1179: Probability
Lecture 9 — Special Discrete Random Variables, Expected Value, and Variance

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Sum of 3 Cubes

- Question: Can we find integers x, y, z such that

$$x^3 + y^3 + z^3 = N, \quad 0 \le N \le 100$$

- $\underline{N = 0 :}$ $\quad (a)^3 + (-a)^3 + 0^3 = 0$
- $\underline{N = 29 :}$ $(3)^3 + (1)^3 + (1)^3 = 29$
- $\underline{N = 9m + 4 \text{ or } 9m + 5 :}$ Not possible
- Unsolved case (since 1954): $N = 42$

Sum of 3 Cubes for $N = 42$

Sum of three cubes for 42 finally solved – using real life planetary computer

Press release issued: 6 September 2019 September 2019

Hot on the heels of the ground-breaking 'Sum-Of-Three-Cubes' solution for the number 33, a team led by the University of Bristol and Massachusetts Institute of Technology (MIT) has solved the final piece of the famous 65-year-old maths puzzle with an answer for the most elusive number of all - 42.

The original problem, set in 1954 at the University of Cambridge, looked for Solutions of the Diophantine Equation x^3+y^3+z^3=k, with k being all the numbers from one to 100.

Beyond the easily found small solutions, the problem soon became intractable as the more interesting answers – if indeed they existed – could not possibly be calculated, so vast were the numbers required.

But slowly, over many years, each value of k was eventually solved for (or proved unsolvable), thanks to sophisticated techniques and modern computers - except the last two, the most difficult of all; 33 and 42.

Professor Andrew Booker
Image credit: University of Bristol

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Andrew Booker
(University of Bristol)

- Answer:
  - $x = -80538738812075974$
  - $y = 80435758145817515$
  - $z = 12602123297335631$

charityengine

1.3 million hours of computation

- A pure math project or a CS-related project?

Quick Review

- PMF of a Poisson random variable?
- Any nice properties of Poisson random variables?

$$P(X=k) = \begin{cases} \frac{e^{-\lambda T} \cdot (\lambda T)^k}{k!}, & k=0,1,2,\cdots \\ 0, & \text{else} \end{cases}$$

- Binomial
- Sum of independent Poisson is still Poisson

$X_1 \sim \text{Poisson}(\lambda_1, T)$

$X_2 \sim \text{Poisson}(\lambda_2, T)$

$X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2, T)$

Recall: An Interview Question

$\frac{e^{-\lambda T}\cdot(\lambda T)^k}{k!}$

$P(X\geq 1)=0.95$

$\boxed{P(X=0)=0.05}$

$= e^{-\lambda T}$

- Example: Suppose we stand at the Fude temple.
  - The probability that we see at least 1 car passing through the temple in 30 minutes is 0.95.
  - What is P(we see at least 1 car in 10 mins) under a Poisson model?

$P(Y\geq 1) = 1-P(Y=0)$

X = # of cars passing through the temple in 30min

Y = # of cars " " in 10min

X, Y are independent

$X \sim \text{Poisson}(\lambda, T = 30\text{min})$

$Y \sim \text{Poisson}(\lambda, T' = 10\text{min})$

$$P(Y=0) = \frac{e^{-\lambda T'}\cdot(\lambda T')^0}{0!} = e^{-\lambda T'} = (e^{-\lambda T})^{\frac{1}{3}} = (0.05)^{\frac{1}{3}}$$

Therefore, we have $P(Y\geq 1) = 1-(0.05)^{\frac{1}{3}}$ □

A, B events

A, B $\ni P(A \cap B) = P(A) \cdot P(B)$
independence

X, Y random variables.

X, Y are independent if $P(A_X \cap A_Y) = P(A_X) \cdot P(A_Y)$

This Lecture

1. Special Discrete Random Variables

2. Expected Value

3. Variance and Moments

- Reading material: Chapter 4.4-4.5 and 5.3

Special Discrete Random Variables

4. Geometric Random Variables

- Example: Play with a claw machine, and each trial is successful with probability 0.7. What is $P(\text{get 1st toy at 10-th trial})$?

- Example: Po-Jung Wang makes a hit with probability 0.28 at each at-bat. What is $P(\text{he makes his 1st hit at 5-th at-bat})$?

