1. (a) ○ (Linear transformation of a standard normal random variable is a normal random variable)

(b) X ($\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} A_n = \{ x : x \in A_n, \text{ for all but finitely many } n \}$

$= B \cap C$

Since $\{F_k\}$ is a countably infinite sequence and hence only the elements in $B \cap C$ would appear in all but finitely many A_n.

(c) ○ (For any $k \in \mathbb{N}$,

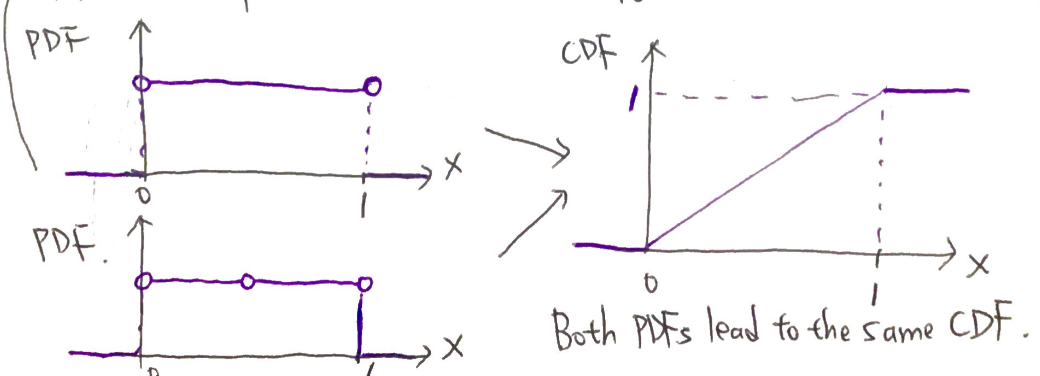
$$P(X>k) = P(\min(X_1,\cdots,X_n)>k)$$
$$= P(X_1>k, X_2>k, \cdots\cdots, X_n>k)$$
$$= (1-p_1)^k \cdot (1-p_2)^k \cdots (1-p_n)^k$$
$$= [(1-p_1)(1-p_2)\cdots(1-p_n)]^k$$
)

(d) X (Define events $E_n = \{ X \in [0, \frac{8}{4+n}] \}$. Then, E_n is a decreasing sequence.

Therefore, $P(X=0) = P(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} P(E_n)$

$$= \lim_{n\to\infty} \frac{1+e^{-n}}{6}$$
$$= \frac{1}{6}$$
)

(e) X (A counterexample mentioned in Lecture 10:



Both PDFs lead to the same CDF.)

(e) X

Recall Lecture 9:

$$E[|Y|] = 2 \cdot \sum_{k=1}^{\infty} \frac{1}{2k(k+1)} \cdot k = \infty.$$

Hence, $E[Y]$ does not exist.

(f) O

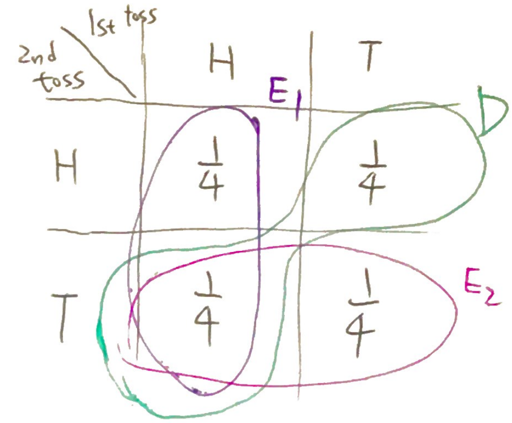
By the Binomial expansion:

$$(x+y)^n = \sum_{k=0}^{n} C_k^n \cdot x^k \cdot y^{n-k} \overset{x=1,\, y=1}{\Rightarrow} 2^n = \sum_{k=1}^{n} C_k^n$$

2. $E_1 = \{\text{1st toss is a head}\}$

$E_2 = \{\text{2nd toss is a tail}\}$

$D = \{\text{2 tosses have different results}\}$



$P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{2}$, $P(D) = \frac{1}{2}$

- To check pairwise independence:

$$\left.\begin{aligned} P(E_1 \cap E_2) &= P(\{HT\}) = \tfrac{1}{4} = P(E_1) \cdot P(E_2) \\ P(E_1 \cap D) &= P(\{HT\}) = \tfrac{1}{4} = P(E_1) \cdot P(D) \\ P(E_2 \cap D) &= P(\{HT\}) = \tfrac{1}{4} = P(E_2) \cdot P(D) \end{aligned}\right\} \quad \underline{E_1, E_2, D \text{ are pairwise independent.}}$$

- To check independence, we need to check both pairwise independence and the following:

$$P(E_1 \cap E_2 \cap D) = P(\{HT\}) = \frac{1}{4} \neq P(E_1) \cdot P(E_2) \cdot P(D).$$

Hence, <u>E_1, E_2, D are not independent (despite the pairwise independence)</u>.

