10.2 Calculus with parametric curves

1. tangent 切線 $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = g'(t)/f'(t)$ if $\frac{dx}{dt} = f'(t) \neq 0$
2. area 面積 $A = \int y\ dx = \int g(t)f'(t)\ dt$
3. arc length 弧長 $L = \int\ ds = \int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$
4. surface area 表面積 $S = \int 2\pi y\ ds = \int 2\pi y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$

Parametric equations $x = f(t)$, $y = g(t)$.

Recall: 如果可以化成 $y = h(x)$ on $[a, b]$.

1. 如果 $h(x)$ 可微分, 切線斜率 $\dfrac{dy}{dx} = h'(x)$.
2. 如果 $h(x)$ 可積分,
   (a) 淨面積 $A = \displaystyle\int_a^b h(x)\ dx$, 面積 $A = \displaystyle\int_a^b |h(x)|\ dx$.
   (b) 繞 x-axis 體積 (disk/washer) $V = \displaystyle\int_a^b \pi[h(x)]^2\ dx$.
   (c) 繞 y-axis 體積 (cylindrical shell) $V = \displaystyle\int_a^b 2\pi x|h(x)|\ dx$.
3. 如果 $h(x)$ smooth ($h'(x)$ 連續),
   (a) 弧長 $L = \displaystyle\int\ ds = \int_a^b \sqrt{1 + [h'(x)]^2}\ dx$,
   (b) 繞 x-axis 表面積 $S = \displaystyle\int 2\pi y\ ds = \int_a^b 2\pi h(x)\sqrt{1 + [h'(x)]^2}\ dx$,
   (c) 繞 y-axis 表面積 $S = \displaystyle\int 2\pi x\ ds = \int_a^b 2\pi x\sqrt{1 + [h'(x)]^2}\ dx$,

Question: 如果沒辦法化成函數, 怎麼求?

0.1 Tangent & derivative

一階導數:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left(= \frac{g'(t)}{f'(t)}\right) \qquad \text{if } \frac{dx}{dt}(= f'(t)) \neq 0}$$

Proof. By Chain Rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$. ■

二階導數:

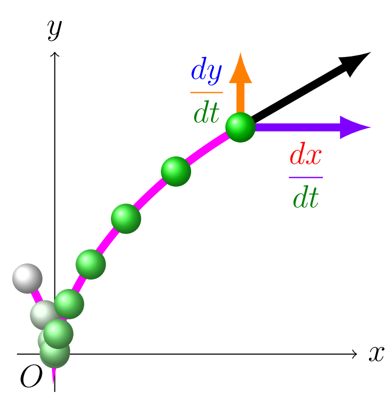
$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \left(= \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{f'(t)}\right) \qquad \text{if } \frac{dx}{dt}(= f'(t)) \neq 0}$$

Proof. By Chain rule $\frac{d}{dt}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{dy}{dx}) \cdot \frac{dx}{dt} = \frac{d^2y}{dx^2} \cdot \frac{dx}{dt}$. ■

Note: 如果把 t 當成時間 (time):

$\frac{dx}{dt} = f'(t)$ 就是 x-axis (往右爲正) 方向的速率,

$\frac{dy}{dt} = g'(t)$ 就是 y-axis (往上爲正) 方向的速率。



Attention: 1. 斜率"剛好"是速率相除。

2. $\frac{dx}{dt} = f'(t) \neq dx \div dt$, $\frac{dy}{dt} = g'(t) \neq dy \div dt$, 是導函數, 不是微分相除。

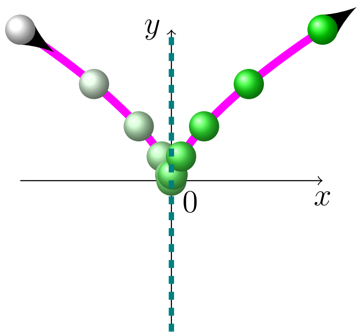
3. $\frac{d^2y}{dx^2} = \frac{(g'/f')'}{f'} \neq \frac{d^2y}{dt^2} \div \frac{d^2x}{dt^2}$ 不是加速度相除。

4. Chain Rule 不是這樣用 $\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx}) \cdot \frac{dt}{dx} \neq \frac{d}{dt}(\frac{dy}{dx}) \div \frac{dx}{dt}$ (~~倒過來~~)。

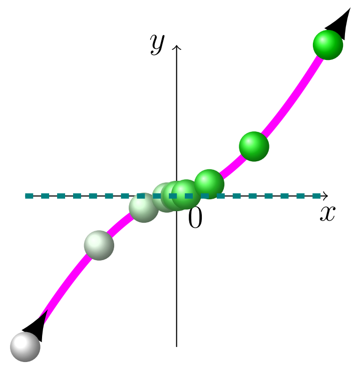
(除非有反函數 $t = f^{-1}(x)$, 才有 $\frac{dt}{dx} = 1 \div \frac{dx}{dt}$.)

