4.4 Indeterminate forms & ℓ'Hospital's rule

微分應用之三: 求未定型的極限。 (★ 授課順序與 §4.3 調換。)

1. Indeterminate forms & ℓ'Hospital's rule 未定型與羅畢達法則
2. Indeterminate product, difference, power 變形的未定型

0.1 Indeterminate forms & ℓ'Hospital's rule

Define: A limit $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is called an ***indeterminate form*** 未定型

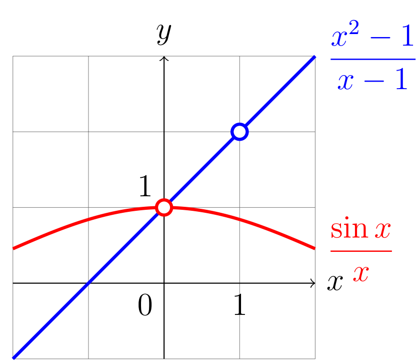
1. ***of type*** $\boxed{\dfrac{\mathbf{0}}{\mathbf{0}}}$ if $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$;
2. ***of type*** $\boxed{\dfrac{\boldsymbol{\infty}}{\boldsymbol{\infty}}}$ if $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$.

Note: "$f(x) \to \pm\infty$ and $g(x) \to \pm\infty$", 都是指: (f and g 都有無限極限)

$$\left.\begin{array}{lll} (1) & f(x) \to \infty & \text{and } g(x) \to \infty \\ (2) & f(x) \to -\infty & \text{and } g(x) \to \infty \\ (3) & f(x) \to \infty & \text{and } g(x) \to -\infty \\ (4) & f(x) \to -\infty & \text{and } g(x) \to -\infty \end{array}\right\} \text{都算是 } \frac{\infty}{\infty} \text{ 的未定型。}$$

Example 0.1

1. $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$ $(= 2,$ §2.3$)$. $(\dfrac{\mathbf{0}}{\mathbf{0}})$
2. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ $(= 1,$ §3.3$)$. $(\dfrac{\mathbf{0}}{\mathbf{0}})$
3. $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1} =?$ $(\dfrac{\mathbf{0}}{\mathbf{0}})$
4. $\lim\limits_{x\to \infty} \dfrac{e^x}{x^2} =?$ $(\dfrac{\boldsymbol{\infty}}{\boldsymbol{\infty}})$



Theorem 1 (L'Hospital's Rule 羅畢達法則) *(求未定型極限)*
Suppose f and g are ***differentiable***, $\boldsymbol{g'(x) \neq 0}$ *near a, (可微, g' 近 a 非零)*
and $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is an ***indeterminate form*** *of $\dfrac{\mathbf{0}}{\mathbf{0}}$ or $\dfrac{\infty}{\infty}$, then ($\dfrac{\mathbf{0}}{\mathbf{0}}$, $\dfrac{\infty}{\infty}$未定型)*

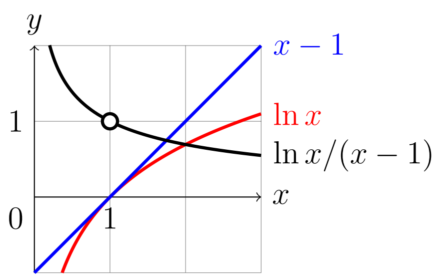
$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

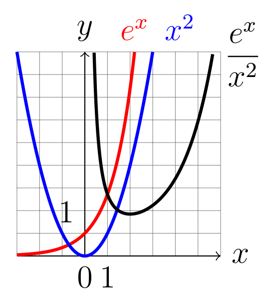
if the limit on the right side ***exists*** *or is $\pm\infty$. (存在或無限)*

Note: 未定型與羅畢達法則中, $x \to a$ 也可以是 $x \to a^+, a^-, \infty, -\infty$。
Note: 只要條件 (未定型) 滿足就可以重複使用。♻
♦: 以法國侯爵羅畢達 (Guillaume François Antoine, Marquis de l'Hôpital) 命名, l'(=la) 句首大寫, H 必大寫, l'hôpital [lopitɑl][法] = the hospital [英]。

