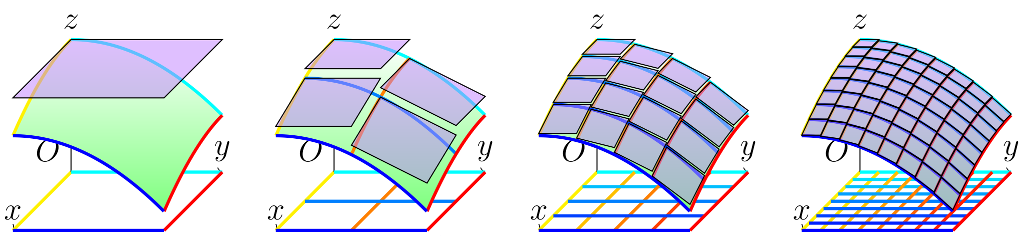
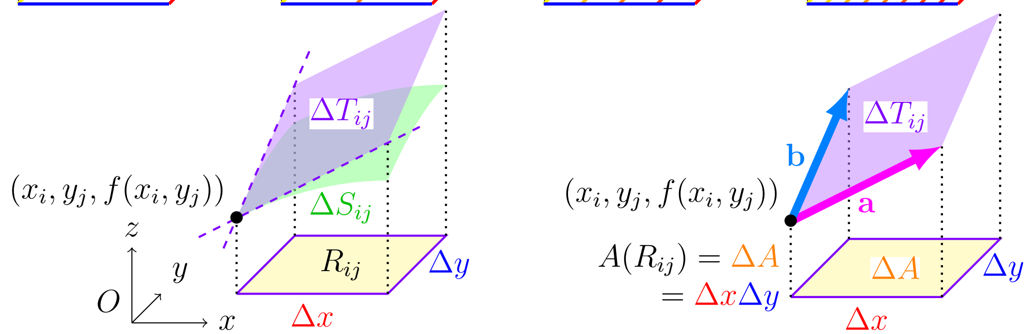
15.5 Surface area

1. $A(S)=\iint\limits_D \sqrt{(f_x)^2+(f_y)^2+1}\,dA$





Define: The ***surface area*** 表面積 of the surface S with equation $z = f(x, y)$ over D is

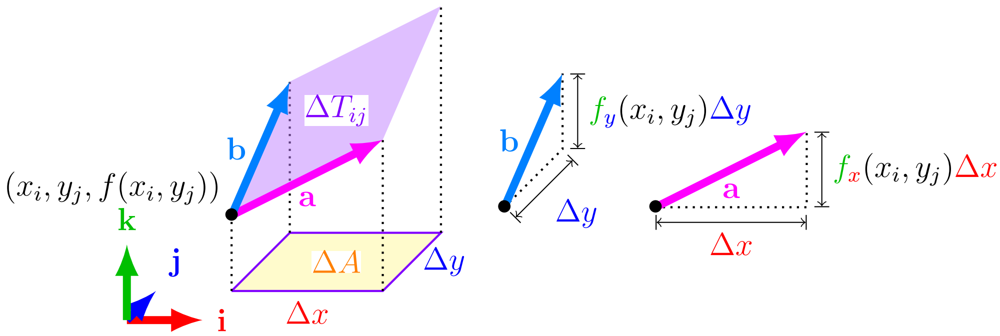
$$A(S)=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta T_{ij}$$

where the area ΔT_{ij} of the tangent plane at $(x_i, y_j, f(x_i, y_j))$ over R_{ij} is an approximation to the area ΔS_{ij} over R_{ij}. (用切平面的截面積近似表面積。)

Theorem 1 *The area of the surface S with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are* ***continuous*** *(連續的偏導數), is*

$$\boxed{\boldsymbol{A(S)=\iint\limits_D \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\;dA}}$$

$$\text{or } A(S)=\iint\limits_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dA$$



