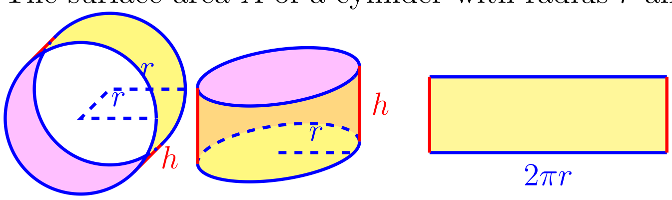
8.2 Area of a surface of revolution

1. surface area formula 表面公式 $S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

Cylinder 圓柱

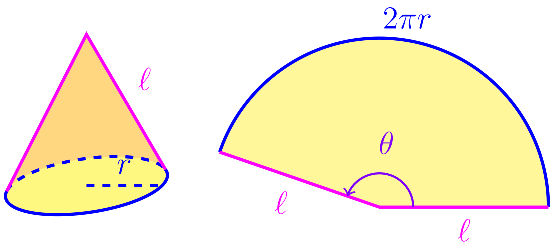
The surface area A of a cylinder with radius r and height h is $2\pi rh$.



$A = 2\pi rh$

Cone 圓錐

The surface area of a circular cone with base radius r and slant height ℓ is $\pi r\ell$.



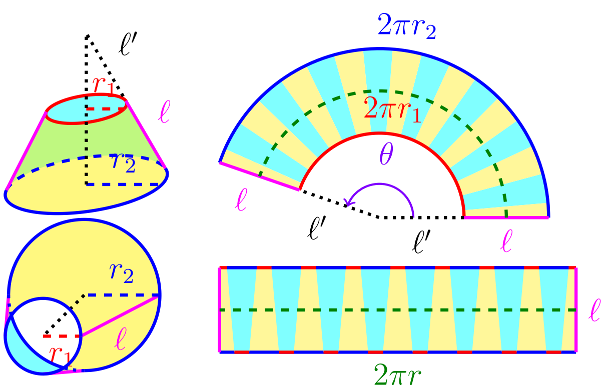
set θ, then $\ell\theta = 2\pi r$,

$A = \pi\ell^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2}\ell^2\theta = \pi r\ell.$

Note: 扇面積$=\frac{1}{2}$半徑$^2\times$夾角。

Band 帶

The surface area of a band (frustum 截頭 of a cone) with upper and lower radii r_1 and r_2 and slant height ℓ is $2\pi r\ell$, where $r = \dfrac{r_1+r_2}{2}$.



set ℓ', then $\dfrac{\ell'}{\ell'+\ell} = \dfrac{r_1}{r_2}$,

$(r_2 - r_1)\ell' = r_1\ell$,

$$\begin{aligned} A &= \pi r_2(\ell'+\ell) - \pi r_1\ell' \\ &= \pi(r_2-r_1)\ell' + \pi r_2\ell \\ &= \pi r_1\ell + \pi r_2\ell = 2\pi r\ell. \end{aligned}$$

$\boxed{A = 2\pi r\ell}$

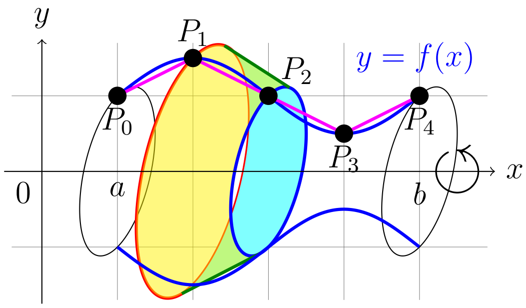
帶面積		(平均) 圓周長		斜邊長
band Area	=	Circumference	×	Length

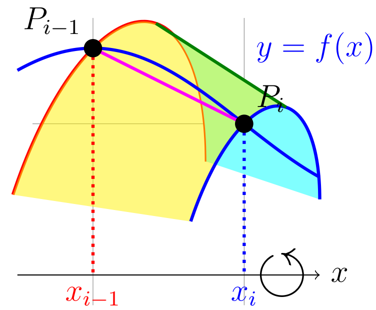
Note: 把圓錐想成 $r_1 = 0$ & $r_2 = r$ 的帶，代入可得圓錐表面積公式。

Revolution 旋轉體

Rotating the curve of $y = f(x)$ from a to b about x-axis.

怎麼算? 切成 n 段用帶子 (band) 來估計。(不是用圓柱!)





把 $[a, b]$ 分成 n 等分, $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta$, $P_i(x_i, y_i = f(x_i))$.