Example 0.2 $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1} =?$

$\lim\limits_{x\to 1} \ln x = \ln 1 = 0$, $\lim\limits_{x\to 1} (x-1) = 0$, $(x-1)' = 1 \neq 0$ *near* 1. $(\dfrac{\mathbf{0}}{\mathbf{0}})$
$\therefore \lim\limits_{x\to 1} \dfrac{\ln x}{x-1} \stackrel{l'H}{=} \lim\limits_{x\to 1} \dfrac{(\ln x)'}{(x-1)'} = \lim\limits_{x\to 1} \dfrac{1/x}{1} = 1$. *(極限存在, 等號($\stackrel{l'H}{=}$)才成立。)* ■





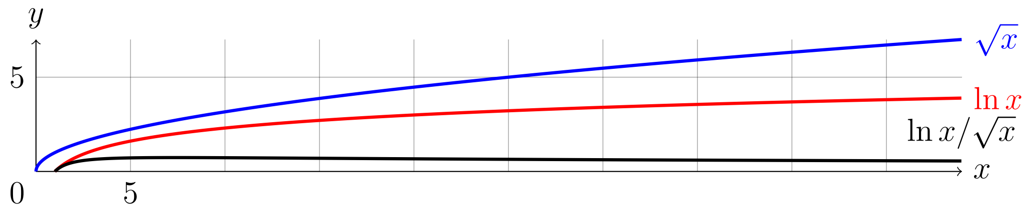
Example 0.3 *(twice)* $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} =?$

$\lim\limits_{x\to\infty} e^x = \infty$, $\lim\limits_{x\to\infty} x^2 = \infty$, $(x^2)' = 2x \neq 0$ *as* $x \to \infty$. $(\dfrac{\infty}{\infty})$
$\therefore \lim\limits_{x\to\infty} \dfrac{e^x}{x^2} \stackrel{l'H}{=} \lim\limits_{x\to\infty} \dfrac{(e^x)'}{(x^2)'} = \lim\limits_{x\to\infty} \dfrac{e^x}{2x}$. *(每次使用都要檢查是不是未定型。)*
$\lim\limits_{x\to\infty} e^x = \infty$, $\lim\limits_{x\to\infty} 2x = \infty$, $(2x)' = 2 \neq 0$ *as* $x \to \infty$. $(\dfrac{\infty}{\infty})$
$\therefore \lim\limits_{x\to\infty} \dfrac{e^x}{2x} \stackrel{l'H}{=} \lim\limits_{x\to\infty} \dfrac{(e^x)'}{(2x)'} = \lim\limits_{x\to\infty} \dfrac{e^x}{2} = \infty$. *(無限極限, 等號($\stackrel{l'H}{=}$)也成立。)* ■

Example 0.4 $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt{x}}=?$

$$\lim_{x\to\infty}\ln x=\infty,\ \lim_{x\to\infty}\sqrt{x}=\infty,\ (\sqrt{x})'=\frac{1}{2\sqrt{x}}\neq 0\ \textit{as}\ x\to\infty.\ (\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}})$$

$$\therefore\ \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}\overset{l'H}{=}\lim_{x\to\infty}\frac{1/x}{1/2\sqrt{x}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0.$$ ■



Example 0.5 *(another method)* $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3}=?$

$$\lim_{x\to 0}(\tan x-x)=0,\ \lim_{x\to 0}x^3=0,\ (x^3)'=3x^2\neq 0\ \textit{near}\ 0.\ (\frac{\mathbf{0}}{\mathbf{0}})$$

$$\therefore\ \lim_{x\to 0}\frac{\tan x-x}{x^3}\overset{l'H}{=}\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}$$ *(路線分歧)*

[Sol 1:] (繼續用羅畢達)

$$\lim_{x\to 0}(\sec^2 x-1)=0,\ \lim_{x\to 0}3x^2=0,\ (3x^2)'=6x\neq 0\ \textit{near}\ 0.\ (\frac{\mathbf{0}}{\mathbf{0}})$$

$$\therefore\ \lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}\overset{l'H}{=}\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}.$$

$$\lim_{x\to 0}2\sec^2 x\tan x=0,\ \lim_{x\to 0}6x=0,\ (6x)'=6\neq 0\ \textit{near}\ 0.\ (\frac{\mathbf{0}}{\mathbf{0}})$$

$$\therefore\ \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}\overset{l'H}{=}\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x}{6}=\frac{4\cdot 1^2\cdot 0^2+2\cdot 1^4}{6}=\frac{1}{3}.$$

