14.7 Maximum and minimum values

名詞: (局部) 極大/小, (絕對) 最大/小, 奇異點, 鞍點, 邊界點, 閉集, 有界集。
方法: 找極值, 找鞍點。

Define: A function f of two variables at (a,b) in its domain D has a

- ***local maximum*** 局部極大值 if $f(x,y) \le f(a,b)$ when (x,y) near (a,b) (in a disk center at (a,b)).
- ***local minimum*** 局部極小值 if $f(x,y) \ge f(a,b)$ when (x,y) near (a,b).
- ***absolute maximum*** 絕對最大值 if $f(x,y) \le f(a,b)$ for all $(x,y) \in D$.
- ***absolute minimum*** 絕對最小值 if $f(x,y) \ge f(a,b)$ for all $(x,y) \in D$.

Recall: Fermat's Theorem: If $f(x)$ has a local maximum or local minimum at a, and $f'(x)$ exists, then $f'(a) = 0$.

Theorem 1 *If f has a local maximum or local minimum at (a,b), and f_x and f_y exist, then $f_x(a,b) = f_y(a,b) = 0$.*

Define: (a,b) is a ***critical point*** 奇異點 (or ***stationary point*** 駐點) of f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or one of them does not exist.
(都是0, 或只要有一個不存在, 可以記成 $\boxed{\nabla f = \mathbf{0} \text{ or 不存在}}$。)

Attention: 極值在奇異點, 奇異點不一定是極值。

★ 相同之三: max, min, 奇異點, Fermat's Theorem (& 不可逆)。

♦ 有些書上分別把 $\nabla f = \mathbf{0}$ 的點稱爲 critical/stationary point 臨界點, 把 ∇f 不存在的點稱爲 singularity point 奇異點; 又有些書上專稱前者爲 stationary point 穩定點, 而把兩者合稱爲 critical point 臨界點。

	$\nabla f = \mathbf{0}$	$\nexists \nabla f$	both
book A	critical 臨界	singularity 奇異	
book B	stationary 臨界	singularity 奇異	
book C	stationary 穩定	singularity 奇異	critical 臨界
Stawrt			$\boxed{\text{critical 奇異}}$

Theorem 2 (Second Derivatives Test)
Suppose f has continuous second partial derivatives near (a,b), and $f_x(a,b) = f_y(a,b) = 0$. Let

$$\boxed{D = D(a,b) = [f_{xx}f_{yy} - (f_{xy})^2](a,b) = \begin{vmatrix} \boldsymbol{f_{xx}} & \boldsymbol{f_{xy}} \\ \boldsymbol{f_{yx}} & \boldsymbol{f_{yy}} \end{vmatrix} \boldsymbol{(a,b)}}$$

(a) *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*

(b) *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*

(c) *If $D < 0$, then $f(a,b)$ is not a local maximum or minimum.*

♦: $H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ / $H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ 稱爲海森 (Hessian) 矩陣/行列式。
♦ **Proof.** For a unit vector $\mathbf{u} = \langle h, k \rangle$, $\boldsymbol{D}_{\mathbf{u}}f = f_x h + f_y k$ and $\boldsymbol{D}_{\mathbf{u}}f(a,b) = 0$.
$\boldsymbol{D}^2_{\mathbf{u}}f = \dfrac{\partial}{\partial x}(\boldsymbol{D}_{\mathbf{u}}f)h + \dfrac{\partial}{\partial y}(\boldsymbol{D}_{\mathbf{u}}f)k = (f_{xx}h + f_{xy}k)h + (f_{yx}h + f_{yy}k)k$
$= f_{xx}(h + \dfrac{f_{xy}}{f_{xx}}k)^2 + k^2\dfrac{f_{xx}f_{yy} - (f_{xy})^2}{f_{xx}} = f_{xx}(h + \dfrac{f_{xy}}{f_{xx}}k)^2 + k^2\dfrac{D}{f_{xx}}$.
$\because$ f_{xx} and $D = f_{xx}f_{yy} - (f_{xy})^2$ are continuous when (x,y) near (a,b).
Case (a): $f_{xx}(a,b) > 0$ and $D(a,b) > 0$. $\implies$ $\boldsymbol{D}^2_{\mathbf{u}}f(x,y) > 0$.
$\therefore$ the curve C on the surface $z = f(x,y)$ in the direction of $\mathbf{u}$ is ***concave upward*** 凹向上, and so $f(x,y) \geq f(a,b)$. (每個方向都一樣凹。)
Case (b): $f_{xx}(a,b) < 0$ and $D(a,b) > 0$. $\implies$ $\boldsymbol{D}^2_{\mathbf{u}}f(x,y) < 0$.
$\therefore$ C is ***concave downward*** 凹向下, and so $f(x,y) \leq f(a,b)$. ■

