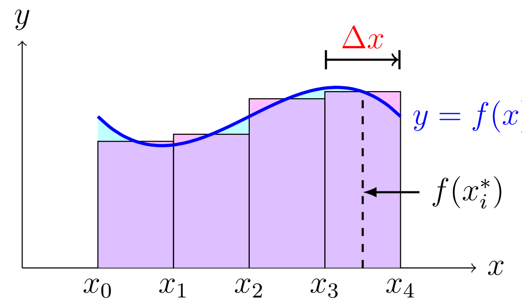
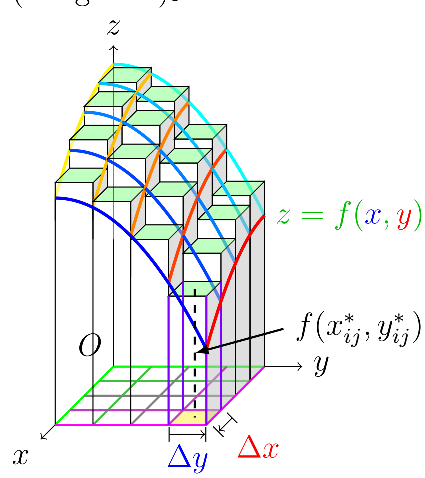
15.1 Double integrals over rectangles

Recall: The definite integral is the limit of the Riemann sum
定積分是黎曼和的極限, $y = f(x)$ 到 x-軸的淨面積 (net area)。

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

把 $[a,b]$ 分成 n 等分 $[x_{i-1},x_i]$, $x_i = a + i\Delta x$, 寬度是 $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$,
高度是每等分裡的樣本點 (sample point) x_i^* 的函數值 $f(x_i^*)$。
當 $f(x) \geq 0$, 等於是用長方形面積和去逼近 $y = f(x)$ 到 x-軸的面積。
當 f 連續, 或是有界並且只有有限多個不連續點
$\Longrightarrow$ 黎曼和極限存在 $\Longleftrightarrow$ f 在 $[a,b]$ 可積分 (integrable)。



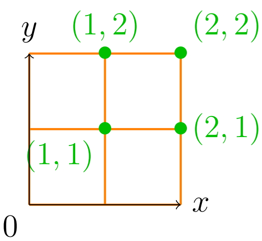


Define: The ***double integral*** is the limit of the ***double Riemann sum***
雙重積分是雙重黎曼和的極限, $z = f(x,y)$ 到 xy-平面的淨體積 (net volume)。

$$\boxed{\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A}$$

把 $R = [a,b]\times[c,d]$ 分成 $m\times n$ 等分, 面積是 $\Delta A = \Delta x\Delta y = \dfrac{b-a}{m}\dfrac{d-c}{n}$,
高度是每等分裡的樣本點 (sample point) (x_{ij}^*, y_{ij}^*) 的函數值 $f(x_{ij}^*, y_{ij}^*)$。
當 $f(x,y) \geq 0$, 等於是用長方柱體積和去逼近 $z = f(x,y)$ 到 xy-平面的體積。
當 f 連續, 或是有界並且只有有限多條線不連續
$\Longrightarrow$ 雙重黎曼和極限存在 $\Longleftrightarrow$ f 在 R 上可積分(***integrable***)。

Example 0.1 *Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$, and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$ by dividing R into four equal squares and choosing the sample point to be the upper right corner of each square.*

