13.1 Vector functions and space curves

1. vector functions
2. space curves

0.1 Vector functions

Define: A ***vector(-valued) function*** 向量函數 is a function

$$\mathbf{r}(t) : Domain \subseteq \mathbb{R} \to Range \subseteq V_n. \text{ (主要在}V_3\text{)}$$

$$\boxed{\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}}$$

where $f(t), g(t), h(t)$ are called ***component functions*** 分量函數 of $\mathbf{r}$.

Note: 1. t 是獨立的變數, 通常用來代表時間 (time)。
2. 定義域 = 分量函數都有定義處 = 分量函數定義域的交集。(看分量函數!)

Define: (**Limit**) If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\boxed{\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle}$$

provided the limit of the component functions exist.

Precise Definition: (equivalent, see Exercise 13.1.54)

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L} \text{ or } \mathbf{r}(t) \to \mathbf{L} \text{ as } t \to a.$$

if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, $\ni 0 < |t - a| < \delta \implies |\mathbf{r}(t) - \mathbf{L}| < \varepsilon$.
(只要 t 離 a 有 δ 這麼近, $\mathbf{r}(t)$ 離 $\mathbf{L}$ 就有 ε 這麼近。)
(兩個向量之差的長度很小 $\iff$ 兩者 (的方向與長度) 差不多一樣。)

Note: 分量函數極限都存在, 整個才有極限存在。(看分量函數!)

Limit Law 極限律: 加減常數倍, 內外積。(Exercise 13.1.53)
If $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$ and $\lim_{t\to a} \mathbf{s}(t) = \mathbf{M}$ exist (極限存在), then

1 $\lim_{t\to a}(\mathbf{r}(t) \pm \mathbf{s}(t)) = \mathbf{L} \pm \mathbf{M}$

2 $\lim_{t\to a} c\mathbf{r}(t) = c\mathbf{L}$

3 $\lim_{t\to a}(\mathbf{r}(t) \bullet \mathbf{s}(t)) = \mathbf{L} \bullet \mathbf{M}$

4 $\lim_{t\to a}(\mathbf{r}(t) \times \mathbf{s}(t)) = \mathbf{L} \times \mathbf{M}$

Define: (**Continuity**) A vector function $\mathbf{r}(t)$ is ***continuous*** at a if

$$\boxed{\lim_{t\to a}\mathbf{r}(t)=\mathbf{r}(a)}.$$

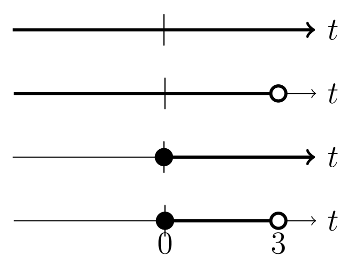
Note: $\lim_{t\to a} f(t)=f(a)$, $\lim_{t\to a} g(t)=g(a)$, $\lim_{t\to a} h(t)=h(a)$, 分量函數都連續。(看分量函數!)

Example 0.1 *Find the domain of* $\mathbf{r}(t)=\langle t^3, \ln(3-t), \sqrt{t}\rangle$.

看分量函數*:* t^3*, domain* $\mathbb{R}=(-\infty,\infty)$*;*

$\ln(3-t)$*, domain* $3>t$*,* $(-\infty,3)$*;*

$\sqrt{t}$*, domain* $t\geq 0$*,* $[0,\infty)$*.*



So the domain of $\mathbf{r}(t)$ *is*
$(-\infty,\infty)\cap(-\infty,3)\cap[0,\infty)=[0,3)$. ■

Example 0.2 *Find* $\lim_{t\to 0}\mathbf{r}(t)$*, where* $\mathbf{r}(t)=(1+t^3)\mathbf{i}+te^{-t}\mathbf{j}+\dfrac{\sin t}{t}\mathbf{k}$.

$$\lim_{t\to 0}\mathbf{r}(t)=\left[\lim_{t\to 0}(1+t^3)\right]\mathbf{i}+\left[\lim_{t\to 0}te^{-t}\right]\mathbf{j}+\left[\lim_{t\to 0}\frac{\sin t}{t}\right]\mathbf{k}=\mathbf{i}+\mathbf{k}.$$ ■

Recall: 向量函數的定義域? 極限? 連續? 分量函數! 分量函數! 分量函數!

