Problem 1(a)

Problem 1

(a) $$f_{Z_2|Z_1} = \frac{f_{Z_1,Z_2}(z_1,z_2)}{f_{Z_1}(z_1)} = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot e^{-\frac{\frac{(z_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}}}{\frac{1}{\sigma_1\sqrt{2\pi}} \cdot e^{\frac{-(z_1-\mu_1)^2}{2\sigma_1^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)\sigma_2^2}} e^{-\frac{\frac{\rho^2(z_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(z_1-\mu_1)(z_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(z_2-\mu_2)^2}{\sigma_2^2}}{2(1-\rho^2)}}$$

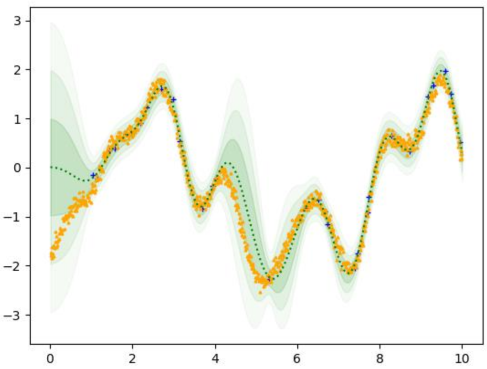
$$= \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)\sigma_2^2}}\cdot e^{-\frac{\left(\frac{\rho(z_1-\mu_1)}{\sigma_1}-\frac{z_2-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)\sigma_2^2}}\cdot e^{\frac{\left(z_2-\left(\mu_2+\frac{\rho\sigma_2(z_1-\mu_1)}{\sigma_1}\right)\right)^2}{2(1-\rho^2)\sigma_2^2}}$$

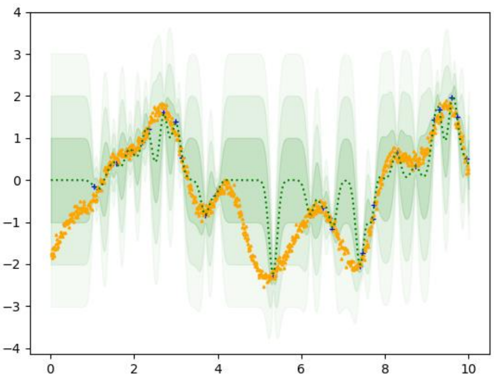
$$\Rightarrow Z_2 \sim \left(\mu_2+\frac{\rho\sigma_2(z_1-\mu_1)}{\sigma_1},\ (1-\rho^2)\sigma_2^2\right)$$ #

Problem 1(b)

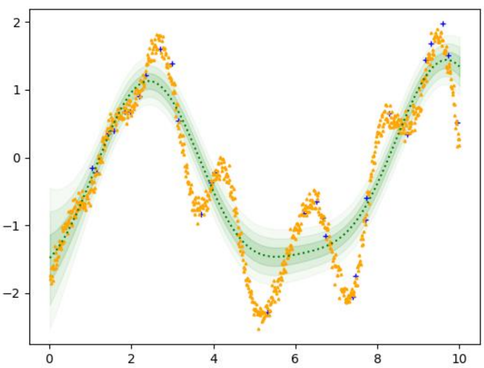
Lengthscale = 0.5 :



Lengthscale = 0.1 :



Lengthscale = 2:



According to the statistics above, when using smaller lengthscale, the standard deviation is larger, but have higher probability that the predicted mean is within 3 predicted standard deviation. On the contrary, when using larger lengthscale, the predicted standard deviation is smaller; however, the predicted mean is not so precise. In conclusion, we should use a proper lengthscale to estimate the predicted mean and predicted standard deviation.

Problem 2(a)

```
size = 1000
iteration 0 : 0.6348217527450568
iteration 1 : 0.612454495553295
iteration 2 : 0.5661897457967261
iteration 3 : 0.5859919096118864
iteration 4 : 0.5671803364982161
iteration 5 : 0.6102783126430272
iteration 6 : 0.5718546857651398
iteration 7 : 0.6064129144736585
iteration 8 : 0.6058152164716489
iteration 9 : 0.6376858992412436
iteration 10 : 0.618821521057024
iteration 11 : 0.6255852865319692
iteration 12 : 0.6340949029398953
iteration 13 : 0.6404297202758101
iteration 14 : 0.5336099083116922
iteration 15 : 0.6152975471986211
iteration 16 : 0.6134224994148427
iteration 17 : 0.6296827575247643
iteration 18 : 0.61698266493156
iteration 19 : 0.6578079568893755
```

```
size = 100000
iteration 0 : 0.6064832324871384
iteration 1 : 0.604256754822873
iteration 2 : 0.6053758149683165
iteration 3 : 0.6059914813924207
iteration 4 : 0.6101755997674125
iteration 5 : 0.6050011409892135
iteration 6 : 0.6059209444535418
iteration 7 : 0.6071239706960739
iteration 8 : 0.6065736922664905
iteration 9 : 0.6047260012078247
iteration 10 : 0.6050203731953211
iteration 11 : 0.6045012281819222
iteration 12 : 0.6051420243858445
iteration 13 : 0.6098311698555873
iteration 14 : 0.6078343175190476
iteration 15 : 0.6074104587961913
iteration 16 : 0.6074500982682195
iteration 17 : 0.6048772064277748
iteration 18 : 0.6040449723255946
iteration 19 : 0.6096097297619937
```

When we take more sample points, the estimation result each time is closer, and also more precise to the real expected value.

Problem 2(b)

```
n = 10 : 1.2
n = 1000 : 0.852
n = 100000 : 0.78356
n = 10000000 : 0.7859304
```

When taking more sample points, the resulting estimation is more precise and closer to the real area of the region A.