Define: If $D(a,b) < 0$, then (a,b) is called a ***saddle point*** 鞍點 of f, and the graph of f crosses its tangent plane ($z = f(a,b)$) at (a,b).
(圖在鞍點穿過切平面)

Note: 如果 $D > 0$ 所有方向都凹向同一邊, 有極大/小; 如果 $D < 0$ 有方向凹不同邊, 沒大沒小有鞍點; 如果 $D = 0$, 什麼都有可能。

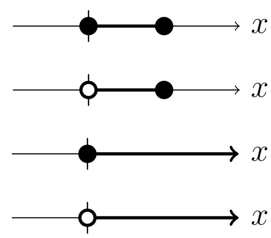
Skill: 1. 如果 f_{xx} 不容易看正負, 看 f_{yy} 也一樣。
2. 奇異點很多: 先算行列式再代; 反之, 先代偏導數再算行列式。
3. By Clairaut's Theorem, $f_{xy} = f_{yx}$ ($\because$ continuous), 只要算一個。

★ 差異之五：二階導數測試法: f'' & 反曲點 v.s. $f_{xx}, f_{xy}, f_{yy}, D$ & 鞍點。

Define:

- A point of a set is called a ***boundary point*** 邊界點 if every disk center at the point contains both some points in the set and some not.
- A set is called ***closed*** 閉 if it contains all its boundary point
- A set is called ***bounded*** 有界 if it contained within some disk.

Example 0.1 (Additional) *in* $\mathbb{R}$*:*



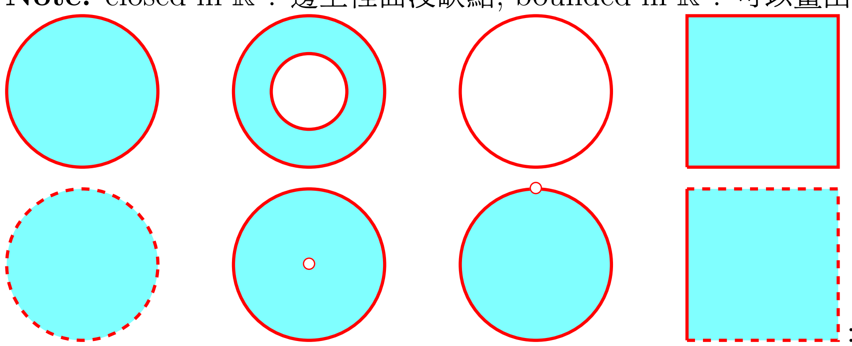
1. closed and bounded: $[0,1]$.

2. not closed and bounded: $(0,1)$, $[0,1)$, $(0,1]$, $\mathbb{Q}\cap[0,1]$.

3. closed and not bounded: $[0,\infty)$, $\cup_{n\in\mathbb{Z}}[2n,2n+1]$.

4. not closed and not bounded: $(0,\infty)$.
(5.) ***open*** $\neq$ *not closed:* $\emptyset$ *and* $\mathbb{R}=(-\infty,\infty)$ *are closed and open.*

Note: closed in $\mathbb{R}^2$: 邊上裡面沒缺點; bounded in $\mathbb{R}^2$: 可以畫出來的有限區域。



:closed sets.

