1179: Probability
Lecture 24 — Correlation Coefficient nd Properties of Bivariate Normal

Ping-Chun Hsieh (謝秉均)

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Quick Review

1. Uniqueness Thm
2. $X_1, X_2 \Rightarrow M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$
3. $E[X^n] = \frac{d^n M_X(t)}{dt^n}\Big|_{t=0}$

r.v. X. $E[e^{tX}] = \underline{M_X(t)}$

$(-\delta, \delta)$

- What are the 3 properties of MGFs?

- What is "covariance"? Any alternative expression

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$

$$= \underline{E[XY]} - \underline{E[X] \cdot E[Y]}$$

This Lecture

1. Correlation Coefficient

2. Nice Properties of Bivariate Normal

- Reading material: Chapter 10.4-10.5

Recall: There are still a few remaining questions about bivariate normal…

(Q1) Is X_2 a normal random variable? What is the PDF?

MGF and sum of independent random variables

(Q2) What is “ρ” in the joint PDF of bivariate normal?

Covariance and correlation coefficient

(Q3) Why is bivariate normal useful? Any nice properties?

4 nice properties

A Property of Covariance

Z_1, Z_2

$E[Z_1^2] \cdot E[Z_2^2] \geq (E[Z_1 Z_2])^2$

- **Property:**

$$\left(\mathrm{Cov}(X, Y)\right)^2 \leq \mathrm{Var}[X] \cdot \mathrm{Var}[Y]$$

$$|\mathrm{Cov}(X,Y)| \leq \sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}$$

- **Question:** How to show this?

$$\left(E[(X-E[X])(Y-E[Y])]\right)^2$$

$$\leq E[(X-E[X])^2] \cdot E[(Y-E[Y])^2]$$

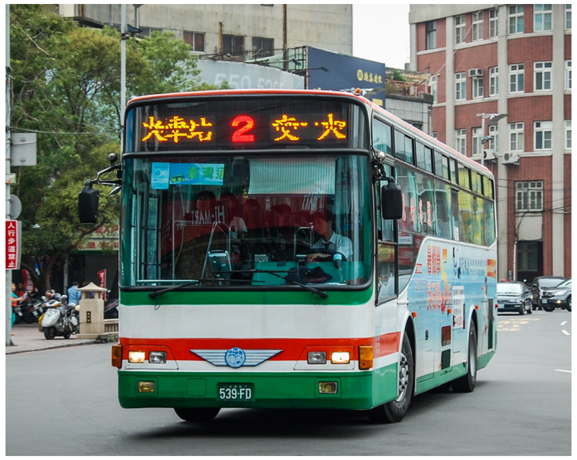
Apply "Cauchy-Schwartz"

Any Issue With Covariance?

X

$Var(a \cdot X) = a^2 \cdot Var(X)$

- Example: Bus #2 (NCTU - Mackay - Train Station)
  - From NCTU to Mackay: X minutes 10
  - From Mackay to Train Station: Y minutes
  - Question: Cov(X, Y) = ? 10
  - Question: What if time is measured in "seconds"? Any change in the covariance?



$Cov(X, Y)$

$$Cov(60 \cdot X, 60 \cdot Y) = E[(60X)(60Y)] - E[60X]E[60Y]$$

$$= 60^2 \cdot Cov(X, Y)$$

Covariance is Sensitive to the Units

- Property: $\underline{\text{Cov}(aX, aY) = a^2 \cdot \text{Cov}(X, Y)}$
  - a: scaling factor due to change of unit

- Question: Any suggested solution?

Correlation Coefficient

- **Correlation Coefficient**: Let X, Y be two random variables with finite variance $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$. Then, the correlation coefficient of X and Y is defined as

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$\sqrt{Var[X] \cdot Var[Y]}$

- Question: Do we have $\rho(X, Y) = \rho(aX, aY)$, for any $a \neq 0$?

$$\rho(aX, aY) = \frac{Cov(aX, aY)}{\sqrt{Var[aX] \cdot Var[aY]}} = \frac{\cancel{a^2} \cdot Cov(X, Y)}{\sqrt{\cancel{a^2} \cdot Var[X] \cdot \cancel{a^2} \cdot Var[Y]}}$$

$$= \rho(X, Y)$$

A Property of Correlation Coefficient

- **Property**:

$$-1 \le \rho(X, Y) \le 1$$

$$-1 \le \frac{Cov(X,Y)}{\sqrt{Var[X] \cdot Var[Y]}} \le 1$$

- Question: How to prove this?

Apply the property

"$(Cov(X,Y))^2 \le Var(X) \cdot Var[Y]$

A Brief Summary of Covariance

$\mathrm{Cov}(X, Y)$

definition

$E[(X-E[X])(Y-E[Y])]$

another expression

$E[XY] - E[X]E[Y]$

"uncorrelated?"

$\mathrm{Cov}(X,Y)=0$

correlation coefficient?

