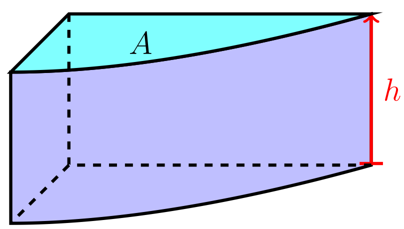
6.2 Volumes

3D立體: 體積篇

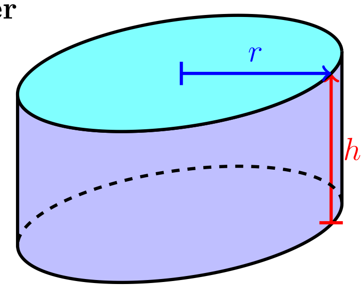
英語教室: solid [ˋsɑlɪd] 立體, volume [ˋvɑljəm] 體積,
cylinder [ˋsɪlɪdɚ] 柱, cone [kɔn] 錐, circular [ˋsɝkjəlɚ] 圓形的,
box [bɑx] 盒, rectangular [rɛkˋtæŋgjəlɚ] 矩形的,
sphere [sfɪr] 球, spherical [ˋsfɛrəkl] 球狀的,
perpendicular [ˏpɝpənˋdɪkjəlɚ] 垂直的, cross-section [krɔs-ˋsɛkʃən] 橫切面,
revolution [ˏrɛvəˋluʃən] 旋轉. disk [dɪsk] 圓盤, washer [ˋwɑʃɚ] 墊圈。

Cylinder



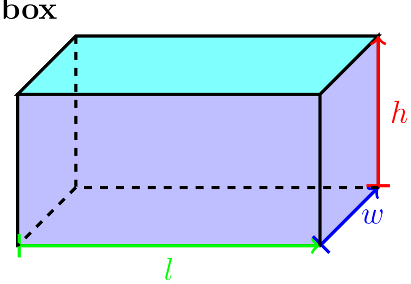
$$V = \boxed{A}\,h$$

Circular cylinder



$$V = \pi r^2 h$$

Rectangular box

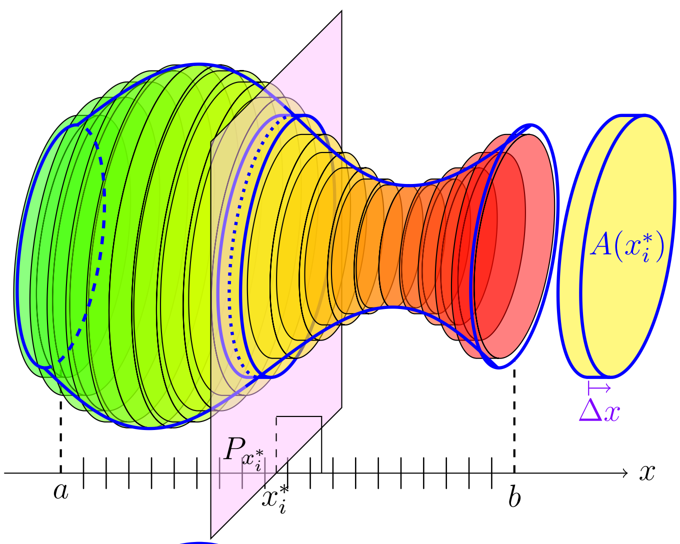


$$V = lwh$$

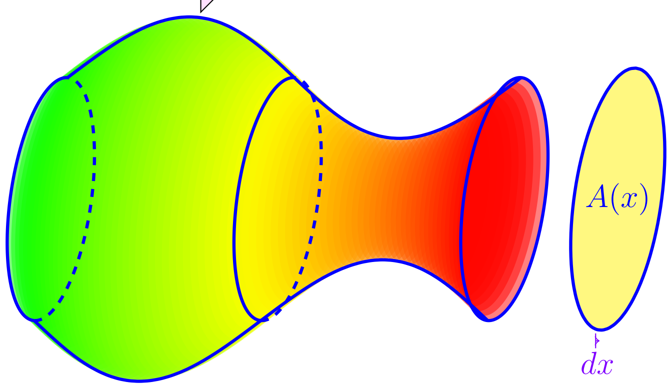
Note: $V(\text{錐}) = \dfrac{1}{3}V(\text{柱})$。

