6.5 Average value of a function

積分版本的均值定理

Recall: Mean Value Theorem (MVT):
f is continuous on $[a,b]$ 閉連續 and differentiable on (a,b) 開可微,
$\Longrightarrow\ \exists\ c \in (a,b),\ \ni\ f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

What is average? $y_1, \cdots, y_n$, then $y_{ave} = \dfrac{y_1+\cdots+y_n}{n} = \dfrac{\sum_{i=1}^{n} y_i}{n}$.

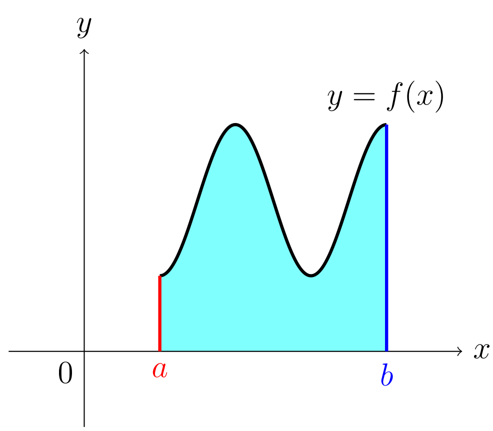
把 $[a,b]$ 分成 n 等分, 挑 n 個樣本 x_i^* 算平均:
$y_i = f(x_i^*)$, $\Delta x = \dfrac{b-a}{n} \iff n = \dfrac{b-a}{\Delta x}$, then

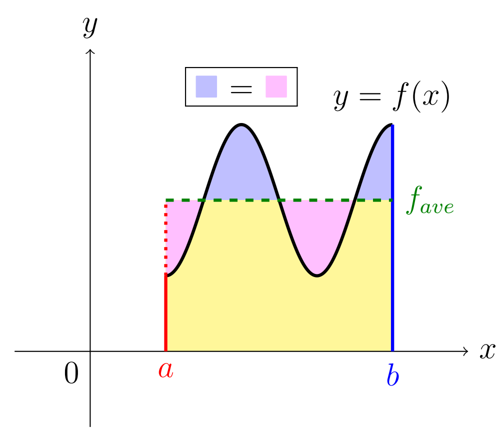
$$\frac{\sum_{i=1}^{n} y_i}{n} = \frac{\sum_{i=1}^{n} f(x_i^*)}{(b-a)/\Delta x} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

When $n \to \infty$, $\displaystyle\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\ dx$.

Define: The ***average value*** 平均值 of f on $[a,b]$ is

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx}.$$





Theorem 1 (The Mean Value Theorem for Integrals)
If f is ***continuous*** *on $[a,b]$, then $\exists\ c \in$ $\underline{(\boldsymbol{a},\boldsymbol{b})\text{(課本寫 }[a,b])}$ $\ni$*

$$\boxed{\boldsymbol{f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx}}.$$

♦ **Proof.** (Exercise 6.5.25) Let $F(x) = \displaystyle\int_a^x f(t)\ dt$.
By **TFTC**, F is continuous on $[a,b]$ and differentiable on (a,b), $F' = f$, and

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

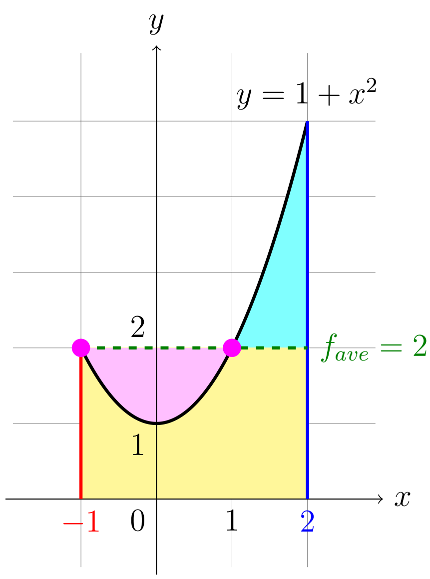
By **MVT**, $\exists\ c \in (a,b)$, $\ni$

$$f(c) = F'(c) = \frac{F(b)-F(a)}{b-a} = \frac{1}{b-a}\int_a^b f(x)\,dx. \qquad \blacksquare$$

Note: c 也有可能發生在端點 ($f(a) = f(c)$ 或 $f(b) = f(c)$, 所以是 $[a,b]$).
因爲定理證明只保證裡面 ((a,b)) 有, 沒說端點不會有.

Example 0.1 *Find f_{ave} of $f(x) = 1 + x^2$ on $[-1,2]$ and c with $f(c) = f_{ave}$.*

$$f_{ave} = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\ dx = \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} = 2.$$
$f(c) = 1 + c^2 = 2,\ c = \pm 1$. *(兩個都要寫)* $\blacksquare$



(定理保證裡面有 ($c = 1 \in (-1,2)$)),
沒說端點 ($c = -1$) 不會有。
題目指定範圍 $[-1,2]$,
所以端點如果有也要寫。)