7.8 Improper integrals

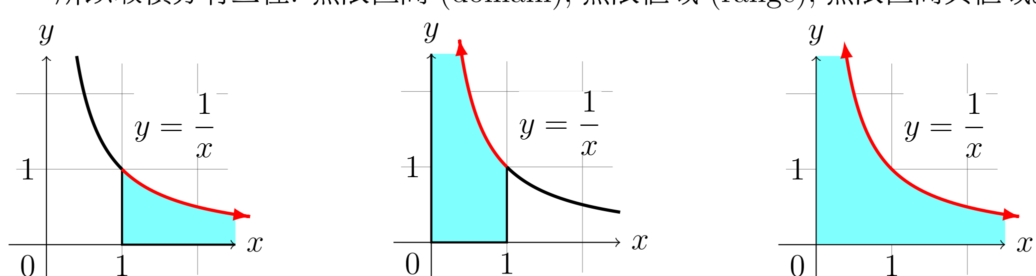
1. infinite interval 無限區間 $[a, \infty), (-\infty, a], (-\infty, \infty)$
2. (infinite) discontinuous integrand (無限) 不連續積分域 (f)
3. Comparison Theorem 比較定理 — 大收就小收, 小發就大發。

Recall: $\displaystyle\int_a^b f(x)\ dx$: definite integral 定積分, 是極限, 是淨面積, 是數字。
$\displaystyle\int f(x)\ dx$: indefinite integral 不定積分, 是 (最一般) 反導數 $(+C)$, 是函數。
TFTC: $\displaystyle\int_a^b f(x)\ dx = \left[\int f(x)\ dx\right]_a^b$: 定積分等於不定積分代上界減代下界。

Observation: 目前看到的定積分都有兩個性質:
1. integration on finite domain 在有限區間 $([a, b])$ 上積分。
2. integrand of finite range 有限值域的積分域 (f)。
這種的叫做 ***proper integral*** 正常積分, 眞積分; 不是的, 叫做不正常積分, 或是:

Define: Definite integral $\displaystyle\int_a^b f(x)\ dx$ is an ***improper integral*** 瑕積分 if
(i) the interval $[a, b]$ is infinite ($(-\infty, b]$ or $[a, \infty)$ or $(-\infty, \infty)$), or
(ii) f has an infinite discontinuity in $[a, b]$ ($\lim_{x\to c^\pm} f(x) = \infty$ or $-\infty$).

所以瑕積分有三種: 無限區間 (domain), 無限值域 (range), 無限區間與值域。



$\displaystyle\int_1^\infty \frac{1}{x}\ dx$ $\qquad$ $\displaystyle\int_0^1 \frac{1}{x}\ dx$ $\qquad$ $\displaystyle\int_0^\infty \frac{1}{x}\ dx$

(無限邊界區域的面積怎麼算? 用極限。怎麼把無限切成 n 等分? 不能切!)

Key Idea: 有限靠近無限, 瑕積分就是眞積分的極限: $\boxed{\text{瑕積分} = \lim \text{眞積分}}$
極限存在叫收斂 (convergent a. 康福聚的; converge v. 康福聚),
不存在叫發散 (divergent a. 歹福聚的; diverge v. 低/歹福聚)。

0.1 Infinite interval

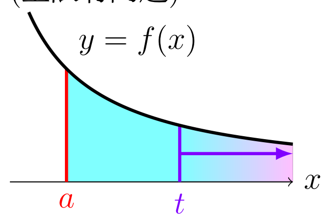
Definition: 有三種

(a) If $\int_a^t f(x)\,dx$ exists for $t \ge a$, then

(上限有問題)

$$\boxed{\int_{a}^{\infty} \boldsymbol{f(x)\,dx} = \lim_{t\to\infty} \int_{a}^{t} \boldsymbol{f(x)\,dx}}$$

provided this limit exists (as a finite number).

