1179: Probability
Lecture 12 — Continuous Random Variables

Ping-Chun Hsieh (謝秉均)

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Announcements

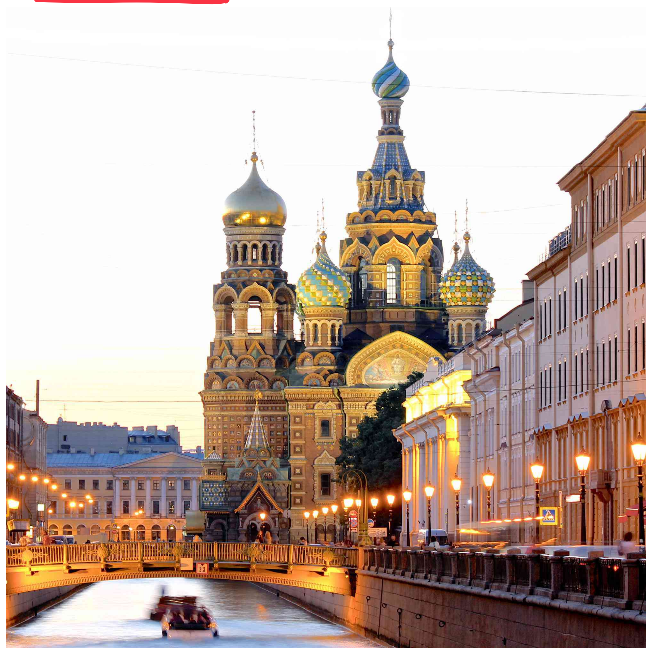
- HW2 has been posted on E3 (Due: 11/1, 9pm)

- Midterm on 11/10 (on Wednesday, in class)
  - 10:10am - 12:10pm
  - Coverage: Lec 1 - Lec 16
  - You are allowed to bring a cheat sheet (A4 size, 2-sided, without any attachments)
  - Locations to be announced

St. Petersburg Paradox

1st toss = win 2 dollars, $p=\frac{1}{2}$
2nd toss = win 2^2 dollars, $p=\frac{1}{4}$
3rd toss = win 2^3 dollars, $p=\frac{1}{8}$

- Example: We are asked to pay 10000 dollars to play a game.
  - We can keep flipping a fair coin until a head is observed.
  - If the 1st head occurs at n-th toss, then we get a prize of 2^n dollars and the game is over.
  - Shall we play this game?



Define X = # of tosses until 1st head

Then, $X \sim \text{Geometric}\left(\frac{1}{2}\right)$

The prize we get $= 2^X$

$$E[2^X] \overset{\text{LOTUS}}{=} \sum_{k=1}^{\infty} 2^k \cdot p_X(k) = \infty$$

$$2^k \cdot \left(\frac{1}{2}\right)^k = 1$$

Quick Review

- Existence of moments?

- Mean and variance of a Bernoulli random variable?

This Lecture

1. Expected Value and Variance of Special Discrete Random Variables

2. Continuous Random Variables

- Reading material: Chapter 6.1

2. Binomial Random Variables

$Var[X] = E[X^2] - (E[X])^2$

PMF: $P(X=k) = \begin{cases} C_k^n \cdot p^k \cdot (1-p)^{n-k}, & k=0,1,\dots,n \\ 0 & \text{otherwise} \end{cases}$

- Example: $X \sim \text{Binomial}(n, p)$
  - How to show that $E[X] = np$?
  - How to show that $\text{Var}[X] = np(1-p)$?

$$E[X^2] = \sum_{k=1}^{n} C_k^n \cdot p^k \cdot (1-p)^{n-k} \cdot k^2$$

$$= \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} p^k \cdot (1-p)^{n-k} \cdot k^2$$

$k^2 = k(k-1) + k$

$$= \left(\sum_{k=1}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \cdot k(k-1) \right) + \left(E[X] \right)$$

$n(n-1)$, $(k-2)!$

$$= n \cdot (n-1) \cdot p^2 \cdot \boxed{\left(\sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2} (1-p)^{n-k} \right)} + E[X] = n(n-1)p^2 + np$$

$Var[X]$
$= E[X^2] - (E[X])^2$
$= n \cdot (n-1)p^2 + np - (np)^2$
$= np(1-p)$

Tricks For Deriving $E[X]$ and $\mathrm{Var}[X]$?

