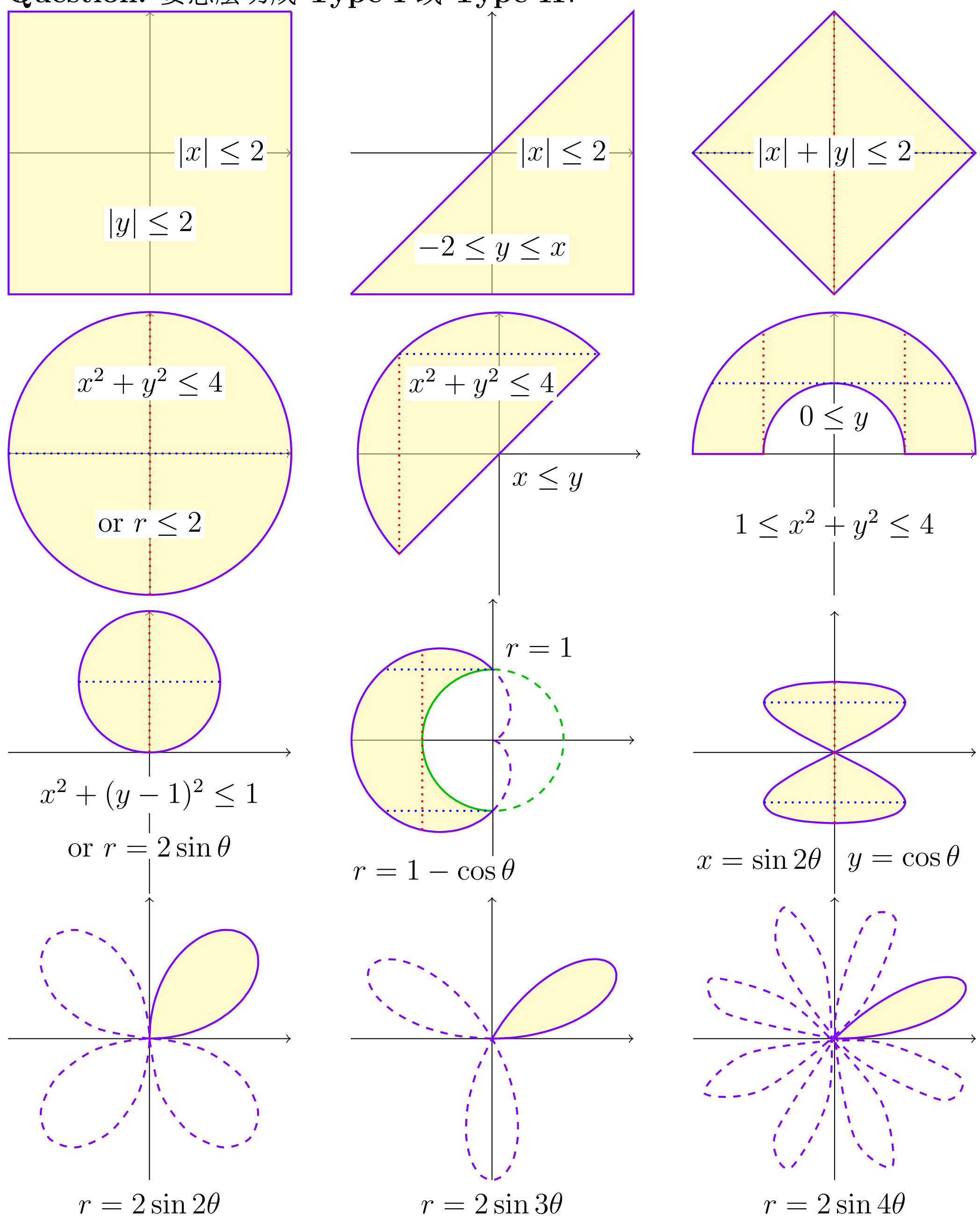
15.3 Double integrals in polar coordinates

Question: 要怎麼切成 **Type I** 或 **Type II**?



