7.3 Trigonometric substitution

1. $\int \sqrt{a^2-x^2}\ dx \implies x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
2. $\int \sqrt{a^2+x^2}\ dx \implies x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
3. $\int \sqrt{x^2-a^2}\ dx \implies x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}, \pi \le \theta < \frac{3\pi}{2}$

變數變換之 — 三角變換法; 時機: 積分域有 $\sqrt{\text{二次式}}$。

Recall: $\displaystyle\int_0^1 \sqrt{1-x^2}\ dx = \frac{\pi}{4}$ 看圖的, 用積分要怎麼算?

1. TFTC: 不認識反導數 (X);
2. 變數變換: 令 $u = 1-x^2$, 則 $du = -2x\ dx$.

如果有一個 x (或奇數次, $x^2 = 1-u$), $\displaystyle\int x\sqrt{1-x^2}\ dx = \frac{-1}{2}\int \sqrt{u}\ du$, 能算;

但是沒有 x (或偶數次) 硬換: $\displaystyle\int \sqrt{1-x^2}\ dx = \frac{-1}{2}\int \sqrt{\frac{u}{1-u}}\ du$, 不能算 (X);

3. 分部積分: $\displaystyle\int_0^1 \sqrt{1-x^2}\ dx = \left[x\sqrt{1-x^2}\right]_0^1 + \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\ dx$ 更複雜 (X)。

How to compute? 遇到有$\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ 的積分 $(a>0)$, 如果不能變數變換, 可以用三角變換: 把 x 換成三角函數 $a\sin\theta$, $a\tan\theta$, $a\sec\theta$.

0.1 $\displaystyle\int \sqrt{a^2-x^2}\ dx$

Let $\boxed{\boldsymbol{x = a\sin\theta},\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.}$ (其實是 Let $\boldsymbol{\theta = \sin^{-1}\frac{x}{a}}$.)

Then $dx = a\cos\theta\ d\theta$ and $\sqrt{a^2-x^2} = a\cos\theta$. (因爲 θ 範圍會是正的。)

$$\begin{aligned}
\int \sqrt{a^2-x^2}\ dx &= \int \sqrt{a^2(1-\sin^2\theta)}\ d(a\sin\theta) = \int a^2\cos^2\theta\ d\theta \\
&= \int a^2\Big(\frac{1+\cos 2\theta}{2}\Big)\ d\theta = \frac{a^2}{2}\theta + \frac{a^2}{4}\sin 2\theta + C \\
&= \frac{a^2}{2}\theta + \frac{1}{2}(a\sin\theta)(a\cos\theta) + C \qquad (\text{換回 } x) \\
&= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{1}{2}x\sqrt{a^2-x^2} + C.
\end{aligned}$$

$$a = 1 \implies \int_0^1 \sqrt{1-x^2}\ dx = \left[\frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2}\right]_0^1 = \frac{\pi}{4}.$$

0.2 $\int \sqrt{a^2+x^2}\ dx$

Let $\boxed{\boldsymbol{x = a\tan\theta},\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$ (其實是 Let $\boldsymbol{\theta = \tan^{-1}\frac{x}{a}}$.)

Then $dx = a\sec^2\theta\ d\theta$ and $\sqrt{a^2+x^2} = a\sec\theta$. (因爲 θ 範圍是正的。)

$$\begin{aligned}
\int \sqrt{a^2+x^2}\ dx &= \int \sqrt{a^2(1+\tan^2\theta)}\ d(a\tan\theta) = \int a^2\sec^3\theta\ d\theta \\
&= \frac{a^2}{2}\sec\theta\tan\theta + \frac{a^2}{2}\ln|\sec\theta+\tan\theta| + C' \\
&= \frac{1}{2}(a\tan\theta)(a\sec\theta) + \frac{a^2}{2}\ln\left|\frac{a\tan\theta + a\sec\theta}{a}\right| + C' \\
&= \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + C,
\end{aligned}$$

where $C = C' - \frac{a^2}{2}\ln a$. ($\because x + \sqrt{a^2+x^2} > 0$, 可以去絕對值。)

0.3 $\int \sqrt{x^2-a^2}\ dx$

Let $\boxed{\boldsymbol{x = a\sec\theta},\ 0 \le \theta < \frac{\pi}{2}, \pi \le \theta < \frac{3\pi}{2}.}$ (其實是 Let $\boldsymbol{\theta = \sec^{-1}\frac{x}{a}}$.)

