♦ 7.5 Strategy for integration (optional)

積分戰略 (5.3 $\sim$ 5.5, 7.1 $\sim$ 7.4)。

- 分開: 加減常數倍, 部分分式。
  - ▶ $\displaystyle\int \sqrt{x}(1+\sqrt{x})\ dx = \int \sqrt{x}\ dx + \int x\ dx.$
  - ▶ $\displaystyle\int \frac{x}{x^2-1}\ dx = \int \frac{1/2}{x-1}\ dx + \int \frac{1/2}{x+1}\ dx.$
- 變形: 三角函數 (定義, 恆等式, 半角)。
  - ▶ $\displaystyle\int \frac{\tan\theta}{\sec^2\theta}\, d\theta = \int \frac{\sin\theta/\cos\theta}{1/\cos^2\theta}\, d\theta = \int \sin\theta\cos\theta\ d\theta = \int \frac{1}{2}\sin 2\theta\ d\theta.$
  - ▶ $\displaystyle\int (\sin x+\cos x)^2\ dx = \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\ dx$
    $\displaystyle= \int\ dx + 2\int \sin x\cos x\ dx = \int\ dx + \int \sin 2x\ dx.$
  - ▶ $\displaystyle\int \tan^2 x\sec x\ dx = \int \sec^3 x\ dx - \int \sec x\ dx.$
  - ▶ $\displaystyle\int \frac{dx}{1-\cos x} = \int \frac{dx}{2\sin^2(x/2)} = \int \csc^2\frac{x}{2}\ d(\frac{x}{2}).$
- 變換: 有理, 三角。
  - ▶ $\displaystyle\int \frac{x}{x^2+1}\ dx = \int \frac{1/2}{x^2+1}\ d(x^2+1) = \int \frac{\tan\theta}{\sec^2\theta}\cdot\sec^2\theta\ d\theta.$
  - ▶ $\displaystyle\int \sin\theta\cos\theta\ d\theta = \int \sin\theta\ d(\sin\theta) = \int -\cos\theta\ d(\cos\theta).$
- 其他: 同乘 (小心 0), 簡化。
  - ▶ $\displaystyle\int \frac{dx}{1-\cos x} = \int \Big(\frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\Big)\ dx = \int \frac{1+\cos x}{1-\cos^2 x}\ dx$
    $\displaystyle= \int \frac{1+\cos x}{\sin^2 x}\ dx = \int \csc^2 x + \cot x\csc x\ dx.$
- 分部: $\displaystyle\int u\ dv = uv - \int v\ du.$

Example 0.1 $\int \frac{\tan^3 x}{\cos^3 x}\ dx$ .... $\left(\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C\right)$

$$\int \frac{\tan^3 x}{\cos^3 x}\ dx = \int \tan^3 x \sec^3 x\ dx \qquad \text{(換成 } \tan, \sec\text{)}$$

$$= \int \tan^2 x \sec^2 x \cdot \sec x \tan x\ dx \boxed{= \int (\sec^2 x - 1)\sec^2 x\ d(\sec x)}$$

$$= \int (v^4 - v^2)\ dv = \frac{v^5}{5} - \frac{v^3}{3} + C;$$

$$\int \frac{\tan^3 x}{\cos^3 x}\ dx = \int \frac{\sin^3 x}{\cos^6 x}\ dx \qquad \text{(換成 } \sin, \cos\text{)}$$

$$= \int \frac{-\sin^2 x}{\cos^6 x}(-\sin x)\ dx \boxed{= \int \frac{\cos^2 x - 1}{\cos^6 x}\ d(\cos x)}$$

$$= \int (u^{-4} - u^{-6})\ du = -\frac{u^{-3}}{3} + \frac{u^{-5}}{5} + C. \qquad \blacksquare$$

