11.8 Power series

Why power series?

1. 提供一個用來表示一些在數學物理化學界最重要的函數的方法。
2. 提供一個對某些難以微積的函數可以簡單計算微分積分的方法。
3. 提供一個逼近無理數或超越 (transcendental) 數的方法。

$$
\begin{aligned}
e &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}, \\
\ln 2 &= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}, \\
\frac{\pi}{4} &= \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}, \\
\frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}.
\end{aligned}
$$

0.1 Power series, radius/interval of convergence

Define: A ***power series*** 冪級數 is a series of the form (從 $n = 0$ 開始)

$$\boxed{\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots}$$

where c_n's are constants called the ***coefficients*** 係數 of the series.

A power series may converge for some x and diverge for others. The sum of the series is a function (可以看成函數值是級數和的函數, 定義域是收斂處。)

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n x^n}$$

whose domain is the set of all x for which the series converges. More generally,

$$\boxed{\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots}$$

is called a ***power series in $(x-a)$***, or ***centered at a*** or ***about a***.

Recall: 你我約定 $x^0 = 1$, 也答應永遠都不爲 $x = 0$ 擔心。

Theorem 1 *For a given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *there are only three possibilities:*

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

Proof. Find R by the Completeness Axiom of Real Numbers. ■

Define: The number $\boldsymbol{R}$ is called the ***radius of convergence*** 收斂半徑 of the power series: in case (i) $R = 0$, (ii) $R = \infty$.

Define: The ***interval of convergence*** 收斂區間 of the power series, also the domain of $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$: (i) $[a,a] = \{a\}$, (ii) $(-\infty, \infty)$, (iii) $|x-a| < R$ can be written as $a - R < x < a + R$.

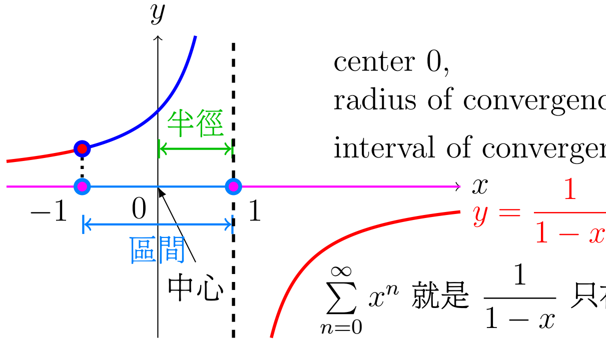
When x is an endpoint, $x = a \pm R$, anything can be happened, so there are four possibilities: $(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, $[a-R, a+R]$. 是哪個? 要檢驗 $x = a \pm R$ 會不會收斂。

Skill: 用 Ratio (或 Root) Test 找 R, 再用其他的檢查 $x = a \pm R$ 的時候。

	Radius	*Interval*	
(i)	0	$\{a\}$	中心收斂, 半徑爲零。
(ii)	∞	$(-\infty, \infty)$	處處收斂, 半徑無限。
(iii)	R	$(a-R, a+R)$ $(a-R, a+R]$ $[a-R, a+R)$ $[a-R, a+R]$	內收外發。

Note: $\sum_{n=0}^{\infty} c_n x^n$ (or $\sum_{n=0}^{\infty} c_n(x-a)^n$) 在中心 0 (or a) 一定會收斂,
總和 $= c_0 + c_1 0^1 + c_2 0^2 + \cdots + c_n 0^n + \cdots = c_0$. 在冪級數中心呼喊收斂。

Recall: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n (:= 1 + x + x^2 + \cdots + x^n + \cdots) \iff |x| < 1.$



center 0,
radius of convergence 1,
interval of convergence $\Big(-1, 1\Big)$.

$\sum_{n=0}^{\infty} x^n$ 就是 $\frac{1}{1-x}$ 只在 $(-1, 1)$ 的部分。

Attention: 冪級數會等於 (可以當成) 函數是有區域限定。(收斂區間)

Ex: take $x = -1$, $\frac{1}{1-x} = \frac{1}{2}$, $\sum_{n=0}^{\infty}(-1)^n$ does not exist (diverges).

