1179: Probability
Lecture 1 — Probability Model

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September 15, 2021

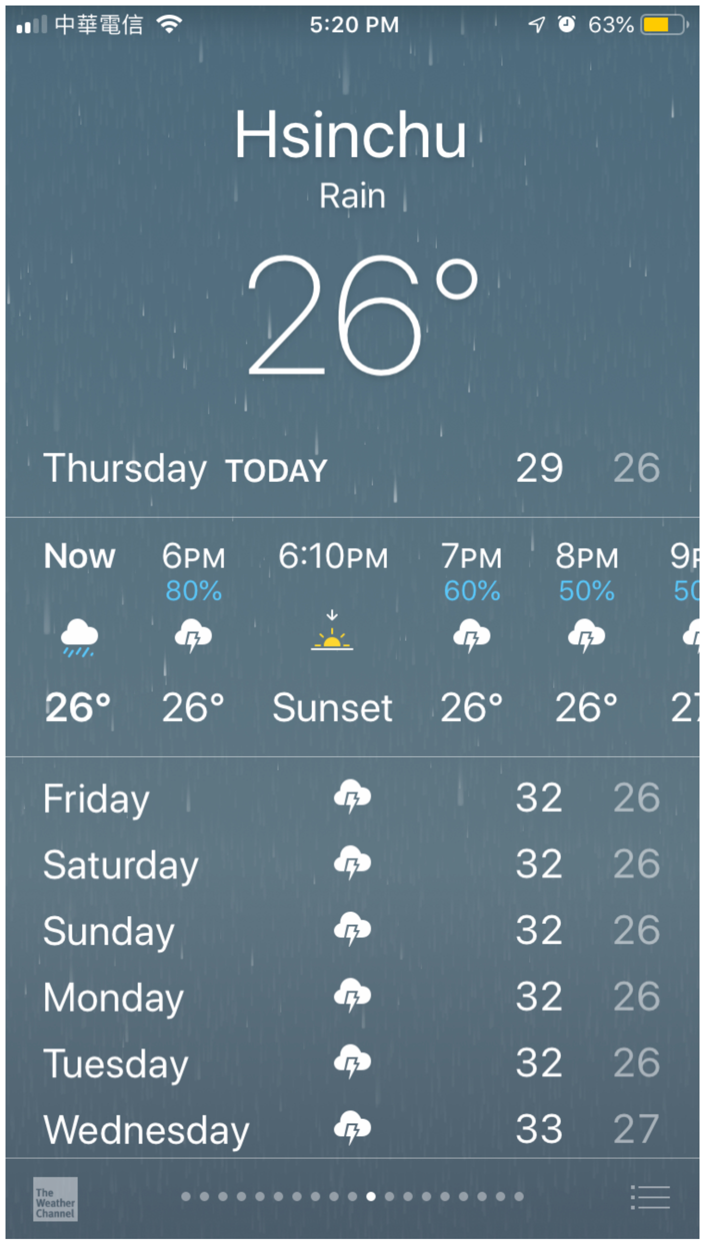
This Lecture

1. Probability model and basic terminology

2. Review: Set operations

- Reading material: Chapter 1.1~1.4

What is Probability?



- "The probability of rain is 60% at 7pm" means?
  1. 60% of historical data?
  2. (On average) 60% of the area?
  3. (On average) 60% of the time between 7pm and 8pm?
- What does 1.87 meters mean?

Probability = a measure of how likely an event would happen

Probability Model

- Experiments
- Outcome / Event / Sample Space

Experiments: What is Random?

