11.9 Representations of functions as power series

逐項微積分是一個解微分方程很強方法的基礎。

0.1 Differentiation and integration of power series

Question: 如果一個函數可以表示成冪級數和, 要怎麼微積分? 逐項微積分。

Theorem 1 (Term-by-term differentiation and integration)

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then the function* f *defined by (*冪級數和的函數*)*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=\boxed{1}}^{\infty} nc_n(x-a)^{n-1}$ 常數項微分歸零

(ii) $\int f(x)\, dx = C + c_0 x + \frac{c_1}{2}(x-a)^2 + \ldots = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$

The radii of convergence of the power series in (i) *and* (ii) *are both* R*.*

Note: 1. (i) and (ii) can be rewritten in the form ($\frac{d}{dx}$, $\int$ 跟 $\sum_{n=0}^{\infty}$ 可交換):

(iii) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right]$$

(iv) $$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int \left[c_n(x-a)^n\right] dx$$

逐項微積分只有收斂的冪級數會成立, 其他的函數級數 ($\sum f_n(x)$) 不一定對。

2. $\frac{d}{dx}f(x)$ & $\int f(x)\, dx$ 對應的冪級數的收斂半徑 (R) 一樣, 但端點 $x = a \pm R$ 可能不會收斂, 收斂區間可能不同, 要另外檢查。

Recall: $\boxed{\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.}$

Example 0.1 (變形) *Express* $\dfrac{1}{1+x^2}$ *as the sum of a power series and find the interval of convergence.*

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \dots$$

The geometric series converges when $|-x^2| < 1 \iff |x| < 1$*, so the interval of convergence is* $(-1, 1)$. ■

Example 0.2 *Find a power series representation for* $\dfrac{1}{x+2}$.

$$\frac{1}{2+x} = \frac{1}{2[1-(-\frac{x}{2})]} = \frac{1}{2}\sum_{n=0}^{\infty}(-\frac{x}{2})^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \dots$$

The geometric series converges when $\left|-\dfrac{x}{2}\right| < 1 \iff |x| < 2$*, so the interval of convergence is* $(-2, 2)$. ■

Attention: 不可同除 x 變成 $\dfrac{\frac{1}{x}}{1-(-\frac{2}{x})} = \dfrac{1}{x}\sum_{n=0}^{\infty}(-\dfrac{2}{x})^n$, 這<u>不是</u>冪級數!

Example 0.3 *Find a power series representation for* $\dfrac{x^3}{x+2}$.

$$\frac{x^3}{2+x} = x^3 \cdot \frac{1}{2+x} = x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}\left(= \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{2^{n-2}}x^n\right)$$

The interval of convergence is also $(-2, 2)$. ■

Shift 平移: $x^{\boxed{n+3\to n}}$: $\boxed{n \to n-3 \geq 0 \to n \geq 3}$. 各自代第一項檢查。

Example 0.4 (微分) *The Bessel function* $J_0(x) = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ *for all* x,

$$J_0'(x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty}\frac{d}{dx}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty}\frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} \quad \textit{for all } x.$$

(注意序號變成從 $n = 1$ *開始)* ■

Example 0.5 *Express* $\dfrac{1}{(1-x)^2}$ *as a power series by differentiating* $\dfrac{1}{1-x}$. *What is the radius of convergence.*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n,$$

$$\downarrow 微分 \qquad \boxed{n-1 \to n:\ n \to n+1 \geq 1 \to n \geq 0}$$

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n.$$

The radius of convergence is the same as $\dfrac{1}{1-x}$, $R = 1$. *(微分半徑一樣)* ■

Example 0.6 (積分) *Find a power series representation for* $\ln(1+x)$ *and its radius of convergence.*

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-x)^n, \qquad |x| < 1,$$

$$\downarrow 積分 \qquad \boxed{n+1 \to n:\ n \to n-1 \geq 0 \to n \geq 1}$$

$$\begin{aligned} \ln(1+x) &= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + C, \qquad |x| < 1. \end{aligned}$$

To determine C, *put* $x = 0$, $\ln(1+0) = 0 = C$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \qquad |x| < 1.$$

The radius of convergence is the same as $\dfrac{1}{1+x}$, $R = 1$. *(積分半徑一樣)* ■

Remark: $x = 1$, $\sum \dfrac{(-1)^{n-1}}{n}$ alternating harmonic series, converges; $x = -1$, $\sum \dfrac{-1}{n} = -\sum \dfrac{1}{n}$ negative harmonic series, diverges.
收斂區間: $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} x^n$ is $(-1, 1]$, $\sum_{n=0}^{\infty} (-x)^n$ is $(-1, 1)$. (區間不同)
Take $x = 1$, then

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Example 0.7 *Find a power series representation for* $f(x) = \tan^{-1} x$.

$$\tan^{-1} x \;=\; \int \frac{1}{1+x^2}\,dx \;=\; \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx \;=\; C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

$$\frac{1}{1+x^2} \;=\; 1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

$$\downarrow\text{積分}$$

$$\tan^{-1} x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C,$$

$|-x^2| < 1 \iff |x| < 1.$
Put $x = 0$, $\tan^{-1} 0 = 0 = C$, *(要代收斂區間內的, 最安全好算就是代中心。)*

$$\tan^{-1} x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad |x| < 1.$$

The radius of convergence is the same as $\frac{1}{1+x^2}$, $R = 1$. ■

Remark: $x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$; $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n+1}$; both converge by the Alternating Series Test.
收斂區間: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ is $\Big[-1, 1\Big]$, $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ is $\Big(-1, 1\Big)$. (區間不同)
The Leibniz formula for π (take $x = 1$):

$$\frac{\pi}{4} \;=\; \tan^{-1} 1 \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Recall: Alternating Series Estimation Theorem: $|R_n| < b_{n+1}$.

