11.7 Strategy for testing series

More exercises. Apply: T4D(§11.2), I.T.(§11.3), C.T., L.C.T.(§11.4), A.S.T.(§11.5), Abs., Ratio T., Root T.(§11.6).

Example 0.1 $\sum \frac{n-1}{2n+1}$.

T4D	*IT*	*CT*	*LCT*	*AST*	*Abs*	*RaT*	*RoT*
$\to \frac{1}{2} \neq 0$ ✓	$+ \nearrow$ ✗	$> \frac{1}{n}$ ○	$\div \frac{1}{2}$ ○	✗	✗	$\to 1$ ✗	$\to 1$ ✗

Example 0.2 $\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$.

T4D	*IT*	*CT*	*LCT*	*AST*	*Abs*	*RaT*	*RoT*
$\to 0$ ✗	$+ \searrow$ ○	$< \frac{1}{3n^{3/2}}$ ○	$\div \frac{1}{n^{3/2}}$ ✓	✗	✗	$\to 1$ ✗	$\to 1$ ✗

Example 0.3 $\sum ne^{-n^2}$. $\left(\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}+C.\right)$

T4D	*IT*	*CT*	*LCT*	*AST*	*Abs*	*RaT*	*RoT*
$\to 0$ ✗	$= \frac{1}{2}$ ✓	✗	✗	✗	✗	$\to 0$ ○	$\to 0$ ○

Example 0.4 $\sum (-1)^n \frac{n^3}{n^4+1}$. $\left(\sum \frac{n^3}{n^4+1} \textit{ diverges.}\right)$

T4D	*IT*	*CT*	*LCT*	*AST*	*Abs*	*RaT*	*RoT*
$\to 0$ ✗	✗	✗	✗	$\rightsquigarrow$ ✓	*C.C.* ✗	$\to 1$ ✗	$\to 1$ ✗

Example 0.5 $\sum \frac{2^n}{n!} (= e^2 - 1)$. $\left(\blacklozenge\text{: } n! \approx \sqrt{2\pi n}(\frac{n}{e})^n.\right)$

T4D	*IT*	*CT*	*LCT*	*AST*	*Abs*	*RaT*	*RoT*
$\to 0$ ✗	✗	$< \frac{8}{n(n-1)}$ ○	✗	✗	✗	$\to 0$ ✓	$\to 0$ ○

Example 0.6 $\sum \frac{1}{2+3^n}$. $\left(\int \frac{dx}{2+3^x} = \frac{1}{2}\log_3 \frac{3^x}{2+3^x} + C.\right)$

T4D	*IT*	*CT*	*LCT*	*AST*	*Abs*	*RaT*	*RoT*
$\to 0$ ✗	$\frac{1}{2}\log_3 \frac{5}{3}$ ○	$< \frac{1}{3^n}$ ✓	$\div \frac{1}{3^n}$ ✓	✗	✗	$\to \frac{1}{3}$ ✓	$\to \frac{1}{3}$ ✓

學了這麼多 Test, 還是有很多級數沒辦法判斷。Ex: $\sum \frac{\sin n}{n}$.

♦ Additional: Dirichlet's Theorem and more tests

Theorem 1 (Dirichlet's Theorem)
$\exists M \ni \left|\sum_{k=1}^{n} a_k\right| \le M$ *for* $n \in \mathbb{N}$ *and* $b_n \searrow 0$*, then* $\sum a_n b_n$ *converges.*

♦ **Fact:** $\sum \frac{\sin n}{n^p}$ and $\sum \frac{\cos n}{n^p}$ are A.C. for $p > 1$, and C.C. for $0 < p \le 1$.

Sketch of Proof. For $p > 1$, $\sum \frac{|\sin n|}{n^p}$ converges by C.T. ($\frac{|\sin n|}{n^p} \le \frac{1}{n^p}$).
For $p > 0$, $\left|\sum_{k=1}^{n} \sin k\right| = \left|\frac{\cos(1/2) - \cos(n + 1/2)}{2\sin(1/2)}\right| \le \frac{1}{\sin(1/2)}$ and $\frac{1}{n^p} \searrow 0$,
$\sum \frac{\sin n}{n^p}$ converges by Dirichlet's Theorem. For $0 < p \le 1$, $\frac{|\sin n|}{n^p} \ge \frac{|\sin n|}{n}$,
$\sum \frac{|\sin n|}{n}$ (harder to prove), and hence $\sum \frac{|\sin n|}{n^p}$, diverges by C.T.. ■

Raabe's Test. $a_n > 0$,

$$\lim_{n\to\infty} \left[n(1 - \frac{a_{n+1}}{a_n})\right] \begin{cases} > 1, & \Longrightarrow \sum a_n \text{ converges,} \\ < 1, & \Longrightarrow \sum a_n \text{ diverges,} \\ = 1, & \Longrightarrow \text{inconclusive.} \end{cases}$$

Kummer's Test. $a_n > 0$, $b_n > 0$,

$$\lim_{n\to\infty} (b_n \frac{a_n}{a_{n+1}} - b_{n+1}) \begin{cases} > 0, & \Longrightarrow \sum a_n \text{ converges,} \\ < 0 \ \& \sum b_n \text{ diverges,} & \Longrightarrow \sum a_n \text{ diverges,} \\ = 0, & \Longrightarrow \text{inconclusive.} \end{cases}$$

Berlrand's Test. $\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{p_n}{n \ln n}$,

$$\begin{cases} \liminf_{n\to\infty} \ p_n > 1 & \Longrightarrow \sum a_n \text{ converges,} \\ \limsup_{n\to\infty} \ p_n < 1 & \Longrightarrow \sum a_n \text{ diverges.} \end{cases}$$

上極限 (limit superior or upper limit):	下極限 (limit inferior or lower limit):
$\overline{\lim_{n\to\infty}} \ p_n = \limsup_{n\to\infty} p_n = \lim_{n\to\infty} \sup_{m\ge n} p_m$,	$\underline{\lim_{n\to\infty}} \ p_n = \liminf_{n\to\infty} p_n = \lim_{n\to\infty} \inf_{m\ge n} p_m$,
supremum = the least upper bound.	infimum = the greatest lower bound.