1179: Probability
Lecture 18 — Joint Distributions

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This Lecture

1. Joint PMF and Marginal PMF

2. Joint PDF and Marginal PDF

3. Independent Random Variables

- Reading material: Chapter 8.1~8.2

Review: Joint PMF and Marginal PMF

Joint PMF: Let X and Y be two discrete random variables defined on the same sample space Ω. The joint PMF $p(x, y)$ is defined as

$$p(x, y) = P(X = x, Y = y) = P(\{\omega : X(\omega) = x, Y(\omega) = y\})$$

"**Marginal** PMF": Let S_X and S_Y be the sets of possible values of X and Y. The marginal PMF of X and Y are

$$P(X = x, Y \in S_Y) = P(X = x) = \sum_{y \in S_Y} p(x, y)$$

$$P(Y = y, X \in S_X) = P(Y = y) = \sum_{x \in S_X} p(x, y)$$

Example: From Joint PMF to Marginals

- Example: Let the joint PMF of X and Y be

$$\underbrace{p_{XY}(x,y)}_{} = \begin{cases} \frac{1}{25}(x^2+y^2) & \text{, if } \underline{x = 1,2}, \underline{y = 0,1,2} \\ 0 & \text{, otherwise} \end{cases}$$

- What is the marginal PMF of X and Y?

$$P_X(x) = \sum_{y \in S_Y} P_{XY}(x,y) \Rightarrow$$

Case 1: $x = 1$ or 2

$$P_X(x) = \sum_{y\in\{0,1,2\}} \frac{1}{25}(x^2+y^2) = \frac{3}{25}x^2 + \frac{5}{25}$$

Case 2: Otherwise

$$P_X(x) = 0$$

$$P_Y(y) = \sum_{x \in S_X} P_{XY}(x,y) \Rightarrow$$

Case 1: $y = 0, 1, 2$

$$P_Y(y) = \sum_{x\in\{1,2\}} \frac{1}{25}(x^2+y^2) = \frac{2}{25}y^2 + \frac{5}{25}$$

Case 2: Otherwise

$$P_Y(y) = 0$$

Example: Bernoulli and Poisson

- Example: Consider $X \sim \text{Bernoulli}(p)$, $Z_0 \sim \text{Poisson}(\lambda_0, T)$ and $Z_1 \sim \text{Poisson}(\lambda_1, T)$ (all are independent). Suppose $Y = Z_0$ if $X = 0$, and $Y = Z_1$ if $X = 1$.
  - What is the joint PMF of X and Y?
  - What is the marginal PMF of Y?

$$Y(\omega) = \begin{cases} Z_0(\omega), & \text{if } X(\omega)=0 \\ Z_1(\omega), & \text{if } X(\omega)=1 \end{cases}$$

$$P_{XY}(x,y) = \begin{cases} p \cdot \dfrac{e^{-\lambda_1 T} \cdot (\lambda_1 T)^y}{y!}, & \text{if } x=1,\ y=0,1,2,\cdots \\ (1-p) \cdot \dfrac{e^{-\lambda_0 T} \cdot (\lambda_0 T)^y}{y!}, & \text{if } x=0,\ y=0,1,2,\cdots \\ 0 & \text{otherwise} \end{cases}$$