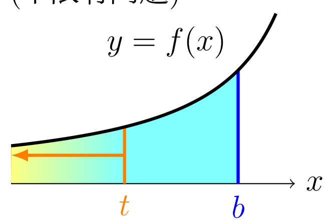


(b) If $\int_t^b f(x)\,dx$ exists for $t \le b$, then

(下限有問題)

$$\boxed{\int_{-\infty}^{b} \boldsymbol{f(x)\,dx} = \lim_{t\to-\infty} \int_{t}^{b} \boldsymbol{f(x)\,dx}}$$

provided this limit exists (as a finite number).



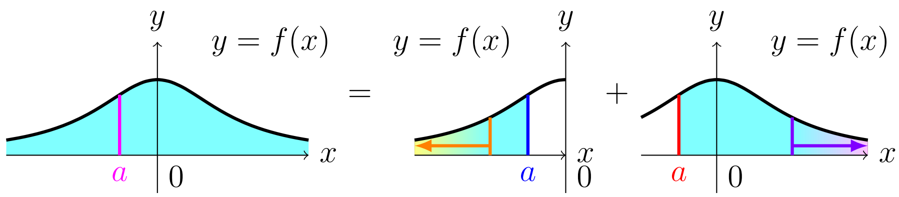
The improper integrals 瑕積分 $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called ***convergent*** 收斂 if the corresponding limit exists 極限存在, and ***divergent*** 發散 if the limit does not exist 極限不存在。

(c) If ***both*** $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are ***convergent***, then

$$\boxed{\int_{-\infty}^{\infty} \boldsymbol{f(x)\,dx} = \int_{-\infty}^{a} \boldsymbol{f(x)\,dx} + \int_{a}^{\infty} \boldsymbol{f(x)\,dx}}$$

meanwhile, any real number a can be used. ($\because \int_a^b f(x)\,dx$ exists)

(上下限都有問題)



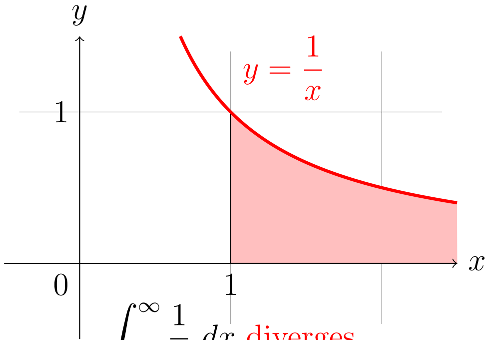
Attention: $(-\infty, \infty)$ 要切! 不管切哪, 會收斂 (極限都存在), 切哪都收斂。

Example 0.1 $\displaystyle\int_1^\infty \frac{1}{x}\,dx$? *Divergent.*

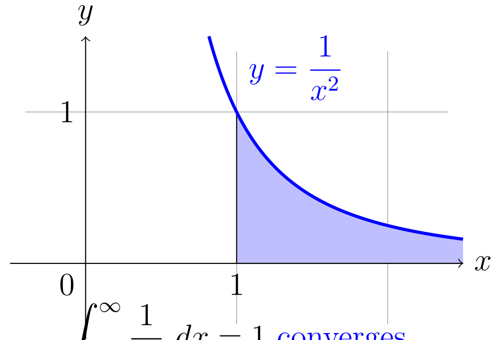
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\Big[\ln|x|\Big]_1^t = \lim_{t\to\infty}\ln|t| = \infty. \qquad \blacksquare$$

Example 0.2 $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$? *Convergent* $(=1)$.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1. \qquad \blacksquare$$



$\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges



$\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = 1$ converges

([右上]可以說無界限區域的面積等於 1, 或是畫得越遠面積越靠近 1。)

Example 0.3 *For what p is* $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ *convergent?*

When $p=1$, $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *diverges.*

When $p\neq 1$, $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t$

$$= \lim_{t\to\infty}\frac{1}{p-1}\left(1-\frac{1}{t^{p-1}}\right) = \begin{cases} \dfrac{1}{p-1} & \text{if } p>1; \\ \infty & \text{if } p<1. \end{cases}$$

$$\therefore \int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \text{ is convergent for } p>1. \qquad \blacksquare$$

