1179: Probability
Lecture 28 — Strong Law of Large Numbers

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Announcements

- HW4 is now available on E3!
  - HW4-Part I will be due on 12/30 (Thursday), 9pm
  - HW4-Part II will be due on 1/3 (Monday), 9pm

- Final exam on 1/5 (on Wednesday, in class)
  - 10:10am - 12:10pm
  - Coverage: Lec 1 - Lec 29
  - You are allowed to bring a cheat sheet (A4 size, 2-sided, without any attachments)
  - Locations: EC015 and EC022

"Convergence in probability":

$Y_1, Y_2, \cdots, Y_n, \cdots$

$$\boxed{Y_n \xrightarrow{P} Y}$$

$$\Updownarrow$$

$$\lim_{n\to\infty} P\Big(\underbrace{\{\omega : |Y_n(\omega) - Y(\omega)| > \varepsilon\}}_{\text{bad event}}\Big) = 0,$$

for any $\varepsilon > 0$.

"WLLN":

$X_1, X_2, X_3, \cdots\cdots, X_n, \cdots$ i.i.d. random variables.

$$Y_n = \frac{X_1 + \cdots + X_n}{n}$$

$$\lim_{n\to\infty} P\big(\{\omega : |Y_n(\omega) - \underset{\text{true mean}}{\underline{\mu}}| > \varepsilon\}\big) = 0, \text{ for any } \varepsilon > 0$$

This Lecture

1. Strong Law of Large Numbers (SLLN) and Almost-Sure Convergence

2. Monte-Carlo Simulation

- Reading material: Chapter 11.4

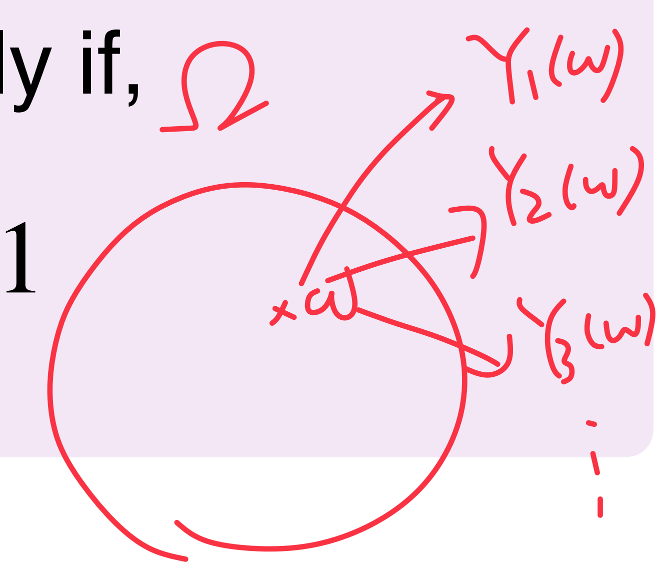
Convergence in Probability and Almost-Sure Convergence

- **Convergence to a Random Variable in Probability**: Let $Y_1, Y_2 \cdots$ be a sequence of random variables. We say that Y_n converges to a random variable Y in probability if $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} P(\{\omega : |Y_n(\omega) - Y(\omega)| > \varepsilon\}) = 0$$

- **Convergence to a Random Variable Almost Surely**: Let $Y_1, Y_2 \cdots$ be a sequence of random variables. We say that Y_n converges to a random variable Y almost surely if,

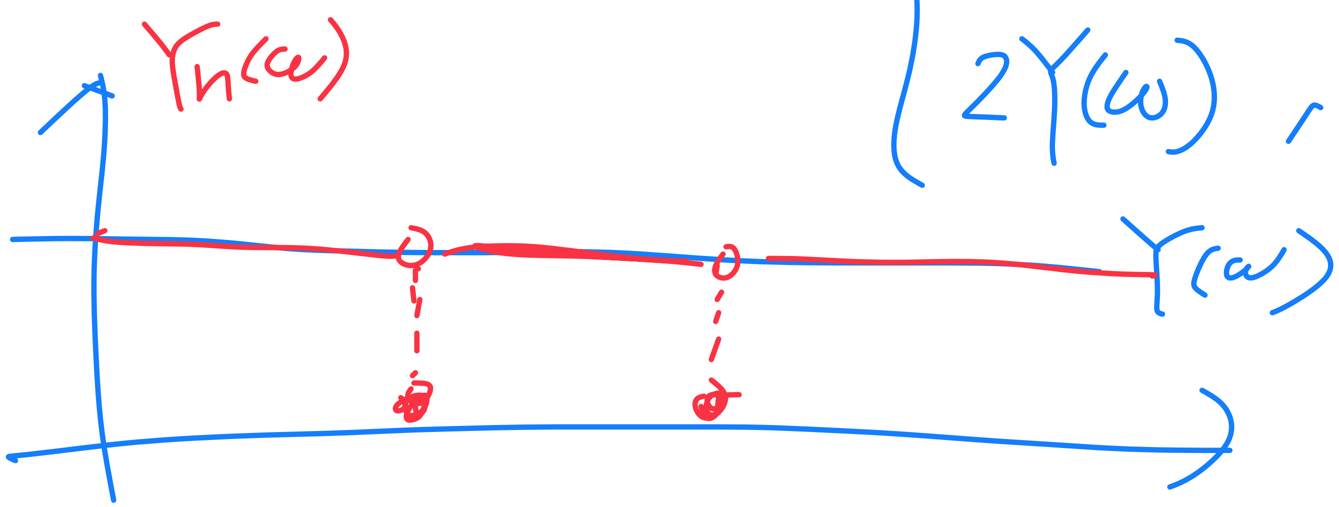
$$P\Big(\underbrace{\{\omega : \lim_{n \to \infty} Y_n(\omega) = Y(\omega)\}}_{\text{Good event}}\Big) = 1$$