Example 0.2 $\int e^{\sqrt{x}}\ dx$ .... $\left(e^{\sqrt{x}}(2\sqrt{x} - 2) + C\right)$

$$\int e^{\sqrt{x}}\ dx = \int e^u (2u)\ du \boxed{= 2\int u\ d(e^u)} = 2ue^u - 2\int e^u\ du$$

$$= 2ue^u - 2e^u + C. \qquad \blacksquare$$

Example 0.3 $\int \frac{x^5 + 1}{x^3 - 3x^2 - 10x}\ dx$

.... $\left(\frac{x^3}{3} + \frac{3x^2}{2} + 19x - \frac{1}{10}\ln|x| + \frac{3126}{35}\ln|x-5| - \frac{31}{14}\ln|x+2| + C\right)$

$$\int \frac{x^5 + 1}{x^3 - 3x^2 - 10x}\ dx = \int \left(x^2 + 3x + 19 + \frac{87x^2 + 190x + 1}{x(x-5)(x+2)}\right)\ dx$$

$$= \int \left(x^2 + 3x + 19 + \frac{-1/10}{x} + \frac{3126/35}{x-5} + \frac{-31/14}{x+2}\right)\ dx. \qquad \blacksquare$$

Example 0.4 $\int \frac{dx}{x\sqrt{\ln x}}$ .... $\left(2\sqrt{\ln x} + C\right)$

$$\int \frac{dx}{x\sqrt{\ln x}} \boxed{= \int \frac{1}{\sqrt{\ln x}}\ d(\ln x)} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C. \qquad \blacksquare$$

Example 0.5 $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$ . . . . $\left(\sin^{-1}x+\sqrt{1-x^2}+C\right)$

$$\int \sqrt{\frac{1-x}{1+x}}\,dx \overset{\times\sqrt{1-x}}{=} \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$\boxed{= \int \frac{1}{\sqrt{1-\sin^2\theta}}\,d(\sin\theta) + \int \frac{1/2}{\sqrt{1-x^2}}\,d(1-x^2)} \quad \textit{(分開做三角/變數變換)}$$

$$= \int d\theta + \int \frac{du}{2\sqrt{u}} = \theta + \sqrt{u} + C. \qquad \blacksquare$$

♦ [Problem Plus, page 540–542]

2. $\displaystyle\int \frac{1}{x^7-x} = ?$ $\qquad (\frac{?}{x}+\frac{?}{x^6-1})$

3. $\displaystyle\int_0^1 (\sqrt[3]{1-x^7}-\sqrt[7]{1-x^3})\,dx = ?$ $\qquad (y=\sqrt[3]{1-x^7})$

8. $n\in\mathbb{N}$, $\displaystyle\int_0^1 (\ln x)^n\,dx = ?$ $\qquad (u=(\ln x)^n,\ dv=dx)$

11. $\displaystyle\lim_{t\to 0}\left\{\int_0^1 [bx+a(1-x)]^t\,dx\right\}^{1/t} = ?$ $\qquad (u=bx+a(1-x))$

13. $\displaystyle\int_{-1}^{\infty}\left(\frac{x^4}{1+x^6}\right)^2 dx = ?$ $\qquad (x^3=u=\tan t)$

14. $\displaystyle\int \sqrt{\tan x}\,dx = ?$ $\qquad (u=\sqrt{\tan x})$

但是, 還是有些積不出來。(或許有其他方法。) 例如:

$$\int e^{x^2}\,dx,\ \int e^{-x^2}\,dx,\ \int \frac{e^x}{x}\,dx = \int \frac{dx}{xe^x},\ \int \frac{e^x}{x^2}\,dx = -\frac{e^x}{x} + \int \frac{e^x}{x}\,dx,$$

$$\int \sin x^2\,dx,\ \int \cos e^x\,dx,\ \int \sqrt{x^3+1}\,dx,\ \int x\sqrt{x^3+1}\,dx,$$