$1 - 1 + 1 - 1 + \cdots + (-1)^n + \cdots \neq \frac{1}{2}$.

Example 0.1 *For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?*

(中心) If $x = 0$, $\sum_{n=0}^{\infty} n!x^n = 0! + 1!0^1 + 2!0^2 + \cdots = 1$, converges.

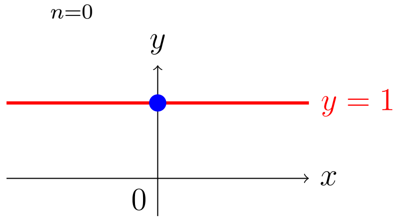
(找半徑) If $x \neq 0$ (才能約分), ($a_n = n!x^n$, 要包含 x^n。)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty. \quad (\not< 1, \text{ 沒半徑。})$$

By the Ratio Test, $\sum_{n=0}^{\infty} n!x^n$ converges only when $x = 0$. ■

(radius and interval of convergence are 0 and $[0, 0] = \Big\{0\Big\}$.)

♦: $\sum_{n=0}^{\infty} n!x^n = 0! = 1 \iff x = 0$. Who is $f(x)$? 1.



Example 0.2 *For what values of x is the series* $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$ *convergent?*

(中心) *If* $x=3$, $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}=0$, *converges.*

(找半徑) *If* $x\neq 3$, $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}}\right|=\lim_{n\to\infty}\frac{|x-3|}{1+\frac{1}{n}}=|x-3|.$

By the Ratio Test, $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}\begin{cases}\text{converges} & \text{if } |x-3|<1, \text{ and}\\ \text{diverges} & \text{if } |x-3|>1.\end{cases}$

(查端點) $|x-3|<1 \iff 2<x<4.$

If $x=4$, *then* $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}\overset{\boxed{\text{代入 } x=4}}{=}\sum_{n=1}^{\infty}\frac{1}{n}$ *(harmonic series) diverges.*

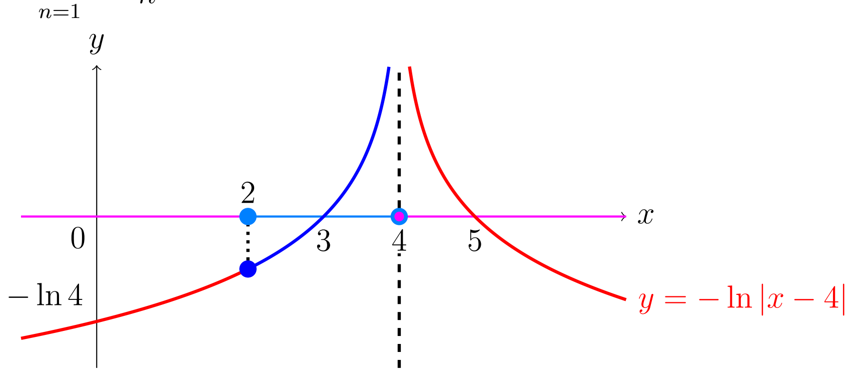
If $x=2$, *then* $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}\overset{\boxed{\text{代入 } x=2}}{=}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ $(=-\ln 2)$ *converges by the Alternating Series Test* $(\frac{1}{n}\searrow 0)$. $\therefore \displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$ *converges for* $2\leq x<4$. ■

(radius and interval of convergence are 1 *and* $\big[2,4\big)$*.)*

Note: $a_n=c_n(x-a)^n$, c_n 有 $n!$ 用比值 Ratio, 有 c^n 用開根 Root。

沒規定只能用一種: $\displaystyle\sqrt[n]{\left|\frac{(x-3)^n}{n}\right|}=\frac{|x-3|}{\sqrt[n]{n}}\to|x-3|$ as $n\to\infty$ $(\sqrt[n]{n}\to 1)$.