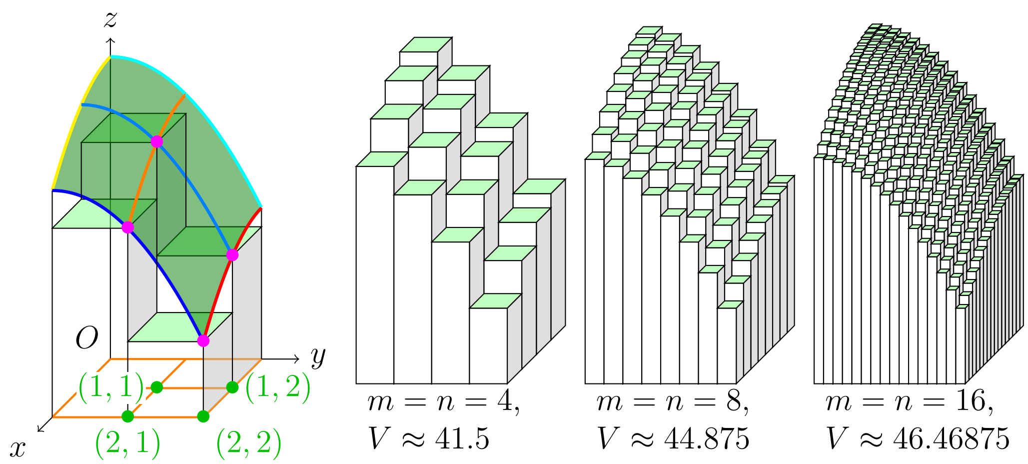


$f(x,y) = z = 16 - x^2 - 2y^2$, $x_k = y_k = k$, $k = 1, 2$,

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\Delta A \qquad (\Delta A = \Delta x \Delta y = 1)$$

$= f(1,1) \times 1 + f(1,2) \times 1 + f(2,1) \times 1 + f(2,2) \times 1$

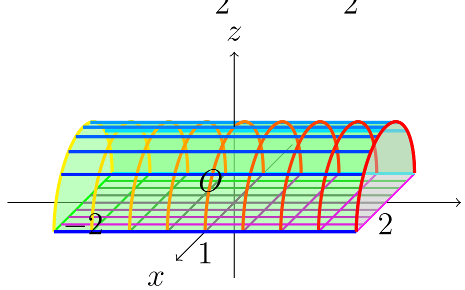
$= 13 + 7 + 10 + 4 = 34$. ■

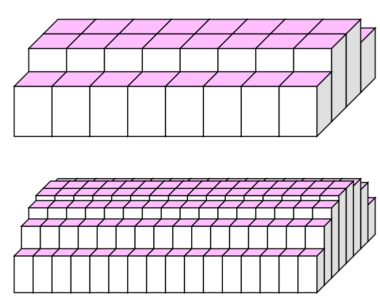


($V \to 48$ as $m, n \to \infty$. 之後會算給你看。)

Example 0.2 *If $R = \{(x,y) : -1 \le x \le 1, -2 \le y \le 2\}$, evaluate the integral* $\displaystyle\iint\limits_R \sqrt{1 - x^2}\, dA$.

The volume of a half circular cylinder with base radius 1 and height 4, (如果看出來) $V = \dfrac{1}{2}\pi r^2 h = \dfrac{1}{2}\pi(1)^2(4) = 2\pi$. ■



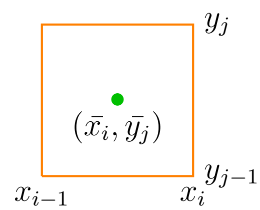


Recall: Approximating Method in definite integral:

- Left-Endpoint 左端 (L_n),
- Right-Endpoint 右端 (R_n),
- Midpoint 中點 (M_n),
- Trapezoidal 梯形 ($T_n = \frac{1}{2}L_n + \frac{1}{2}R_n$)
- Simpson's 辛普森 ($S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$) Rules.

樣本點 (sample point) 可以選四個角落, 也可以選正中間。

Midpoint Rule 中點法: $\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$, $\bar{y}_j = \dfrac{y_{j-1} + y_j}{2}$,



$$\iint_R f(x)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A.$$

Recall: $f_{ave} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\ dx$. (定積分除以積分範圍長度。)

Average Value 平均值: $A(R)$ is the area of R,

$$f_{ave} = \frac{1}{A(R)}\iint_R f(x, y)\ dA.$$

(雙重積分除以積分範圍面積。)

Property: f and g are integrable functions on R and c is constant.