$$\begin{array}{llllll}
n = 1: & \left|\ln 2 - 1\right| & < \frac{1}{2}, & \left|\frac{\pi}{4} - 1\right| & < & \frac{1}{3}, \\
n = 2: & \left|\ln 2 - \left(1 - \frac{1}{2}\right)\right| & < \frac{1}{3}, & \left|\frac{\pi}{4} - \left(1 - \frac{1}{3}\right)\right| & < & \frac{1}{5}, \\
n = 3: & \left|\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3}\right)\right| & < \frac{1}{4}, & \left|\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5}\right)\right| & < & \frac{1}{7}.
\end{array}$$

可以任意準確地逼近 $\ln 2$ & π 的方法。

Example 0.8 *(a) Evaluate* $\displaystyle\int \frac{1}{1+x^7}\,dx$ *as a power series.*
(b) Approximate $\displaystyle\int_0^{0.5} \frac{1}{1+x^7}\,dx$ *correct to within* 10^{-7}.

(a)
$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty}(-x^7)^n = \sum_{n=0}^{\infty}(-1)^n x^{7n},$$
$$\int \frac{1}{1+x^7}\,dx = \sum_{n=0}^{\infty}\int (-1)^n x^{7n}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{7n+1}x^{7n+1}+C.$$

The series converges for $|-x^7|<1 \iff |x|<1$.

*(b) By the Fundamental Theorem of Calculus, (*反導數找加零 $(C=0)$*;* $[0,0.5]$ 在收斂範圍 $(-1,1)$ 內, 才可以用逐項微積分算。*)*

$$\begin{aligned}\int_0^{0.5}\frac{1}{1+x^7}\,dx &= \left[x-\frac{x^8}{8}+\frac{x^{15}}{15}-\frac{x^{22}}{22}+\dots\right]_0^{0.5}\\ &= \frac{1}{2}_{(b_0)}-\frac{1}{8\cdot 2^8}_{(b_1)}+\frac{1}{15\cdot 2^{15}}_{(b_2)}-\frac{1}{22\cdot 2^{22}}_{(b_3)}+\dots\\ &= \sum_{n=0}^{\infty}(-1)^n b_n,\ \textit{(alternating series)}\end{aligned}$$

$b_3=\dfrac{1}{22\cdot 2^{22}}\approx 1.1\times 10^{-8}$, $b_4=\dfrac{1}{29\cdot 2^{29}}\approx 6.4\times 10^{-11}$.
By the Alternating Series Estimation Theorem, $|R_4|\le b_4<6.5\times 10^{-11}$.

$$\begin{aligned}\int_0^{0.5}\left[\frac{1}{1+x^7}\right]dx &\approx s_4=\frac{1}{2}-\frac{1}{8\cdot 2^8}+\frac{1}{15\cdot 2^{15}}-\frac{1}{22\cdot 2^{22}}\\ &\approx 0.49951374.\end{aligned}$$

*(*注意*!* 從 $n=0$ 算起, b_4 是第五項, R_4 *&* s_4 是第四個剩餘項 *&* 部分和。*)* ■

Additional:
$s_1=0.49999999999$, $|R_1|\le b_1\approx 4.8828\times 10^{-4}$;
$s_2=0.49951171875$, $|R_2|\le b_2\approx 2.0345\times 10^{-6}$;
$s_3=0.49951375325$, $|R_3|\le b_3\approx 1.0837\times 10^{-8}$; 因為不會進位, 這個也可以。
$s_4=0.49951374241$, $|R_4|\le b_4\approx 6.4229\times 10^{-11}$;
$s_5=0.49951374248$, $|R_5|\le b_5\approx 4.0422\times 10^{-13}$.

♦ Additional: Application on Differential Equation

解微分方程: $f'(x) = f(x)$, $f(0) = 1$.

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$, $c_0 = f(0) = 1$.

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n,$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n.$$

比較係數: $c_n = \dfrac{c_{n-1}}{n} = \dfrac{c_{n-2}}{n(n-1)} = \cdots = \dfrac{c_0}{n!} = \dfrac{1}{n!}$,

$$\Longrightarrow f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

....

解微分方程: $f''(x) + f(x) = 0$.

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$.

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n,$$

$$\begin{aligned} f''(x) = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n. \end{aligned}$$

$$\begin{aligned} c_{2n} &= \frac{-c_{2n-2}}{(2n)(2n-1)} = \frac{-(-c_{2n-4})}{(2n)(2n-1)(2n-2)(2n-3)} = \cdots = \frac{(-1)^n}{(2n)!} c_0, \\ c_{2n+1} &= \frac{-c_{2n-1}}{(2n+1)(2n)} = \frac{-(-c_{2n-3})}{(2n+1)(2n)(2n-1)(2n-2)} = \cdots = \frac{(-1)^n}{(2n+1)!} c_1, \end{aligned}$$

$$\begin{aligned} \Longrightarrow f(x) &= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= c_0 \cos x + c_1 \sin x. \end{aligned}$$