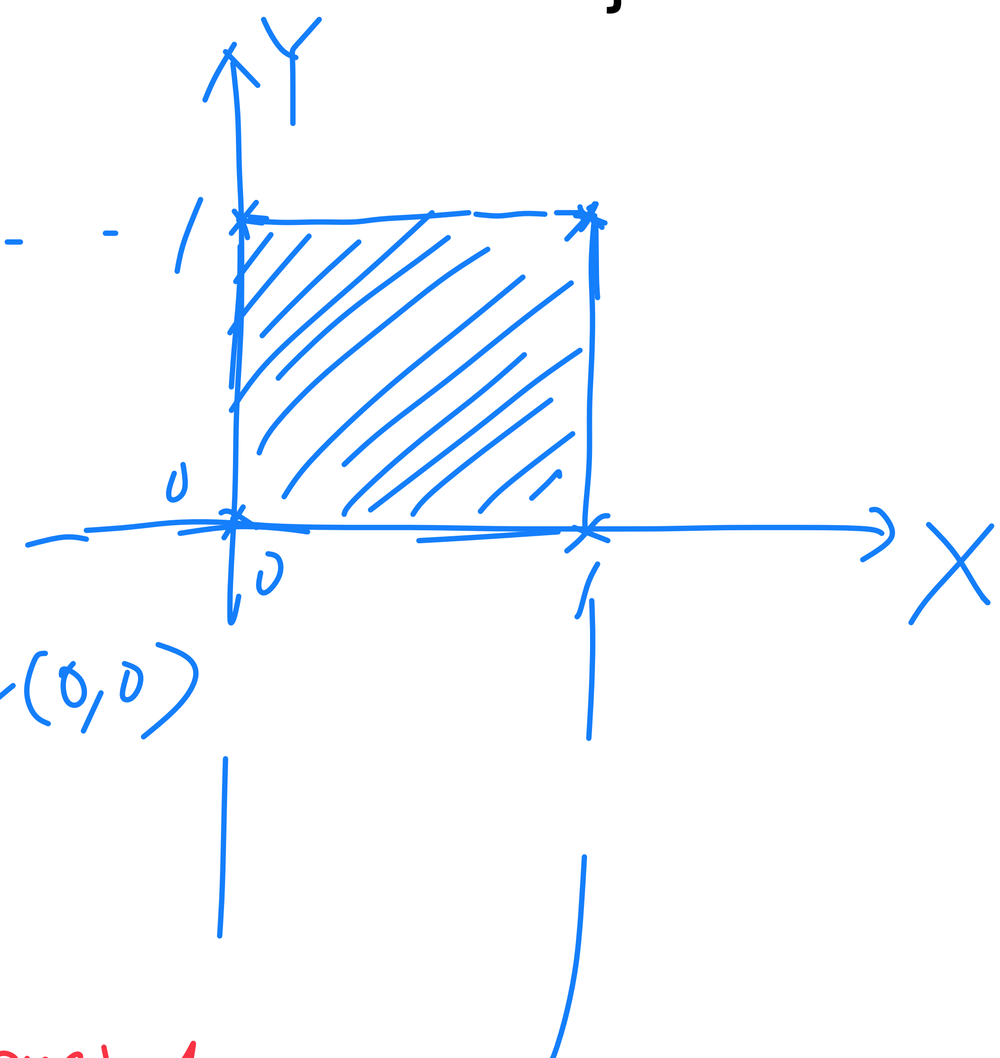
$$P_Y(y) = \begin{cases} \underline{\qquad\qquad} + \underline{\qquad\qquad}, & y=0,1,2,\cdots \\ 0, & \text{otherwise} \end{cases}$$

