7.4 Integration of rational functions by partial fractions

變數變換之 — 部份分式法。

Type 理解: 有理函數 $\dfrac{P(x)}{Q(x)}$ 的積分。
Idea 分解: 分成會積的分式 (proper fraction) 相加, 使用公式個別積分。
Formula 再構成:

$$\boxed{\int \frac{dx}{x-a} = \ln|x-a| + C}$$

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C}$$

$$\boxed{\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2) + C}$$

$$\boxed{\int \frac{dx}{x^n} = \frac{-1}{(n-1)x^{n-1}} + C}$$

Additional 1.: 代數基本定理 (**TFTA**): n 次多項式有 n 個根 (in $\mathbb{C}$)。因此可以因式分解(polynomial factorization) 成一次式 $(x-a)$ 或無法再化簡的(irreducible)二次式 (x^2+bx+c with $b^2-4c<0$, or $(x-b)^2+c^2$) 的乘積:

$$p(x) = K\prod_{i=1}^{r}(x-a_i)^{d_i}\prod_{j=1}^{s}[(x-b_j)^2+c_j^2]^{e_j},$$

where $K, a_i, b_j, c_j \in \mathbb{R}, d_i, e_j \in \mathbb{N}\cup\{0\}$, $\displaystyle\sum_{i=1}^{r} d_i + 2\sum_{j=1}^{s} e_j = n.$

Note: $\prod$ (大寫 π) 是乘積 (product) 符號, 用法與 $\sum$ (summation) 一樣。

Additional 2.: 整係數多項式 ($\mathbb{Z}[x]$) 的因式分解技巧: 一次因式檢驗法。

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{Z}, \quad a_n \neq 0.$$

考慮所有滿足 $k \mid a_n$(最高次係數) 與 $\ell \mid a_0$(常數項) 的 $kx-\ell$.
Ex: $p(x) = 2x^n + \cdots + 4$, $a_n = 2$, $a_0 = 4$, $\implies$ $2x\pm1$, $x\pm1$, $x\pm2$, $x\pm4$.
♦: 牛頓 (有理根) 定理: ℓ/k 是 $p(x)\in\mathbb{Z}[x]$ 的有理根 $\implies$ $k \mid a_n$ & $\ell \mid a_0$.

Partial Fractions Method 部分分式法: $\displaystyle\int \frac{P(x)}{Q(x)}\,dx.$

Step 1. If $\deg(P) < \deg(Q)$: ***proper*** 眞分式, let $R(x) = P(x)$ & goto **Step 2**.

If $\deg(P) \geq \deg(Q)$: ***improper*** 假分式, $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$,

用 $\boxed{\text{長除法}}$ (long division) 求商式 $S(x)$: 用冪次律的積分公式;

而餘式 $R(Q)$: $\dfrac{R(x)}{Q(x)}$ is proper, goto **Step 2**.

Step 2. $\boxed{\text{因式分解}}$ $Q(x)$ 成一次式與 (irreducible)二次式的乘積: (當首係數是 1.)

$$Q(x) = \prod_{i=1}^{r}(x-a_i)^{d_i}\prod_{j=1}^{s}[(x-b_j)^2+c_j^2]^{e_j}.$$

$\boxed{\text{假設未知數}}$ A_{i_k}'s, B_{j_ℓ}'s, C_{j_ℓ}'s 滿足: (每項都是眞分式)

$$\begin{aligned}\frac{R(x)}{Q(x)} &= \sum_{i=1}^{r}\left[\frac{A_{i_1}}{x-a_i}+\frac{A_{i_2}}{(x-a_i)^2}+\cdots+\frac{A_{i_{d_i}}}{(x-a_i)^{d_i}}\right]\\ &+ \sum_{j=1}^{s}\left[\frac{B_{j_1}x+C_{j_1}}{(x-b_j)^2+c_j^2}+\frac{B_{j_2}x+C_{j_2}}{[(x-b_j)^2+c_j^2]^2}+\cdots+\frac{B_{j_{e_j}}x+C_{j_{e_j}}}{[(x-b_j)^2+c_j^2]^{e_j}}\right],\end{aligned}$$

$\boxed{\text{通分}}$ 右式(只看分子), 與 $R(x)$ $\boxed{\text{比較}}$ (讓兩邊 x 相同冪次的係數相同),
得到 A_{i_k}'s, B_{j_ℓ}'s, C_{j_ℓ}'s 的聯立方程組, $\boxed{\text{解聯立方程組}}$。
(方程式與未知數的個數一定一樣多 $= \deg(Q)$.)