Define: Let S be a ***solid*** 立體 that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the ***volume*** 體積 of S is (體積是近似柱體積和的極限)

$$\boxed{\boldsymbol{V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\ dx}}$$

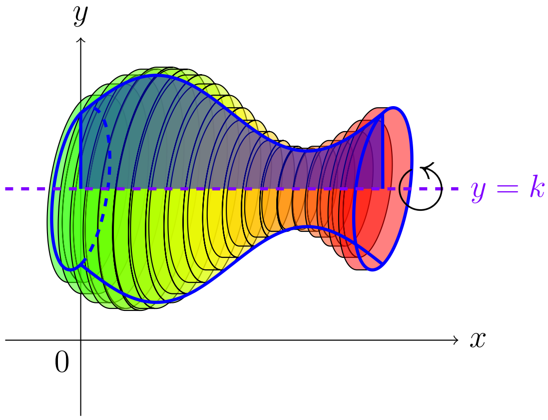


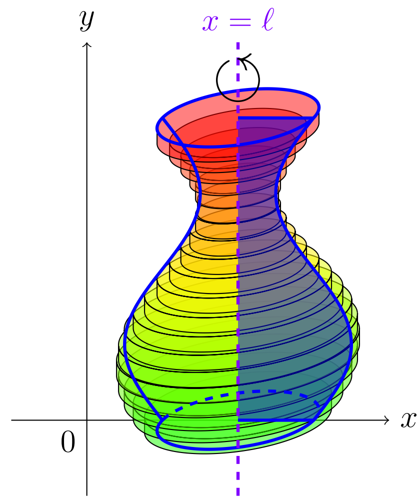
雪
切, 一片一片一片一片一片,
緣
拼出你我的積分。
我的愛因你而生,
手 摸
你的錯寫出我的心疼。



Note: 如果選擇的橫切面垂直 y-軸, 用 $V = \displaystyle\int_c^d A(y)\ dy$.

Define: The ***solid of revolution*** 旋轉體 is obtained by revolving a region about a line.





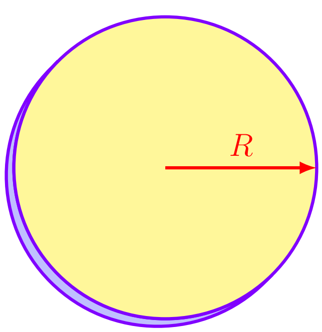
Note: 如果繞:

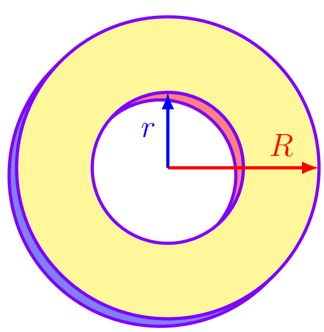
- horizontal line 水平線 (x-軸, $y = k$), 用 $V = \displaystyle\int_a^b A(x)\,dx$;
- vertical line 垂直線 (y-軸, $x = \ell$), 用 $V = \displaystyle\int_c^d A(y)\,dy$.

(Why? 切面積好算!)

If the cross-sectional area $A(x)$ or $A(y)$ is:

- a ***disk*** 圓盤, then $\boxed{\boldsymbol{A = \pi R^2}}$, or
- a ***washer*** 墊圈, then $\boxed{\boldsymbol{A = \pi R^2 - \pi r^2}}$





Skill: 如何列式: $\boxed{\boldsymbol{V} = \int_{③}^{③} \underbrace{\boldsymbol{A}(②)}_{\text{截面積}} \underbrace{\boldsymbol{d}①}_{\text{厚度}}}$

① 厚度(thickness) 是往 x/y-軸方向就 dx/dy。
② 截面積(cross-sectional area) 跟著變成 x/y 的函數。
③ 上下限(upper/lower limits) 找 x/y 的範圍。

