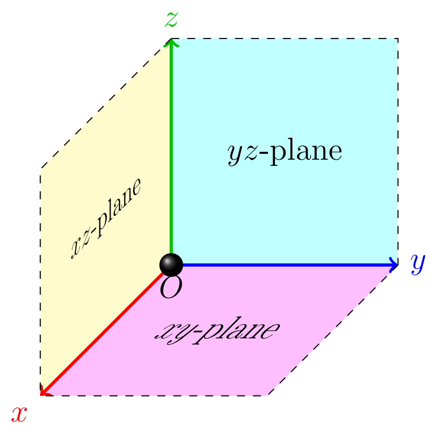
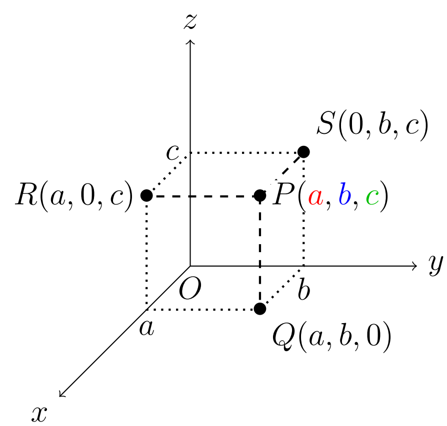
12.1 Three-dimensional coordinate systems

Define: 3D 座標系:

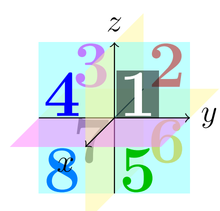
- O: ***origin*** 原點; x-, y- and z-axis: ***coordinate axes*** 坐標軸.
- xy-, yz- and xz-plane: ***coordinate planes*** 坐標平面.
- 方向順序: $\boxed{\text{Right-Hand Rule}}$.





- 8 ***octant*** 卦限 (***quadrant*** 象限), the first octant $(x > 0, y > 0, z > 0)$.

	1	2	3	4	5	6	7	8
x	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
y	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
z	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$



- $P(a, b, c)$: ***coordinate*** 座標, a, b, c, are x-, y-, z-coordinate, resp.
- $Q(a, b, 0)$, $R(0, b, c)$, $S(a, 0, c)$: projections 投影 of P onto the xy-, yz- and xz-plane, resp.
- The distance 距離 between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$|P_1P_2| = \boxed{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}.$$

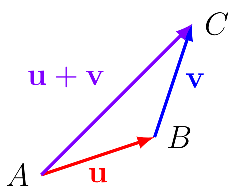
- The equation of a sphere 球面 with center $C(h, k, \ell)$ and radius r:

$$\boxed{(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2}.$$

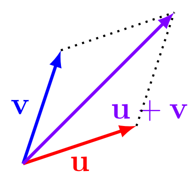
12.2 Vectors

Define: A ***vector*** 向量 has direction 方向 and magnitude 度量. (一稱矢量)

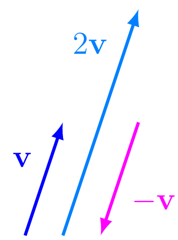
- $\mathbf{v} = \overrightarrow{AB}$, A ***initial point*** or tail, B ***terminal point*** or tip.
- $\mathbf{u} = \mathbf{v}$: ***equivalent*** or ***equal*** 相等. • $\mathbf{0}$: the ***zero vector*** 零向量.
- **Vector Addition** 向量加法: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Scalar Multiplication** 純量乘法: c scalar, $\mathbf{v}$ vector,
  $c\mathbf{v}$: $|c|$ 倍長, 與 $\mathbf{v}$ 同 (反) 向 if $c > 0 (< 0)$.
- Vector Difference: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$, $-\mathbf{v} = -1\mathbf{v}$.



Triangle Law



Parallelogram Law



- Properties of vectors ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ vectors, d, e scales):

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$	4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $d(\mathbf{a} + \mathbf{b}) = d\mathbf{a} + d\mathbf{b}$	6. $(d + e)\mathbf{a} = d\mathbf{a} + e\mathbf{a}$
7. $(de)\mathbf{a} = d(e\mathbf{a})$	8. $1\mathbf{a} = \mathbf{a}$

Representation: $\boxed{\mathbf{a} = \langle \boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3} \rangle}$, a_1, a_2, a_3: ***components*** 分量 of $\mathbf{a}$.