The area of i-th band is

$$S_i = \underbrace{2\pi\frac{y_{i-1}+y_i}{2}}_{\text{[均圓周長]}}\underbrace{|P_{i-1}P_i|}_{\text{[斜邊長]}}$$

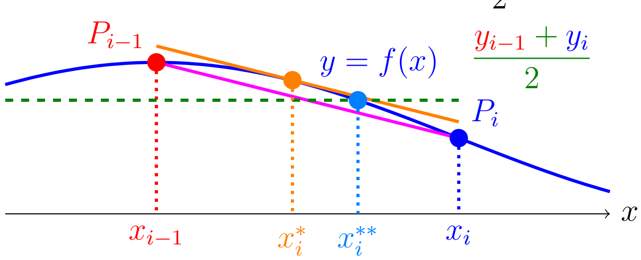
$\because |P_{i-1}P_i| = \sqrt{1+[f'(x_i^*)]^2}\Delta x$ (by MVT, $\exists\ x_i^* \in [x_{i-1}, x_i]$),

and when Δx small, $y_{i-1} \approx f(x_i^*) \approx y_i$, $\dfrac{y_{i-1}+y_i}{2} \approx f(x_i^*)$. $\quad (*)$

Then the surface area S of the revolution is

$$\begin{aligned} S \approx \sum_{i=1}^{n} S_i &= \sum_{i=1}^{n} 2\pi f(x_i^{**})\sqrt{1+[f'(x_i^*)]^2}\Delta x \quad \boxed{\text{不是黎曼和}} \\ &\approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\sqrt{1+[f'(x_i^{**})]^2}\Delta x. \quad \boxed{\text{是黎曼和}} \end{aligned}$$

Note: $(*)$ 更嚴僅的來說, $\because f$ is continuous, by Locating Root (勘根定理), $\exists\ x_i^{**} \in [x_{i-1}, x_i] \ni f(x_i^{**}) = \dfrac{f(x_{i-1})+f(x_i)}{2} = \dfrac{y_{i-1}+y_i}{2}$. x_i^{**} 不一定等於 x_i^*.



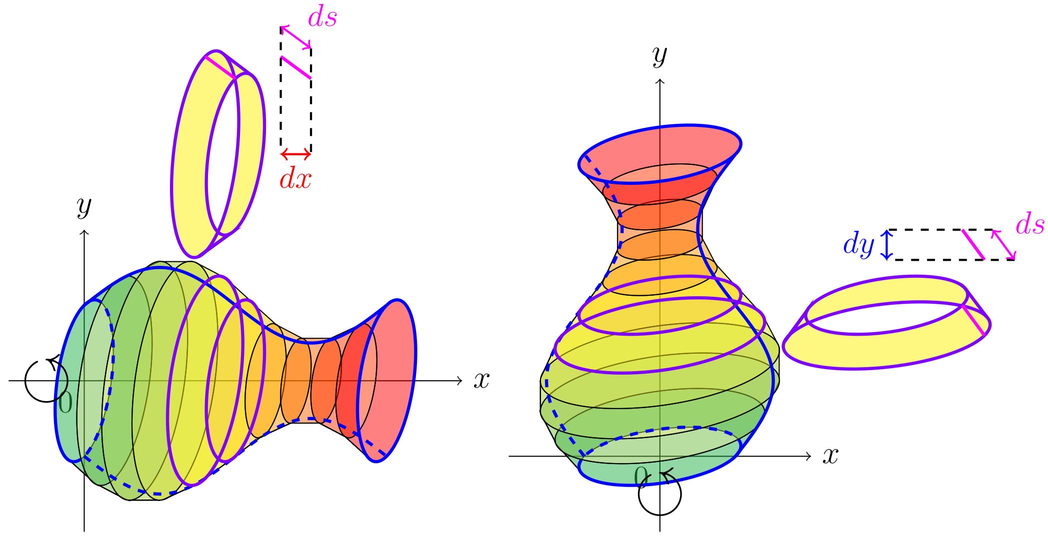
But $\because f'$ is continuous,
when $n \to \infty$,
$\implies \Delta x \to 0$,
$\implies x_i^* \to x_i^{**}$,
$\implies f'(x_i^*) \to f'(x_i^{**})$.

