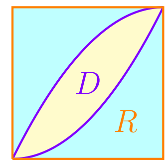
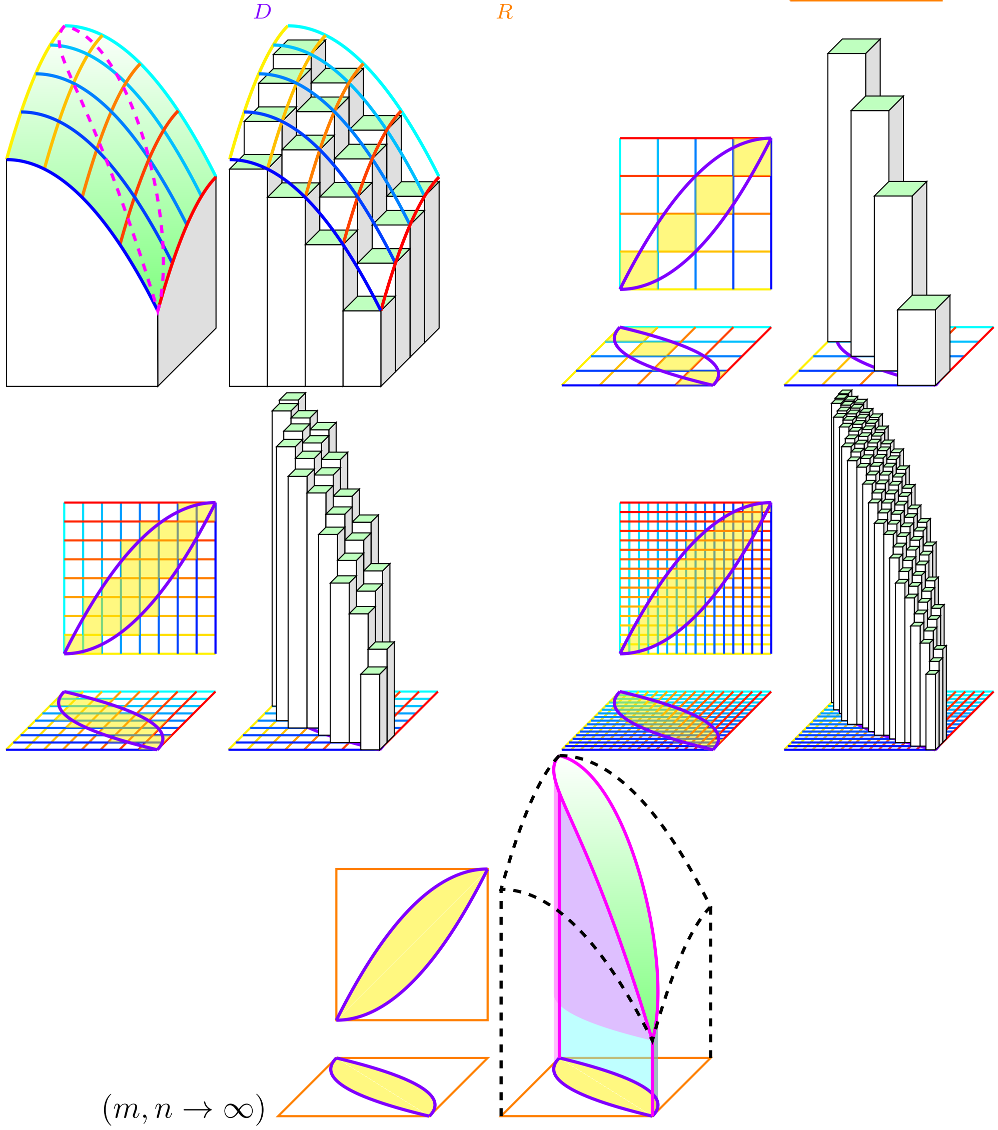
15.2 Double integrals over general regions

Question: (可積分) $f(x,y)$ 要怎麼在不規則的形狀 D 上雙重積分?