Answer: 不用切, 也不用加辣, 用極座標。

Recall: polar coordinate (r,θ) and Cartesian (rectangle) coordinate (x,y):

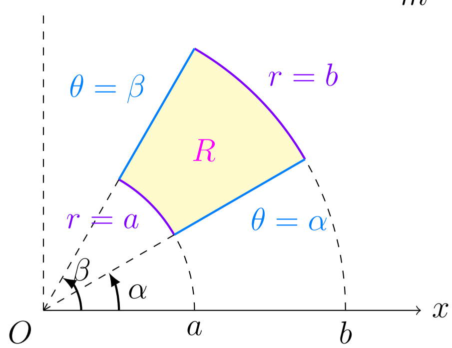
$$\boxed{r^2 = x^2 + y^2 \quad x = r\cos\theta \quad y = r\sin\theta}$$

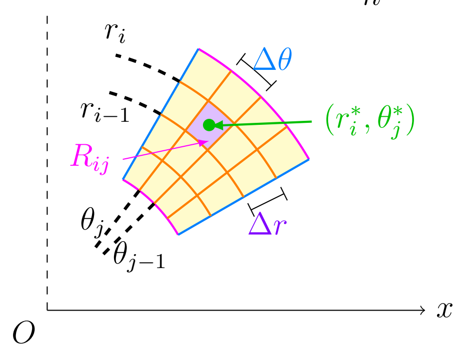
Define: A ***polar rectangle*** 極矩形

$$R = \{(r,\theta) : (0 \le) a \le r \le b,\ \alpha \le \theta \le \beta\}$$

圓心 O 半徑 a 與 b 的圓, 與正 x 軸夾角 α 與 β 的直線之間。

把 $[a,b]$ 分成 m 等分, $\Delta r = \dfrac{b-a}{m}$, 把 $[\alpha,\beta]$ 分成 n 等分, $\Delta\theta = \dfrac{\beta-\alpha}{n}$。





$\Delta A_i = A(R_{ij})$ (不是每塊都一樣大, 但同一圈的一樣大。)

$$\Delta A_i = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta,$$

where $r_i^* = \dfrac{1}{2}(r_i + r_{i-1})$ and $\theta_j^* = \dfrac{1}{2}(\theta_j + \theta_{j-1})$. (扇形面積 $= \dfrac{1}{2}r^2\theta$.)

(其實 r_i^* 就是中點 $\bar{r}_i$(midpoint), 而 θ_j^* 可以隨便選, 就一起選中點。)

Let $g(r,\theta) = f(r\cos\theta, r\sin\theta)\cdot r$, then

$$\begin{aligned}
\iint_R f(x,y)\ dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i \\
&= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\cdot r_i^*\Delta r\Delta\theta \\
&= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*,\theta_j^*)\Delta r\Delta\theta \ = \int_\alpha^\beta\int_a^b g(r,\theta)\ dr\ d\theta \\
&= \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\cdot r\ dr\ d\theta
\end{aligned}$$

Theorem 1 (Change to Polar Coordinates in a Double Integral) *If f is continuous on a polar rectangle*

$$R = \{(r,\theta) : (0 \le) a \le r \le b,\ \alpha \le \theta \le \beta\},$$

where $0 \le \beta - \alpha \le 2\pi$ *(不要重疊), then*

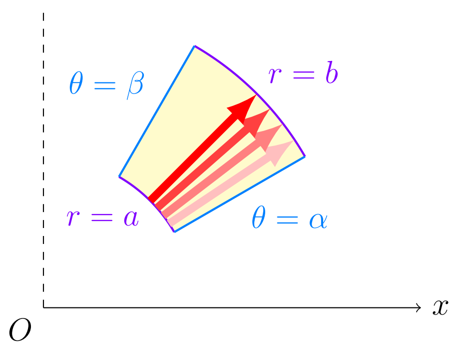
$$\boxed{\iint\limits_{R} f(x,y)\ dA = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)\cdot r\ dr\ d\theta}$$