- **Experiment**: Procedure, model, and outcome

- Example: Roll 2 four-sided dice
  - Procedure: Pick up the dice and roll them without manipulation
  - Model: All pairs (1,1),…,(4,4) are equally likely
  - Outcome: (2,4)

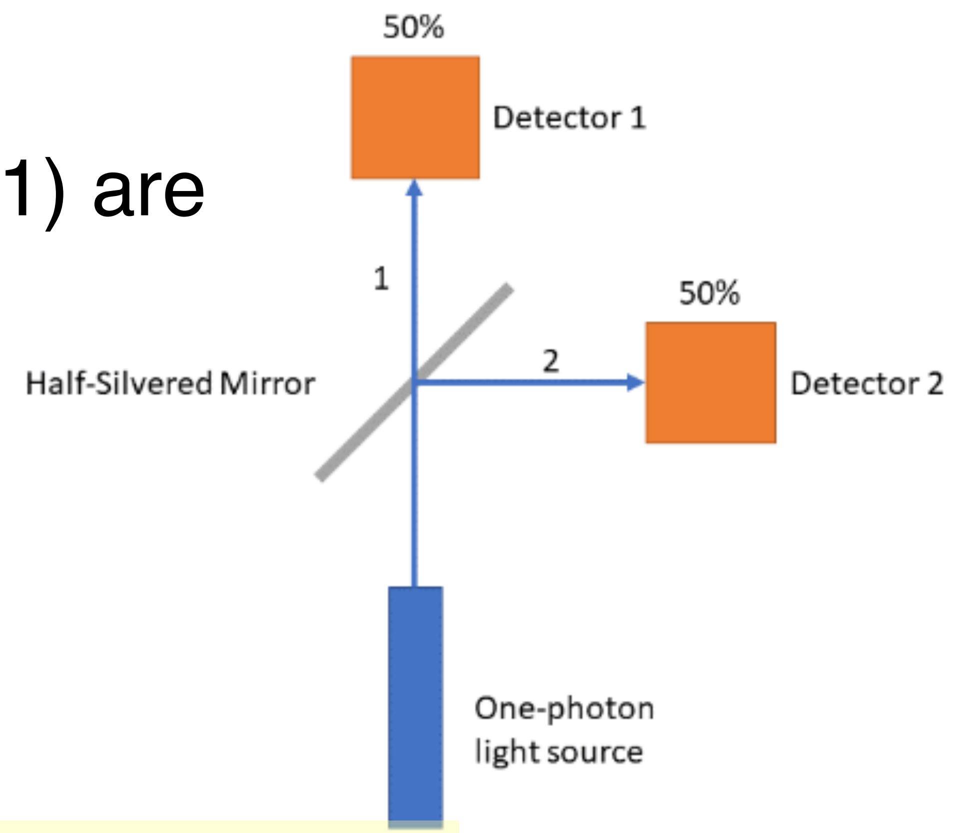
- What is random? → Principle of indifference

cover your eyes so that you'll not see the initial condition

Experiments: What is Random?

- **Experiment**: Procedure, model, and outcome

- Example: Photons & a semi-transparent mirror
  - Procedure: Shoot a photon to the mirror
  - Model: Reflection (0) and transmission (1) are equally likely
  - Outcome: 1



- What is random?

→ Intrinsic randomness in quantum physics

Sample Space Ω

- Sample space = set of all possible outcomes (for an experiment)
  - Use Ω to denote sample space

- Example: Sorting hat in Harry Potter, Ω = ?

Gryffindor, Slytherin, Ravenclaw, Hafflepuff

- Example: Toss a coin for 2 times, Ω = ?

HH,HT,TH,TT

H=Head

T=Tail

- Example: Student's grade for a course, Ω = ?

0<=x<=100

Sample Space: Countably Infinite Case

有某映射可使 N 双射到 Ω 就是 countably infinite

- Example: Toss a coin for infinite number of times and observe the number of heads. What is Ω?

