A Bound for the 4-th moment of S_n :

- Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with mean μ and $E[X_1^4] < \infty$.
- Define $S_n = X_1 + X_2 + \cdots + X_n$

Then, there exists a constant $K < \infty$ such that

$$E\left[(S_n - n\mu)^4\right] \leq Kn^2$$

Pf : Define $Y_i = X_i - \mu$, for all i

Then, $E\left[(S_n - n\mu)^4\right]$

$$= E\left[\left(Y_1 + Y_2 + \cdots + Y_n\right)^4\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} E\left[\underbrace{X_i X_j X_k X_l}_{}\right]$$

Consider all possible combinations (in the next page)

Consider all possible combinations:

① all i, j, k, l are equal $\Rightarrow E[X_1^4]$

② 3 out of i, j, k, l are equal $\Rightarrow E[X_1 \cdot X_2^3] = E[X_1] \cdot E[X_2^3]$
(e.g. $j = k = l \neq i$) $= 0$

③ i, j, k, l can be divided into 2 groups $\Rightarrow E[X_1^2 \cdot X_2^2]$
(e.g. $i = j$ and $k = l$, but $i \neq k$)

④ 2 out of i, j, k, l are equal, and $\Rightarrow E[X_1^2 X_2 X_3]$
the others are all different $= 0$
(e.g. $i = j$, and $i \neq k$, $i \neq l$)

⑤ All i, j, k, l are different $\Rightarrow E[X_1 X_2 X_3 X_4] = 0$

So, only ① and ③ lead to non-zero values.

Therefore,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} E[X_i X_j X_k X_l]$$

$$= n\cdot E[X_1^4] + 3n\cdot(n-1)\cdot E[X_1^2 X_2^2]$$

X_1, X_2 are independent

$$= n\cdot E[X_1^4] + 3n(n-1)\cdot E[X_1^2]\cdot E[X_2^2]$$

Since $E[X_1^4]<\infty$, we also know $E[X_1^2]<\infty$

Then, there must exist some constant $C<\infty$

such that $E[X_1^4]\le C$ and $E[X_1^2]\le C$.

Hence,

$$E[(S_n-n\mu)^4] = n\cdot E[X_1^4] + 3n(n-1)E[X_1^2]E[X_2^2]$$

$$\le n\cdot C + 3n\cdot(n-1)\cdot C^2$$

$$\le (3C^2+C)\cdot n^2$$

$\square$