0.1 Surface area formula

Define: Let S denote the ***surface area*** of the surface obtained by rotating the curve $y = f(x)$ from a to b, assuming f is positive and has a continuous derivative (smooth) on $[a, b]$, about the ***x-axis***, is

$$\boxed{\boldsymbol{S} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx} \quad \boxed{\text{不建議背}}$$

$$\boxed{= \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi y\, ds}$$



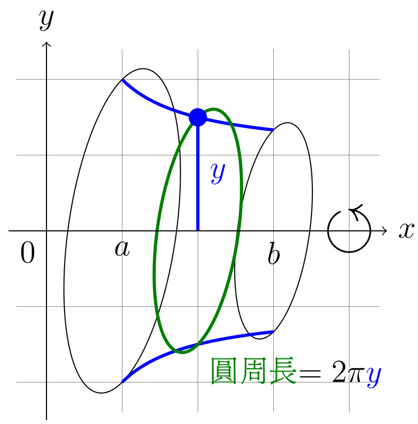
Note: If $x = g(y)$ from c to d about ***y-axis***, it is

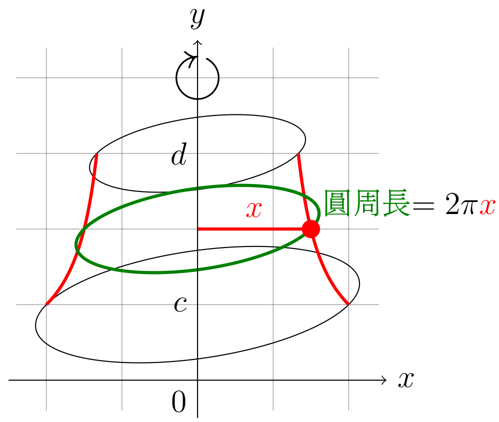
$$\boxed{\boldsymbol{S} = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy} \quad \boxed{\text{不建議背}}$$

$$\boxed{= \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi x\, ds}$$

Recall: $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$

Attention: 不管繞 x-axis 或 y-axis, 跟弧長一樣, 可以對 x 積分, 也可以對 y 積分, 重點在半徑: 繞 x-axis, 半徑是 y; 繞 y-axis, 半徑是 x。
課本的公式 $\int 2\pi f(x)\sqrt{\cdots}\,dx$ 只針對繞 x-axis, 繞其他線就不對; 所以不建議背。

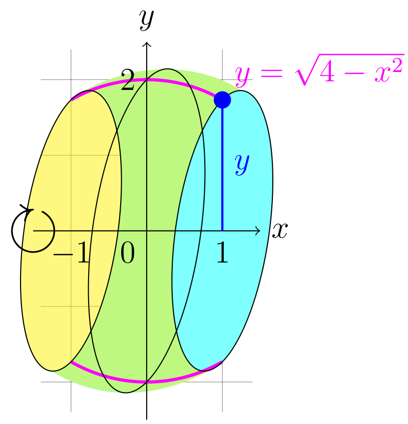




Skill: 表面積 $= \int$ 圓周長 d斜邊, 再變成 $\int \cdots\, dx$ 或 $\int \cdots\, dy$.

Example 0.1 *Find the area of surface obtained by rotating the curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$ about the x-axis.*

$S = \int 2\pi y\, ds$ .... (繞 x-axis, 半徑是 y.)
(對 x 積分, 要寫成 $y = f(x)$, ds 變出 dx.)



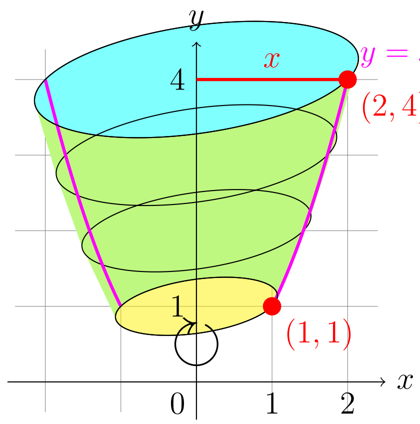
$y = \sqrt{4-x^2}$, $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{4-x^2}}$,

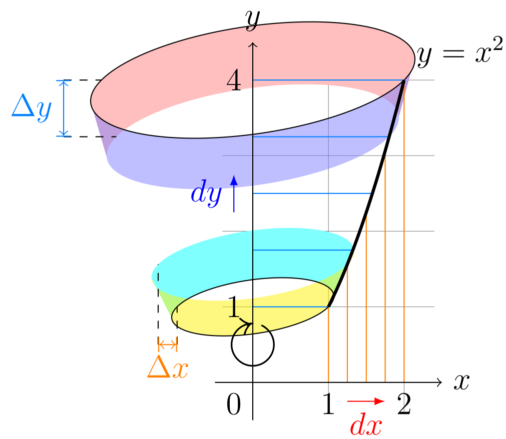
$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \sqrt{1 + \frac{x^2}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}\,dx,$$

$$\therefore S = \int_{-1}^{1} \underbrace{2\pi\sqrt{4-x^2}}_{2\pi y}\,\underbrace{\frac{2}{\sqrt{4-x^2}}\,dx}_{ds} = 4\pi\int_{-1}^{1} dx = 4\pi(2) = 8\pi. \qquad \blacksquare$$

