1179: Probability
Lecture 8 — Special Discrete Random Variables

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Quick Review

$X \quad F_X(t) = P(X \le t) = P(\{\omega : X(\omega) \le t\})$

- What is a CDF? How about PMFs? discrete $P(X = k_i)$
- What are the properties of CDFs?
  1. non-decreasing in t
  2. $F_X(t) \to 1$ as $t \to \infty$
  3. $F_X(t) \to 0$ as $t \to -\infty$
  4. right-continuous
- Bernoulli random variables?
- Binomial random variables?

$X \sim \text{Bernoulli}(p)$

$X \sim \text{Binomial}(n, p) \xRightarrow{\text{PMF}} P(X = k) = \begin{cases} C_k^n \, p^k (1-p)^{n-k}, & k = 0, 1, 2, 3, \cdots, n \\ , & \text{otherwise} \end{cases}$

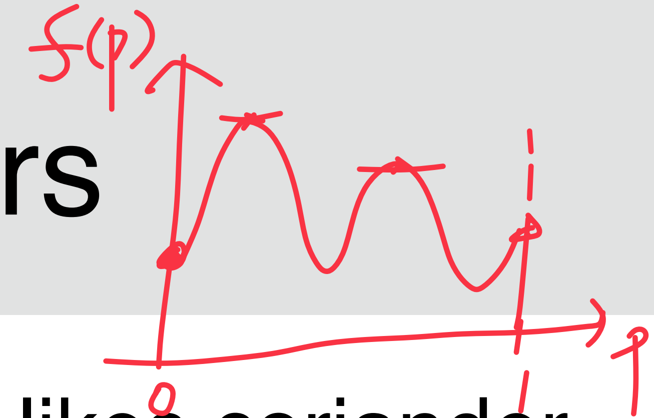
This Lecture

1. Special Discrete Random Variables

- Reading material: Chapter 5.2

Special Discrete Random Variables

Example: A Poll of Coriander Lovers



- Example: Let p = probability that a random person likes coriander.
  - Suppose we randomly sample N people and define a random variable X = {number of coriander lovers in N people}
  - For a fixed integer k, under what value of p is $P(X = k)$ maximized?

$X \sim \text{Binomial}(N, p)$

$$P(X=k) = C_k^N \, p^k \, (1-p)^{N-k} \triangleq f(p)$$

$$g(p) = \ln f(p) = \ln C_k^N + k \cdot \ln p + (N-k)\ln(1-p)$$

$$\frac{d\,g(p)}{dp} = 0 + \frac{k}{p} + \frac{N-k}{1-p}(-1) = 0 \Rightarrow p = \frac{k}{N}$$

$p = 0$

$p = 1$

泊松

3. Poisson Random Variables

- Example: On average, 20 people stop by Shinemood every hour. What is P(exactly 100 people visit Shinemood in 3 hours)?

- Example: On average, 1000 MayDay's concert tickets are sold every second. What is P(all 50k tickets are sold out in 1 min)?

PMF

CDF

- What are the common features?
  - Average rate is known and static
  - Want: how many occurrences in an observation window?

Poisson: Limiting Case of Binomial

$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$e = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}$

$n \cdot p = \lambda$

- Example: Consider $X \sim \text{Binomial}(n, p = \lambda/n)$, λ is a constant
  - What is $P(X = k)$?
  - What if $n \to \infty$?

$\boxed{\text{If } n\to\infty,\ P(X=k) \to \frac{e^{-\lambda}\cdot\lambda^k}{k!}}$

If $k = 0, 1, 2, \cdots, n$:

$$P(X=k) = C_k^n\, p^k \cdot (1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k \cdot \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n \cdot n \cdots n}}_{(1)} \cdot \underbrace{\frac{\lambda^k}{k!}}_{(2)} \cdot \frac{\overbrace{\left(1-\frac{\lambda}{n}\right)^n}^{(3)}}{\underbrace{\left(1-\frac{\lambda}{n}\right)^k}_{(4)}}$$

1, $1-\frac{1}{n}$, $1-\frac{k-1}{n}$

If $n\to\infty$:

(1) $\longrightarrow 1$

(2) $\longrightarrow \frac{\lambda^k}{k!}$

(3): $\left(1-\frac{\lambda}{n}\right)^n = \left(\underbrace{\left(1-\frac{\lambda}{n}\right)^{\frac{n}{\lambda}}}_{\to e}\right)^{\lambda} \longrightarrow e^{-\lambda}$

(4) $\longrightarrow 1$

3. Poisson Random Variables (Formally)

$e^x = 1 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \cdots_\infty = \sum_{n=0} \frac{1}{n!}x^n$

$\sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} = e^{\lambda T}$

Poisson Random Variables: Given parameters

- λ: average rate
- T: duration of the observation window

A random variable X is Poisson with parameter λT if its PMF is given by

$$P(X = n) = \begin{cases} \frac{e^{-\lambda T}(\lambda T)^n}{n!}, & n = 0,1,2,3,\cdots \\ 0 & \text{otherwise} \end{cases}$$

- Do we have $\sum_{n=0}^{\infty} P(X = n) = 1$?