(♦ 書上: $=\lim\limits_{x\to 0}\dfrac{\sec^2 x}{3}\lim\limits_{x\to 0}\dfrac{\tan x}{x}=\dfrac{1}{3}\lim\limits_{x\to 0}\dfrac{\tan x}{x}\overset{l'H}{=}\dfrac{1}{3}\lim\limits_{x\to 0}\dfrac{\sec^2 x}{1}=\dfrac{1}{3}.$*)*

[Sol 2:] (變形用極限律)

$$\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}=\lim_{x\to 0}\frac{\tan^2 x}{3x^2}=\lim_{x\to 0}\left(\frac{1}{3x^2}\frac{\sin^2 x}{\cos^2 x}\right)=\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\frac{1}{3\cos^2 x}\right)$$

$$=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\lim_{x\to 0}\frac{1}{3\cos^2 x}=1^2\cdot\frac{1}{3\cdot 1^2}=\frac{1}{3}.$$ ■

Example 0.6 *(If blindly using ℓ'Hospital's rule)* $\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=?$

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}\overset{l'H}{\cancel{=}}\lim_{x\to\pi^-}\frac{\cos x}{\sin x}=-\infty,\ (\cos x\to -1,\ \sin x\to 0)\ \textit{(}\textbf{Wrong!}\textit{)}$$

$\lim_{x\to\pi^-}\sin x=0,\ \lim_{x\to\pi^-}1-\cos x=2.$ *(不是未定型, 不能用!)*

$$\therefore \lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\frac{\lim_{x\to\pi^-}\sin x}{\lim_{x\to\pi^-}(1-\cos x)}=\frac{0}{2}=0.\ \textit{(不要瞎用!)}\qquad\blacksquare$$

Attention: 1. $\frac{f}{g}$ 要 $\frac{\mathbf{0}}{\mathbf{0}}$ 或 $\frac{\infty}{\infty}$ 未定型, 才可以使用羅畢達律改算 $\frac{f'}{g'}$ 的極限。

2. $\frac{f'}{g'}$ 的極限要存在或是 $\pm\infty$ 才能相等。

3. 注意! 不要把 $\frac{f'}{g'}$ 跟 $\left(\frac{f}{g}\right)'\left(=\frac{f'g-fg'}{g^2}\right)$ 搞錯。

0.2 Indeterminate product, difference, power

Product: $\mathbf{0}\cdot\infty$; Difference: $\infty-\infty$; Power: $\mathbf{0^0},\infty^{\mathbf{0}},\mathbf{1}^\infty$. ($\mathbf{0}^\infty$ 不是)

(a) $\lim_{x\to a}fg$ of type $\boxed{\mathbf{0}\cdot\infty}$
if $f\to 0$ and $g\to\pm\infty$ as $x\to a$: 挑一個除到下面去。

$$\begin{aligned}\lim_{x\to a}fg &= \lim_{x\to a}\frac{f}{1/g}\overset{l'H}{=}\lim_{x\to a}\frac{(f)'}{(1/g)'}, && (\mathbf{0}\cdot\infty\to\frac{\mathbf{0}}{\mathbf{0}})\\ \text{or} &= \lim_{x\to a}\frac{g}{1/f}\overset{l'H}{=}\lim_{x\to a}\frac{(g)'}{(1/f)'}. && (\mathbf{0}\cdot\infty\to\frac{\infty}{\infty})\end{aligned}$$

Example 0.7 $\lim_{x\to0^+}x\ln x=?$

$$\lim_{x\to0^+}x=0,\ \lim_{x\to0^+}\ln x=-\infty,\ (\frac{1}{x})'=-\frac{1}{x^2}\neq 0\ \textit{near}\ 0.\ (\mathbf{0}\cdot\infty)$$

先試 $(\frac{\mathbf{0}}{\mathbf{0}})$ $\lim_{x\to0^+}x\ln x=\lim_{x\to0^+}\frac{x}{\frac{1}{\ln x}}\overset{l'H}{=}\lim_{x\to0^+}\frac{1}{\frac{-1}{(\ln x)^2}\frac{1}{x}}=\lim_{x\to0^+}-x(\ln x)^2$, 變複雜, 但別放棄。

改用 $(\frac{\infty}{\infty})$ $\lim_{x\to0^+}x\ln x=\lim_{x\to0^+}\frac{\ln x}{1/x}\overset{l'H}{=}\lim_{x\to0^+}\frac{1/x}{-1/x^2}=\lim_{x\to0^+}-x=0.\qquad\blacksquare$