$$\int \frac{1}{\ln x}\,dx,\ \int \frac{\sin x}{x}\,dx, \cdots$$

推薦做一做這節的習題作爲綜合練習。

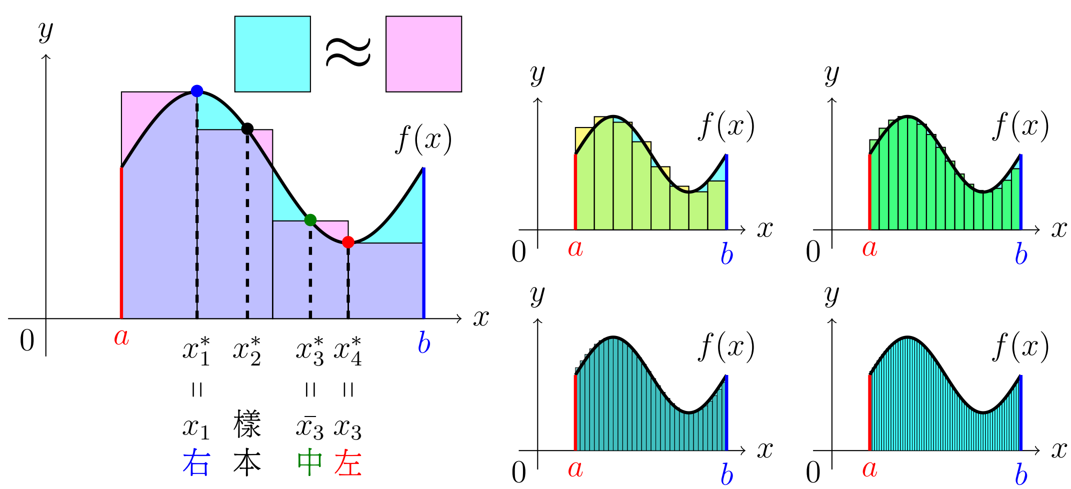
♦ 7.7 Approximate integration

1. Right endpoint rule 右端法 R_n
2. Left endpoint rule 左端法 L_n
3. Trapezoidal rule 梯形法 T_n
4. Midpoint rule 中點法 M_n
5. Simpson's rule 辛普森法 S_{2n}
6. Error bounds 誤差

Ex: $\displaystyle\int_0^1 e^{x^2}\,dx$, $\displaystyle\int_{-1}^1 \sqrt{x^3+1}\,dx$: 求不出來。
Ex: 有時候只是測量所得, 不見得是個函數。

Idea: 用黎曼和 (Riemann sum) 求近似值。
Recall: $f(x)$ is integrable on $[a,b]$,
sample points $x_i^* \in [x_{i-1}, x_i]$, $x_i = a + i\Delta x$, $i = 1, \ldots, n$, $\Delta x = \dfrac{b-a}{n}$.

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x \;\approx\; \sum_{i=1}^n f(x_i^*)\Delta x.$$



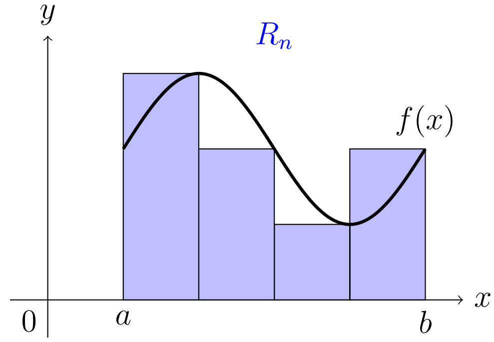
0.1 Right/Left endpoint rule

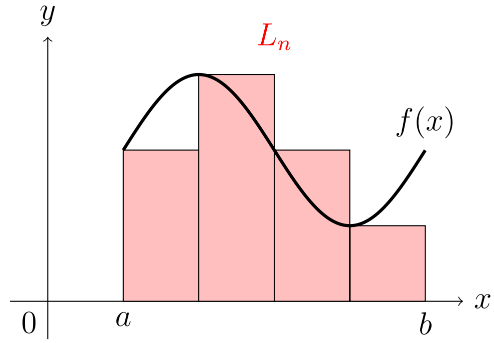
$$\int_a^b f(x)\,dx \;\approx\; \boxed{\boldsymbol{R_n}} \quad (右端點)$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\int_a^b f(x)\,dx \;\approx\; \boxed{\boldsymbol{L_n}} \quad (左端點)$$