Example 0.1 *Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.*

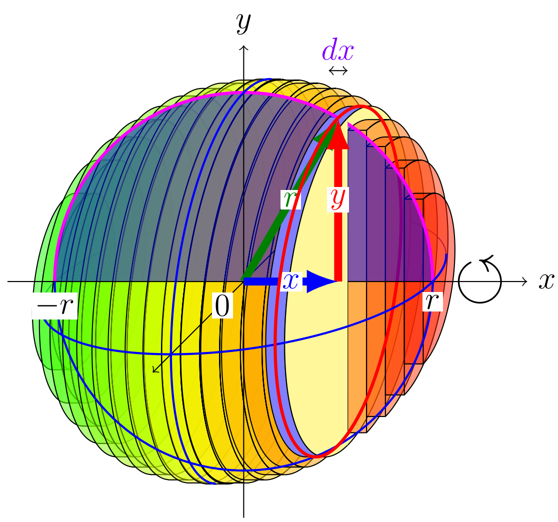
(把球想像成半圓繞直徑的旋轉體。)

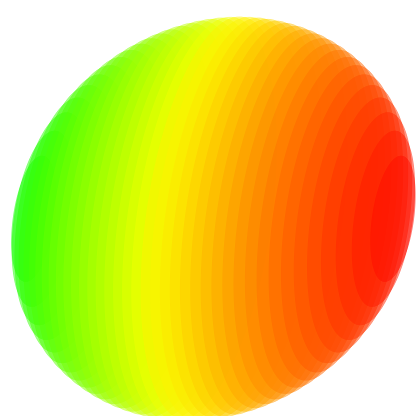
$x^2 + y^2 = r^2$, $A(x) = \pi y^2 = \pi(r^2 - x^2)$. *(對 x 積分, 要換成 x 的函數。)*

$$V = \int_{-r}^{r} A(x)\ dx = \int_{-r}^{r} \pi(r^2 - x^2)\ dx$$

$$= \left\langle \begin{array}{ll} \left[\pi r^2 x - \pi\frac{x^3}{3}\right]_{-r}^{r} = [\pi r^3 - \pi\frac{r^3}{3}] - [\pi r^2(-r) - \pi\frac{(-r)^3}{3}] & \text{(直接算)} \\ 2\pi\int_0^r (r^2 - x^2)\ dx = 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) & \text{(偶函數)} \end{array} \right\rangle$$

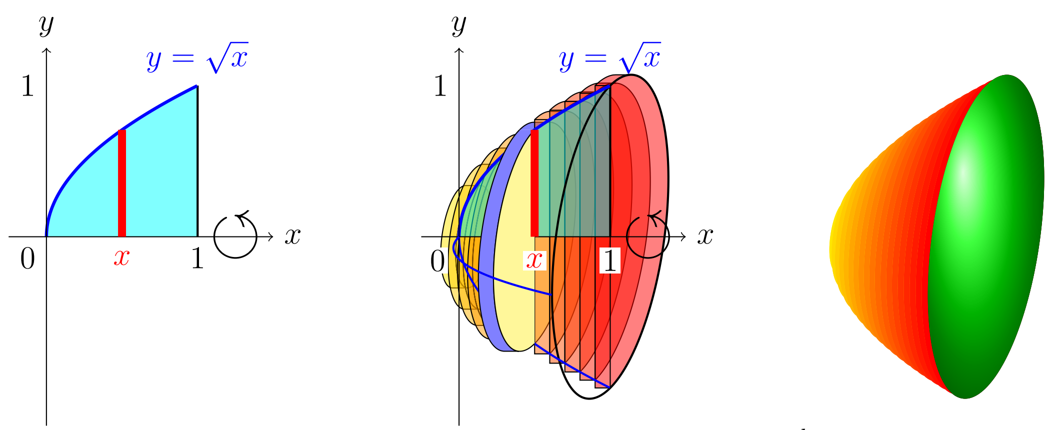
$$= \frac{4}{3}\pi r^3. \qquad \blacksquare$$





