7.2 Trigonometric integrals

1. $\int \sin^m x \cos^n x \; dx$
2. $\int \tan^m x \sec^n x \; dx$
3. $\int \sin mx \cos nx \; dx$, $\int \sin mx \sin nx \; dx$, $\int \cos mx \cos nx \; dx$

三角函數的積分攻略: 換換換→變數變換→分部積分。

0.1 $\int \sin^m x \cos^n x \; dx$

Recall: $\int \sin x \; dx = -\cos x + C$, $\int \cos x \; dx = \sin x + C$.

- **Case a.** $n = 2k+1$ is odd.
  (Let $\boldsymbol{u = \sin x}$, $du = \cos x \; dx$, use $\cos^2 x = 1 - \sin^2 x$.)

$$\begin{aligned}
&\int \sin^m x \cos^n x \; dx \\
&= \int \sin^m x \cos^{2k+1} x \; dx \;=\; \int \sin^m x (\cos^2 x)^k \cdot \cos x \; dx \\
&= \boxed{\int \sin^m x (1 - \sin^2 x)^k \; d\sin x} \;=\; \int u^m (1-u^2)^k \; du.
\end{aligned}$$

- **Case b.** $m = 2k+1$ is odd.
  (Let $\boldsymbol{u = \cos x}$, $du = -\sin x \; dx$, use $\sin^2 x = 1 - \cos^2 x$.)

$$\begin{aligned}
&\int \sin^m x \cos^n x \; dx \\
&= \int \sin^{2k+1} x \cos^n x \; dx \;=\; \int (\sin^2 x)^k \cos^n x \cdot \sin x \; dx \\
&= \boxed{\int (1 - \cos^2 x)^k \cos^n x (-\; d\cos x)} \;=\; \int -(1-u^2)^k u^n \; du.
\end{aligned}$$

- **Case c.** m and n are even.
  使用 $\cos^2 x = 1 - \sin^2 x$ or $\sin^2 x = 1 - \cos^2 x$ 換成只由 $\sin^2 x$ or $\cos^2 x$ 組成的多項式, 再用 **Half/double angle formula** 半/倍角公式:

$$\boxed{\boldsymbol{\sin^2 x = \frac{1 - \cos 2x}{2}}}, \quad \boxed{\boldsymbol{\cos^2 x = \frac{1 + \cos 2x}{2}}}, \quad \sin x \cos x = \frac{\sin 2x}{2}.$$

Example 0.1 $\int \cos^3 x \; dx.$

Let $u = \sin x,\ du = \cos x\ dx.$

$$\int \cos^3 x\ dx = \int \cos^2 x \cdot \cos x\ dx = \boxed{\int (1-\sin^2 x)\ d\sin x} = \int 1-u^2\ du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C. \qquad \blacksquare$$

Example 0.2 $\int \sin^5 x \cos^2 x\ dx.$

Let $u = \cos x,\ du = -\sin x\ dx,\ \sin x\ dx = -\ du.$

$$\int \sin^5 x \cos^2 x\ dx = \int \sin^4 x \cos^2 x \cdot \sin x\ dx$$

$$= \boxed{\int (1-\cos^2 x)^2 \cos^2 x (-\ d\cos x)} = \int -u^2 + 2u^4 - u^6\ du$$

$$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C. \qquad \blacksquare$$

Example 0.3 $\int_0^\pi \sin^2 x\ dx.$

$$\int \sin^2 x\ dx = \int \frac{1}{2}(1-\cos 2x)\ dx = \frac{1}{2}\int\ dx - \frac{1}{2}\int \cos 2x\ dx$$

$$= \frac{1}{2}\int\ dx - \frac{1}{2\times 2}\int \cos 2x\ d(2x) = \frac{x}{2} - \frac{1}{4}\sin 2x + C,\ (u = 2x,\ dx = \frac{1}{2}\ du.)$$

$$\int_0^\pi \sin^2 x\ dx = \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^\pi = \frac{\pi}{2}. \qquad \blacksquare$$

Example 0.4 $\int \sin^4 x\ dx.$

$$\int \sin^4 x\ dx = \int (\sin^2 x)^2\ dx = \int \left[\frac{1}{2}(1-\cos 2x)\right]^2\ dx$$

$$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\ dx = \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}(1+\cos 4x)\ dx$$

$$= \int \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\ dx$$

$$= \frac{3}{8}\int\ dx - \frac{1}{2}\int \frac{1}{2}\cos 2x\ d(2x) + \frac{1}{8}\int \frac{1}{4}\cos 4x\ d(4x)$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. \qquad \blacksquare$$

0.2 $\displaystyle\int \tan^m x \sec^n x \; dx$

Recall: $\int \sec^2 x \; dx = \tan x + C$, $\int \sec x \tan x \; dx = \sec x + C$.