0.2 Space curves

Define: A **space curve** 空間曲線 C is the set of all point

$$(x,y,z)=(f(t),g(t),h(t)).$$

$$x=f(t),\quad y=g(t),\quad z=h(t),$$

are called ***parametric equations*** 參數方程式 of C and t is called a ***parameter*** 參數. Then

$$\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$$

is the ***position vector*** 位置向量 of the point $P(f(t),g(t),h(t))$ on C.
(用向量函數表示空間曲線。)

Example 0.3 *Describe the curve defined by* $\mathbf{r}(t) = \langle 1+t, 2+5t, -1+6t\rangle$.

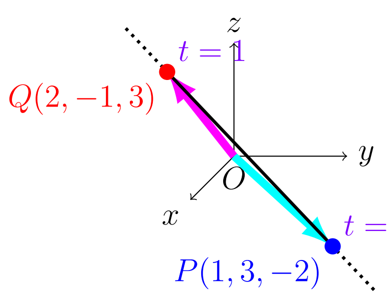
A line through $(1,2,-1)$ *and parallel to* $\langle 1,5,6\rangle$. ■

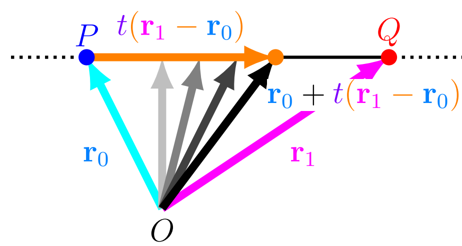
Example 0.4 *Find a vector equation and parametric equations for the line segment joining* $P(1,3,-2)$ *and* $Q(2,-1,3)$.

Let $\mathbf{r}_0$, $\mathbf{r}_1$ *and* $\mathbf{r}$ *be position vectors of* P, Q *and point* (x,y,z) *on* $\overline{PQ}$. *Then* $\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 = (1-t)\langle 1,3,-2\rangle + t\langle 2,-1,3\rangle$,

向量方程式*:* $\mathbf{r}(t) = \langle 1+t, 3-4t, -2+5t\rangle$, $0 \le t \le 1$.

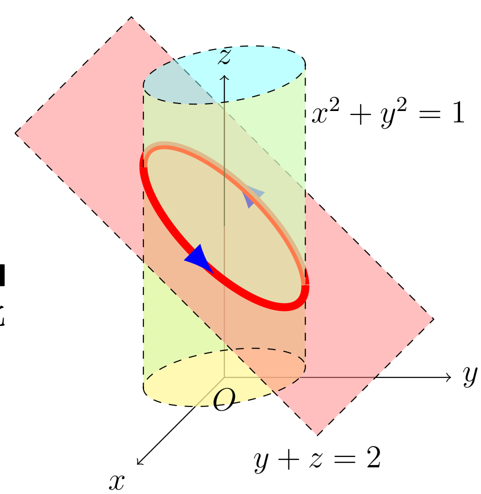
參數方程式*:* $x = 1+t$, $y = 3-4t$, $z = -2+5t$, $0 \le t \le 1$. *(*注意範圍*)* ■





Example 0.5 *Find a vector function representing the curve of intersection of the cylinder* $x^2+y^2=1$ *and the plane* $y+z=2$.

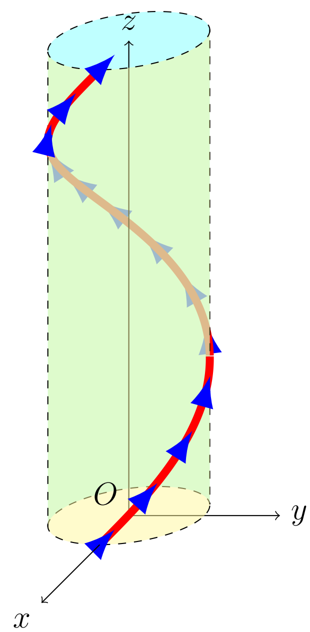
$x^2+y^2=1 \implies x=\cos t,\ y=\sin t,$
$y+z=2 \implies z = 2-\sin t.$
$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (2-\sin t)\mathbf{k}$
$(= \langle \cos t, \sin t, 2-\sin t\rangle)$. ■
*(*這個過程稱爲 ***parametrization*** 參數化 *of the curve.)*