$\sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} \stackrel{?}{=} 1$

Taylor's expansion

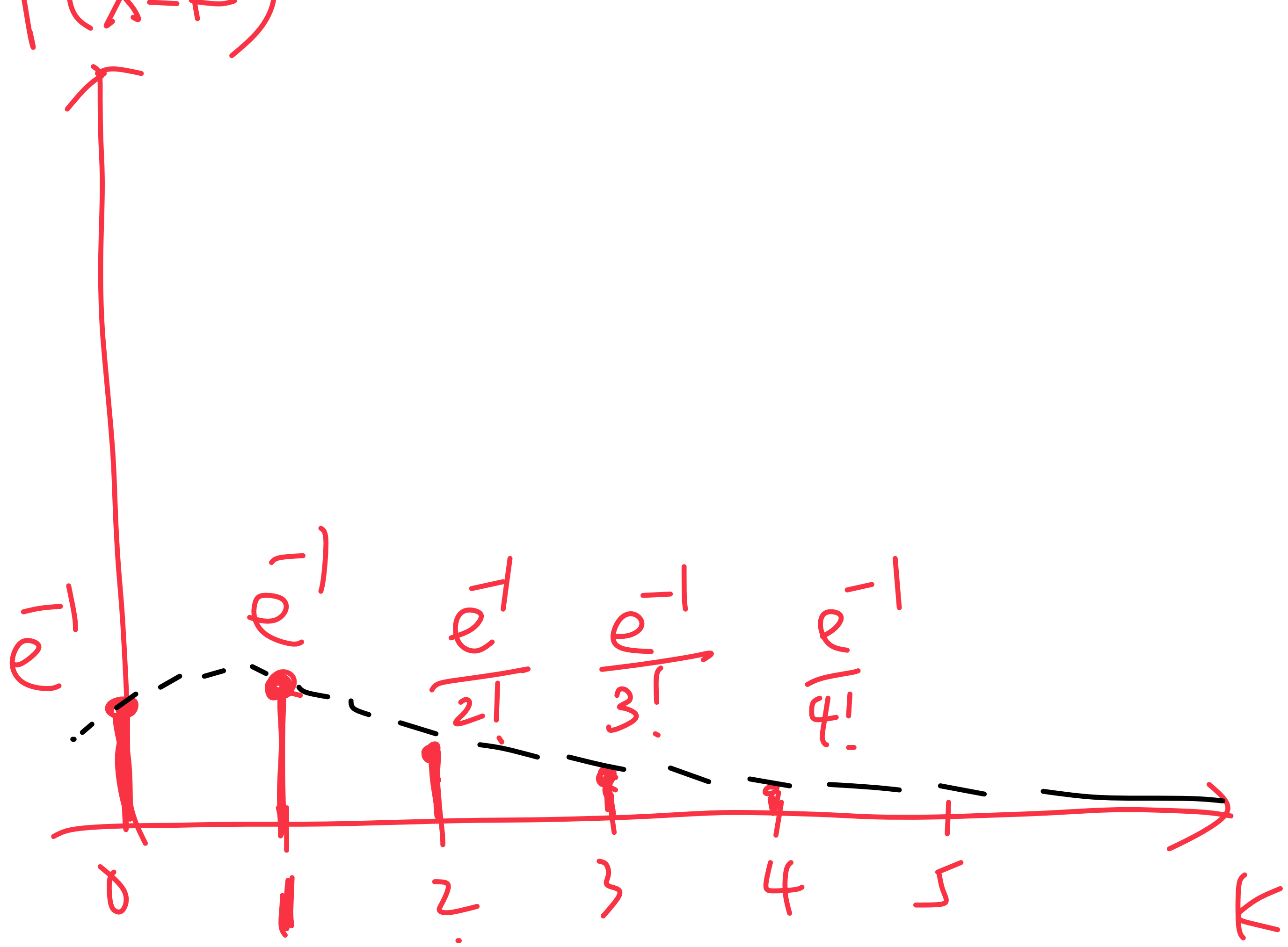
PMFs of Poisson Random Variables

