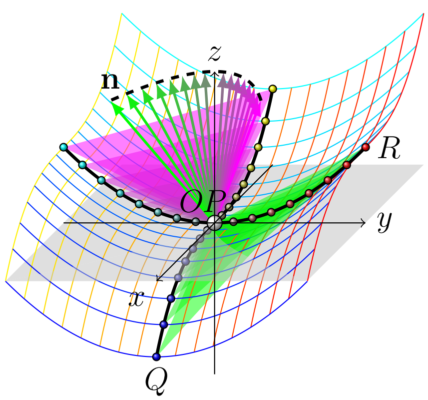
曲面 S 上一點 P, 考慮面上不共線另兩點 Q & R, 會決定唯一一個 (割) 平面通過 P, Q & R, 法向量平行 $\mathbf{n}_P(Q, R)(:= \overrightarrow{PQ} \times \overrightarrow{PR})$。定義對 S 在 P 的切平面 (***tangent plane*** to S at P) 是通過 P 法向量平行 $\lim\limits_{Q\to P, R\to P} \mathbf{n}_P(Q, R)$ 的平面, 若且爲若極限存在。(lim 割面 = 切面。)

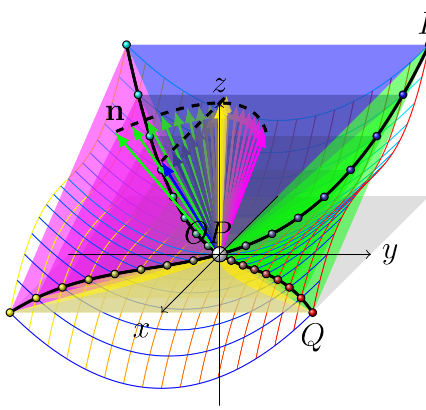
函數曲面 $z = f(x, y)$, f 在 a 可微分,

$\implies E : z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

等高曲面 $F(x, y, z) = k$, 存在 $\nabla F(a, b, c) = \langle F_x, F_y, F_z\rangle(a, b, c) \neq \mathbf{0}$,

$\implies E : F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$.





從幾何觀點定義的切平面, 也是包含所有通過該點的曲線在該點的切線的平面。如此一來, 切平面/線的存在 $\iff$ 可微分 $\implies$ 連續性。

如果定義成包含該點在x-,y-軸方向(如書上, 或是所有方向) 的曲線在該點的切線的平面, 就不保證可微分或連續; 要有連續的偏導數才可微分與連續。