11.6 Absolute convergence and the ratio and root tests

1. absolutely and conditionally convergence $\sum |a_n|$
2. Ratio Test $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}|$
3. Root Test $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$
4. rearrangement 重排

不是正項 (非負), 也不是交錯, 考慮用絕對值變成正項 (非負) 來檢驗。
如果不能用積分 (非負遞減), 找不到別人比較, 也不是交錯的, 只好跟自己比。

0.1 Absolutely and conditionally convergence

For $\sum a_n$, the series of absolute values is $\boxed{\sum |\boldsymbol{a_n}|} = |a_1| + |a_2| + |a_3| + \cdots$.

Example 0.1 $\sum \frac{(-1)^{n-1}}{n^2}$ *converges by the Alternating Series Test.*

$$\sum \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum \frac{1}{n^2},\ p\text{-series with } p = 2 > 1,\ \textit{converges}. \qquad \blacksquare$$

Example 0.2 $\sum \frac{(-1)^{n-1}}{n}$, *alternating harmonic series, is converges.*

$$\sum \left|\frac{(-1)^{n-1}}{n}\right| = \sum \frac{1}{n},\ \textit{harmonic series, diverges}. \qquad \blacksquare$$

(收斂加了絕對值不一定還是收斂。)

♦ **Fact:**

$$\sum \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12},$$

$$\sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6},$$

$$\sum \frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

$$\left(\sum \frac{(-1)^{n-1}}{n^2} = \sum \frac{1}{n^2} - 2\sum \frac{1}{(2n)^2} = \sum \frac{1}{n^2} - \frac{2}{2^2}\sum \frac{1}{n^2} = \frac{1}{2}\sum \frac{1}{n^2} = \frac{\pi^2}{12}.\right)$$

Define: A series $\sum a_n$ is ***absolutely convergent*** 絕對收斂 if $\sum |a_n|$ is convergent. (加了會收斂。)

Define: A series $\sum a_n$ is ***conditionally convergent*** 條件收斂 if it is convergent but **not** absolutely convergent. (本身收斂, 加了不收斂。)

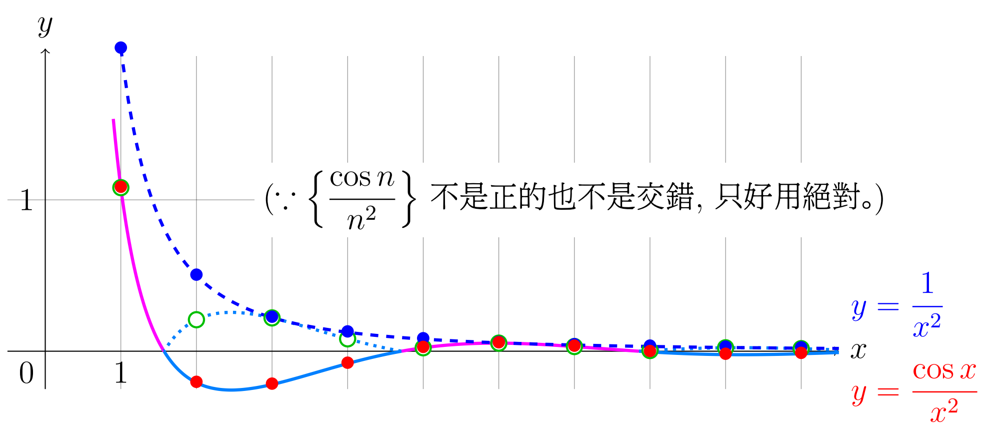
$\sum a_n \backslash \sum \lvert a_n\rvert$	*Conv.*	*Div.*	級數分三種:
Conv.	*Abs.*	*Cond.*	會收斂, 絕對與條件。
Div.	$\nexists$	*Div.*	不收斂, 絕對不收斂。

Theorem 1 *If a series $\sum a_n$ is absolutely convergent, then it is convergent. (絕對收斂會收斂。青出於藍勝於藍。)*

Proof. $\because 0 \le a_n + |a_n| \le 2|a_n|$ (非負), and $\sum 2|a_n|$ converges (大收), so $\sum (a_n + |a_n|)$ converges (小收) by the Comparison Test.
$\therefore \sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ is convergent. ∎

Example 0.3 *Determine whether the series $\sum \dfrac{\cos n}{n^2}$ is convergent or divergent.*

$\left|\dfrac{\cos n}{n^2}\right| \le \dfrac{1}{n^2}$, $\sum \dfrac{1}{n^2}$, p-series with $p = 2 > 1$, converges,
$\sum \left|\dfrac{\cos n}{n^2}\right|$ is convergent by the Comparison Test,
$\sum \dfrac{\cos n}{n^2}$ is absolutely convergent and hence convergent by the Theorem. ∎