Note: Type $\mathbf{0}\cdot\infty$ 不一定都是 0。Ex: $\lim_{x\to0}x\cdot\frac{c}{x}=c.$

(b) $\lim_{x\to a}(f-g)$ of type $\boxed{\infty - \infty}$

if $\begin{cases} f\to\infty \\ g\to\infty \end{cases}$ or $\begin{cases} f\to-\infty \\ g\to-\infty \end{cases}$ as $x\to a$: 合併成 $\dfrac{\mathbf{0}}{\mathbf{0}}$ 或 $\dfrac{\infty}{\infty}$。

Example 0.8 *(7th ed.)* $\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)=?$

$\lim_{x\to\frac{\pi}{2}^-}\sec x=\infty$, $\lim_{x\to\frac{\pi}{2}^-}\tan x=\infty$. $(\infty-\infty)$

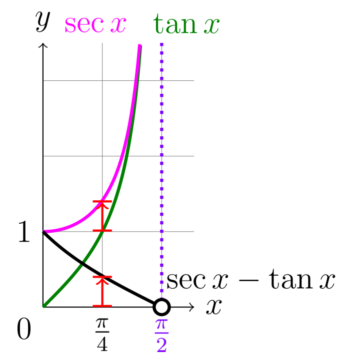
變形: $\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)$

$=\lim_{x\to\frac{\pi}{2}^-}(\dfrac{1}{\cos x}-\dfrac{\sin x}{\cos x})=\lim_{x\to\frac{\pi}{2}^-}\dfrac{1-\sin x}{\cos x}$

$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x)=0$, $\lim_{x\to\frac{\pi}{2}^-}\cos x=0$,

$(\cos x)'=-\sin x\neq 0$ *near* $\dfrac{\pi}{2}^-$. $(\dfrac{\mathbf{0}}{\mathbf{0}})$

$\implies \overset{l'H}{=}\lim_{x\to\frac{\pi}{2}^-}\dfrac{-\cos x}{-\sin x}=\dfrac{0}{1}=0.$ ∎



Note: Type $\infty-\infty$ 不一定都是 0。Ex: $\lim_{x\to\infty}[(x+c)-x]=c$.

Example 0.9 $\lim_{x\to1^+}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)=?$

$\lim_{x\to1^+}\dfrac{1}{\ln x}=\infty$, $\lim_{x\to1^+}\dfrac{1}{x-1}=\infty$. $(\infty-\infty)$

通分: $\lim_{x\to1^+}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)=\lim_{x\to1^+}\dfrac{x-1-\ln x}{(x-1)\ln x}$.

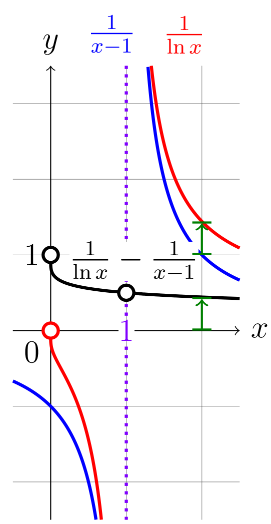
$\lim_{x\to1^+}(x-1-\ln x)=0$, $\lim_{x\to1^+}[(x-1)\ln x]=0$,

$[(x-1)\ln x]'=\ln x+1-\dfrac{1}{x}\neq 0$ *near* 1^+. $(\dfrac{\mathbf{0}}{\mathbf{0}})$

$\overset{l'H}{=}\lim_{x\to1^+}\dfrac{1-1/x}{\ln x+1-1/x}\overset{(\times x)}{=}\lim_{x\to1^+}\dfrac{x-1}{x\ln x+x-1}$ $(\dfrac{\mathbf{0}}{\mathbf{0}})$

$\overset{l'H}{=}\lim_{x\to1^+}\dfrac{1}{\ln x+x/x+1}=\dfrac{1}{0+1+1}=\dfrac{1}{2}.$

(or $\overset{l'H}{=}\lim_{x\to1^+}\dfrac{1/x^2}{1/x+1/x^2}=\dfrac{1}{1+1}=\dfrac{1}{2}$.*)* ∎



(c) $\lim\limits_{x\to a} f^g$ of type $\boxed{\mathbf{0^0, \infty^0, 1^\infty}}$:

取<u>自然對數 ln</u> 求出極限, 再取<u>自然指數 e</u> : (不限未定型)

(Step 1) Let $y = f^g$, $\ln y = g \ln f$, $\quad (\mathbf{0^0, \infty^0, 1^\infty}$ 會變成 $\mathbf{0 \bullet \infty})$

