11.10 Taylor and Maclaurin Series

1. Taylor and Maclaurin series
2. e^x, $\sin x$, $\cos x$, $(1+x)^k$
3. multiplication and division of power series

函數表示成冪級數可以做逐項微積分, 可以逼近無理數或超越數; 要怎麼做?

給定函數 $f(x)$, 怎麼找到 $c_0, c_1, \ldots$, 讓 $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ (在 $|x-a| < R$)?

0.1 Taylor and Maclaurin series

逆向思考**:** If f can be represented by a power series at a.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots, \qquad |x-a| < R.$$

先找出 c_n 跟 f 有甚麼關係? When $|x-a| < R$, 由逐項微積分:

$$\begin{array}{rlr}
f(x) = & c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, & f(a) = c_0, \\
f'(x) = & c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots, & f'(a) = c_1, \\
f''(x) = & 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \cdots, & f''(a) = 2c_2, \\
f'''(x) = & 6c_3 + 24c_4(x-a) + \cdots, & f'''(a) = 6c_3, \\
\vdots & & \vdots \\
f^{(n)}(x) = & n!c_n + \cdots + \frac{(n+k)!}{k!}(x-a)^k + \cdots, & f^{(n)}(a) = n!c_n.
\end{array}$$

$\Longrightarrow$ We get $c_n = \dfrac{f^{(n)}(a)}{n!}$. Adopt the conventions $\boxed{\mathbf{0! = 1}}$ and $\boxed{\boldsymbol{f^{(0)} = f}}$.

Theorem 1 *If f has a power series representation (expansion) at a, that is, if (函數 f 可以在 a 寫成冪級數 (或“有冪級數的表示法”, 或簡稱“展開”),)*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad |x-a| < R,$$

*then its **coefficients** are given by the formula (係數與函數的關係公式)*

$$\boxed{\boldsymbol{c_n = \frac{f^{(n)}(a)}{n!}}} \text{(} f \text{ 在 } a \text{ 的 } n \text{ 階導數除以 } n! \text{)}$$

♡ **Important:** 要除以 $n!$, 要除以 $n!$, 要除以 $n!$。

Define: The ***Taylor Series*** 泰勒級數 of the function f at a (centered at a or about a) (f 在 a (以 a 為中心/ a 附近) 的泰勒級數):

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots}$$

Define The ***Maclaurin Series*** 馬克勞林級數 of the function f:

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots}$$

Note: f 的馬克勞林級數就是 f 在 0 的泰勒級數; 泰勒級數要講在哪 (at a), 馬克勞林級數不用。

Attention: 每個函數都有他的泰勒級數, 但<u>不一定</u>相等. (Exercise 11.10.84)

Skill: 怎麼找泰勒/馬可勞林級數: 寫出公式, 微分 f, 代入 a, 放進公式。

Ex: Let $f(x) = \sqrt{2} + ex^2 + \pi x^3$. **Q1:** M.S.=?

	(微分)	(代入中心)	(放進公式)
$n=0:$	$f(x) = \sqrt{2} + ex^2 + \pi x^3,$	$f(0) = \mathbf{\sqrt{2}},$	$\mathbf{\sqrt{2}}$
$n=1:$	$f'(x) = 2ex + 3\pi x^2,$	$f'(0) = \mathbf{0},$	$+\mathbf{0} \cdot x/1!$
$n=2:$	$f''(x) = 2e + 6\pi x,$	$f''(0) = \mathbf{2e},$	$+\mathbf{2e} \cdot x^2/2!$
$n=3:$	$f'''(x) = 6\pi,$	$f'''(0) = \mathbf{6\pi},$	$+\mathbf{6\pi} \cdot x^3/3!$
$n \geq 4:$	$f^{(n)}(x) = 0,$	$f^{(n)}(0) = \mathbf{0},$	$+\mathbf{0} \cdot x^n/n! + \cdots.$

A1: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \mathbf{\sqrt{2}} + \frac{\mathbf{0}}{1!}x + \frac{\mathbf{2e}}{2!}x^2 + \frac{\mathbf{6\pi}}{3!}x^3(+\mathbf{0} + \cdots) = \underline{\sqrt{2} + ex^2 + \pi x^3}.$
($\because$ finite, converges for all x, $R = \infty$.)