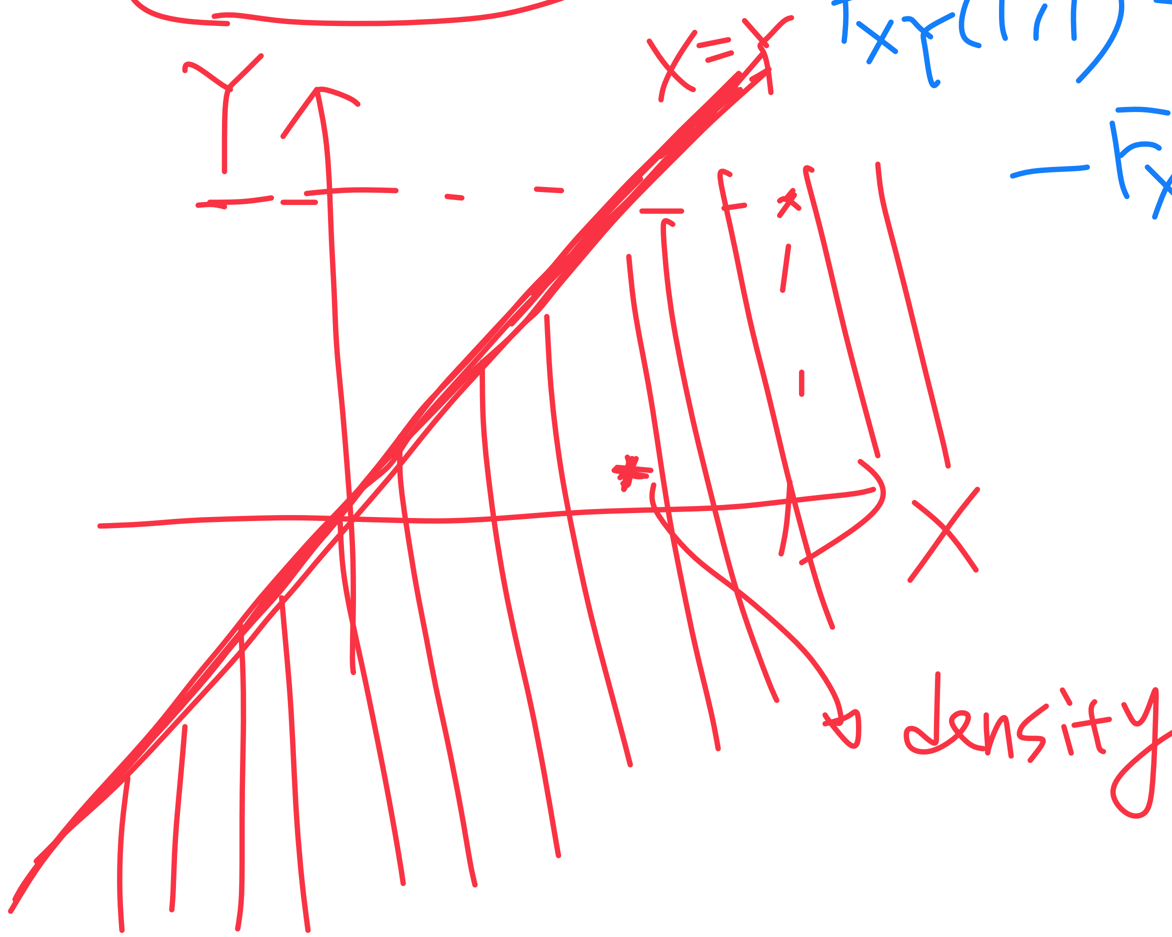
Joint PDF and Marginal PDF

Why Studying Joint PDF?

$F_{XY}(t,u) = P(X \le t \text{ and } Y \le u)$

- Question: X, Y are continuous random variables with the joint CDF $F_{XY}(x,y)$
  - $P(X \le 0, Y \le 1)$? $= F_{XY}(0,1)$
  - $P(0 < X \le 1, 0 < Y \le 1)$? $= F_{XY}(1,1) - F_{XY}(1,0) - F_{XY}(0,1) + F_{XY}(0,0)$
  - $P(X \ge Y)$?