Vertical/Horizontal tangent line:

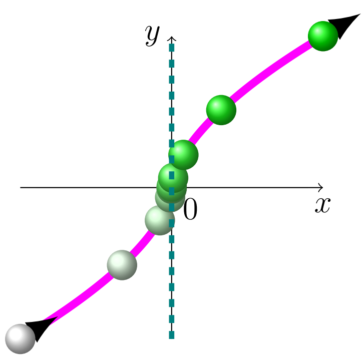
1. 如果 $\boxed{\dfrac{dx}{dt} = \mathbf{0}\ \&\ \dfrac{dy}{dt} \neq \mathbf{0}}$, $\Longrightarrow$ 有垂直切線;
2. 如果 $\boxed{\dfrac{dx}{dt} \neq \mathbf{0}\ \&\ \dfrac{dy}{dt} = \mathbf{0}}$, $\Longrightarrow$ 有水平切線;
3. 如果 $\boxed{\dfrac{dx}{dt} = \mathbf{0} = \dfrac{dy}{dt}}$, 要看 $\displaystyle\lim_{t\to a^{\pm}} \frac{dy}{dx} = \lim_{t\to a^{\pm}} \frac{g'(t)}{f'(t)}$ $(\frac{\mathbf{0}}{\mathbf{0}})$, 什麼都有可能:



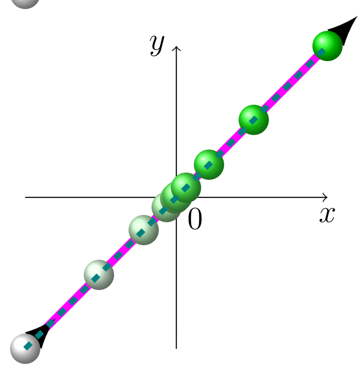
$x = t^3$,
$y = t^2$,
$\displaystyle\lim_{t\to 0^{\pm}} \frac{dy}{dx}$
$\displaystyle = \lim_{t\to 0^{\pm}} \frac{2t}{3t^2}$
$= \pm\infty$:
垂直切線;



$x = t^3$,
$y = t^5$,
$\displaystyle\lim_{t\to 0} \frac{dy}{dx}$
$\displaystyle = \lim_{t\to 0} \frac{5t^4}{3t^2}$
$= 0$:
水平切線;

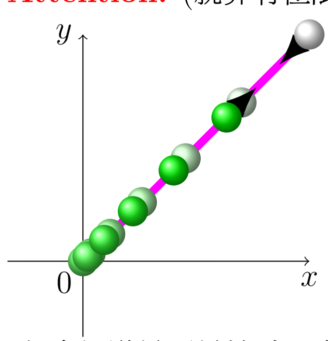


$x = t^5$,
$y = t^3$,
$\displaystyle\lim_{t\to 0} \frac{dy}{dx}$
$\displaystyle = \lim_{t\to 0} \frac{3t^2}{5t^4}$
$= \infty$:
垂直切線;



$x = t^3$,
$y = t^3$,
$\displaystyle\lim_{t\to 0} \frac{dy}{dx}$
$\displaystyle = \lim_{t\to 0} \frac{3t^2}{3t^2}$
$= 1$:
斜切線率1;

Attention: (就算有極限)<u>端點沒切線</u>。



$x = t^2$, $y = t^2$, $\displaystyle\lim_{t\to 0} \frac{dy}{dx} = \lim_{t\to 0} \frac{2t}{2t} = 1$:
除了 $(0, 0)$ 端點沒切線, 其他點都有。

(怎麼知道是不是端點? 畫圖!)