- A point $P(a, b, c)$, the ***position vector*** 位置向量 $\mathbf{a} = \overrightarrow{OP} = \langle a, b, c \rangle$.
- $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, $\overrightarrow{AB} = \mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$
- ***length*** 長度 or ***magnitude***, $|\cdot|$ or $||\cdot||$: $\boxed{|\mathbf{a}| = \sqrt{\boldsymbol{a_1}^2 + \boldsymbol{a_2}^2 + \boldsymbol{a_3}^2}}$.
- $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, c constant:
  $\mathbf{a} + \mathbf{b} = \boxed{\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle}$, $c\mathbf{a} = \boxed{\langle ca_1, ca_2, ca_3 \rangle}$, $\mathbf{0} = \boxed{\langle 0, 0, 0 \rangle}$.

- ***standard basis vectors*** 標準基底向量:

$$\boxed{\mathbf{i} = \langle \mathbf{1}, \mathbf{0}, \mathbf{0} \rangle,\ \mathbf{j} = \langle \mathbf{0}, \mathbf{1}, \mathbf{0} \rangle,\ \mathbf{k} = \langle \mathbf{0}, \mathbf{0}, \mathbf{1} \rangle.}$$

  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ (唯一且充分表示).

- 推廣到 n-dimensional vectors: $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ in V_n n-維向量空間。

- A ***unit vector*** 單位向量 is a vector of length 1.

  The unit vector in the direction of $\mathbf{a}(\neq \mathbf{0})$ is $\boxed{\dfrac{\mathbf{a}}{|\mathbf{a}|}}$.

12.3 Dot product

Define: ***Dot*** (or ***inner***, ***scalar***) ***product*** 內積 of verctors:

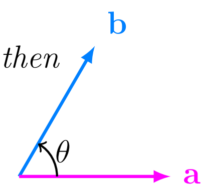
- $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$,

$$\boxed{\mathbf{a} \bullet \mathbf{b} = \boldsymbol{a_1 b_1} + \boldsymbol{a_2 b_2} + \boldsymbol{a_3 b_3} = \mathbf{b} \bullet \mathbf{a}}$$

- Properties of Dot Product:
  1. $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$
  2. $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$
  3. $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$
  4. $(c\mathbf{a}) \bullet \mathbf{b} = c(\mathbf{a} \bullet \mathbf{b}) = \mathbf{a} \bullet (c\mathbf{b})$
  5. $\mathbf{0} \bullet \mathbf{a} = 0$

Theorem 1 *If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then*

$$\boxed{\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \boldsymbol{\theta},\ \cos \boldsymbol{\theta} = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.}$$



- $\mathbf{a}$ and $\mathbf{b}$ are ***perpendicular*** 垂直 or ***orthogonal*** 直角 (多記爲: $\mathbf{a} \perp \mathbf{b}$)
  $\iff \theta = \dfrac{\pi}{2} \iff \boxed{\mathbf{a} \bullet \mathbf{b} = 0}$.

- ***direction angles*** 方向角 α, β, γ of $\mathbf{a}$: $\mathbf{a}$ 與 x-, y-, z-axis 的夾角.

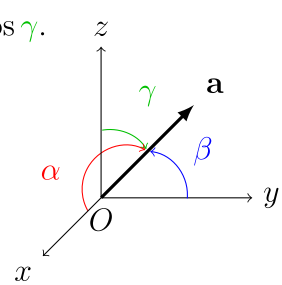
- ***direction cosines*** 方向餘弦 of $\mathbf{a}$: $\cos \alpha, \cos \beta, \cos \gamma$.