$\int \text{density}$

1 R.V.

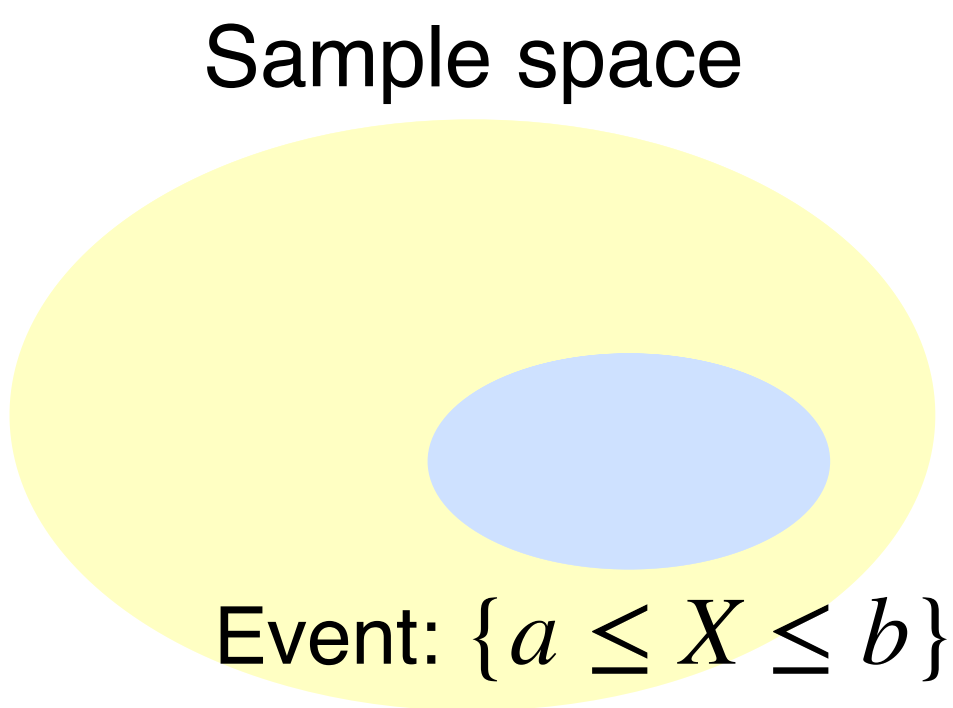
$$P(B) = \int_B \underline{f(x)}\,dx$$

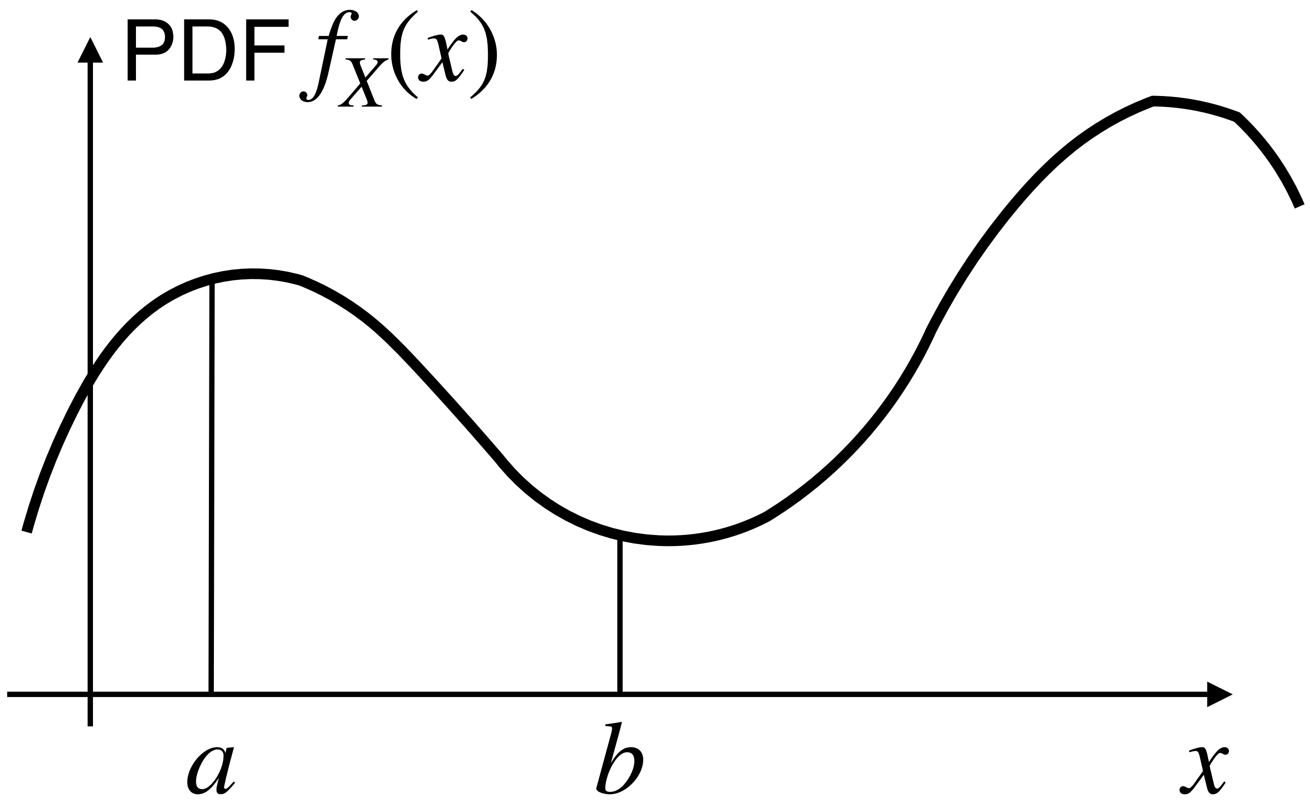
2 R.V.s

For every event B, $f_{XY}(x,y)$

$$P(B) = \underbrace{\iint}_{B} f_{XY}(x,y)\,dx\,dy$$

Recall: Probability Density Function (PDF)





Probability Density Function (PDF):
Let X be a random variable. Then, $f_X(x)$ is the PDF of X if for every subset B of the real line, we have