Example 0.2 *(x-axis) Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.*

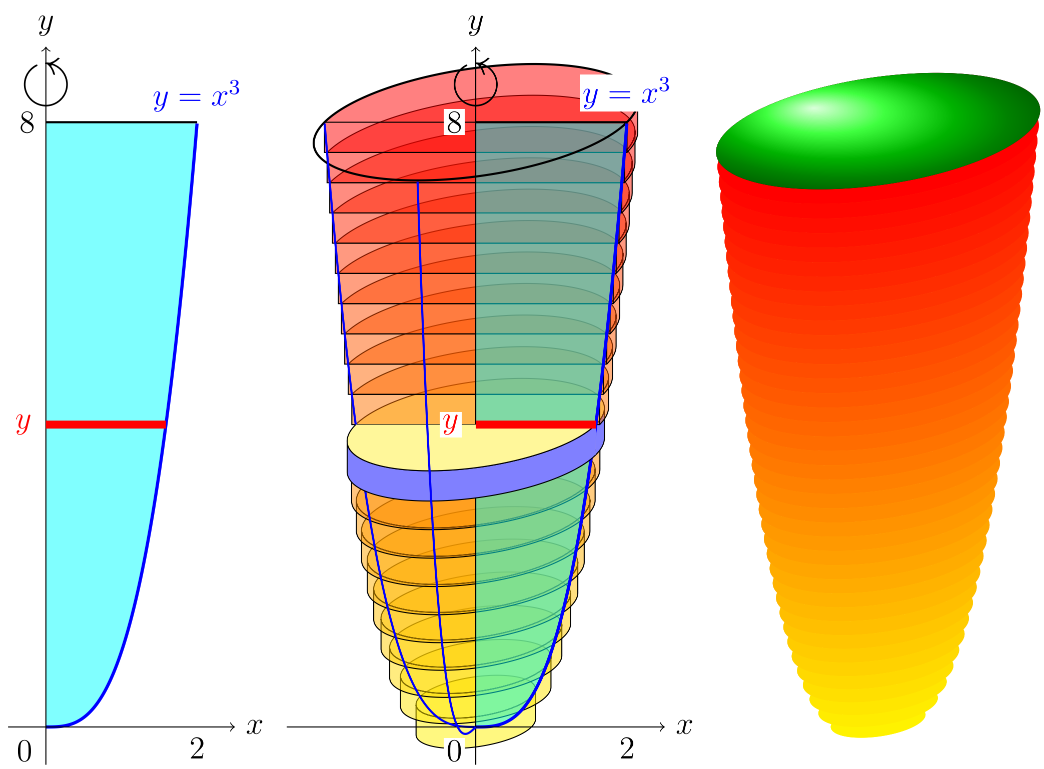
$A(x) = \pi(\sqrt{x})^2 = \pi x.$
the volume of the approximating cylinder is $A(x)\Delta x = \pi x \Delta x$.
$$V = \int_0^1 \pi x \; dx = \left[\frac{\pi x^2}{2}\right]_0^1 = \frac{\pi}{2}. \qquad \blacksquare$$



Note: $y = f(x)$ 繞 x-軸 旋轉體: 面積是 πy^2, 體積是 $\displaystyle\int_a^b \pi[f(x)]^2 \; dx$.
不要背! 用畫圖找圓半徑。

Example 0.3 *(y-axis) Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.*

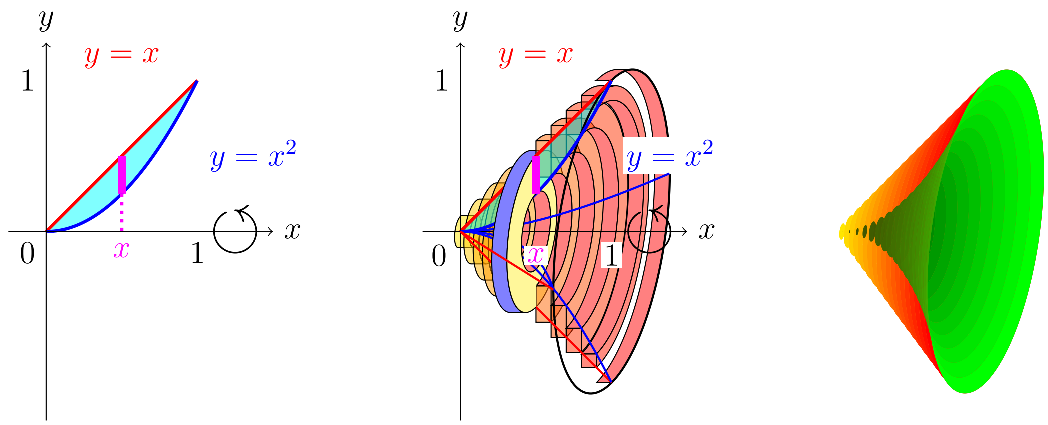
$A(y) = \pi x^2 = \pi y^{2/3}$, (解反函數: $y = x^3 \to x = \sqrt[3]{y} = y^{1/3}$.)
the volume of the approximating cylinder is $A(x)\Delta x = \pi y^{2/3} \Delta y$.
$V = \int_0^8 \pi y^{2/3}\, dy = \left[\frac{3}{5}\pi y^{5/3}\right]_0^8 = \frac{96\pi}{5}$. ■