  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle = |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$,

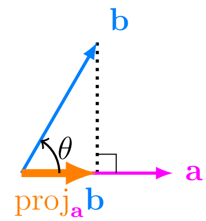
  $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$,

  $\cos \alpha = \dfrac{\mathbf{a} \bullet \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \dfrac{a_1}{|\mathbf{a}|}$, $\cos \beta = \dfrac{a_2}{|\mathbf{a}|}$, $\cos \gamma = \dfrac{a_3}{|\mathbf{a}|}$,

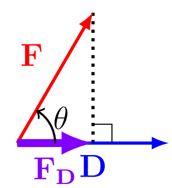
  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.



- The ***scalar projection*** 投影量 of $\mathbf{b}$ onto $\mathbf{a}$ (***component*** of $\mathbf{b}$ along $\mathbf{a}$)
  $\boxed{\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|}}$ $(= |\mathbf{b}| \cos\theta)$.



- The ***vector projection*** 投影向量 of $\mathbf{b}$ onto $\mathbf{a}$
  (等於投影長乘單位向量)
  $\boxed{\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}}$.



In physics, force $\mathbf{F}$, displacement $\mathbf{D}$, work $W = \mathbf{F} \bullet \mathbf{D}$.

12.4 Cross product

Define: ***Cross*** (or ***vector***) ***product*** 外積 of vectors:

- $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$,

$$\boxed{\begin{aligned}\mathbf{a} \times \mathbf{b} &= \langle \boldsymbol{a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1} \rangle = -\mathbf{b} \times \mathbf{a} \\ &= \left\langle \begin{vmatrix} \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix}, \begin{vmatrix} \boldsymbol{a_3} & \boldsymbol{a_1} \\ \boldsymbol{b_3} & \boldsymbol{b_1} \end{vmatrix}, \begin{vmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} \\ \boldsymbol{b_1} & \boldsymbol{b_2} \end{vmatrix} \right\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix} = \begin{vmatrix} \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix} \mathbf{i} + \begin{vmatrix} \boldsymbol{a_3} & \boldsymbol{a_1} \\ \boldsymbol{b_3} & \boldsymbol{b_1} \end{vmatrix} \mathbf{j} + \begin{vmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} \\ \boldsymbol{b_1} & \boldsymbol{b_2} \end{vmatrix} \mathbf{k}\end{aligned}}$$

Note: 注意, 課本第二項是 $-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}$. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

Theorem 2 *$\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ ($\mathbf{a} \times \mathbf{b} \perp \mathbf{a}$, $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$).*

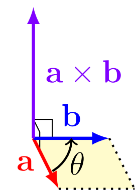
Proof. 計算得到 $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{b} = 0$. ■

Theorem 3 *If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then*

$$\boxed{|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\boldsymbol{\theta}}.$$

- The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

  Note: 大小看面積, 方向看右手 (Right-Hand Rule)。



- $\mathbf{a}$ and $\mathbf{b}$ are ***parallel*** 平行 (多記爲: $\mathbf{a} \parallel \mathbf{b}$)
  $\iff \theta = 0$ or $\pi \iff \boxed{\mathbf{a} \times \mathbf{b} = \mathbf{0}}$.

  Recall: orthogonal $\iff \mathbf{a} \bullet \mathbf{b} = 0$. 內積 0 垂直, 外積 $\mathbf{0}$ 平行.

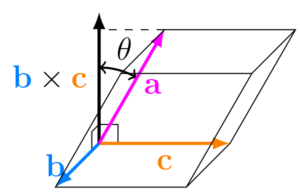
- $\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}$, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

- Properties of Cross Product:
  1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
  2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
  3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
  4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
  5. $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$
  6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$

- Scalar Triple Products $\boxed{\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\|}$

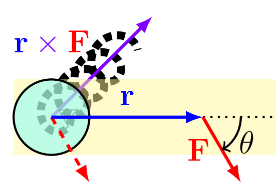
- The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is the magnitude of their scalar triple product $V = |\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$.

  Note: 大小看體積.



- $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are ***coplanar*** 共平面 $\iff \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = 0$.