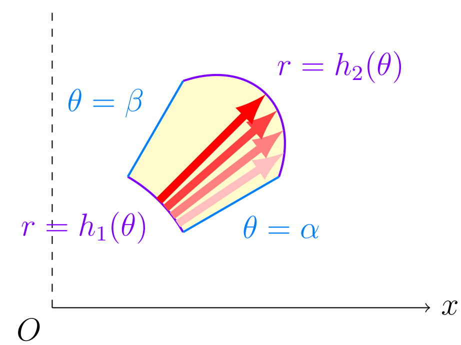
Theorem 2 *If f is continuous on a polar region*

$$D = \{(r,\theta) : (0 \le) h_1(\theta) \le r \le h_2(\theta),\ \alpha \le \theta \le \beta\},$$

where $0 \le \beta - \alpha \le 2\pi$ *(不要重疊), then*

$$\boxed{\iint\limits_{D} f(x,y)\ dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\ dr\ d\theta}$$

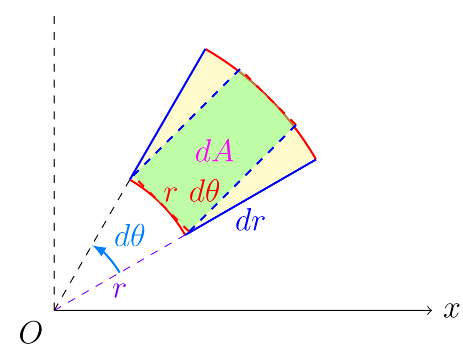




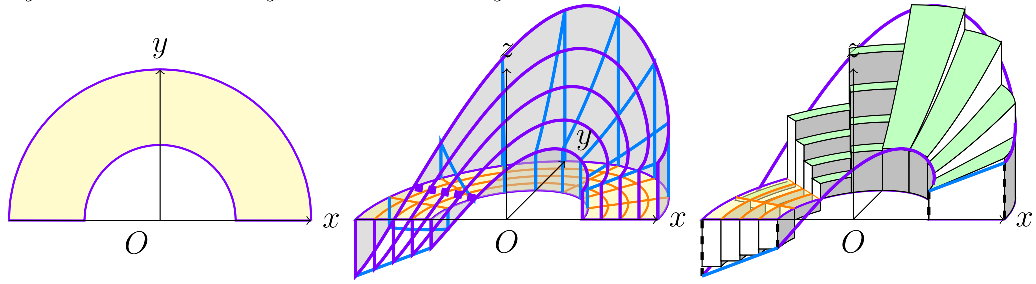
Attention: 1. r 的範圍要正的;
2. θ 的範圍不超過 2π;
3. 轉換: $x \to r\cos\theta$, $y \to r\sin\theta$,
$\boxed{dA \text{ (or } dx\,dy \text{ or } dy\,dx) \to \boldsymbol{r}\ dr\ d\theta}$.
不要忘記乘 r!

$$dA = dx \cdot dy \approx r\ d\theta \cdot dr.$$

當切得很細時, 極矩形跟矩形差不多。



Example 0.1 $\displaystyle\iint_R (3x+4y^2)\ dA$, *where R is the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.*



$R=\{(r,\theta): 1\le r\le 2,\ 0\le\theta\le\pi\}$,

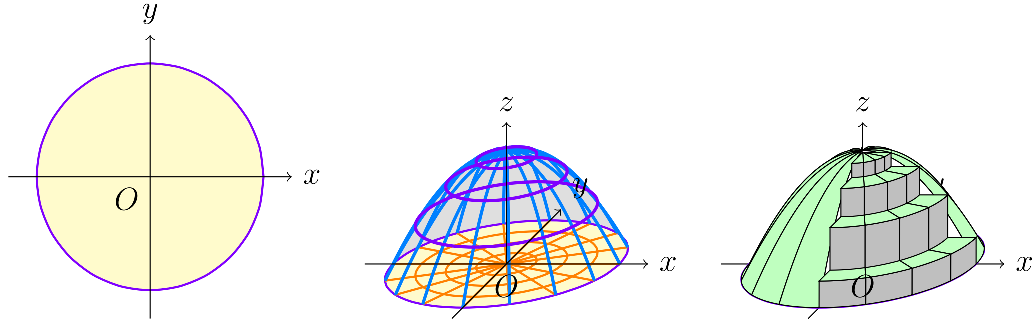
$$\iint_R (3x+4y^2)\ dA=\int_0^{\pi}\int_1^2 (3r\cos\theta+4r^2\sin^2\theta)\cdot r\ dr\ d\theta$$

$$=\int_0^{\pi}\Big[r^3\cos\theta+r^4\sin^2\theta\Big]_{r=1}^{r=2}\ d\theta=\int_0^{\pi}\Big(7\cos\theta+15\sin^2\theta\Big)\ d\theta$$

$$\overset{\text{倍角}}{=}\int_0^{\pi}\Big(7\cos\theta+\frac{15}{2}-\frac{15}{2}\cos 2\theta\Big)\ d\theta=\Big[7\sin\theta+\frac{15}{2}\theta-\frac{15}{4}\sin 2\theta\Big]_0^{\pi}=\frac{15\pi}{2}.$$

