Part I

◎ 單選擇題 (單選十題, 每題五分, 共五十分, 答錯不倒扣。)

2. Let $f(x) = (1 + \frac{a}{x})^{1/x}$, where $a \neq 0$. Then $f'(a) =$ 47:53

(A) $\frac{2^{1/a}}{2a}$; (B) $-\frac{2^{1/a}}{2a^2}$; (C) $-\frac{2^{1/a}}{a^2}(\ln 2 - \frac{1}{2})$; **(D)** $\boxed{\boldsymbol{-\frac{2^{1/a}}{a^2}(\ln 2 + \frac{1}{2})}}$.

Solution: $y = f_a(x)$, $\ln y = \frac{1}{x}\ln(1 + \frac{a}{x})$,
$\frac{y'}{y} = \frac{-1}{x^2}\ln(1 + \frac{a}{x}) + \frac{1}{x}\frac{-a/x^2}{1 + a/x} = \frac{-1}{x^2}(\ln(1 + \frac{a}{x}) + \frac{a}{x+a})$,
$f'(a) = f(a)\frac{-1}{a^2}(\ln(1 + \frac{a}{a}) + \frac{a}{a+a}) = \frac{-2^{1/a}}{a^2}(\ln 2 + \frac{1}{2})$.

◎ 多選擇題 (多選五題, 每題五分, 共二十五分。答錯一個選項扣兩分, 錯兩個選項以上不給分, 分數不倒扣。)

11. Which of the following statements are **true**? 21:27:52

(A) If both $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$ hold, then it follows that $\lim_{x\to 0} \big(f(x) - g(x)\big) = 0$.

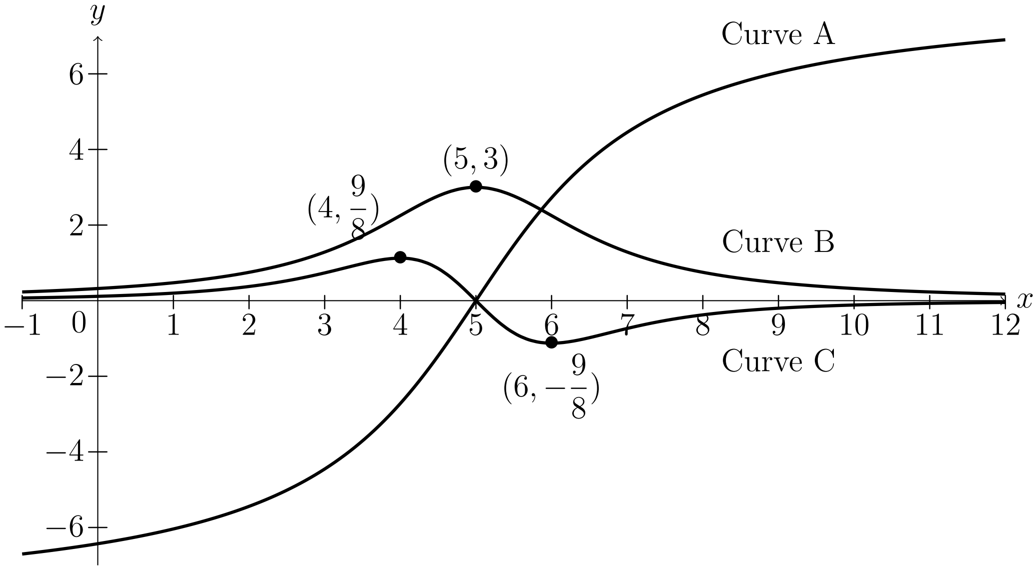
(B) $\boxed{\textbf{The equation } \boldsymbol{x^{10} - 10x^2 + 8 = 0} \textbf{ has a root in } \boldsymbol{(0, 10)}.}$

(C) If $|f|$ is integrable on $[0, 1]$, then so is f.