Example 0.1 *A curve C is defined by $x = t^2$, $y = t^3 - 3t$. (沒說就是 $t \in \mathbb{R}$。)*
(a) Show C has two tangent lines at $(3, 0)$ and find their equations.
(b) Find the points on C where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

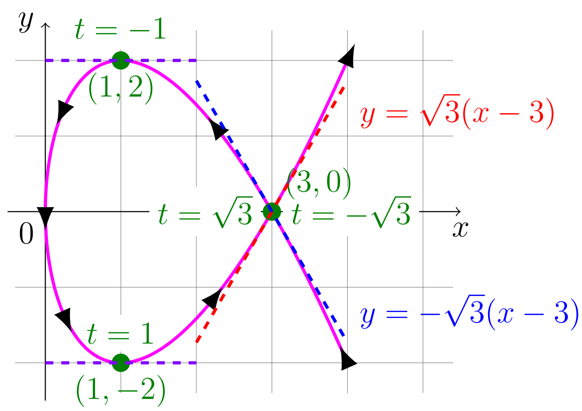
(a) $x = t^2 = 3$ *and* $y = t^3 - 3t = t(t^2 - 3) = 0$ *only when* $t = \pm\sqrt{3}$,
通過 $(3, 0)$ 只有 $t = \pm\sqrt{3}$ 兩個, 切線最多兩條 (可能同一條)。

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t} = \frac{3}{2}(t - \frac{1}{t}), \quad \frac{dy}{dx}\Big|_{t=\pm\sqrt{3}} = \pm\sqrt{3}.$$

$\Longrightarrow$ *Two tangent line* $y = \sqrt{3}(x - 3)$ *and* $y = -\sqrt{3}(x - 3)$.

(b) $\frac{dy}{dt} = 3(t^2 - 1) = 0$ *when* $t = \pm 1$, *and* $\frac{dx}{dt}\Big|_{t=\pm 1} = 2t\Big|_{t=\pm 1} = \pm 2 \neq 0$.
C has horizontal tangent at $(1, -2)$ (when $t = 1$) and $(1, 2)$ (when $t = -1$).
$\frac{dx}{dt} = 2t = 0$ *when* $t = 0$, *and* $\frac{dy}{dt}\Big|_{t=0} = 3(t^2 - 1)\Big|_{t=0} = -3 \neq 0$.
C has vertical tangent at $(0, 0)$ (when $t = 0$).

(c) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{\frac{3}{2}(1 + \frac{1}{t^2})}{2t} = \frac{3(t^2 + 1)}{4t^3},$$ *has critical number* $t = 0$.
C is CU ($\frac{d^2y}{dx^2} > 0$) when $t > 0$ and CD ($\frac{d^2y}{dx^2} < 0$) when $t < 0$.

(d) 

$\blacksquare$

Example 0.2 *(a) Find the tangent to the cycloid*

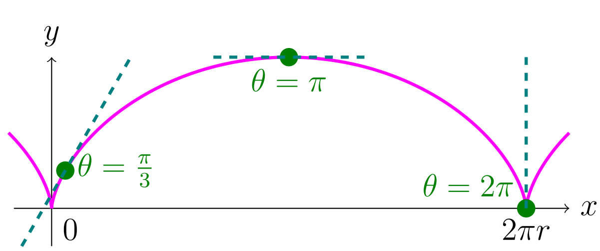
$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$$

at the point where $\theta = \dfrac{\pi}{3}$.
(b) At what points is the tangent horizontal? When is it vertical?

(a) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{r\sin\theta}{r(1-\cos\theta)} = \dfrac{\sin\theta}{1-\cos\theta}$,

When $\theta = \dfrac{\pi}{3}$, $x = r(\dfrac{\pi}{3} - \sin\dfrac{\pi}{3}) = r(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2})$, $y = r(1 - \cos\dfrac{\pi}{3}) = \dfrac{r}{2}$,
$\dfrac{dy}{dx}\Big|_{\theta=\pi/3} = \dfrac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \dfrac{\sqrt{3}/2}{1-1/2} = \sqrt{3}$.
$\Longrightarrow$ *tangent line* $y = \sqrt{3}(x - r(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2})) + \dfrac{r}{2}$ *or* $\sqrt{3}x - y = r(\dfrac{\pi}{\sqrt{3}} - 2)$.