$$\underbrace{\lim_{n\to\infty} P\left(\left\{\omega : Y_n(\omega) \neq Y(\omega) \right\}\right)}= 0$$

$$P\left(\left\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\right\}\right) = 1$$

ω $\quad Y_n(\omega) = \begin{cases} Y(\omega), & \text{if } n \text{ is odd} \\ 2Y(\omega), & \text{if } n \text{ is even} \end{cases}$



Convergence in Probability, But Not Almost Surely

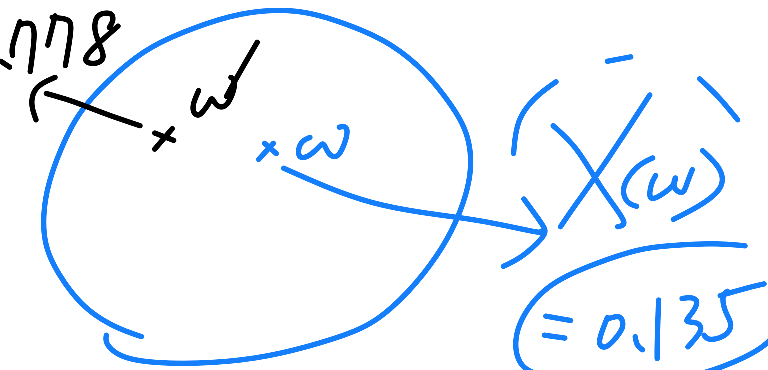
- **Example**: Let X be a continuous uniform r.v. on $(0,1)$
  - indicator function
  - Consider a sequence of r.v.s $X_1, X_2, \cdots$ as follows: $\Omega \to$ Real numbers

$X_1 = \mathbb{I}\{X \in [0,1]\}$

$X_1(\omega) = 1$

$X_2(\omega) = 1$, $X_2(\omega') = 0$

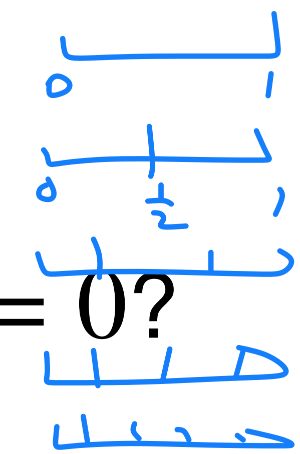
$X(\omega) = 0.778$



$X_2 = \mathbb{I}\{X \in [0,\frac{1}{2}]\}$ $\quad X_3 = \mathbb{I}\{X \in [\frac{1}{2},1]\}$

$X_4 = \mathbb{I}\{X \in [0,\frac{1}{3}]\}$ $\quad X_5 = \mathbb{I}\{X \in [\frac{1}{3},\frac{2}{3}]\}$ $\quad X_6 = \mathbb{I}\{X \in [\frac{2}{3},1]\}$

$\cdots \qquad \cdots \qquad \cdots$



- **Question**: Do we have $\lim_{n\to\infty} P(\underbrace{\{\omega : |X_n(\omega) - 0| > \varepsilon\}}_{\text{bad event}}) = 0$?