Step 3. 每項各自積分, 利用 $\boxed{\text{變數變換}}$ 以及

a. (一次式) $\displaystyle\int \frac{dx}{x-a} \overset{(u=x-a)}{=} \ln|x-a| + C.$ $\quad (a \in \mathbb{R})$

b. (二次式) $\displaystyle\int \frac{dx}{x^2+a^2} \overset{(x=a\tan\theta)}{=} \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$ $\quad (a > 0)$

c. (二次式) $\displaystyle\int \frac{x\,dx}{x^2+a^2} \overset{(u=x^2+a^2)}{=} \frac{1}{2}\ln(x^2+a^2) + C.$ $\quad (a > 0)$

d. (冪次律) $\displaystyle\int \frac{dx}{x^n} = \frac{-1}{(n-1)x^{n-1}} + C.$ $\quad (a \in \mathbb{R},\ n > 1)$

Example 0.1 *(Improper, 一個一次式)* $\displaystyle\int \frac{x^3+x}{x-1}\,dx.$

用長除法:

$$
\begin{array}{rllll}
 & x^2 & +x & +2 & \\
\cline{2-5}
x-1) & x^3 & +0 & +x & +0 \\
-) & x^3 & -x^2 & & \\
\cline{2-3}
 & & x^2 & +x & \\
 & -) & x^2 & -x & \\
\cline{3-4}
 & & & 2x & +0 \\
 & & -) & 2x & -2 \\
\cline{4-5}
 & & & & 2
\end{array}
$$

(2. 消去最高次項)
(1. 降冪, 缺項要補 0)
(3. 次數不夠就停)

$$
\begin{aligned}
\int \overbrace{\frac{x^3+x}{x-1}}^{\frac{P(x)}{Q(x)}}\,dx &= \int \Big(\overbrace{x^2+x+2}^{S(x)} + \overbrace{\frac{2}{\underline{x-1}}}^{\frac{R(x)}{Q(x)}} \Big)\,dx \\
&= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\,\underline{\ln|x-1|}+C.
\end{aligned}
$$

$(\displaystyle\int \frac{dx}{x-1} = \ln|x-1|+C.)$ ■

Example 0.2 *(多個一次式)* $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$

$2x^3+3x^2-2x = x(2x-1)(x+2),$ …………………… *(分母因式分解)*

Assume $\displaystyle\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2},$ ………… *(假設未知數)*

$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$ …… *(通分右式)*

$= (2A+B+2C)x^2+(3A+2B-C)x+(-2A),$ *(只看分子部分)*

(比較係數, 解聯立方程組)

$$
\left\{
\begin{array}{lrcrcrcr}
(x^2:) & 2A & + & B & + & 2C & = & 1 \\
(x^1:) & 3A & + & 2B & - & C & = & 2 \\
(x^0:) & -2A & & & & & = & -1
\end{array}
\right.
\implies A=\frac{1}{2},\ B=\frac{1}{5},\ C=\frac{-1}{10}.
$$

$$
\begin{aligned}
\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx &= \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2} \right) dx \\
\text{(注意係數!)} \quad &= \frac{1}{2}\ln|x| + \boxed{\frac{1}{10}}\ln|2x-1| - \frac{1}{10}\ln|x+2| + \underline{K}.
\end{aligned}
$$

(C 用過了改用 K, 每項各自變數變換 $\left\{ \begin{array}{ll} u=2x-1, & du=2\,dx; \\ v=x+2, & dv=dx. \end{array} \right.$ *)* ■

Example 0.3 $\int \frac{1}{x^2-a^2}\,dx$, *where* $a \neq 0$.

$x^2-a^2=(x-a)(x+a)$.

Assume $\frac{1}{x^2-a^2}=\frac{A}{x-a}+\frac{B}{x+a}$,

$1=A(x+a)+B(x-a)$ .... *(∗)*

$=(A+B)x+(A-B)a$,

$$\begin{cases} A+B=\boxed{0} \\ A-B=\frac{1}{a} \end{cases} \implies A=\frac{1}{2a},\ B=-\frac{1}{2a}.$$ *(左邊 x 缺項, 1 當成 $\boxed{0}x+1$.)*

$$\begin{aligned}\int \frac{1}{x^2-a^2}\,dx &= \int\left(\frac{1}{2a}\frac{1}{x-a}-\frac{1}{2a}\frac{1}{x+a}\right)dx \\ &= \frac{1}{2a}(\ln|x-a|-\ln|x+a|)+C \\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C.\end{aligned}$$ ∎