:not closed sets

Theorem 3 (Extreme Value Theorem for Functions of Two Variables) *If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some points (x_1,y_1) and (x_2,y_2) in D. (連續閉有界, 有極值。)*

★ 差異之六：極值定理: 閉區間 (closed interval [?, ?]) v.s. 閉且有界。

Question: 怎麼找極值? 單變數的 Closed Interval Method 的延伸:

Theorem 4 *If f is continuous on a closed, bounded set D in $\mathbb{R}^2$.*
1. 找 *critical point* (a,b). *(*$\nabla f=\mathbf{0}$ *or* $\nexists$, 不用二階導數測試檢查極大/小*)*
2. 找 *boundary* 上的極值。*(*先找極值省時間*)*
3. 比大小。

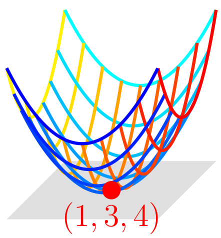
Example 0.2 *Let* $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.

1. 找奇異點: $f_x(x,y) = 2x - 2 = 0$, $x = 1$, $f_y(x,y) = 2y - 6 = 0$, $y = 3$.
critical point: $(1,3)$.

(二階? 先不用, 配平方。) $f(x,y) = 4 + (x-1)^2 + (y-3)^2 \geq 4$.

$\therefore$ $f(1,3) = 4$ *is a local minimum, and in fact a absolute minimum of* f. ■

(二階偏導: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$, $f_{xx}(1,3) = 2 > 0$.)



$z = (x-1)^2 + (y-3)^2 + 4$
(elliptic paraboloid 橢圓拋物面)

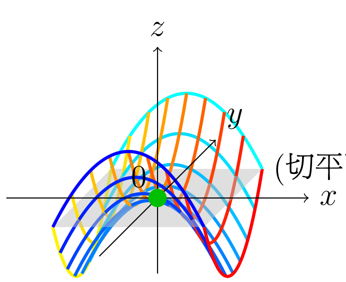
Attention: 極值要完整寫出點跟數值 $f(?,?) = ?$, 奇異/鞍點要寫座標 $(?,?)$。

Example 0.3 *Find the extreme value of* $f(x,y) = y^2 - x^2$.

1. 找奇異點: $f_x = -2x = 0$, $x = 0$, $f_y = 2y = 0$, $y = 0$.
But $f(x,0) = -x^2 < 0$ *for* $x \neq 0$ *and* $f(0,y) = y^2 > 0$ *for* $y \neq 0$, *every disk centered at* $(0,0)$ *contains points with* $f > 0$ *and some with* $f < 0$.

$\therefore$ $f(0,0) = 0$ *is not an extreme value for* f, *so* f *has no extreme value.* ■

(In fact, $(0,0)$ *is a saddle point.* 二階偏導: $D = ?$*)*



(切平面與圖在鞍點交錯)

$z = y^2 - x^2$
(hyperbolic paraboloid 雙曲拋物面)

Example 0.4 *Find the local maximum and minimum values and saddle points of* $f(x,y) = x^4 + y^4 - 4xy + 1$.

1. 找奇異點*:* $f_x = 4x^3 - 4y = 0,\ f_y = 4y^3 - 4x = 0,\ \Longrightarrow\ x = y^3 = x^9,$ $x^9 - x = x(x-1)(x+1)(x^2+1)(x^4+1) = 0,\ x = 0, 1, -1.$
critical points: $(0,0),\ (1,1),\ (-1,-1)$.

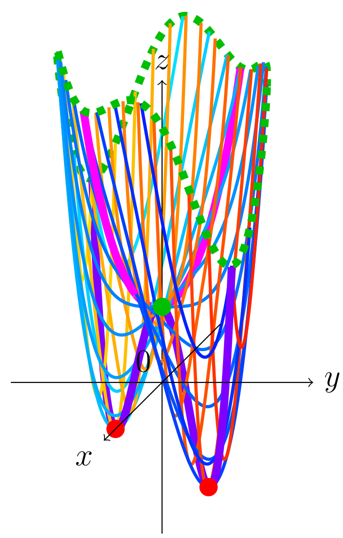
2. 二階偏導*:* $f_{xx} = 12x^2,\ f_{xy} = -4 = f_{yx},\ f_{yy} = 12y^2.$

$D = f_{xx}f_{yy} - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 16(9x^2y^2 - 1).$