- **Case a-1.** $n = 2k \geq 2$ is even.
  (Let $\boldsymbol{u = \tan x}$, $du = \sec^2 x \; dx$, use $\sec^2 x = 1 + \tan^2 x$.)

$$\begin{aligned}
&\int \tan^m x \sec^n x \; dx \\
&= \int \tan^m x \sec^{2k} x \; dx = \int \tan^m x (\sec^2 x)^{k-1} \cdot \sec^2 x \; dx \\
&= \int \tan^m x (1 + \tan^2 x)^{k-1} \; d \tan x = \int u^m (1 + u^2)^{k-1} \; du.
\end{aligned}$$

- **Case a-2.** $n = 0$, $m = 1$. $\boxed{\displaystyle\int \boldsymbol{\tan x \; dx = \ln |\sec x| + C}}$.

- **Case a-3.** $n = 0$, $m \geq 2$. (Exercise 7.1.53)

$$\begin{aligned}
\int \tan^m x \; dx &= \int \tan^{m-2} x (\sec^2 x - 1) \; dx \\
&= \int \tan^{m-2} x \cdot \sec^2 x \; dx - \int \tan^{m-2} x \; dx \\
&= \int \tan^{m-2} x \; d \tan x - \int \tan^{m-2} x \; dx \\
&= \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \; dx. \quad \text{(降兩次)}
\end{aligned}$$

  Reduction formula, 最後是 $\displaystyle\int \tan x \; dx = \ln |\sec x| + C$ (if m is odd) or $\displaystyle\int \; dx = x + C$ (if m is even).

- **Case b.** $m = 2k + 1$ is odd and $n \geq 1$.
  (Let $\boldsymbol{u = \sec x}$, $du = \sec x \tan x \; dx$, use $\tan^2 x = \sec^2 x - 1$.)

$$\begin{aligned}
&\int \tan^m x \sec^n x \; dx \\
&= \int \tan^{2k+1} x \sec^n x \; dx = \int (\tan^2 x)^k \sec^{n-1} x \cdot \sec x \tan x \; dx \\
&= \int (\sec^2 x - 1)^k \sec^{n-1} x \; d \sec x = \int (u^2 - 1)^k u^{n-1} \; du.
\end{aligned}$$

- **Case c-1.** $m = 2k \geq 2$ is even and n is odd.

$$\begin{aligned}
&\int \tan^m x \sec^n x \; dx \\
&= \int \tan^{2k} x \sec^n x \; dx = \int (\tan^2 x)^k \sec^n x \; dx \\
&= \int (\sec^2 x - 1)^k \sec^n x \; dx = \sum_{i=0}^{\ell} C_i \int \sec^{2i+1} x \; dx \ldots (\text{續})
\end{aligned}$$

(使用 $\tan^2 x = \sec^2 x - 1$ 變成 $\sec^{\text{奇數次}} x$ 的積分。)

- **Case c-2.** $m = 0$ and $n = 1$. $\boxed{\int \sec x \; dx = \ln|\sec x + \tan x| + C}$.

- **Case c-3.** $m = 0$ and $n \geq 3$ is odd.
用分部積分法: $u = \sec^{n-2} x$, $dv = \sec^2 x \; dx$.

$$\begin{aligned}
\int \sec^n x \; dx &= \int \sec^{n-2} x \cdot \sec^2 x \; dx \\
&= \boxed{\int \sec^{n-2} x \; d\tan x = \sec^{n-2} x \tan x - \int \tan x \; d\sec^{n-2} x} \\
&= \sec^{n-2} x \tan x - \int \tan^2 x \, (n-2) \sec^{n-2} x \; dx \\
&= \sec^{n-2} x \tan x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x \; dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \; dx + (n-2) \int \sec^{n-2} x \; dx,
\end{aligned}$$

$$\begin{aligned}
(n-1) \int \sec^n x \; dx &= \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \; dx, \\
\int \sec^n x \; dx &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \; dx.
\end{aligned}$$

Reduction formula, 最後是 $\int \sec x \; dx = \ln|\sec x + \tan x| + C$.
(降兩次: $n \to (n-2) \to \cdots \to 5 \to 3 \to 1$, 用公式。)

Note: $\int \cot^m x \csc^n x \; dx$ 方法類似。

Example 0.5 $\displaystyle\int \tan^6 x \sec^4 x \; dx.$

Let $u = \tan x$, $du = \sec^2 x \; dx$.

$$\int \tan^6 x \sec^4 x \; dx = \int \tan^6 x \sec^2 x \cdot \sec^2 x \; dx$$

$$= \int \tan^6 x (1 + \tan^2 x) \; d\tan x = \int u^6 + u^8 \; du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C. \qquad \blacksquare$$