(Fix $\varepsilon = 0.01$)

$P(\{\omega : |X_1(\omega) - 0| > \varepsilon\}) = 1$

$P(\{\omega : |X_2(\omega) - 0| > \varepsilon\}) = \frac{1}{2}$

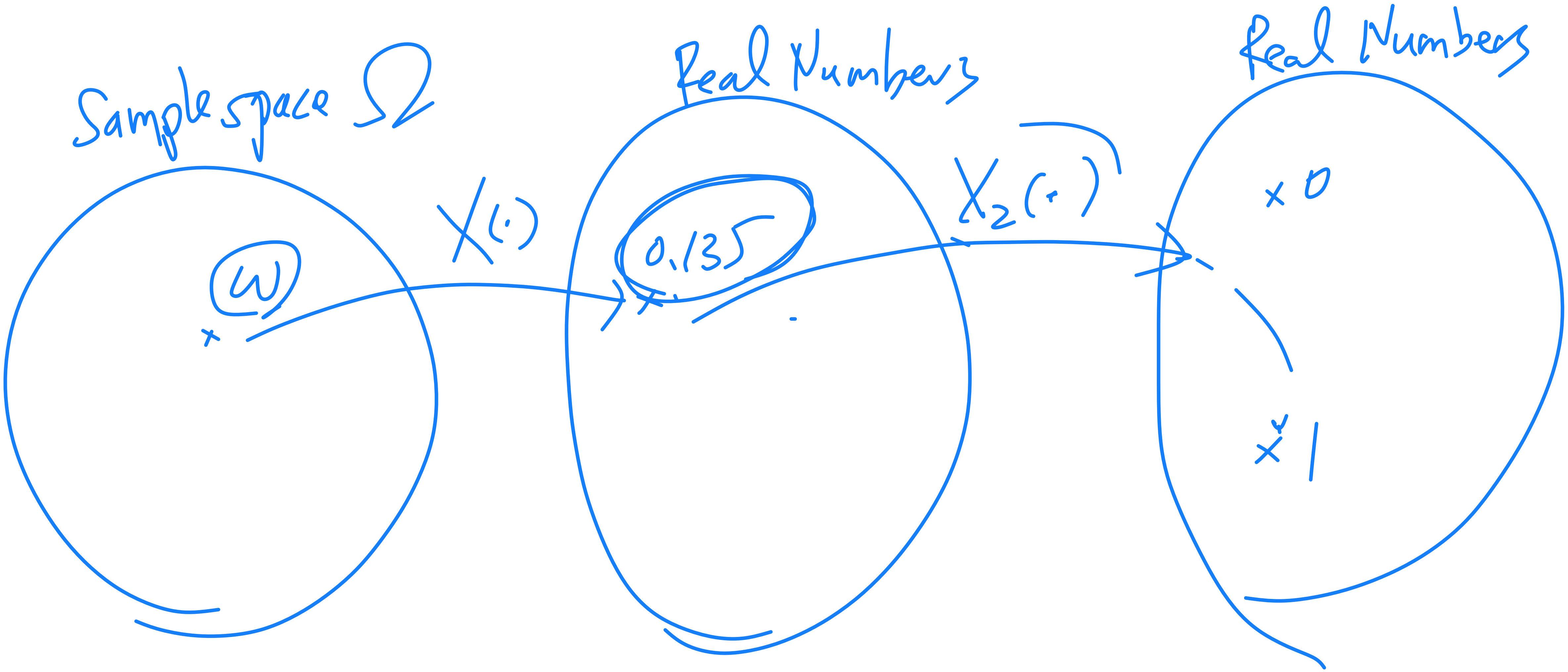
$P(\{\omega : |X_3(\omega) - 0| > \varepsilon\}) = \frac{1}{2}$

$P(\{\omega : |X_4(\omega) - 0| > \varepsilon\}) = \frac{1}{3}$

$P(\{\omega : |X_5(\omega) - 0| > \varepsilon\}) = \frac{1}{3}$

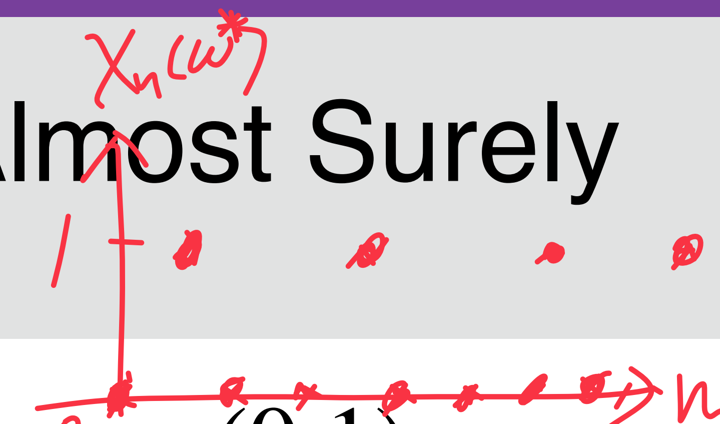
$\boxed{\text{Conclusion: } X_n \xrightarrow{P} 0}$

$\Rightarrow \lim_{n\to\infty} P(\{\omega : |X_n(\omega) - 0| > \varepsilon\}) = 0$ for any $\varepsilon > 0$



Sample space Ω
ω
X(·)
Real Numbers
0.135
X₂(·)
Real Numbers
0
1

Convergence in Probability, But Not Almost Surely (Cont.)