$$P(X \in B) = \int_B f_X(x)dx$$

Joint PDF

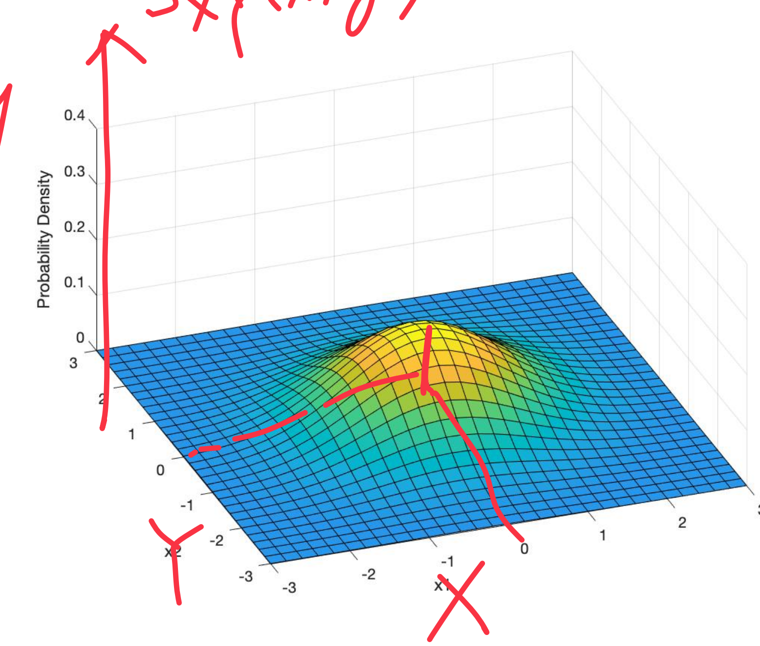
Joint PDF: Let X and Y be two continuous random variables. Then, $f_{XY}(x, y)$ is the joint PDF of X and Y if for every subset B of $\mathbb{R}^2$, we have

$$P((X, Y) \in B) = \iint_B f_{XY}(x, y)dxdy$$

- $P(X \in B_X, Y \in B_Y) = \int_{B_Y}\int_{B_X} f_{XY}(x,y)dxdy$

  $[0,1]$ $[3,5]$

- $P(X \in \mathbb{R}, Y \in \mathbb{R}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)dxdy$

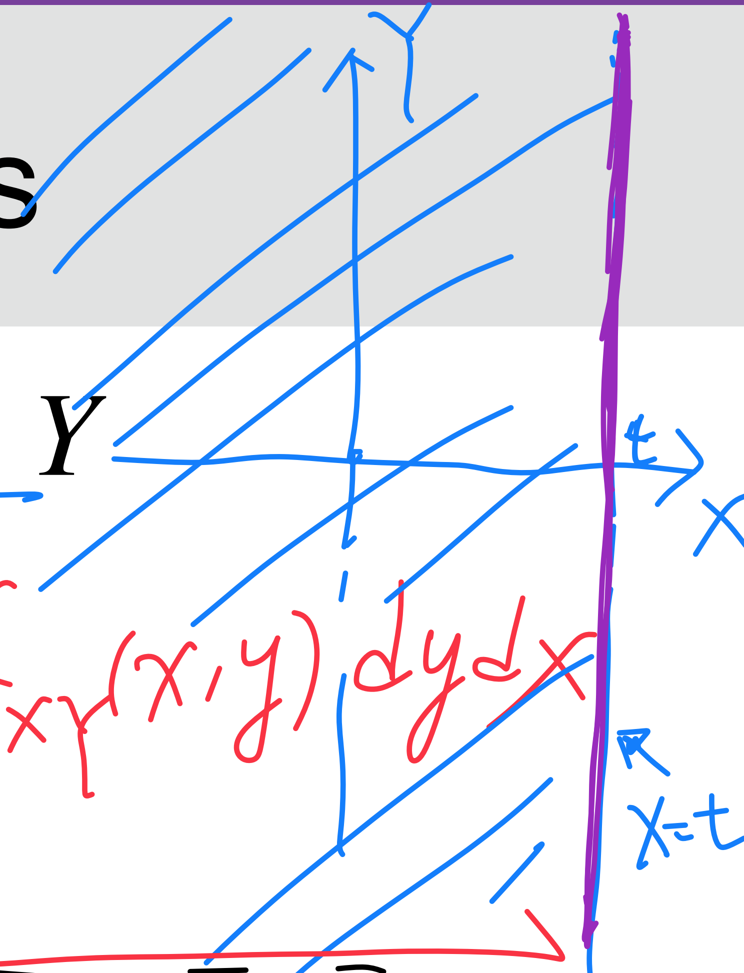


Joint PDF and Event Probabilities

$f_{XY}(x, y)$ is the joint PDF of X and Y

marginal CDF

- $P(X \leq t) = P(X \leq t \text{ and } Y \in \mathbb{R}) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy\, dx$



marginal PDF

- $f_X(x) = \left. \frac{d\, P(X \leq t)}{dt} \right|_{t=x} = \left. \frac{d \left(\int_{-\infty}^{t} \boxed{\int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy} \, dx \right)}{dt} \right|_{t=x} = \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy$