1. Reuse $\underbrace{\sum_x p(x) = 1}$ and $\underbrace{E[X] = \sum_x xp(x)}$

2. View X as a sum of independent random variables

3. Moment generating functions $E[e^{tX}]$

$E[X^n]$

3. Poisson Random Variables

$E[X^n]$

$$\sum_{m=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^m}{m!} = 1$$

- Example: $X \sim \text{Poisson}(\lambda, T)$
  - How to show that $E[X] = \lambda T$?
  - How to show that $\text{Var}[X] = \lambda T$?

PMF: $P(X=k) = \begin{cases} \frac{e^{-\lambda T} \cdot (\lambda T)^k}{k!}, & k=0,1,\ldots \\ 0, & \text{otherwise} \end{cases}$

$$E[X] = \sum_{k=1}^{\infty} \cancel{k} \cdot \frac{e^{-\lambda T}(\lambda T)^k}{\cancel{k!}\,(k-1)!} = \lambda T \cdot \boxed{\sum_{k=1}^{\infty} \frac{e^{-\lambda T}(\lambda T)^{k-1}}{(k-1)!}} = \lambda T$$

k^3

$$E[X^2] = \sum_{k=1}^{\infty} \underbrace{k^2}_{k(k-1)+k} \cdot \frac{e^{-\lambda T}(\lambda T)^k}{k!}$$

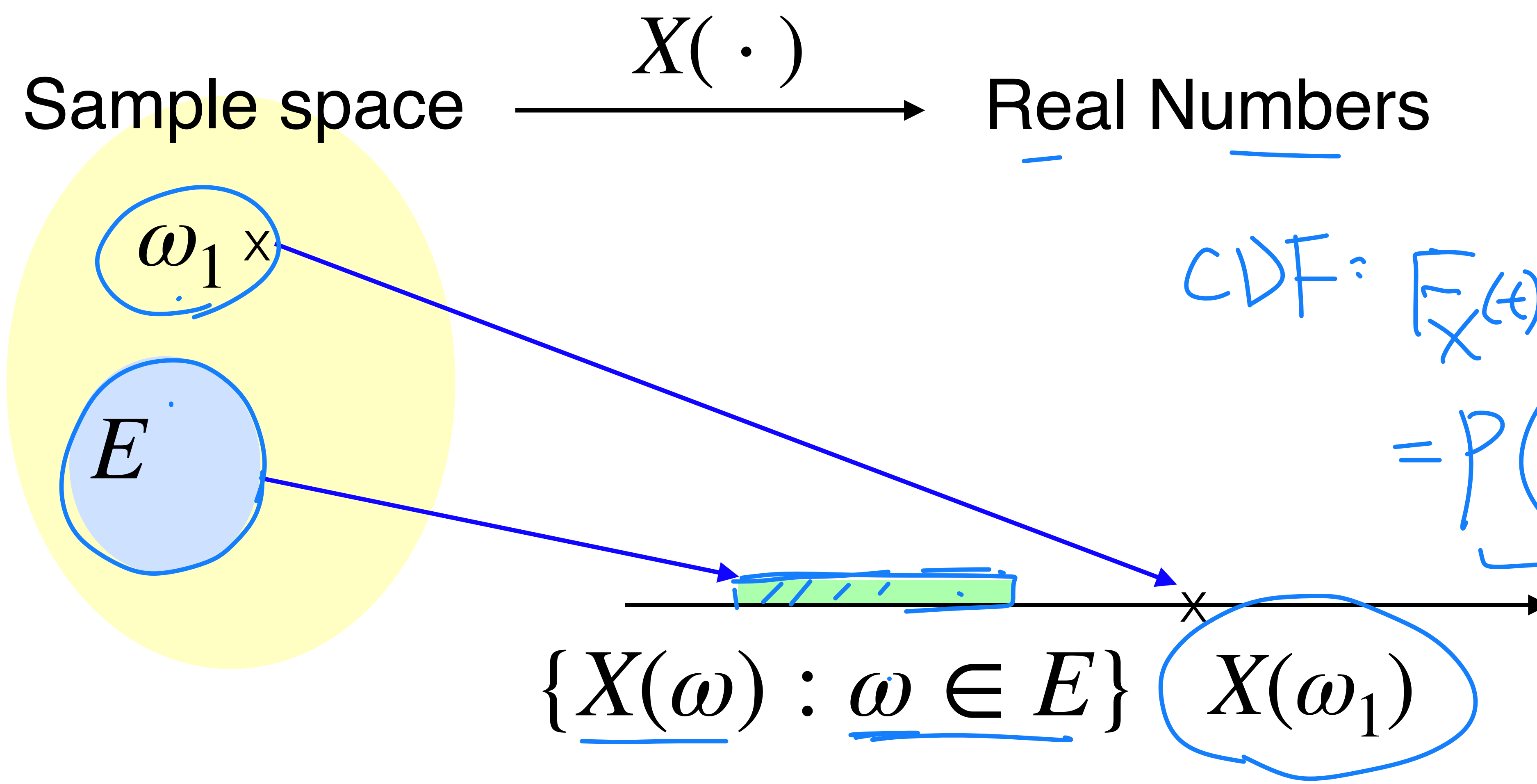
$$= \underbrace{\left(\sum_{k=1}^{\infty} \cancel{k(k-1)} \frac{e^{-\lambda T}(\lambda T)^k}{\cancel{k!}\,(k-2)!}\right)}_{(\lambda T)^2} + \left(E[X]\right) = (\lambda T)^2 + \lambda T$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = \lambda T$$