$\lim X_n(\omega^*)$

- **Example**: Let X be a continuous uniform r.v. on $(0,1)$
  - Consider a sequence of r.v.s $X_1, X_2, \cdots$ as follows:

$X(\omega^*) = 0.135$

$$X_1 = \mathbb{I}\{X \in [0,1]\}$$

$$X_2 = \mathbb{I}\{X \in [0,\tfrac{1}{2}]\} \qquad X_3 = \mathbb{I}\{X \in [\tfrac{1}{2},1]\}$$

$$X_4 = \mathbb{I}\{X \in [0,\tfrac{1}{3}]\} \qquad X_5 = \mathbb{I}\{X \in [\tfrac{1}{3},\tfrac{2}{3}]\} \qquad X_6 = \mathbb{I}\{X \in [\tfrac{2}{3},1]\}$$

X_7 $\cdots$ $[0,\frac{1}{4}]$ $\cdots$ $\cdots$

Good event

- **Question**: Do we have $\boxed{P(\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\}) = 1}$?

$X_1(\omega^*) = 1$

$X_2(\omega^*) = 1 \quad X_3(\omega^*) = 0$

$X_4(\omega^*) = 1 \quad X_5(\omega^*) = 0 \quad X_6(\omega^*) = 0$

$X_7(\omega^*) = 1 \quad X_8(\omega^*) = 0 \quad X_9(\omega^*) = 0, \quad X_{10}(\omega^*) = 0$

Conclusion:

$X_n \overset{a.s.}{\not\to} 0$

Equivalent Definition of Almost-Sure Convergence

Def ① $Y_n \xrightarrow{a.s.} Y$

- Almost-Sure Convergence: $\boxed{P(\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\}) = 1}$

Good event

- **Equivalent Definition of Almost-Sure Convergence**: Let $Y_1, Y_2 \cdots$ be a sequence of random variables. We say that Y_n converges to a random variable Y almost surely if for all $\varepsilon > 0$,

Def ②

$$P\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\underbrace{\{\omega : |Y_n(\omega) - Y(\omega)| > \varepsilon\}}_{A_n}\Big) = 0$$

$$\underbrace{\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n}$$

$$P\Big(\underbrace{\Big\{\omega : \lim_{n\to\infty} Y_n(\omega) \neq Y(\omega)\Big\}}_{\text{bad event}}\Big) = 0$$

(Fix $\varepsilon > 0$) $|Y_n(\omega) - Y(\omega)| > \varepsilon$ for infinitely many times

WLLN vs SLLN

- **The Weak Law of Large Numbers (Khinchin's Law)**: Let $X_1, \cdots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean μ. Define $S_n = (X_1 + \cdots + X_n)$. Then, for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} P\left(\{\omega : \left| \frac{S_n(\omega)}{n} - \mu \right| \geq \varepsilon \right) = 0$$

- **The Strong Law of Large Numbers**: Let $X_1, \cdots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean μ. Define $S_n = (X_1 + \cdots + X_n)$.