(D) **Every continuous function defined on $\mathbb{R}$ has at most two horizontal asymptotes.**

Solution: Indeterminant form of type $\infty - \infty$. . . . . (A)
$f(0)f(1) = 8 \cdot (-1) < 0$, $x^{10} - 10x^2 + 8$ has a root on $[0, 1]$. . . . . (B)
Let $f(x) = \pm 1$ if $x \in$ or $\notin \mathbb{Q}$. . . . . (C)
$\lim_{x\to\pm\infty} f(x)$ exist or not. . . . . (D)

12. The figure below shows graphs of f, f' and f''. Which of the following statements are **true**? 64:9:27



(A) The curve B represents the graph of f.

(B) f attains a local maximum at $x = 5$.

(C) $\boxed{\textbf{\textit{f} is concave upward on } (0, 5).}$

(D) **The largest slope of the graph of *f* on $[0, 10]$ is happened at $x = 5$.**

Solution: $f = A$, $f' = B$, $f'' = C$. .... (A)

$f' > 0$ no critical number. .... (B)

$f'' > 0$ on $(0, 5)$. .... (C)

max f' at 5. .... (D)

14. Which of the following statements are **true**? 7:46:47

(A) If f is differentiable and increasing on $\mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

(B) If $f(x)$ has a critical point at $x = c$, then $f'(c) = 0$.

(C) If $f(t)$ is differentiable on $(0, \infty)$ and has a limit as $t \to \infty$, then $\lim_{t\to\infty} f'(t) = 0$.

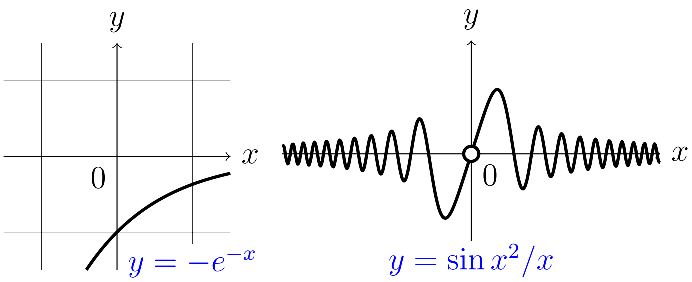
(D) $\boxed{\textbf{A continuous function on } [a, b] \textbf{ attains its absolute maximum.}}$

Solution: $f(x) = -e^{-x}$. .... (A)

$f'(c)$ may not exist. .... (B)

Let $f(t) = \dfrac{\sin t^2}{t}$, then $\lim_{t\to\infty} f(t) = 0$, but $f'(t) = 2\cos t^2 - \dfrac{\sin t^2}{t^2}$ which $\lim_{t\to\infty} f'(t)$ does not exist. .... (C)

Extreme Value Theorem. .... (D)



15. Consider the function $f(x) = \begin{cases} x\cos x, & \text{if } x \text{ is rational} \\ \sin x, & \text{if } x \text{ is irrational} \end{cases}$.
Which of the following statements are **true**? 7:59:33

(A) $\boxed{\text{\textbf{$f(x)$ is continuous at $x = 0$.}}}$

(B) $\boxed{\text{\textbf{$f(x)$ is differentiable at $x = 0$.}}}$

(C) $\boxed{\text{\textbf{$f(x)$ is continuous at infinite many points.}}}$

(D) $f(x)$ is differentiable at infinite many points.

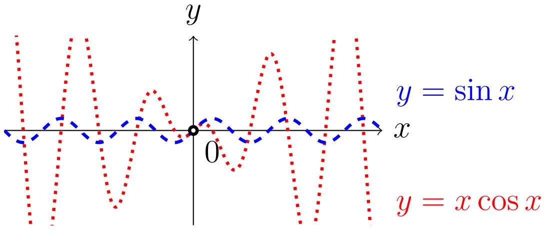
Solution: When $x = 0$, $x\cos x = 0 = \sin x$. .... (A)

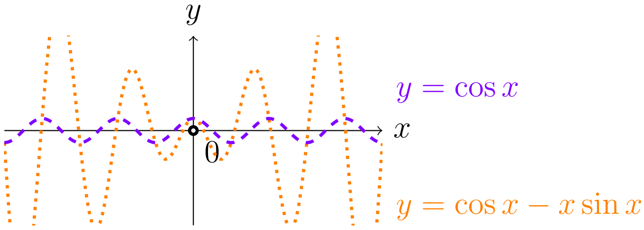
When $x = 0$, $(x\cos x)' = \cos x - x\sin x = 1 = \cos x = (\sin x)'$. . (B)