Note: 如果函數 f 可以表示成冪級數 (在某個以 a 為中心的收斂區間內):

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \implies c_n = \frac{f^{(n)}(a)}{n!} \implies f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

這個冪級數就會是 f (在 a) 的泰勒級數.

Question: 函數什麼時候可以表示成冪級數 (power series representation)?

稱為: f $\left\{\begin{array}{ll} \text{is the sum of its Taylor series} & \text{等於他的泰勒級數和} \\ \text{has a Taylor expansion} & \text{有泰勒展開式} \\ \text{can be Taylor expanded} & \text{能被泰勒展開} \end{array}\right\}$ at a.

Define: The ***n*th-degree *Taylor polynomial*** 第 n 階泰勒多項式 of f at a (n 是指 x 的最高次數):

$$\boxed{\begin{aligned} \boldsymbol{T_n(x)} &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}}$$

(可以看成泰勒級數的部分和, $T_n(x)$ 是前 (最多) $n+1$ 項和。)

Define: The ***n*th *remainder*** 第 n 個剩餘項 (函數):

$$\boxed{\boldsymbol{R_n(x) = f(x) - T_n(x)}.}$$

Attention: 注意, 跟級數剩餘項 ($R_n = s - s_n$)不一樣!
課本上稱 $R_n(x)$ 爲泰勒級數的剩餘項 (the remainder of the Taylor series), 並不是級數扣掉前 n 個非零項, 而是 $f(x)$ 與 f 的第 n 階泰勒多項式 $T_n(x)$ 的差。

Theorem 2 *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and $\lim\limits_{n\to\infty} R_n(x) = 0$ for $|x-a| < R'$, then $f(x) = \lim\limits_{n\to\infty} T_n(x)$ the sum of its Taylor series on the interval $|x-a| < R'$.*
(當剩餘項歸零, 函數就是他的泰勒級數 (可以表示成冪級數, 或可以泰勒展開)。)

Attention: 注意! 不是泰勒級數收斂, R' 是 $R_n(x) \to 0$ 的收斂半徑, 不是泰勒級數的收斂半徑 R。(同學們容易混淆, 所以我用 R' 做出區別, 跟課本上不同。)

EX: (Continuous) **Q2**: Find $T_n(x)$ & $R_n(x)$ of $f(x)$ at 0=?

A2:

M.S. $= \sqrt{2} + ex^2 + \pi x^3$,	$f(x) = \sqrt{2} + ex^2 + \pi x^3$,
$T_0(x) = \sqrt{2}$,	$R_0(x) = ex^2 + \pi x^3$,
$T_1(x) = \sqrt{2}$,	$R_1(x) = ex^2 + \pi x^3$,
$T_2(x) = \sqrt{2} + ex^2$,	$R_2(x) = \pi x^3$,
$T_{n\geq 3}(x) = \sqrt{2} + ex^2 + \pi x^3$,	$R_{n\geq 3}(x) = 0$.

Question: When $\lim\limits_{n\to\infty} R_n(x) = 0$? For $|x-a| < R'$, $R' =$?

Ex: (Continuous) **Q3**: Each x gives a sequence $\{R_n(x)\}_{n=0}^{\infty}$.
For what value of x, $\lim\limits_{n\to\infty} R_n(x) = 0$? ($\iff$ $f(x) =$ its M.S.)
$\{R_n(x)\}_{n=0}^{\infty} = \{ex^2 + \pi x^3, ex^2 + \pi x^3, \pi x^3, 0, 0, \ldots\}$, $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x.
A3: For all x ($R' = \infty$).

Theorem 3 (Taylor's Inequality) *If* $|f^{(n+1)}(x)| \le M$ *for* $|x-a| \le d$*, then the remainder* $R_n(x)$ *of the Taylor series satisfies the inequality*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}, \qquad \textit{for } |x-a| \le d.$$

♦: **Proof.** By integrating $f^{(n+1)}(x)$ (from a to x) $n+1$ times and the Fundamental Theorem of Calculus. (從 a 到 x 定積分不等式兩邊 $n+1$ 次。)

1^{st}: $f^{(n)}(x) - f^{(n)}(a) = \displaystyle\int_a^x f^{(n+1)}(t)\,dt \le \int_a^x M\,dt = M(x-a),$