- What are the common features?
  - Repetitions of the same Bernoulli experiment
  - Want: how many trials needed until the 1st success?

4. PMF of Geometric Random Variables

- Example: Play with a claw machine, and each trial is successful with probability 0.7. All trials are independent.
  - X = the number of trials until we get the first toy
  - What is the PMF of X?

$P(X=1) = 0.7^1$

$P(X=2) = (0.3)^1 \times (0.7)^1$

$P(X=3) = (0.3)^2 \times (0.7)^1$

$\vdots$

$P(X=k) = (0.3)^{k-1} \times (0.7)^1$

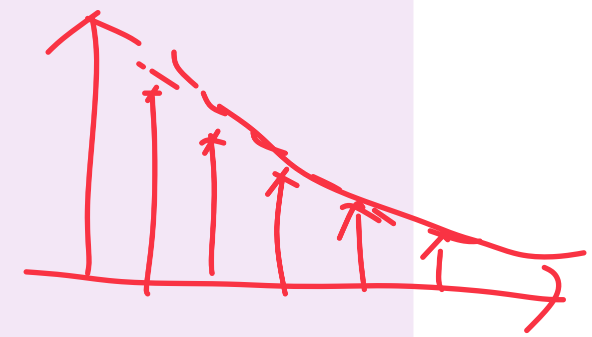
$k \in \mathbb{N}$

geometric sequence

4. Geometric Random Variables (Formally)

Geometric Random Variables: A random variable X is Geometric with parameters p if its PMF is given by

$$P(X=k) = \begin{cases} (1-p)^{k-1}p, & k=1,2,3,\cdots \\ 0, & \text{otherwise} \end{cases}$$



- Do we have $\sum_{k=1}^{\infty} P(X=k) = 1$? $\Longrightarrow \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p = 1$

$$= \frac{p}{1-(1-p)}$$

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CDF of Geometric Random Variables

$$P(X = k) = (1-p)^{k-1}p, \; k = 1,2,3,\cdots \quad t$$

- CDF: $F_X(t) = P(X \le t)$, $t \in \mathbb{R}$

1. $t < 1$: $F_X(t) = 0$

2. $t \in \mathbb{N}$ ($= t \in \{1,2,3,\cdots\}$): $F_X(t) = \sum_{k=1}^{t} (1-p)^{k-1} \cdot p$

3. $t \geq 1$ but $t \notin \mathbb{N}$: $F_X(t) = \sum_{k=1}^{\lfloor t \rfloor} (1-p)^{k-1} \cdot p$

$t = 1.87$

Geometric r.v.: Memoryless Property

$B=\{X>m\}$

$A=\{X=n+m\}$

$P(A\cap B)=P(A)$

- Example: Suppose $X \sim \text{Geometric}(p)$, $p \in (0,1)$
  - $(1-p)^{n-1}\cdot p$
  - What is $P(X=n+m \mid X>m)$? $(n,m \in \mathbb{N})$
  - What is $P(X>n+m \mid X>m)$? $(n,m \in \mathbb{N})$

$$P(X=n+m \mid X>m) = \frac{P(X>m \text{ and } X=n+m)}{P(X>m)} = \frac{P(X=n+m)}{P(X>m)} = \frac{(1-p)^{n+m-1}\cdot p}{(1-p)^m} = (1-p)^{n-1}\cdot p$$

- $P(X=n+m) = (1-p)^{n+m-1}\cdot p$
- $P(X>m) = (1-p)^m$

$$= 1-P(X\le m) = 1-\sum_{k=1}^{m}(1-p)^{m-1}\cdot p = 1-\frac{\cancel{p}\cdot(1-(1-p)^m)}{(1-(1-p))\times \cancel{p}} = (1-p)^m$$

$$P(\underline{X>n+m} \mid \underline{X>m}) = (1-p)^n = P(X>n)$$

$$\frac{P(X>n+m \text{ and } X>m)}{P(X>m)} = \frac{P(X>n+m)}{P(X>m)} = (1-p)^n$$

$P(X>n+m) \to (1-p)^{n+m}$

$P(X>m) \to (1-p)^m$

Memoryless Property:

$$P(X>n+m \mid X>m) = P(X>n)$$

$$P(X=n+m \mid X>m) = P(X=n)$$

5. Discrete Uniform Random Variables

- Example: Roll a 4-sided die, and the numbers 1, 2, 3, 4 are equally likely to occur

