15.8 Triple integrals in spherical coordinates

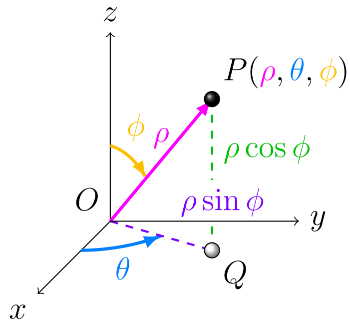
1. spherical coordinates

2. triple integrals in spherical coordinates

$$x \to \rho\sin\phi\cos\theta,\ y \to \rho\sin\phi\sin\theta,\ z \to \rho\cos\phi,\ dV \to \rho^2\sin\phi\ d\rho\ d\theta\ d\phi.$$

0.1 Spherical Coordinates

Spherical ([ˈsfɛrəkl]) ***coordinate system*** 球面坐標系: $P(\rho, \theta, \phi)$.



ρ (“rho” [ro]): P 到 O 的距離;
θ: OP 在 xy-平面的投影 OQ 與正 x-軸夾角;
ϕ (“phi” [faɪ]): OP 與正 z-軸夾角。

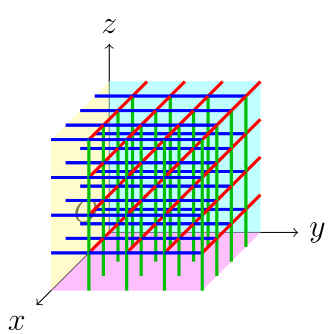
Attention: 表示法唯一 (except possibly on the boundary):

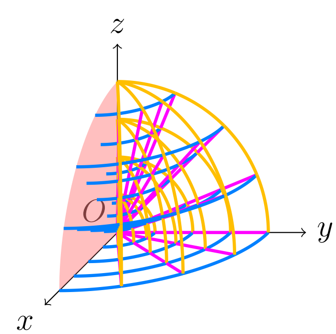
$$\rho \ge 0,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

座標變換: spherical coordinate $\longleftrightarrow$ Cartesian (rectangular) coordinate:

$$\boxed{\boldsymbol{x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi}}$$

$$\boxed{\rho = \sqrt{x^2+y^2+z^2} \ge 0 \qquad \phi = \cos^{-1}\left(\frac{z}{\rho}\right) \in [0,\pi]}$$



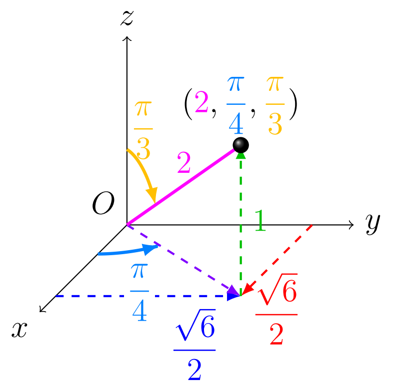


Example 0.1 *(a) Plot the point with spherical coordinates* $(2, \frac{\pi}{4}, \frac{\pi}{3})$ *and find its rectangular coordinates.*
(b) Find spherical coordinates of the point with rectangular coordinates $(0, 2\sqrt{3}, -2)$.

(a)

$x = \rho \sin\phi \cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4}$
$= 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$;
$y = \rho\sin\phi\sin\theta = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4}$
$= 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$;
$z = \rho\cos\phi = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1.$

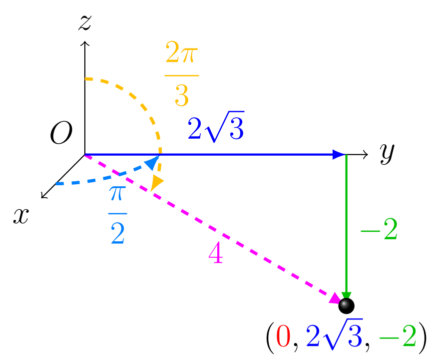
The point is $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$ *in rectangular coordinates.*



