2.8 The derivative as a function

1. derivative of $f(x)$ 導函數 $f'(x)$
2. differentiable function 可微函數
3. higher derivatives & other notations 高階導數與其他寫法 $\frac{df}{dx}$

0.1 Derivative of $f(x)$

Recall: The derivative of f at a, f 在 a 的導數:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

收集 $\{(a, f'(a)) : a \in \text{domain of } f, \text{ and } f'(a) \text{ exists}\}$, 可以看做一個函數:

Define: The ***derivative*** 導函數 of f is the function $\boxed{\boldsymbol{f'}}$ defined by

$$\boxed{\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}}$$

if these limits exist.

Note: f' 的定義域包含在 f 的裡面, f' 的值域跟 f 的無關。

0.2 Differentiable function

Define: 單點可微: A function f is ***differentiable*** 可微分 at a if $f'(a)$ exists. (可微分 = 有導數 = $f'(x)$ 有定義 = 有極限。)

Define: 區間可微: A function f is differentiable on an open interval if f is differentiable at every number in the interval.

Note: 整塊開區間只有四種: (a, b), (a, ∞), $(-\infty, b)$, $(-\infty, \infty)$.

Note: 極限有左右, 連續有左右, 可微沒有左右; $\therefore$ 可微分的定義域不含端點。

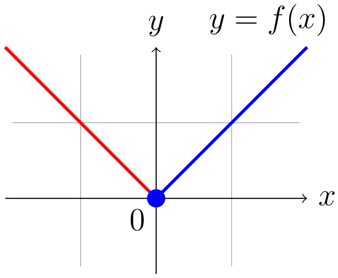
Theorem 1 (可微就連續)
*If f is **differentiable** at a, then f is **continuous** at a.*

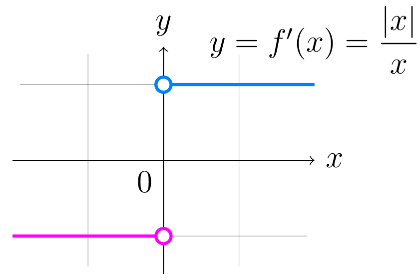
Proof. By definition, the limit exists $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$. Then

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}[f(x) - f(a) + f(a)] \\
&= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a) + f(a)\right] \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a) + \lim_{x\to a} f(a) \\
&= f'(a)\cdot 0 + f(a) = f(a). \qquad (\text{極限律: 加乘}c\&x)
\end{aligned}$$ ■

Note: 可微就連續, 但反之不對, 連續不一定可微。(很常考觀念!)
怎麼說明反過來不對? 找一個反例。去哪找? 多認識一些函數。

Example 0.1 *Where is $f(x) = |x|$ differentiable?*





*If $x > 0$, $|x| = x$, and choose† h near 0 enough such that $x + h > 0$,
then* $f'(x) = \lim\limits_{h\to 0}\dfrac{|x+h|-|x|}{h} = \lim\limits_{h\to 0}\dfrac{x+h-x}{h} = \lim\limits_{h\to 0} 1 = 1.$

*If $x < 0$, $|x| = -x$, and choose† h near 0 enough such that $x + h < 0$,
then* $f'(x) = \lim\limits_{h\to 0}\dfrac{|x+h|-|x|}{h} = \lim\limits_{h\to 0}\dfrac{-(x+h)-(-x)}{h} = \lim\limits_{h\to 0} -1 = -1.$

For $x = 0$, $f'(0) = \lim\limits_{h\to 0}\dfrac{|0+h|-|0|}{h} = \lim\limits_{h\to 0}\dfrac{|h|}{h}$, *(要用左右極限)*
but $\lim\limits_{h\to 0^-}\dfrac{|h|}{h} = -1 \neq 1 = \lim\limits_{h\to 0^+}\dfrac{|h|}{h}$, *the limit does not exist.*

$\because \lim\limits_{x\to 0} f(x) = 0 = f(0)$, *$f$ is continuous at 0.*

Therefore, $f(x)$ is differentiable for $x \neq 0$ (or $(-\infty, 0)\cup(0,\infty)$). ■

(想想看: †: For $x \gtrless 0$, find $\delta > 0$, $\ni 0 < |h| < \delta \implies x + h \gtrless 0$.)

Remark: 連續函數: 不斷&傳極限, 可微函數: 長得很柔順。($|x|$ 在 0 不順。)

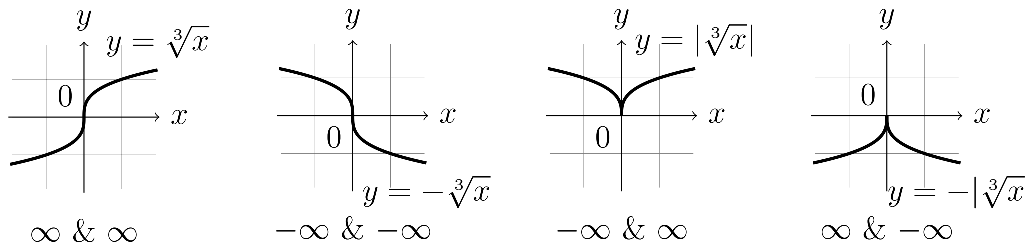
Question: 何時不可微? 切勿明知其不可微而微之。

1. discontinuous: 由定理的等價論述, 不連續就不可微。ex: $\sin\dfrac{1}{x}$ at 0.