Note: $y = f(x)$ 繞 y-軸 旋轉體: 面積是 πx^2, 體積是 $\int_a^b \pi[f^{-1}(y)]^2\, dy$.
還是不要背! 用畫圖找圓半徑。

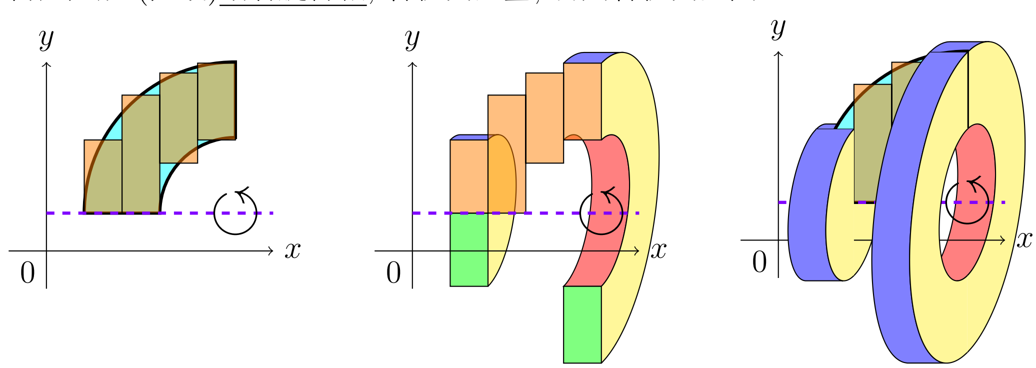
Example 0.4 *(washer) The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.*

Solve $x = y = x^2$, $(x, y) = (0,0), (1,1)$. (上下界是從 0 到 1。)
$x \geq x^2$ *on* $[0,1]$, 外圈半徑是 $y = x$, 內圈半徑是 $y = x^2$。
$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$, (繞 x-軸, 對 x 積。)
$$V = \int_0^1 \pi(x^2 - x^4)\ dx = \left[\pi(\frac{x^3}{3} - \frac{x^5}{5})\right]_0^1 = \frac{2\pi}{15}. \qquad \blacksquare$$



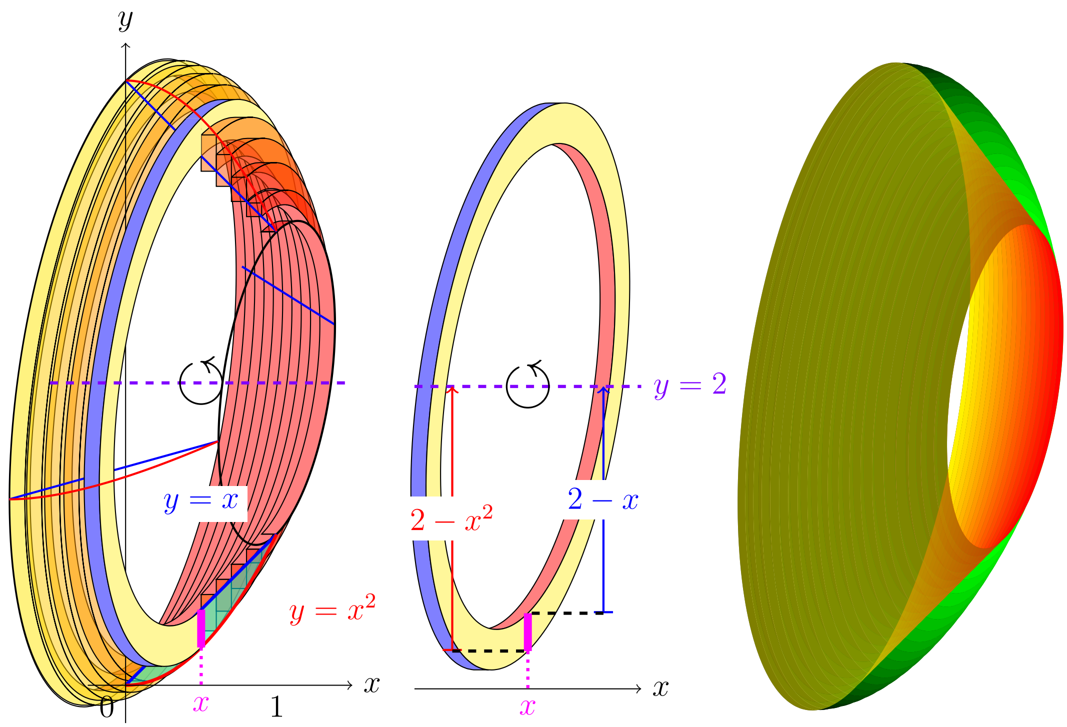
Question: 什麼時候會是圓盤/墊圈?