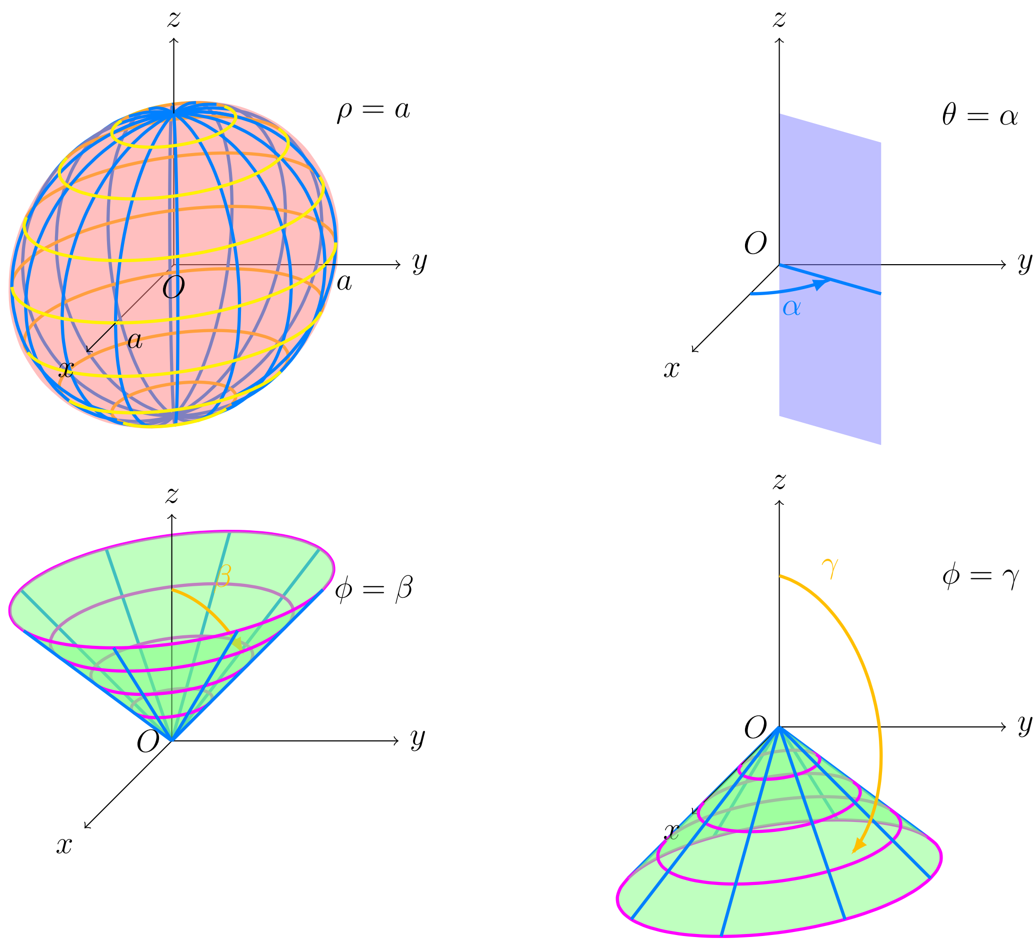
(b)

$\rho = \sqrt{x^2+y^2+z^2} = 4(\geq 0)$;
$\phi = \cos^{-1}\frac{z}{\rho} = \cos^{-1}\frac{-2}{4} = \frac{2\pi}{3}(\in[0,\pi])$;
$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{0}{2\sqrt{3}} = 0,$
$\sin\theta = \frac{y}{\rho\sin\phi} = \frac{2\sqrt{3}}{2\sqrt{3}} = 1,$
$\theta = \frac{\pi}{2}(\in[0,2\pi]).$

The point is $\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)$ *in spherical coordinates.* ■