♦: $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}=-\ln|x-4| \iff x\in[2,4).$



Example 0.3 *Find the domain of the Bessel function of order 0 defined by*

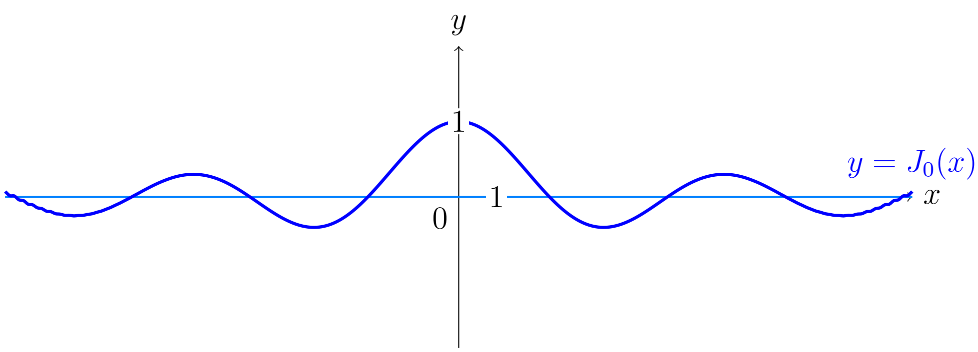
$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

If $x = 0$, $J_0(0)$ *converges.*

If $x \neq 0$, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}}{\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}} \right| = \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1.$

By the Ratio Test, $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ *converges for all* x*, the domain of the Bessel function* J_0 *is* $(-\infty, \infty) = \mathbb{R}$. ■

(radius and interval of convergence are ∞ *and* $\Big(-\infty, \infty\Big)$*.)*



Skill: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \left(\text{or } \sqrt[n]{|a_n|}\right) = \begin{cases} \infty, & \Longrightarrow R = 0; \\ 0, & \Longrightarrow R = \infty; \\ \boxed{?}, & \boxed{?} \lessgtr 1 \iff |x-a| \lessgtr R. \end{cases}$

Example 0.4 *Find the radius of convergence and interval of convergence of the series* $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$.

(中心) If $x=0$*, converges.*
(找半徑) If $x\neq 0$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}}{\dfrac{(-3)^n x^n}{\sqrt{n+1}}}\right| = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\dfrac{1}{n}}{1+\dfrac{2}{n}}} = 3|x|.$$

By the Ratio Test, $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$ *converges if* $(3|x|<1 \iff)|x|<\dfrac{1}{3}$ *and diverges if* $(3|x|>1 \iff)|x|>\dfrac{1}{3}$.

So the radius of convergence is $\dfrac{1}{3}$.

(查端點) $|x|<\dfrac{1}{3} \iff -\dfrac{1}{3}<x<\dfrac{1}{3}$.

If $x=-\dfrac{1}{3}$*, then* $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}} \overset{\boxed{平移}}{=} \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$,

p-series with $p=\dfrac{1}{2}\leq 1$*, diverges.*

If $x=\dfrac{1}{3}$*, then* $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}} \overset{\boxed{平移}}{=} \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$,

converges by the Alternating Series Test ($\dfrac{1}{\sqrt{n}}\searrow 0$).

So the interval of convergence is $\left(-\dfrac{1}{3},\dfrac{1}{3}\right]$. ■

Note: 不一定每個冪級數收斂時的函數都能寫得出來。

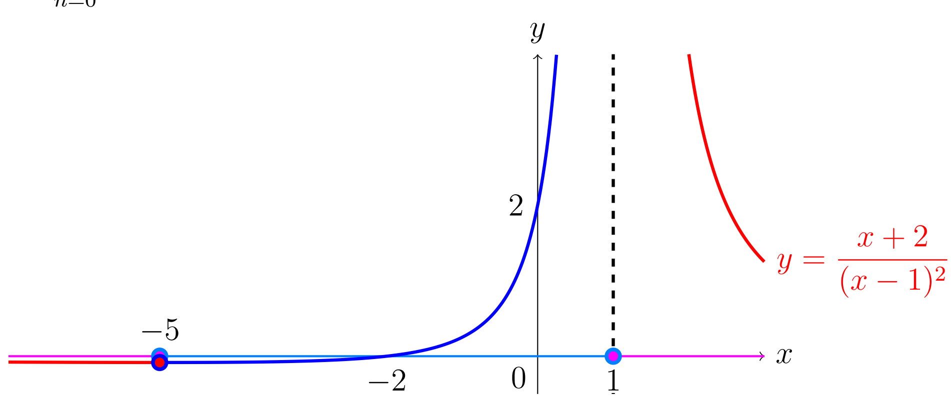
Example 0.5 *Find the radius of convergence and interval of convergence of the series* $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