Answer: 用長方形去近似區域, 長方形繞出的體積近似區域繞出的體積。當長方形 (區域)貼著旋轉軸, 會繞出圓盤; 否則會繞出墊圈。



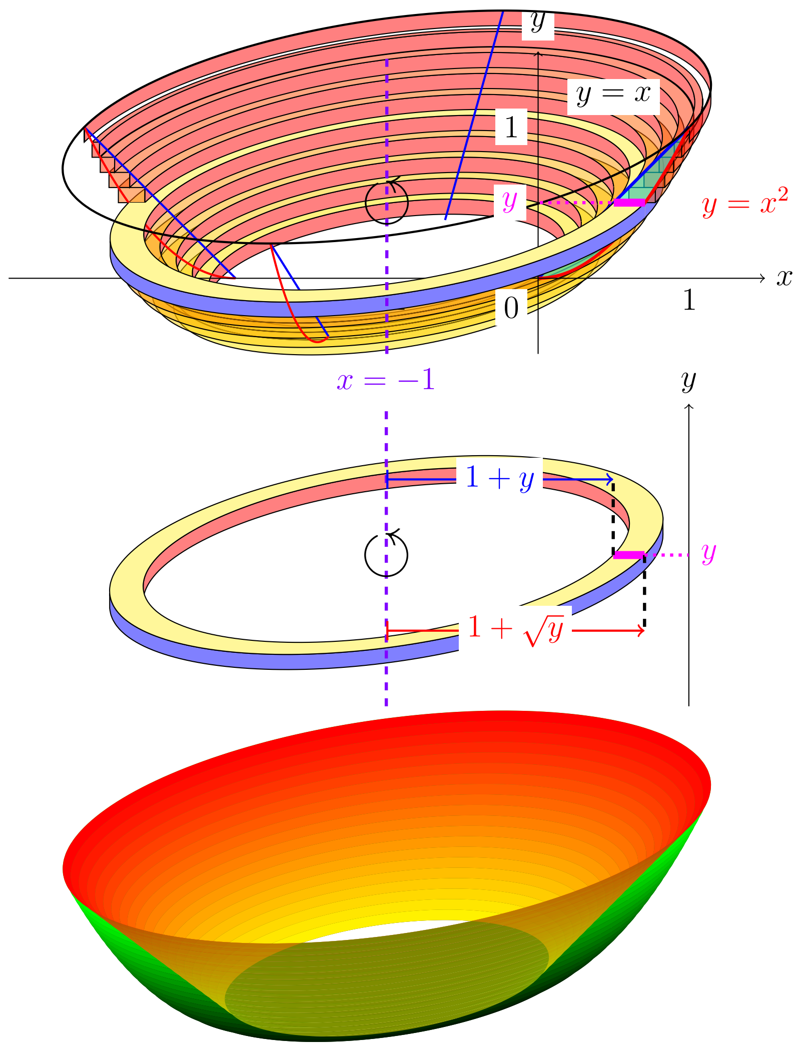
Example 0.5 *(horizontal line) Find the volume of the solid obtained by rotating the region in Example 4 about the line $y = 2$.*