Example 0.2 *Graphs: (a) $\rho = a$; (b) $\theta = \alpha$; (c) $\phi = \beta$, $0 < \beta < \dfrac{\pi}{2}$; (d) $\phi = \gamma$, $\dfrac{\pi}{2} < \gamma < \pi$.*



♦ *xz-plane:* $(\theta = 0) \cup (\theta = \pi)$,

yz-plane: $(\theta = \dfrac{\pi}{2}) \cup (\theta = \dfrac{3\pi}{2})$,

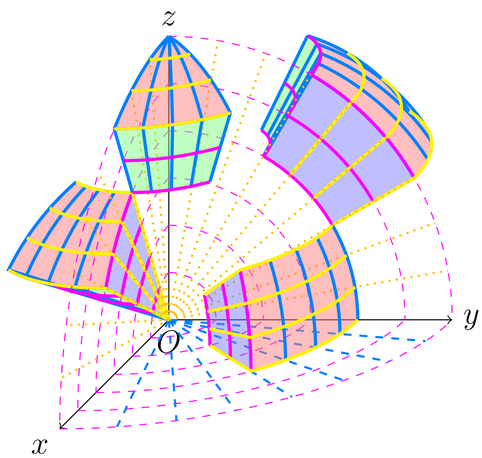
xy-plane: $\phi = \dfrac{\pi}{2}$.

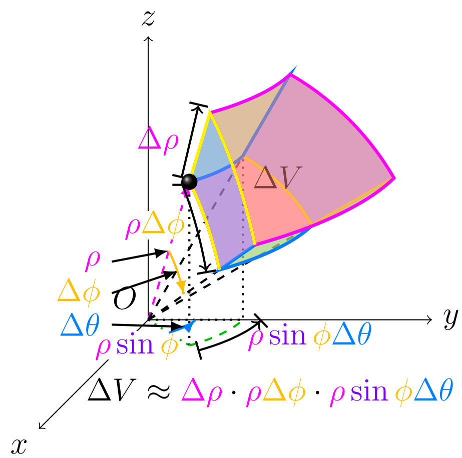
0.2 Triple Integrals in Spherical Coordinates

Define: A ***spherical wedge***([wɛdʒ]畏懼) 球楔 is a rectangular box in spherical coordinate system.

$$E = \{(\rho, \theta, \phi) : (0 \leq) a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

where $\beta - \alpha \leq 2\pi$, $d - c \leq \pi$.





把 $[a, b] \times [\alpha, \beta] \times [c, d]$ 分成 $\ell \times m \times n$ 個小球楔 E_{ijk}, 體積是

$$\Delta V_{ijk} \approx \Delta\rho \cdot \rho_i \Delta\phi \cdot \rho_i \sin\phi_k \Delta\theta = {\rho_i}^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$$

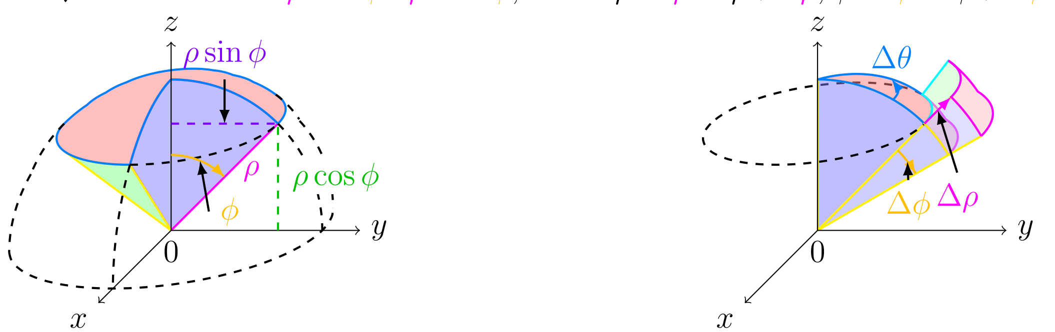
In fact, by Mean Value Theorem (exercise 15.8.49), $\exists\ (\widetilde{\rho_i}, \widetilde{\theta_j}, \widetilde{\phi_k}) \in E_{ijk} \ni$