2^{nd}: $f^{(n-1)}(x) - f^{(n-1)}(a) - \dfrac{f^{(n)}(a)}{1!}(x-a) \le \dfrac{M}{2!}(x-a)^2, \ldots,$

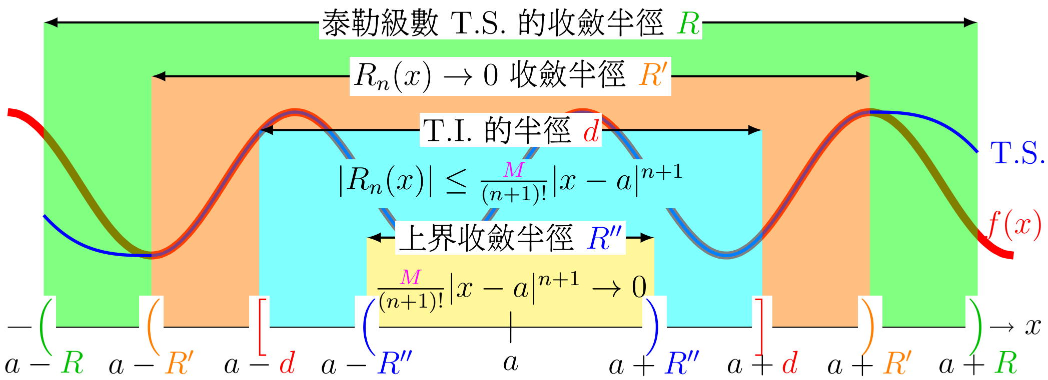
n^{th}: $f'(x) - f'(a) - \dfrac{f''(a)}{1!}(x-1) - \cdots - \dfrac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} \le \dfrac{M}{n!}(x-a)^n,$

$R_n(x) = f(x) - f(a) - \dfrac{f'(a)}{1!}(x-1) - \cdots - \dfrac{f^{(n)}(a)}{n!}(x-a)^n \le \dfrac{M}{(n+1)!}(x-a)^{n+1}.$

Similarly, $R_n \ge \dfrac{-M}{(n+1)!}(x-a)^{n+1}$. ■

Note: Why Taylor's inequality? 直接用 $R_n(x)$ 不容易找到 R':

1. 利用 Taylor's inequality, 找到 $|R_n(x)|$ 的上界 (冪次函數)。(d 不是收斂半徑)
2. 找到這個上界數列收斂的收斂半徑 R''。(比較大比較難收斂, 範圍比較小。)
3. 利用 Squeeze Theorem, $\lim\limits_{n\to\infty} |R_n(x)| = 0 \implies \lim\limits_{n\to\infty} R_n(x) = 0$.
4. 定理只保證在 $|x-a| < R''$ 內會收斂, 但 R'' 不一定是 R'. In fact, $R'' \le R'$.



♦ **Additional:** Lagrange's form of the remainder term:

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1},\ z \in (x,a) \text{ or } x \in (a,x).$$

An extension of Mean Value Theorem ($n=0$). (精準, 但是難找收斂半徑。)

0.2 e^x, $\sin x$, $\cos x$, $(1+x)^k$

先找 e^x, $\sin x$, $\cos x$ 的馬克勞林級數, 再用泰勒不等式證明可以展開。

Example 0.1 *Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.*

(微分) $f^{(n)}(x) = e^x$ and (代入) $f^{(n)}(0) = e^0 = 1$ for all n,

(放進公式) $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$

(用比/根值測試) $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1,$

by the Ratio Test, $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)(= \mathbb{R})$. ■

Recall: $\displaystyle\sum_{n=0}^{\infty} a_n$ converges $\implies \displaystyle\lim_{n\to\infty} a_n = 0$. (Test for Divergence)

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad \text{for all } x.}$$

Example 0.2 *Prove that e^x is equal to the sum of its Maclaurin series.*

For x, consider d with $|x| \le d$, $|f^{(n+1)}(x)| = e^x \le e^d$, choose $M = e^d$.

By Taylor's inequality, $\displaystyle |R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.