$2 - x^2 \geq 2 - x$ *on* $[0, 1]$, 外圈半徑變成 $2 - x^2$, 內圈半徑變成 $2 - x$。
$A(x) = \pi(2 - x^2)^2 - \pi(2 - x)^2 = \pi(x^4 - 5x^2 + 4x)$, *(*繞 $y = 2$, 對 x 積。*)*
$V = \int_0^1 \pi(x^4 - 5x^2 + 4x)\ dx = \left[\pi(\frac{x^5}{5} - \frac{5x^3}{3} + 2x^2)\right]_0^1 = \frac{8\pi}{15}$. ■



Example 0.6 *(vertical line) Find the volume of the solid obtained by rotating the region in Example 4 about the line $x = -1$.*

$x = \sqrt{y}$ *and* $x = y$. (解反函數)
$1 + \sqrt{y} \geq 1 + y$ *on* $[0, 1]$, 外圈半徑變成 $1 + \sqrt{y}$, 內圈半徑變成 $1 + y$。
$A(y) = \pi(1 + \sqrt{y})^2 - \pi(1 + y)^2 = \pi(2\sqrt{y} - y - y^2)$, (繞 $x = -1$, 對 y 積。)
$$V = \int_0^1 \pi(2\sqrt{y} - y - y^2)\, dy = \left[\pi(\frac{4}{3}y^{3/2} - \frac{1}{2}y^2 - \frac{1}{3}y^3)\right]_0^1 = \frac{\pi}{2}.$$ ■



Example 0.7 *A solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral(等邊) triangles. Find the volume of the solid.*

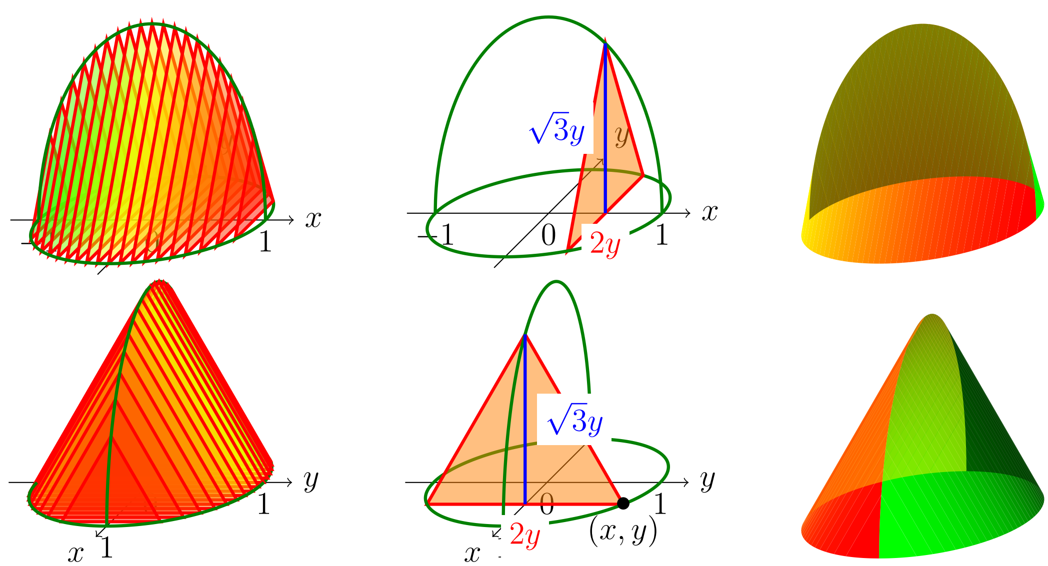
$$x^2 + y^2 = 1,\ A(x) = \frac{1}{2} \cdot {\color{red}2y} \cdot {\color{blue}\sqrt{3}y} = \sqrt{3}(1 - x^2),$$

$$V = \int_{-1}^{1} A(x)\ dx = \int_{-1}^{1} \sqrt{3}(1 - x^2)\ dx$$

$$= \left\langle \begin{array}{ll} \left[\sqrt{3}(x - \frac{x^3}{3})\right]_{-1}^{1} = [\sqrt{3}(1 - \frac{1}{3})] - [\sqrt{3}(-1 + \frac{1}{3})] & \textit{(直接算)} \\ 2\int_{0}^{1} \sqrt{3}(1 - x^2)\ dx = 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_{0}^{1} = 2\sqrt{3}(1 - \frac{1}{3}) & \textit{(偶函數)} \end{array} \right\rangle$$