Let $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \setminus D \end{cases}$,

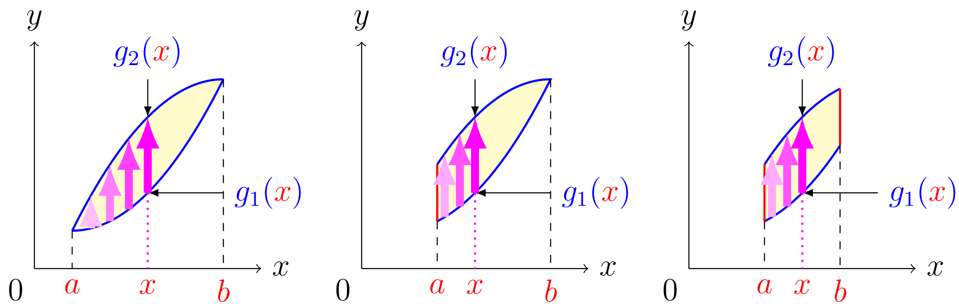
and define $\displaystyle\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA.$





Type I $D=\{(x,y): a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$ (先積 y)

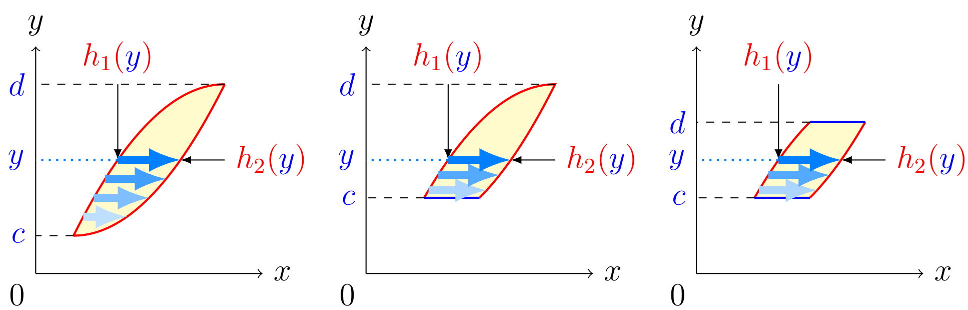
$$\boxed{\iint\limits_D \boldsymbol{f(x,y)\ dA}=\int_a^b\int_{g_1(x)}^{g_2(x)}\boldsymbol{f(x,y)\ dy\ dx}}$$



(兩條垂直線與兩條 y 寫成 x 的函數的曲線夾住的區域。)

Type II $D=\{(x,y): h_1(y)\le x\le h_2(y),\ c\le y\le d\}$ (先積 x)

$$\boxed{\iint\limits_D \boldsymbol{f(x,y)\ dA}=\int_c^d\int_{h_1(y)}^{h_2(y)}\boldsymbol{f(x,y)\ dx\ dy}}$$

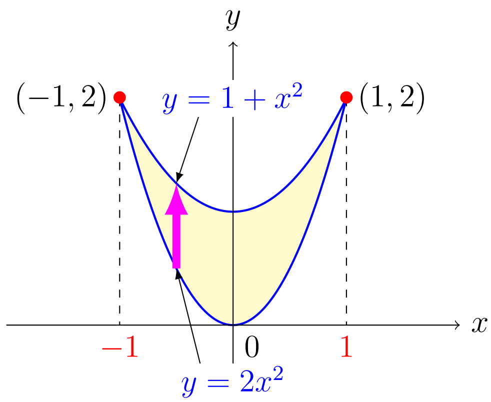


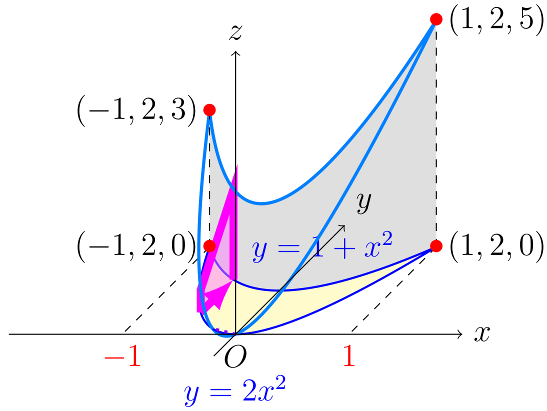
(兩條水平線與兩條 x 寫成 y 的函數的曲線夾住的區域。)