Then $dx = a\sec\theta\tan\theta\ d\theta$ and $\sqrt{x^2-a^2} = a\tan\theta$. (因爲 θ 範圍是正的。)

$$\begin{aligned}
\int \sqrt{x^2-a^2}\ dx &= \int \sqrt{a^2(\sec^2\theta-1)}\ d(a\sec\theta) = \int a^2\tan^2\theta\sec\theta\ d\theta \\
&= \int a^2(\sec^3\theta - \sec\theta)\ d\theta \qquad (\tan^2 x\sec x \text{ 不能變換。}) \\
&= \frac{a^2}{2}\sec\theta\tan\theta + (\frac{1}{2}-1)a^2\ln|\sec\theta+\tan\theta| + C' \\
&= \frac{1}{2}(a\sec\theta)(a\tan\theta) - \frac{a^2}{2}\ln\left|\frac{a\sec\theta + a\tan\theta}{a}\right| + C' \\
&= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C,
\end{aligned}$$

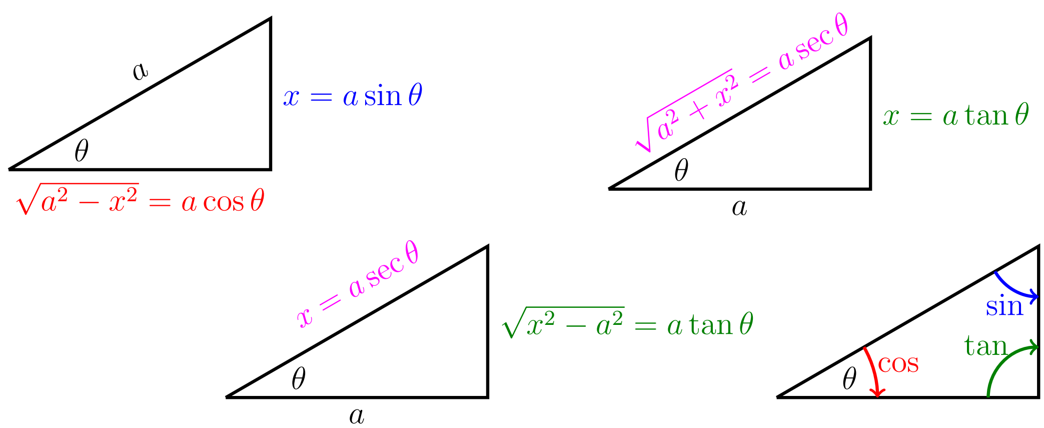
where $C = C' + \frac{a^2}{2}\ln a$. ($\because x + \sqrt{x^2-a^2} < 0$ when $x < -a$, 不可去絕對值。)

0.4 Remark

$$
\begin{aligned}
\int \sqrt{a^2-x^2}\,dx &= \frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C\\
\int \sqrt{a^2+x^2}\,dx &= \frac{1}{2}x\sqrt{a^2+x^2}+\frac{a^2}{2}\ln(x+\sqrt{a^2+x^2})+C\\
\int \sqrt{x^2-a^2}\,dx &= \frac{1}{2}x\sqrt{x^2-a^2}-\frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+C
\end{aligned}
$$

Note: 不要背公式, 會背錯. 應該記的是方法: 把 x 換成三角函數消去根號。其實還是在做變數變換, 只是換個好記的方式, 所以要定 θ 的範圍。

Skill: 畫圖有助於快速把 θ 的三角函數換回 x 的函數。
(夾角是 θ, 一邊是 a, 一邊是 x, 一邊是...)

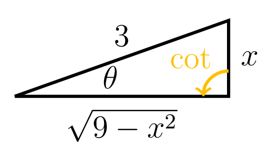


Example 0.1 $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$

Let $x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. $dx = 3\cos\theta\, d\theta$ *and* $\sqrt{9-x^2} = 3\cos\theta$.