$$P(X=k) = \frac{e^{-\lambda T}\cdot(\lambda T)^{k}}{k!}$$

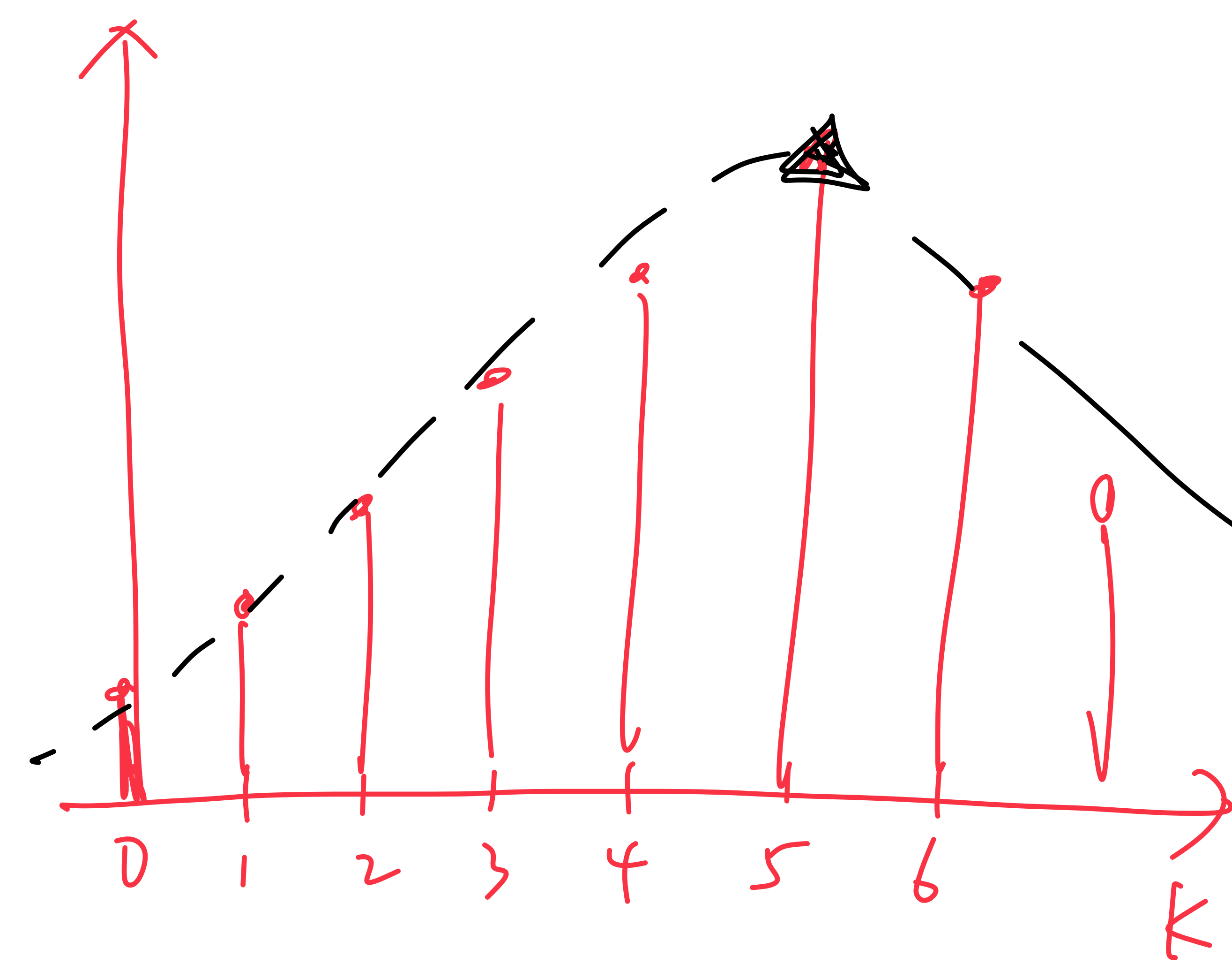
if $k=0,1,2,\cdots$

- Example: Let's plot the PMF of $X \sim$ Poisson(λT)