$$= \sum_{i=1}^{n} f(x_{i-1})\Delta x$$





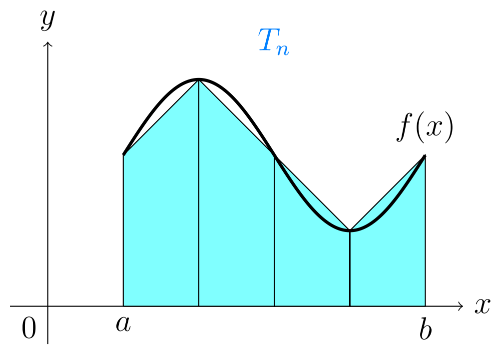
0.2 Trapezoidal rule

$$\int_a^b f(x)\,dx \;\approx\; \boxed{\boldsymbol{T_n}} \quad (梯形)$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Note: 係數是: 1,2,2,...,2,1.

$$\boxed{\boldsymbol{T_n} = \frac{\boldsymbol{R_n} + \boldsymbol{L_n}}{\boldsymbol{2}}} \;(梯形 = 左右端平均)$$

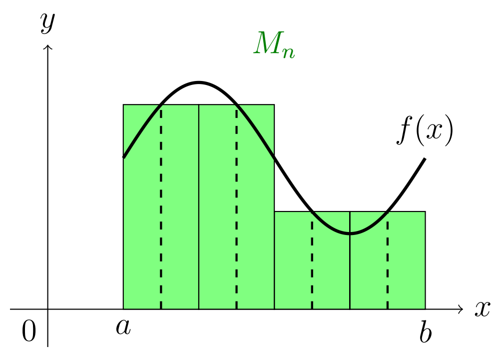


0.3 Midpoint rule

$$\int_a^b f(x)\,dx \;\approx\; \boxed{\boldsymbol{M_n}} \quad (中點)$$

$$= \sum_{i=1}^{n} f(\bar{x}_i)\Delta x,$$

where $\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$.



0.4 Simpson's rule

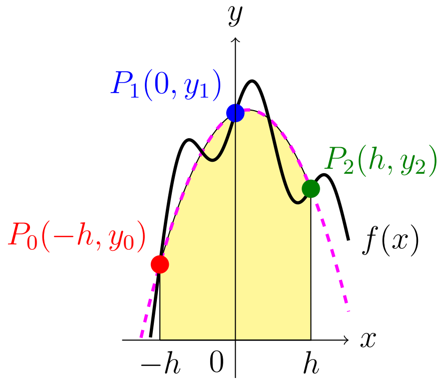
Simpson 考慮偶數 n, 用通過 $(x_{2i-2}, f(x_{2i-2})), (x_{2i-1}, f(x_{2i-1})), (x_{2i}, f(x_{2i}))$ 的拋物線逼近第 $(2i-1)$ 與第 $(2i)$ 段。

(方便計算面積, 把 x_{2i-1} 平移到 0, let $h = \Delta x$.)

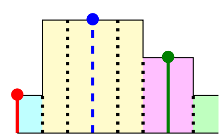
假設拋物線 $y = Ax^2 + Bx + C$ 通過 $P_0(-h, y_0)$, $P_1(0, y_1)$, $P_2(h, y_2)$,

$$\Longrightarrow \begin{cases} y_0 = Ah^2 - Bh + C, \\ y_1 = C, \\ y_2 = Ah^2 + Bh + C. \end{cases}$$

$$\begin{aligned} &\int_{-h}^{h} (Ax^2 + Bx + C)\, dx \\ &= 2\int_0^h (Ax^2 + C)\, dx \\ &= \frac{h}{3}(2Ah^2 + 6C) \\ &= \frac{h}{3}(y_0 + 4y_1 + y_2), \end{aligned}$$