$\mathbb{N} \cup \{0\}$

> **Definition**: Ω is said to be **countably infinite** if there exists a **1-to-1 correspondence** between Ω and the set of all positive integers $\mathbb{N}$

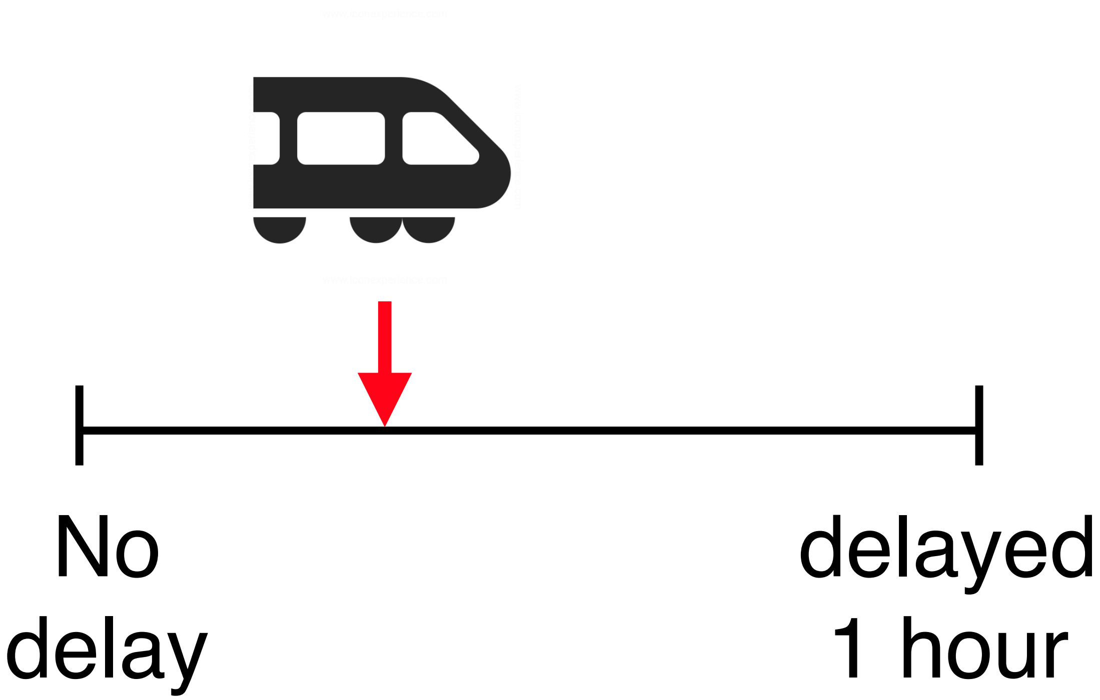
- Example: Ω = the set of all positive odd integers $\{1,3,5,\ldots\}$?

$\{2k-1 : k \in \mathbb{N}\}$

- Example: Ω = the set of tuples of positive integers $\{(p,q) : p,q \in \mathbb{N}\}$? $\{(1,1),(1,2),(1,3),\cdots\}$

Sample Space: Continuous Case

- Example: Train arrival time



- Possible outcome = ? $[0, 1]$
- Sample space = ? $[0, 1]$

cross interval means uncountably infinite

- Is the sample space finite? No
- Is the sample space countably infinite? No

Events

- Event → to describe the results of an experiment
- (Math) Event = a set of outcomes
- (Math) Event = a subset of sample space
- Probability → describe how likely an event will happen

- Example: tossing a coin 4 times
    - (1) The event of having at least 3 heads?

      {HHHT, HHTH, HTHH, THHH, HHHH}

    - (2) The event of having 0 head?

      {TTTT}

2. Review: Set Theory

Review: Set Operations

- Let's set up the notations!