Example 0.2 *Find the area of surface obtained by rotating the parabola $y = x^2$ from $(1,1)$ to $(2,4)$ about the y-axis.*





$$\boxed{S} = \int 2\pi x \, ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(繞 } y\text{-axis, 半徑是 } x.)$$

[Sol 1] (對 y 積分: $y = x^2$ 不是 $x = g(y)$ 型式, 要解反函數; ds 變出 dy.)

$$x = \sqrt{y},\ 1 \le y \le 4,\ \frac{dx}{dy} = \frac{1}{2\sqrt{y}},\ ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{1}{4y}}\, dy.$$

$$\therefore S = \int_1^4 2\pi\sqrt{y}\sqrt{1 + \frac{1}{4y}}\, dy = \pi \int_1^4 \sqrt{4y+1}\, dy$$

(變數變換 Let $u = 4y + 1$, $5 \le u \le 17$, $du = 4\ dy$.)

$$= \frac{\pi}{4}\int_5^{17} \sqrt{u}\, du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{17} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}).$$

[Sol 2] (對 x 積分: ds 變出 dx.)

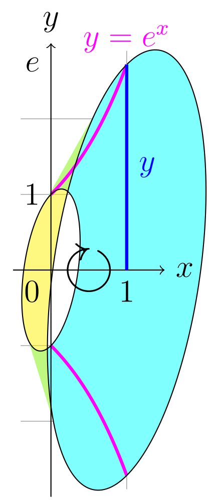
$$y = x^2,\ 1 \le x \le 2,\ \frac{dy}{dx} = 2x,\ ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + 4x^2}\, dx.$$

$$\therefore S = \int_1^2 2\pi x\sqrt{1 + 4x^2}\, dx \qquad \text{(剛好是對 } x \text{ 積分, } 2\pi x \text{ 不用變。)}$$

(變數變換 Let $u = 1 + 4x^2$, $5 \le u \le 17$, $du = 8x\ dx$.)

$$= \frac{\pi}{4}\int_5^{17} \sqrt{u}\, du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{17} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}). \qquad \blacksquare$$

Example 0.3 *Find the area of surface obtained by rotating the curve $y = e^x$, $0 \le x \le 1$ about the x-axis.*



$$\boxed{S} = \int 2\pi y\ ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (\text{繞 } x\text{-axis, 半徑 } y.)$$

(對 x 積分:) $y = e^x$, $0 \le x \le 1$, $\dfrac{dy}{dx} = e^x$,

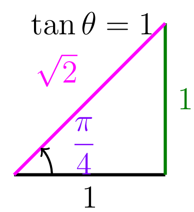
$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \sqrt{1+e^{2x}}\ dx,$$

$$S = \int_0^1 2\pi e^x \sqrt{1+e^{2x}}\ dx$$

Let $y = e^x$, $1 \le y \le e$, $dy = e^x\ dx$. $\ldots\ldots\ldots\ldots$ (變數變換)

$$S = 2\pi \int_1^e \sqrt{1+y^2}\ dy$$

(對 y 積分:) $x = \ln y$, $1 \le y \le e$, $\dfrac{dx}{dy} = \dfrac{1}{y}$,

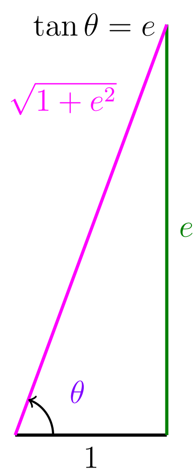


$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy = \sqrt{1+\frac{1}{y^2}}\ dy,$$

$$S = \int_1^e 2\pi y \sqrt{1+\frac{1}{y^2}}\ dy = 2\pi \int_1^e \sqrt{1+y^2}\ dy \quad (\text{一樣!?})$$

Let $y = \tan\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *then* $\frac{\pi}{4} \le \theta \le \tan^{-1} e$,

$dy = \sec^2\theta\ d\theta$, $\sqrt{1+y^2} = \sec\theta$. $\ldots\ldots\ldots\ldots$ (三角變換)



$$S = 2\pi \int_{\pi/4}^{\tan^{-1} e} \sec^3\theta\ d\theta$$

$$= 2\pi \cdot \frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\pi/4}^{\tan^{-1} e} \ \ldots (\text{用圖直接代})$$