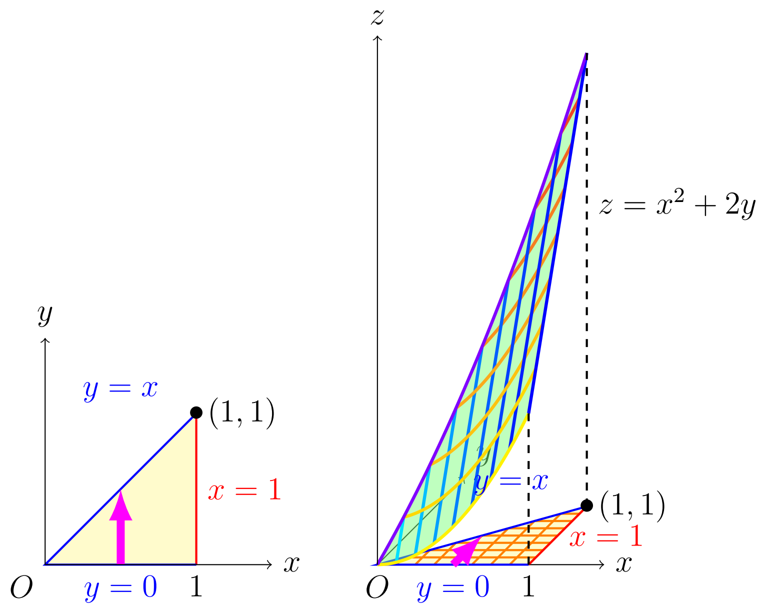
Proof. $\because$ f_x and f_y are continuous $\Longrightarrow$ 有切平面

$$
\begin{aligned}
\Delta z &= f_x(x_i, y_j)\Delta x + f_y(x_i, y_j)\Delta y \qquad \text{(tangent plane)} \\
\mathbf{a} &= \Delta x\ \mathbf{i} \qquad\qquad + f_x(x_i, y_j)\Delta x\ \mathbf{k} \qquad (\Delta y = 0) \\
\mathbf{b} &= \qquad\quad \Delta y\ \mathbf{j} + f_y(x_i, y_j)\Delta y\ \mathbf{k} \qquad (\Delta x = 0) \\
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\Delta y \end{vmatrix} \\
&= -f_x(x_i, y_j)\Delta A\ \mathbf{i} - f_y(x_i, y_j)\Delta A\ \mathbf{j} + \Delta A\ \mathbf{k} \\
\Delta T_{ij} &= |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\ \Delta A \\
A(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} \qquad \text{(definition)} \\
&= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\ \Delta A \\
&= \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\ dA
\end{aligned}
$$

(the limit of the double Riemann sum of $\sqrt{(f_x)^2 + (f_y)^2 + 1}$.)
(不選 (x_i, y_j) 改選樣本點 (x_{ij}^*, y_{ij}^*) 結果也一樣。) ■

(另一個觀點) Consider $F(x, y, z) = f(x, y) - z$, S is also the level surface $F(x, y, z) = 0$, then $\nabla F = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle$,
the tangent plane to S at $\mathbf{x_0} = (x_0, y_0, z_0 = f(x_0, y_0))$ is $\nabla F(\mathbf{x_0}) \bullet (\mathbf{x} - \mathbf{x_0}) = 0$,
and the surface area of S over D is $A(S) = \iint_D |\nabla F|\ dA$.

Example 0.1 *Find the surface area of the part of the surface $z = x^2 + 2y$ lying above the triangular region T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.*

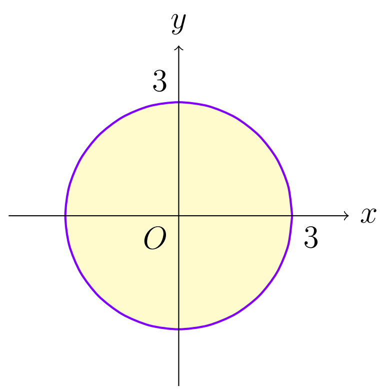


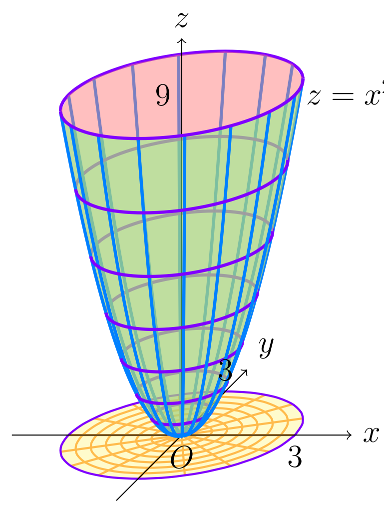
Let $f(x, y) = (z =)x^2 + 2y$, $T = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}$.