$$\Delta V_{ijk} = \widetilde{\rho_i}^2 \sin\widetilde{\phi_k} \Delta\rho\Delta\theta\Delta\phi,$$

where $\rho_i < \widetilde{\rho_i} < \rho_i + \Delta\rho$, $\theta_j < \widetilde{\theta_j} < \theta_j + \Delta\theta_j$, $\phi_k < \widetilde{\phi_k} < \phi_k + \Delta\phi$.

$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk} \\
&= \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\widetilde{\rho_i}\sin\widetilde{\phi_k}\cos\widetilde{\theta_j}, \widetilde{\rho_i}\sin\widetilde{\phi_k}\sin\widetilde{\theta_j}, \widetilde{\rho_i}\cos\widetilde{\phi_k}) \\
&\qquad \cdot \widetilde{\rho_i}^2 \sin\widetilde{\phi_k}\Delta\rho\Delta\theta\Delta\phi \\
&= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi
\end{aligned}$$

♦ **Proof.** $\Delta V = \widetilde{\rho}^2 \sin\widetilde{\phi}\Delta\rho\Delta\theta\Delta\phi$, where $\rho < \widetilde{\rho} < \rho + \Delta\rho$, $\phi < \widetilde{\phi} < \phi + \Delta\phi$



spherical sector 球扇形 = spherical cap 球蓋 + cone 圓錐

$$\text{cap} = \int_{\rho\cos\phi}^{\rho} \pi(\rho^2 - z^2)\, dz = \frac{1}{3}\pi\rho^3 \left(2 + \cos^3\phi - 3\cos\phi\right),$$

$$\text{cone} = \frac{1}{3}\pi(\rho\sin\phi)^2 \cdot \rho\cos\phi = \frac{1}{3}\pi\rho^3\sin^2\phi\cos\phi,$$

$$\text{sector} = \text{cap} + \text{cone} = \frac{2}{3}\pi\rho^3(1 - \cos\phi).$$

$$\text{one piece (increment in } \theta) = \frac{2}{3}\pi\rho^3(1-\cos\phi)\cdot\frac{\Delta\theta}{2\pi} = \frac{1}{3}\rho^3\Delta\theta(1-\cos\phi).$$

$$\text{one piece (increment in } \phi) = \frac{1}{3}\rho^3\Delta\theta(1-\cos(\phi+\Delta\phi)) - \frac{1}{3}\rho^3\Delta\theta(1-\cos\phi)$$

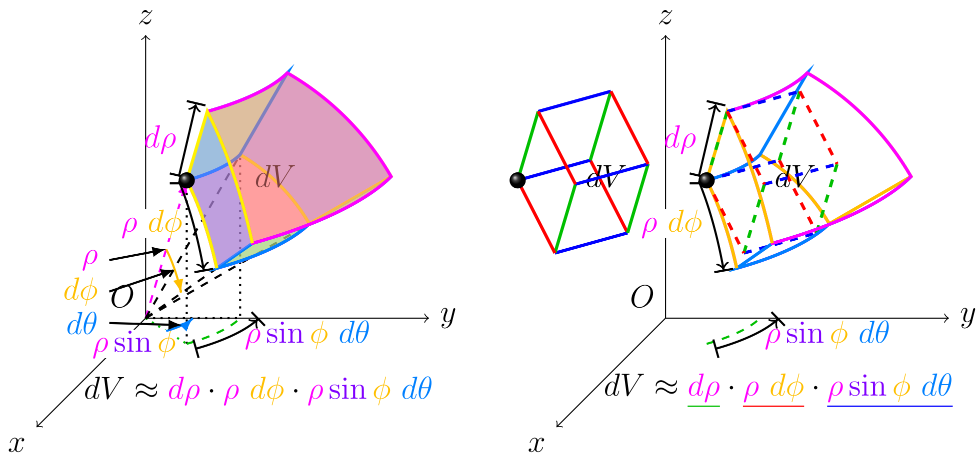
$$= \frac{1}{3}\rho^3\Delta\theta(\cos\phi - \cos(\phi+\Delta\phi)).$$

$$\text{(increment in } \rho)\ \Delta V = \frac{(\rho+\Delta\rho)^3 - \rho^3}{3}\Delta\theta(\cos\phi - \cos(\phi+\Delta\phi))$$