- $\displaystyle\iint_R (f \pm g)\ dA = \iint_R f\ dA \pm \iint_R g\ dA$, (加減)
- $\displaystyle\iint_R cf\ dA = c\iint_R f\ dA$, (常數倍)
- $\displaystyle f \geq g \implies \iint_R f\ dA \geq \iint_R g\ dA$. (大小)

Additional: 1. $f(x,y)$ 在 R 上的雙重積分,

$$\iint\limits_{R} f(x,y)\ dA.$$

$\iint\limits_{?} ?\ dA$ 是固定寫法, A 是指面積 (Area), R 是指長方形 (Rectangle), 其表示法:
(1) 點集合: $R = \{(x,y) : a \le x \le b, c \le y \le d\}$;
(2) 區間乘積: $R = [a,b] \times [c,d]$。

2. 雙重無限處極限:

$$\lim_{m,n\to\infty} f(m,n) = L \text{ 或}$$

$$f(m,n) \to L \text{ as } m,n \to \infty$$

的嚴格定義是:

$$\forall\, \varepsilon > 0, \exists\, M > 0, \ni m > M \,\&\, n > M \implies |f(m,n) - L| < \varepsilon.$$

(只要 m,n 夠大, $f(m,n)$ 就會靠近 L。有多大? 比 M 大; 有多近? ε 那麼近。)

Note: $m,n \to \infty$ 是同時夠大, 不是分開來看, 也沒有先後,
跟 $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} f(m,n)$ 或 $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} f(m,n)$ 都不同!

Example 0.3 (extra) $f(m,n) = \dfrac{n}{m+n}$, $\lim\limits_{m,n\to\infty} f(m,n) =?$

Take $m = n$, $\displaystyle\lim_{m,n\to\infty} \frac{n}{m+n} = \lim_{n\to\infty} \frac{n}{n+n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$;

Take $m = 2n$, $\displaystyle\lim_{m,n\to\infty} \frac{n}{m+n} = \lim_{n\to\infty} \frac{n}{2n+n} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$.

$\implies$ $f(m,n) \to \dfrac{1}{2}$ *when along* $x = y$ *and* $\to \dfrac{1}{3}$ *when along* $x = 2y$.

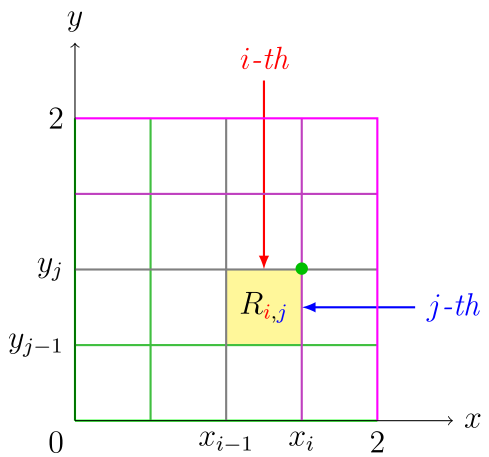
$\implies$ 極限不同 *(*殊途不同歸*)*, 所以極限不存在*! (does not exist)*

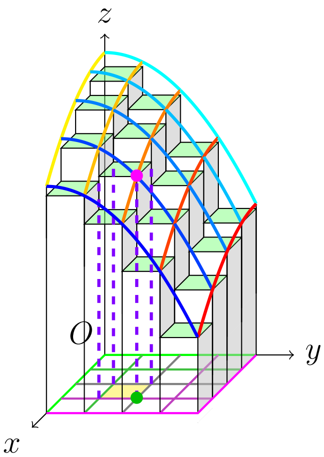
$$\lim_{n\to\infty}\lim_{m\to\infty} \frac{n}{m+n} \overset{\div m}{=} \lim_{n\to\infty}\lim_{m\to\infty} \frac{n/m}{1+n/m} \overset{n\text{ 當常數}}{=} \lim_{n\to\infty} 0 = 0.$$

$$\lim_{m\to\infty}\lim_{n\to\infty} \frac{n}{m+n} \overset{\div n}{=} \lim_{m\to\infty}\lim_{n\to\infty} \frac{1}{m/n+1} \overset{m\text{ 當常數}}{=} \lim_{m\to\infty} 1 = 1. \qquad \blacksquare$$

Example 0.4 (extra) *(用雙重黎曼和的極限算)*

Evaluate $\displaystyle\iint_R f(x,y)\,dA$, *where* $f(x,y) = 16 - x^2 - 2y^2$, $R = [0,2]\times[0,2]$.