Kuan-Yu A-Shin Masa

Stone Monster

- Universal set Ω ($\Leftrightarrow$ sample space)
  - Example: MayDay, Ω = {Monster, A-Shin, Masa, Stone, Kuan-Yu}
- Element ($\Leftrightarrow$ outcome)
  - Example: A-Shin is an element of MayDay → A-Shin $\in$ MayDay
- Subset ($\Leftrightarrow$ event)
  - Example: {A-Shin, Masa} is a subset of MayDay
  - {A-Shin, Masa} $\subseteq$ MayDay
- Empty set $\emptyset = \{\,\}$

Review: Set Operations

- Complement (S^c)
  - Example: $S = \{\text{Monster, A-Shin}\}$ is the complement of $S^c = \{\text{Masa, Stone, Kuan-Yu}\}$
- Union ($S \cup T$)
  - Example: $S = \{\text{Masa, A-Shin}\}$, $T = \{\text{Stone}\}$, then $S \cup T = \{\text{Masa, A-Shin, Stone}\}$
- Intersection ($S \cap T$)
  - Example: $S = \{\text{Masa, A-Shin, Stone}\}$, $T = \{\text{Stone}\}$, then $S \cap T = \{\text{Stone}\}$

Review: Set Operations

$T - S = \emptyset$

- Difference ($S - T$)
  - Example: $S =$ {Masa, Stone, Kuan-Yu}, $T =$ {Stone, Kuan-Yu}, then $S - T =$ {Masa}
- Disjoint: S, T are disjoint if $S \cap T = \emptyset$
  - Example: $S =$ {Masa, Stone}, $T =$ {A-shin} $\Rightarrow S, T$ are disjoint
- Mutually exclusive: $S_1, S_2, S_3, \cdots$ are mutually exclusive if $S_i \cap S_j = \emptyset$, for every pair i, j with $i \neq j$
  - Example: $S_1 =$ {Masa, Stone}, $S_2 =$ {A-shin}, $S_3 =$ {Monster} $\Rightarrow S_1, S_2, S_3$ are mutually exclusive

Review: Set Operations in Math

- Let S, T be two sets
  1. (Union): $S \cup T = \{x : x \in S \text{ or } x \in T\}$
  2. (Intersection): $S \cap T = \{x : x \in S \text{ and } x \in T\}$
  3. (complement): $S^c = \{x : x \notin S \text{ and } x \in \Omega\}$
  4. (subset): $S \subseteq T \Leftrightarrow$ For every $x \in S$, we have $x \in T$
  5. (equal): $S = T \Leftrightarrow S \subseteq T$ and $T \subseteq S$

Set Operations: Countable Union/Intersection

- Let $S_1, S_2, S_3 \cdots$ be a sequence of sets

1. $\bigcup_{n=1}^{\infty} S_n = \{x : x \in S_n, \text{ for some } n \in \mathbb{N}\}$

big cup

2. $\bigcap_{n=1}^{\infty} S_n = \{x : x \in S_n, \text{ for every } n \in \mathbb{N}\}$

A Useful Set Operation: Finding Elements that Appears in Infinitely Many Sets?

Let's consider "infinitely many"!

- Example: Avengers — Infinity War
  - Let $S_1, S_2, S_3 \cdots$ be an infinite sequence of sets
  - $S_n :=$ {Avenger members who survive in the n-th possible outcome}
  - Question: How to represent the set $\{x$: Avenger member x who survives in infinitely many possible outcomes$\}$?

- Example: Avengers — Infinity War
  - Let $S_1, S_2, S_3 \cdots$ be an infinite sequence of sets
  - $S_n :=$ {Avenger members who survive in the n-th possible outcome}
  - Question: How to represent the set {x: Avenger member x who survives in infinitely many possible outcomes}?
- Idea:

$y \in A$

$$\left\{\begin{array}{lll} \bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup S_3 \cup \cdots & y \in \bigcup_{n=1}^{\infty} S_n ? & \text{Yes!} \\ \bigcup_{n=2}^{\infty} S_n = S_2 \cup S_3 \cup S_4 \cup \cdots & y \in \bigcup_{n=2}^{\infty} S_n ? & \text{Yes!} \\ \bigcup_{n=100}^{\infty} S_n = S_{100} \cup S_{101} \cup S_{102} \cup \cdots & y \in \bigcup_{n=100}^{\infty} S_n ? & \text{Yes!} \end{array}\right.$$

$$\boxed{A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \quad !}$$

Set Operations: Countable Union/Intersection

- Let $S_1, S_2, S_3 \cdots$ be a sequence of sets

3. $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n = \{ x : x \in S_n, \text{ for infinitely many } n \}$