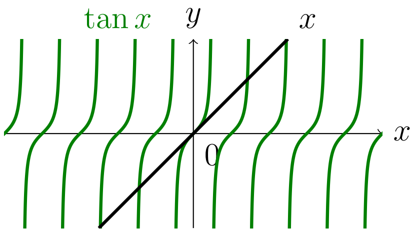
$x\cos x = \sin x \iff x = \tan x$ at ∞ many. .... (C)

$\cos x - x\sin x = \cos x \iff x\sin x = 0 \iff x = n\pi$,

but $n\pi = \tan n\pi$ only for $n = 0$. .... (D)







◎ 填空題 (填空五題, 每題五分, 共二十五分, 答錯不倒扣。)

17. The equation of the **tangent line** to the curve $y\sin(2x) = x\cos(2y)$ at the point $(\pi/2, \pi/4)$ is $y = mx + b$. Then $(m, b) =$ ___(17)___. 48:43

Solution: $(\frac{1}{2}, 0)$.

. . . .

$y'\sin 2x + 2y\cos 2x = \cos 2y - 2xy'\sin 2y$,
$y'\cdot 0 + 2\cdot\frac{\pi}{4}\cdot(-1) = 0 - 2\cdot\frac{\pi}{2}\cdot y'\cdot 1$, $y' = \frac{1}{2}$, $y = \frac{1}{2}(x - \frac{\pi}{2}) + \frac{\pi}{4} = \frac{x}{2}$.

18. Let $f(x)$ and $g(x)$ be polynomials of the **third** degree, and $f(x) - g(x) = x^3 + ax^2 + bx + c$, where a, b, c are real numbers. Assume that $f(x)$ and $g(x)$ are **tangent** to each other at $x = 1$. Moreover, $f(x)$ and $g(x)$ **only intersect** at $x = 1$. Then $(a, b, c) =$ ___(18)___. 24:52

Solution: $(-3, 3, -1)$.

. . . .

$f(x) - g(x) = (x-1)(x^2 + d^2)$, $d > 0$, or $(x-1)^3$;
$f'(1) - g'(1) = 1^2 + d^2 + 2\cdot 1(1-1) \neq 0$ or $3(1-1)^2 = 0$.
$f - g = (x-1)^3 = x^3 - 3x^2 + 3x - 1$.

19. The limit $\lim_{x\to 0} \frac{|6x-1| - |6x+1|}{x} =$ ___(19)___. 69:27

Solution: -12.

. . . .

$= \lim_{x\to 0} \frac{(1-6x) - (6x+1)}{x} = \lim_{x\to 0} \frac{-12x}{x} = -12$.

End

Part II

◎ 單選擇題 (單選十題, 每題五分, 共五十分, 答錯不倒扣。)

1. For $x > 0$, let $f(x)$ be the **average** value of e^{-t} on $[0, x]$. How many **critical numbers** on $(0, \infty)$ does f have? 50:50

   (A) $\boxed{\mathbf{0;}}$ (B) 1; (C) 2; (D) 3.

   Solution: $f(x) = \dfrac{1}{x}\displaystyle\int_0^x e^{-t}\,dt = -\dfrac{e^{-x}}{x}$, $f'(x) = \dfrac{(x+1)e^{-x}}{x^2} = 0$ when $x = -1$ and does not exist when $x = 0$, both not in $(0, \infty)$.

3. Let $f(x) = \displaystyle\int_{2x}^{3-x} e^{t^2}\,dt$. Then $(f^{-1})'(0) =$ 50:50

   (A) $\boxed{\mathbf{\dfrac{-1}{3e^4};}}$ (B) $\dfrac{1}{e^9 - 1}$; (C) $\dfrac{-1}{e^9 + 2}$; (D) $\dfrac{1}{e^4}$.