把 R 分成 $m\times n$ 個格子 $R_{i,j} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]$, 其中
$\Delta x = \dfrac{2}{m}$, $x_i = \dfrac{2i}{m}$, $0\le i\le m$, $\Delta y = \dfrac{2}{n}$, $y_j = \dfrac{2j}{n}$, $0\le j\le n$.
找 (x_i, y_j) *(右上角)* 當樣本點 *(如果可積分, 找誰都一樣)*, $\Delta A = \Delta x\,\Delta y$:

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta A &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[16 - \left(\frac{2i}{m}\right)^2 - 2\left(\frac{2j}{n}\right)^2\right]\frac{2}{m}\,\frac{2}{n}\\
&= \frac{64}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} 1 - \frac{16}{m^3n}\sum_{i=1}^{m}\sum_{j=1}^{n} i^2 - \frac{32}{mn^3}\sum_{i=1}^{m}\sum_{j=1}^{n} j^2\\
&= \frac{64}{mn}\cdot mn - \frac{16}{m^3n}\cdot\frac{m(m+1)(2m+1)}{6}n - \frac{32}{mn^3}\cdot m\frac{n(n+1)(2n+1)}{6}\\
&= 64 - \frac{16}{3}\left(1+\frac{1}{m}\right)\left(1+\frac{1}{2m}\right) - \frac{32}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right),
\end{aligned}$$

$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta x\,\Delta y = 64 - \frac{16}{3} - \frac{32}{3} = 48. \qquad \blacksquare$$

積積復積積, 怎麼算重積?
不問 Riemann sum, 唯問 Fubini。
積 x 腳撲朔, 積 y 眼迷離。
換個座標做, Jacobian 是雌雄?

Iterated integral

Supposer that $f(x,y)$ is integrable on $[a,b]\times[c,d]$. Let

$$A(x)=\int_c^d f(x,y)\ dy$$

the ***partial integration with respect to*** y from c to d.
f 對 y (從 c 到 d) 偏積分後, 可以看成是 x 的函數。

$$\int_a^b A(x)\ dx=\int_a^b\left[\int_c^d f(x,y)\ dy\right]\ dx$$

integrate $A(x)$ with respect to x from a to b. 再對 x (從 a 到 b) 積分。

Define: The ***iterated integral*** 迭代積分

$$\boxed{\int_a^b\int_c^d \boldsymbol{f(x,y)\ dy\ dx}=\int_a^b\left[\int_c^d \boldsymbol{f(x,y)\ dy}\right]\ \boldsymbol{dx}}$$

先對 y (偏) 積分 [y 沒了], 再對 x 積分 (中括號可省略)。

$$\boxed{\int_c^d\int_a^b \boldsymbol{f(x,y)\ dx\ dy}=\int_c^d\left[\int_a^b \boldsymbol{f(x,y)\ dx}\right]\ \boldsymbol{dy}}$$

先對 x (偏) 積分 [x 沒了], 再對 y 積分。

Skill: 1. 積分順序<u>由內而外</u>。
2. 偏積分時, 跟偏微分一樣, 把其他變數<u>當常數</u>。
3. 偏積分完代上下界時, 記得<u>註明變數</u>才不會代錯:
$\displaystyle\int_a^b f(x,y)\ dx=F(x,y)\Big|_{x=a}^{x=b},\ \int_c^d f(x,y)\ dy=G(x,y)\Big|_{y=c}^{y=d}.$

Example 0.5 *Evaluate (a)* $\displaystyle\int_0^3\int_1^2 x^2y\ dy\ dx$*, (b)* $\displaystyle\int_1^2\int_0^3 x^2y\ dx\ dy$.