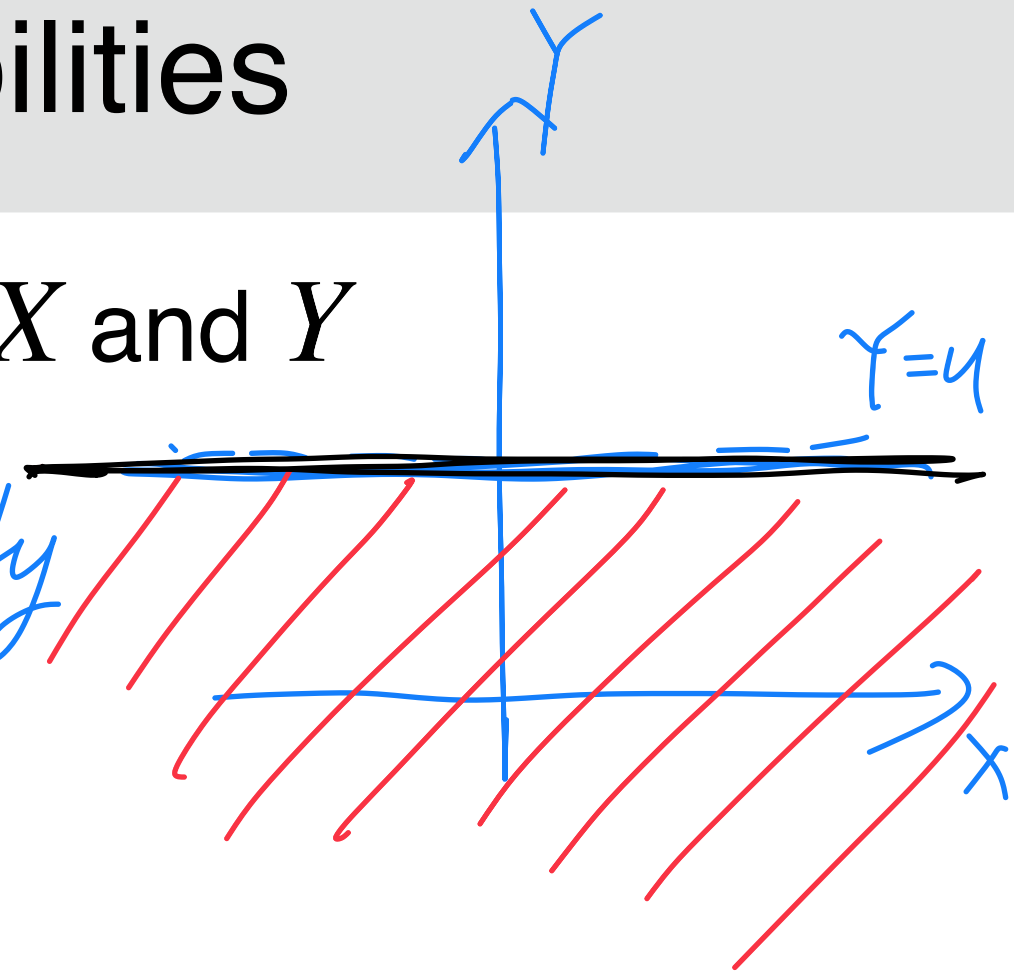
Fundamental Theorem of Calculus

Joint PDF and Event Probabilities

$f_{XY}(x, y)$ is the joint PDF of X and Y

- $P(Y \leq u) = \int_{-\infty}^{u} \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy$

  marginal CDF



- $f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$

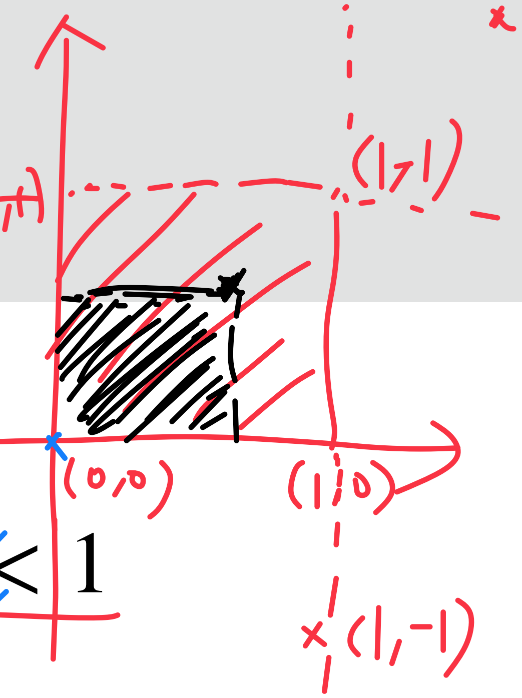
Marginal PDF

"**Marginal PDF**": Let X and Y be two continuous random variables, and $f_{XY}(x, y)$ is the joint PDF of X and Y. The marginal PDF of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

