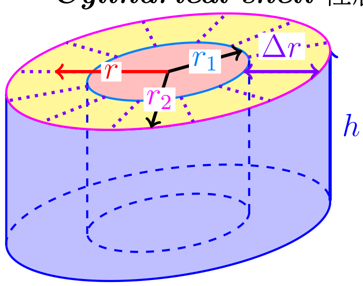
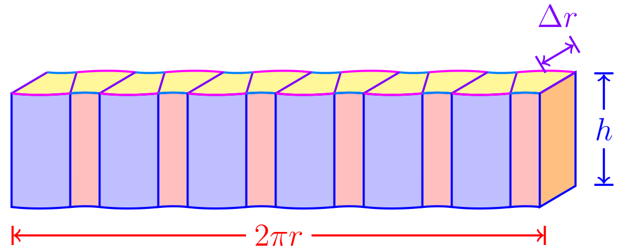
6.3 Volumes by cylindrical shells

另一種求體積法: 剝殼法 (洋蔥)

英語教室: cylindrical [səˈlɪdrɪkl] 柱狀的, shell [ʃɛl] 殼。

Cylindrical shell 柱狀殼:





$$V = \pi {r_2}^2 h - \pi {r_1}^2 h = 2\pi \frac{r_2 + r_1}{2}\, h(r_2 - r_1) = 2\pi r h \Delta r,$$

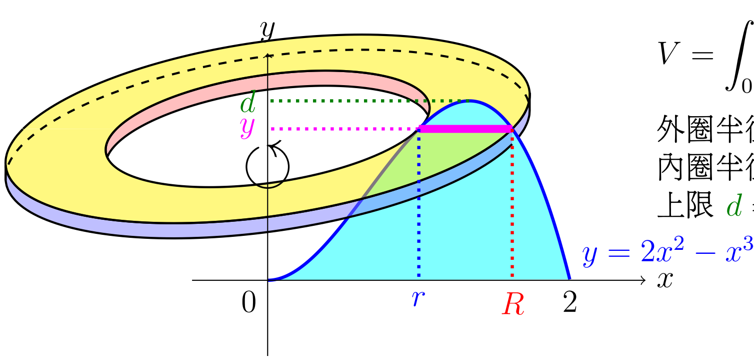
where $r = \dfrac{r_2 + r_1}{2}$ the average radius 平均半徑 of the shell

and $\Delta r = r_2 - r_1$ the thickness 厚度 of the shell.

體積		(平均) 圓周長		高度		厚度
Volume	=	Circumference	×	Height	×	Thickness

當 $y = f(x)$ 繞著 y-軸 or 垂直線 $x = a$, 體積用 disk/washer 對 y 積分: 圓盤法 $V = \int_c^d A(y)\, dy$ 但是有時候很難去算出 $x = f^{-1}(y)$ 來得到內/外半徑。

Example 0.1 *Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.*



$$V = \int_0^{?} \pi (R^2 - r^2)\, dy$$

外圈半徑 $R =?$

內圈半徑 $r =?$

上限 $d =?$

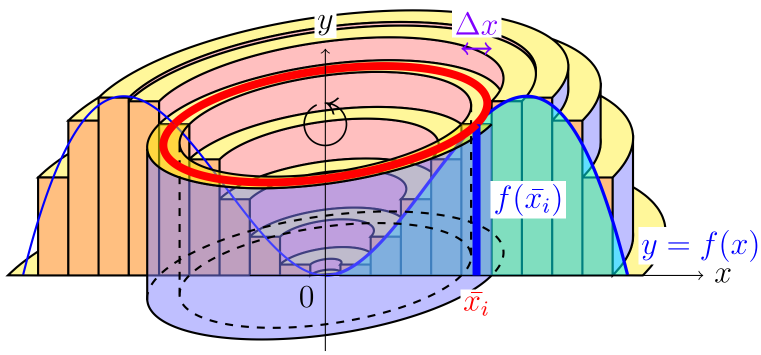
Note: 算出 $x = f^{-1}(y)$ 要解三次方程, 還要算出上下限 (極值 0, $d = \dfrac{32}{27}$), 判斷左右的函數, 是非常的複雜。

The ***method of cylindrical shells*** 剝殼法:

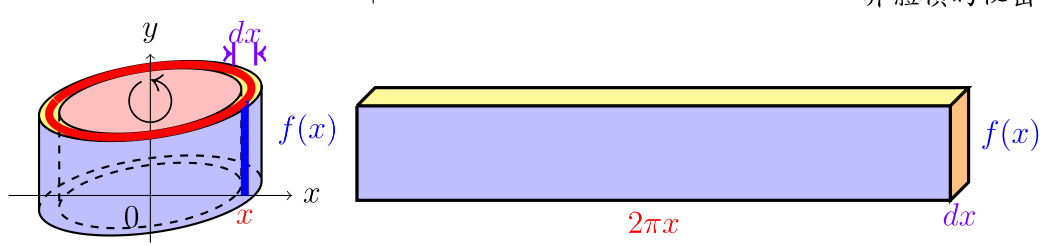
Let S 是由 $y = f(x)(\geq 0)$, $y = 0$, $x = a$, $x = b(> a \geq 0)$ 所圍區域, 繞 y-軸所成。

把 $[a, b]$ 分成 n 等分, $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$. 考慮中點 $\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$. Let V_i 是面積 $f(\bar{x}_i)\Delta x$ 的第 i 條長方形繞 y-軸的體積, then

$$\begin{aligned} V_i &= 2\pi \bar{x}_i f(\bar{x}_i)\Delta x, \\ V \approx \sum_{i=1}^{n} V_i &= \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x, \\ V &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x. \end{aligned}$$



如果你願意
一層一層一層
的剝開我的心
你會發現
你會訝異
你　　最壓抑
這是我剝殼法
最深處
算體積的秘密



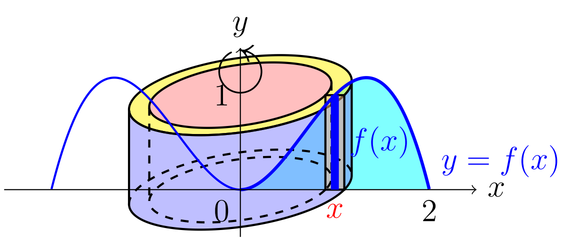
Theorem 1 (Method of cylindrical shells) *The volume of the solid obtained by rotating about the y-axis the region bounded under the curve $y = f(x)$ from a to b, where $b > a \geq 0$, is (*體積是近似柱狀殼體積和的極限*)*

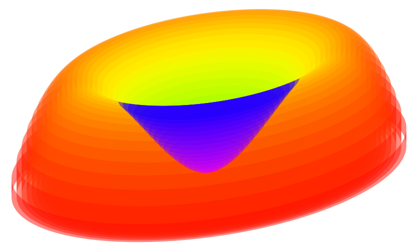
$$\boxed{\boldsymbol{V = \int_a^b 2\pi x f(x)\ dx}}$$

Note:
$y = f(x)$ 繞 y-軸, $V = \int 2\pi x f(x)\ dx$. 橫著剝, 對 x 積。
$x = f(y)$ 繞 x-軸, $V = \int 2\pi y f(y)\ dy$. 縱著剝, 對 y 積。

Example 0.2 *(Continuous)* $f(x) = 2x^2 - x^3$.

$$V = \int_0^2 2\pi x(2x^2 - x^3)\ dx = 2\pi \int_0^2 2x^3 - x^4\ dx = 2\pi\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = \frac{16\pi}{5}. \qquad \blacksquare$$





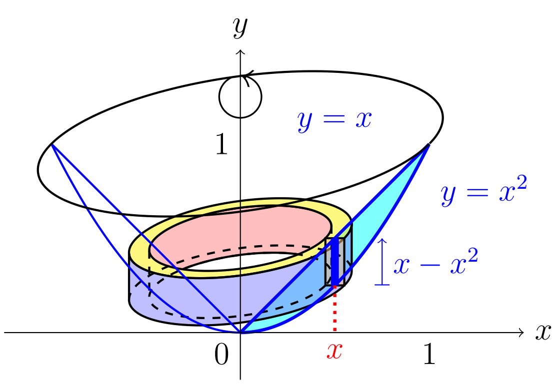
Skill: 畫圖找半徑 $r(x)/r(y)$ 與高度 $h(x)/h(y)$:

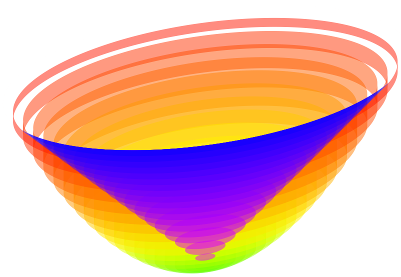
$$\boxed{V = \int_a^b \underbrace{2\pi r(x)}_{\text{圓周長}}\underbrace{h(x)}_{\text{高度}}\underbrace{dx}_{\text{厚度}} \qquad V = \int_c^d 2\pi r(y) h(y)\ dy}$$