   Solution: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

   $\because e^{t^2} > 0$, $f(x) = 0 \iff 3 - x = 2x \iff x = 1$, $f^{-1}(0) = 1$.

   $f'(x) = -e^{(3-x)^2} - 2e^{(2x)^2}$, $f'(1) = -3e^4$,

   $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))} = \dfrac{1}{f'(1)} = \dfrac{-1}{3e^4}$.

4. The **value** of $\displaystyle\int_1^{\sqrt{3}} \frac{x^4 - x^3 + 2x^2 + 1}{x(x^2+1)^2}\,dx$ is 43:56

(A) $\ln\sqrt{3} - \dfrac{\pi}{24} + \dfrac{\sqrt{3}-2}{4}$; (B) $\ln\sqrt{3} - \dfrac{\pi}{12} + \dfrac{\sqrt{3}-2}{4}$;

(C) $\boxed{\boldsymbol{\ln\sqrt{3} - \dfrac{\pi}{24} + \dfrac{\sqrt{3}-2}{8};}}$ (D) $\ln\sqrt{3} - \dfrac{\pi}{12} + \dfrac{\sqrt{3}-2}{8}$.

Solution: $\dfrac{x^4 - x^3 + 2x^2 + 1}{x(x^2+1)^2}\,dx = \dfrac{1}{x} - \dfrac{1}{x^2+1} + \dfrac{1}{(x^2+1)^2}$,

$$\int \frac{1}{(x^2+1)^2}\,dx \overset{x=\tan t}{=} \int \frac{\sec^2 t}{\sec^4 t}\,dt = \int \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}\,dt$$
$$= \frac{t}{2} + \frac{\sin 2t}{4} + C = \frac{\tan^{-1} x}{2} + \frac{x}{2(1+x^2)} + C,$$
$$\int_1^{\sqrt{3}} \frac{x^4 - x^3 + 2x^2 + 1}{x(x^2+1)^2}\,dx = \left[\ln|x| - \frac{\tan^{-1} x}{2} + \frac{x}{2(1+x^2)}\right]_1^{\sqrt{3}}$$
$$= (\ln\sqrt{3} - \frac{\pi}{6} + \frac{\sqrt{3}}{8}) - (\ln 1 - \frac{\pi}{8} + \frac{1}{4}) = \ln\sqrt{3} - \frac{\pi}{24} + \frac{\sqrt{3}-2}{8}.$$

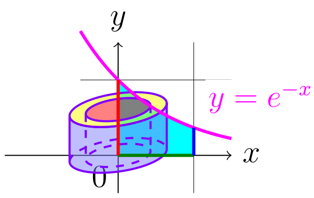
5. The region bounded by curves $y = e^{-x}$, $y = 0$, $x = 0$ and $x = 1$ is rotated about the y-axis. Then the **volume** of the resulting solid of revolution is 64:36

(A) $\dfrac{\pi}{2}(1 - e^{-2})$; **(B)** $\boxed{\boldsymbol{2\pi(1 - 2e^{-1});}}$

(C) $2\pi(1 - e^{-1})$; (D) $\pi\big(\sqrt{2} + \ln(1+\sqrt{2})\big)$.

Solution: $V = \displaystyle\int_0^1 2\pi x e^{-x}\,dx$

$$= 2\pi\left[x(-e^{-x})\Big|_0^1 - \int_0^1 -e^{-x}\,dx\right]$$
$$= 2\pi\Big[-(x+1)e^{-x}\Big]_0^1 = 2\pi(1 - 2e^{-1}).$$



6. The limit $\lim\limits_{x\to 0} \dfrac{\displaystyle\int_0^x \left(\int_0^{\sin t} \sqrt{1+u^2}\, du \right) dt}{\tan^2 x} =$ 60:39

(A) 0; **(B)** $\boxed{\mathbf{\frac{1}{2}};}$ (C) 1; (D) Does not exist.

