1179: Probability
Lecture 16 — Expected Values of Continuous Random Variables and Joint Distributions

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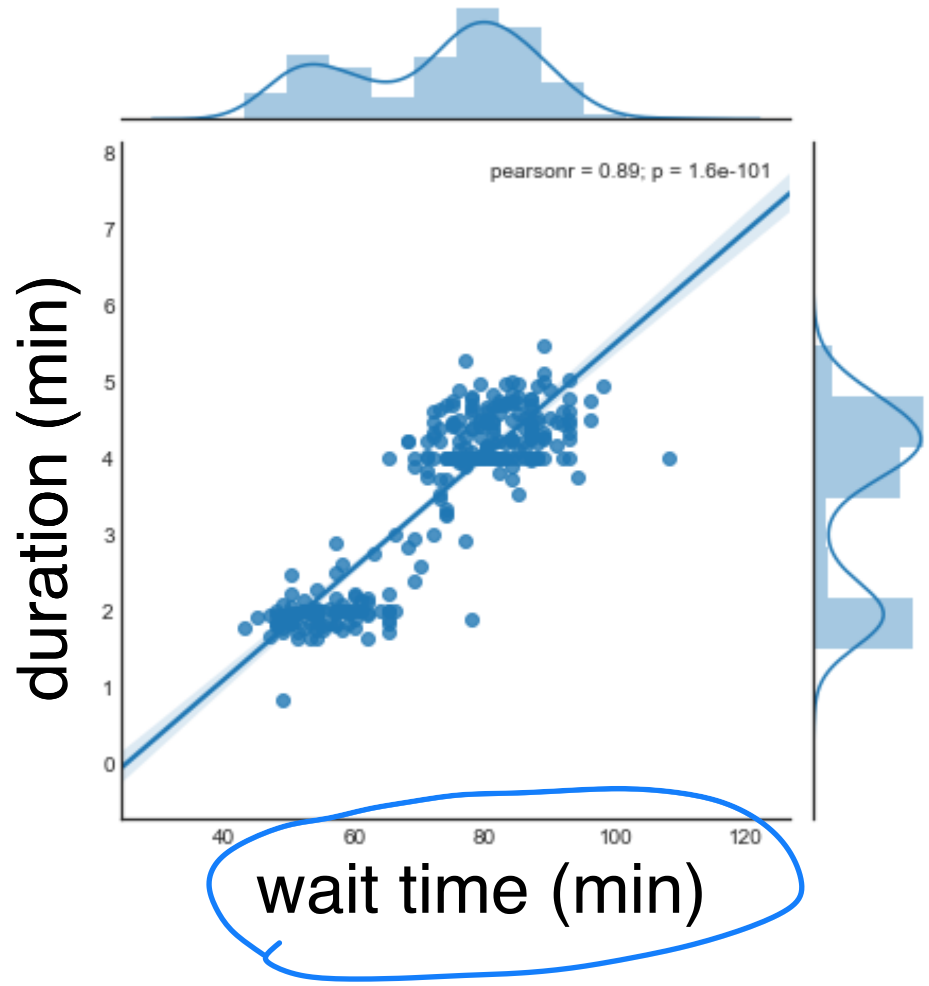
Announcements

- TA Hour: 11/5 (Fri.), 6:30pm-8pm @ EC513 and Google Meet
  - https://meet.google.com/nap-bwvz-fft

Why Jointly Study 2 Random Variables?