Note: 其實只能積有一邊是平行的區域, 但是這就夠了。
~~有時~~常常不會跟你講要用哪種 type, ~~有時~~常常題目給的 type 不一定好積。
難積就換人積: **Type I** $\leftrightarrow$ **Type II**, $y=g(x)\leftrightarrow x=g^{-1}(y)$ (要解反函數)。

Skill: 解方程式找邊界函數與交點, 畫圖畫箭頭決定順序與 type, 箭頭端的座標決定上下界; 如果兩種 types 都能算, 找上下界函數簡單一點的。

Example 0.1 *Evaluate* $\displaystyle\iint_D (x+2y)\ dA$, *where* D *is the region bounded by the parabolas* $y=2x^2$ *and* $y=1+x^2$.





找邊界: $2x^2=y=1+x^2$, $(x,y)=(\pm 1,2)$.

Type I $D=\{(x,y):-1\le x\le 1,\ 2x^2\le y\le 1+x^2\}$.

$$\iint_D (x+2y)\ dA=\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)\ dy\ dx=\int_{-1}^{1}\Big[xy+y^2\Big]_{y=2x^2}^{y=1+x^2}\ dx$$

$$=\int_{-1}^{1}\Big[x(1+x^2)+(1+x^2)^2-x(2x^2)-(2x^2)^2\Big]\ dx \qquad \text{(函數照樣帶入)}$$

$$=\int_{-1}^{1}\Big(-3x^4-x^3+2x^2+x+1\Big)\ dx$$

$$=\Big[-3\frac{x^5}{5}-\frac{x^4}{4}+2\frac{x^3}{3}+\frac{x^2}{2}+x\Big]_{-1}^{1}=\frac{32}{15}. \qquad \blacksquare$$

Attention: 注意! 不能寫成 $\displaystyle\int_{2x^2}^{1+x^2}\int_{-1}^{1}(x+2y)\ dx\ dy$ (順序錯誤)!

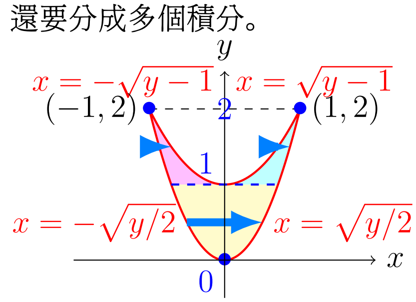
怎麼記? 規則: 裡面變數的上下界必須是外面變數的函數。

♦ 換 **Type II**: 不僅僅只是上下界解反函數, 還要分成多個積分。

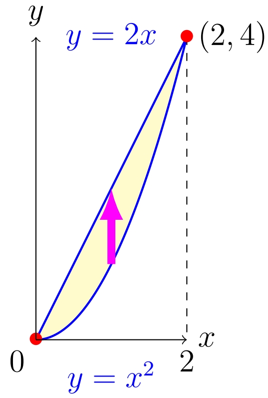
$$\int_0^1\int_{-\sqrt{y/2}}^{\sqrt{y/2}}(x+2y)\ dx\ dy$$