(其實可以分開算: $\displaystyle\int_1^2 3r^2\ dr\int_0^{\pi}\cos\theta\ d\theta+\int_1^2 4r^3\ dr\int_0^{\pi}\sin^2\theta\ d\theta$.) ■

Example 0.2 *Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.*

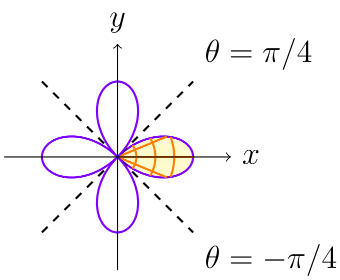


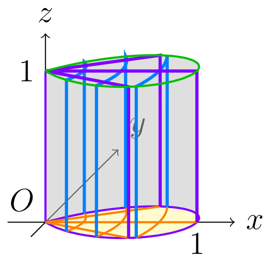
When $z=0$, $x^2+y^2=1$, $\Longrightarrow$ $D=\{(r,\theta): 0\le r\le 1,\ 0\le\theta\le 2\pi\}$.

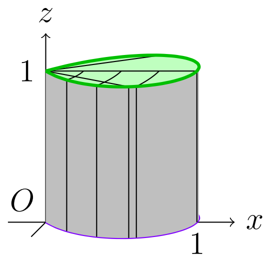
$$V=\iint_D (1-x^2-y^2)\ dA=\int_0^{2\pi}\int_0^1 (1-r^2)\cdot r\ dr\ d\theta=\int_0^{2\pi}\Big[\frac{r^2}{2}-\frac{r^4}{4}\Big]_{r=0}^{r=1}\ d\theta$$

$$=\int_0^{2\pi}\frac{1}{4}\ d\theta=\Big[\frac{\theta}{4}\Big]_0^{2\pi}=\frac{\pi}{2}.$$ *(這也可以分開算: $\displaystyle\int_0^1 (r-r^3)\ dr\int_0^{2\pi}\ d\theta$.)* ■

Example 0.3 *Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.*







(面積等於 $f(x,y) = 1$ 的體積。)

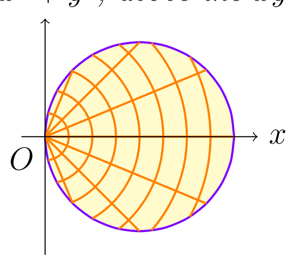
$D = \{(r,\theta) : 0 \le r \le \cos 2\theta,\ -\pi/4 \le \theta \le \pi/4\}$.

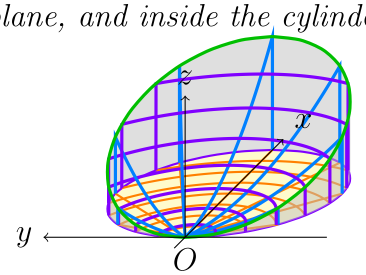
$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r\ dr\ d\theta = \int_{-\pi/4}^{\pi/4} \left[\frac{r^2}{2}\right]_{r=0}^{r=\cos 2\theta} d\theta$$

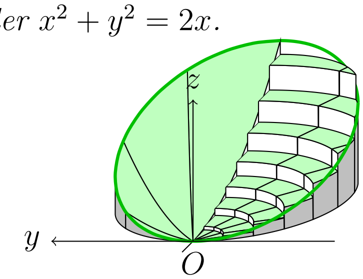
$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2 2\theta\ d\theta \overset{\text{倍角}}{=} \int_{-\pi/4}^{\pi/4} \left(\frac{1}{4} + \frac{1}{4}\cos 4\theta\right) d\theta = \left[\frac{\theta}{4} + \frac{1}{16}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}.$$