- Example: The correct answer to an exam question: A, B, C, D are equally likely

- What are the common features?
  - 1 experiment trial (no repetition) with n equally-likely outcomes
  - Want: Whether a specific outcome occurs

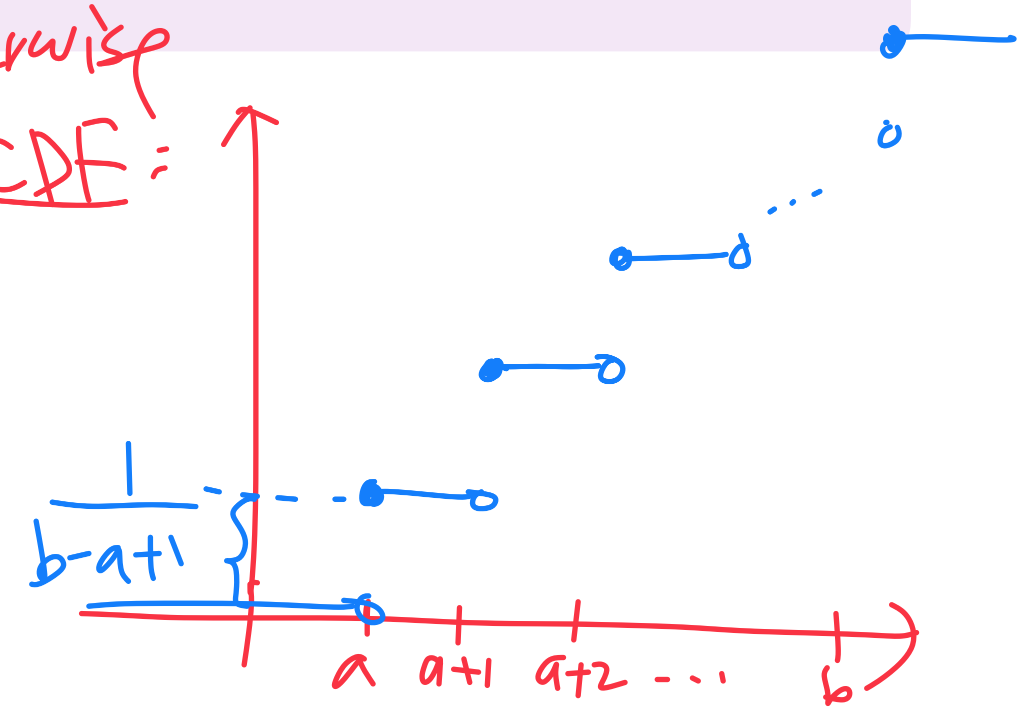
5. Discrete Uniform Random Variables (Formally)

Discrete Uniform Random Variables: A random variable X is discrete uniform with parameters (a, b) ($a, b \in \mathbb{Z}$ with $a \leq b$), if its PMF is given by

$$P(X = k) = \begin{cases} \dfrac{1}{b-a+1}, & k = a, a+1, \cdots, b \\ 0 & \text{otherwise} \end{cases}$$

PMF:

CDF:



Expected Value

Motivation: Guidelines for Decision Making?

- Suppose we are given 2 options:

Option A

$500 \times 10^6 \times \frac{1}{1000}$

$-50 \times 10^3 \times \frac{999}{1000}$

Win 500 millions with prob. 0.1%

Lose 50k dollars, otherwise

Risk-seeking

Option B

Win 50k dollars, with prob. 1

Risk-averse

- Which option will you choose?
- Could you come up with a creative way to get a reward higher than B but with a lower risk than A?

Why Expected Value?

- Example: Imagine you are an investor of Shinemood
  - $X =$ # of waffles sold by Shinemood today
  - Suppose $X \sim$ Poisson($\lambda = 150$, $T =$ 1 day)
- Question 1: How many waffles are expected to be sold today?