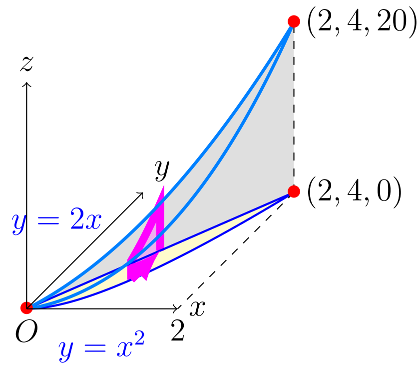
$$+\int_1^2\int_{\sqrt{y-1}}^{\sqrt{y/2}}(x+2y)\ dx\ dy$$

$$+\int_1^2\int_{-\sqrt{y/2}}^{-\sqrt{y-1}}(x+2y)\ dx\ dy.$$



Example 0.2 *Find the volume of the solid lying under the paraboloid* $z = x^2 + y^2$ *and above the region* D *in the* xy*-plane bounded by the line* $y = 2x$ *and the parabola* $y = x^2$.



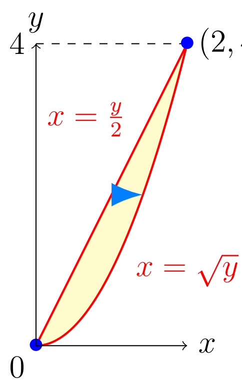


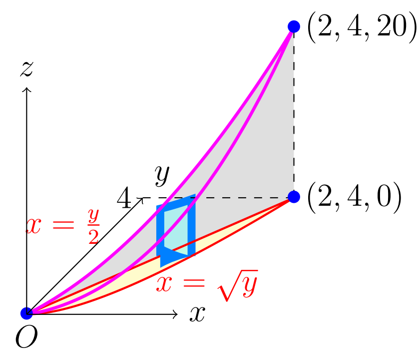
找邊界: $2x = y = x^2$, $(x, y) = (0, 0), (2, 4)$.

Type I $D = \{(x, y) : 0 \le x \le 2,\ x^2 \le y \le 2x\}$.

$$V = \iint_D (x^2 + y^2)\ dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\ dy\ dx = \int_0^2 \Big[x^2 y + \frac{y^3}{3}\Big]_{y=x^2}^{y=2x}\ dx$$

$$= \int_0^2 \Big(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\Big)\ dx = \Big[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Big]_0^2 = \frac{216}{35}.$$



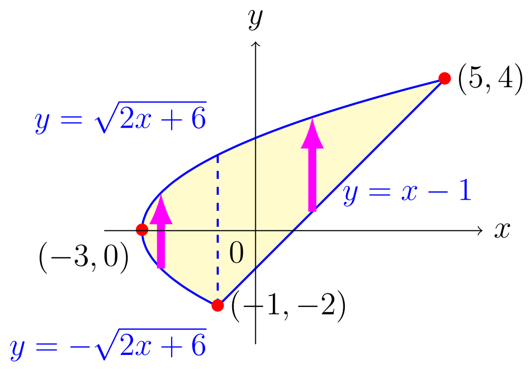


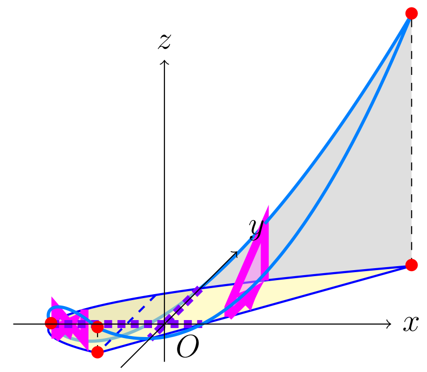
Type II $D = \{(x, y) : \frac{y}{2} \le x \le \sqrt{y},\ 0 \le y \le 4\}$.

$$V = \iint_D (x^2 + y^2)\ dA = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2 + y^2)\ dx\ dy = \int_0^4 \Big[\frac{x^3}{3} + xy^2\Big]_{x=y/2}^{x=\sqrt{y}}\ dy$$

$$= \int_0^4 \Big(\frac{y^{3/2}}{3} + y^{5/2} - \frac{13y^3}{24}\Big)\ dy = \Big[\frac{2y^{5/2}}{15} + \frac{2y^{7/2}}{7} - \frac{13y^4}{96}\Big]_0^4 = \frac{216}{35}. \qquad \blacksquare$$