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\cdot 3\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta = \int(\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C,$$

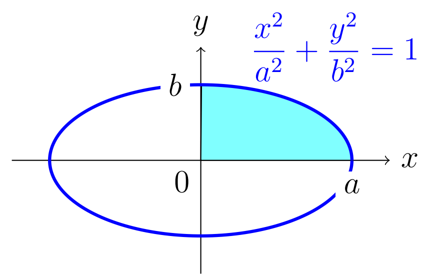


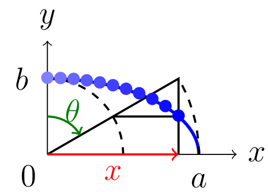
$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sqrt{9-x^2}}{x}$ *or see diagram:*

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C. \text{ (單項可以不加括號。)} \qquad \blacksquare$$

Example 0.2 *Find the area enclosed by the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

$y=\pm\dfrac{b}{a}\sqrt{a^2-x^2}$. *Let* $x=a\sin\theta$, *(範圍...).*
When $x=0$, $\theta=0$, *and when* $x=a$, $\theta=\dfrac{\pi}{2}$.





橢圓面積 $=4\displaystyle\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$
$=4\displaystyle\int_0^{\pi/2}\frac{b}{a}\cdot a\cos\theta\cdot a\cos\theta\ d\theta$
$=4ab\displaystyle\int_0^{\pi/2}\cos^2\theta\ d\theta \qquad (\cos^2\theta=\frac{1+\cos 2x}{2})$
$=4ab\Big[\dfrac{\theta}{2}+\dfrac{\sin 2\theta}{4}\Big]_0^{\pi/2}=ab\pi.$ ■

Note: 定積分版的變數變換, 可以上下界跟著換過去直接代入, 不用換回來。

Additional: 橢圓 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with $a>b$, a 與 b 分別稱爲長軸與短軸, 當 $a=b=r$ 就是半徑 r 的圓, 面積公式依然適用 $ab\pi=\pi r^2$.

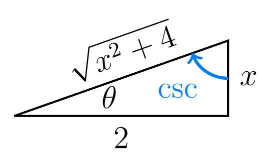
Example 0.3 $\displaystyle\int\frac{1}{x^2\sqrt{x^2+4}}\,dx.$

Let $x=2\tan\theta$, *(範圍...).*

$$\int\frac{1}{x^2\sqrt{x^2+4}}\,dx=\int\frac{2\sec^2\theta}{4\tan^2\theta\cdot 2\sec\theta}\,d\theta=\int\frac{\sec\theta}{4\tan^2\theta}\,d\theta$$

(路線分歧)
[Sol 1] 看出來了! (Recall: §5.4 Ex 3.)
$=\dfrac{1}{4}\displaystyle\int\csc\theta\cot\theta\ d\theta=-\dfrac{1}{4}\csc\theta+C=-\dfrac{\sqrt{x^2+4}}{4x}+C.$

[Sol 2] 變數變換!
Let $u=\sin\theta$, *then* $du=\cos\theta\ d\theta$.



$=\dfrac{1}{4}\displaystyle\int\frac{\cos\theta}{\sin^2\theta}\,d\theta=\frac{1}{4}\int\frac{du}{u^2}=-\frac{1}{4u}+C$
$=-\dfrac{1}{4\sin\theta}+C=-\dfrac{\sqrt{x^2+4}}{4x}+C.$ ■

Example 0.4 $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\ dx.$

[Sol 1] (三角代換)

Let $x = 2\tan\theta$, *(範圍...).*

$$\int \frac{x}{\sqrt{x^2+4}}\ dx = \int \frac{2\tan\theta}{2\sec\theta} 2\sec^2\theta\ d\theta = \int 2\sec\theta\tan\theta\ d\theta$$
$$= 2\sec\theta + C = \sqrt{x^2+4} + C.$$

[Sol 2] (變數變換 A)

Let $u = x^2+4$, *then* $du = 2x\ dx$.

$$\int \frac{x}{\sqrt{x^2+4}}\ dx = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C.$$

[Sol 3] (變數變換 B)

Let $v = \sqrt{x^2+4}$, *then* $dv = \dfrac{x}{\sqrt{x^2+4}}\ dx$.

$$\int \frac{x}{\sqrt{x^2+4}}\ dx = \int\ dv = v + C = \sqrt{x^2+4} + C.$$ ■

Note: 先試變數變換, 不行再用三角代換 (較複雜)。

Example 0.5 $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}},\ a>0.$

Let $x = a\sec\theta$, *(範圍...).*

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\ d\theta = \int \sec\theta\ d\theta = \ln|\sec\theta+\tan\theta| + C$$
$$\left(= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C = \ln|x+\sqrt{x^2-a^2}|\ \underline{-\ln a} + C\right)$$
$$= \ln|x+\sqrt{x^2-a^2}| + C.\ \ (-\ln a \text{ 被 } C \text{ 吃掉了。})$$ ■