Let $f(x) = \dfrac{x^3}{3}$ and $g(y) = -\cos y$.

By Mean Value Theorem $f(b) - f(a) = f'(c)(b-a)$ for some $a < c < b$.

So $f(\rho+\Delta\rho) - f(\rho) = \widetilde{\rho}^2\Delta\rho$ for some $\rho < \widetilde{\rho} < \rho + \Delta\rho$,
and $g(\phi+\Delta\phi) - g(\phi) = \sin\widetilde{\phi}\Delta\phi$ for some $\phi < \widetilde{\phi} < \phi + \Delta\phi$.
Therefore, $\Delta V = \widetilde{\rho}^2\Delta\rho\cdot\Delta\theta\cdot\sin\widetilde{\phi}\Delta\phi = \widetilde{\rho}^2\sin\widetilde{\phi}\Delta\rho\Delta\theta\Delta\phi$. ■



$$\boxed{\begin{aligned}&\iiint\limits_E f(x,y,z)\ dV\\ &= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot\rho^2\sin\phi\ d\rho\ d\theta\ d\phi\end{aligned}}$$

$$E = \{(\rho,\theta,\phi) : (0\le)a\le\rho\le b,\ \alpha\le\theta\le\beta,\ c\le\phi\le d\}$$

Note: $x\to\rho\sin\phi\cos\theta$, $y\to\rho\sin\phi\sin\theta$, $z\to\rho\cos\phi$,

$$\star\ \boxed{dV\to \boldsymbol{\rho^2\sin\phi}\ d\rho\ d\theta\ d\phi}\ \star$$

Note: If "box" and
$f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot\rho^2\sin\phi = g(\rho)h(\theta)u(\phi)$, 可以分開

$$\iiint\limits_E f(x,y,z)\ dV = \int_a^b g(\rho)\ d\rho\int_\alpha^\beta h(\theta)\ d\theta\int_c^d u(\phi)\ d\phi$$

可以推廣到 general spherical region:

$$E = \{(\rho,\theta,\phi) : \alpha\le\theta\le\beta,\ c\le\phi\le d,\ (0\le)g_1(\theta,\phi)\le\rho\le g_2(\theta,\phi)\}$$

$$\iiint\limits_E f(x,y,z)\ dV$$

$$= \int_c^d\int_\alpha^\beta\int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

Timing: 積分由圓錐 (cone: $\phi=c$) 與球面 (sphere: $\rho=a$) 包圍的區域.

Example 0.3 *Evaluate* $\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$*, where* B *is the unit ball:*

$$B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$$

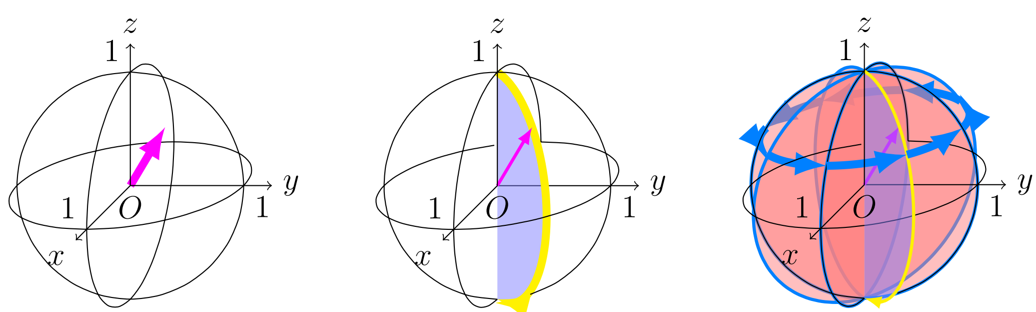
$$B = \{(\rho,\theta,\phi) : 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\},\ e^{(x^2+y^2+z^2)^{3/2}} = e^{\rho^3}.$$