Expected Value

PMF of X

- Question 2: In the coming year, how many waffles will be sold per day on average?

Empirical average

$$\frac{X_1 + X_2 + \cdots + X_{365}}{365}$$

- Q1 and Q2 are closely related: Law of Large Numbers

Example: Expected Value and Poisson

- Example: X = # of waffles sold by Shinemood today
  - Suppose $X \sim$ Poisson($\lambda = 150$, $T = 1$ day)
  - PMF of X? How to define the expected value of X?

$$P(X=k) = \begin{cases} \frac{e^{-\lambda T} \cdot (\lambda T)^k}{k!}, & k = 0, 1, 2, \cdots \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{k=0}^{\infty} \underline{k} \cdot P(X=k) = \sum_{k=1}^{\infty} k \cdot P(X=k) = \sum_{k=1}^{\infty} \cancel{k} \cdot \frac{e^{-\lambda T} \cdot (\lambda T)^k}{\cancel{k!}\ (k-1)!}$$

$$= \lambda T \cdot \sum_{k=1}^{\infty} \frac{e^{-\lambda T} \cdot (\lambda T)^{k-1}}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda T} \cdot (\lambda T)^k}{k!} = 1$$

$$= \lambda T$$

Expected Value of a Discrete R.V. (Formally)

Expected Value (or Mean / Expectation):

Let X be a discrete random variable with

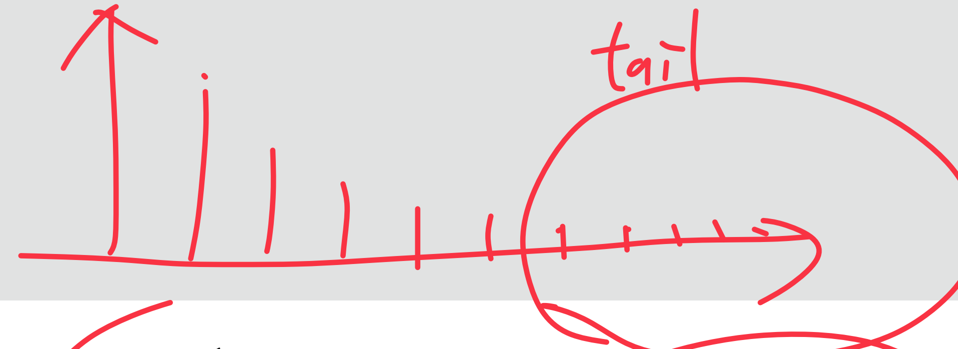
- the set of possible values S
- PMF of X is $p_X(x)$

The expected value of X is defined as

$$E[X] := \sum_{x \in S} x \cdot p_X(x)$$

- Sometimes we use the notation: $\mu_X \equiv E[X]$

Example: Expected Value



- Example: Suppose X has a PMF: $p_X(n) = \dfrac{1}{n(n+1)}, \; n = 1,2,\cdots$
  - What is $E[X]$?

$$E[X] = \sum_{n=1}^{\infty} n \cdot p_X(n) = \sum_{n=1}^{\infty} \cancel{n} \cdot \frac{1}{\cancel{n}\cdot(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$$