$$\begin{aligned}
A &= \iint\limits_T \sqrt{(f_x)^2 + (f_y)^2 + 1}\ dA \\
&= \int_0^1 \int_0^x \sqrt{(2x)^2 + (2)^2 + 1}\ dy\ dx \qquad \text{(type II 不好積)} \\
&= \int_0^1 \left[y\sqrt{4x^2+5} \right]_{y=0}^{y=x} dx \ = \ \int_0^1 x\sqrt{4x^2+5}\ dx \\
&\overset{\text{變換}}{=} \left[\frac{1}{8} \cdot \frac{2}{3} (4x^2+5)^{3/2} \right]_0^1 \ = \ \frac{27 - 5\sqrt{5}}{12}. \qquad \blacksquare
\end{aligned}$$

(變數變換*: Let* $u = 4x^2 + 5$, $du = 8x\ dx$,
$\int x\sqrt{4x^2+5}\ dx = \int \frac{1}{8}\sqrt{u}\ du = \frac{1}{8}\frac{2}{3}u^{3/2} + C = \frac{1}{12}(4x^2+5)^{3/2} + C$.)

Example 0.2 *Find the surface area of the part of the paraboloid $z = x^2 + y^2$ lying under the plane $z = 9$.*



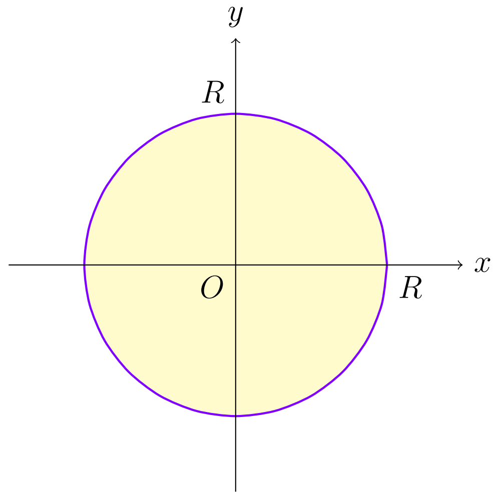


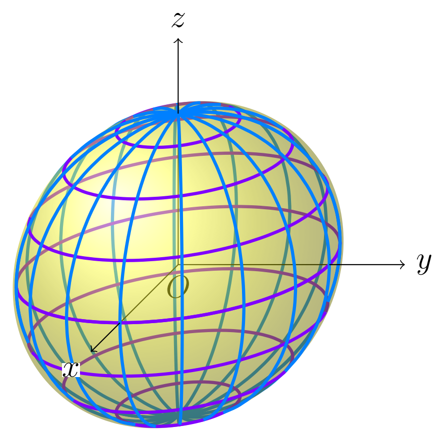
Let $f(x, y) = (z =)x^2 + y^2$. $z = 9$ 帶入 $z = x^2 + y^2 \implies x^2 + y^2 = 9$.
$D = \{(x, y) : x^2 + y^2 \le 3^2\} = \{(r, \theta) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$.

$$\begin{aligned}
A &= \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\ dA \\
&= \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\ dA \\
&= \iint_D \sqrt{4(x^2 + y^2) + 1}\ dA \\
&= \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} \cdot r\ dr\ d\theta \qquad \textit{(use polar coordinates)} \\
&= \int_0^{2\pi} d\theta \int_0^3 r\sqrt{4r^2 + 1}\ dr \qquad \textit{(polar rectangle, 可以分開)} \\
&= 2\pi \cdot \frac{1}{8} \cdot \frac{2}{3} \Big[(4r^2 + 1)^{3/2}\Big]_0^3 = \frac{\pi}{6}(37\sqrt{37} - 1).
\end{aligned}$$
■