(b) horizontal:
$\dfrac{dy}{d\theta} = r\sin\theta = 0$ *and* $\dfrac{dx}{d\theta} = r(1-\cos\theta) \neq 0$, *when* $\theta = (2n-1)\pi$,
$x = r((2n-1)\pi - \sin(2n-1)\pi) = (2n-1)\pi r$, $y = r(1 - \cos(2n-1)\pi) = 2r$,
and the points are $((2n-1)\pi r, 2r)$.
vertical:
when $\theta = 2n\pi$, $\dfrac{dx}{d\theta} = r(1-\cos\theta) = 0$ *and* $\dfrac{dy}{d\theta} = r\sin\theta = 0$. *(要看極限)*
$\lim\limits_{\theta\to 2n\pi^\pm} \dfrac{dy}{dx} = \lim\limits_{\theta\to 2n\pi^\pm} \dfrac{\sin\theta}{1-\cos\theta} \stackrel{l'H}{=} \lim\limits_{\theta\to 2n\pi^\pm} \dfrac{\cos\theta}{\sin\theta} = \pm\infty$ $(\dfrac{\mathbf{0}}{\mathbf{0}})$,
$x = r(2n\pi - \sin 2n\pi) = 2n\pi r$, $y = r(1 - \cos 2n\pi) = 0$,
and the points are $(2n\pi r, 0)$. ■



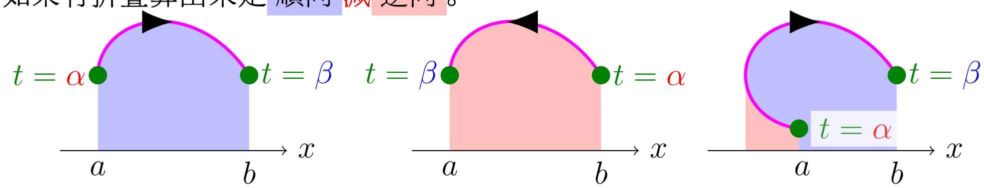
0.2 Area

$x = f(t)$, $y = g(t)$. If $y \geq 0$ and $f(\alpha) = a \leq b = f(\beta)$.

$$\boxed{\boldsymbol{A = \int_a^b y\ dx = \int_\alpha^\beta g(t) f'(t)\ dt}}$$

Note: 如果是逆向, $f(\beta) = a$ and $f(\alpha) = b$, 則 $A = \displaystyle\int_\beta^\alpha g(t) f'(t)\ dt$.

如果有折疊算出來是順向減逆向。



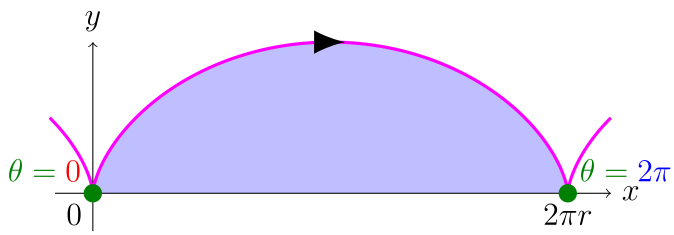
Example 0.3 *Find the area under one arch [artʃ] 拱 of the cycloid:*

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

One arch of the cycloid: $0 \leq \theta \leq 2\pi$, $dx = r(1 - \cos\theta)\ d\theta$, $0 \leq x \leq 2\pi r$. *(不一定容易算出變數變換的對應範圍。)*

$$\begin{aligned}
A &= \int_0^{2\pi r} y\ dx = \int_0^{2\pi} r(1-\cos\theta) \cdot r(1-\cos\theta)\ d\theta \\
&= r^2 \int_0^{2\pi} (1-\cos\theta)^2\ d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\ d\theta \\
&= r^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta \quad (\because \cos^2 x = \frac{1+\cos 2x}{2}) \\
&= r^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi r^2. \qquad \textit{(1634 Roberval)}
\end{aligned}$$

■



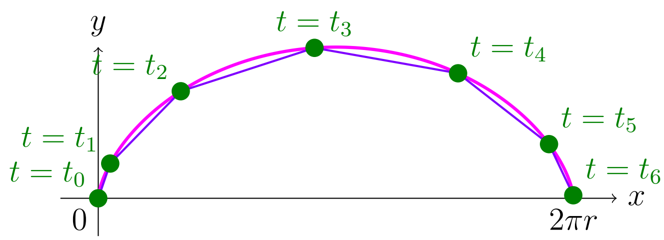
0.3 Arc length

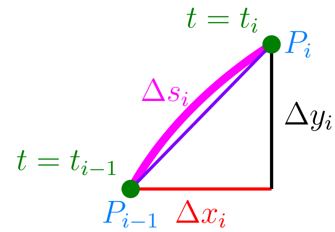
If $\frac{dx}{dt} > 0$, then C is traversed once from left to right (由左走到右沒回頭), and

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\frac{dx}{dt}\,dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt,$$

where $f(\alpha) = a$ and $f(\beta) = b$.