① $\frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \cdots$

② $\sum_{n=1}^{\infty} \frac{1}{n+1} \geq \int_1^{\infty} \frac{1}{x}\,dx$

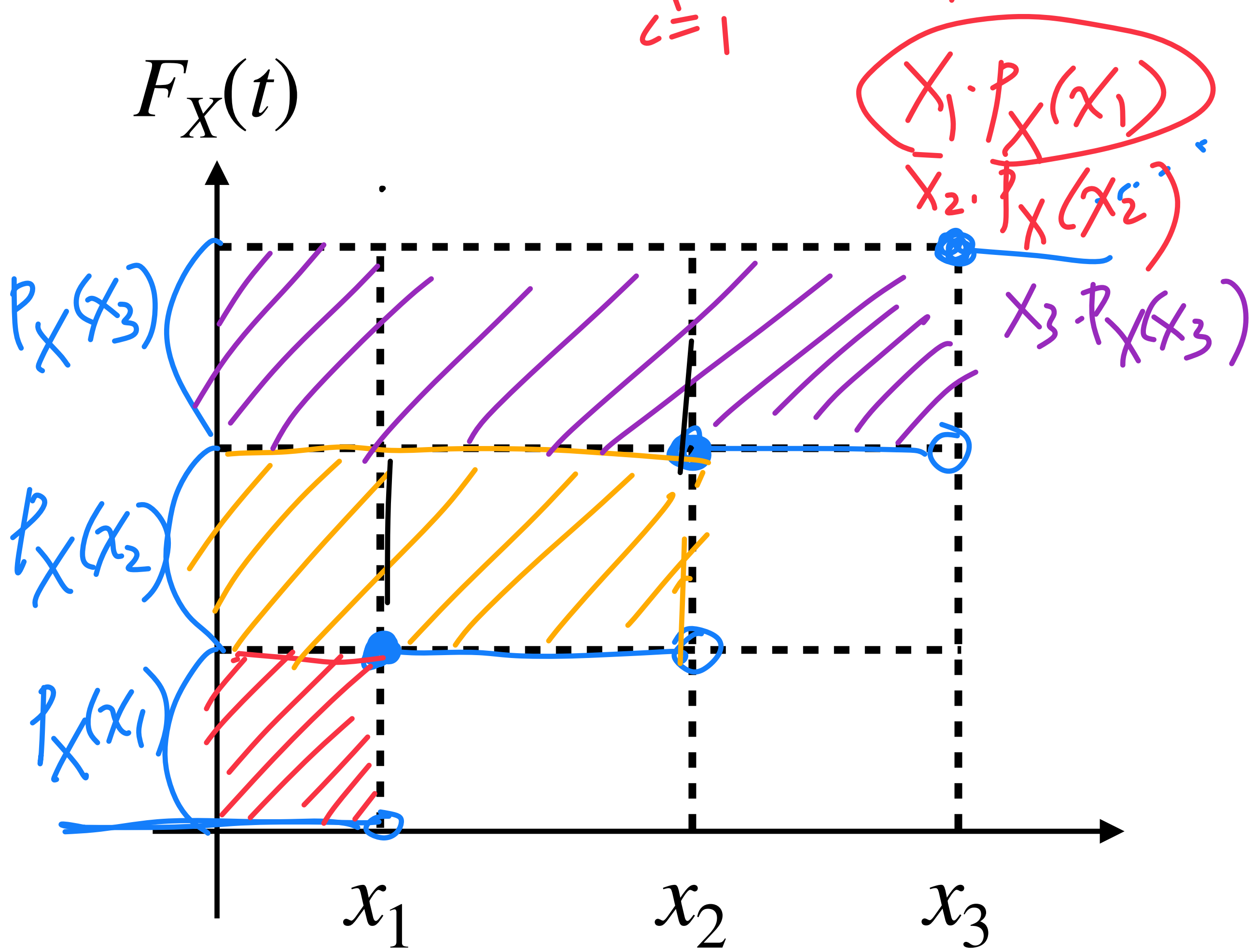
Example: St. Petersburg Paradox

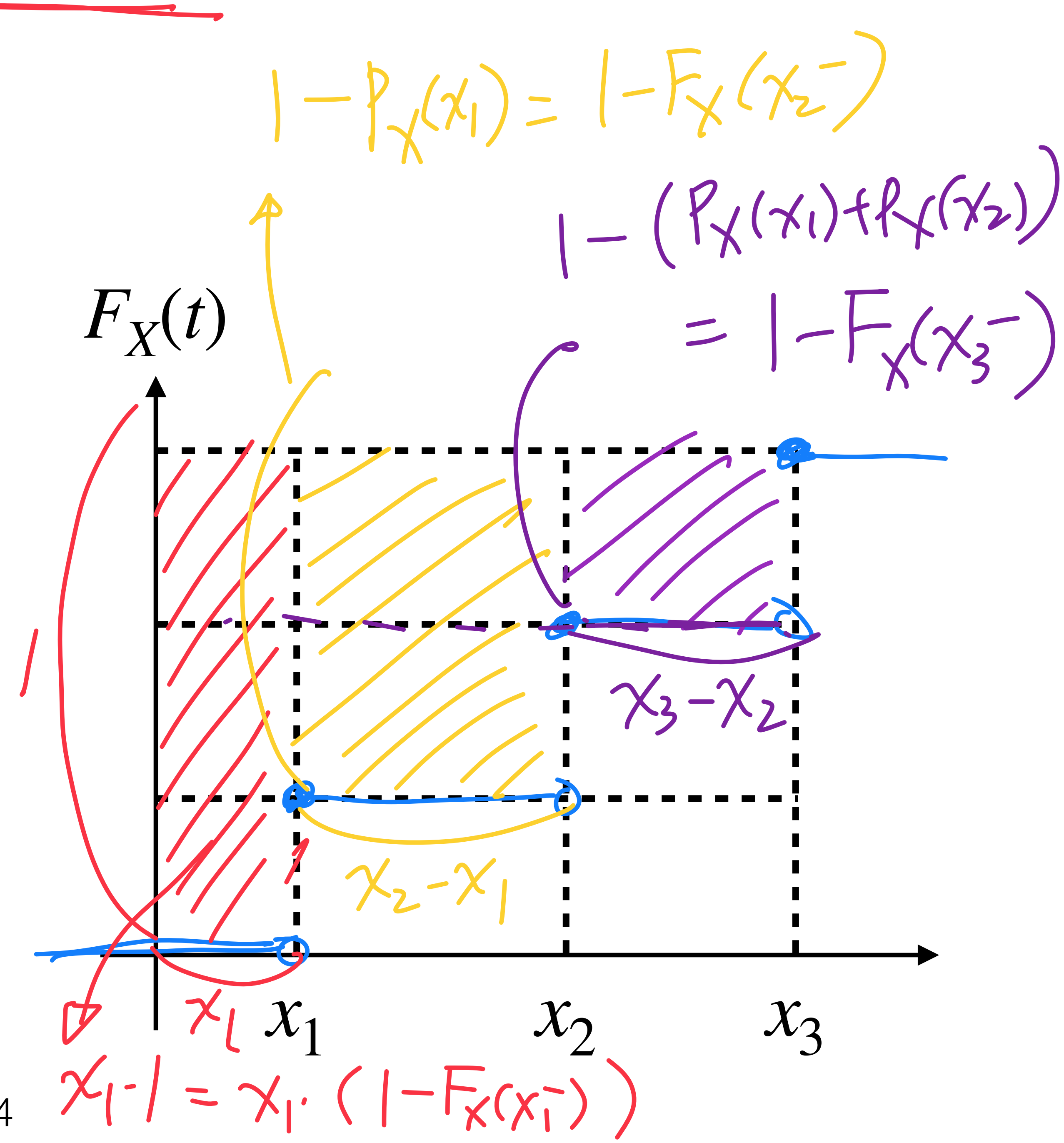
- Example: We are asked to pay 10000 dollars to play a game.
  - We can keep flipping a fair coin until a head is observed.
  - If the 1st head occurs at n-th toss, then we get a prize of 2^n dollars and the game is over.
  - Shall we play this game?

Visualize the Expected Value Using CDF

- Suppose X is a non-negative discrete random variable with
  - The set of possible values $\{x_1, x_2, x_3, \cdots\}$ (assume $x_i < x_{i+1}$)
  - Denote $x_0 = 0$

$$E[X] = \sum_{i=1}^{\infty} x_i \cdot P_X(x_i)$$





Expected Value of a Discrete Random Variable: An Alternative Expression

Expected Value (or Mean / Expectation):

Let X be a non-negative discrete random variable with

- the set of possible values $S = \{x_1, x_2, x_3 \cdots\}$
- CDF of X is $F_X(t)$

Denote $x_0 = 0$. The expected value of X is

$$E[X] = \sum_{i=1}^{\infty} (x_i - x_{i-1}) \cdot \left(1 - F_X(x_i^-)\right)$$

- What if $S = \{1,2,3 \cdots\}$?
- How about continuous cases?

$$E[X] = \int_0^{\infty} P(X > t)\,dt$$

Example: Using the Alternative Expression

- Example: Suppose X is a discrete random variable
  - For X, the set of possible values $A = \{2,4,6,8\cdots\}$
  - The CDF of X is $F_X(t) = 1 - \dfrac{1}{t^2}, t \in A$
  - What is $E[X]$?