- Example: Old Faithful Geyser

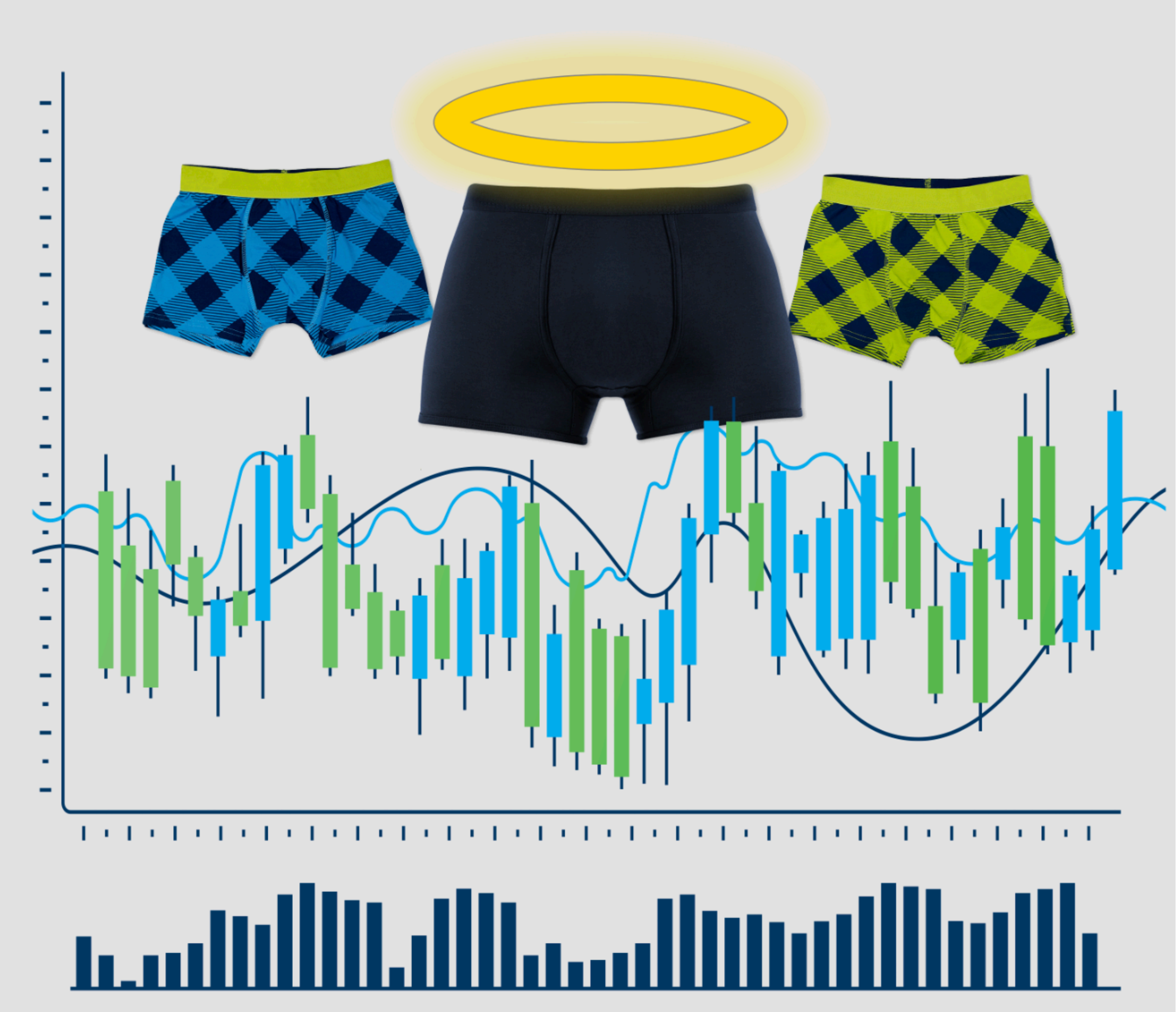


- Eruption duration could help predict the next wait time

http://www.stat.cmu.edu/~larry/all-of-statistics/=data/faithful.dat

Men's Underwear Index (MUI)