Question: When $\frac{dx}{dt} < 0$? 還是可以得到一樣的公式。





回到原點: 把 $[\alpha, \beta]$ 分成 n 段 $[t_{i-1}, t_i]$, $\Delta t = t_i - t_{i-1} = \frac{\beta-\alpha}{n}$, $t_i = \alpha + i\Delta t$.

$$L = \lim_{n\to\infty}\sum_{i=1}^{n} |P_{i-1}P_i|, \quad P_i(f(t_i), g(t_i)).$$

Let $\Delta x_i = f(t_i) - f(t_{i-1})$, $\Delta y_i = g(t_i) - g(t_{i-1})$.
By Mean Value Theorem, $\exists\ t_i^*, t_i^{**} \in (t_{i-1}, t_i)$ such that

$$\begin{aligned}\Delta x_i &= f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1}) = f'(t_i^*)\Delta t, \text{ and}\\ \Delta y_i &= g(t_i) - g(t_{i-1}) = g'(t_i^{**})(t_i - t_{i-1}) = g'(t_i^{**})\Delta t.\end{aligned}$$

Then

$$\begin{aligned}|P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}\\ &= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\Delta t.\end{aligned}$$

When Δt small, $t_i^* \approx t_i^{**}$. ($\because f'$ and g' continuous, $\begin{array}{l} f'(t_i^*) \approx f'(t_i^{**}) \\ g'(t_i^{**}) \approx g'(t_i^*) \end{array}$.)

$$\begin{aligned}\therefore L &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{[f'(t_i^*)]^2 + [g'(t_i^*)]^2}\Delta t\\ &= \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt.\end{aligned}$$

Theorem 1 *If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous (f and g are smooth) on $[\alpha, \beta]$ and C is* ***traversed exactly once*** 只走一次 *as t increases from α to β, then the length of C is and*

$$\boxed{\boldsymbol{L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \left(= \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt \right)}}$$

Skill: 記成 $ds = \sqrt{(dx)^2 + (dy)^2}$, 則 $L = \displaystyle\int ds$ 與 §8.1 的公式一致。

Example 0.4 *Find the arc length of $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.*

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \\ &= \int_0^{2\pi} dt = 2\pi. \end{aligned}$$

■

Example 0.5 *Find the arc length of $x = \sin 2t$, $y = \cos 2t$, $0 \le t \le 2\pi$.*

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(2\cos 2t)^2 + (-2\sin 2t)^2}\, dt \\ &= \int_0^{2\pi} 2\, dt = 4\pi. \quad (???) \end{aligned}$$

But! 因爲轉兩圈, 答案是 $4\pi \div 2 = 2\pi$.

(或是考慮 $0 \le t \le \pi$, $L = \displaystyle\int_0^{\boxed{\pi}} \cdots\, dt = 2\pi$). ■

Attention: $L = \displaystyle\int ds$ 會是實際走的長度, 求弧長要找走一次的範圍, 或試算出來再除以走的次數。

Example 0.6 *Find the arc length of one arch of the cycloid:*

$$x = r(\theta - \sin\theta), \quad y = r(1-\cos\theta).$$

One arch of the cycloid: $0 \le \theta \le 2\pi$, $\dfrac{dx}{d\theta} = r(1-\cos\theta)$, $\dfrac{dy}{d\theta} = r\sin\theta$.

$$\begin{aligned}
L &= \int ds = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{[r(1-\cos\theta)]^2 + (r\sin\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{r^2(\sin^2\theta + \cos^2\theta - 2\cos\theta + 1)}\, d\theta \\
&= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta \qquad (\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}) \\
&= r\int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \\
&= 2r\int_0^{2\pi} \left|\sin\frac{\theta}{2}\right|\, d\theta \\
&= 2r\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta \qquad (\because \sin\frac{\theta}{2} \ge 0 \textit{ for } 0 \le \theta \le 2\pi) \\
&= 2r\Big[-2\cos\frac{\theta}{2}\Big]_0^{2\pi} = 8r. \qquad \textit{(1658 Wren)}
\end{aligned}$$