Solution: $\lim\limits_{x\to 0} \dfrac{\displaystyle\int_0^x \int_0^{\sin t} \sqrt{1+u^2}\, du\; dt}{\tan^2 x}$ $(\frac{\mathbf{0}}{\mathbf{0}})$ twice

$\overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{\displaystyle\int_0^{\sin x} \sqrt{1+u^2}\, du}{2\tan x \sec^2 x} \overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin^2 x}\cos x}{2\sec^4 x + 4\tan^2 x \sec^2 x} = \dfrac{1}{2}.$

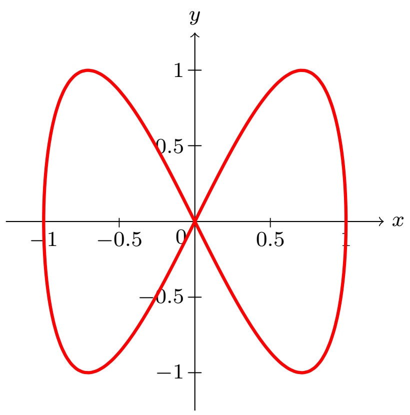
7. Which pair of parametric equations represents the graph below? 78:22

(A) $\begin{cases} x = \cos\theta \\ y = \theta + \sin\theta \end{cases}$;

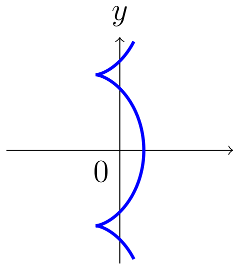
(B) $\begin{cases} x = \sin(3\theta) \\ y = \cos(4\theta) \end{cases}$;

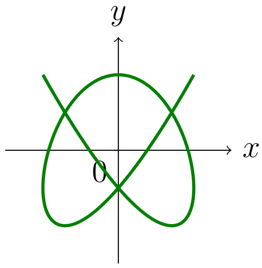
(C) $\begin{cases} x = \theta - \sin\theta \\ y = \cos\theta \end{cases}$;

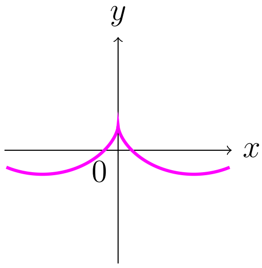
(D) $\boxed{\begin{cases} \boldsymbol{x = \sin\theta} \\ \boldsymbol{y = \sin(2\theta)} \end{cases}.}$



Solution:



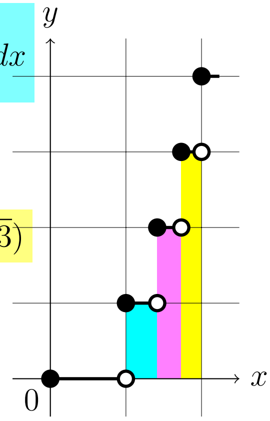




8. The greatest integer function $[\![x]\!]$ is a function from $\mathbb{R}$ to $\mathbb{Z}$ with $x-1 < [\![x]\!] \le x$. The **value** of $\displaystyle\int_0^2 [\![x^2]\!]\,dx$ is 39:61

(A) $\frac{8}{3}$; (B) 1; (C) $7-\sqrt{2}-\sqrt{3}$; **(D)** $\boxed{\mathbf{5-\sqrt{2}-\sqrt{3}.}}$

Solution: $\displaystyle\int_0^2 [\![x^2]\!]\,dx = \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3\,dx$
$= 0 + (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3})$
$= 5-\sqrt{2}-\sqrt{3}.$



9. Let f be a **continuous** function on $\mathbb{R}$ satisfying $f(x) = x^{-5}\displaystyle\int_0^x \big(1-\cos(t^2)\big)\,dt$ for $x \ne 0$. Then $f(0)$ equals 63:37

(A) $\frac{1}{2}$; (B) $\frac{1}{5}$; **(C)** $\boxed{\mathbf{\frac{1}{10};}}$ (D) $\frac{1}{20}$.