Example: (1) Joint PDF → Joint CDF



- Example: The joint PDF of X and Y is

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{, if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

- Joint CDF of X and Y? $F_{XY}(t,u)$

$F_{XY}(2,2) = P(X \le 2, Y \le 2)$

$t \backslash u$	$u \le 0$	$0 < u < 1$	$u \ge 1$
$t \le 0$	0	0	0
$0 < t < 1$	0	$\frac{1}{2}ut(t^2+u^2)$	
$t \ge 1$	0		

- $0 < t < 1$, $0 < u < 1$:

$$F_{XY}(t,u) = \int_{-\infty}^{u}\int_{-\infty}^{t} f(x,y)\,dx\,dy$$

$$= \int_0^u \int_0^t \frac{3}{2}(x^2+y^2)\,dx\,dy$$

$$= \int_0^u \frac{1}{2}t^3 + \frac{3}{2}t\cdot y^2 \,dy$$

$$= \frac{1}{2}t^3\cdot y + \frac{1}{2}ty^3\Big|_0^u = \frac{1}{2}ut(t^2+u^2)$$

Example: (2) Joint PDF → Marginal PDF

- Example: X and Y be

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{, if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

- Marginal PDF of X?

$$\boxed{f_X(x) = \int_{(-\infty)}^{(+\infty)} f(x,y)\,dy}$$

Case 1: $x \le 0$

$$f_X(x) = 0.$$

Case 2: $0 < x < 1$

$$f_X(x) = \int_0^1 \frac{3}{2}(x^2+y^2)\,dy$$

$$= \frac{3}{2}x^2 y + \frac{1}{2}y^3 \Big|_0^1$$

$$= \frac{3}{2}x^2 + \frac{1}{2}$$

Case 3: $x \ge 1$

$$f_X(x) = 0$$

Review: From CDF to PDF (Formally)

Derivative of CDF is PDF:
Let X be a random variable with a CDF $F_X(\cdot)$ and a PDF $f_X(\cdot)$. If $f_X(\cdot)$ is continuous at x_0, then

$$F'_X(x_0) = f_X(x_0)$$

- Question: Do we have any similar property regarding joint CDF and joint PDF?

Given Joint CDF: Find Joint PDF

Partial Derivative of Joint CDF is Joint PDF:

X and Y are two continuous random variables.

Let $F_{XY}(x, y)$ be the joint CDF of X and Y.

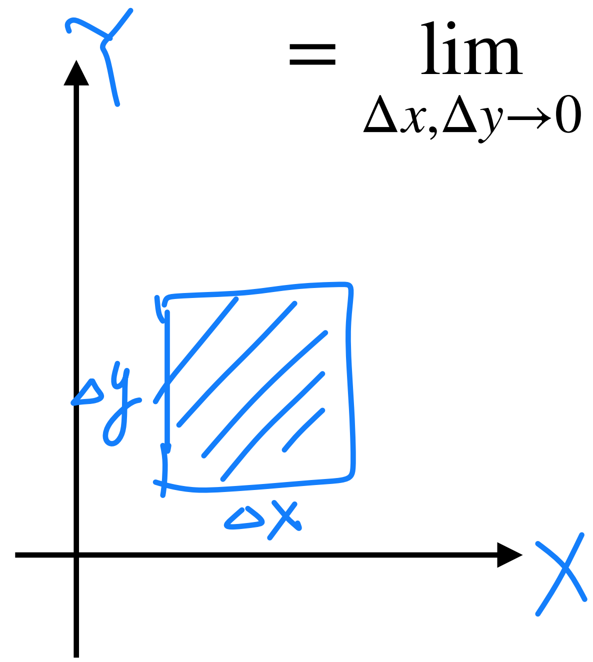
Assume the partial derivatives of $F_{XY}(x, y)$ exist. Then, one valid choice of PDF can be