(中心) *If* $x=-2$, *converges.*
(找半徑) *If* $x \neq -2$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}}{\frac{n(x+2)^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x+2|}{3} = \frac{|x+2|}{3}.$$

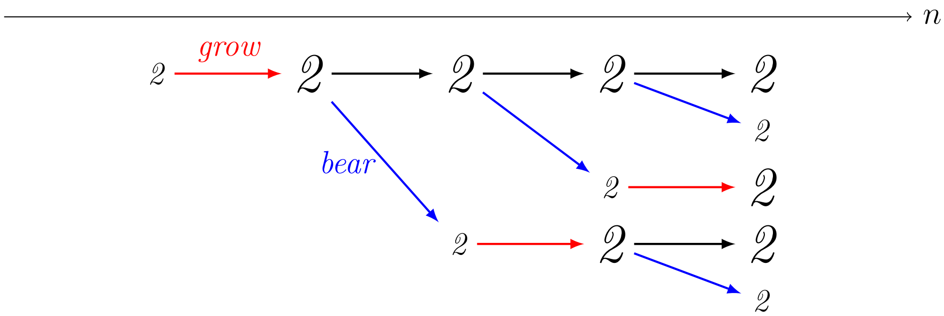
By the Ratio Test, $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ *converges if* $(\frac{|x+2|}{3} < 1 \iff)|x+2| < 3$ *and diverges if* $(\frac{|x+2|}{3} > 1 \iff)|x+2| > 3$.
So the radius of convergence is 3.
(查端點) $|x+2| < 3 \iff -5 < x < 1$.
If $x=-5$, *then* $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=1}^{\infty}(-1)^n n$,
diverges by the Test for Divergence ($\lim_{n\to\infty}(-1)^n n \nexists$).
If $x=1$, *then* $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=1}^{\infty} n$,
diverges by the Test for Divergence ($\lim_{n\to\infty} n = \infty \nexists$).
So the interval of convergence is $\left(-5, 1\right)$. ■

♦: $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \frac{x+2}{(x-1)^2} \iff x \in (-5,1)$.



♦ Additional: Application of power series

斐波那契數列: *Fibonacci sequence:* $1, 1, 2, 3, 5, 8, 13, 21, \ldots$



$$f_n = f_{n-1} + f_{n-2}, \qquad f_1 = f_2 = 1.$$

Let $f(x) = \sum_{n=0}^{\infty} f_n x^n = x + x^2 + 2x^3 + \ldots$, *where* $f_0 = 0$.

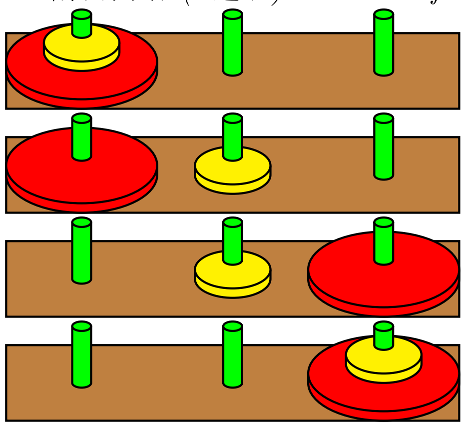
$$\begin{aligned}
f(x) &= x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{n=0}^{\infty} f_n x^n \\
-)\ xf(x) &= \phantom{x +{}} x^2 + x^3 + 2x^4 + \cdots = \sum_{n=0}^{\infty} f_n x^{n+1} = \sum_{n=1}^{\infty} f_{n-1} x^n \\
-)\ x^2 f(x) &= \phantom{x + x^2 +{}} x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} f_n x^{n+2} = \sum_{n=2}^{\infty} f_{n-2} x^n
\end{aligned}$$