empirical mean

Then, we have

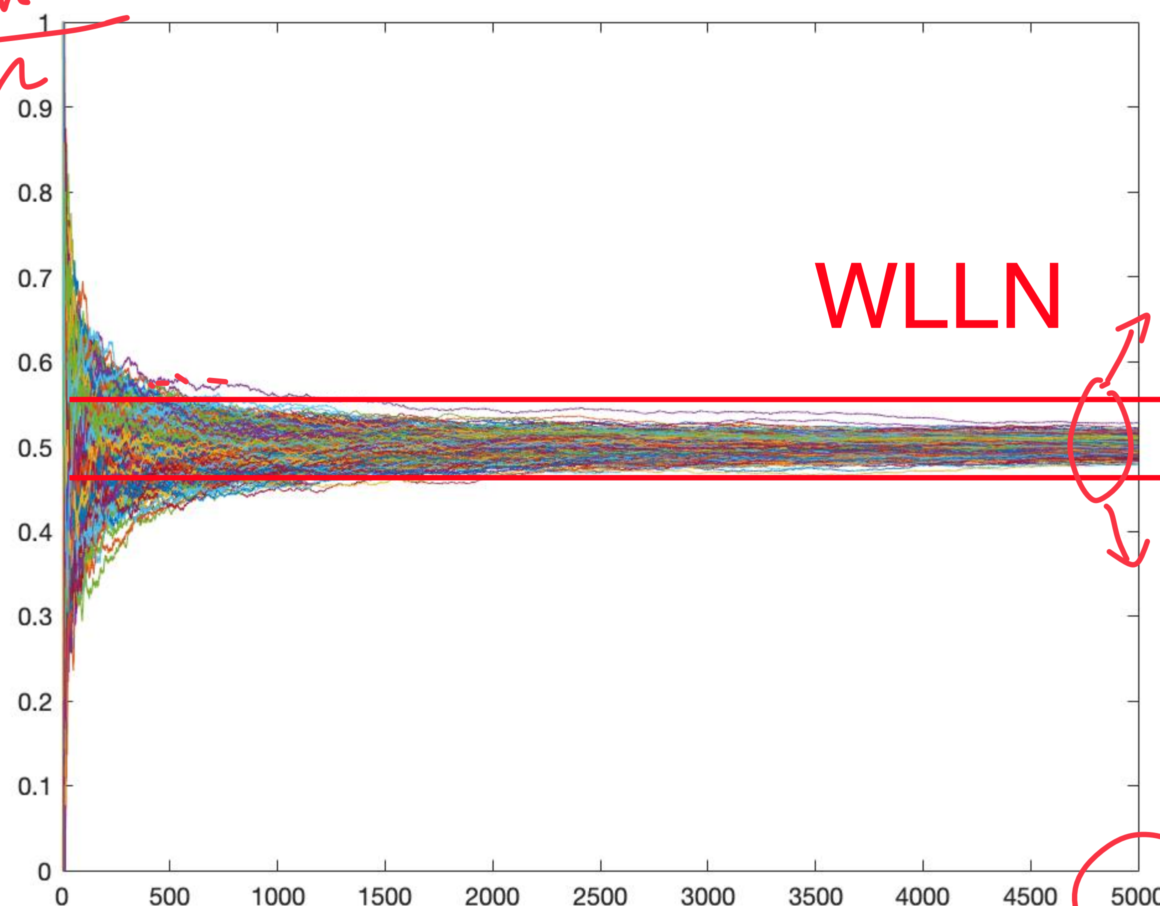
$$P\left(\{\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \mu\} \right) = 1$$