A Property of Expected Value

Theorem (Expectation of a Function of r.v.):

1. Let X be a discrete random variable with
   - the set of possible values S
   - PMF of X is $p_X(x)$
2. Let $g(\cdot)$ be a real-valued function

The expectation of $g(X)$ is

$$E[g(X)] = \sum_{x \in S} g(x) \cdot p_X(x)$$

- Is this intuitive? Do we need a proof?
- Also called Law of the unconscious statistician

Proof of Law of the Unconscious Statistician

$$E[g(X)] := \sum_{x \in S} g(x) \cdot p_X(x)$$

Linearity of Expected Values (I)

Linearity Property (I):
Let X be a discrete random variable and α, β be real numbers. Then, we have

$$E[\alpha X + \beta] = \alpha \cdot E[X] + \beta$$

- How to show this?

Linearity of Expected Values (II)

Linearity Property (II):

Let X be a discrete random variable and $g(\cdot), h(\cdot)$ be real numbers. Then, we have

$$E[g(X) + h(X)] = E[g(X)] + E[h(X)]$$

- How to show this?

Conditional Expectation

- Example: Roll a fair 6-sided die once
  - Define $X =$ the number that we observe
  - Given that $X \geq 4$, what is the expected value of X?

Conditional Expectation:

Let X be a discrete random variable with the set of possible values $S = \{x_1, x_2, x_3 \cdots\}$. Let A be an event. The expected value of X conditioned on A

$$E[X \mid A] := \sum_{x \in S} x \cdot P(X = x \mid A)$$

Example: Taiwan Receipt Lottery

- Example: Suppose we have a receipt at hand
  - Define X = the prize we get
  - What is $E[X]$?
  - Given that the last digit is 7, what is the expected value of X?

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Variance and Moments

Moments and Others

$$E[g(X)] := \sum_{x \in S} g(x) \cdot p(x)$$

- Example: $g(X) = X^2$
- Example: $g(X) = X^n$
- Example: $g(X) = (X - \mu_X)^2$
- Example: $g(X) = (X - \mu_X)^n$
- Example: $g(X) = e^{tX}$

Variance

Variance (2nd central moment):
Let X be a discrete random variable with the set of possible values S and PMF $p_X(x)$. The variance of X is

$$\text{Var}[X] := E[(X - \mu_X)^2] = \sum_{x \in S} (x - \mu_X)^2 \cdot p_X(x)$$

- Sometimes we use the notation: $\sigma_X^2 \equiv \text{Var}[X]$
- Variance captures the <u>variability</u> of a random variable

Variance: An Alternative Explanation

- Example: Suppose we are given a random variable X
  - We need to output a prediction of X (denoted by z)
  - Penalty of prediction is $(X - z)^2$
  - What is the minimum expected penalty?

Another Way for Calculating Variance

Theorem:
Let X be a random variable. Then, we have

$$\mathsf{Var}[X] := E[X^2] - (E[X])^2$$

- How to show this?

Properties of Variance and Moments (I)

1. $\text{Var}(X + c) = \text{Var}(X)$?

2. $\text{Var}(aX) = a \cdot \text{Var}(X)$?

3. $\text{Var}(\,|X|\,) = \text{Var}(X)$?

4. $E(X^2) \geq (E(X))^2$?

5. Can $\text{Var}(X)$ be infinite?

When are Higher Moments Useful?

Berry-Esseen Theorem:

Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with $E[X_1] = 0$, $E[X_1^2] = \sigma^2$ and $E[\,|X_1|^3\,] < \infty$. Define $Y = (X_1 + X_2 + \cdots + X_n)/n$. Then, we have

$$|F_Y(t) - \Phi(t)| \le \frac{C\rho}{\sigma^3\sqrt{n}}$$

- Usually higher moments are used as <u>technical conditions</u>
  - Hence, we usually care about <u>whether $E(X^n) < \infty$</u>

1-Minute Summary

1. Special Discrete Random Variables

- Geometric / Uniform

2. Expected Value

- Definition / alternative expression

3. Variance and Moments

- Definition / alternative explanation using penalty / properties