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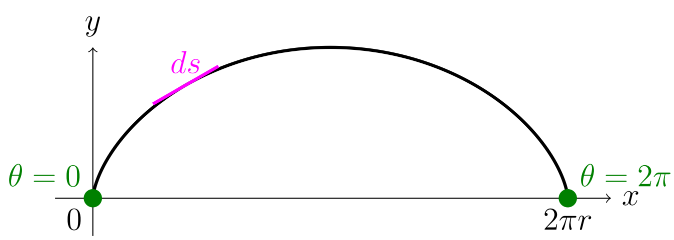

Skill: $\sqrt{1-\cos\theta} = \sqrt{2\sin^2\frac{\theta}{2}} = \sqrt{2}\left|\sin\frac{\theta}{2}\right|$, 去掉絕對值時要注意正負。

0.4 Surface area

f' and g' are continuous, $g(t) \geq 0$, rotating about x-axis.

$$\boxed{\begin{aligned} \boldsymbol{S} &= \int_{\alpha}^{\beta} \boldsymbol{2\pi y} \sqrt{\left(\frac{\boldsymbol{dx}}{\boldsymbol{dt}}\right)^2 + \left(\frac{\boldsymbol{dy}}{\boldsymbol{dt}}\right)^2}\, \boldsymbol{dt} \\ \Big(&= \int_{\alpha}^{\beta} \boldsymbol{2\pi g(t)} \sqrt{[\boldsymbol{f'(t)}]^2 + [\boldsymbol{g'(t)}]^2}\, \boldsymbol{dt} \Big) \end{aligned}}$$

Note: $ds = \sqrt{(dx)^2 + (dy)^2}$, $S = \int 2\pi y\ ds$ 與 §8.2 的公式一致。

Note: 如果是繞 y-axis 就是 $S = \int 2\pi x\ ds$.

Example 0.7 *Show that the surface area of a sphere of radius r is $4\pi r^2$.*

Rotating the semicircle(半圓) about the x-axis:

$$x = r\cos t, \quad y = r\sin t, \quad 0 \leq t \leq \pi.$$

$$\begin{aligned} S &= \int 2\pi y\ ds = \int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \\ &= \int_0^{\pi} 2\pi r\sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\ dt \\ &= 2\pi r^2 \int_0^{\pi} \sin t\ dt = 2\pi r^2 \Big[-\cos t\Big]_0^{\pi} = 4\pi r^2. \end{aligned}$$

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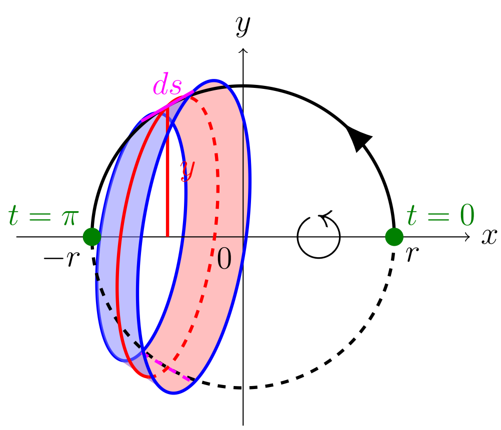


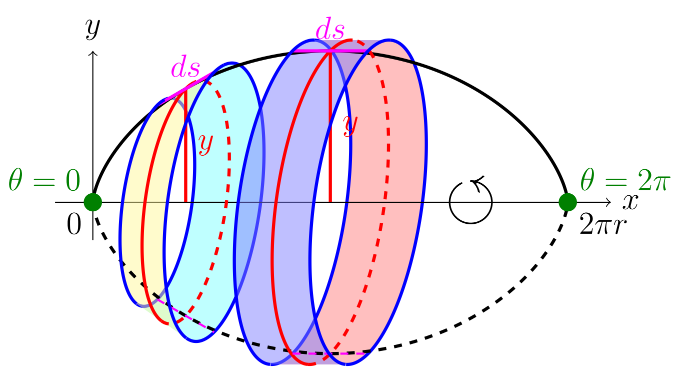
Example 0.8 (Extra) *Find the surface area obtained by rotating about the x-axis one arch of the cycloid:*

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

One arch of the cycloid: $0 \le \theta \le 2\pi$, $\dfrac{dx}{d\theta} = r(1-\cos\theta)$, $\dfrac{dy}{d\theta} = r\sin\theta$.