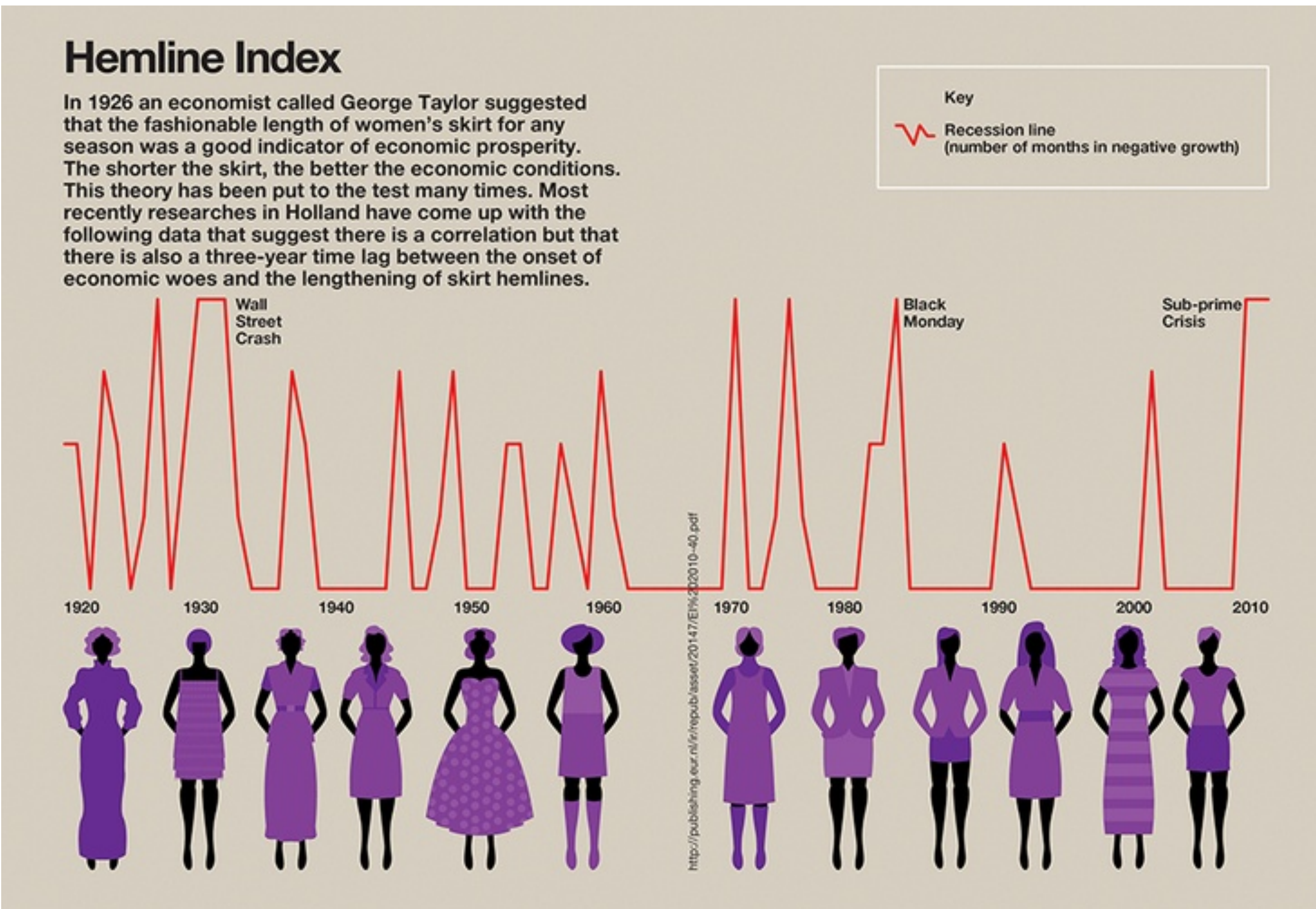
- "MUI is an economic index that can supposedly detect the beginnings of a recovery during an economic slump"… (Wiki)



http://www.washingtonpost.com/wp-dyn/content/article/2009/08/30/AR2009083002761.html
https://medium.com/@nadexexchange/the-me-too-movement-says-the-underwear-index-stinks-a34d7c5c1430

Hemline Index

- “ …hemlines on women's dresses rise along with stock prices.”



https://imarknaden.se/2018/02/26/258/

This Lecture

1. Expected Value and Variance of Continuous Random Variables

2. Joint Distributions of Two Random Variables

- Reading material: Chapter 6.3 and 8.1

Review: Expected Value and Variance of a Continuous R.V.

Let X be a continuous random variable with a PDF $f_X(x)$. Then, we have

1. $E[X] := \int_{-\infty}^{+\infty} x \cdot f_X(x)dx$

2. $E[g(X)] := \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)dx$

3. $\text{Var}[X] := \int_{-\infty}^{+\infty} (x - E[X])^2 \cdot f_X(x)dx$

Exponential: Mean and Variance

- Example: $X \sim \text{Exp}(\lambda)$
  - What is $E[X]$?
  - How about $\text{Var}[X]$?

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{, if } x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx$$

$$= \int_0^\infty \underbrace{x}_{f} \cdot \underbrace{\lambda \cdot e^{-\lambda x}}_{g'} dx = x \cdot -e^{-\lambda x}\Big|_0^\infty - \int_0^\infty 1 \cdot -e^{-\lambda x} dx$$

$$= (0 - 0) - \frac{-1}{\lambda}\int_0^\infty \underbrace{\lambda e^{-\lambda x} dx}_{=1}$$

$$= \frac{1}{\lambda}$$

- λ = rate
- $\frac{1}{\lambda}$: "mean lifetime"

$$\mathrm{Var}[X] = \underline{E[X^2]} - (E[X])^2$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f_X(x)\,dx = \int_0^{\infty} \underbrace{x^2}_{= f} \cdot \underbrace{\lambda e^{-\lambda x}}_{= g'}\,dx$$

$$= \underline{x^2 \cdot -e^{-\lambda x} \Big|_0^{\infty}} - \underline{\int_0^{\infty} 2x \cdot -e^{-\lambda x}\,dx}$$

$$= (0 - 0) - \frac{-2}{\lambda} \underbrace{\int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx}_{= E[X] = \frac{1}{\lambda}}$$

$$= \frac{2}{\lambda^2}$$

Therefore, $\mathrm{Var}[X] = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Properties of Discrete R.V. Still Hold for Continuous R.V.?