(a) $\displaystyle\int_1^2 x^2y\ dy=\left[x^2\frac{y^2}{2}\right]_{y=1}^{y=2}=\frac{3}{2}x^2,\ \int_0^3\frac{3}{2}x^2\ dx=\left[\frac{x^3}{2}\right]_0^3=\frac{27}{2}.$

(b) $\displaystyle\int_0^3 x^2y\ dx=\left[y\frac{x^3}{3}\right]_{x=0}^{x=3}=9y,\ \int_1^2 9y\ dy=\left[\frac{9y^2}{2}\right]_1^2=\frac{27}{2}.$ ■

Theorem 1 (Fubini's Theorem)
If f is ***continuous*** *on the rectangle* $R=\{(x,y): a\le x\le b, c\le y\le d\}$, *then*

$$\boxed{\iint\limits_{R} f(x,y)\ dA=\int_a^b\int_c^d f(x,y)\ dy\ dx=\int_c^d\int_a^b f(x,y)\ dx\ dy}$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist. 如果 f 連續 *(* $\Longrightarrow$ 可積分 *&* 迭代積分存在*)*, 或是 f 有界且只在有限多條線不連續 *(*就是可積分*)* 而且迭代積分存在, 則雙重積分等於迭代積分。

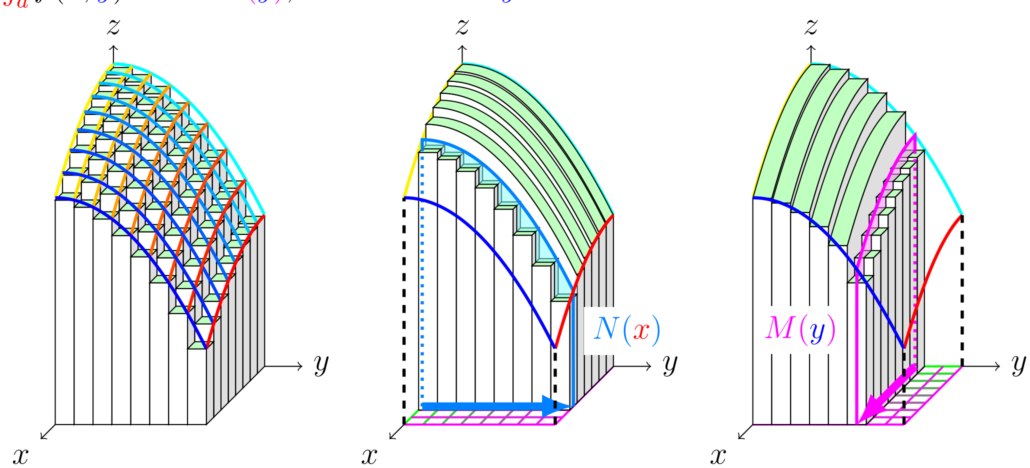
♦: 證明太難, 概念就是: 要切丁要先切片, 從哪邊先切都一樣。
如果可積分, 選中心點 (midpoint) 當樣本點 (sample point) 的答案都一樣。

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\Delta x\Delta y=\sum_{i=1}^{m}\Big[\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\Delta y\Big]\Delta x=\sum_{j=1}^{n}\Big[\sum_{i=1}^{m} f(\bar{x}_i,\bar{y}_j)\Delta x\Big]\Delta y$$

$$\sum_{i=1}^{m}\Big[\int_c^d f(\bar{x}_i,y)\ dy\Big]\Delta x \approx \sum_{i=1}^{m}\Big[\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\Delta y\Big]\Delta x, \qquad \sum_{j=1}^{n}\Big[\int_a^b f(x,\bar{y}_j)\ dx\Big]\Delta y \approx \sum_{j=1}^{n}\Big[\sum_{i=1}^{m} f(\bar{x}_i,\bar{y}_j)\Delta x\Big]\Delta y$$

$$\iint\limits_R f(x,y)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\Delta x\Delta y, \qquad \int_a^b\int_c^d f(x,y)\ dy\ dx \approx \sum_{i=1}^{m}\Big[\int_c^d f(\bar{x}_i,y)\ dy\Big]\Delta x, \qquad \int_c^d\int_a^b f(x,y)\ dx\ dy \approx \sum_{j=1}^{n}\Big[\int_a^b f(x,\bar{y}_j)\ dx\Big]\Delta y$$

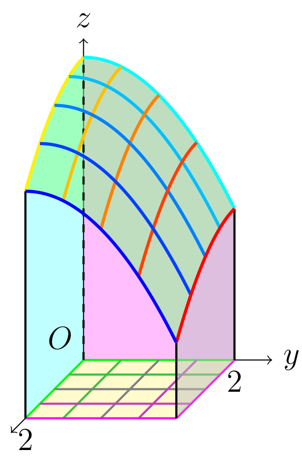
$\int_c^d f(x,y)\ dy=N(x)$, 往 y-軸方向在 x 切片的面積;
$\int_a^b f(x,y)\ dx=M(y)$, 往 x-軸方向在 y 切片的面積。



Example 0.6 *Find the volume of the solid bounded by elliptic paraboloid* $x^2+2y^2+z=16$, *the planes* $x=2$, $y=2$, $x=0$, $y=0$ *and* $z=0$.