(r 的上下界有 θ 不能分開算!) ■

Example 0.4 *Find the volume of the solid lying under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.*







$r^2 = x^2 + y^2 = 2x = 2r\cos\theta$, $D = \{(r,\theta) : 0 \le r \le 2\cos\theta,\ -\pi/2 \le \theta \le \pi/2\}$.

$$V = \iint_D (x^2 + y^2)\ dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 \cdot r\ dr\ d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\ d\theta \overset{\text{對稱}}{=} \int_0^{\pi/2} 8(\cos^2\theta)^2\ d\theta \quad \left(\int_{-a}^{a} \text{偶函數}\ d\theta = 2\int_0^a \text{偶函數}\ d\theta\right)$$

$$\overset{\text{倍角}}{=} \int_0^{\pi/2} 2(1 + \cos 2\theta)^2\ d\theta = \int_0^{\pi/2} (2 + 4\cos 2\theta + 2\cos^2 2\theta)\ d\theta$$

$$\overset{\text{倍角}}{=} \int_0^{\pi/2} (3 + 4\cos 2\theta + \cos 4\theta)\ d\theta = \left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{3\pi}{2}.$$ ■

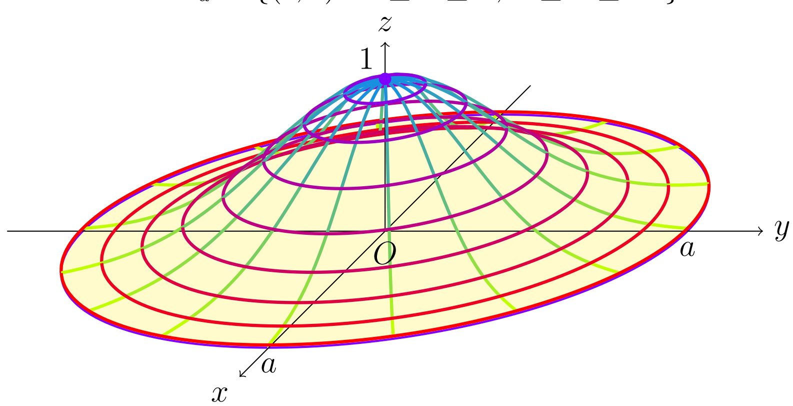
Additional: Gaussian/Euler-Poisson integral

The Gaussian/Euler-Poisson integral 高斯/歐拉-帕松積分: (Exercise 15.3.40)

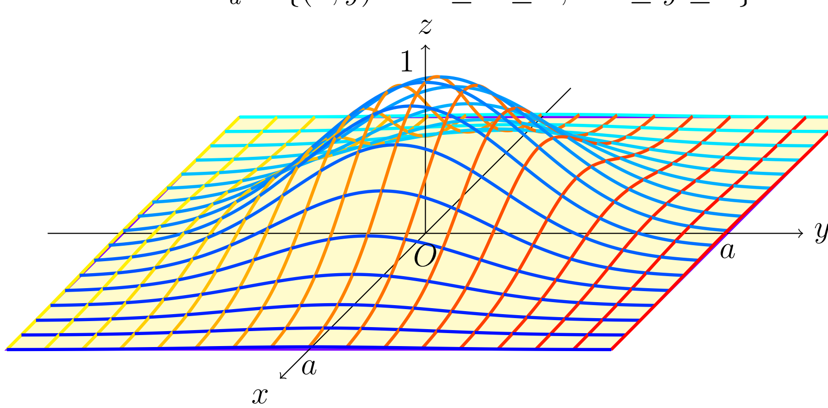
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Consider $\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy.$

Disk of radius a: $D_a = \{(r,\theta) : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$.



Square of side $2a$: $S_a = \{(x,y) : -a \le x \le a,\ -a \le y \le a\}$.



$$\begin{aligned}
\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\, dA &= \lim_{a\to\infty} \iint\limits_{D_a} e^{-(x^2+y^2)}\, dA = \pi \quad \text{(polar)} \\
&= \lim_{a\to\infty} \iint\limits_{S_a} e^{-(x^2+y^2)}\, dA = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2.
\end{aligned}$$