Skill: 好用的瑕積分: (下限是任何正數都適用, 只是收斂時值不同。)

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{convergent } \left(=\dfrac{1}{p-1}\right) \text{ for } p>1, \\ \text{divergent for } p\le 1. \end{cases}}$$

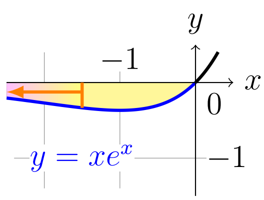
Example 0.4 $\displaystyle\int_{-\infty}^{0} xe^x\,dx.$

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx, \quad\ldots\ldots\ldots\ldots\ldots \text{(哪邊有問題, 哪邊取極限。)}$$

$$\int_t^0 x\,e^x\,dx = x\,e^x\Big|_t^0 - \int_t^0 e^x\,dx = -te^t - 1 + e^t, \quad\ldots\ldots\ldots\ldots \text{(分部積分。)}$$

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} \overset{l'H}{=} \lim_{t\to-\infty}\frac{1}{-e^{-t}} = \lim_{t\to-\infty}(-e^t) = 0, \ \left(\mathbf{0\cdot\infty \to \frac{\infty}{\infty}}\right)$$

$$\therefore \int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\left(-te^t - 1 + e^t\right) = 0 - 1 + 0 = -1. \qquad \blacksquare$$



Example 0.5 $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_0^{\infty}\frac{1}{1+x^2}\,dx, \quad\ldots\ldots\ldots\ldots \text{(切在 0。)}$$

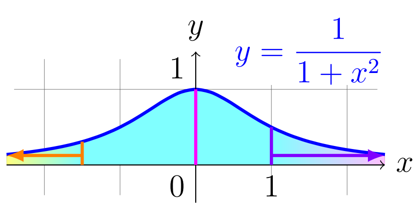
$$\text{(左)} \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0\frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\Big[\tan^{-1}x\Big]_t^0$$

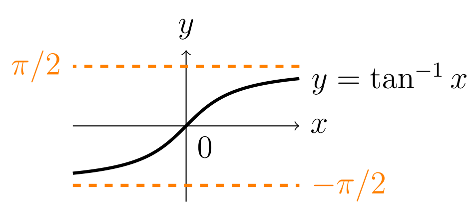
$$= \lim_{t\to-\infty}(\tan^{-1}0 - \tan^{-1}t) = 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}.$$

$$\text{(右)} \int_0^{\infty}\frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t\frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\Big[\tan^{-1}x\Big]_0^t$$

$$= \lim_{t\to\infty}(\tan^{-1}t - \tan^{-1}0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

$$\therefore \int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \ \text{(要兩個都收斂合起來才會收斂。)} \qquad \blacksquare$$





Skill: $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ 切哪都一樣, 那就切在 0。

WARNING: 常見錯誤 Part I

1. 自創寫法, 把符號 ∞ 當成數字:

$$\int_0^\infty \frac{1}{1+x^2}\,dx \overset{!}{=} \Big[\tan^{-1}x\Big]_0^{\boxed{\infty}} \overset{!!}{=} \tan^{-1}\boxed{\infty} - \tan^{-1}0 \overset{!!!}{=} \boxed{\frac{\pi}{2}}.$$

!: ∞ 是符號, 不能用 TFTC;
!!: $\tan^{-1}\infty$ 沒定義。
!!!: $\tan^{-1}\infty$ 更不會是 $\frac{\pi}{2}$, 只有 $\lim\limits_{t\to\infty}\tan^{-1}t = \frac{\pi}{2}$。

2. 自創定義的幻覺:

$$\int_{-\infty}^{\infty} x\,dx \overset{!}{=} \lim_{t\to\infty}\int_{\boxed{-t}}^{\boxed{t}} x\,dx = \lim_{t\to\infty}\left[\frac{x^2}{2}\right]_{-t}^{t} = \lim_{t\to\infty}\left[\frac{t^2}{2} - \frac{(-t)^2}{2}\right] = \lim_{t\to\infty} 0 = 0.$$