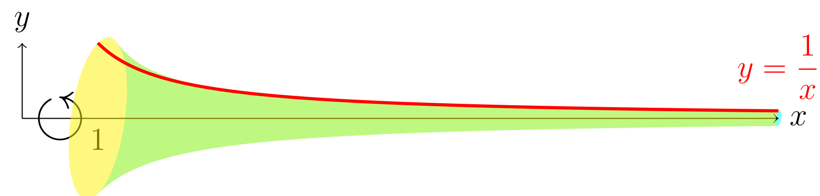
$$\left(= \pi\Big[y\sqrt{1+y^2} + \ln|y + \sqrt{1+y^2}|\Big]_1^e\right) \ \ldots\ldots (\text{換回 } y \text{ 再代})$$

$$\left(= \pi\Big[e^x\sqrt{1+e^{2x}} + \ln|e^x + \sqrt{1+e^{2x}}|\Big]_0^1\right) \ \ldots (\text{換回 } x \text{ 再代})$$

$$= \pi[e\sqrt{1+e^2} + \ln(\sqrt{1+e^2}+e) - \sqrt{2}(\times 1) - \ln(\sqrt{2}+1)]. \quad \blacksquare$$

Skill: 畫圖找出半徑列式 $S = \int 2\pi y\ ds$ (直繞) 或 $S = \int 2\pi x\ ds$ (平繞), 再看要積誰把 ds (把 y 變成 x 的函數) 變出 dx 或 (把 x 變成 y 的函數) dy。

♦ Additional: Gabriel's Horn



曲線 $\boldsymbol{y = \frac{1}{x}, x \geq 1}$ 繞 x-軸的旋轉體稱爲加百列的號角/托里拆利小號 (***Gabriel's Horn***/***Torricelli's trumpet***), 由義大利物理&數學家托里拆利 (Evangelista Torricelli) 所發明 (也發明氣壓計)。

♠《啓示錄 (Revelation)》中寫到: 大天使加百列 (Archangel Gabriel) 吹響號角宣告審判日 (Judgment Day) 的到來。

♡ 有限體積: (Exercise 7.8.63)

$$V = \int_1^\infty \frac{\pi}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\pi}{x^2}\,dx = \lim_{t\to\infty}\Big[-\frac{\pi}{x}\Big]_1^t = \pi - \lim_{t\to\infty}\frac{\pi}{t} = \pi.$$

♢ 無限面積: (Exercise 8.2.27)

$$S = \int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = \int_1^\infty 2\pi\frac{\sqrt{1+x^4}}{x^3}\,dx$$

$$= \int_1^\infty \pi\frac{\sqrt{1+(x^2)^2}}{(x^2)^2}\cdot 2x\,dx = \int_1^\infty \pi\frac{\sqrt{1+v^2}}{v^2}\,dv \qquad (v = x^2,\ dv = 2x\,dx.)$$

$$= \int_{\pi/4}^{\pi/2} \pi\frac{\sec\theta}{(\tan\theta)^2}\cdot\sec^2\theta\,d\theta \qquad (v = \tan\theta, \sqrt{1+v^2} = \sec\theta,\ dv = \sec^2\theta\,d\theta)$$

$$= \int_{\pi/4}^{\pi/2} \pi\frac{\sec\theta}{\tan^2\theta}(\tan^2\theta+1)\,d\theta = \int_{\pi/4}^{\pi/2}\pi(\sec\theta + \frac{\sec\theta}{\tan^2\theta})\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\pi(\sec\theta + \csc\theta\cot\theta)\,d\theta \qquad (\frac{\sec\theta}{\tan^2\theta} = \frac{\cos\theta}{\sin^2\theta} = \csc\theta\cot\theta)$$

$$= \lim_{t\to\pi/2}\int_{\pi/4}^t \pi(\sec\theta + \csc\theta\cot\theta)\,d\theta = \lim_{t\to\pi/2}\pi\Big[\ln|\sec\theta + \tan\theta| - \csc\theta\Big]_{\pi/4}^t$$

$$= \lim_{t\to\pi/2}\pi\Big(\underbrace{\ln|\sec t + \tan t|}_{\to\infty} - \underbrace{\csc t}_{\to 1}\Big) - \pi\Big(\ln|\sqrt{2}+1| - \sqrt{2}\Big) = \infty.$$

♣ 漆匠的矛盾 (***Painter's paradox***): 裝得滿 Gabriel's horn 的油漆卻塗不滿它的內壁表面。 — 那五顆檸?!