Visualization of WLLN and SLLN

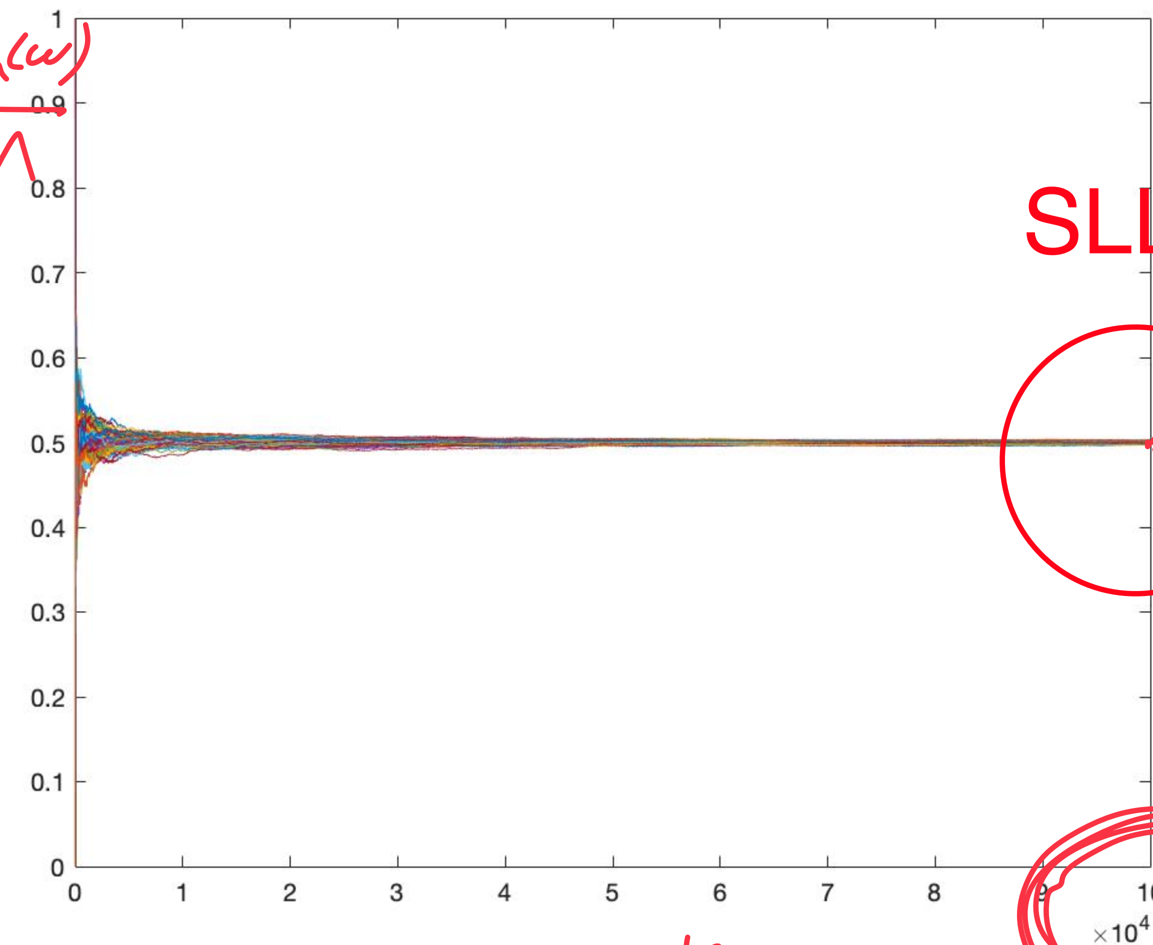
p=0.5

- Example: $X_i \sim$ Bernoulli(0.5) and $S_n = X_1 + \cdots + X_n$

$\frac{S_n(\omega)}{n}$



$\frac{S_n(\omega)}{n}$



n

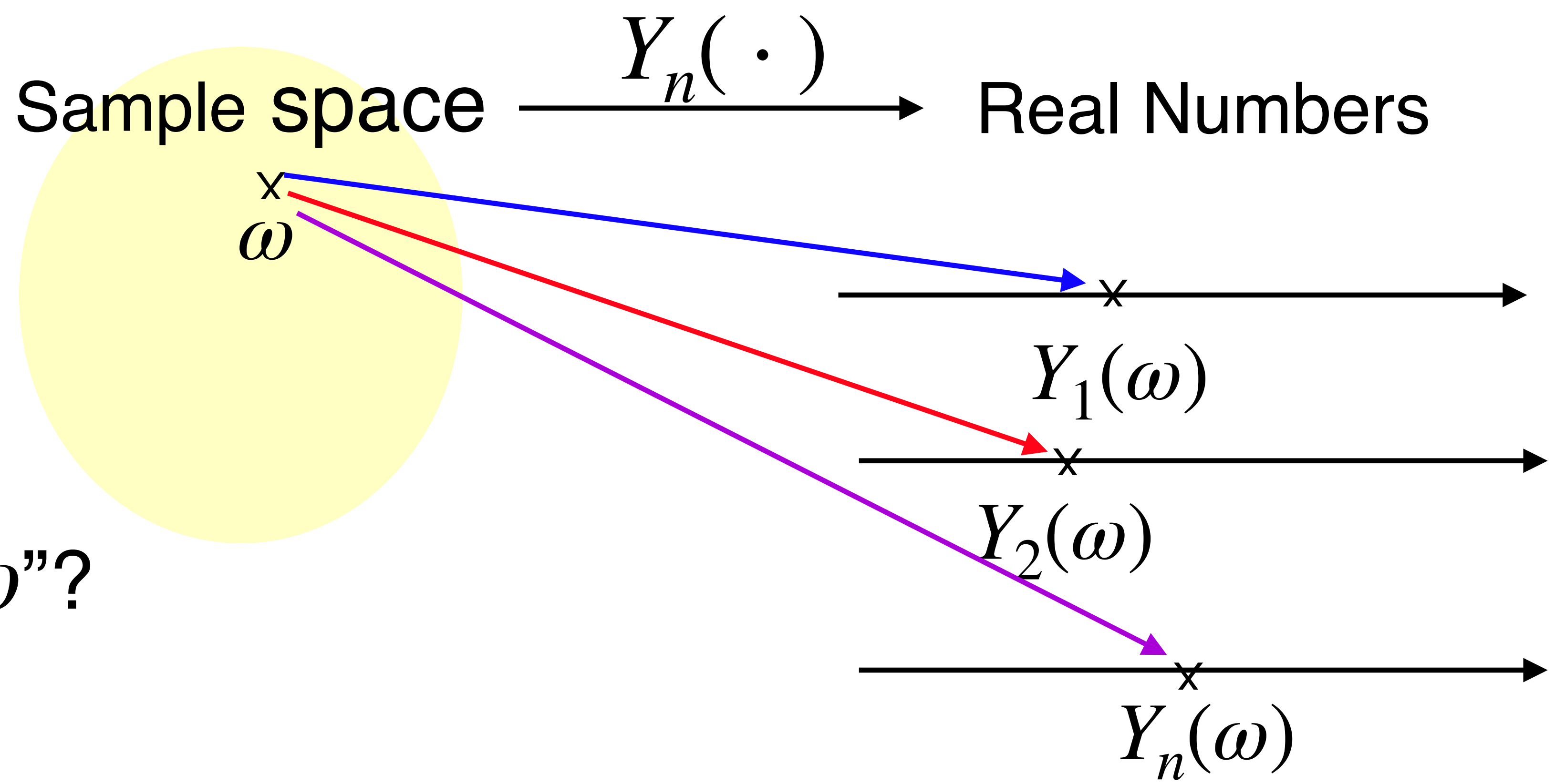
$$\lim_{n\to\infty} P\Big(\underbrace{\{\omega : |\frac{S_n(\omega)}{n} - \mu| \geq \varepsilon\}}_{\text{bad event}}\Big) = 0$$