1. $E[\alpha X + \beta] = \alpha \cdot E[X] + \beta$? $(\alpha, \beta \in \mathbb{R})$

$$= \int_{-\infty}^{+\infty} (\alpha x + \beta) \cdot f_X(x)\, dx = \int_{-\infty}^{+\infty} \alpha x f_X(x)\, dx + \int_{-\infty}^{+\infty} \beta \cdot f_X(x)\, dx$$

2. $E[g(X) + h(X)] = E[g(X)] + E[h(X)]$?
(g, h real-valued functions)

(LOTUS + linearity of integration)

Properties of Discrete R.V. Still Hold for Continuous R.V.?

3. $\text{Var}[X] := E[X^2] - (E[X])^2$?

4. $\text{Var}(X + c) = \text{Var}(X)$?

$$\text{Var}[X+c] = E\left[\left(X+c - E[X+c]\right)^2\right]$$

5. $\text{Var}(aX) = a^2 \cdot \text{Var}(X)$?

(follows directly from the def. of Var)

Recall: Expected Value of a Discrete Random Variable Using CDF

Expected Value (or Mean / Expectation):

Let X be a non-negative discrete random variable with

- the set of possible values $S = \{x_1, x_2, x_3 \cdots\}$
- CDF of X is $F_X(t)$

Denote $x_0 = 0$. The expected value of X is

$$E[X] = \sum_{i=1}^{\infty} (x_i - x_{i-1}) \cdot \left(1 - F_X(x_i^-)\right)$$

- How about continuous cases?

Alternative Expression for Expected Value of a Continuous Random Variable Using CDF

Expected Value via CDF:

Let X be a continuous random variable with CDF $F_X(t)$.

The expected value of X is

$$E[X] = \int_0^\infty (1 - F_X(t))dt - \int_0^\infty F_X(-t)dt$$

- What if X is a non-negative random variable?

$F_X(t) = 0$, for all $t < 0$ $\qquad E[X] = \int_0^\infty (1 - F_X(t))dt$

- How to prove this?

Proof: Expected Value of a Continuous Random Variable Using CDF

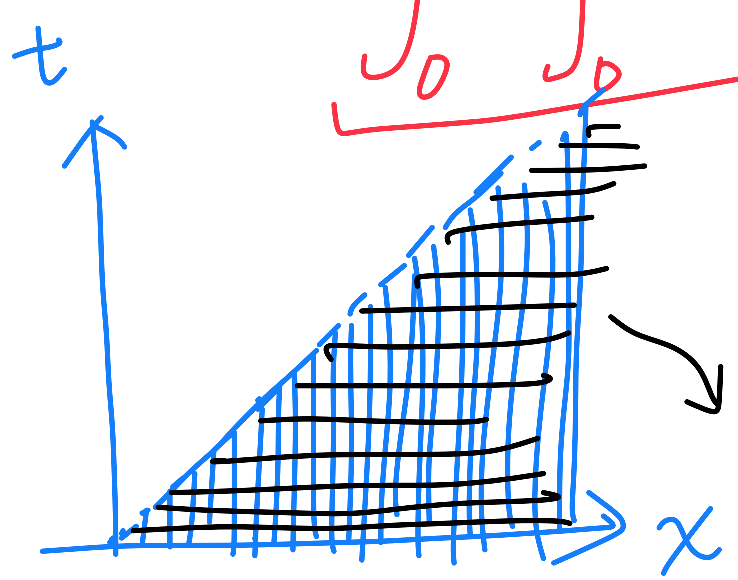
① change of variables

② double integration

$$E[X] = \int_0^\infty (1 - F_X(t))dt - \int_0^\infty F_X(-t)dt$$

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx, \quad x = \int_0^x 1 \cdot dt$$