- Vector Triple Products $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$



In physics, position $\mathbf{r}$, force $\mathbf{F}$, torque 扭力 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

12.5 Equations of lines and planes

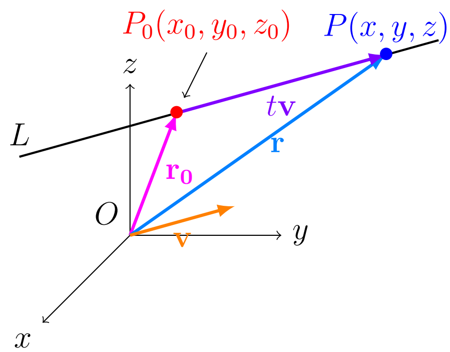
0.1 Lines

Define: A ***line*** L in V_3 is determined by a point $P_0(x_0, y_0, z_0)$ and the direction of L. Let $P(x, y, z)$ be an arbitrary point on L.

- A ***vector equation*** of the line L:

  $$\boxed{\mathbf{r} = \mathbf{r_0} + t\mathbf{v}}$$

  where $\mathbf{v}$ is a vector parallel L, t is a ***parameter***, $\mathbf{r}$ and $\mathbf{r_0}$ are position vector of P and P_0.



- The ***parametric equations*** of the line L:

  $$\boxed{x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct}$$

  a, b, c (also $-a, -b, -c$) are called ***direction numbers*** of L.
  Let $\mathbf{v} = \langle a, b, c\rangle$, then $\mathbf{r} = \langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle = \mathbf{r_0} + t\mathbf{v}$.

- The ***symmetric equations*** of the line L:

  $$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} (= t),}$$

  if none of a, b, c is 0; if $a = 0$, written $x = x_0$, $\dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

- The line L through points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$, then $\mathbf{v} = \overrightarrow{P_0P_1} = \mathbf{r_1} - \mathbf{r_0}$, direction numbers of L are $x_1 - x_0$, $y_1 - y_0$, $z_1 - z_0$.

  The vector equation of L is $\mathbf{r} = \mathbf{r_0} + t\mathbf{v} = (1 - t)\mathbf{r_0} + t\mathbf{r_1}$.

  The parametric equation of L is
  $x = x_0 + t(x_1 - x_0)$, $y = y_0 + t(y_1 - y_0)$, $z = z_0 + t(z_1 - z_0)$.

  The symmetric equation of L is
  $\dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0} = \dfrac{z - z_0}{z_1 - z_0}$ (if none of direction numbers is 0)

- The ***line segment*** 線段 from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation

$$\boxed{\mathbf{r}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1}, \quad 0 \le t \le 1.}$$

- Two lines are **skew** 歪斜 if they do not intersect and are not parallel (and therefore do not lie in the same plane). 不相交不平行不共面

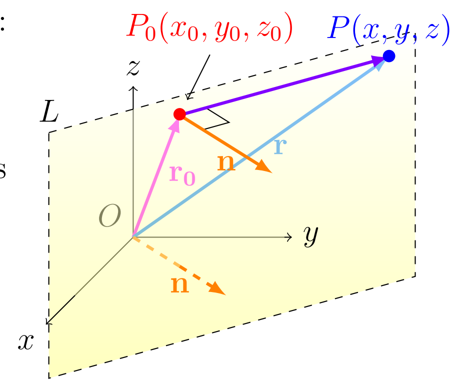
0.2 Plane

Define: A plane E in V_3 is determined by a point $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{n}$, called ***normal vector*** 法向量, orthogonal to E. Let $P(x, y, z)$ be an arbitrary point on E.

- A ***vector equation*** of the plane E:

$$\boxed{\mathbf{n} \bullet (\mathbf{r} - \mathbf{r_0}) = 0}$$

where $\mathbf{r}$ and $\mathbf{r_0}$ are position vectors of P and P_0.



- The ***scalar equation*** of the plane E through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$:

$$\boxed{\boldsymbol{a(x - x_0) + b(y - y_0) + c(z - z_0)} = 0}$$

- A ***linear equation*** in x, y, z:

$$\boxed{\boldsymbol{ax + by + cz + d} = 0}$$

where $d = -ax_0 - by_0 - cz_0$.