Example 0.4 (extend) *(Exercise 11.5.34)* $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$

Let $a_n = (-1)^{n-1}\frac{(\ln n)^p}{n}$, *and* $b_n = |a_n| = \frac{(\ln n)^p}{n}$. ....*(交錯)*

(檢查 $\sum_{n=2}^{\infty} a_n$ *的收發)*

Let $f(x) = \frac{(\ln x)^p}{x}$, $f'(x) = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ *when* $x > e^p > 0$,
so $b_n = f(n) \geq f(n+1) = b_{n+1}$ *for* $n \geq \lceil e^p \rceil$. ....*($b_n \searrow$ 遞減)*
For $p \leq 0$, $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n(\ln n)^{-p}} = 0$. *For* $p > 0$,

[Sol 1] $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{(\ln n)^p}{n} \overset{l'H}{=} \lim_{n\to\infty}\frac{p(\ln n)^{p-1}}{n} \overset{l'H}{=} \cdots$ $\qquad (\frac{\infty}{\infty})$
$\overset{l'H}{=} \lim_{n\to\infty}\frac{p(p-1)\cdots(p-\lceil p\rceil+1)}{n(\ln n)^{\lceil p\rceil - p}} = 0.$ $\qquad$ *(ℓ'Hospital rule $\lceil p \rceil$ 次)*

[Sol 2] $1 < \ln n < 2pn^{1/2p}$, $\frac{1}{n} = \frac{1^p}{n} < \frac{(\ln n)^p}{n} < \frac{(2pn^{1/2p})^p}{n} = \frac{2^p p^p}{\sqrt{n}}$,
$\lim_{n\to\infty}\frac{1}{n} = 0 = \lim_{n\to\infty}\frac{2^p p^p}{\sqrt{n}}$, *by the Squeeze Theorem,* $\lim_{n\to\infty} b_n = 0$. ....*($b_n \to 0$)*

$\therefore \sum_{n=2}^{\infty} a_n$ *converges for all* p *by the Alternating Series Test.*

(檢查 $\sum_{n=2}^{\infty} |a_n|$ *的收發)*

$f(x)$ *continuous, positive, decreasing on* $[\lceil e^p \rceil, \infty)$ *(including* $p = 0$*).*
$\int_e^{\infty} f(x)\, dx = \int_e^{\infty}\frac{(\ln x)^p}{x}\, dx = \int_1^{\infty} u^p\, du = \int_1^{\infty}\frac{1}{u^{-p}}\, du$ *converges* $\iff$
$-p > 1 \iff p < -1$. *(應該積* $[e^p, \infty)$, *但是* $[e, e^p]$ *&* $\sum_{n=2}^{\lfloor e^p \rfloor}|a_n|$ *有限不影響。)*

$\therefore \sum_{n=2}^{\infty} |a_n|$ *converges* $\iff p < -1$ *by the Integral Test.*

	$\sum a_n$	$\sum \vert a_n \vert$	
$p < -1$	*Conv.*	*Conv.*	*Abs.*
$p \geq -1$	*Conv.*	*Div.*	*Cond.*

$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$ *is* $\begin{cases} \text{absolutely convergent for } p < -1, \text{ and} \\ \text{conditionally convergent for } p \geq -1. \end{cases}$ ∎

Example 0.5 (extend) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}\ln n + n\sqrt{\ln n}}$ *is conditionally convergent.*

Let $a_n = \dfrac{(-1)^{n-1}}{\sqrt{n}\ln n + n\sqrt{\ln n}}$, *and* $b_n = |a_n| = \dfrac{1}{\sqrt{n}\ln n + n\sqrt{\ln n}}$.