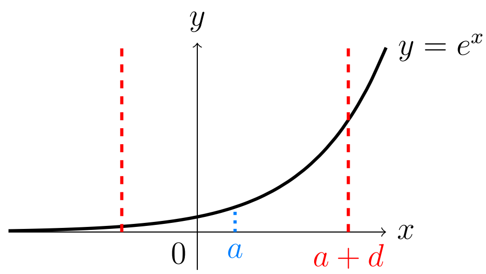
$\displaystyle\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = e^d \cdot 0 = 0$ *for all x, $(R'' = \infty)$ by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} |R_n(x)| = 0$ and hence $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ for all x.*

By the Theorem, $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. $(R' = R = \infty.)$ ■

Attention: f 的泰勒級數收斂<u>不代表</u>級數和的函數就會是 f。
級數的收斂半徑 R <u>不一定</u>等於 $(R_n(x) \to 0)$ 的收斂半徑 R'。In fact, $R' \le R$。

Example 0.3 *Find the Taylor series for $f(x) = e^x$ at $a = 2$.*

$$f^{(n)}(2) = e^2 \text{ for all } n, \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n. \qquad \blacksquare$$



Remark: 類似的證明過程可以得到兩個收斂半徑一樣 ($R' = R = \infty$):

1. e^x 在 a 的泰勒級數 $\displaystyle\sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$ converges for all x. ($R = \infty$)
2. By Taylor's inequality, $|R_n(x)| \le \dfrac{e^{a+d}}{(n+1)!}|x-a|^{n+1}$ for $|x-a| < d$.
3. 利用 $\displaystyle\lim_{n\to\infty} \frac{|x-a|^n}{n!} = 0$ ($R'' = \infty$), 再用夾擠定理 $\displaystyle\lim_{n\to\infty} |R_n(x)| = 0$, 以及絕對收斂到零 $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ for all x. ($R' = \infty$)
4. $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$ for all x.

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n \qquad \text{for all } a, x}$$

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots} \text{ (take } x = 1.)$$

(對冪級數做逐項微積分練習證明 $(e^x)' = e^x$。)

Skill: (阿雄師超快速展開法) 只記 M.S. $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (n 階分之 n 次),

$$e^x = e^a \cdot e^{x-a} = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n.$$

Example 0.4 *Find the Maclaurin series for* $\sin x$ *and prove that it represents* $\sin x$ *for all* x*.*

$f^{(4k)}(0) = \sin 0 = 0$, $f^{(4k+1)}(0) = \cos 0 = 1$,
$f^{(4k+2)}(0) = -\sin 0 = 0$, $f^{(4k+3)}(0) = -\cos 0 = -1$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

For x, $|f^{(n+1)}(x)| \le 1$, *choose* $M = 1$,
by Taylor inequality, $|R_n(x)| \le \dfrac{M}{(n+1)!}|x|^{n+1} = \dfrac{|x|^{n+1}}{(n+1)!}$.
$\because \lim_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$ *for all* x*, by the Squeeze Theorem,* $\lim_{n\to\infty} |R_n(x)| = 0$ *and hence* $\lim_{n\to\infty} R_n(x) = 0$ *for all* x*. By the Theorem,* $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$.
($R' = \infty$. $\sin x$ 的馬克勞林級數的收斂半徑 $R =$?) ■

Example 0.5 *Find the Maclaurin series for* $\cos x$*.*

可以仿造 $\sin x$, 或是利用逐項微分:

$$\begin{aligned} \cos x &= \frac{d}{dx}\sin x = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} \frac{d}{dx}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$

($\cos x$ 等於他的馬克勞林級數的收斂半徑 $R' =$?) ■

Remark: M.S. $\sin x$ 交錯奇階分之奇次, $\cos x$ 交錯偶階分之偶次。

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x}$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x}$$

Example 0.6 *Find the Maclaurin series for the function $f(x) = x\cos x$.*

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!}. \quad (R'=? \ R=?) \qquad \blacksquare$$

Example 0.7 *Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.*

$f^{(4k)}(\frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $f^{(4k+1)}(\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}$,

$f^{(4k+2)}(\frac{\pi}{3}) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$, $f^{(4k+3)}(\frac{\pi}{3}) = -\cos\frac{\pi}{3} = -\frac{1}{2}$.