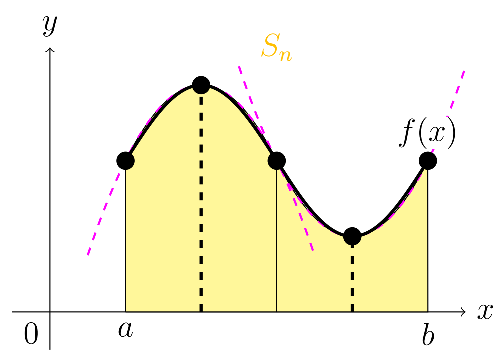
$$\begin{aligned} \int_a^b f(x)\, dx \quad &\approx \quad \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) \\ &\qquad + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \quad \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \end{aligned}$$



Simpson's Rule

$$\int_a^b f(x)\, dx \approx \boxed{S_n}$$

$$\begin{aligned} = \frac{\Delta x}{3} \big[\quad & f(x_0) \quad +4f(x_1) \quad +2f(x_2) \\ & +4f(x_3) \quad +2f(x_4) \quad + \cdots \\ & +2f(x_{n-2}) +4f(x_{n-1}) +f(x_n) \big], \end{aligned}$$



where n is even.

Note: 係數是: 1,4,2,4,2,...,2,4,1.

$$\boxed{S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n}$$ (辛普森 $= \frac{1}{3}$梯形 $+ \frac{2}{3}$中點, 注意下標不同。)

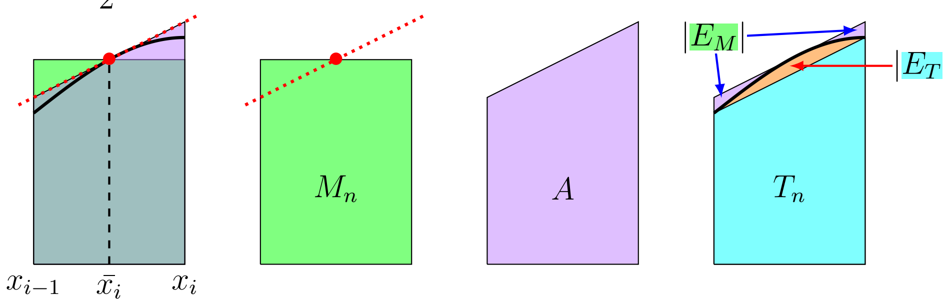
0.5 Error bounds

誤差 (error) 就是: 眞正的數值減去逼近的數值。
(>0 低估 (under-estimate), <0 高估 (over-estimate).)

$$E_T = \int_a^b f(x)\,dx - T_n,\quad E_M = \int_a^b f(x)\,dx - M_n,\quad E_S = \int_a^b f(x)\,dx - S_n.$$

Observation:

1. The larger n, the more accurate approximation. n 越大, 近似値越準。
2. 左右端點法的誤差 $\pm$ 相反 (R_n 多算 $\iff$ L_n 少算, 反之亦然);
當 n 加倍, 誤差剩 $\frac{1}{2}$。
3. T_n & M_n 比 R_n & L_n 精確。
4. T_n & M_n 的誤差 $\pm$ 相反 (T_n 多算 $\iff$ M_n 少算, 反之亦然);
當 n 加倍, 誤差剩 $\frac{1}{4}(=\frac{1}{2^2})$。
5. $|E_M| \approx \frac{1}{2}|E_T|$, 中點比梯形準 (誤差小) 一倍。



Note: $M_n = A$, E_T =最右圖中的橙色>0, $E_M = -$最右圖中的紫色<0; 所以差負號 ($\pm$ 相反), 而且紫色面積約橙色的一半 (數値一半)。

6. S_n 比 T_n & M_n 精確 ($\because S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$ and $E_M \approx -\frac{1}{2}E_T$);
當 n 加倍, 誤差剩 $\frac{1}{16}(=\frac{1}{2^4})$。

Additional: 估計法還有很多, 但是要在計算複雜度與精準度上做選擇。

估計法 approximation	R_n/L_n	T_n	M_n	S_n
複雜度 complexity	small	$<$	$<$	large
精準度 accuracy	rough	$>$	$>$	fine
誤差正比 error $\propto$	$1/n$	$1/n^2$	$1/n^2$	$1/n^4$