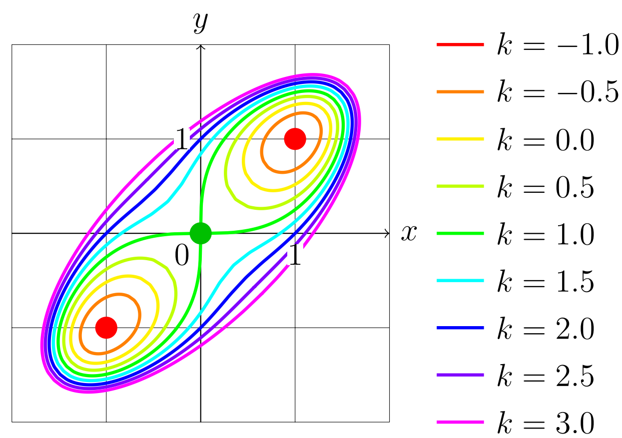
$D(0,0) = -16 < 0,\ \therefore (0,0)$ *is a saddle point.*

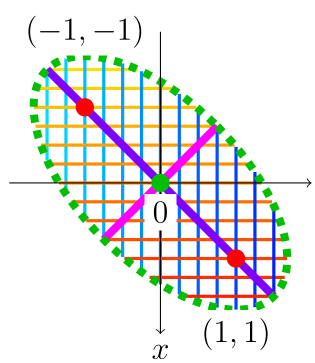
$D(1,1) = 128 = D(-1,-1) > 0,\ f_{xx}(1,1) = 12 = f_{xx}(-1,-1) > 0,$

$\therefore f(1,1) = -1 = f(-1,-1)$ *are local minima(minimum* 複數形*).* ■



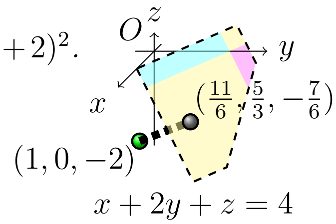
$z = x^4 + y^4 - 4xy + 1$





Example 0.5 *Find the shortest distance from the point* $(1, 0, -2)$ *to the plane* $x + 2y + z = 4$.

$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$, 代入 $z = 4 - x - 2y$ 減少變數;
d^2 最小 d 也會一樣最小, 但 d^2 比較好微分。
Minimize: $d^2 = f(x, y) = (x-1)^2 + y^2 + (4 - x - 2y + 2)^2$.

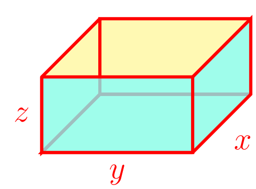


1. 找奇異點:
$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$,
$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$,
critical point: $(\frac{11}{6}, \frac{5}{3})$.

2. 二階導數: $f_{xx} = 4 > 0$, $f_{xy} = 4 = f_{yx}$, $f_{yy} = 10$, $D = f_{xx}f_{yy} - f_{xy}^2 = 24 > 0$, f *has a local minimum* $f(\frac{11}{6}, \frac{5}{3}) = \frac{25}{6}$, $z = 4 - (\frac{11}{6}) - 2(\frac{5}{3}) = -\frac{7}{6}$.
$d = \sqrt{f(\frac{11}{6}, \frac{5}{3})} = \frac{5}{6}\sqrt{6}$, *and the projection point is* $(\frac{11}{6}, \frac{5}{3}, -\frac{7}{6})$. ■
(Recall: 點到平面距離 $\frac{|1(1) + 2(0) + 1(-2) - 4|}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{5}{6}\sqrt{6}$, 但不知道投影點。*)*

Example 0.6 *A rectangular box without a lid (*無蓋*) is to be made from* 12m^2 *of cardboard. Find the maximum volume of such a box.*

Let the length, width, and height be x, y, z.
Then the volume is $V = xyz$,
and the area is $2xz + 2yz + xy = 12$.
代入 $z = \frac{12 - xy}{2(x+y)}$ 減少變數。
Maximize: $V(x, y) = xy\frac{12 - xy}{2(x+y)}$.