$\square$

3. $U \sim \mathrm{Unif}(0,1)$.

$$G(z) = \ln\left(\frac{z}{1-z}\right) \Rightarrow G^{-1}(t) = \frac{e^t}{1+e^t}$$

By the Inverse transform sampling, we know the CDF of $G(U)$ is characterized by $G^{-1}(t)$.

That is, $P(G(U) \leq t) = G^{-1}(t) = \dfrac{e^t}{1+e^t}$, for all $t \in \mathbb{R}$.

$\square$

4. $X \sim N(0,1)$, The PDF of X: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, for all $x \in \mathbb{R}$.

(a).

$$E[e^X] = \int_{-\infty}^{+\infty} e^x \cdot f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}+x}\,dx$$

by LOTUS

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2+\frac{1}{2}}\,dx$$

$$= e^{\frac{1}{2}} \cdot \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\,dt}_{=1}$$

...... by change of variable $\begin{cases} t = x-1. \\ dt = dx \end{cases}$

$$= e^{\frac{1}{2}}$$

$\square$

$$E[(e^X)^2] = \int_{-\infty}^{+\infty} e^{2x} \cdot f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}+2x}\,dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2+2}\,dx$$

$$= e^2 \cdot \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\,du}_{=1}$$

...... by change of variable $\begin{cases} u = x-2 \\ du = dx \end{cases}$

$$= e^2$$

Therefore, $\text{Var}[e^X] = E[(e^X)^2] - (E[e^X])^2$

$$= e^2 - (e^{\frac{1}{2}})^2$$

$$= e^2 - e^1$$

$\square$

(b). $Y = \begin{cases} 2X+3, & \text{if } X>1 \\ X-5, & \text{otherwise} \end{cases}$

To find the CDF of Y, we consider the following cases:

(i) <u>$t \le -4$:</u>

$$P(Y\le t) = P\Big(\overbrace{\{X-5\le t \text{ and } X\le 1\}}^{\{X-5\le t\} \text{ since } t\le -4} \cup \overbrace{\{2X+3\le t \text{ and } X>1\}}^{\text{empty set since } t\le -4}\Big)$$

$$= P(\{X-5\le t\})$$

$$= \Phi(t+5)$$

(ii) <u>$-4 < t \le 5$:</u>

$$P(Y\le t) = P(X\le 1) = \Phi(1).$$

(iii) <u>$t>5$:</u>

$$P(Y\le t) = P\Big(\overbrace{\{X-5\le t \text{ and } X\le 1\}}^{\{X\le 1\}} \cup \overbrace{\{2X+3\le t \text{ and } X>1\}}^{\{1<X\le \frac{t-3}{2}\}}\Big)$$

$$= P\left(\{X\le 1\}\cup\left\{1<X\le \tfrac{t-3}{2}\right\}\right)$$

$$= P\left(\left\{X\le \tfrac{t-3}{2}\right\}\right)$$

$$= \Phi\left(\tfrac{t-3}{2}\right).$$

Hence, $F_Y(t) \equiv P(Y\le t) = \begin{cases} \Phi(t+5), & t\le -4 \\ \Phi(1), & -4 < t\le 5 \\ \Phi\left(\frac{t-3}{2}\right), & t>5. \end{cases}$

By $F_Y(t)$, we know Y is <u>not</u> a normal random variable. □

5.

(a). $Y \sim Unif(-2, 2)$, $Z = X + Y$.

$$P(Z<0 \mid X=+1) = P(Y<-1 \mid X=+1)$$ ...... by the fact that $Z = X+Y$

$$= P(Y<-1)$$ ...... Y is independent from X.

$$= \frac{1}{4}$$ ...... Y is continuous uniform between -2 and 2.