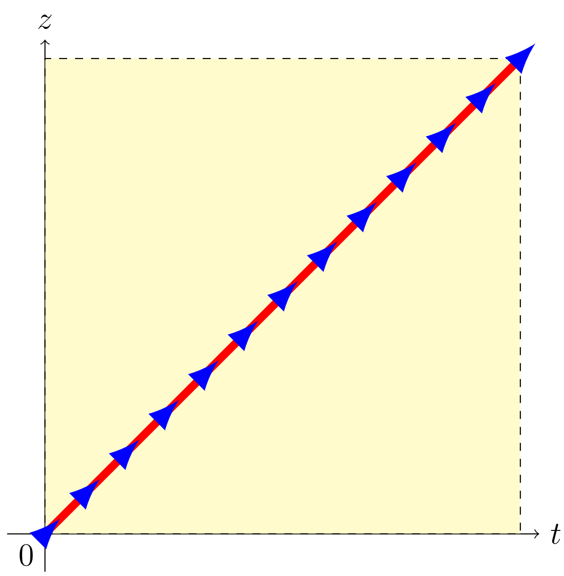


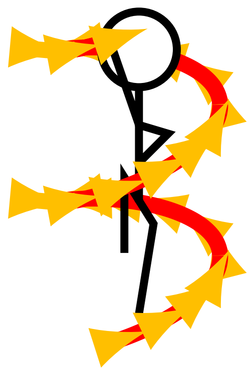
Example 0.6 *Sketch the curve whose vector equation is*

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

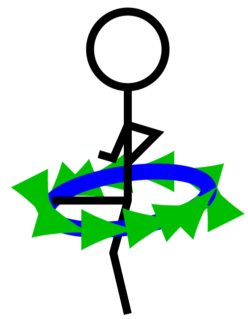
*A **helix** [ˈhilɪx] 螺旋, the curve spirals upward around the (circular) cylinder $x^2 + y^2 = 1$ as t increases.* ∎