Example 0.3 *Evaluate* $\displaystyle\iint_D xy\ dA$*, where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.*





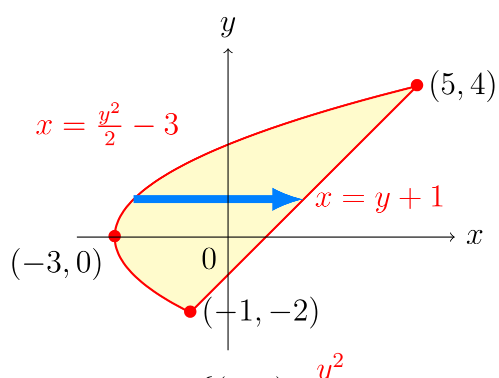
找邊界: $(x-1)^2 = 2x+6$, $(x,y) = (-1,-2), (5,4)$.

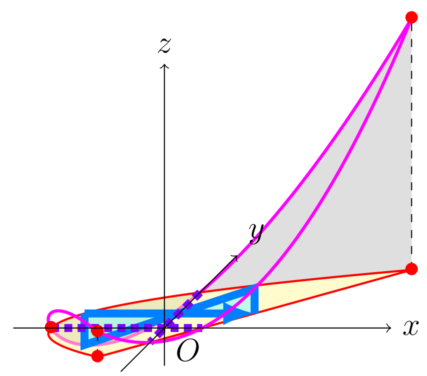
Type I $D = \{(x,y) : -1 \le x \le 5,\ x-1 \le y \le \sqrt{2x+6}\}$? 錯!

(只解交點可能會漏看。)

(眞正的) $D = \{(x,y) : -3 \le x \le -1,\ -\sqrt{2x+6} \le y \le \sqrt{2x+6}\}$
$\cup \{(x,y) : -1 \le x \le 5,\ x-1 \le y \le \sqrt{2x+6}\}$.

$$\iint_D xy\ dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\ dy\ dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\ dy\ dx = \cdots \text{(有點難)}$$



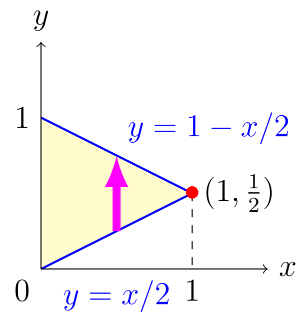


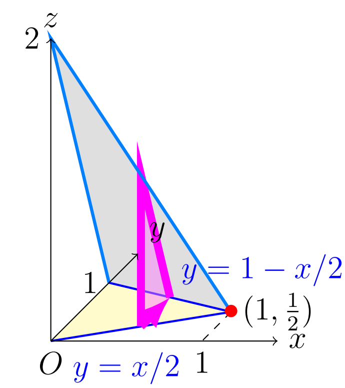
Type II $D = \{(x,y) : \dfrac{y^2}{2} - 3 \le x \le y+1,\ -2 \le y \le 4\}$.

$$\iint_D (x^2+y^2)\ dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\ dx\ dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \int_{-2}^{4}\frac{1}{2}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy = \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + \frac{2y^3}{3} - 4y^2\right]_{-2}^{4} = 36. \quad \blacksquare$$

Example 0.4 *Find the volume of the tetrahedron(四面體) bounded by the planes* $x+2y+z=2$, $x=2y$, $x=0$ *and* $z=0$.