Skill: 解未知數技巧: 不要乘開, x 代入使某式變零的值。

$$\begin{array}{lrcll} \text{Ex: } (*) & 1 &=& A(x+a)+B(x-a) & \\ (\text{代入 } x=a) & 1 &=& A(a+a)+B(a-a) &= 2aA, \\ & \implies A &=& \frac{1}{2a}; & \\ (\text{代入 } x=-a) & 1 &=& A(-a+a)+B(-a-a) &= -2aB, \\ & \implies B &=& -\frac{1}{2a}. & \end{array}$$

Additional: (不好背, 用部分分式直接推。)

$$\boxed{\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C} \overset{\blacklozenge}{=} \begin{Bmatrix} -\frac{1}{a}\tanh^{-1}\frac{x}{a}, & \text{for } |x|<a \\ -\frac{1}{a}\coth^{-1}\frac{x}{a}, & \text{for } |x|>a \end{Bmatrix}+C.$$

Example 0.4 *(重複的一次式)* $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx.$

$$\begin{array}{rrrrrr} & & x & +1 & & \\ \cline{2-6} x^3-x^2-x+1) & x^4 & +0 & -2x^2 & +4x & +1 \\ -) & x^4 & -x^3 & -x^2 & +x & \\ \cline{2-5} & & x^3 & -x^2 & +3x & +1 \\ -) & & x^3 & -x^2 & -x & +1 \\ \cline{3-6} & & & & 4x & \end{array}$$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}.$$

$x^3-x^2-x+1=(x-1)^2(x+1).$

假設分子少分母一次:
$$\frac{Ax+B'}{(x-1)^2} \text{ (很難積)} = \frac{A(x-1)+A+B'}{(x-1)^2} = \frac{A}{x-1}+\frac{A+B'}{(x-1)^2} = \frac{A}{x-1}+\frac{B}{(x-1)^2} \text{ (會積)}$$

$$Assume\ \frac{4x}{x^3-x^2-x+1} = \boxed{\frac{A}{x-1}+\frac{B}{(x-1)^2}}+\frac{C}{x+1},\ (Why?\ \uparrow)$$

$$\begin{aligned} 4x &= A(x-1)(x+1)+B(x+1)+C(x-1)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (**) \\ &= (A+C)x^2+(B-2C)x+(-A+B+C), \end{aligned}$$

$$\left\{\begin{array}{rcl} A \quad\ \ + \ C &=& 0 \\ B - 2C &=& 4 \\ -A+B+\ C &=& 0 \end{array}\right. \implies A=1,\ B=2,\ C=-1.\ \ (\text{當作 } 0x^2+4x+0.)$$

$$\begin{aligned} \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int \left(x+1+\frac{1}{x-1}+\boxed{\frac{2}{(x-1)^2}}-\frac{1}{x+1}\right)\,dx \\ \text{(變數變換冪次律)} &= \frac{x^2}{2}+x+\ln|x-1|\boxed{-\frac{2}{x-1}}-\ln|x+1|+K \\ &= \frac{x^2}{2}+x-\frac{2}{x-1}+\ln\left|\frac{x-1}{x+1}\right|+K. \end{aligned}$$

(Let $u=x-1$, $\displaystyle\int\frac{dx}{(x-1)^2}=\int\frac{du}{u^2}=-\frac{1}{u}+C=-\frac{1}{x-1}+C;$

代入比解聯立快: $(**)$ $4x=A(x-1)(x+1)+B(x+1)+C(x-1)^2$,

$$\left\{\begin{array}{llll} \text{把}(x-1)\text{變零}: x=1 & \implies 4=2B, & B=2; & \\ \text{把}(x+1)\text{變零}: x=-1 & \implies -4=4C, & C=-1; &) \\ \text{代其他好算的}: x=0 & \implies 0=-A+B+C, & A=B+C=1. & \end{array}\right.$$ ■

Attention: $Q(x)$ 有 d 重的一次因式 $(x-a)^d$, 就要假設 d 個未知數 $A_1, A_2, \ldots, A_d$:

$$\boxed{Q(x)=(x-a)^d\times\cdots \stackrel{\text{假設}}{\Longrightarrow} \frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_d}{(x-a)^d}}.$$

Example 0.5 *(二次式)* $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx.$

$x^3+4x = x(x^2+4)$ *irreducible.*

Assume $\dfrac{2x^2-x+4}{x^3+4x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}, \ldots$ *(二次式分母的分子要假設一次式)*

