Problem 1

(a)

1. We first prove that $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n \subseteq \underbrace{\{x \mid x\in S_n, \text{for infinitely many } n\}}_{=A}$

Suppose there's $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n$, then $x\in \bigcup_{n=k}^{\infty} S_n$ for all k, which means $x \in A$

2. We prove that $A \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n$,

Suppose there's $y \in A$, then "y appears in S_k for uncountable ks"

which means for every M, we can always find $y \in \bigcup_{n=k}^{\infty} S_n$ where $k \geq M$

$\Rightarrow y \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n$

combine 1 and 2. $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n = \{x \mid x \in S_n, \text{for infinitely many } n\}$

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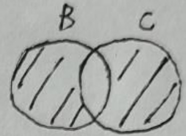
Problem 1

(b) As given a countably infinite Fibonacci sequence, we know there's infinitely many n that n is in the $\{F_k\}$, while there's also infinitely many n that n is not in the $\{F_k\}$, which means there are both infinitely many $(B-C)$ and $(C-B)$ in $\{A_n\}$

$$\Rightarrow \bigcap_{n=1}^{\infty} A_n = (B-C) \cap (C-B) = \emptyset \quad \#$$



$$\Rightarrow \bigcup_{n=1}^{\infty} A_n = (B-C) \cup (C-B) = (B \cup C) - (B \cap C) \quad \#$$

$\Rightarrow$ as there are both infinitely many $(B-C)$ and $(C-B)$, we cannot find any M such that A_k equals to only $(B-C)$ or $(C-B)$ for every k where $k \geq M$, therefore $\bigcap_{n=k}^{\infty} A_n = \emptyset$ and also $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \emptyset$ #

$\Rightarrow$ as there are both infinitely many $(B-C)$ and $(C-B)$, for every M we can discover that $\bigcup_{n=k}^{\infty} A_n = (B-C) \cup (C-B)$, where $k \geq M$, therefore, $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = (B-C) \cup (C-B) = (B \cup C) - (B \cap C)$ #

Problem 1

(c) We first assume that there are countably infinite real numbers in $(0,1)$, and we list them all:

$x_1 = 0.\cdots\cdots$

$x_2 = 0.\cdots\cdots$ and we take $r_i = x_i$'s i^{th} digit

$x_3 = 0.\cdots\cdots$

$\vdots$

and we now construct a number $y = 0.f_1f_2\cdots\cdots$, where

$f_i = \begin{cases} 1, & \text{when } r_i \neq 1 \\ 2, & \text{when } r_i = 1 \end{cases}$, so that $y \neq x_i \ \forall i$, which reaches the contradiction as we already list them all

$\Rightarrow$ there are uncountably infinite real numbers in $(0,1)$ #

Problem 2

(a) 1. Let $N=1$, $P(\bigcup_{n=1}^{1} A_n) = P(A_1) \leq \sum_{n=1}^{1} P(A_n) = P(A_1)$ is true

2. Suppose $N=k$ is true $\Rightarrow P(\bigcup_{n=1}^{k} A_n) \leq \sum_{n=1}^{k} P(A_n)$, then,

when $N=k+1$, $P(\bigcup_{n=1}^{k+1} A_n) = P((\bigcup_{n=1}^{k} A_n) \cup A_{k+1})$, we assume A_{k+1} and $\bigcup_{n=1}^{k} A_n$ are mutually exclusive (as it maximize the value of $P(\bigcup_{n=1}^{k+1} A_n))$ $\Rightarrow P((\bigcup_{n=1}^{k} A_n) \cup A_{k+1}) = P(\bigcup_{n=1}^{k} A_n) + P(A_{k+1})$ (By Axiom 3)

$$\leq \sum_{n=1}^{k} P(A_n) + P(A_{k+1})$$

$$= \sum_{n=1}^{k+1} P(A_n) \text{ is true}$$

proved by mathematical induction #

Problem 2

(b) $P(\{1,2,3,4\}) = P(\{1,2,4\}) + P(\{3\}) = 0.4 + 0.3 = 0.7$

$P(\{5\}) = \Omega - P(\{1,2,3,4\}) = 1 - 0.7 = 0.3$ (By Axiom 2)

$P(\{1\}) = P(\{1,5\}) - P(\{5\}) = 0.5 - 0.3 = 0.2$

$P(\{2,4\}) = P(\{1,2,4\}) - P(\{1\}) = 0.4 - 0.2 = 0.2$

$\Rightarrow P(\{2\}) + P(\{4\}) = 0.2$ and $P(\{2\}) \geq 0$, $P(\{4\}) \geq 0$ (By Axiom 1)

possible probability assignments :

$$\begin{cases} P(\{1\}) = 0.2 ,\ P(\{2\}) = 0 ,\ P(\{3\}) = 0.3 ,\ P(\{4\}) = 0.2 ,\ P(\{5\}) = 0.3 \\ P(\{1\}) = 0.2 ,\ P(\{2\}) = 0.1 ,\ P(\{3\}) = 0.3 ,\ P(\{4\}) = 0.1 ,\ P(\{5\}) = 0.3 \\ P(\{1\}) = 0.2 ,\ P(\{2\}) = 0.2 ,\ P(\{3\}) = 0.3 ,\ P(\{4\}) = 0 ,\ P(\{5\}) = 0.3 \end{cases}$$