Continuous Random Variables and Probability Density Functions

Continuous Random Variables

- Continuous random variable: A random variable that takes values over a continuous range



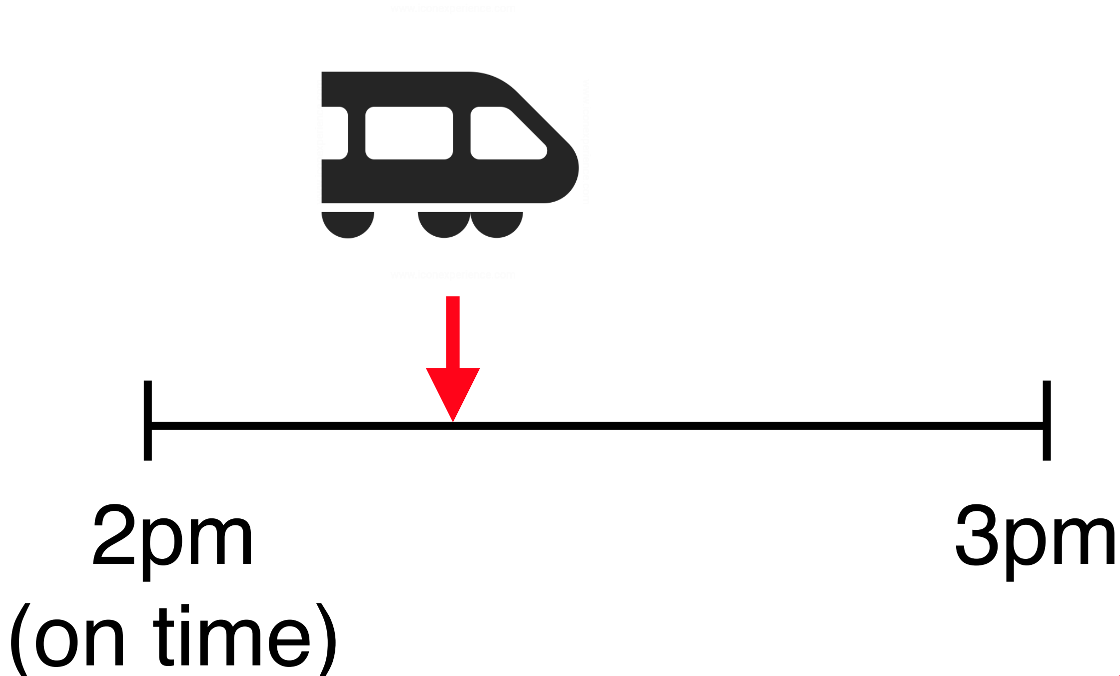
CDF: $F_X(t) = P(X \le t)$

$= P(\underbrace{\{\omega : X(\omega) \le t\}})$

- CDF is still available for a continuous random variable
- How about PMF?