$2x^2-x+4 = A(x^2+4)+(Bx+C)x = (A+B)x^2+Cx+4A,$

$$\begin{cases} A+B & = & 2 \\ \qquad\quad C & = & -1 \\ 4A & = & 4 \end{cases} \implies A=1,\ B=1,\ C=-1.$$

$$\begin{aligned}
\int \frac{2x^2-x+4}{x^3+4x}\,dx &= \int\left(\frac{1}{x}+\frac{x-1}{x^2+4}\right)dx \qquad \text{(再分開)} \\
&= \int\left(\frac{1}{x}+\frac{x}{x^2+4}-\frac{1}{x^2+4}\right)dx \\
&= \ln|x| + \frac{1}{2}\ln(\underline{x^2+4}) - \frac{1}{2}\tan^{-1}\frac{x}{2} + K.
\end{aligned}$$

(Let $u = x^2+4$, $du = 2x\ dx$; $x^2+4>0$, 絕對值可以換掉。) ■

Example 0.6 *(要配方的二次式.)* $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx.$

$\dfrac{4x^2-3x+2}{4x^2-4x+3} = 1 + \dfrac{x-1}{4x^2-4x+3},$

$4x^2-4x+3$ *is irreducible* $(\because b^2-4ac = [(-4)^2-4\cdot4\cdot3]<0)$.

配方: $4x^2-4x+3 = (2x-1)^2+2$, *let* $u = 2x-1$, $du = 2\ dx$.

$$\begin{aligned}
\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx &= \int 1 + \frac{x-1}{4x^2-4x+3}\,dx \\
&= x + \int \frac{\frac{u+1}{2}-1}{u^2+2}\cdot\frac{1}{2}\,du \qquad \text{(變數變換)} \\
&= x + \frac{1}{4}\int\left(\frac{u}{u^2+2}-\frac{1}{u^2+2}\right)du \\
&= x + \frac{1}{4}\frac{1}{2}\ln(\underline{u^2+2}) - \frac{1}{4}\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C \\
&= x + \frac{1}{8}\ln(\underline{4x^2-4x+3}) - \frac{1}{4\sqrt{2}}\tan^{-1}\frac{2x-1}{\sqrt{2}} + C.
\end{aligned}$$

($4x^2-4x+3>0$, 絕對值可以拿掉; 最後的 u^2+2 直接換回 $4x^2-4x+3$.) ■

Observation: 不能分解的二次式 x^2+bx+c 一定可以配方成 $u^2+a^2>0$, 所以 $\ln|x^2+bx+c|$ 的絕對值都可以換成小括號 $\ln(x^2+bx+c)$; 最後換回 x 的時候若有 u^2+a^2 也可以直接換成 x^2+bx+c (用代的也一樣)。

Example 0.7 *(重複的二次式.)* $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx.$

$$\textit{Assume}\ \frac{1-x+2x^2-x^3}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2},\qquad (\text{設少了會算錯})$$

$$\begin{aligned}1-x+2x^2-x^3 &= A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x \quad \ldots (***)\\ &= (A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A,\end{aligned}$$

$$\left\{\begin{array}{rcccl} A+B & & & = & 0\\ & C & & = & -1\\ 2A+B & & +D & = & 2\\ & +C & +E & = & -1\\ A & & & = & 1\end{array}\right. \Longrightarrow \begin{array}{l} A=1\\ B=-1\\ C=-1\\ D=1\\ E=0\end{array}$$

直接解 *(easy)*, 或是代入 *(***)(hard, but learn it)*

$$\left\{\begin{array}{lll} x=0 & \Longrightarrow & 1=A;\\ x^2=-1 & \Longrightarrow & -1=-D+Ex,\ D=1,\ E=0; \quad (\text{用比較係數})\\ x=\pm1 & \Longrightarrow & -2=B+C,\ 0=B-C,\ B=C=-1.\end{array}\right.$$

$$\begin{aligned}&\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx\\ =\ &\int\left(\frac{1}{x}-\frac{x}{x^2+1}-\frac{1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx\\ =\ &\ln|x|-\frac{1}{2}\ln(x^2+1)-\tan^{-1}x-\frac{1}{2(x^2+1)}+K.\end{aligned}$$

■

Attention: $Q(x)$ 有 e 重的二次因式 $[(x-b)^2+c^2]^e$, 就要假設 $2e$ 個未知數 $B_1,C_1,B_2,C_2,\ldots,B_e,C_e$:

$$\boxed{Q(x)=[(x-b)^2+c^2]^e\times\cdots \overset{\text{假設}}{\Longrightarrow} \frac{B_1x+C_1}{(x-b)^2+c^2}+\cdots+\frac{B_ex+C_e}{[(x-b)^2+c^2]^e}}.$$

Example 0.8 *(不要放棄嘗試變數變換)* $\displaystyle\int \frac{x^2+1}{x(x^2+3)}\,dx.$

Let $u = x(x^2+3) = x^3+3x$, *then* $du = 3(x^2+1)\,dx$, $(x^2+1)\,dx = \frac{1}{3}\,du$.