(Step 2) $\lim\limits_{x\to a} \ln y \stackrel{\dagger}{=} L/\infty/-\infty$, $\quad$ (†: 能變成 $\dfrac{\mathbf{0}}{\mathbf{0}}, \dfrac{\infty}{\infty}$ 才可試用 $l'H$)

(Step 3) $\lim\limits_{x\to a} y = \lim\limits_{x\to a} e^{\ln y} \stackrel{*}{=} e^{\lim\limits_{x\to a} \ln y} \stackrel{!}{=} e^L/\infty/0$. $\quad$ (∗: 因爲 e^x 處處連續)

(!: 不可以寫 $\cancel{= e^\infty = \infty}$, $\cancel{= e^{-\infty} = 0}$.)

f^g	$f \to$	$g \to$	$\ln f \to$	$g \cdot \ln f$
$\mathbf{0^0}$	0^+	0	$-\infty$	$\mathbf{0 \cdot \infty}$
$\mathbf{\infty^0}$	∞	0	∞	$\mathbf{0 \cdot \infty}$
$\mathbf{1^\infty}$	1	$\pm\infty$	0	$\mathbf{\infty \cdot 0}$
$\mathbf{0^\infty}$	0^+	$\pm\infty$	$-\infty$	$\mp\infty$

這型極限是 0 或 ∞, 不可用羅畢達。

Example 0.10 $\lim\limits_{x\to 0^+} (1+\sin 4x)^{\cot x} = ?$

$\lim\limits_{x\to 0^+} (1+\sin 4x) = 1$, $\lim\limits_{x\to 0^+} \cot x = \infty$. $(\mathbf{1^\infty})$

Let $y = (1+\sin 4x)^{\cot x}$, $\ln y = \cot x \ln(1+\sin 4x) = \dfrac{\ln(1+\sin 4x)}{\tan x}$.

$\lim\limits_{x\to 0^+} \ln(1+\sin 4x) = 0$, $\lim\limits_{x\to 0^+} \tan x = 0$, $(\tan x)' = \sec^2 x \neq 0$ *near* 0^+. $(\dfrac{\mathbf{0}}{\mathbf{0}})$

$$\therefore \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}$$

$$\stackrel{l'H}{=} \lim_{x\to 0^+} \left(\frac{4\cos 4x}{1+\sin 4x} \frac{1}{\sec^2 x} \right) = \frac{4\cdot 1}{1+0} \cdot \frac{1}{1^2} = 4. \text{ (還沒完!)}$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim\limits_{x\to 0^+} \ln y} = e^4. \quad \blacksquare$$

Example 0.11 $\lim\limits_{x\to 0^+} x^x = ?$

$\lim\limits_{x\to 0^+} x = 0$. $(\mathbf{0^0})$

Let $y = x^x$, $\ln y = x \ln x = \dfrac{\ln x}{1/x}$. $(\mathbf{0^0} \to \mathbf{0 \cdot \infty} \to \dfrac{\infty}{\infty})$

$$\therefore \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{1/x} \stackrel{l'H}{=} \ldots \text{(前面剛講過)} = 0. \text{ (還沒完!)}$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim\limits_{x\to 0^+} \ln y} = e^0 = 1. \quad \blacksquare$$

Attention: 注意! 不要跟對數微分法 $(y' = y(g \ln f)')$ 搞混。

Example 0.12 *(If ℓ'Hospital's rule fails)* $\lim_{x\to\infty} \frac{x+\cos x}{x} = ?$

Indeterminate form of type $\frac{\infty}{\infty}$. *(自己檢查)*

$\lim_{x\to\infty} \frac{x+\cos x}{x} \overset{l'H}{=} \lim_{x\to\infty} \frac{1-\sin x}{1}$ *does not exist nor infinite limit.*

$\therefore \lim_{x\to\infty} \frac{x+\cos x}{x}$ *does not exist. (***Wrong!***)*

這題要用 *Squeeze Theorem:*

Consider $x > 0$ *since* $x \to \infty$, *then* $1 - \frac{1}{x} \leq \frac{x+\cos x}{x} \leq 1 + \frac{1}{x}$,

$\lim_{x\to\infty} (1 - \frac{1}{x}) = \lim_{x\to\infty} (1 + \frac{1}{x}) = 1, \implies \lim_{x\to\infty} \frac{x+\cos x}{x} = 1.$ ■

Attention: 如果 $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ 不存在也不是 $\pm\infty$, 則 ℓ'Hospital's rule 不能用 ($\overset{l'H}{=}$ 不成立); 但並不代表極限就不存在! 這時候要改用其他方法。

Question: $\overset{l'H}{=}$ 是啥? 一定要寫嗎?
Answer: 只是解釋這一步是用 ℓ'Hospital's rule, 計算證明要寫。

Question: 每次都要這麼麻煩嗎?
Answer: 不用!