[♦ Optional solution]

$\because \cosh^2\theta - \sinh^2\theta = 1$, *let* $x = a\cosh\theta$, *then* $dx = a\sinh\theta\ d\theta$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh\theta}{a\sinh\theta}\ d\theta = \int\ d\theta = \theta + C = \cosh^{-1}\frac{x}{a} + C.$$ ■

Question: 一定要用 $x = a\sin\theta / a\tan\theta / a\sec\theta$?

Answer: 可以用 $x = a\cos\theta / a\cot\theta / a\csc\theta$, 但是 dx 會有負號。

Example 0.6 *(三角變換→變數變換)* $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$

$(4x^2+9)^{3/2} = 8\sqrt{x^2+(3/2)^2}^3$, *let* $x = \frac{3}{2}\tan\theta$, *then* $\sqrt{4x^2+9} = 3\sec\theta$.

When $x = 0$, $\theta = 0$, *and when* $x = \frac{3\sqrt{3}}{2}$, $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$.

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{\pi/3} \Big(\frac{\frac{3}{2}\tan\theta}{3\sec\theta}\Big)^3 \frac{3}{2}\sec^2\theta\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

Let $u = \cos\theta$, *then* $du = -\sin\theta\,d\theta$. (cos 偶數次, sin 奇數次, 令 $u=\cos$。)

When $\theta = 0$, $u = 1$, *and when* $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$.

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\cos^2\theta - 1}{\cos^2\theta}(-\sin\theta)\,d\theta = \frac{3}{16}\int_1^{1/2} \frac{u^2-1}{u^2}\,du = \frac{3}{16}\int_1^{1/2} 1 - \frac{1}{u^2}\,du$$

$$= \frac{3}{16}\Big[u + \frac{1}{u}\Big]_1^{1/2} = \frac{3}{16}\Big[(\frac{1}{2}+2) - (1+1)\Big] = \frac{3}{32}. \qquad \blacksquare$$

[Another](其實用變數變換比較簡單: Try $w = 4x^2+9$.*)* . . . .

Let $v = \sqrt{4x^2+9}$, *then* $dv = \frac{4x}{\sqrt{4x^2+9}}\,dx$, $x^2 = \frac{v^2-9}{4}$.

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{3\sqrt{3}/2} \frac{x^2}{4(4x^2+9)}\frac{4x}{\sqrt{4x^2+9}}\,dx = \int_3^6 \frac{v^2-9}{16v^2}\,dv$$

$$= \frac{1}{16}\int_3^6 1 - \frac{9}{v^2}\,dv = \frac{1}{16}\Big[v + \frac{9}{v}\Big]_3^6 = \frac{1}{16}\Big[(6+\frac{3}{2}) - (3+3)\Big] = \frac{3}{32}. \qquad \blacksquare$$

Note: 不要放棄嘗試變數變換, 可以換了又換。

Example 0.7 *(配方→變數變換→三角變換)* $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx.$

先配方變換*:* $3-2x-x^2 = 4-(x+1)^2$.

Let $u = x+1$, *then* $du = dx$, $x = u-1$ *and* $3-2x-x^2 = 4-u^2$.

再三角變換 *(已試過變數變換): Let* $u = 2\sin\theta$, *then* $\sqrt{4-u^2} = 2\cos\theta$.

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx \overset{\boxed{\to u}}{=} \int \frac{u-1}{\sqrt{4-u^2}}\,du \overset{\boxed{\to \theta}}{=} \int \frac{2\sin\theta - 1}{2\cos\theta}2\cos\theta\,d\theta$$

$$= \int 2\sin\theta - 1\,d\theta = -2\cos\theta - \theta + C \qquad \text{(換回 } \theta \to u \to x.\text{)}$$

$$\overset{\boxed{\to u}}{=} -\sqrt{4-u^2} - \sin^{-1}\frac{u}{2} + C \overset{\boxed{\to x}}{=} -\sqrt{3-2x-x^2} - \sin^{-1}\frac{x+1}{2} + C. \qquad \blacksquare$$

Note: 相當於變換 $x(=u-1) = 2\sin\theta - 1$, 但是不容易看出來。