- 平行: Two planes are parallel $\iff$ their normal vectors are parallel. $(\mathbf{n_1} = c\mathbf{n_2})$ 面平行 $\iff$ 法向量平行。

- 夾角: The angle between two planes is equal to the angle between their normal vectors. $(\cos\theta = \dfrac{\mathbf{n_1} \bullet \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|})$ 面夾角=法向量夾角。

- If two planes are not parallel, then intersect in a line with direction parallel to the cross product of their normal vectors. ($\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2}$) 交線方向是法向量外積。

- Distance from a point $P_1(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$ is

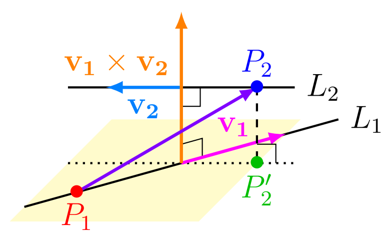
$$\boxed{\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Proof. $\mathbf{n} = \langle a, b, c\rangle$ and let $P_0(x_0, y_0, z_0)$ on the plane and $\mathbf{b} = \overrightarrow{P_0P_1}$.
Distance $= |\text{comp}_{\mathbf{n}}\mathbf{b}| = \dfrac{|\mathbf{n} \bullet \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$
$= \dfrac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$ ■

Note: 歪斜線 $\begin{cases} L_1 : \mathbf{r} = t\mathbf{v_1} + \mathbf{r_1}, \\ L_2 : \mathbf{r} = s\mathbf{v_2} + \mathbf{r_2}, \end{cases}$

距離 $= \dfrac{|(\mathbf{r_2} - \mathbf{r_1}) \bullet (\mathbf{v_1} \times \mathbf{v_2})|}{|\mathbf{v_1} \times \mathbf{v_2}|}$,

$\overrightarrow{P_1P_2}$ 在 $\mathbf{v_1} \times \mathbf{v_2}$ 的投影量 $|\overline{P_1P_2'}|$。



12.6 Cylinders and quadric surfaces

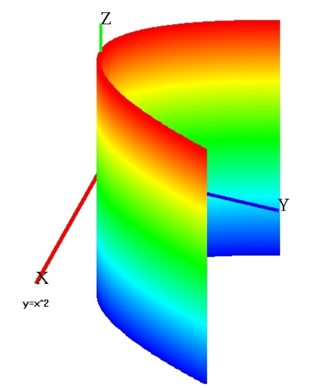
1. cylinders
2. quadric surfaces

Cylinders 柱面

$$Ax^2 + By^2 + Cxy + Dx + Ey + G = 0$$

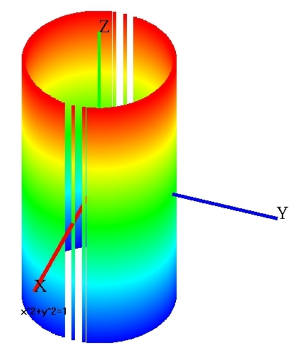
***parabolic cylinder* 抛物柱面:**

$$\frac{x^2}{a^2} = \frac{y}{b}$$



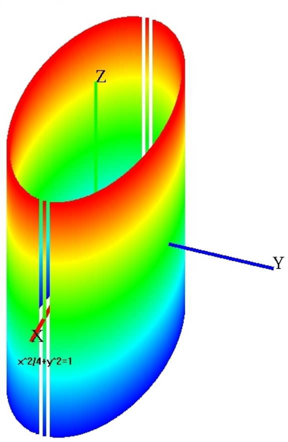
***(circular) cylinder* 圓柱面:**

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$



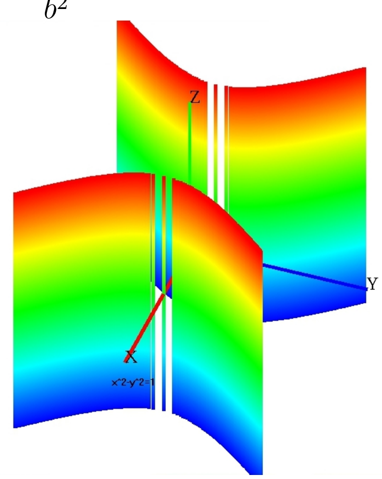
***elliptic cylinder* 橢圓柱面:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