For $n \geq 1$, $\because \ln n < n$, $\sqrt{\ln n} < \sqrt{n}$,

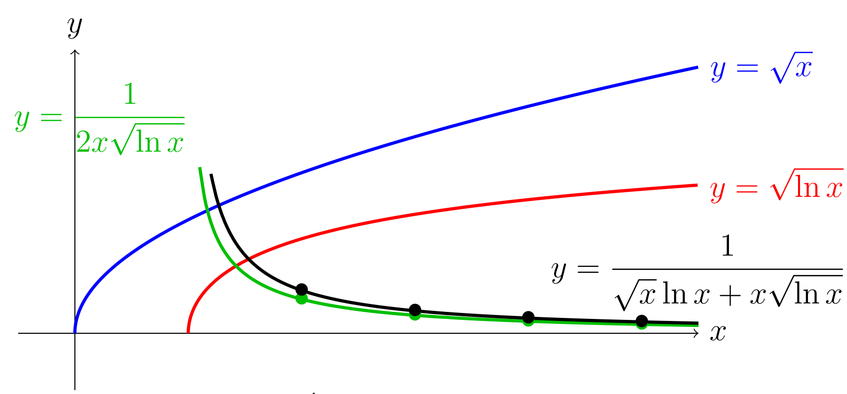
$\sqrt{n}\ln n = \sqrt{n}\sqrt{\ln n}\sqrt{\ln n} < \sqrt{n}\sqrt{n}\sqrt{\ln n} = n\sqrt{\ln n}$, $b_n > \dfrac{1}{2n\sqrt{\ln n}}$.

$$\int_2^{\infty} \frac{1}{2x\sqrt{\ln x}}\,dx = \lim_{n\to\infty}\int_2^{n} \frac{1}{2x\sqrt{\ln x}}\,dx \overset{u=\ln x}{=} \lim_{n\to\infty}\int_{\ln 2}^{\ln n} \frac{1}{2\sqrt{u}}\,du$$
$$= \lim_{n\to\infty} \sqrt{u}\Big|_{\ln 2}^{\ln n} = \lim_{n\to\infty} \sqrt{\ln x}\Big|_2^{n} = \infty.$$

$\sum b_n$ *diverges by the Integral Test & the Comparison Test.*

$\because b_n \searrow 0$, $\sum a_n$ *converges by the Alternating Series Test.*

$\therefore \sum a_n$ *is conditionally convergent.* ■



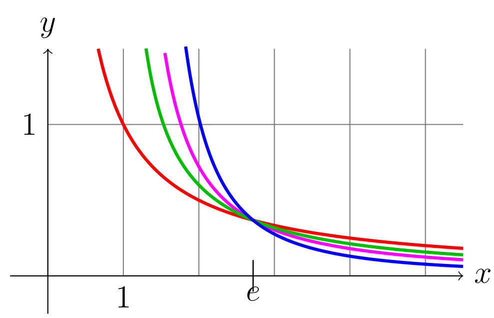
Note: 不是比 $\sum \frac{1}{n}$ 小就一定收斂。

For $x > e$,

$\dfrac{1}{x} > \dfrac{1}{x\sqrt{\ln x}} > \dfrac{1}{x\ln x} > \dfrac{1}{x(\ln x)^2}$.

$\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}, \sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}, \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverge,

$\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges.



0.2 Ratio test and root test

Theorem 2 (The Ratio Test 比值測試)

> *(i) If* $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$, *then* $\sum a_n$ *is (absolutely) convergent.*
>
> *(ii) If* $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1$ *or* $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \infty$, *then* $\sum a_n$ *is divergent.*
>
> *(iii) If* $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, *the Ratio Test is inconclusive* 未確定.

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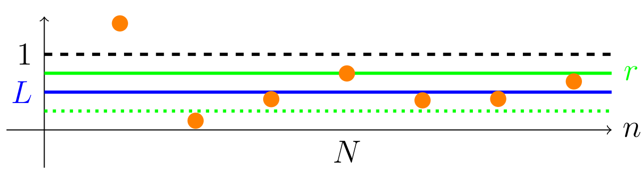
Proof. (i) Say $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L$.



$$\begin{array}{ccccc}
|a_N| + & |a_{N+1}| & + |a_{N+2}| & + \cdots \\
\shortparallel & \wedge & \wedge & \cdots \\
|a_N| & |a_N| r & + |a_{N+1}| r + & \cdots \\
 & & \wedge & \cdots \\
 & & |a_N| r^2 & + \cdots
\end{array}$$

Choose $L < r < 1$, then $\exists\ N > 0 \ni n > N \implies \left|\dfrac{a_{n+1}}{a_n}\right| < r$. $(\varepsilon = r - L)$

$|a_n| \le |a_{n-1}| r \le |a_{n-2}| r^2 \le \ldots \le |a_N| r^{n-N}$ for $n \ge N$.

$\sum\limits_{n=N}^{\infty} |a_N| r^{n-N}$, geometric series with $|r| < 1$, converges.

$\sum\limits_{n=N}^{\infty} |a_n|$ and hence $\sum |a_n|$ converges by the Comparison Test. $(b_n = |a_N| r^{n-N})$

$\therefore \sum a_n$ is absolutely convergent. (有限項不影響。)

(ii) $\exists\ N > 0 \ni n > N \implies \left|\dfrac{a_{n+1}}{a_n}\right| > 1$. $|a_{n+1}| > |a_n|$, $\lim\limits_{n\to\infty} a_n \ne 0$.