Example 0.3 *Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.*

When radius x from 0 to 1, the circumference $2\pi x$ and height $x - x^2$.

$$V = \int_0^1 2\pi x(x - x^2)\ dx = 2\pi \int_0^1 x^2 - x^3\ dx = 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}. \qquad \blacksquare$$



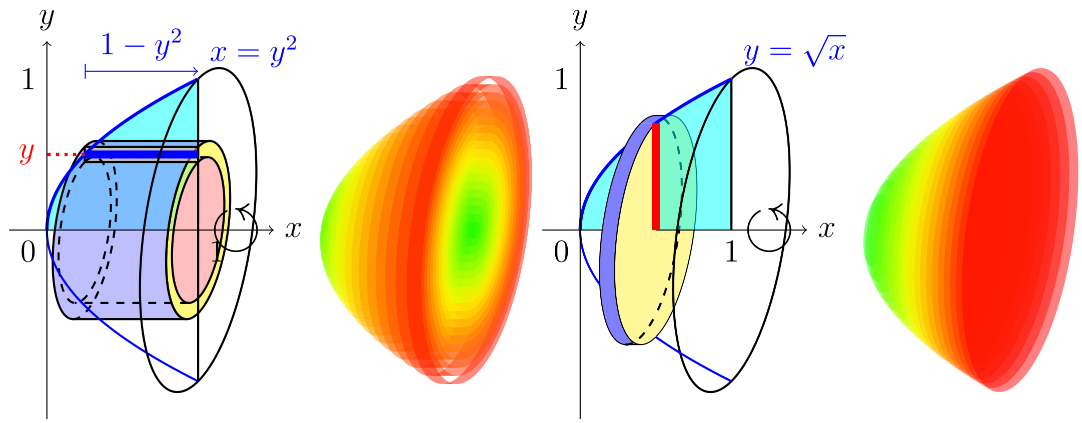


Example 0.4 *Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y=\sqrt{x}$ from 0 to 1.*

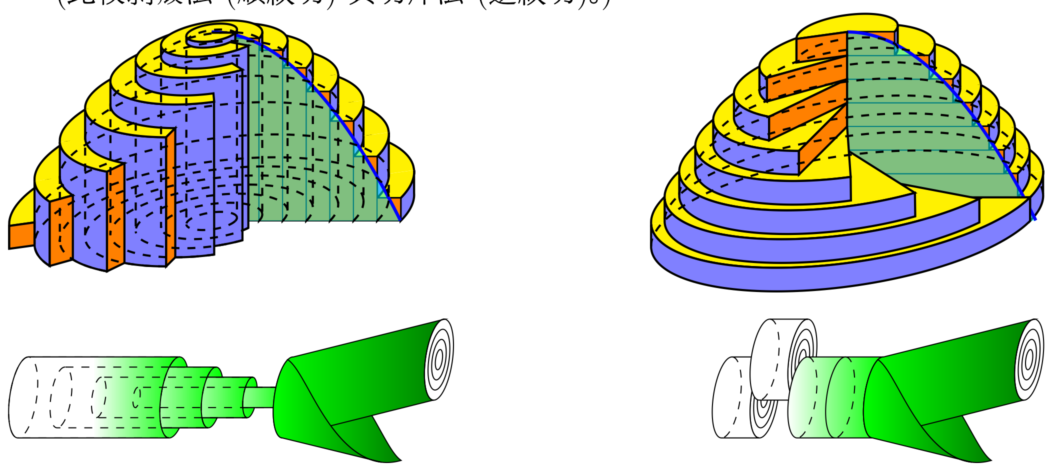
$x=y^2$ (解反函數: $y=\sqrt{x}\to x=y^2$), *circumference* $2\pi y$, *height* $1-y^2$.

$V=\int_0^1 2\pi y(1-y^2)\,dy=2\pi\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1=\frac{\pi}{2}$. ■

Recall (§ 6.2.ex2) disk: $V=\int_0^1 \pi x\,dx=\frac{\pi}{2}$ is simpler.



(比較剝殼法 (順紋切) 與切片法 (逆紋切)。)



Observation: 計算旋轉體體積的剝殼法與切片法, 一個對 x 積分, 另一個就對 y 積分。