Skill: 先把 $\sqrt{(f_x)^2 + (f_y)^2 + 1}$ 整理好再積分, 需要換座標再換。

Example 0.3 (Extra) *The surface area of a sphere of radius R is $4\pi R^2$.*



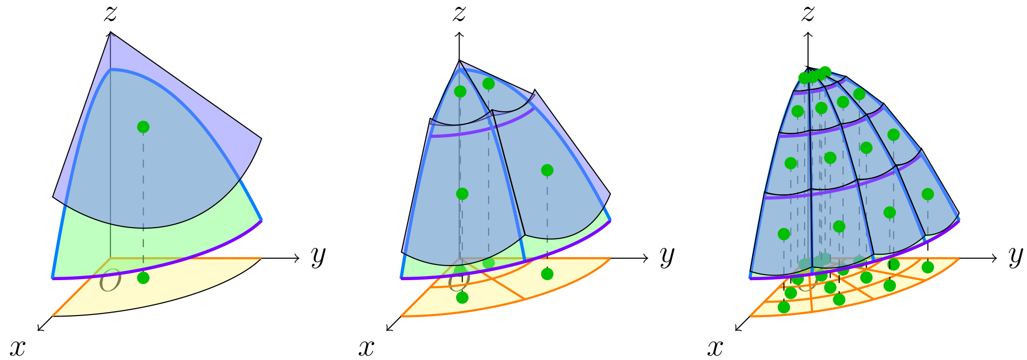


Sphere formula: $x^2+y^2+z^2=R^2$, *let* $f(x,y)=z=\sqrt{R^2-x^2-y^2}$,
$D=\{(x,y):x^2+y^2\le R^2\}=\{(r,\theta):0\le r\le R,\ 0\le\theta\le 2\pi\}$.
Then the surface area of the sphere $=2\times$ *the surface area of* f *over* D.

$$
\begin{aligned}
A &= 2\iint_D \sqrt{(f_x)^2+(f_y)^2+1}\,dA \\
&\quad \Big(\text{let } u=R^2-x^2-y^2,\ f_x=\frac{df}{du}\frac{\partial u}{\partial x}=\frac{d\sqrt{u}}{du}\frac{\partial}{\partial x}(R^2-x^2-y^2) \\
&\quad =\frac{1}{2\sqrt{u}}\cdot(-2x)=\frac{-x}{\sqrt{R^2-x^2-y^2}}.\Big) \\
&= 2\iint_D \sqrt{\Big(\frac{-x}{\sqrt{R^2-x^2-y^2}}\Big)^2+\Big(\frac{-y}{\sqrt{R^2-x^2-y^2}}\Big)^2+1}\,dA \\
&= 2\iint_D \sqrt{\frac{(-x)^2+(-y)^2+(R^2-x^2-y^2)}{R^2-x^2-y^2}}\,dA \\
&= 2R\iint_D \frac{dA}{\sqrt{R^2-x^2-y^2}} \;=\; 2R\int_0^{2\pi}\int_0^R \frac{1}{\sqrt{R^2-r^2}}\cdot r\,dr\,d\theta \\
&= 2R\int_0^{2\pi} d\theta\int_0^R \frac{r}{\sqrt{R^2-r^2}}\,dr \quad \Big(\int\frac{x}{\sqrt{a^2-x^2}}\,dx=-\sqrt{a^2-x^2}+C\Big) \\
&= 2R\cdot 2\pi\cdot\Big[-(R^2-r^2)^{1/2}\Big]_0^R \;=\; 4\pi R^2. \qquad \blacksquare
\end{aligned}
$$

⧫ Additional: Surface area formula in polar coordinates system 極座標下的表面積公式

用切平面逼近? No, 在極矩形上的截面不是平行四邊形, 很難算!



$$
\begin{aligned}
g(r,\theta) &= f(x,y) = f(r\cos\theta, r\sin\theta),\\
g_r &= f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x \cos\theta + f_y \sin\theta,\\
g_\theta &= f_x\frac{\partial x}{\partial \theta} + f_y\frac{\partial y}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta,\\
(g_r)^2 + \left(\frac{g_\theta}{r}\right)^2 &= (f_x)^2\cos^2\theta + \cancel{2f_xf_y\cos\theta\sin\theta} + (f_y)^2\sin^2\theta\\
&\quad + (f_x)^2\sin^2\theta - \cancel{2f_xf_y\sin\theta\cos\theta} + (f_y)^2\cos^2\theta\\
&= (f_x)^2 + (f_y)^2,\\
A(S) &= \iint\limits_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\; dA\\
&= \boldsymbol{\int\int \sqrt{(g_r)^2 + \left(\frac{g_\theta}{r}\right)^2 + 1}\cdot r\; dr\; d\theta}.
\end{aligned}
$$

Attention: 不是 $\iint\limits_D \sqrt{(g_r)^2 + \underline{(g_\theta)^2} + 1}\; dA.$