Continuous Random Variables and PMF?

- Example: Train arrival time is between 2pm-3pm (equally likely)



- How to define a random variable?

Define $X = \dfrac{\text{the time difference between arrival time and 2pm (in mins)}}{60 \text{ minutes}}$

Possible values of X: $[0, 1]$

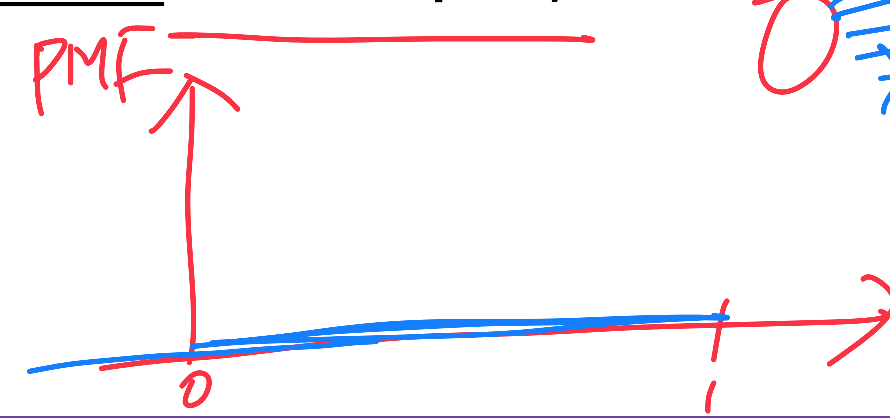
$P(X = \frac{1}{3}) = 0$

0.3333333

- P(arrives at exactly 20 min 31.537 sec after 2pm) = ?

0

$(\frac{1}{10})^n$



Density / Concentration

- Example: B-52 Cocktail

orange liqueur (40%): 10 ml

milk wine (17%): 10 ml

coffee liqueur (23%): 10 ml

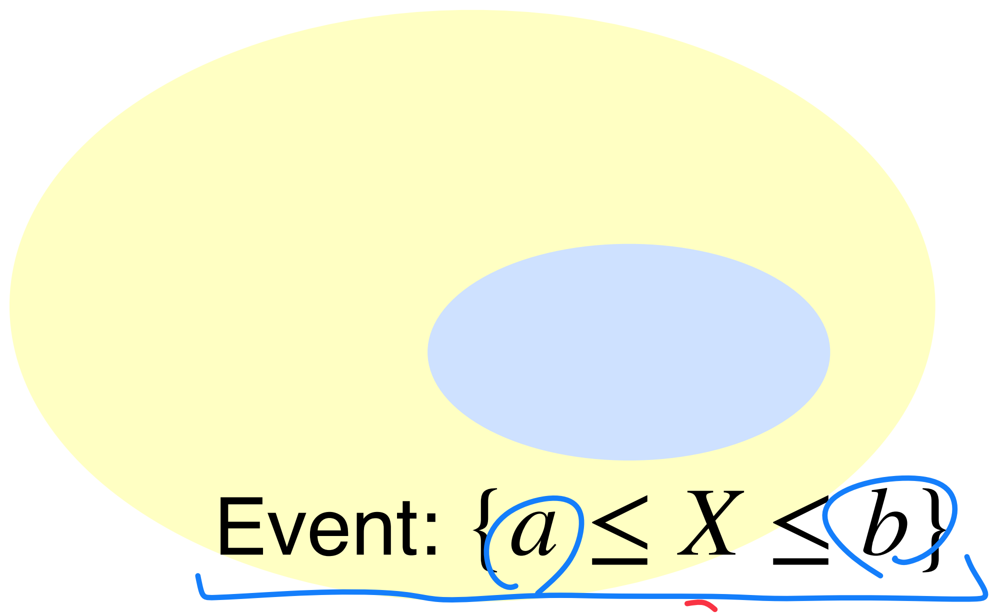
$10 \times 40\% + 10 \times 17\% + 10 \times 23\%$
$= 8ml.$