n

$$P\Big(\{\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \mu\}\Big) = 1$$

How to Interpret SLLN?

- Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with mean μ
- Define $Y_n = (X_1 + X_2 \cdots, + X_n)/n$
- SLLN: $P\left(\left\{\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \mu\right\}\right) = 1$



- Question: What is an "ω"?

How to Prove SLLN (Under a Mild Condition)?

1. Borel-Cantelli Lemma

2. A Bound for the 4-th Moment Condition

3. Markov's Inequality

1. Borel-Cantelli Lemma

- **Recall:** HW1, Problem 3

Problem 3 (Continuity of Probability Functions) (12+12=24 points)

(a) Let $A_1, A_2, A_3, \cdots$ be a countably infinite sequence of events. Prove that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = 0$. This property is known as the *Borel-Cantelli Lemma.* (Hint: Consider the continuity of probability function for $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$ and then apply the union bound)

(b) Consider a countably infinite sequence of coin tosses. The probability of having a head at the k-th toss is p_k, with $p_k = 100 \cdot k^{-N}$ (Note: different tosses are NOT necessarily independent). We use I to denote the event

- **Borel-Cantelli Lemma**: Let $\{A_n\}$ be any sequence of events. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then we have

$$P\Big(\{\omega : \omega \in A_n \text{ for infinitely many } n\}\Big) = P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = 0$$

Review: Proof of Borel-Cantelli Lemma

- **Borel-Cantelli Lemma**: Let $\{A_n\}$ be any sequence of events. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then we have
$$P\Big(\{\omega : \omega \in A_n \text{ for infinitely many } n\}\Big) = P(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n) = 0$$

- Proof:

2. A Bound For 4-th Moment

- **A Bound on 4-th Moment**: Let $X_1, \cdots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean μ and $E[X_1^4] < \infty$. Define $S_n = (X_1 + \cdots + X_n)$. Then, there exists a constant $K < \infty$ such that

$$E[(S_n - n\mu)^4] \leq Kn^2$$

- Proof: Please see the supplemental on E3
- Question: How about $E[(\frac{S_n}{n} - \mu)^4] \leq$?