1. 找奇異點: $V_x = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} = 0$, $V_y = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2} = 0$,
If $x = 0$ *or* $y = 0$ *then* $V = 0$, *not a maximum.*
So $12 - 2xy - x^2 = 12 - 2xy - y^2 = 0$, $x^2 = y^2$, $x = y$ *(*負不合*)*,
$12 - 3x^2 = 0$, $x = 2$ *(*負不合*)*.
critical point: $(2, 2)$ *and* $z = \frac{12 - 2 \cdot 2}{2(2+2)} = 1$.

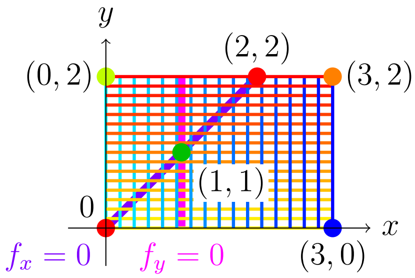
2. 二階導數: *(*略*)*; 其實這個問題很自然的一定有最大值並且發生在奇異點,
so the maximum is $V = 2 \cdot 2 \cdot 1 = 4\text{m}^3$. ■

Example 0.7 (closed bounded) *Find the absolute maximum and minimum values of* $f(x,y) = x^2 - 2xy + 2y$ *on the rectangle* $D = \{(x,y) : 0 \le x \le 3, 0 \le y \le 2\}$.

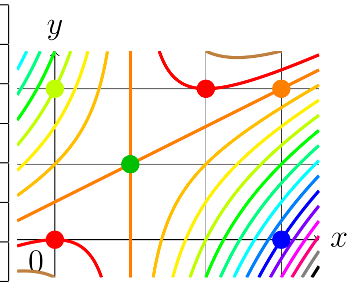
Step 1. 找奇異點*:* $f_x = 2x - 2y = 0,\ f_y = -2x + 2 = 0,\ x = y = 1.$
critical point: $(1,1)$ *and* $f(1,1) = 1^2 - 2\cdot 1\cdot 1 + 2\cdot 1 = 1$. *(*$D(x,y) = -4$*!)*

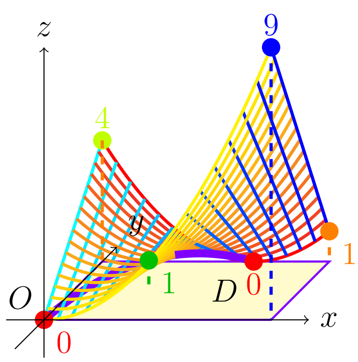
Step 2. 找邊點*:*
$y = 0,\ 0 \le x \le 3,\ f(x,0) = x^2 \implies$ *min* $f(0,0) = 0$ *and max* $f(3,0) = 9$.
$y = 2,\ 0 \le x \le 3,\ f(x,2) = (x-2)^2 \implies$ *min* $f(2,2) = 0$ *and max* $f(0,2) = 4$.
$x = 0,\ 0 \le y \le 2,\ f(0,y) = 2y \implies$ *min* $f(0,0) = 0$ *and max* $f(0,2) = 4$.
$x = 3,\ 0 \le y \le 2,\ f(3,y) = 9 - 4y \implies$ *min* $f(3,2) = 1$ *and max* $f(3,0) = 9$.

Step 3. 比大小*:*

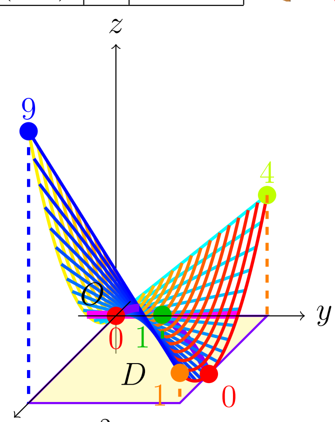


point	f	
$(0,0)$	0	*min*
$(0,2)$	4	
$(1,1)$	1	*saddle*
$(2,2)$	0	*min*
$(3,0)$	9	*max*
$(3,2)$	1	





$z = x^2 - 2xy + 2y$



$z = x^2 - 2xy + 2y$

The *absolute maximum* *value is* $f(3,0) = 9$,
and the *absolute minimum* *value is* $f(0,0) = f(2,2) = 0$. *(*兩個都要寫*)* ■