$$\int \frac{x^2+1}{x(x^2+3)}\,dx = \int \frac{du}{3u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+3x| + C. \qquad \blacksquare$$

(Try yourself: $\displaystyle\frac{x^2+1}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3} = \frac{1}{3}\frac{1}{x} + \frac{2}{3}\frac{x}{x^2+3}$.*)*

Rationalizing substitutions 有理代換: 分式裡有開 n 次根的函數 $\sqrt[n]{g(x)}$, let $u = \sqrt[n]{g(x)}$, 然後換成沒有根式的有理函數再積分。
例如: 積分時看到 $\sqrt{x^2+1}$, 變數變換用 $u = x^2+1$ 或許沒有 $u = \sqrt{x^2+1}$ 來得簡化, 平平是變數變換, 撇步不同, 過程不同, 雖然答案是一樣的。

Example 0.9 *(有理代換)* $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx.$

Let $u = \sqrt{x+4}$, *then* $du = \frac{1}{2u}\,dx$, $x = u^2-4$.

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\,dx &= \int \frac{u}{u^2-4}\cdot 2u\,du = \int \left(2 + \frac{2}{u-2} - \frac{2}{u+2}\right)du \\
&= 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C \\
&= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.
\end{aligned}$$

(Let $u = x+4$ *好做嗎? Try yourself.)* $\blacksquare$

♦ **Additional:** $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$ by substitution rule:

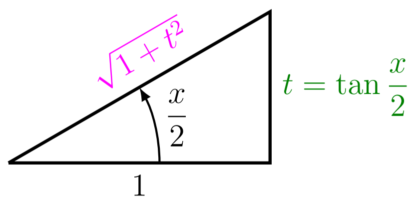
$$\int \sec x\,dx = \int \frac{dx}{\cos x} \overset{\text{同乘 } \cos x}{=} \int \frac{\cos x\,dx}{\cos^2 x} = \int \frac{d(\sin x)}{1-\sin^2 x} \overset{u=\sin x}{=} \int \frac{du}{1-u^2}$$
$$= \frac{1}{2}\int \frac{1}{1+u} + \frac{1}{1-u}\,du \overset{\text{中略}}{=} \frac{1}{2}(\ln|1+u| - \ln|1-u|) + C$$
$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| \overset{\text{同乘 } (1+u)}{=} \frac{1}{2}\ln\left|\frac{(1+u)^2}{1-u^2}\right| \overset{\text{換回}}{=} \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right|$$
$$= \ln\sqrt{\left|\frac{(1+\sin x)^2}{\cos^2 x}\right|} = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| = \ln|\sec x + \tan x| + C.$$

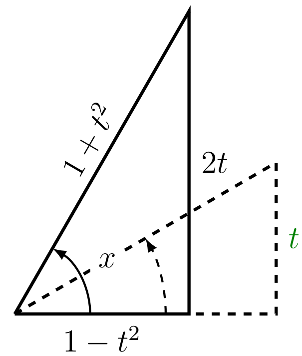
Additional: Weierstrass substitution 魏爾斯特拉斯變換

又稱 Tangent half-angle substitution 正切半角變換, 把三角函數換成有理函數。以德國數學家 Karl Theodor Wilhelm Weierstraß (1815–1897) 命名。

> The world's sneakiest substitution is undoubtedly.
> 無庸置疑的是世界上最卑鄙的變換。 — Michael Spivak

Example 0.10 (Ex 7.4.59.) *Let* $t = \tan\frac{x}{2}$, $x \in (-\pi, \pi)$. *Then*





(a) $\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$, $\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$.

(b) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$,

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = (\frac{1}{\sqrt{1+t^2}})^2 - (\frac{t}{\sqrt{1+t^2}})^2 = \frac{1-t^2}{1+t^2}$,

$\tan x = \frac{2t}{1-t^2}$, $\cot x = \frac{1-t^2}{2t}$, $\sec x = \frac{1+t^2}{1-t^2}$, $\csc x = \frac{1+t^2}{2t}$.

(c) $x = 2\tan^{-1} t$, $dx = \frac{2}{1+t^2}\ dt$. ■

Example 0.11 (Ex 7.4.60) $\int \frac{dx}{1-\cos x}$.

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\ dt = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\cot\frac{x}{2} + C.$$

■

(Try yourself: Exercise 7.4.61–63:
$\int \frac{dx}{3\sin x - 4\cos x}$, $\int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x - \cos x}$, $\int_0^{\pi/2} \frac{\sin 2x}{2+\cos x}\ dx$.)