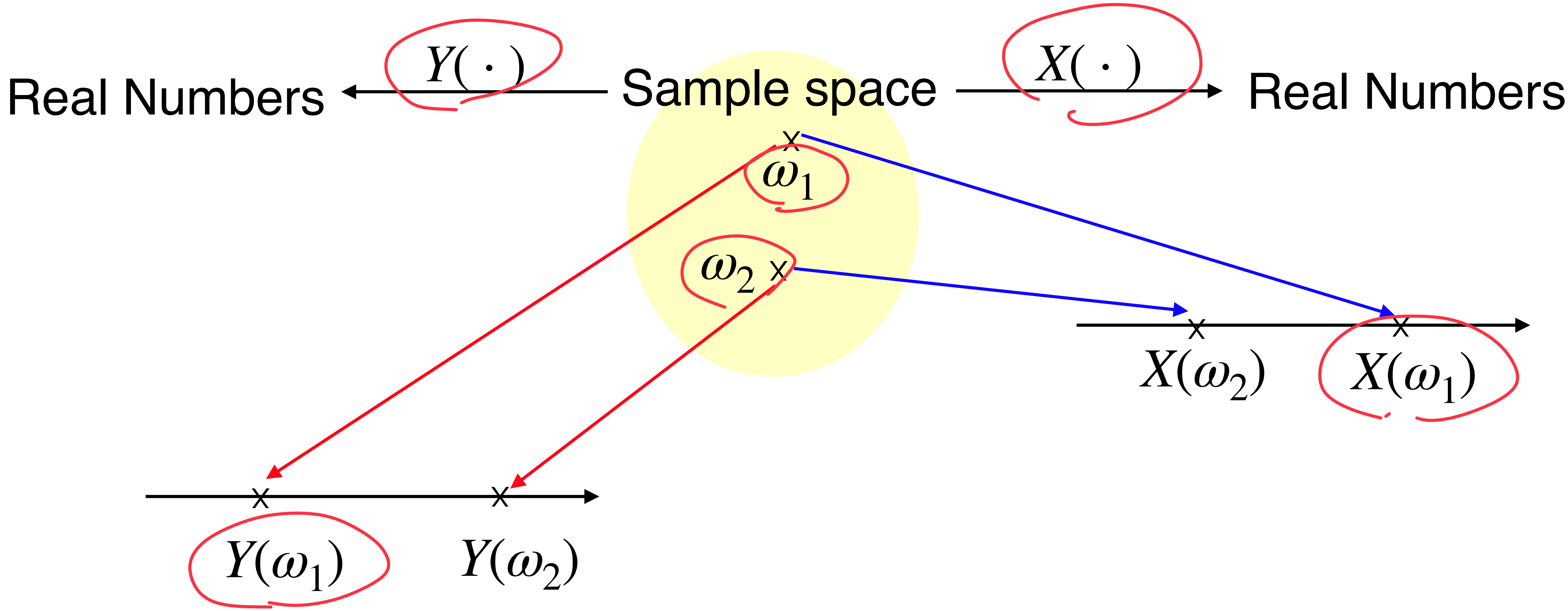
$$= \underbrace{\int_0^{+\infty}\int_0^x 1 \cdot dt \cdot f_X(x)\,dx}_{} + \underbrace{\int_{-\infty}^{0}\int_0^x 1 \cdot dt \cdot f_X(x)\,dx}_{}$$



$$\int_0^\infty \underbrace{\left(\int_t^\infty f_X(x)\,dx\right)}_{=1-F_X(t)} dt = \int_0^\infty (1 - F_X(t))\,dt$$

Joint CDF of Two Random Variables

Recall: Random Variables Defined on Ω



- Could we study the "CDF" regarding both X and Y?

Joint CDF

$F_X(t) = P(X \le t)$

Joint CDF: Let X and Y be two random variables defined on the same sample space Ω. The joint CDF $F_{XY}(t, u)$ is defined as

$$F_{XY}(t, u) = P(X \le t, Y \le u), \quad \forall t, u \in \mathbb{R}$$

$= P(\{\omega : X(\omega) \le t \text{ and } Y(\omega) \le u\})$

- $0 \le F_{XY}(t, u) \le 1$? (Axioms of probability)
- Suppose $t_1 \le t_2$ and $u_1 \le u_2$, then $F_{XY}(t_1, u_1) \le F_{XY}(t_2, u_2)$? Yes!
- What is $F_{XY}(\infty, \infty)$? How about $F_{XY}(-\infty, -\infty)$?