!: 沒有這樣定義; 否則會變成:

$$0 = \int_{-\infty}^{\infty} x\,dx = \lim_{t\to\infty}\int_{1-t}^{1+t} x\,dx = \lim_{t\to\infty}\left[\frac{(1+t)^2}{2} - \frac{(1-t)^2}{2}\right] = \lim_{t\to\infty} 2t = \infty.$$

3. 不照定義靠直覺, 其實是錯覺:

$\because f(x) = x$ is odd, by symmetry, $\therefore \int_{-\infty}^{\infty} x\,dx \overset{!}{=} 0$. (其實是發散。)

!: 對稱性只對眞積分保證成立, 對瑕積分不一定。 無駄無駄無駄!

. . . .

WARNING: 常見錯誤 Part II

4. 看到積分就算, 沒注意到是眞積分還是瑕積分 (Trap!):

$$\int_0^3 \frac{1}{x-1}\,dx \overset{!}{=} \Big[\ln|x-1|\Big]_0^3 = (\ln 2 - \ln 1) = \ln 2.$$

!: (熊出) 沒注意到 1 有問題 (其實是發散)。

5. 偷渡不連續:

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx \overset{!}{=} \Big[2\sqrt{x-2}\Big]_{\boxed{2}}^{5} = 2(\sqrt{3}-\sqrt{0}) = 2\sqrt{3}.$$

!: 在 2 不連續, 不能用 TFTC (閉區間連續函數)。

0.2 discontinuous integrand

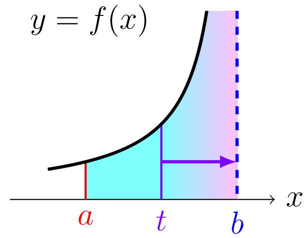
Definition:

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\boxed{\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx}$$

if this limit exists (as a finite number).

(上限有問題)

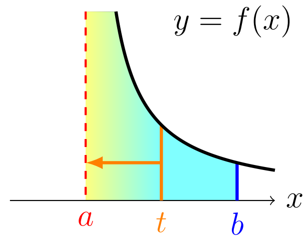


(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\boxed{\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx}$$

if this limit exists (as a finite number).

(下限有問題)

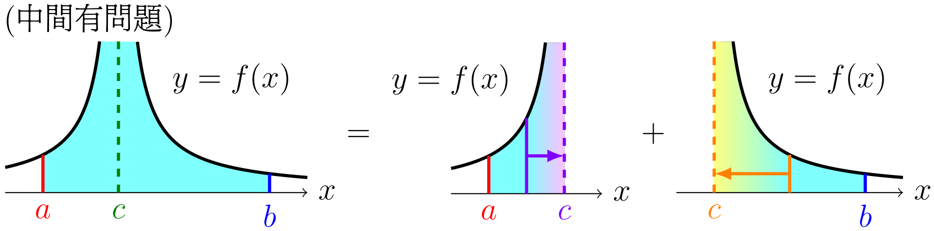


The improper integral 瑕積分 $\int_a^b f(x)\, dx$ is called ***convergent*** 收斂 if the corresponding limit exists 極限存在, and ***divergent*** 發散 if the limit does not exist 極限不存在。

(c) If f has a discontinuity at c, where $a < c < b$, and ***both*** $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ are ***convergent***, then

$$\boxed{\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx}$$

(中間有問題)



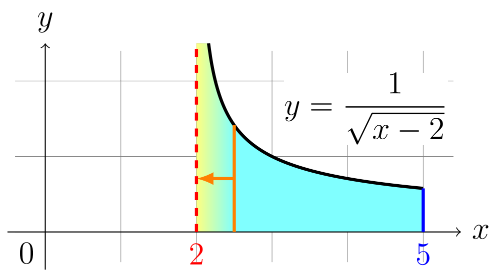
Attention: 要檢查是不是瑕積分, 要切在 (有問題)不連續點。

Example 0.6 $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx.$

$\because \displaystyle\lim_{x\to 2^+} \frac{1}{\sqrt{x-2}} = \infty,$ *improper.*