***hyperbolic cylinder* 雙曲柱面:**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

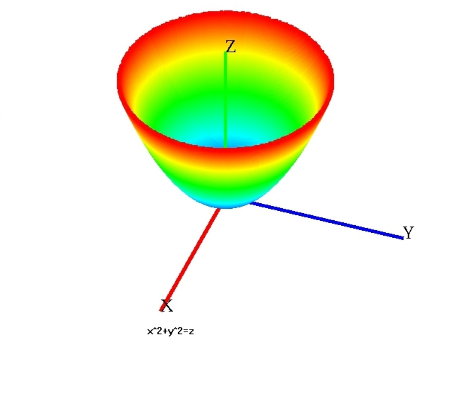


Quadric surfaces 二次曲面

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$$

elliptic paraboloid 橢圓拋物面***:***

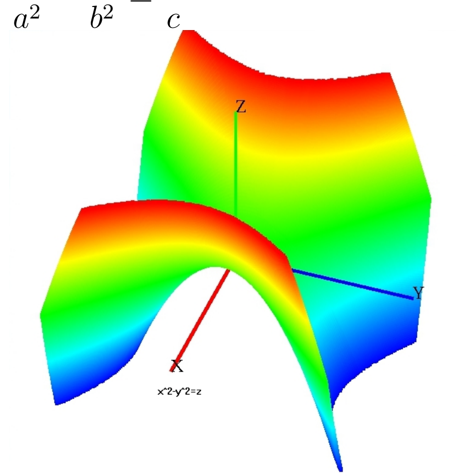
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{z}{c}$



circular paraboloid $(a=b)$

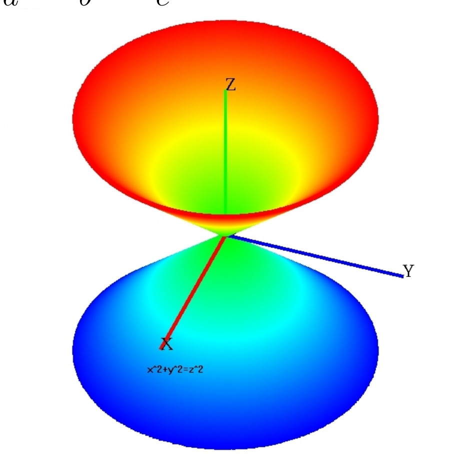
hyperbolic paraboloid 雙曲拋物面***:***

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=\dfrac{z}{c}$



cone 錐面***:***

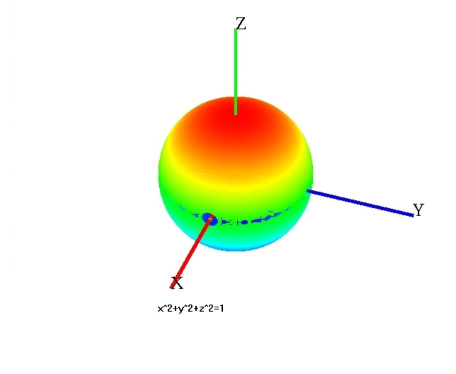
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{z^2}{c^2}$



circular cone 圓錐面 $(a=b)$

ellipsoid 橢球面***:***

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

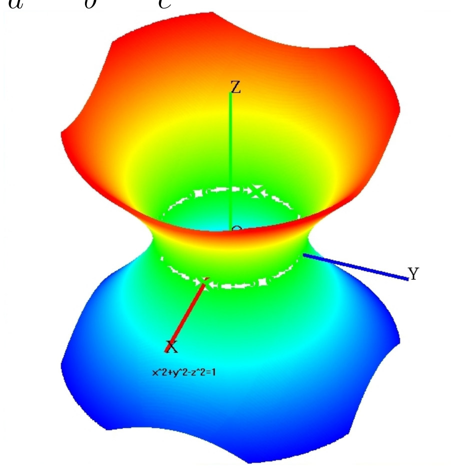


sphere 球面 $(a=b=c)$

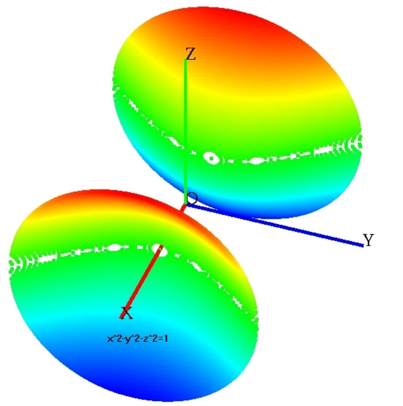
spheroid 類球面 $(a=b\neq c)$

oblate 扁 $(a>c)$, prolate 長 $(a<c)$

hyperboloid of one sheet
單葉雙曲面***:***
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$



hyperboloid of two sheets
雙葉雙曲面***:***
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$