$$\begin{aligned}
\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\, dV &= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{\rho^3}\cdot \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi \\
&= \int_0^{\pi} \sin\phi\ d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3}\ d\rho \qquad \textit{(可以分開)} \\
&= \Big[-\cos\phi\Big]_0^{\pi}\Big[\theta\Big]_0^{2\pi}\Big[\frac{1}{3}e^{\rho^3}\Big]_0^1 \\
&= [-(-1)-(-1)]\cdot 2\pi\cdot\frac{1}{3}(e-1) \\
&= \frac{4}{3}\pi(e-1).
\end{aligned}$$

■

Note: 球面座標系迭代積分 $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{\rho^3}\cdot\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$ 的過程:

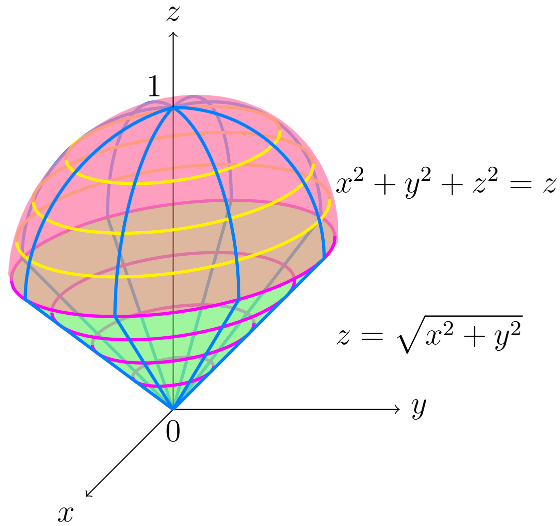
$$\int * \, d\rho \to \int * \, d\phi \to \int * \, d\theta$$

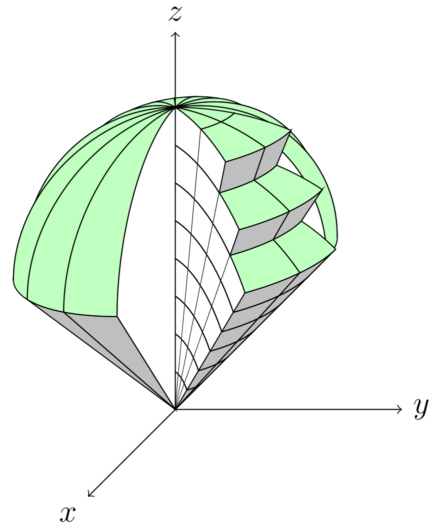


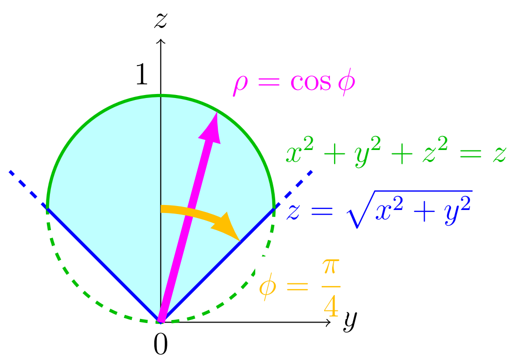
Question: $\int\ d\phi$ 跟 $\int\ d\theta$ 誰先誰後?

看上下界, 裡面的是<u>外面變數的函數</u>; 如果不是就沒差 (還可以分開).