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

Joint PDF: Interpret "Density" Using Limits

$$f_{XY}(x, y) \equiv \lim_{\Delta x, \Delta y \to 0} \frac{P(x < X \leq x + \Delta x, y < Y \leq y + \Delta y)}{\Delta x \Delta y}$$

$$= \lim_{\Delta x, \Delta y \to 0} \frac{F_{XY}(x + \Delta x, y + \Delta y) - F_{XY}(x, y + \Delta y) - F_{XY}(x + \Delta x, y) + F_{XY}(x, y)}{\Delta x \Delta y}$$



density = $\dfrac{\text{Probability mass on}}{\text{Area of}}$ $= \Delta x \cdot \Delta y$

Technical Issues With Joint PDF

1. Given joint CDF $F_{XY}(x, y)$, the joint PDF is <u>NOT unique</u>

2. Suppose the partial derivatives of $F_{XY}(x, y)$ exist, then $\frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$ is a valid joint PDF

3. In this class, we usually assume (unless stated otherwise):
   1. The partial derivatives of $F_{XY}(x, y)$ exist
   2. $\frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$ is the joint PDF to operate with

A Quick Summary

$P(X \leq t) = P(X \leq t \text{ and } Y \in \mathbb{R})$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Joint CDF}\\ F_{XY}(t,u)\end{array} \arrow[d, shift right, "\begin{array}{c}\text{(second-order}\\ \text{derivative??)}\end{array}"'] & \begin{array}{c}\text{Marginal CDF}\\ F_X(t), F_Y(u)\end{array} \arrow[d, shift right, "\frac{dF_X(t)}{dt}"'] \\
\begin{array}{c}\text{Joint PDF}\\ f_{XY}(x,y)\end{array} \arrow[u, shift right, "\int\int f_{XY}(x,y)\,dx\,dy"'] \arrow[ur, "\int\int f_{XY}(x,y)\,dy\,dx"] \arrow[r, "\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy"'] & \begin{array}{c}\text{Marginal PDF}\\ f_X(x), f_Y(y)\end{array} \arrow[u, shift right, "\int f_Y(y)\,dy"']
\end{tikzcd}
$$

Marginal CDF/PDF to Joint CDF/PDF?

- Question: Could we get joint CDF/PDF from marginal CDF/PDF?

$$\begin{array}{ccc} \text{Joint CDF} & & \text{Marginal CDF} \\ F_{XY}(x, y) & \longleftarrow & F_X(x), F_Y(y) \end{array}$$

$$\begin{array}{ccc} \text{Joint PDF} & & \text{Marginal PDF} \\ f_{XY}(x, y) & \longleftarrow & f_X(x), f_Y(y) \end{array}$$

Independent Random Variables

Recall: Independence of 2 Random Variables

Definition: Two random variables X, Y are said to be **independent** if for arbitrary sets of real numbers A, B, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent, i.e.

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

- Remark: Same definition for both discrete and continuous random variables

- Question: What if we choose the sets as $A = (-\infty, t]$ and $B = (-\infty, u]$?

Property: Independence of 2 Random Variables

Independence $\equiv$ joint CDF is the product of the marginal CDFs:
Two random variables X, Y are **independent** if and only if

$$F_{XY}(t,u) = F_X(t) \cdot F_Y(u)$$

- Remark: This property holds for both discrete and continuous random variables

Example: 2 Discrete Uniform Random Variables

- Example: X, Y are two independent discrete uniform random variables with the same range $\{0,1,2\}$.
  - CDF of X? How about Y?
  - Joint CDF of X and Y?

Example: Continuous Uniform and Exponential

- Example: $X \sim$ Unif$(0,1)$ and $Y \sim$ Exp$(\lambda = 1)$ be two independent continuous random variables.
  - Joint CDF of X and Y?

Property: Independence of 2 Discrete Random Variables

Joint PMF is the product of the marginal PMFs under independence:

If two discrete random variables X, Y are **independent**, then the joint PMF satisfies that

$$p_{XY}(x, y) = p_X(x) \cdot p_Y(y)$$

- Proof:

Property: Independence of 2 Continuous Random Variables

Joint PDF is the product of the marginal PDFs under independence:

If two continuous random variables X, Y are **independent**, then the joint PDF satisfies that

$$f_{XY}(t, u) = f_X(t) \cdot f_Y(u)$$

- Proof:

15-Minutes Brain Workout

Joint PMF $P(X = x, Y = y)$	Marginal PMF $P(X = x), P(Y = y)$
Joint CDF $F_{XY}(x, y)$	Marginal CDF $F_X(x), F_Y(y)$
Joint PDF $f_{XY}(x, y)$	Marginal PDF $f_X(x), f_Y(y)$