4. $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n = \{ x : x \in S_n, \text{ for almost all } n \text{ except finitely many } n \}$

↳ ex. $x \in S_1, S_3, S_4, S_5, S_6 \ldots\ldots$

- Show: $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_n = \{x : x \in S_k, \text{ for all except for finitely many } k\}$

De Morgan's Laws

- Let S_1, S_2 be two sets

1. $\left(S_1 \cup S_2\right)^c = S_1^c \cap S_2^c$

2. $\left(S_1 \cap S_2\right)^c = S_1^c \cup S_2^c$

- Prove this by Venn diagram

De Morgan's Laws (General Case)

- Let $S_1, S_2, S_3 \cdots$ be a sequence of sets

1. $\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$

2. $\left(\bigcap_n S_n\right)^c = \bigcup S_n^c$

3. Axioms of Probability

Probability Axioms

- In a probabilistic model, we assign probability to events (How?)
- Axioms: rules to verify a probabilistic model
- Example: 8 axioms of vector space in linear algebra

Axiom	Meaning
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 2]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

- Why are axioms useful?
- Can we prove axioms?

3 Axioms of Probability

A probability assignment is valid if:

1. $P(A) \geq 0$, for any event A
2. $P(\Omega) = 1$
3. $A_1, A_2, \cdots$ is an infinite sequence of <u>mutually exclusive</u> events, then $$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

- Can we find $P(\emptyset) = ?$
- $A_1, \cdots, A_n$ are disjoint events, then $P(\bigcup_{i=1}^{n} A_i) = ?$
- Do we have $P(A) \leq 1$, for any A?

Examples: Probability Assignment

- Example: $\Omega = \{1,2,3,4\}$
  - $P(\{1,2\}) = 3/4$
  - $P(\{1,3,4\}) = 7/8$
  - $P(\{1,3\}) = 1/2$
  - Can this be made a valid probability assignment?

- Example: $\Omega = \{0,1,2,3\cdots\}$
  - $P(\{k\}) = 2^{-k} \cdot |\cos(k\pi + \frac{\pi}{3})|$, for all k
  - Is this a valid probability assignment?

Discrete Uniform Probability Law

Theorem: Let Ω be the sample space of an experiment. If Ω has N elements that are equally likely to occur, then for any event A of Ω, we have

$$P(A) = \frac{\text{Number of elements in A}}{N}$$

- How to verify this using the axioms?

Recap: Probability of Rain?

- Experiment for probability of rain forecast:

例	降水量
1	0.1mm
2	0.0mm
3	4.8mm
4	0.3mm
5	0.0mm
6	1.2mm
7	0.0mm
8	2.4mm
9	0.9mm
10	0.5mm

- Procedure: Collect all historical data points of similar weather condition
- Model: All data points are equally likely to occur
- The rainy event = {rainfall $\geq$ 1mm}
- P(rainy event) = ?

Useful Properties

- $P(A^c) = 1 - P(A)$
- $P(A) = P(A - B) + P(A \cap B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If $A \subseteq B$, then $P(A) \leq P(B)$

Union Bound

- For any events $A_1, A_2, \cdots, A_n$, we have

$$P(\bigcup_{n=1}^{N} A_n) \leq \sum_{n=1}^{N} P(A_n)$$

1-Minute Summary

1. Probability model and basic terminology

- Experiment / outcome / sample space / event
- Probability axioms

2. Review: Set operations

- Definitions (union / intersection / disjoint / mutually exclusive)
- Algebra of sets and De Morgan's laws