Example 0.4 *Use spherical coordinates to find the volume of the solid lying above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.*







$$\rho^2 = x^2+y^2+z^2 = z = \rho\cos\phi,$$
$$\Longrightarrow \quad \rho = \cos\phi.$$
$$\rho\cos\phi = z = \sqrt{x^2+y^2} = \rho\sin\phi,$$
$$\cos\phi = \sin\phi,$$
$$\Longrightarrow \quad \phi = \frac{\pi}{4}.$$

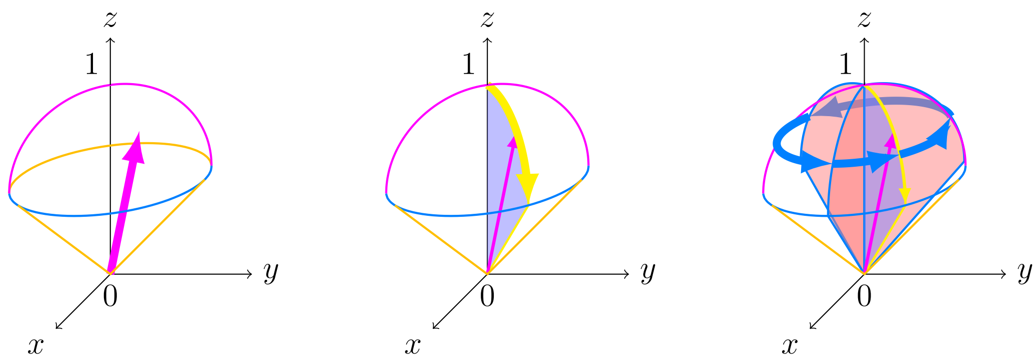
$$E = \{(\rho,\theta,\phi) : 0 \le \phi \le \frac{\pi}{4},\ 0 \le \theta \le 2\pi,\ 0 \le \rho \le \cos\phi\}.$$

$$\begin{aligned}
V(E) &= \iiint_E dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi \qquad (\text{沒 } \theta \text{ 的事, 可以先分開}) \\
&= \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi\cos^3\phi \, d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\pi/4} \\
&= \frac{2\pi}{3}\left[-\frac{1}{16} - \left(-\frac{1}{4}\right)\right] = \frac{\pi}{8}. \qquad \left(\frac{1}{2}\frac{4}{3}\pi\left(\frac{1}{2}\right)^3 + \frac{1}{3}\pi\left(\frac{1}{2}\right)^2\frac{1}{2} = \frac{\pi}{8}\right) \blacksquare
\end{aligned}$$

(注意, 課本上倫換順序。)

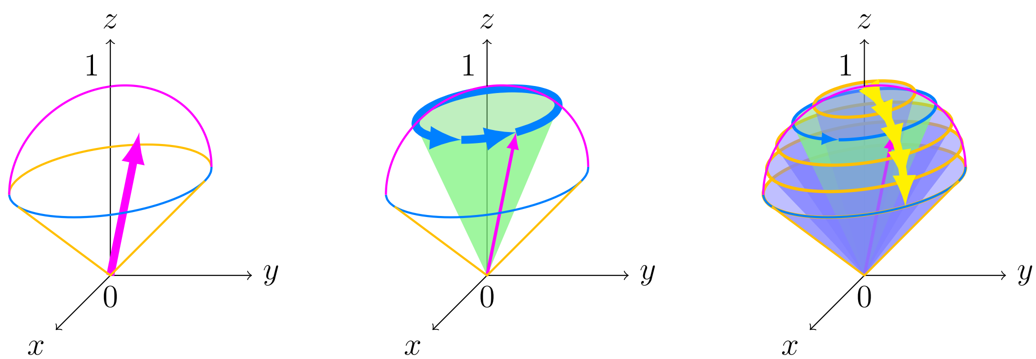
$$\int\int\int * \ d\rho \ d\phi \ d\theta$$