$f(x) = \sin x$ 在 $a = \frac{\pi}{3}$ 的泰勒級數 $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ *(後面的 a 也要代!)*

$$\begin{aligned}
&= \frac{\sqrt{3}}{2} + \frac{1}{2}(x-\frac{\pi}{3}) - \frac{\sqrt{3}}{2}\frac{1}{2!}(x-\frac{\pi}{3})^2 - \frac{1}{2}\frac{1}{3!}(x-\frac{\pi}{3})^3 + \cdots \\
&= \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x-\frac{\pi}{3})^{2n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x-\frac{\pi}{3})^{2n+1}. \\
\sin x &= \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{3}}{2(2n)!}(x-\frac{\pi}{3})^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2(2n+1)!}(x-\frac{\pi}{3})^{2n+1}.
\end{aligned}$$

$(R'=? \ R=?)$ $\blacksquare$

Skill: (阿雄師超快速展開法)

只記 M.S. $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ and $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$.

$$\begin{aligned}
\sin x &= \sin(\frac{\pi}{3} + x - \frac{\pi}{3}) && \text{(合角公式)} \\
&= \sin\frac{\pi}{3}\cos(x-\frac{\pi}{3}) + \cos\frac{\pi}{3}\sin(x-\frac{\pi}{3}) && \text{(換成 M.S.)} \\
&= \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(x-\frac{\pi}{3})^{2n}}{(2n)!} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(x-\frac{\pi}{3})^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{3}}{2(2n)!}(x-\frac{\pi}{3})^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2(2n+1)!}(x-\frac{\pi}{3})^{2n+1}.
\end{aligned}$$

Fact: e^x, $\sin x$, $\cos x$ 在每個地方都能展開 (expand) (等於他的泰勒級數), 而且收斂半徑 (相等的 R' 與級數的 R) 都是 ∞。

Example 0.8 *Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.*

$$
\begin{aligned}
f(x) &= (1+x)^k,\ f(0) = (1+0)^k = 1,\\
f'(x) &= k(1+x)^{k-1}(1+x)' = k(1+x)^{k-1},\ f'(0) = k(1+0)^{k-1} = k,\\
f''(x) &= k(k-1)(1+x)^{k-2},\ f''(0) = k(k-1)(1+0)^{k-2} = k(k-1),\ \ldots,\\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n},\ f^{(n)}(0) = k(k-1)\cdots(k-n+1).
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 1 + \overbrace{\sum_{n=\boxed{1}}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n}^{\boxed{n=0\text{ 時的公式不同，需分開寫}}} \left(= \overbrace{\sum_{n=0}^{\infty}\binom{k}{n}x^n}^{\boxed{\text{用 }\binom{k}{n}\text{ 時可合併}}} \right)
$$

called ***binomial series*** 二項式級數. ■

Notation: The (general) ***binomial coefficients*** (廣義的) 二項式係數 (唸作 "k choose n" k 取 n)

$$
\boxed{\binom{\boldsymbol{k}}{\boldsymbol{n}}} = \frac{\overbrace{k(k-1)\cdots(k-n+1)}^{n\text{ 項}}}{\underbrace{n(n-1)\cdots 1}_{n\text{ 項}}} \boxed{= \frac{\boldsymbol{k(k-1)\cdots(k-n+1)}}{\boldsymbol{n!}}},
$$

where $k \in \mathbb{R}$ and $n \in \mathbb{N}$, and denote $\boxed{\binom{\boldsymbol{k}}{\boldsymbol{0}} = \boldsymbol{1}}$, $\boxed{\binom{\boldsymbol{k}}{\boldsymbol{-n}} = \boldsymbol{0}}$.

Theorem 4 (Binomial theorem 二項式定理)
If k is a positive integer, then for all x,

$$
(1+x)^k = \sum_{n=0}^{k}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + kx^{k-1} + x^k.
$$

♦: **Proof.** By Pascal's formula $\binom{k}{n} = \binom{k-1}{n} + \binom{k-1}{n-1}$, $1 \le n \le k$, and the Mathematical Induction on k. (因爲有限 $(k+1)$ 項必定收斂。) ■

Theorem 5 (Newton's binomial theorem 牛頓的二項式定理)
If k is any real number and $|x|<1$, then

$$\boxed{(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\cdots.}$$

Proof. If $k\in\mathbb{N}$, then $\binom{k}{n}=0$ when $n>k$, and hence the series is finite.