Solution: $f(0) = \displaystyle\lim_{x\to 0} \frac{\int_0^x \big(1-\cos(t^2)\big)\,dt}{x^5}$ $\quad(\frac{\mathbf{0}}{\mathbf{0}})$ twice

$\overset{l'H}{=} \displaystyle\lim_{x\to 0} \frac{1-\cos(x^2)}{5x^4} \overset{l'H}{=} \lim_{x\to 0} \frac{2x\sin(x^2)}{20x^3} = \frac{1}{10}\lim_{x^2\to 0}\frac{\sin(x^2)}{x^2} = \frac{1}{10}.$

10. The base of a solid S is the region enclosed by curves $y = \sec x$, $y = \tan x$, $x = 0$ and $x = \pi/4$. The cross-sections perpendicular to the x-axis are **squares**. Then the **volume** of S is 59:40

(A) $4 - 2\sqrt{2} - \pi/2$; (B) $4 - \sqrt{2} - \pi/2$;

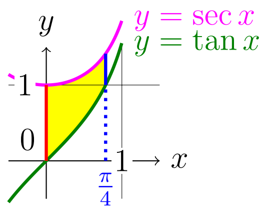
(C) $\boxed{\mathbf{4 - 2\sqrt{2} - \pi/4;}}$ (D) $4 - \sqrt{2} - \pi/4$.

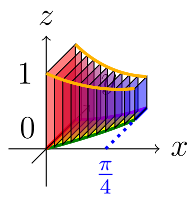
Solution: $V = \displaystyle\int_0^{\pi/4} (\sec x - \tan x)^2 \, dx$

$= \displaystyle\int_0^{\pi/4} 2\sec^2 x - 2\sec x \tan x - 1 \, dx$

$= \Big[2\tan x - 2\sec x - x\Big]_0^{\pi/4}$

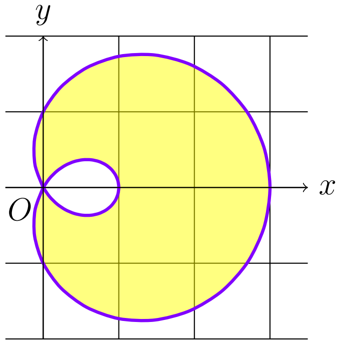
$= 4 - 2\sqrt{2} - \pi/4.$





10. (107-2) Consider the polar curve $r = 1 + 2\cos\theta$ and let R be the region inside the large loop but outside the small loop. Then, the area is 51:49

(A) $\pi + \sqrt{3}$;
(B) $\pi + 2\sqrt{3}$;
(C) $\boxed{\boldsymbol{\pi + 3\sqrt{3};}}$
(D) $\pi + 4\sqrt{3}$.



Solution: $r = 1 + 2\cos\theta = 0$, $\cos\theta = -\dfrac{1}{2}$, $\theta = \pm\dfrac{2}{3}\pi$.

$$A = \int_{-2\pi/3}^{2\pi/3} \frac{r^2}{2}\, d\theta - \int_{2\pi/3}^{4\pi/3} \frac{r^2}{2}\, d\theta$$
$$= 2\int_0^{2\pi/3} \frac{(1+2\cos\theta)^2}{2}\, d\theta - 2\int_{2\pi/3}^{\pi} \frac{(1+2\cos\theta)^2}{2}\, d\theta$$
$$= \int_0^{2\pi/3} (1+4\cos\theta+4\cos^2\theta)\, d\theta - \int_{2\pi/3}^{\pi} (1+4\cos\theta+4\cos^2\theta)\, d\theta$$
$$= \int_0^{2\pi/3} (3+4\cos\theta+2\cos 2\theta)\, d\theta - \int_{2\pi/3}^{\pi} (3+4\cos\theta+2\cos 2\theta)\, d\theta$$
$$= \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_0^{2\pi/3} - \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{2\pi/3}^{\pi}$$
$$= \Big[(2\pi+2\sqrt{3}-\frac{\sqrt{3}}{2})-(0+0+0)\Big] - \Big[(3\pi+0+0)-(2\pi+2\sqrt{3}-\frac{\sqrt{3}}{2})\Big]$$
$$= \pi + 3\sqrt{3}.$$

◎ 多選擇題 (多選五題, 每題五分, 共二十五分。答錯一個選項扣兩分, 錯兩個選項以上不給分, 分數不倒扣。)