$$\int * \ d\rho \to \int * \ d\phi \to \int * \ d\theta$$



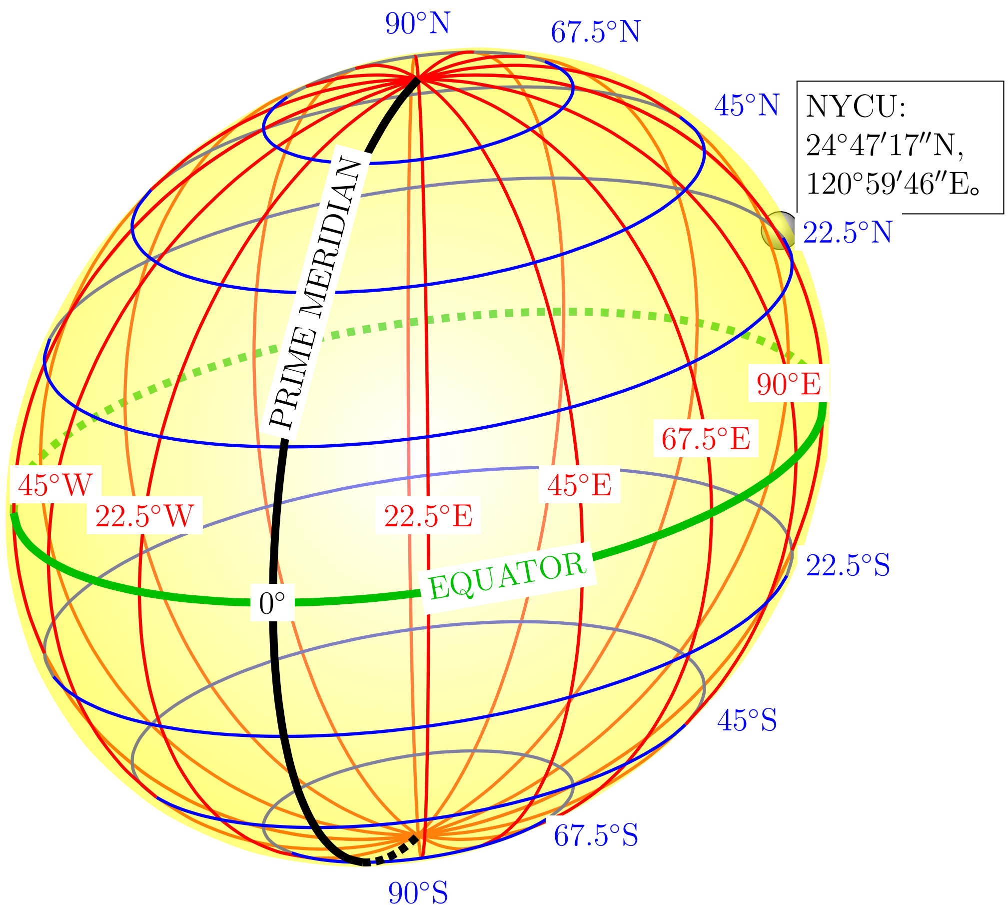
$$\int\int\int * \ d\rho \ d\theta \ d\phi$$

$$\int * \ d\rho \to \int * \ d\theta \to \int * \ d\phi$$



Additional: Geographic Coordinate System

Geographic Coordinate System 地理座標系 (經緯度):



- **Longitude** [ˈlɑndʒətjud] 經度, east- 東-, west- 西- (0° ∼ 180°)。
- **Latitude** [ˈlætətjud] 緯度, north- 北-, south- 南 (0° ∼ 90°)。
- **Equator** [ɪˈkwetɚ] 赤道, 0° 緯線。
- **Meridian** [məˈrɪdɪən] 子午線, 經線 (line of longitude)。
- **Prime Meridian** 本初子午線, 0° 線, 格林威治子午線或本初經線, 經過英國皇家格林威治天文台 (Royal Greenwich Observatory, RGO) 的經線。
- **International Date Line** 國際換日線, 約爲 180° 經線。