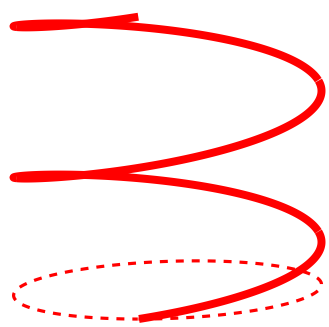




$\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + ct\,\mathbf{k}$



$\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\,\mathbf{k}$

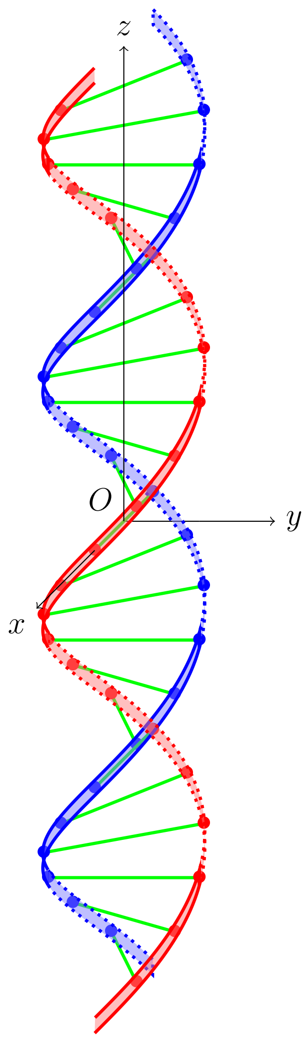


$a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\,\mathbf{k}$

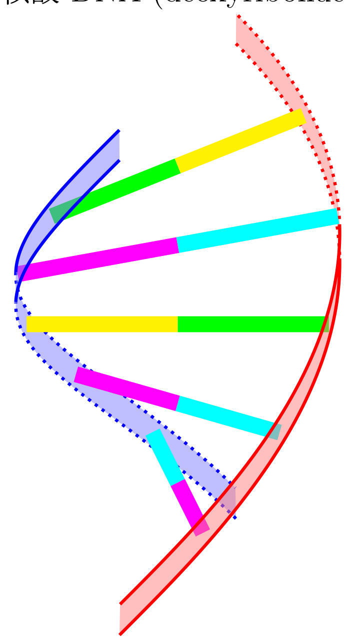
double helix 雙螺旋

$$x = \cos t \quad y = \sin t \quad z = t$$

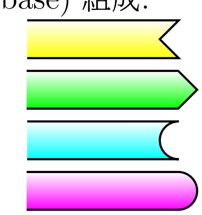
$$x = -\cos t \quad y = -\sin t \quad z = t$$



♦ 脫氧核糖核酸 DNA (deoxyribonucleic acid)

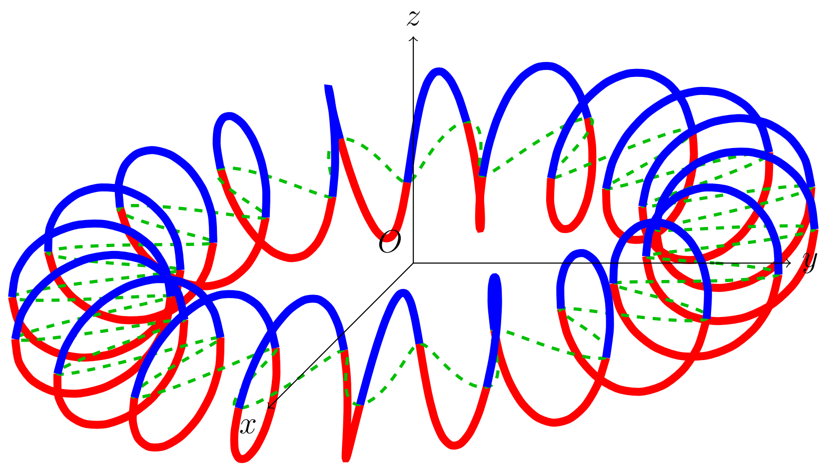


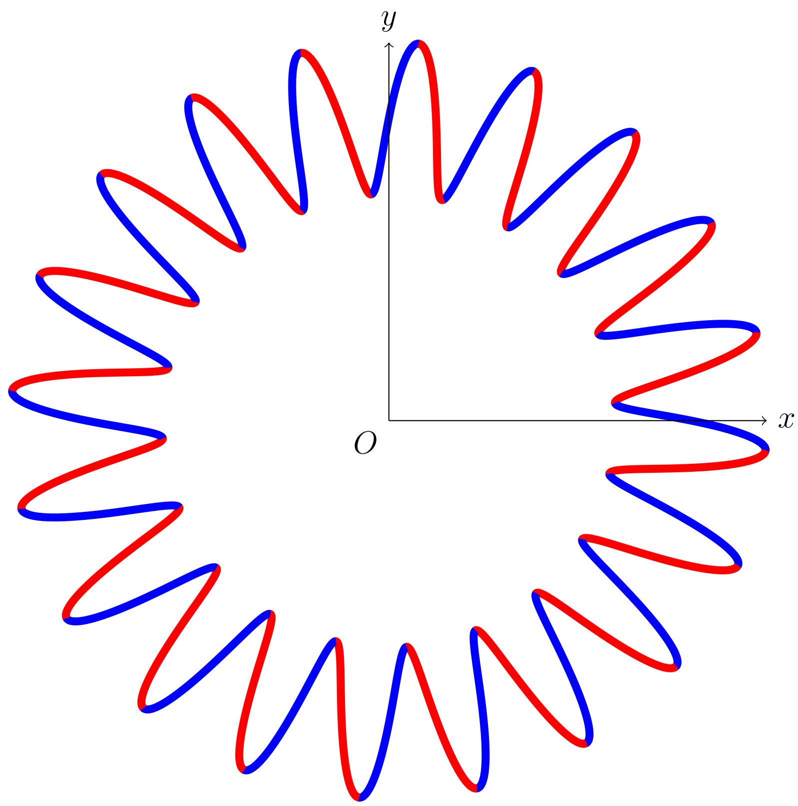
雙螺旋結構
由四種 (ATGC) 核鹼基 (nucleobase) 組成:
腺嘌呤 (Adenine)
胸腺嘧啶 (Thymine)
鳥嘌呤 (Guanine)
胞嘧啶 (Cytosine)