$\underline{\lambda T = 1}$



$\underline{\lambda T = 5}$



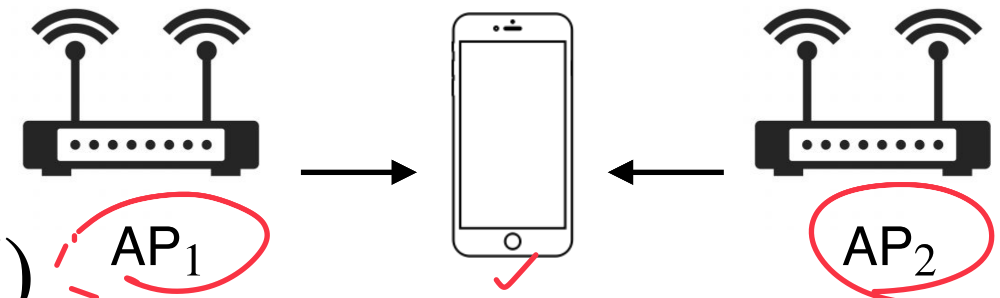
Recall: An Interview Question by Google

- Example: Suppose we stand at the Fude temple.
  - The probability that we see at least 1 car passing through the temple in 30 minutes is 0.95.
  - What is $P(\text{we see at least 1 car in 10 mins})$?

Sum of Independent Poisson Random Variables

is still Poisson

- Let N_1 and N_2 be the number of total bits transmitted by AP$_1$ and AP$_2$ in a time interval T, respectively
  - Suppose N_1 and N_2 are independent
  - $N_1 \sim$ Poisson(λ_1, T), $N_2 \sim$ Poisson(λ_2, T)
  - Let Z be the total received bits. Then, what is the PMF of Z?



Convolution

$Z = N_1 + N_2$

If $k = 0, 1, 2, \ldots$

$$P(Z=k) = P(N_1+N_2=k) = \sum_{j=0}^{k} P(N_1=j)\,P(N_2=k-j)$$

$$= \sum_{j=0}^{k} \frac{e^{-\lambda_1 T}(\lambda_1 T)^j}{j!} \cdot \frac{e^{-\lambda_2 T}(\lambda_2 T)^{k-j}}{(k-j)!} \cdot \frac{k!}{k!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)T}}{k!} \sum_{j=0}^{k} (\lambda_1 T)^j (\lambda_2 T)^{k-j} C_j^k = (\lambda_1 T + \lambda_2 T)^k$$

$$\frac{e^{-(\lambda_1+\lambda_2)T}((\lambda_1+\lambda_2)T)^k}{k!} = \frac{e^{-(\lambda_1+\lambda_2)T}}{k!} \sum_{j=0}^{k} (\lambda_1 T)^j (\lambda_2 T)^{k-j} C_j^k$$

4. Geometric Random Variables

- Example: Play with a claw machine, and each trial is successful with probability 0.7. What is $P(\text{get 1st toy at 10-th trial})$?

- Example: Po-Jung Wang makes a hit with probability 0.28 at each at-bat. What is $P(\text{he makes his 1st hit at 5-th at-bat})$?

- What are the common features?
  - Repetitions of the same Bernoulli experiment
  - Want: how many trials needed until the 1st success?

4. PMF of Geometric Random Variables

- Example: Play with a claw machine, and each trial is successful with probability 0.7. All trials are independent.
  - X = the number of trials until we get the first toy
  - What is the PMF of X?

4. Geometric Random Variables (Formally)

Geometric Random Variables: A random variable X is Geometric with parameters p if its PMF is given by

$$P(X = k) = (1 - p)^{k-1}p, \; k = 1,2,3,\cdots$$

- Do we have $\displaystyle\sum_{k=1}^{\infty} P(X = k) = 1$?

CDF of Geometric Random Variables

$$P(X = k) = (1 - p)^{k-1}p, \ k = 1,2,3,\cdots$$

- CDF: $F_X(t) = P(X \leq t)$

Geometric r.v.: Memoryless Property

- Example: Suppose $X \sim$ Geometric(p), $p \in (0,1)$
  - What is $P(X = n + m \,|\, X > m)$? ($n, m \in \mathbb{N}$)
  - What is $P(X > n + m \,|\, X > m)$? ($n, m \in \mathbb{N}$)

5. Discrete Uniform Random Variables

- Example: Roll a 4-sided die, and the numbers 1, 2, 3, 4 are equally likely to occur

- Example: The correct answer to an exam question: A, B, C, D are equally likely

- What are the common features?
  - 1 experiment trial (no repetition) with n equally-likely outcomes
  - Want: Whether a specific outcome occurs

5. Discrete Uniform Random Variables (Formally)

Discrete Uniform Random Variables: A random variable X is discrete uniform with parameters (a, b) ($a, b \in \mathbb{Z}$ with $a \leq b$), if its PMF is given by

$$P(X = k) = \frac{1}{b - a + 1}, \ k = a, a + 1, \cdots, b$$

1-Minute Summary

1. Special Discrete Random Variables

- Poisson / Geometric / Uniform