Otherwise, $k\notin\mathbb{N}$, (先證明二項式級數收斂:) assume $n>k$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\binom{k}{n+1}x^{n+1}}{\binom{k}{n}x^n}\right|=\lim_{n\to\infty}\left|\frac{\frac{k(k-1)\cdots(k-n+1)(k-n)}{n!(n+1)}}{\frac{k(k-1)\cdots(k-n+1)}{n!}}\right||x|$$

$=\lim\limits_{n\to\infty}\dfrac{n-k}{n+1}|x|=\lim\limits_{n\to\infty}\dfrac{1-\frac{k}{n}}{1+\frac{1}{n}}|x|=|x|$, by the Ratio Test, the binomial series converges if $|x|<1$ and diverges if $|x|>1$. ($R=1$, $|x|=1$ 未決定。)

♦: (證明等於函數, 用 T.I. 證明 $\lim\limits_{n\to\infty}R_n(x)=0$ 不容易, 改用這個方式:)

Let $g(x)=\sum\limits_{n=0}^{\infty}\binom{k}{n}x^n$ for $|x|<1$. Then $[(1+x)^{-k}g(x)]'\overset{*}{=}0$ and hence is constant with $(1+x)^{-k}g(x)=(1+0)^{-k}g(0)=1\implies g(x)=(1+x)^k$. ■
(*: see Exercise 11.10.85 for details.)

♦ **Additional:** In fact $R'=R=1$, 二項式級數的收斂區間 (不好證):
$[-1,1]$ for $k\ge 0$, $(-1,1]$ for $-1<k<0$, and $(-1,1)$ for $k\le -1$.

Example 0.9 *Find the Maclaurin series for the function $f(x)=\dfrac{1}{\sqrt{4-x}}$ and its radius of convergence.*

$$\begin{aligned}\frac{1}{\sqrt{4-x}}&=\frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n\\&=\frac{1}{2}+\frac{1}{2\cdot 8}x+\frac{1\cdot 3}{2\cdot 2!8^2}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 3!8^3}x^3+\ldots\end{aligned}$$

converges when $\left|-\dfrac{x}{4}\right|<1$, $|x|<4$, so the radius of convergence is $R=4$. ■

Table of Maclaurin series of functions: ($R' = R$) (♡考)

$\dfrac{1}{1-x}$	$= 1 + x + x^2 + x^3 + \dots$	$= \displaystyle\sum_{n=0}^{\infty} x^n$	$R = 1$
e^x	$= 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots$	$= \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$R = \infty$
$\sin x$	$= x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$R = \infty$
$\cos x$	$= 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	$R = \infty$
$\tan^{-1} x$	$= x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	$R = 1$
$\ln(1+x)$	$= x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots$	$= \displaystyle\sum_{n=\boxed{1}}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$	$R = 1$
$(1+x)^k$	$= 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dots$	$= \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$	$R = 1$

Note: $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ 是減號, $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$ 是加號。

$\ln(1+x) = \displaystyle\sum_{n=\boxed{1}}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$ 從 $\boxed{1}$ 開始。

Example 0.10 *(級數變函數, 101, 102 會考考過)*
Find the sum of the series $\dfrac{1}{1\cdot 2} - \dfrac{1}{2\cdot 2^2} + \dfrac{1}{3\cdot 2^3} - \dfrac{1}{4\cdot 2^4} + \dots$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{1}{2}\right)^n \overset{*}{=} \ln\left(1 + \frac{1}{2}\right) = \ln \frac{3}{2}.$$

(∗: ∵ $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ *for* $|x| < 1$, *and* $\left|\dfrac{1}{2}\right| < 1$.*)* ∎

Example 0.11 *(a) Evaluate* $\int e^{-x^2}\ dx$ *as an infinite series.*
(b) Evaluate $\int_0^1 e^{-x^2}\ dx$ *correct to within an error of 0.001.*

(a) Integrate term by term

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \\ \int e^{-x^2}\ dx &= \int \left[\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\right]\ dx = \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{2n}}{n!}\ dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + C \qquad \textit{converges for all } x. \\ (b) \int_0^1 e^{-x^2}\ dx &= 1 - \frac{1}{3\cdot 1!} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} + \dots \\ &\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475 \end{aligned}$$