13. Which ones are **convergent**? 12:40:47

(A) $\boxed{\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\,dx;}$ (B) $\displaystyle\int_0^{1/2} \frac{1}{x^2(\ln x)^4}\,dx;$

(C) $\boxed{\displaystyle\int_{-1}^1 \sqrt[3]{\frac{\sin x}{x^2}}\,dx;}$ (D) $\boxed{\displaystyle\int_0^1 \frac{\sin\sqrt{x}}{x}\,dx.}$

Solution: $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^3} = \int_{\ln 2}^\infty \frac{du}{u^{3(>1)}}$, conv. .... (A)

$\displaystyle\int_0^{1/2} \frac{dx}{x^2(\ln x)^4} \overset{(u=1/x)}{=} \int_2^\infty \frac{du}{(\ln u)^4} \overset{(\ln x<4\sqrt[4]{x})}{>} \int_2^\infty \frac{du}{4^4 u}$, div. ... (B)

$\displaystyle\int_0^1 \sqrt[3]{\frac{\sin x}{x^2}}\,dx < \int_0^1 \frac{dx}{x^{2/3(<1)}}$, conv. .... (C)

$\displaystyle\int_0^1 \frac{\sin\sqrt{x}}{x}\,dx \overset{\sin x<x}{<} = \int_0^1 \frac{\sqrt{x}}{x}\,dx = \int_0^1 \frac{dx}{x^{1/2(<1)}} = 2$, conv. . (D)

◎ 填空題 (填空五題, 每題五分, 共二十五分, 答錯不倒扣。)

16. The **length** of the parametric curve $x = \cos\theta$, $y = \theta + \sin\theta$, $\theta \in [0, \pi]$ is ___(16)___. 41:44

Solution: 4.

....

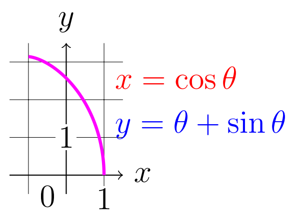
$\sqrt{(\cos\theta)'^2 + (\theta+\sin\theta)'^2} = \sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}$

$\displaystyle = \sqrt{4\cos^2\frac{\theta}{2}} = 2|\cos\frac{\theta}{2}| = 2\cos\frac{\theta}{2},\ \cos\frac{\theta}{2} \ge 0 \text{ for } 0 \le \theta \le \pi$

$\displaystyle L = \int_0^\pi \sqrt{(\cos\theta)'^2 + (\theta+\sin\theta)'^2}\,d\theta$

$\displaystyle = \int_0^\pi 2\cos\frac{\theta}{2}\,d\theta \overset{t=\theta/2}{=} \int_0^{\pi/2} 4\cos t\,dt$

$\displaystyle = 4\sin t\Big|_0^{\pi/2} = 4.$



20. The line $y = mx$ cuts the region bounded above by the curve $y = x(1-x)$ and below by the x-axis into two parts. Then, the areas of the two parts are **equal** when m is ______(20)______. 26:62

Solution: $1 - 2^{-1/3}$ or $1 - \dfrac{1}{\sqrt[3]{2}}$ or $1 - \dfrac{\sqrt[3]{4}}{2}$.

. . . .

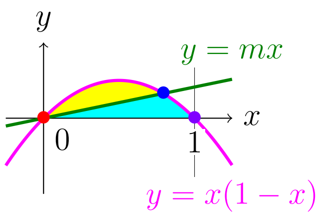
$mx = y = x(1-x)$, $x = 0, 1-m$; $x(1-x) = 0$, $x = 0, 1$.

$$\int_0^{1-m} x(1-x) - mx \; dx = \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1-m} = \frac{(1-m)^3}{6},$$

$$\int_0^{1} x(1-x) \; dx = \left[\frac{x^2}{2} - \frac{x^3}{3} + \right]_0^{1} = \frac{1}{6},$$

$$\frac{(1-m)^3}{6} = \frac{1}{2}\,\frac{1}{6},\ (1-m) = 2^{-1/3},$$

$m = 1 - 2^{-1/3}$.



End