$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\cdot \mathrm{Var}(Y)}}$

independence?

X, Y independent $\overset{?}{\Longrightarrow}$ $\mathrm{Cov}(X,Y)=0$

$\Longleftarrow$ ✗

Connection with $\mathrm{Var}(X+Y)$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$?

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\cdot \mathrm{Cov}(X,Y)$

(Q3) Nice Properties of Bivariate Normal

Properties of Bivariate Normal R.V.

- **Suppose the joint PDF of X_1, X_2 is bivariate normal as**

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}{2(1-\rho^2)} \right]$$

$$\frac{1}{2\pi\sqrt{\det}} e^{-(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

Then we have:

1. Marginal: $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

2. Conditional: $X_2 \,|\, X_1 = x_1 \sim \mathcal{N}\left(\mu_2 + \frac{\rho\sigma_2(x_1-\mu_1)}{\sigma_1}, (1-\rho^2)\sigma_2^2\right)$

3. Correlation coefficient: $\rho(X_1, X_2) = \rho$

4. If X_1, X_2 are uncorrelated ($\rho = 0$), then X_1, X_2 are independent

1. Marginal: $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

$\checkmark$ $$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}{2(1-\rho^2)}\right]$$

$$\frac{\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}{2(1-\rho^2)} = \frac{(x_1-\mu_1)^2}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\left(\frac{(x_2-\mu_2)-\rho(x_1-\mu_1)}{\sqrt{1-\rho^2}}\right)^2$$

Take X_1 for example (X_2 would be similar)

$$f_{X_1}(x_1) = \int_{-\infty}^{+\infty} f_{X_1X_2}(x_1,x_2)\,dx_2 = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \int \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_2^2}\left(\frac{(x_2-\mu_2)-\rho(x_1-\mu_1)}{\sqrt{1-\rho^2}}\right)^2} dx_2$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}}$$

density of $\mathcal{N}(\mu_2+\rho(x_1-\mu_1), \sigma_2^2(1-\rho^2))$

$Z \sim \mathcal{N}(0,1)$ and $W \sim \mathcal{N}(0,1)$

$X_1 = \sigma_1 \cdot Z + \mu_1$

$X_2 = \sigma_2 \cdot [\underbrace{\rho Z + \sqrt{1-\rho^2}\,W}_{\mathcal{N}(0,1)}] + \mu_2$

2. Conditional: $X_2 | X_1 = x_1 \sim \mathcal{N}\left(\mu_2 + \frac{\rho\sigma_2(x_1 - \mu_1)}{\sigma_1}, (1-\rho^2)\sigma_2^2\right)$

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[- \frac{\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right)}{2(1-\rho^2)} \right]$$

$$f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[- \frac{(x_1-\mu_1)^2}{2\sigma_1^2} \right]$$

$$f_{X_2|X_1}(x_2 | x_1) = \frac{f_{X_1X_2}(x_1, x_2)}{f_{X_1}(x_1)}$$

3. Correlation Coefficient: $\rho(X_1, X_2) = \rho$

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}{2(1-\rho^2)} \right]$$

$$\mathrm{Cov}(X_1, X_2) = E[(X_1-\mu_1)(X_2-\mu_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1-\mu_1)(x_2-\mu_2) f_{X_1X_2}(x_1, x_2)\,dx_1 dx_2$$

Hint: $f_{X_2|X_1} = \dfrac{f_{X_1X_2}}{f_{X_1}} \Rightarrow f_{X_1X_2} = f_{X_2|X_1} f_{X_1}$

$\mathrm{Cov}(X_1, X_2) =$

$$\rho(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{\mathrm{Var}(X_1)\cdot \mathrm{Var}(X_2)}}$$

$$X_1 = \sigma_1 Z + \mu_1$$

$$X_2 = \sigma_2\left(\rho Z + \sqrt{1-\rho^2}\, W\right) + \mu_2$$

$$\mathrm{Cov}(X_1, X_2) = E\left[(\sigma_1 \cdot Z)\cdot \sigma_2\left(\rho Z + \sqrt{1-\rho^2}\, W\right)\right]$$

$$= \sigma_1\sigma_2\cdot\rho$$

4. Uncorrelated ($\rho = 0$) Implies Independence

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\underbrace{\sqrt{1-\rho^2}}_{=1}} \exp\left[- \frac{\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} \cancel{- 2\rho \frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right)}{\underbrace{2(1-\rho^2)}_{2}} \right]$$

- **If $\rho = 0$:**

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

Final Remark: X_1, X_2 Normal $\not\Rightarrow$ X_1, X_2 Bivariate Normal

- Example: Let Y and Z be two independent standard normal r.v.s
  - $X_1 = |Y| \cdot \text{sign}(Z)$
  - $X_2 = Y$

- Question:
  - Are X_1 and X_2 normal?
  - Are X_1 and X_2 bivariate normal?