$\therefore \sum a_n$ is divergent by the Test for Divergence. ■

Timing: a_n 有 $n!$ (n factorial 階乘) 很好用。

Note: $\sum \dfrac{1}{n^2}$ and $\sum \dfrac{1}{n}$ both have $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, but the former converges while the latter diverges. 要用別的方法。

Example 0.6 *Test the series* $\sum(-1)^n\dfrac{n^3}{3^n}$ *for absolute convergence.*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\dfrac{(n+1)^3}{3^{n+1}}}{(-1)^n\dfrac{n^3}{3^n}}\right| = \lim_{n\to\infty}\frac{1}{3}(1+\frac{1}{n})^3 = \frac{1}{3} < 1,$$

$\sum(-1)^n\dfrac{n^3}{3^n}$ *is absolutely convergent by the Ratio Test.* ■

Example 0.7 *Test the convergence of the series* $\sum\dfrac{n^n}{n!}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1,$$

$\sum\dfrac{n^n}{n!}$ *is divergent by the Ratio Test.*
*(*其實 $a_n \geq n$, $\lim\limits_{n\to\infty} a_n = \infty(\neq 0)$, *by the Test for Divergence.)* ■

Theorem 3 (The Root Test 根值測試)

(i) If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$, *then* $\sum a_n$ *is (absolutely) convergent.*

(ii) If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} > 1$ *or* $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, *then* $\sum a_n$ *is divergent.*

(iii) If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1$, *the Root Test is inconclusive.*

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Proof. (Exercise 11.6.49) (Hint: $\sqrt[n]{|a_n|} \gtreqless r \gtreqless 1$, $|a_n| \gtreqless r^n$.)

Timing: a_n 有 n 冪次很好用, 如果是 (iii) 一樣要用別的方法。

Example 0.8 *Test the convergence of the series* $\sum\left(\dfrac{2n+3}{3n+2}\right)^n$.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty}\frac{2n+3}{3n+2} = \lim_{n\to\infty}\frac{2+\frac{3}{n}}{3+\frac{2}{n}} = \frac{2}{3} < 1,$$

$\sum\left(\dfrac{2n+3}{3n+2}\right)^n$ *is absolutely convergent by the Root Test.* ■

0.3 Rearrangement

一個收斂的級數不管是絕對收斂還是條件收斂, 先加跟後加一不一樣? infinite sum 會不會跟 finite sum 行爲一樣? 也就是說, sum 與 partial sums 的關係。

Define: A ***rearrangement*** 重排 of an infinite series $\sum a_n$ is a series obtained by changing the order of the terms (換順序).

Question: 順序有差嗎? Alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$:

$$\begin{aligned}
\ln 2 &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\ldots,\\
\text{(重排)} &= \underbrace{1-\frac{1}{2}}-\frac{1}{4}+\underbrace{\frac{1}{3}-\frac{1}{6}}-\frac{1}{8}+\underbrace{\frac{1}{5}-\frac{1}{10}}-\frac{1}{12}+\ldots\\
&= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\ldots\\
&= \frac{1}{2}\Big(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\Big) = \frac{1}{2}\ln 2.
\end{aligned}$$

$\Longrightarrow$ $\ln 2 = 0$ or $2 = 1$!?

$$\begin{array}{rl}
1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\ldots = & \ln 2,\\
+)\ 0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+0+\frac{1}{10}+\ldots = & \frac{1}{2}\ln 2,\\
\hline
=)\ 1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+0+\ldots = & \frac{3}{2}\ln 2,
\end{array}$$

$$\Longrightarrow\ 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}+\ldots = \frac{3}{2}\ln 2.$$

Theorem 4 (Riemann's Rearrangement Theorem)
If $\sum a_n$ is absolutely convergent, then any rearrangement of $\sum a_n$ has the same sum as $\sum a_n$. (絕對收斂, 怎麼排都一樣。)
If $\sum a_n$ is conditionally convergent, then for any real number r, there exists a rearrangement of $\sum a_n$ that has the sum r. (條件收斂, 什麼都排得到。)

♦: **Proof.** (Exercise 11.6.51 & 11.6.52)

♦: More about Riemann's Rearrangement Theorem:
There exists a divergent rearrangement of a conditionally convergent series.
條件收斂級數還能重排出發散($\to\infty$, $\to -\infty$, or oscillation(震盪)) 級數。