$f(x,y)=(z=)16-x^2-2y^2$, $R=[0,2]\times[0,2]$, (自己找函數跟區域)

$$V=\iint_R f(x,y)\,dA=\iint_R (16-x^2-2y^2)\,dA.$$



[Sol 1] 先積 y: $V=\int_0^2\int_0^2 (16-x^2-2y^2)\,dy\,dx$

$=\int_0^2\left[16y-x^2y-\frac{2}{3}y^3\right]_{y=0}^{y=2}dx$

$=\int_0^2\left(\frac{80}{3}-2x^2\right)dx=\left[\frac{80}{3}x-\frac{2}{3}x^3\right]_0^2=48.$

[Sol 2] 先積 x: $V=\int_0^2\int_0^2 (16-x^2-2y^2)\,dx\,dy$

$=\int_0^2\left[16x-\frac{x^3}{3}-2y^2x\right]_{x=0}^{x=2}dy$

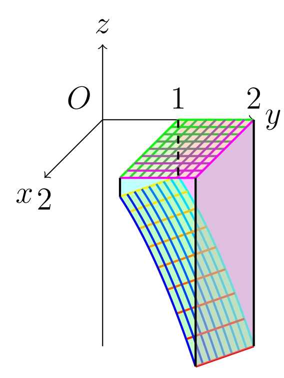
$=\int_0^2\left(\frac{88}{3}-4y^2\right)dy=\left[\frac{88}{3}y-\frac{4}{3}y^3\right]_0^2=48.$ ■

Example 0.7 *Evaluate* $\iint_R (x-3y^2)\,dA$, *where* $R=[0,2]\times[1,2]$.

[Sol 1] 先積 y: $\iint_R (x-3y^2)\,dA$

$=\int_0^2\int_1^2 (x-3y^2)\,dy\,dx=\int_0^2\left[xy-y^3\right]_{y=1}^{y=2}dx$

$=\int_0^2 (x-7)\,dx=\left[\frac{x^2}{2}-7x\right]_0^2=-12.$



[Sol 2] 先積 x: $\iint_R (x-3y^2)\,dA$

$=\int_1^2\int_0^2 (x-3y^2)\,dx\,dy=\int_1^2\left[\frac{x^2}{2}-3y^2x\right]_{x=0}^{x=2}dy$

$=\int_1^2 (2-6y^2)\,dy=\left[2y-2y^3\right]_1^2=-12.$ ■

Note: 雙重積分等於淨體積 (net volume) = 地上積 − 地下積。

Example 0.8 *Evaluate* $\displaystyle\iint_R y\sin(xy)\ dA$, *where* $R = [1,2]\times[0,\pi]$.

[Sol 1] 先積 y: (先不要[2]。)

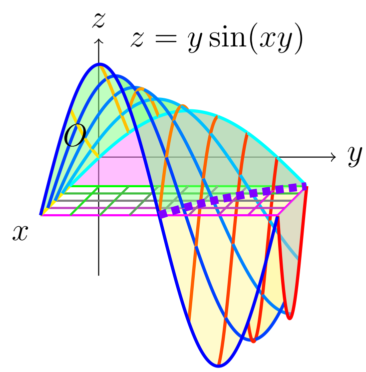
$$\iint_R y\sin(xy)\ dA = \int_1^2\int_0^\pi y\sin(xy)\ dy\ dx = ?$$

[Sol 2] 先積 x:

$$\iint_R y\sin(xy)\ dA = \int_0^\pi\int_1^2 y\sin(xy)\ dx\ dy$$

$$= \int_0^\pi \Big[-\cos(xy)\Big]_{x=1}^{x=2} dy = \int_0^\pi (\cos y - \cos 2y)\ dy$$

$$= \left[\sin y - \frac{\sin 2y}{2}\right]_0^\pi = 0. \quad \blacksquare$$



♦ 先積 y 要怎麼積? 要用分部積分:

Let $u = y$, $dv = \sin(xy)\ dy \implies du = dy$, $v = -\dfrac{\cos(xy)}{x}$.

$$\int y\sin(xy)\ dy = -\frac{y\cos(xy)}{x} + \int \frac{\cos(xy)}{x}\ dy = -\frac{y\cos(xy)}{x} + \frac{\sin(xy)}{x^2} + C,$$

$$\int_0^\pi y\sin(xy)\ dy = \left[-\frac{y\cos(xy)}{x} + \frac{\sin(xy)}{x^2}\right]_0^\pi = -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}.$$

再用分部積分積 x:

Let $U = -\dfrac{1}{x}$, $dV = \pi\cos\pi x\ dx \implies dU = \dfrac{1}{x^2}\ dx$, $V = \sin\pi x$.

$$\int -\frac{\pi\cos\pi x}{x}\ dx = -\frac{\sin\pi x}{x} - \int \frac{\sin\pi x}{x^2}\ dx,$$

$$\int\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)\ dx = -\frac{\sin\pi x}{x} + D.$$

$$\int_1^2\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)\ dx = \left[-\frac{\sin\pi x}{x}\right]_1^2 = 0.$$

Skill: 先試試看對 x 跟對 y 偏積分找反導數, 既然都一樣, 挑好算的先積。

Ex: 比較 $\int y\sin(cy)\ dy$ and $\int c\sin(cx)\ dx$.

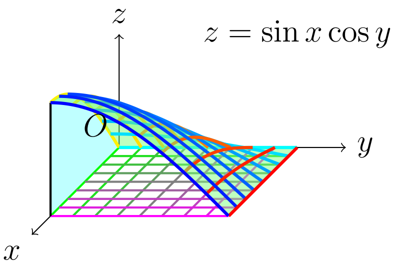
Note: 特殊情況: 當 $f(x,y) = g(x)h(y)$ 可以整個分開 (天時), 在長方形 $R = [a,b] \times [c,d]$ 上雙重積分 (地利), 則(人和):

$$\boxed{\iint\limits_R \boldsymbol{g(x)h(y)\ dA = \int_a^b g(x)\ dx \int_c^d h(y)\ dy}}.$$

Example 0.9 $\displaystyle\iint\limits_R \sin x \cos y\ dA$, *where* $R = [0, \pi/2] \times [0, \pi/2]$.

$$\begin{aligned}
&\iint\limits_R \sin x \cos y\ dA \\
&= \int_0^{\pi/2} \sin x\ dx \int_0^{\pi/2} \cos y\ dy \\
&= \Big[-\cos x\Big]_0^{\pi/2} \Big[\sin y\Big]_0^{\pi/2} \\
&= 1 \cdot 1 = 1.
\end{aligned}$$

■



Note: 不分開也可以: $\displaystyle\iint\limits_R \sin x \cos y\ dA = \int_0^{\pi/2}\int_0^{\pi/2} \sin x \cos y\ dy\ dx$

$$= \int_0^{\pi/2} \sin x \Big[\sin y\Big]_0^{\pi/2} dx = \int_0^{\pi/2} \sin x \cdot 1\ dx = \Big[-\cos x\Big]_0^{\pi/2} = 1.$$

Attention: 要天時地利俱備才可以分開, 能分開就分開。

別人　　　　遲早
— 武雄都說, 我們最好要分開。

♦ Additional: More about Fubini's Theorem

Theorem 2 (Fubini's Theorem)
If f is continuous on the rectangle $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$, then

$$\boxed{\iint\limits_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy}$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Question: 爲什麼要連續 (有界, 可積分&迭代積分存在)?

考慮 $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ on $R = [0,1] \times [0,1]$.