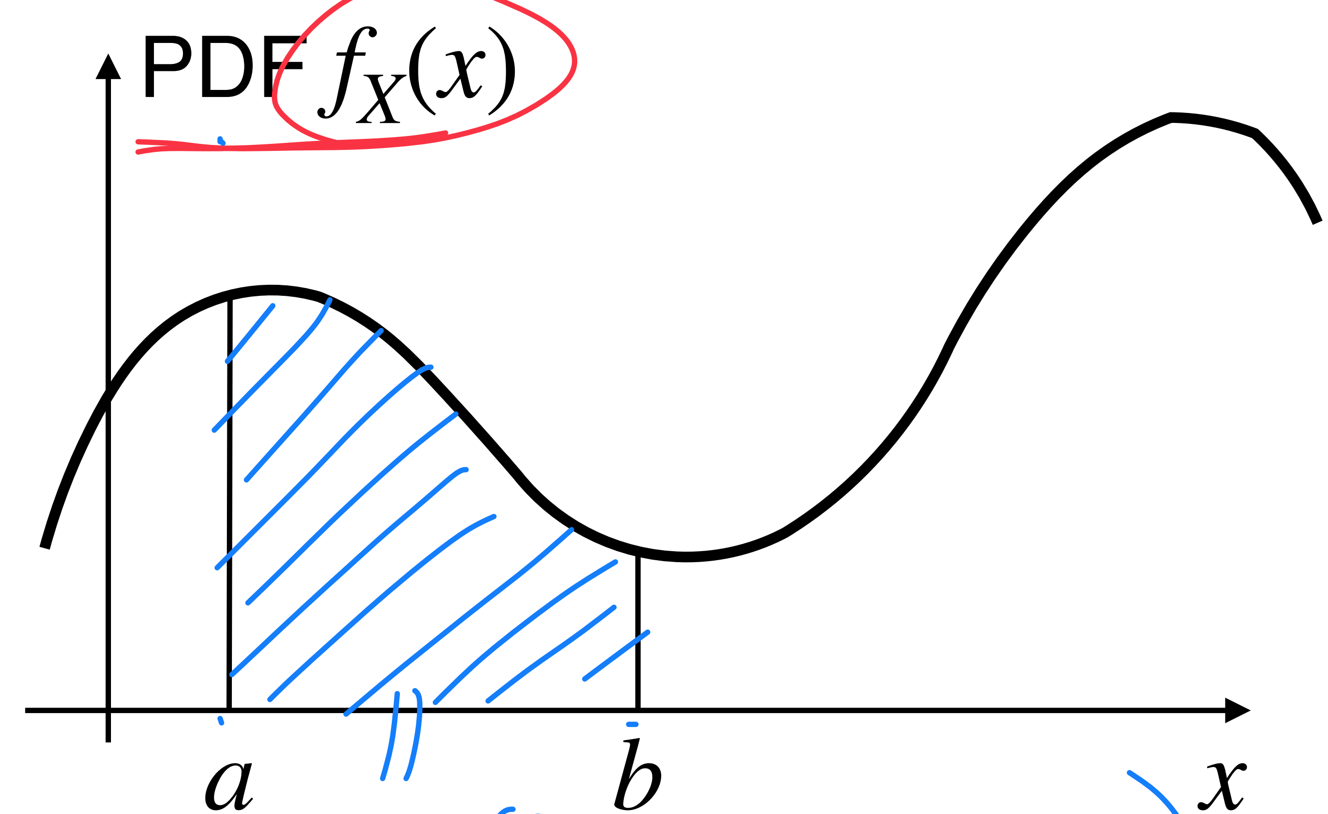
- How much alcohol in total (in ml)?

- How much alcohol in the <u>green</u> cross section?

Probability Density Function (PDF)

Sample space





$P(\{\omega : a \leq X(\omega) \leq b\})$

Probability Density Function (PDF):
Let X be a random variable. Then, $f_X(x)$ is the PDF of X if for every subset B of the real line, we have

$$P(\{\omega : X(\omega) \in B\}) \quad P(X \in B) = \int_B f_X(x)dx$$

$B = [1, 3]$

Express Other Quantities Using PDF

1. $P(X \in \mathbb{R}) =$

2. $P(X \leq t) =$

3. $P(a \leq X \leq b) =$

4. $P(a \leq X < b) =$

5. $P(a < X < b) =$

How to Check if a PDF is Valid?