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+}\left[2\sqrt{x-2}\right]_t^5$$

$$= \lim_{t\to 2^+}(2\sqrt{3}-2\sqrt{t-2}) = 2\sqrt{3}. \quad \ldots\ldots\ldots\ldots (\because \lim_{x\to 0^+}\sqrt{x}=0.) \qquad \blacksquare$$

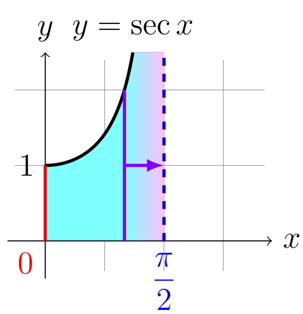


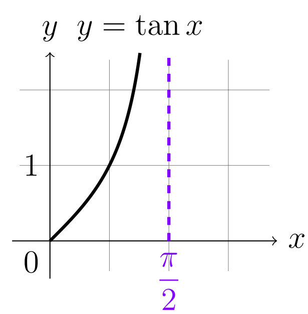
Example 0.7 $\displaystyle\int_0^{\pi/2} \sec x\ dx?$

$\because \displaystyle\lim_{x\to \frac{\pi}{2}^-} \sec x = \infty,$ *improper.*

$$\int_0^{\pi/2} \sec x\ dx = \lim_{t\to \frac{\pi}{2}^-}\int_0^t \sec x\ dx = \lim_{t\to \frac{\pi}{2}^-}\Big[\ln|\sec x+\tan x|\Big]_0^t$$

$$= \lim_{t\to \frac{\pi}{2}^-}[\ln(\sec t+\tan t)-\ln 1] = \infty. \quad \ldots\ldots\ldots (\lim_{t\to \frac{\pi}{2}^-}\tan t = \infty.) \qquad \blacksquare$$





Note: 都是取單邊極限 $\displaystyle\lim_{t\to c^\pm}$, 無限處極限 $\displaystyle\lim_{t\to\pm\infty}$ 也可以看成是單邊極限。

Example 0.8 *Evaluate* $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$.

$\because \displaystyle\lim_{x\to1^-}\frac{1}{x-1}=-\infty$ *and* $\displaystyle\lim_{x\to1^+}\frac{1}{x-1}=\infty$, *improper.*

$\displaystyle\int_{0}^{3}\frac{1}{x-1}\,dx=\int_{0}^{1}\frac{1}{x-1}\,dx+\int_{1}^{3}\frac{1}{x-1}\,dx,\ \ldots\ldots$ *(1 有問題, 從 1 切開。)*

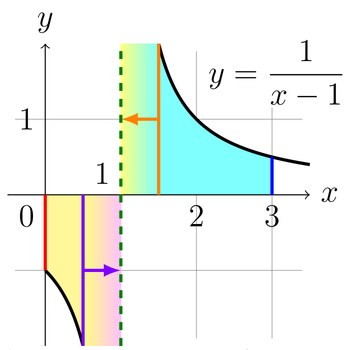
(左) $\displaystyle\int_0^1\frac{1}{x-1}\,dx=\lim_{t\to1^-}\int_0^t\frac{1}{x-1}\,dx=\lim_{t\to1^-}\Big[\ln|x-1|\Big]_0^t$

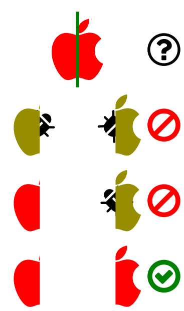
$\displaystyle=\lim_{t\to1^-}(\ln|t-1|-0)=-\infty$, *diverges;*

[Or] (右) $\displaystyle\int_1^3\frac{1}{x-1}\,dx=\lim_{t\to1^+}\int_t^3\frac{1}{x-1}\,dx=\lim_{t\to1^+}\Big[\ln|x-1|\Big]_t^3$

$\displaystyle=\lim_{t\to1^+}(\ln 2-\ln|t-1|)=\infty$, *diverges; (只要其中一塊發散就發散。)*

$\therefore \displaystyle\int_0^3\frac{1}{x-1}\,dx$ *diverges. (Need not to evaluate 說明發散就不用算。)* ■





有問題就要切開。

一邊發散就發散。

這邊收斂看那邊。

要都收斂才收斂。

$\left(\displaystyle\int_0^1\frac{dx}{x-1}\right.$ 區域跟 $\displaystyle\int_1^2\frac{dx}{x-1}$ 相似, 但是不可以相消, $\because \infty-\infty\neq0$, 是未定型.