Let $g(x,y) = -\tan^{-1}\dfrac{y}{x}$. Then

$$g_x = \frac{y}{x^2+y^2}, \quad g_y = \frac{-x}{x^2+y^2},$$

$$g_{xy} = \frac{x^2-y^2}{(x^2+y^2)^2} = g_{yx}$$

$$\Longrightarrow f(x,y) = \frac{\partial^2}{\partial x \partial y} g(x,y) = \frac{\partial^2}{\partial y \partial x} g(x,y).$$

$$\begin{aligned}
\int_0^1 \int_0^1 f(x,y)\ dy\ dx &= \int_0^1 \Big[g_x(x,y)\Big]_{y=0}^{y=1} dx = \int_0^1 \left[\frac{y}{x^2+y^2}\right]_{y=0}^{y=1} dx \\
&= \int_0^1 \frac{1}{x^2+1}\ dx = \tan^{-1} x\Big|_0^1 = \frac{\pi}{4}; \\
\int_0^1 \int_0^1 f(x,y)\ dx\ dy &= \int_0^1 \Big[g_y(x,y)\Big]_{x=0}^{x=1} dy = \int_0^1 \left[\frac{-x}{x^2+y^2}\right]_{x=0}^{x=1} dy \\
&= \int_0^1 \frac{-1}{1+y^2}\ dy = -\tan^{-1} y\Big|_0^1 = -\frac{\pi}{4}.
\end{aligned}$$

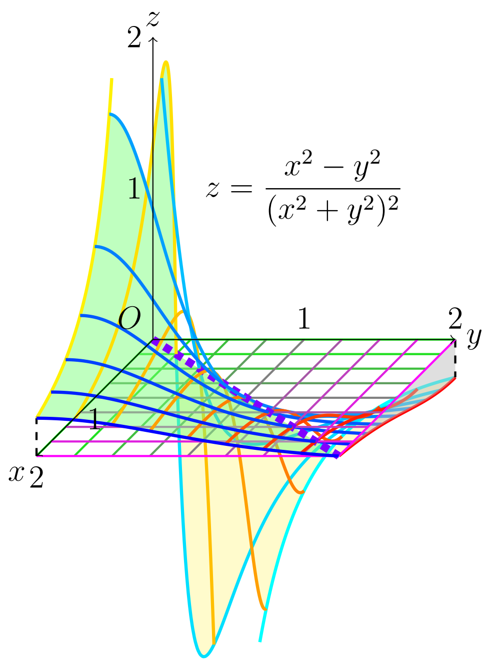
迭代積分都存在, 爲什麼會不一樣? 因爲在 $(0,0)$ 有問題!

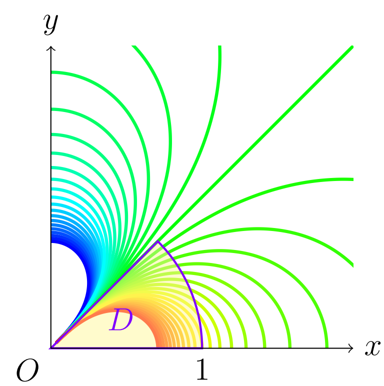
Question: 在 $(0,0)$ 發生什麼事? 不連續。

$f(x,0)=\dfrac{1}{x^2}>0,\ f(0,y)=-\dfrac{1}{y^2}<0,\ \displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Question: 在 $(0,0)$ 不連續有關係嗎? 沒有。

Question: 那問題是? 不是有界, <u>不可積分</u>。





Prove by using polar coordinate: $x=r\cos\theta,\ y=r\sin\theta$.

$f\ge 0 \iff x\ge y \iff 0\le\theta\le\dfrac{\pi}{4}$.

Consider $D=\{(r,\theta): 0\le r\le 1, 0\le\theta\le\dfrac{\pi}{4}\}$.

$$\begin{aligned}\iint_D \frac{x^2-y^2}{(x^2+y^2)^2}\,dA &= \int_0^{\pi/4}\int_0^1 \frac{r^2(\cos^2\theta-\sin^2\theta)}{(r^2)^2}\cdot r\ dr\ d\theta \\ &= \int_0^{\pi/4}\cos 2\theta\ d\theta\int_0^1\frac{1}{r}\ dr = \frac{1}{2}\int_0^1\frac{1}{r}\ dr = \infty.\end{aligned}$$

$\displaystyle\iint_R f(x,y)\,dA$ diverges (the double Riemann sum does not exist). ■