*By the Alternating Series Estimation Theorem (*誤差 $\le$ 第一個忽略項*),*
error $\le \left|-\frac{1}{11\cdot 5!}\right| = \frac{1}{1320} < 0.001.$ ■

Example 0.12 *Evaluate* $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Apply l'Hospital's Rule twice:
$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \stackrel{l'H}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \stackrel{l'H}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}, \qquad \left(\frac{\mathbf{0}}{\mathbf{0}}\right)$$

Or:
$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{(\cancel{1} + \cancel{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots) - \cancel{1} - \cancel{x}}{x^2} = \lim_{x\to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \dots}{x^2}$$
$$\stackrel{(\div x^2)}{=} \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \dots\right) = \frac{1}{2} + \lim_{x\to 0} \frac{x}{3!} + \lim_{x\to 0} \frac{x^2}{4!} + \dots = \frac{1}{2}.$$ ■

*(*因爲冪級數收斂, 可以逐項 $\div x^2$ 與 $\lim_{x\to 0}$。*)*

0.3 Multiplication and division of power series

收斂的冪級數的“加減常數倍微積分”都可以 term-by-term, 但是乘除不行。
不過還是可以利用前幾項的乘/除得到積/商的前幾項, 用來做估計。

Note: 收斂半徑: 加減取小的 (區間交集), 常數倍微積分不變, 乘除不一定。

Example 0.13 *Find the first three **nonzero** terms (前三非零項) in the Maclaurin series for $e^x \sin x$ and $\tan x$.*

可以從 $n = 0, 1, 2, \ldots$ 找 $\dfrac{f^{(n)}(0)}{n!}$ *or* 用乘/除算前幾項*:*

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \ldots\right)$$

$$\begin{array}{rl}
 & 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots \\
\times) & x \qquad\qquad\qquad - \tfrac{1}{6}x^3 + \cdots \\
\hline
 & x + x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{6}x^4 + \cdots \\
+) & \qquad\qquad - \tfrac{1}{6}x^3 - \tfrac{1}{6}x^4 - \cdots \\
\hline
 & x + x^2 + \tfrac{1}{3}x^3 + \cdots
\end{array}$$

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \cdots$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots}$$

$$\begin{array}{rl}
 & x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\
\cline{2-2}
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots \Big) & x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 + \cdots \\
-) & x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 + \cdots \\
\cline{2-2}
 & \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\
-) & \tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots \\
\cline{2-2}
 & \tfrac{2}{15}x^5 + \cdots
\end{array}$$

$\boxed{\text{已經三非零項, } \textit{stop!}}$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

■

♦ Additional: More representation of function as power series

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right]$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}, \qquad R = \infty.$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{d}{dx}\sinh x = \sum_{n=0}^{\infty}\frac{d}{dx}\frac{x^{2n+1}}{(2n+1)!}$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}, \qquad R = \infty.$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \int\frac{dx}{1-x^2} = \sum_{n=0}^{\infty}\int x^{2n}\ dx = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}, \qquad R = 1.$$

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n x^n = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}x^n, \qquad R = 1.$$

$$\left[\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!} = (-1)^n\frac{1\times 2\times 3\times 4\times\cdots\times(2n-1)\times(2n)}{2^n n!\times 2^n n!}\right]$$

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}x^n, \qquad R = 1.$$

$$\sin^{-1} x = \int\frac{dx}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\int x^{2n}\ dx$$

$$= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\frac{x^{2n+1}}{2n+1}, \qquad R = 1.$$

$$\sinh^{-1} x = \int\frac{dx}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\int x^{2n}\ dx$$

$$= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\frac{(-1)^n x^{2n+1}}{2n+1}, \qquad R = 1.$$

$$\tan x = \sum_{n=1}^{\infty}\frac{B_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1}, \qquad R = \frac{\pi}{2}.$$

$$\sec x = \sum_{n=0}^{\infty}\frac{E_{2n}(-1)^n}{(2n)!}x^{2n}, \qquad R = \frac{\pi}{2}.$$

∗: B_n 白努利數 (Bernoulli numbers), E_n 歐拉數 (Euler numbers)。