2. corner: 左右極限不同。

3. ***vertical tangent line***: 垂直切線 $x = a$ if $\boxed{\lim\limits_{x\to a} |f'(x)| = \infty}$.

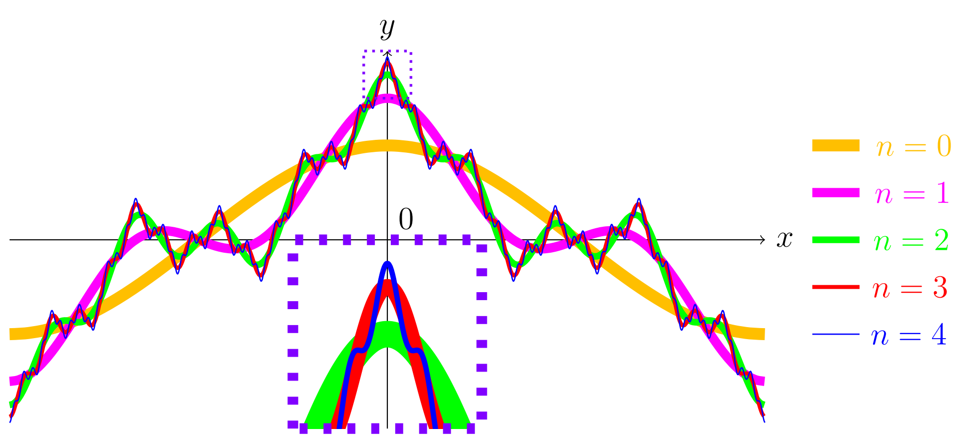
無限極限兩邊可能不同 $(\infty/-\infty)$:



此處不可微, 自有可微處, 處處不可微, ~~薇薇宮中在~~ Weierstrass function.

♦: 起初很多數學家以爲連續函數不可微的地方有限 (或可數無限多 $= \aleph_0$)。
1872, Karl Theodor Wilhelm Weierstraß 提出“處處連續處處不可微的函數”:
魏爾施特拉斯函數 Weierstrass function

$$\sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \text{ where } 0 < a < 1,\ b \text{ positive odd integer},\ ab > 1 + \frac{3}{2}\pi.$$



(第 n 條曲線沿著前一條振盪, 振幅成等比變小, 頻率成等比變快, $n \to \infty$。)

0.3 Higher derivatives & other notations

1. Derivative: $\boxed{f'(x)}$, $\frac{df}{dx}$, $\boxed{\frac{d}{dx}f(x)}$, $Df(x)$, $D_x f(x)$,

   where $\frac{d}{dx}$, D, D_x: differentiation operators 微分算子。

2. When $y = f(x)$: $\boxed{y'}$, $\boxed{\frac{dy}{dx}}$.

   Leibniz: $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, where $\Delta y = f(x + \Delta x) - f(x)$.

3. $\boxed{f'(a)}$, $\boxed{\frac{d}{dx}f(x)\Big|_{x=a}}$, $\boxed{\frac{dy}{dx}\Big|_{x=a}}$, $\frac{dy}{dx}\Big]_{x=a}$.

   Attention: 注意! $\boxed{f'(a) = \frac{d}{dx}f(x)\Big|_{x=a} \neq \frac{d}{dx}f(a)(=0)}$
   左邊是先微分再代入 (導數), 右邊是先代入再微分 (零)。

4. 高階導數 (second derivative, third derivative, ..., n-th derivative)

   $(f')' = \boxed{f''}$, $(f'')' = \boxed{f'''}$, $(f''')' = f^{(4)}$, ..., $(f^{(n-1)})' = \boxed{f^{(n)}}$.

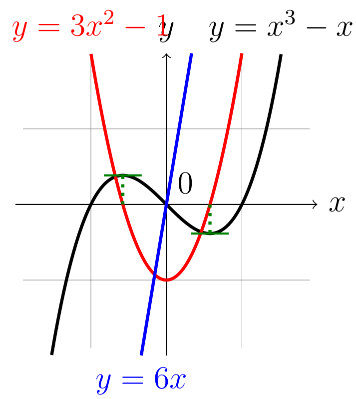
   $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$, ..., $\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \boxed{\frac{d^ny}{dx^n}}$.