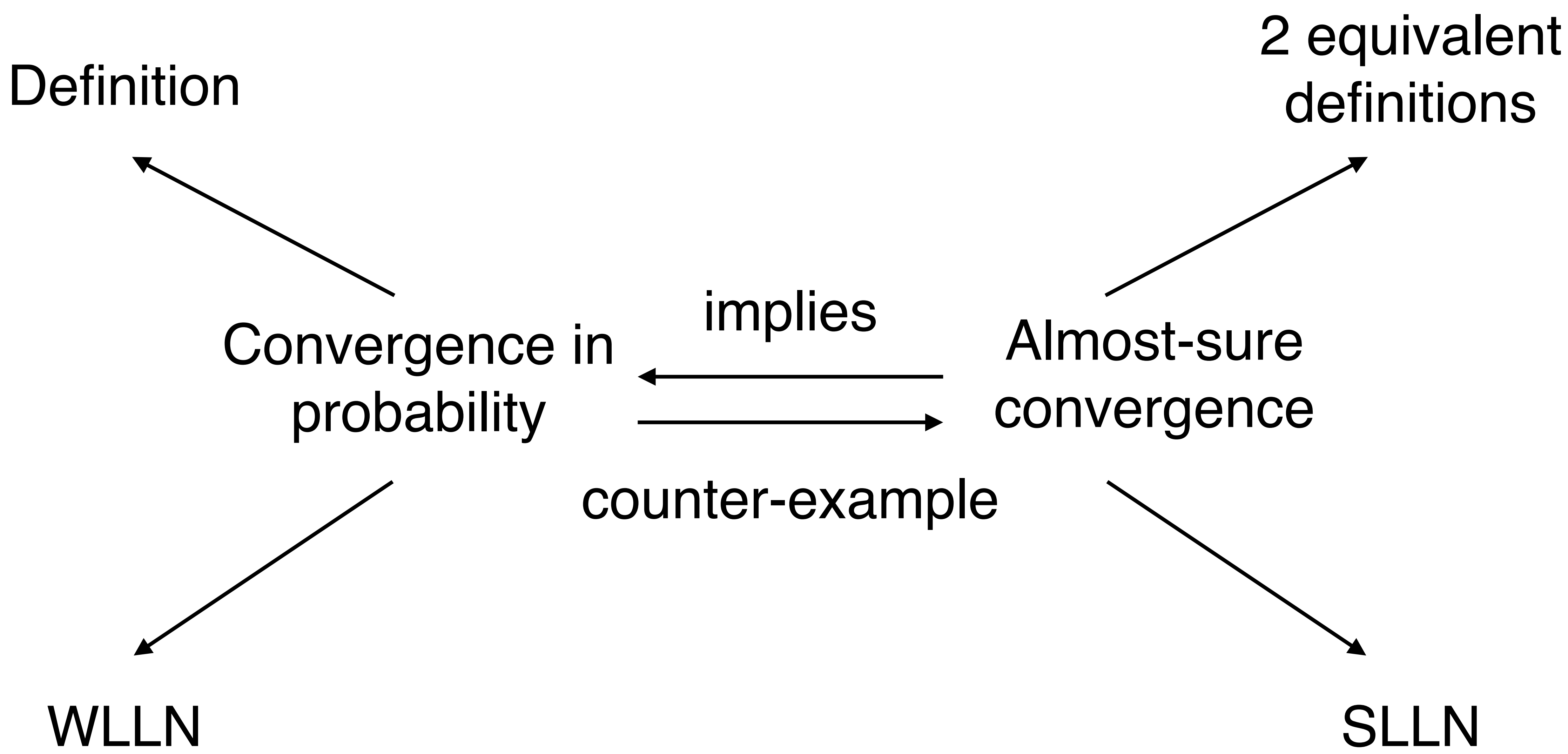
Put Everything Together: Proof of SLLN

- SLLN: $P\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\omega : \Big|\frac{S_n(\omega)}{n} - \mu\Big| > \varepsilon\}\Big) = 0, \forall \varepsilon > 0$

- Proof:

$$P\Big(\underbrace{\{\Big|\frac{S_n}{n} - \mu\Big| \geq n^{-\gamma}\}}_{A_n}\Big) = P\Big(\Big|\frac{S_n}{n} - \mu\Big|^4 \geq n^{-4\gamma}\}\Big) \leq$$

A Quick Summary



Application of SLLN: Monte-Carlo Simulation

- A Motivating Example: Find the following integration

$$I = \int_0^1 e^{-x^3} dx$$

- Question: Is there a closed-form expression for I?
- Question: How about Riemann integration?

 http://faculty.washington.edu/yenchic/18A_stat516/Lec7_MCMC.pdf

Application of SLLN: Monte-Carlo Simulation (Cont.)

- A Motivating Example: Find the following integration

$$I = \int_0^1 e^{-x^3} dx$$

- Monte-Carlo method: Let $U \sim \text{Unif}(0,1)$

1. Let $U \sim \text{Unif}(0,1)$. Rewrite $\displaystyle\int_0^1 e^{-x^3} dx = E\left[e^{-U^3}\right]$

2. Draw K i.i.d. random variables $U_1, \cdots, U_K \sim \text{Unif}(0,1)$

$$E\left[e^{-U^3}\right] \approx \frac{1}{K} \sum_{i=1}^{K} e^{-U_i^3} \qquad \text{(Why?)}$$

 http://faculty.washington.edu/yenchic/18A_stat516/Lec7_MCMC.pdf

Monte-Carlo Simulation (Formally)

- Objective: Find the integration $I = \int g(x)dx$
- Monte-Carlo Simulation:

1. Let X be a random variable with PDF $p(x)$. Rewrite I as

$$I = \int \frac{g(x)}{p(x)} p(x)dx = E_{p(x)}\left[\frac{g(X)}{p(X)}\right]$$

2. Draw K i.i.d. random variables $X_1, \cdots, X_K$ with PDF $p(x)$

Construct $$\hat{I}_K = \frac{1}{K}\sum_{i=1}^{K}\frac{g(X_i)}{p(X_i)} \approx I$$

- Question: $E[\hat{I}_K]$ = ? Var$[\hat{I}_K]$ = ?
- Question: How to choose K and $p(x)$?

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