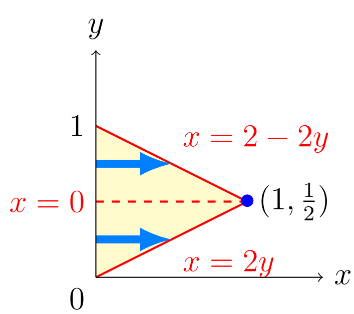


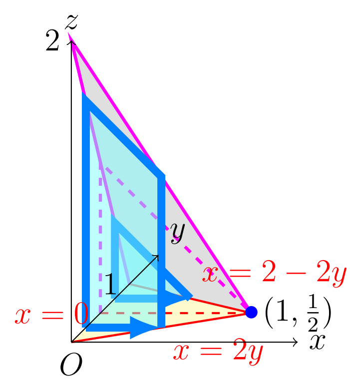
找邊界: 找平面 $z=2-x-2y$ (& $x=2y$) 與 xy-plane ($z=0$) 的交線 $y=1-\dfrac{x}{2}$ (& $x=2y$); $\dfrac{x}{2}=y=1-\dfrac{x}{2}$, $(x,y)=(1,\dfrac{1}{2})$.

Type I $D=\{(x,y): 0\le x\le 1,\ x/2\le y\le 1-x/2\}$.

$$V=\iint_D z\,dA=\int_0^1\int_{x/2}^{1-x/2}(2-x-2y)\,dy\,dx=\int_0^1\Big[2y-xy-y^2\Big]_{y=x/2}^{y=1-x/2}dx$$

$$=\int_0^1\left(x^2-2x+1\right)dx=\left[\frac{x^3}{3}-x^2+x\right]_0^1=\frac{1}{3}.$$





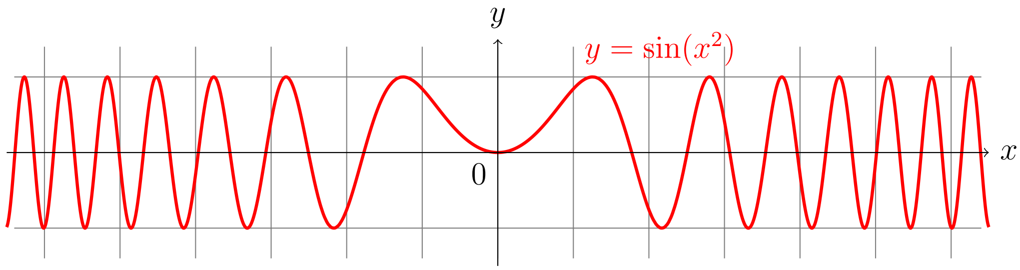
Type II $D=\{(x,y): 0\le x\le 2y,\ 0\le y\le 1/2\}$
$\cup\{(x,y): 0\le x\le 2-2y,\ 1/2\le y\le 1\}$. (要分兩塊)

$$V=\iint_D z\,dA=\int_0^{1/2}\int_0^{2y}(2-x-2y)\,dx\,dy+\int_{1/2}^1\int_0^{2-2y}(2-x-2y)\,dx\,dy$$

$=\cdots=\dfrac{1}{3}$. (*In fact,* 用三角錐體積 $=\dfrac{\text{底面積}\times\text{高}}{3}=\dfrac{1\times1\times2}{3\times2}=\dfrac{1}{3}$ 更快。) ■

Example 0.5 *Evaluate the iterated integral* $\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx$.

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx = \cdots \text{ 不會積 } \int \sin(y^2)\ dy!$$