Example 0.6 $\displaystyle\int \tan^5 x \sec^7 x \; dx.$

Let $u = \sec x$, $du = \sec x \tan x \; dx$.

$$\int \tan^5 x \sec^7 x \; dx = \int \tan^4 x \sec^6 x \cdot \sec x \tan x \; dx$$

$$= \int (\sec^2 x - 1)^2 \sec^6 x \; d\sec x = \int u^{10} - 2u^8 + u^6 \; du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C = \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C. \qquad \blacksquare$$

Example 0.7 $\displaystyle\int \tan^3 x \; dx.$

$$\begin{aligned}
\int \tan^3 x \; dx &= \int \tan x \tan^2 x \; dx \\
&= \int \tan x (\sec^2 x - 1) \; dx \\
&= \int \tan x \cdot \sec^2 x \; dx - \int \tan x \; dx \\
&= \int \tan x \; d\tan x - \int \tan x \; dx \\
&= \frac{1}{2}\tan^2 x - \ln|\sec x| + C. \qquad \blacksquare
\end{aligned}$$

Note: $\displaystyle\int \tan^3 x \; dx = \frac{1}{2}\sec^2 x - \ln|\sec x| + C$ 也對。

$$\because \int \tan x \sec^2 x \; dx = \int \sec x \cdot \tan x \sec x \; dx = \int \sec x \; d\sec x \qquad (u = \sec x)$$

$= \dfrac{1}{2}\sec^2 x + C = \dfrac{1}{2}\tan^2 x + \underline{\dfrac{1}{2} + C}$. (常數($\frac{1}{2}$)通通被 C(任意常數) 吸收。)

Example 0.8 (♡考) $\int \sec^3 x \, dx$.

Let $u = \sec x$ *and* $dv = \sec^2 x \, dx$*, then* $du = \sec x \tan x \, dx$ *and* $v = \tan x$.

$$
\begin{aligned}
\int \sec^3 x \, dx &= \int \sec x \cdot \sec^2 x \, dx \\
&= \boxed{\int \sec x \, d\tan x = \sec x \tan x - \int \tan x \, d\sec x} \\
&= \sec x \tan x - \int \tan x \sec x \tan x \, dx \\
&= \sec x \tan x - \int \tan^2 x \sec x \, dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \ln|\sec x + \tan x|,
\end{aligned}
$$

$$
\begin{aligned}
2\int \sec^3 x \, dx &= \sec x \tan x + \ln|\sec x + \tan x|, \\
\int \sec^3 x \, dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.
\end{aligned}
$$
■

加入你的不定積分表:

$$\boxed{\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C}.$$

(背身體健康, 背萬事如意。有背有保庇, 沒背要會積。)

Question: 記不住策略怎麼辦?

Answer:

1. 三角恆等式換換換, $\sin^{\text{偶數次}} x$ 或 $\cos^{\text{偶數次}} x$ 要用倍角。
2. 變數變換變變變: $\begin{cases} \sin^m x \cos^n x, & \text{猜 } u = \sin x \text{ 或 } \cos x; \\ \tan^m x \sec^n x, & \text{猜 } u = \tan x \text{ 或 } \sec x. \end{cases}$
3. 分部積分分分分: $\sec^{\text{奇數次}} x$; $\tan x$, $\sec x$, $\sec^3 x$ 的最好背起來。

0.3 $\int \sin mx \cos nx \; dx$, $\int \sin mx \sin nx \; dx$, $\int \cos mx \cos nx \; dx$

Recall: Sum/difference formula 和/差角公式:

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\cos(A-B) &= \cos A \cos B + \sin A \sin B
\end{aligned}$$

Product to sum formula 積化和差:

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}[\sin(A-B) + \sin(A+B)] \\
\sin A \sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)] \\
\cos A \cos B &= \frac{1}{2}[\cos(A-B) + \cos(A+B)]
\end{aligned}$$

Example 0.9 $\int \sin 4x \cos 5x \; dx$.

$$\begin{aligned}
\int \sin \underbrace{4x}_{A} \cos \underbrace{5x}_{B} \; dx &= \int \frac{1}{2}[\sin(4x-5x) + \sin(4x+5x)] \; dx \\
&= \int \frac{1}{2}[\sin(-x) + \sin 9x] \; dx \\
&= \frac{1}{2}\int (-\sin x + \sin 9x) \; dx \\
&= \frac{1}{2}\int -\sin x \; dx + \frac{1}{2}\int \frac{1}{9} \sin 9x \; d(9x) \\
&= \frac{1}{2}\cos x - \frac{1}{18}\cos 9x + C. \qquad \blacksquare
\end{aligned}$$

♦ ***Fourier series*** 傅立葉級數: $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \sin nx + b_n \cos nx)$,

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \; dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \; dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \; dx$.