3.6 Derivatives of logarithmic functions

1. derivative of logarithmic function 對數函數的微分
   $(\log_a x)' = \frac{1}{x\ln a}$, $(\ln x)' = \frac{1}{x}$
2. logarithmic differentiation & power rule 對數的微分與冪次律
   $(x^n)' = nx^{n-1}$, $n \in \mathbb{R}$
3. number e as a limit e 是極限 $e = \lim_{x\to 0}(1+x)^{1/x} = \lim_{n\to\infty}(1+\frac{1}{n})^n$

0.1 Derivative of logarithmic function

Recall: 對數函數是指數函數的反函數.

1. $\dfrac{d}{dx}\log_a x = \dfrac{1}{x\ln a}$ .... $\boxed{\boldsymbol{(\log_a x)' = \dfrac{1}{x\ln a}}}$

   Let $y = \log_a x \iff a^y = x$.

   Apply implicit differentiation, $a^y \ln a \dfrac{dy}{dx} = 1$, $\dfrac{dy}{dx} = \dfrac{1}{a^y \ln a} = \dfrac{1}{x\ln a}$.

2. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$ .... $\boxed{\boldsymbol{(\ln x)' = \dfrac{1}{x}}}$

   By (1.) and $\ln e = 1$. ($\ln x$ domain is $(0,\infty)$ (or $x > 0$).)

3. $\dfrac{d}{dx}\ln g(x) = \dfrac{g'(x)}{g(x)}$, $\dfrac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$ .... $\boxed{(\ln g)' = \dfrac{g'}{g}, (e^f)' = f'e^f}$

   Use chain rule:

   let $u = g(x)$, $\dfrac{d}{dx}\ln g(x) = \dfrac{d}{du}\ln u \dfrac{du}{dx} = \dfrac{1}{u}u' = \dfrac{g'(x)}{g(x)}$;

   let $v = f(x)$, $\dfrac{d}{dx}e^{f(x)} = \dfrac{d}{dv}e^v \dfrac{dv}{dx} = e^v v' = f'(x)e^{f(x)}$.

4. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$ .... $\boxed{\boldsymbol{(\ln|x|)' = \dfrac{1}{x}}}$

   $$\ln|x| = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}, \quad \frac{d}{dx}(\ln|x|) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}.$$

   ($\ln|x|$ domain is $(-\infty,0)\cup(0,\infty)$ (or $x \neq 0$).)

Example 0.1 $y = \ln(x^3+1)$, $y' = ?$

Let $u = x^3+1$, *then* $y = \ln u$ *and*
$$y' = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}\ln u \frac{d}{dx}(x^3+1) = \frac{1}{u}\cdot 3x^2 = \frac{3x^2}{x^3+1}. \qquad \blacksquare$$

0.2 Logarithmic differentiation & power rule

當函數是 composition of product, quotient, power, 使用對數微分。

Logarithmic Differentiation: 對數微分 (取自然對數做隱微分)

Step 1. Let $y = f(x)$, $\ln y = \ln f(x)$. 等式兩邊取自然對數。

Step 2. Implicit differentiation and chain rule. 隱微分與連鎖律。

Step 3. Solve y'. 解 y': y 換回 $f(x)$, y' 寫成 x 的函數。

Power rule: 乘冪律
$f(x) = x^n$, $\boxed{n \in \mathbb{R}}$, (§3.1 只有 $n \in \mathbb{N}$ or $\mathbb{Z} \cup \{\frac{1}{2}\}$) then
$$\boxed{f'(x) = nx^{n-1}}.$$

Proof. Let $y = x^n$, $\ln|y| = \ln|x|^n = n\ln|x|$, $x \neq 0$.
(隱微分) $\frac{y'}{y} = \frac{n}{x}$, $y' = \frac{n}{x}y = \frac{n}{x}x^n = nx^{n-1}$. $\blacksquare$

Note: 1. 如果只取 ln 只能證明 $x > 0$, 所以要取 $\ln|\cdot|$, 或另外討論 $x < 0$。
2. When $x = 0$:
if $n > 1$ $f'(0) = \lim_{x\to 0}\frac{x^n - 0^n}{x-0} = \lim_{x\to 0} x^{n-1} = 0 = n0^{n-1}$;
if $n = 1$, $f'(0) = 1 \neq 0^0 = 1\cdot 0^{1-1}$ (0^0 is undetermined);
if $n < 1$, $f'(0)$ 不存在 (無限極限), $n0^{n-1} = \frac{n}{0^{1-n}}$ 未定義。
Ex: $n = \frac{1}{2}$ and $f(x) = x^{1/2} = \sqrt{x}$,
$f'(x) = \frac{1}{2\sqrt{x}}$, $f'(0)$ does not exist; $\frac{1}{2}0^{1/2-1} = \frac{1}{2\sqrt{0}}$ is undefined.

Remark: $a>0$, b constant, $f(x), g(x)$ functions.

1. $\dfrac{d}{dx}(a^b)=0$. (constant)

2. $\dfrac{d}{dx}[f(x)]^b=b[f(x)]^{b-1}f'(x)$. (Chain rule & Power rule)

3. $\dfrac{d}{dx}[a^{g(x)}]=a^{g(x)}\ln a\cdot g'(x)$. (Chain rule & exponential)

4. $\dfrac{d}{dx}[f(x)]^{g(x)}$, use logarithmic differentiation.