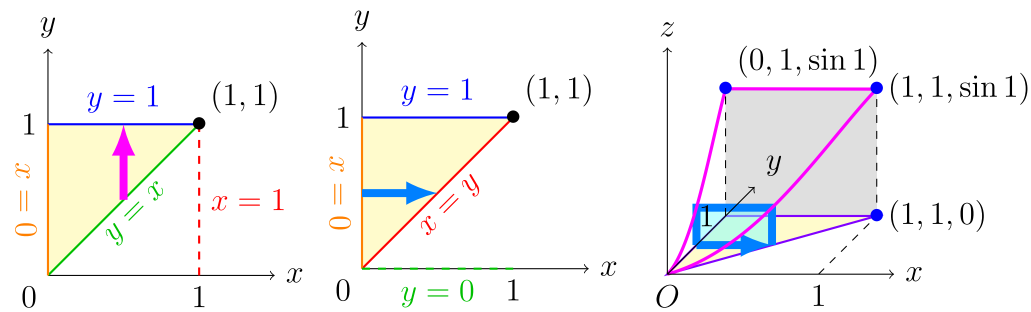


現在放棄的話, 比賽就結束了。 — 安西光義 *(Mitsuyoshi Anzai)*

Skill: *How to change types? Draw!* 歐雷諾疼!

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx \implies \begin{array}{ll} y = 1, & x = 1, \\ y = x, & x = 0. \end{array} \quad \text{(從積分上下界找邊界函數)}$$

$\implies$ **Type I** $D = \{(x, y) : 0 \le x \le 1,\ x \le y \le 1\}$.



Type II $D = \{(x, y) : 0 \le x \le y,\ 0 \le y \le 1\}$.

$$\int_0^1 \int_0^y \sin(y^2)\ dx\ dy = \int_0^1 \Big[x \sin(y^2)\Big]_{x=0}^{x=y} dy = \int_0^1 y \sin(y^2)\ dy$$

$$= \Big[-\frac{1}{2}\cos(y^2)\Big]_0^1 = -\frac{1}{2}\cos 1 - (-\frac{1}{2}) = \frac{1}{2}(1 - \cos 1).$$ ■

Attention: 注意! 換 types 不可以直接交換!

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx \neq \int_x^1 \int_0^1 \sin(y^2)\ dx\ dy.$$

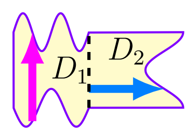
Note: 有些題目不換 type 真的會積不出來。

(Recall) Property: f, g are integrable functions on D and c is constant.

- $\displaystyle\iint_D (f+g)\, dA = \iint_D f\, dA + \iint_D g\, dA,$
- $\displaystyle\iint_D cf\, dA = c\iint_D f\, dA,$
- $\displaystyle f \ge g \implies \iint_D f\, dA \ge \iint_D g\, dA.$

Property:

- $\displaystyle\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA,$



  if $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$ except perhaps on their boundaries.
  不是 **Type I** 也不是 **Type II**: 切割成 **Type I** 或 **Type II** 分開算。

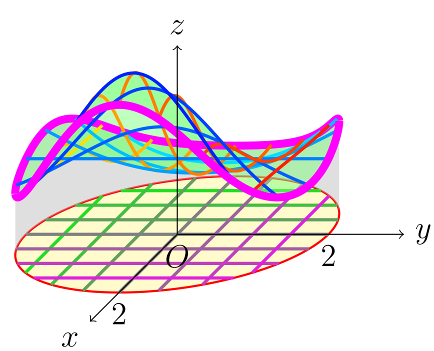
- $\displaystyle\iint_D 1\, dA = A(D),$

  where $A(D)$ is the area of D. 要算 D 的面積, 讓 $f(x,y)=1$.

- $\displaystyle mA(D) \le \iint_D f(x,y)\, dA \le MA(D),$

  if $m \le f(x,y) \le M$ for $(x,y) \in D$. 可以用來估計體積。

Example 0.6 *Estimate* $\displaystyle\iint_D e^{\sin x \cos y}\, dA$, *where D is the disk with center the origin and radius 2.*



$-1 \le \sin x \le 1,\ -1 \le \cos y \le 1,$
$-1 \le \sin x \cos y \le 1,\ e^x$ *is increasing,*
$\implies\ e^{-1} \le e^{\sin x \cos y} \le e,\ A(D) = 4\pi.$
$\displaystyle\frac{4\pi}{e} \le \iint_D e^{\sin x \cos y}\, dA \le 4\pi e.$ ■