   $\frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}f(x)$, ..., $\frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}f(x)\right) = \boxed{\frac{d^n}{dx^n}f(x)}$.

Example 0.2 *$f(x) = x^3 - x$, find and draw f' and find f''.*

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - (x+h) - x^3 + x}{h} \\
&= \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3 - h}{h} \\
&= \lim_{h\to 0} (3x^2 + 3hx + h^2 - 1) = 3x^2 - 1.
\end{aligned}
$$

$$
\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2 - 1 - 3x^2 + 1}{h} \\
&= \lim_{h\to 0} \frac{6hx + 3h^2}{h} = \lim_{h\to 0}(6x + 3h) = 6x. \qquad \blacksquare
\end{aligned}
$$

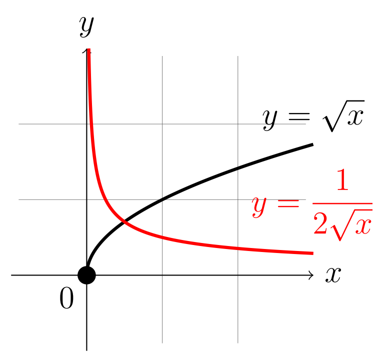


Observation: $y = f(x)$ 在 $x = a$ 水平 $\iff$ 切線斜率 $f'(a) = 0$。

Example 0.3 *$f(x) = \sqrt{x}$, find derivative of f, f' and state its domain.*

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}},
\end{aligned}
$$

and the limit exists only for $x > 0$.

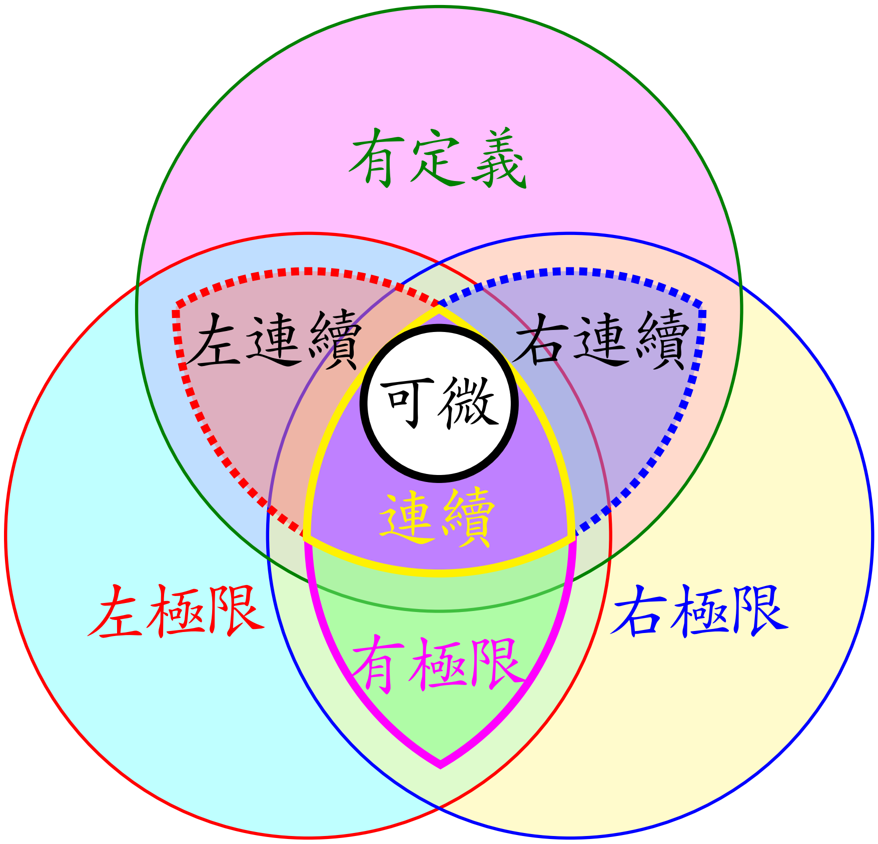


Therefore, $f' = \dfrac{1}{2\sqrt{x}}$ with domain $(0, \infty)$. ($\sqrt{x}$ 的 domain 是 $[0, \infty)$.) $\blacksquare$

(這例子也說明開根函數在 $x > 0$ 是 [有導數=可微分 $\Longrightarrow$]連續函數。)

♦: A function f is called ***symmetrically differentiable***(對稱可微) at a number a if the limit exists:

$$\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}$$



極限 $\iff$ 左極 = 右極

可微 $\implies$ 連續 $\iff$ 極限 = 函數值

次節預告: 用極限去算導數太辛苦了, Sect 3 介紹能幫助快速計算的微分法則 (differentiation rule): 加減乘除常數倍, 冪次 & 多項式, 指數 & 對數, 三角 & 反三角, 合成函數 (chain rule), 隱函數 & 反函數 (implicit differentiation)。