   Let $y=f^g$, $\ln y=g\ln f$, $\dfrac{y'}{y}=g'\ln f+g\dfrac{f'}{f}$, $y'=f^g(g'\ln f+g\dfrac{f'}{f})$.

(這幾個公式都不要背, 會背錯; 應該記的是方法: 連鎖律, 隱微分, 對數微分。)

Example 0.2 $y=x^{\sqrt{x}}$, $y'=?$

[Sol 1]: (取自然對數) $\ln y=\ln x^{\sqrt{x}}=\sqrt{x}\ln x$,
$\dfrac{y'}{y}=\dfrac{1}{2\sqrt{x}}\ln x+\sqrt{x}\dfrac{1}{x}=\dfrac{\ln x+2}{2\sqrt{x}}$, $y'=y\dfrac{\ln x+2}{2\sqrt{x}}=x^{\sqrt{x}}\dfrac{\ln x+2}{2\sqrt{x}}$.
[Sol 2]: (取自然指數) $y=x^{\sqrt{x}}=e^{\ln x^{\sqrt{x}}}=e^{\sqrt{x}\ln x}$,
$y'=e^{\sqrt{x}\ln x}\dfrac{d}{dx}(\sqrt{x}\ln x)=x^{\sqrt{x}}\dfrac{\ln x+2}{2\sqrt{x}}$. ■

Note: Use logarithmic differentiation to solve product/quotient:
$y=fg$, $\ln y=\ln f+\ln g$, $\dfrac{y'}{y}=\dfrac{f'}{f}+\dfrac{g'}{g}$, $y'=fg(\dfrac{f'}{f}+\dfrac{g'}{g})=f'g+fg'$.
$y=\dfrac{f}{g}$, $\ln y=\ln f-\ln g$, $\dfrac{y'}{y}=\dfrac{f'}{f}-\dfrac{g'}{g}$, $y'=\dfrac{f}{g}(\dfrac{f'}{f}-\dfrac{g'}{g})=\dfrac{f'g-fg'}{g^2}$.

Example 0.3 *Differentiate* $y=\dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$. *(Use product/quotient rule?)*

$\ln y=\dfrac{3}{4}\ln x+\dfrac{1}{2}\ln(x^2+1)-5\ln(3x+2)$, $\dfrac{y'}{y}=\dfrac{3}{4}\dfrac{1}{x}+\dfrac{1}{2}\dfrac{2x}{x^2+1}-5\dfrac{3}{3x+2}$,
$y'=\dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\Big(\dfrac{3}{4x}+\dfrac{x}{x^2+1}-\dfrac{15}{3x+2}\Big)$. ■

0.3 Number e as a limit

$$\boxed{\boldsymbol{e = \lim_{x\to 0}(1+x)^{1/x} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}}$$

Recall: e is defined by $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x} = 1$.

Proof. Consider $f(x) = \ln x$, $f'(x) = \dfrac{1}{x}$, $f'(1) = 1$.

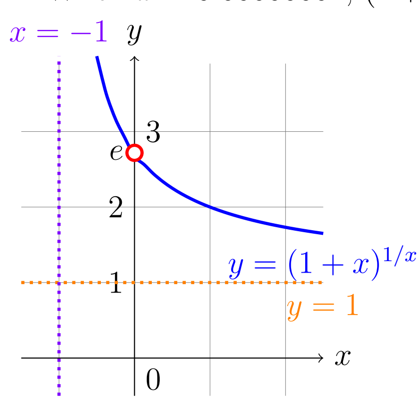
$$\begin{aligned} f'(1) &= \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x} \\ &= \lim_{x\to 0}(\frac{1}{x}\ln(1+x)) \\ &= \lim_{x\to 0}\ln(1+x)^{1/x}. \quad \text{(Exercise 3.6.55)} \end{aligned}$$

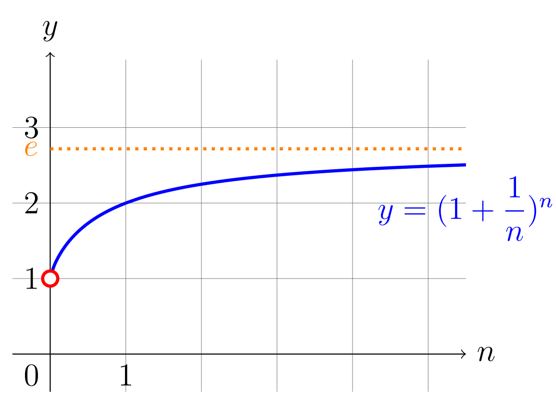
$\because$ e^x 是連續函數 (在 1 連續), 可以傳遞存在的極限 $(=1)$。

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = e^{\lim\limits_{x\to 0}\ln(1+x)^{1/x}} = e^{f'(1)} = e^1 = e.$$

Since $n = \dfrac{1}{x} \to \infty$ as $x \to 0^+$, 又可以寫成 $e = \displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

When $x = 0.00000001$, $(1+x)^{1/x} \approx 2.71828181$.





(Try to verify:
$e^x = \displaystyle\lim_{n\to\infty}(1+\frac{x}{n})^n$ (Exercise 3.6.56) & $\ln x = \displaystyle\lim_{n\to\infty} n(\sqrt[n]{x}-1)$.)