$$(1 - x - x^2) f(x) = x + \sum_{n=2}^{\infty} (f_n - f_{n-1} - f_{n-2}) x^n = x.$$

$$\begin{aligned}
f(x) &= \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{1+\sqrt{5}}{2} x} - \frac{1}{1 - \frac{1-\sqrt{5}}{2} x} \right) \\
&= \frac{1}{\sqrt{5}} \left[\sum_{n=0}^{\infty} \left(\frac{1+\sqrt{5}}{2} \right)^n x^n - \sum_{n=0}^{\infty} \left(\frac{1-\sqrt{5}}{2} \right)^n x^n \right] \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] x^n, \\
f_n &= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].
\end{aligned}$$

(radius of convergence ?)

解河內塔 *(n*連環*): Towers of Hanoi:* $1, 3, 7, 15, \ldots$



$$h_n = 2h_{n-1} + 1, \qquad h_1 = 1.$$

Let $h(x) = \sum_{n=0}^{\infty} h_n x^n = x + 3x^2 + 7x^3 + \ldots$, *where* $h_0 = 0$.

$$\begin{array}{rl}
h(x) = x + 3x^2 + 7x^3 + 15x^4 + \cdots &= \sum_{n=0}^{\infty} h_n x^n \\
-)\,2xh(x) = \quad 2x^2 + 6x^3 + 14x^4 + \cdots &= \sum_{n=0}^{\infty} 2h_n x^{n+1} = \sum_{n=1}^{\infty} 2h_{n-1} x^n \\
-)\,\frac{x}{1-x} = x + x^2 + x^3 + x^4 + \cdots &= \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=1}^{\infty} x^n \\
\hline
\end{array}$$

$$\begin{aligned}
(1-2x)h(x) &= \frac{x}{1-x}, \\
h(x) &= \frac{x}{(1-x)(1-2x)} = \frac{1}{1-2x} - \frac{1}{1-x} \\
&= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} (2^n - 1) x^n, \\
h_n &= 2^n - 1.
\end{aligned}$$

(radius of convergence ?)

⧫ Extra: Some series sums

- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}.$

  Proof. [Euler, 1735] $\because \sin n\pi=0,\ \forall\ n\in\mathbb{Z}$, and

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \\
&= x\prod_{n\in\mathbb{Z}}(1-\frac{x}{n\pi}) = x(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})\cdots \\
&= x-\sum_{n=0}^{\infty}\frac{1}{n^2}\frac{x^3}{\pi^2}+\sum_{1\le n<k}\frac{1}{n^2k^2}\frac{x^5}{\pi^4}+\cdots, \\
\frac{1}{3!} &= \sum_{n=1}^{\infty}\frac{1}{n^2}\frac{1}{\pi^2} \quad \text{(compare coefficient of } x^3\text{)}, \\
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{6}; \\
\frac{1}{5!} &= \sum_{1\le n<k}\frac{1}{n^2k^2}\frac{1}{\pi^4} \quad \text{(compare coefficient of } x^5\text{)}, \\
\sum_{n=1}^{\infty}\frac{1}{n^4} &= (\sum_{n=1}^{\infty}\frac{1}{n^2})^2-2\sum_{1\le n<k}\frac{1}{n^2k^2} = (\frac{\pi^2}{6})^2-2\frac{\pi^4}{120} = \frac{\pi^4}{90}. \qquad \blacksquare
\end{aligned}$$

- $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}.$

  Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
&= \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{3}{4}\times\frac{\pi^2}{6} = \frac{\pi^2}{8}. \qquad \blacksquare
\end{aligned}$$

- $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\approx 0.915965594177,$

  卡塔蘭常數 (Catalan's constant), 尚不知是否爲無理數, 是否爲超越數。