所以不可以變成 $\displaystyle\int_0^3\frac{dx}{x-1} \cancel{=} \int_2^3\frac{dx}{x-1}=\ln 2.)$

Note: 有問題點切開後, 哪邊有問題, 哪邊取極限。

Additional: 想想看, 如果很多點有問題怎麼辦? 要怎麼切? 怎麼取極限?

Attention: 切開後, 如果一邊發散就發散; 如果一邊收斂, 還要檢查另一邊。

$\boxed{\text{go } \textbf{WARNING} \text{ PART II.}}$

Example 0.9 *Evaluate* $\int_0^1 \ln x \; dx$.

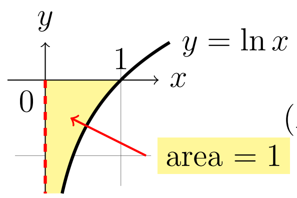
$\because \lim_{x\to 0^+} \ln x = -\infty$, *improper.*

$\int_0^1 \ln x \; dx = \lim_{t\to 0^+} \int_t^1 \ln x \; dx = \lim_{t\to 0^+} \Big[x\ln x - x \Big]_t^1$ (分部積分。)

$= \lim_{t\to 0^+} (1\ln 1 - 1 - t\ln t + t) = \lim_{t\to 0^+} (-1 - t\ln t + t)$,

$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \overset{l'H}{=} \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} (-t) = 0$, $\left(\mathbf{0\cdot\infty \to \frac{\infty}{\infty}}\right)$

$\therefore \int_0^1 \ln x \; dx = \lim_{t\to 0^+} (-1 - t\ln t + t) = -1 - 0 + 0 = -1.$ ∎



(Area = 1, why $\int_0^1 \ln x \; dx = -1$? $\because$ Net area.)

如果一邊是無限區間一邊是無限值域呢? 一樣切開分兩塊。

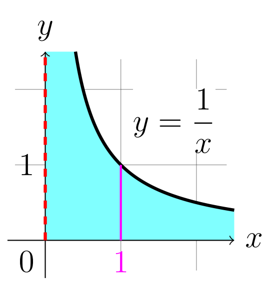
Example 0.10 $\int_0^\infty \frac{1}{x} \; dx$.

$\frac{1}{x}$ *is continuous on* $(0,\infty)$ *and* $\lim_{x\to 0^+} \frac{1}{x} = \infty$, *improper* 中的 *improper.*

$\int_0^\infty \frac{1}{x} \; dx = \int_0^1 \frac{1}{x} \; dx + \int_1^\infty \frac{1}{x} \; dx$, (從 1 切開。)

$\int_0^1 \frac{1}{x} \; dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x} \; dx = \infty$, *or* $\int_1^\infty \frac{1}{x} \; dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} \; dx = \infty$,

$\therefore \int_0^\infty \frac{1}{x} \; dx$ *diverges.* ∎



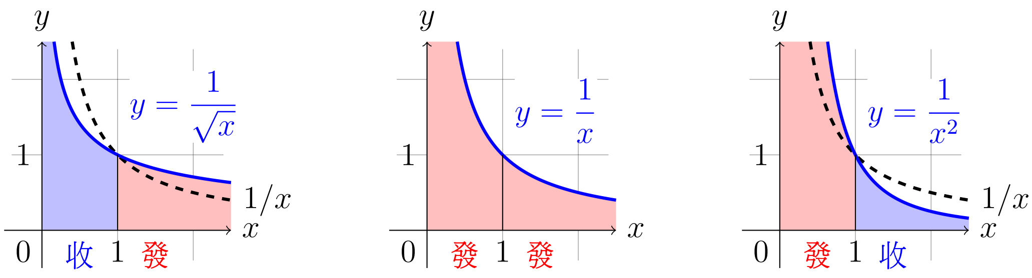
哪邊有問題, 哪邊取極限;
兩邊有問題, 就要切中間;
如果會收斂, 切哪都收斂;
一邊若發散, 整個都發散。