toroidal spiral [ˈtorɔɪdəl ˈspaɪrəl] 圓環螺線

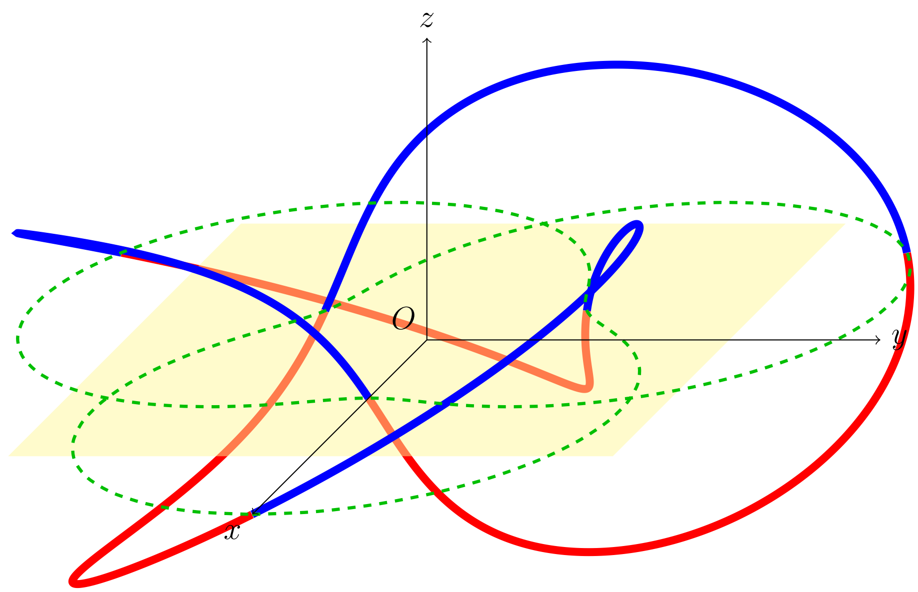
$$x = (4 + \sin 20t)\cos t \quad y = (4 + \sin 20t)\sin t \quad z = \cos 20t$$

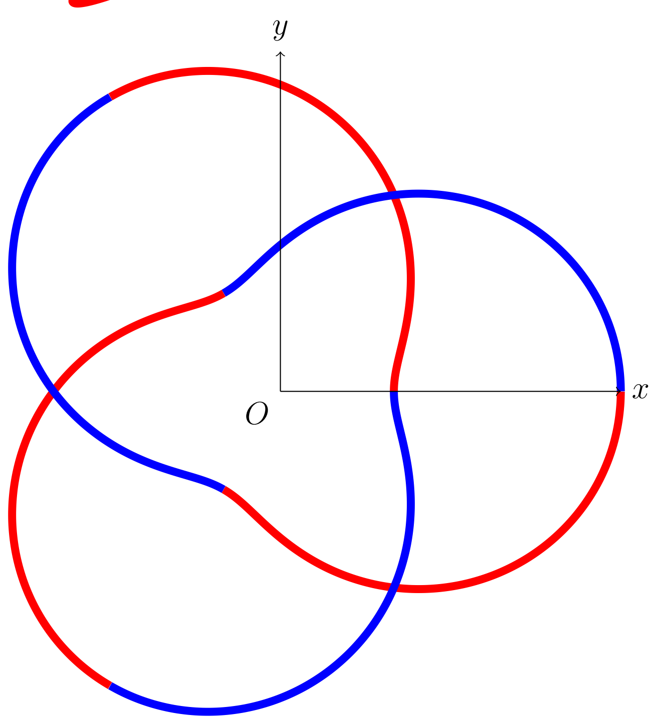




trefoil knot [ˋtrifɔɪl nɑt] 三葉型紐結

$$x = (2 + \cos 1.5t)\cos t \quad y = (2 + \cos 1.5t)\sin t \quad z = \sin 1.5t$$





twisted cubic [ˋtwɪstɪd ˋkjubɪk] 三次繞線

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$