$F_{XY}(\infty, \infty) = P(\Omega) = 1$

$F_{XY}(-\infty, -\infty) = P(\emptyset) = 0$

Event Probabilities and Joint CDF (I)

$$F_{XY}(t, u) = P(X \leq t, Y \leq u), \;\; \forall t, u \in \mathbb{R}$$

- $P(X \leq t)$ = ?

- $P(Y \leq u)$ = ?

Marginal CDF

Marginal CDF: Let X and Y be two random variables defined on the same sample space Ω, and the joint CDF is $F_{XY}(t, u)$. The marginal CDF of X and Y are

$$F_X(t) = P(X \leq t) = F_{XY}(t, \infty)$$

$$F_Y(t) = P(Y \leq t) = F_{XY}(\infty, t)$$

Event Probabilities and Joint CDF (II)

$$F_{XY}(t, u) = P(X \leq t, Y \leq u), \ \forall t, u \in \mathbb{R}$$

- $P(t_1 < X \leq t_2)$ = ?

- $P(u_1 < Y \leq u_2)$ = ?

Event Probabilities and Joint CDF (III)

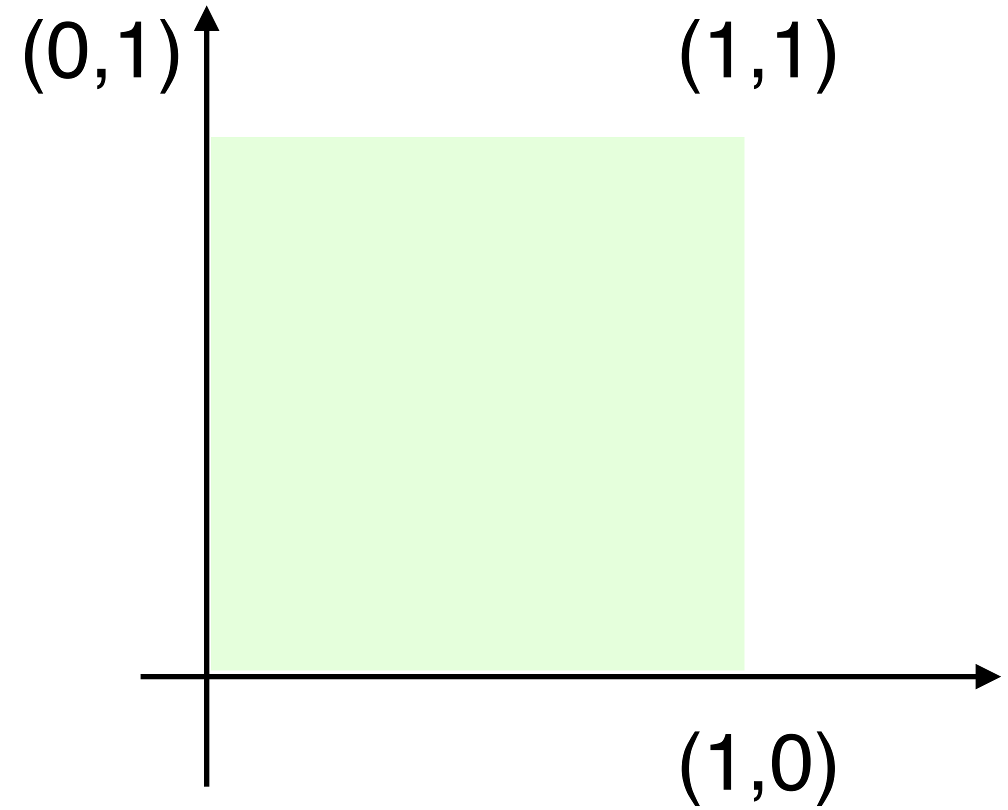
$$F_{XY}(t, u) = P(X \leq t, Y \leq u), \ \forall t, u \in \mathbb{R}$$

- $P(t_1 < X \leq t_2, u_1 < Y \leq u_2)$ = ?

- $P(t_1 < X < t_2, u_1 < Y \leq u_2)$ = ?

Example: A Random Point in a Unit Square

- Example: Suppose a point (X, Y) is selected randomly from the unit square $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.
  - What is the joint CDF of X and Y, i.e. $F_{XY}(t, u)$?



1-Minute Summary

1. Expected Value and Variance of Continuous Random Variables

- Mean and variance of exponential r.v.s
- Alternative expression of expected value via CDF

2. Joint Distributions of Two Random Variables

- Two random variables defined on the same Ω
- Definitions of joint CDF and marginal CDF