$$= \frac{4\sqrt{3}}{3}.$$

■

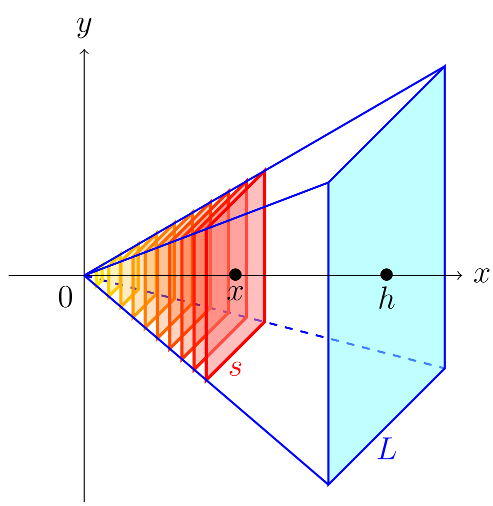


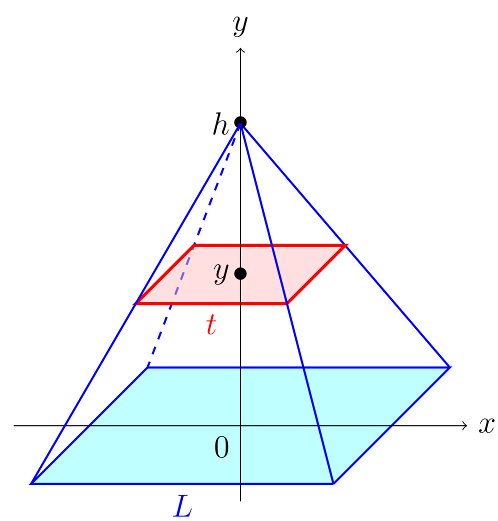
Example 0.8 *Find the volume of a pyramid whose base is a square with side L and whose height is h.*

[Sol 1] 把頂點放在原點, x-軸是中心軸, 假設在 x 時的截方形邊長是 s。
$\frac{x}{h} = \frac{s/2}{L/2}$, $A(x) = s^2 = \frac{L^2}{h^2}x^2$.

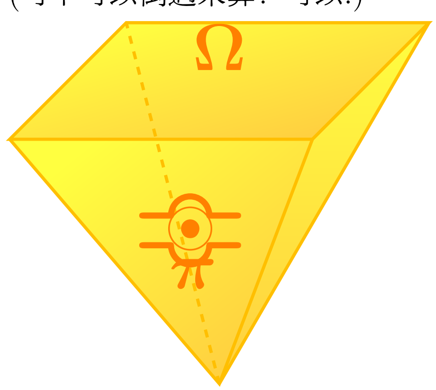
$$V = \int_0^h A(x)\ dx = \int_0^h \frac{L^2}{h^2}x^2\ dx = \frac{L^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{L^2h}{3}.$$

[Sol 2] 把底部中心放在原點, 假設 y 高時的截方形邊長是 t。
$\frac{h-y}{h} = \frac{t/2}{L/2}$, $A(y) = t^2 = \frac{L^2}{h^2}(h-y)^2$. $V = \int_0^h A(y)\ dy = \cdots = \frac{L^2h}{3}$. ■

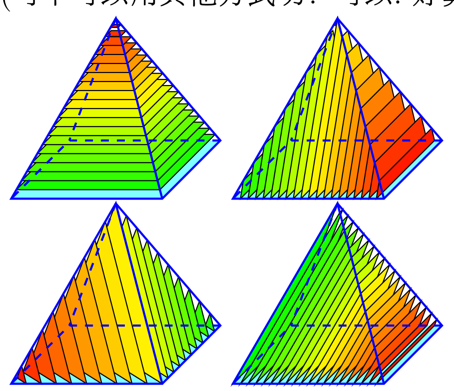




(可不可以倒過來算? 可以!)



(可不可以用其他方式切? 可以! 好算嗎?)



♦: Volume of cone $= \frac{1}{3}$ volume of cylinder

錐體積 $= \lim\limits_{n\to\infty} \sum$ 四角錐體積 $= \lim\limits_{n\to\infty} \sum \frac{1}{3}$ 四角柱體積 $= \frac{1}{3}$ 柱體積。