the minimum possible value of $P(\{2,3,5\}) = 0 + 0.3 + 0.3 = 0.6$ #

Problem 3

(a) Let $B_k = \bigcup_{n=k}^{\infty} A_n$, So $\{B_k\}_1^{\infty}$ is a decreasing sequence

$$\Rightarrow P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) = P\left(\bigcap_{k=1}^{\infty} B_k\right) = P\left(\lim_{k\to\infty} B_k\right) \text{ (as } B_k \text{ is a decreasing sequence)}$$

$$= \lim_{k\to\infty} P(B_k) \text{ (By continuity of probability function)}$$

$$= \lim_{k\to\infty} P\left(\bigcup_{n=k}^{\infty} A_n\right)$$

$$\leq \lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n)$$

as $\sum_{n=1}^{\infty} P(A_n) < \infty$, which means $\{P(A_n)\}$ converges when $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} P(A_n) = 0$$

$$\Rightarrow P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) \leq \lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n) = \lim_{n\to\infty} P(A_n) = 0$$

$$\Rightarrow P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) \leq 0,$$

combined with Axiom 1, $P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) \geq 0$, therefore, $P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) = 0$ #

Problem 3

(b) $\sum_{k=1}^{\infty} P_k = 100 \cdot \sum_{k=1}^{\infty} k^{-N}$ and $N > 1$

we know that $\sum_{k=1}^{\infty} k^{-N}$ converges when $N > 1$

$\Rightarrow \sum_{k=1}^{\infty} P_k < \infty$

then by the Borel-Cantelli Lemma, $P\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right) = 0$. which means the probability of observing infinitely many numbers of head is 0, i.e. $P(I) = 0$ #

Problem 4

(a)
$$P(A_1|B) = \frac{P(A_1)\cdot P(B|A_1)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + P(A_3)\cdot P(B|A_3)}$$

$$= \frac{\frac{1}{3}\cdot 0.1}{\frac{1}{3}\cdot 0.1 + \frac{1}{3}\cdot 0.3 + \frac{1}{3}\cdot 0.6} = \frac{1}{10}$$

$$P(A_2|B) = \frac{P(A_2)\cdot P(B|A_2)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + P(A_3)\cdot P(B|A_3)}$$

$$= \frac{\frac{1}{3}\cdot 0.3}{\frac{1}{3}\cdot 0.1 + \frac{1}{3}\cdot 0.3 + \frac{1}{3}\cdot 0.6} = \frac{3}{10}$$

$$P(A_3|B) = \frac{P(A_3)\cdot P(B|A_3)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + P(A_3)\cdot P(B|A_3)}$$

$$= \frac{\frac{1}{3}\cdot 0.6}{\frac{1}{3}\cdot 0.1 + \frac{1}{3}\cdot 0.3 + \frac{1}{3}\cdot 0.6} = \frac{6}{10}$$

#

Problem 4

$P(C|A_1) \doteq 1.31\times10^{-8}$

$P(C|A_2) \doteq 2.27\times10^{-5}$

$P(C|A_3) \doteq 2.83\times10^{-6}$

(b) $$P(A_1|C) = \frac{P(A_1)\cdot P(C|A_1)}{P(A_1)\cdot P(C|A_1)+P(A_2)\cdot P(C|A_2)+P(A_3)\cdot P(C|A_3)}$$

$$\doteq \frac{\frac{1}{3}\cdot 1.31\times10^{-8}}{\frac{1}{3}\cdot 1.31\times10^{-8}+\frac{1}{3}\cdot 2.27\times10^{-5}+\frac{1}{3}\cdot 2.83\times10^{-6}}$$

$$\doteq 5.13\times10^{-4}$$

$$P(A_2|C) = \frac{P(A_2)\cdot P(C|A_2)}{P(A_1)\cdot P(C|A_1)+P(A_2)\cdot P(C|A_2)+P(A_3)\cdot P(C|A_3)}$$

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$$\doteq 0.89$$

$$P(A_3|C) = \frac{P(A_3)\cdot P(C|A_3)}{P(A_1)\cdot P(C|A_1)+P(A_2)\cdot P(C|A_2)+P(A_3)\cdot P(C|A_3)}$$

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$$\doteq 0.11$$

Given the experimental results, the most probable value for θ is $\{\theta_Y = 0.3,\ \theta_L = 0.6,\ \theta_N = 0.1\}$ #

Problem 4

(c) $$P(A_1|C) = \frac{P(A_1)\cdot P(C|A_1)}{P(A_1)\cdot P(C|A_1)+P(A_2)\cdot P(C|A_2)+P(A_3)\cdot P(C|A_3)}$$

$$\doteq \frac{\frac{3}{5}\cdot 1.31\times10^{-8}}{\frac{3}{5}\cdot 1.31\times10^{-8}+\frac{1}{5}\cdot 2.27\times10^{-5}+\frac{1}{5}\cdot 2.83\times10^{-6}}$$

$$\doteq 1.54\times10^{-3}$$

$$P(A_2|C) = \frac{P(A_2)\cdot P(C|A_2)}{P(A_1)\cdot P(C|A_1)+P(A_2)\cdot P(C|A_2)+P(A_3)\cdot P(C|A_3)}$$

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$$\doteq 0.89$$

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$$\doteq 0.11$$

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