Skill: 判斷圖形: 代入 $x/y/z=$ 數字, 用圓錐曲線 (拋物/橢圓/雙曲) 判斷。

⧫ Additional: Classification of quadric surfaces

$Ax^2+By^2+Cz^2+2Fxy+2Gyz+2Hxz+2Px+2Qy+2Rz+D=0.$

$$e=\begin{pmatrix}A&F&H\\F&B&G\\H&G&C\end{pmatrix},\ E=\begin{pmatrix}A&F&H&P\\F&B&G&Q\\H&G&C&R\\P&Q&R&D\end{pmatrix}.$$

Invariant: $\rho_3=\operatorname{rank}(e)$, $\rho_4=\operatorname{rank}(E)$, $\Delta=\det(E)$,
eigenvalues λ_i: solutions of $\det(\lambda I-e)=0$,
$k=1$ if the signs of nonzero λ_i's are the same, and 0 otherwise.

Standardize: 1. Diagonalize: unit eigenvector $\mathbf{u}_i$: $(\lambda_i I-e)\mathbf{u}_i=\mathbf{0}$,
$T=\begin{pmatrix}\mathbf{u}_1&\mathbf{u}_2&\mathbf{u}_3\end{pmatrix}$, $T^\perp=T^{-1}$. Replace $X=\begin{pmatrix}x&y&z\end{pmatrix}^\perp$ by TX.
$\Longrightarrow\ \lambda_1x^2+\lambda_2y^2+\lambda_3z^2+2\mu_1x+2\mu_2y+2\mu_3z+d$

$$=\begin{pmatrix}x&y&z&1\end{pmatrix}\begin{pmatrix}\lambda_1&0&0&\mu_1\\0&\lambda_2&0&\mu_2\\0&0&\lambda_3&\mu_3\\\mu_1&\mu_2&\mu_3&d\end{pmatrix}\begin{pmatrix}x\\y\\z\\1\end{pmatrix}=0.$$

2. Shift (complete square): replace X by $\begin{pmatrix}x-\frac{\mu_1}{\lambda_1}&y-\frac{\mu_2}{\lambda_2}&z-\frac{\mu_3}{\lambda_3}\end{pmatrix}^\perp$,
then $\mu_i=0$ if $\lambda_i\neq0$.

Table 1: 17 different (canonical) classes of the quadric surfaces.

	surface	equation	ρ_3	ρ_4	Δ	k
1	ellipsoid (real)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	3	4	$-$	1
2	ellipsoid (imaginary)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$	3	4	$+$	1
3	hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	3	4	$+$	0
4	hyperboloid of two sheets	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$	3	4	$-$	0
5	elliptic cone (real)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	3	3	0	0
6	elliptic cone (imaginary)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$	3	3	0	1
7	elliptic paraboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$	2	4	$-$	1
8	hyperbolic paraboloid	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$	2	4	$+$	0
9	elliptic cylinder (real)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	2	3	0	1
10	elliptic cylinder (imaginary)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$	2	3	0	1
11	hyperbolic cylinder	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$	2	3	0	0
12	intersecting planes (real)	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	2	2	0	0
13	intersecting planes (imaginary)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$	2	2	0	1
14	parabolic cylinder	$\frac{x^2}{a^2}=\frac{z}{c}$	1	3	0	0
15	parallel planes (real)	$\frac{x^2}{a^2}=1$	1	2	0	0
16	parallel planes (imaginary)	$\frac{x^2}{a^2}=-1$	1	2	0	0
17	coincident planes	$x^2=0$	1	1	0	0