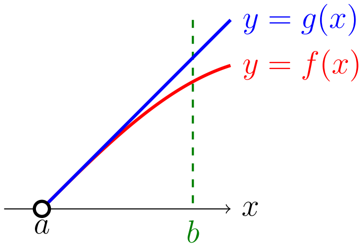
1. 檢查是不是未定型$(\mathbf{0^0}, \mathbf{\infty^0}, \mathbf{1^\infty} \to \mathbf{0} \cdot \boldsymbol{\infty}, \boldsymbol{\infty} - \boldsymbol{\infty} \to \frac{\mathbf{0}}{\mathbf{0}}, \frac{\boldsymbol{\infty}}{\boldsymbol{\infty}})$。
2. 直接 $\overset{l'H}{=} \lim_{x\to a} \frac{f'}{g'}$.
3. 還是未定型: goto 2.
4. 極限存在 或是 $\infty / -\infty$ $\implies$ (如果有取 $\ln x$ 要再取 e^x) 答案。
5. 否則 $\implies$ 劃掉並找別的方法 (換另一型或用夾擠定理)。

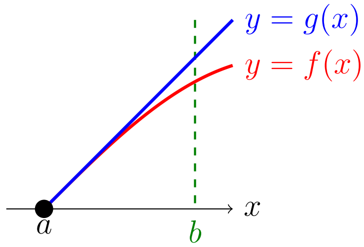
Question: 老師你沒檢查 $g'(x) \neq 0$ near a!
Answer: 不用! 如果 $g(x)$ 可微分且 $g'(x) = 0$ near a,
則 $g(x)$ 是常數函數, $g(x) \not\to 0$ 除非 $g(x) = 0$ (完全不能求極限)。
所以在檢查是不是未定型時就會排除。

♦ Additional: The Proof of L'Hospital's Rule

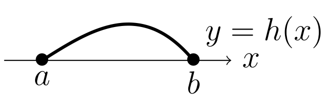
For $\dfrac{\mathbf{0}}{\mathbf{0}}$: $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$.

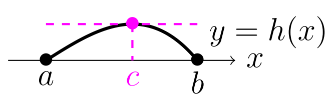
$\because$ f and g differentiable, $\exists\, b > a$, $\ni$ f and g are continuous on $(a, b]$.
Assume $f(a) = g(a) = 0$, (otherwise, consider and replace by:
$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$ and $G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$)
then f and g are continuous on $[a, b]$ and differentiable on (a, b).

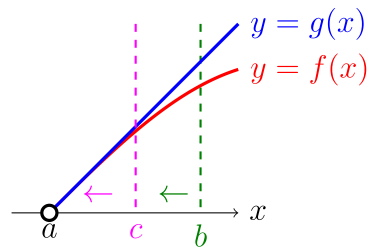




Let $h(x) = f(x) - \dfrac{f(b)}{g(b)} g(x)$, ($\because$ $g(b) \neq 0$) then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = 0$. By Rolle's Theorem, $\exists\, c \in (a, b)$, $\ni$ $h'(c) = f'(c) - \dfrac{f(b)}{g(b)} g'(c) = 0$, ($\because$ $g'(c) \neq 0$) $\implies \dfrac{f'(c)}{g'(c)} = \dfrac{f(b)}{g(b)}$. $(*)$







When $x = b \to a^+ \implies y = c \to a^+$, $(\dagger)$

$$\implies \lim_{x\to a^+} \frac{f(x)}{g(x)} \overset{(*)}{=} \lim_{x\to a^+} \frac{f'(y)}{g'(y)} \overset{(\dagger)}{=} \lim_{y\to a^+} \frac{f'(y)}{g'(y)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

For $\dfrac{\boldsymbol{\infty}}{\boldsymbol{\infty}}$, consider $\dfrac{1/g}{1/f}$ $(\dfrac{\mathbf{0}}{\mathbf{0}})$. ■