Skill: 好用的瑕積分: (上限是任何正數都適用, 只是收斂時值不同。)

$$\boxed{\int_{0}^{1} \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{convergent } \left(= \dfrac{1}{1-p}\right) \text{ for } p < 1, \\ \text{divergent for } p \geq 1. \end{cases}}$$

(注意與 $\displaystyle\int_{1}^{\infty} \frac{1}{x^p}\,dx$ 積分範圍($0 \to a$, $a \to \infty$) 與收發範圍($p \gtreqless 1$) 的差異。)

Fact: (Example 4 & Exercise 7.8.57) $\displaystyle\int_{0}^{\infty} \frac{1}{x^p}\,dx$ diverges for all p.

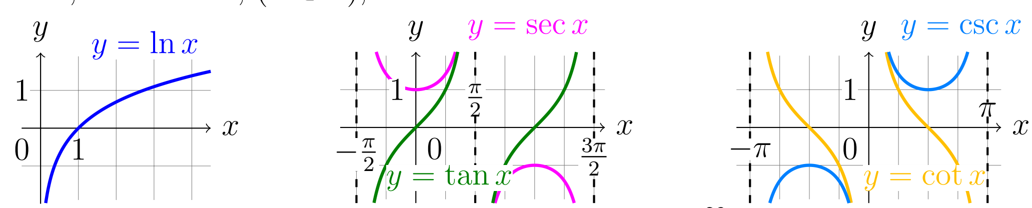


$p < 1$ $\quad$ $p = 1$ $\quad$ $p > 1$

Skill: 記憶法, 以 $p = 1$ 爲界, 比 $\dfrac{1}{x}$ 大的就發散, 比 $\dfrac{1}{x}$ 小的就收斂。

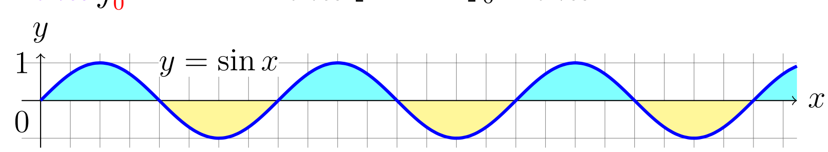
Question: 什麼函數會有無限值域?

Answer: 分母爲 0; $\ln x$, $\log_a x$ at 0; $\tan x$, $\sec x$ at $(n + \frac{1}{2})\pi$, $(n \in \mathbb{Z})$; $\cot x$, $\csc x$ at $n\pi$, $(n \in \mathbb{Z})$; ... etc.



Note: 不是只有無限面積的時候才會發散, Ex: $\displaystyle\int_{0}^{\infty} \sin x\,dx$ (diverges)

$$= \lim_{t\to\infty} \int_{0}^{t} \sin x\,dx = \lim_{t\to\infty} \Big[-\cos x\Big]_{0}^{t} = \lim_{t\to\infty} (1 - \cos t), \text{ does not exist.}$$



0.3 Comparison test for improper integral

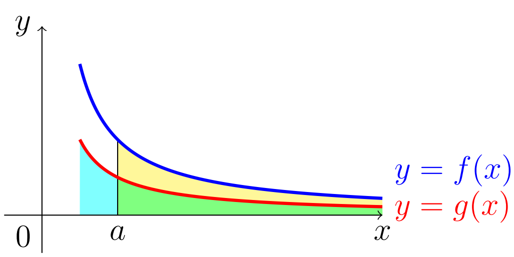
瑕積分常用在計算無界限區域的面積。有些瑕積分很難積, 但是可以用比較來知道發散或是收斂。爲什麼要知道是收斂還是發散? 收斂, 用其他方法積分或是計算近似值; 發散, 就不用算了。