- Recall: 3 Axioms of Probability

1. $P(X \in \mathbb{R}) = 1$

2. $P(X \in A) \geq 0$, for all A

3. Let $A_1, A_2, \cdots$ be mutually exclusive sets of real numbers, then

$$P(X \in \bigcup_{i \geq 1} A) = \sum_{i \geq 1} P(X \in A_i)$$

Example: From PDF to CDF (I)

- Example: Consider the following PDF

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}|x-3| & ,1 \leq x \leq 5 \\ 0 & \text{, otherwise} \end{cases}$$

- Is $f(x)$ a valid PDF of some random variable?

Example: From PDF to CDF (II)

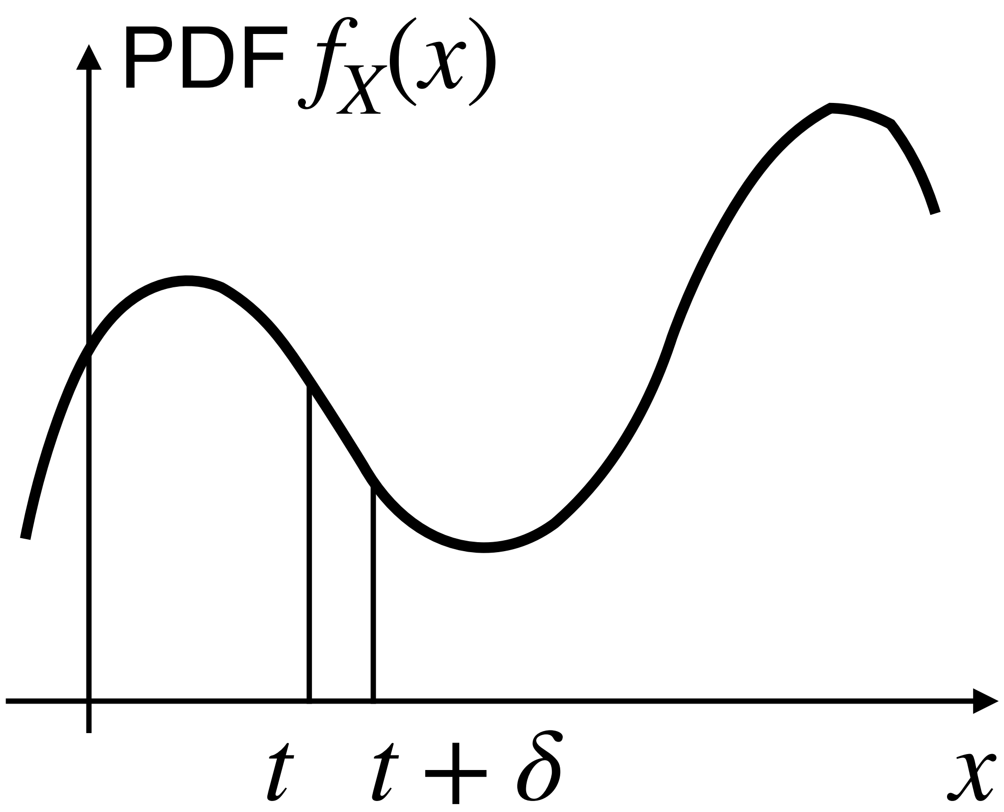
- Example: Consider the following PDF

$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & ,0 < x \le 1 \\ 0 & \text{, otherwise} \end{cases}$$

- Is $f(x)$ a valid PDF of some random variable?

From CDF to PDF

- CDF: $F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(x)dx$



- Suppose PDF is continuous
- $F_X(t + \delta) - F_X(t)$ = ?

From CDF to PDF (Formally)

Derivative of CDF is PDF:
Let X be a random variable with a CDF $F_X(\cdot)$ and a PDF $f_X(\cdot)$. If $f_X(\cdot)$ is continuous at x_0, then

$$F'_X(x_0) = f_X(x_0)$$

- Any similar results in calculus?

1-Minute Summary

2. Expected Value and Variance of Special Discrete Random Variables

- Bernoulli / Binomial / Poisson

3. Continuous Random Variables

- Probability density function
- PDF and CDF