$\square$

Similarly, we know

$$P(Z \geq 0 \mid X=-1) = P(Y \geq 1 \mid X=-1)$$

$$= P(Y \geq 1)$$

$$= \frac{1}{4}.$$

$\square$

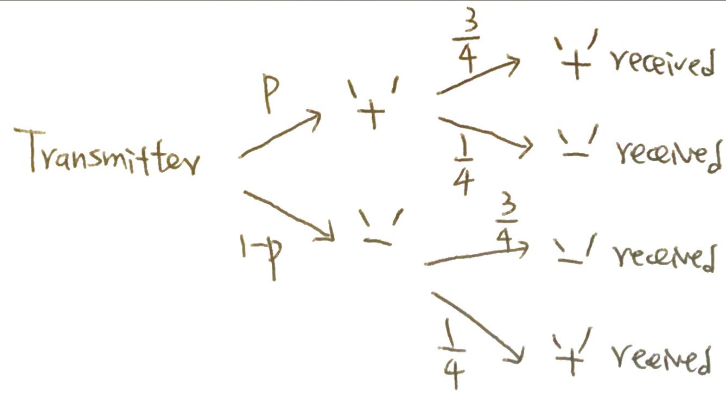
(b). $$P(\text{'-' sent} \mid \text{'+' recv}) = \frac{P(\text{'-' sent and '+' recv})}{P(\text{'-' sent}) \cdot P(\text{'+' recv} \mid \text{'-' sent}) + P(\text{'+' sent}) \cdot P(\text{'+' recv} \mid \text{'+' sent})}$$

$$= \frac{(1-p) \cdot \frac{1}{4}}{(1-p) \cdot \frac{1}{4} + p \cdot \frac{3}{4}}$$

$$= \frac{1-p}{2p+1}$$

$\square$.

(c).



Q = # of '+'s transmitted in the observation window T.

V = # of transmissions in the observation window T.

For any $k \in \mathbb{N}$:

$$P(Q=k) = \sum_{n=0}^{\infty} P(Q=k \mid V=k+n) \cdot P(V=k+n) \quad \text{... by total probability theorem.}$$

$$= \sum_{n=0}^{\infty} \left(C_k^{k+n} \cdot p^k \cdot (1-p)^{(k+n)-k} \right) \cdot \frac{e^{-\lambda T} \cdot (\lambda T)^{k+n}}{(k+n)!}$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{(k+n)!}}{k!\, n!} p^k \cdot (1-p)^n \cdot \frac{e^{-\lambda T} \cdot (\lambda T)^{k+n}}{\cancel{(k+n)!}}$$

$$= \frac{e^{-\lambda p T} \cdot p^k (\lambda T)^k}{k!} \sum_{n=0}^{\infty} \frac{e^{-\lambda(1-p)T} \cdot (1-p)^n \cdot (\lambda T)^n}{n!}$$

PMF of a Poisson random variable with rate $\lambda(1-p)$ and observation window T

$$= \frac{e^{-\lambda p T} \cdot ((\lambda p) T)^k}{k!}$$

Hence, $Q \sim \text{Poisson}(\lambda p, T)$.

$\square$

6.

(a) $X \sim \text{Binomial}(n=20,\ p=0.8)$.

$$P_k = P(X=k) = C_k^{20} \cdot p^k \cdot (1-p)^{20-k}, \quad \text{with } p=0.8$$

$$\frac{P_{k+1}}{P_k} = \frac{C_{k+1}^{20} \cdot p^{k+1} \cdot (1-p)^{20-(k+1)}}{C_k^{20} \cdot p^k \cdot (1-p)^{20-k}} = \frac{\frac{\cancel{20!}}{(k+1)!\,(19-k)!} \cdot p^{k+1}(1-p)^{20-(k+1)}}{\frac{\cancel{20!}}{k!\,(20-k)!} \cdot p^k \cdot (1-p)^{20-k}}$$

$$= \frac{(20-k) \cdot p}{(k+1) \cdot (1-p)}$$

Then, $\frac{P_{k+1}}{P_k} \geqslant 1 \iff \frac{(20-k) \cdot p}{(k+1)(1-p)} \geqslant 1$

$$\iff k \leqslant 21p - 1 = 15.8$$

Hence, the k that maximizes P_k is 16.

$\square$

(b).

$$L(\alpha) = P(E)$$

$$= C_2^{20} \cdot \alpha^2 \cdot (1-\alpha)^{18}$$

$$\frac{dL(\alpha)}{d\alpha} = C_2^{20} \cdot \left(2\alpha \cdot (1-\alpha)^{20-2} + \alpha^2 \cdot 18 \cdot (1-\alpha)^{17} \cdot (-1) \right)$$

$$= C_2^{20} \cdot \alpha \cdot (1-\alpha)^{17} \cdot \left(2 \cdot (1-\alpha) - 18\alpha \right)$$

Then, $\frac{dL(\alpha)}{d\alpha} = 0$ if $\alpha = 0, 1$, or $\frac{1}{10}$.

It is easy to check that the maximizer of $L(\alpha)$ is $\alpha = \frac{1}{10}$.

$\square$