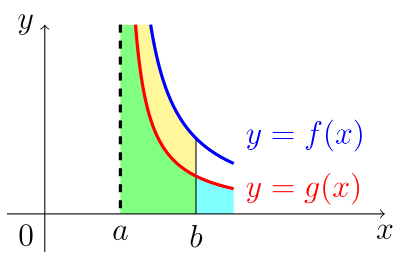
Theorem 1 (Comparison Theorem) 比較定理
Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

> (a) *If* $\int_a^\infty f(x)\ dx$ *is convergent, then* $\int_a^\infty g(x)\ dx$ *is convergent.*
>
> (b) *If* $\int_a^\infty g(x)\ dx$ *is divergent, then* $\int_a^\infty f(x)\ dx$ *is divergent.*

(面積)大的收斂 $\Longrightarrow$ 小的收斂; 小的發散 $\Longrightarrow$ 大的發散。

其他型的也一樣: $\int_{-\infty}^b f(x)\ dx$, $\int_{-\infty}^\infty f(x)\ dx$, $\int_a^b f(x)\ dx$ (不連續積分域).





(因爲畫越遠面積越大, 極限只有存在(L)或是發散至無限(∞)。)

Attention: Converse is <u>not necessarily true</u>(反過來<u>不保證對</u>).

(a) $\int_a^\infty g(x)\ dx$ (小) 收斂 $\overset{\text{不保證}}{\not\Longrightarrow}$ $\int_a^\infty f(x)\ dx$ (大) 收斂或發散;

(b) $\int_a^\infty f(x)\ dx$ (大) 發散 $\overset{\text{不保證}}{\not\Longrightarrow}$ $\int_a^\infty g(x)\ dx$ (小) 收斂或發散。

Timing: 問收斂發散, 積不出來。

Skill: 找誰比? 找 $\frac{1}{x^p}$, e^{-x}, ... 來比。

Example 0.11 *Show that* $\displaystyle\int_0^\infty e^{-x^2}\,dx$ *is convergent.*

$\displaystyle\int e^{-x^2}\,dx$ 不會算, 用比的; 跟誰比? e^{-x}; 能比嗎? *No. How?*

$\displaystyle\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$ (切在 1), *and* $\displaystyle\int_0^1 e^{-x^2}\,dx$ *is proper.*

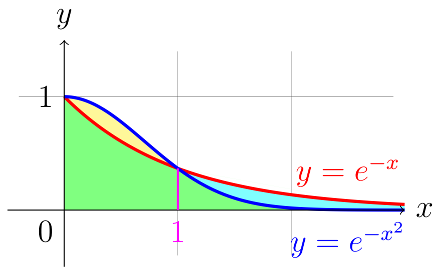
For $x \geq 1$, $x^2 \geq x$, $-x \geq -x^2$, $e^{-x} \geq e^{-x^2} > 0$, *and*

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}\Big[-e^{-x}\Big]_1^t = \lim_{t\to\infty}(-e^{-t}-(-e^{-1})) = \frac{1}{e}.$$

$\because \displaystyle\int_1^\infty e^{-x}\,dx$ *is convergent, by the Comparison Theorem,* (大收就小收。)

$\displaystyle\int_1^\infty e^{-x^2}\,dx$ *is convergent,* $\therefore \displaystyle\int_0^\infty e^{-x^2}\,dx$ *is convergent.* ■

(有限不影響無限, 真積分不影響收斂發散, [0, 1] 不能比就不用比。)



♦: $\boxed{\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$

Example 0.12 *Show that* $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ *is divergent by Comparison Theorem.*

$\because$ *For* $x \geq 1$, $e^{-x} > 0$, $\dfrac{1+e^{-x}}{x} > \dfrac{1}{x} > 0$, *and* $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *is divergent,*

$\therefore$ *by the Comparison Theorem,* $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ *is divergent.* (小發就大發。) ■