$$\begin{aligned}
S &= \int 2\pi y \; ds = \int_0^{2\pi} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \; d\theta \\
&= \int_0^{2\pi} 2\pi r(1-\cos\theta)\sqrt{[r(1-\cos\theta)]^2 + (r\sin\theta)^2} \; d\theta \\
&= 2\pi r^2 \int_0^{2\pi} (1-\cos\theta)\sqrt{2(1-\cos\theta)} \; d\theta \qquad (\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}) \\
&= 8\pi r^2 \int_0^{2\pi} \sin^3\frac{\theta}{2} \; d\theta \\
&= 8\pi r^2 \int_0^{2\pi} 2(\cos^2\frac{\theta}{2} - 1)\cdot\frac{1}{2}(-\sin\frac{\theta}{2}) \; d\theta \qquad (\textit{Let } u = \cos\frac{\theta}{2}) \\
&= 8\pi r^2 \int_1^{-1} 2(u^2-1) \; du \\
&= 16\pi r^2 \Big[\frac{u^3}{3} - u\Big]_1^{-1} \Big(= 16\pi r^2 \Big[\frac{1}{3}\cos^3\frac{\theta}{2} - \cos\frac{\theta}{2}\Big]_0^{2\pi}\Big) \\
&= \frac{64}{3}\pi r^2.
\end{aligned}$$

■



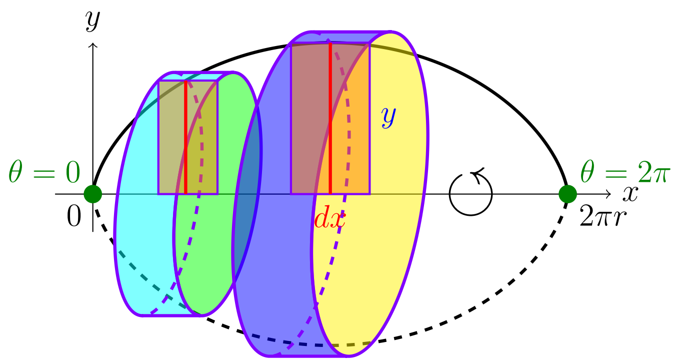
Example 0.9 (Extra) *Find the volume of the solid obtained by rotating about the x-axis the region under one arch of the cycloid:*

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

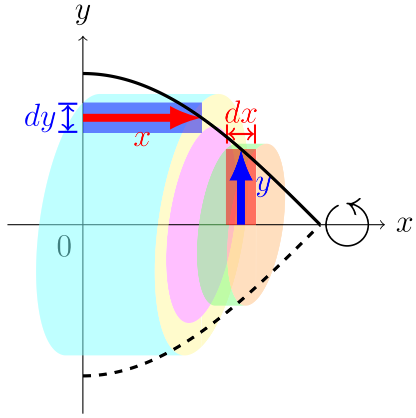
One arch of the cycloid: $0 \le \theta \le 2\pi$, $dx = r(1 - \cos\theta)\ d\theta$.

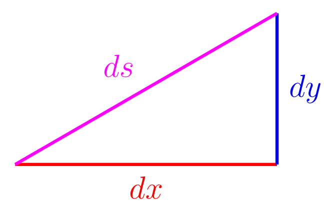
$$\begin{aligned}
V &= \int \pi y^2\ dx \\
&= \int_0^{2\pi} \pi[r(1-\cos\theta)]^2 \cdot r(1-\cos\theta)\ d\theta \\
&= \pi r^3 \int_0^{2\pi} (1-\cos\theta)^3\ d\theta \\
&= \pi r^3 \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)\ d\theta \\
&= \pi r^3 \int_0^{2\pi} (1 + \frac{3}{2}(1 + \underbrace{\cos 2\theta}_{u=2\theta}) - \underbrace{(3 + 1 - \sin^2\theta)\cos\theta}_{v=\sin\theta})\ d\theta \\
\Big(&= \pi r^3 \Big[\int_0^{2\pi} \frac{5}{2}\ d\theta + \int_0^{2\pi} \frac{3}{4}\cos 2\theta\ d(2\theta) + \int_0^{2\pi} (\sin^2\theta - 4)\ d(\sin\theta)\Big]\Big) \\
&= \pi r^3 \Big[\frac{5}{2}\theta + \frac{3}{4}\sin 2\theta + \frac{1}{3}\sin^3\theta - 4\sin\theta\Big]_0^{2\pi} \\
&= 5\pi^2 r^3.
\end{aligned}$$

■



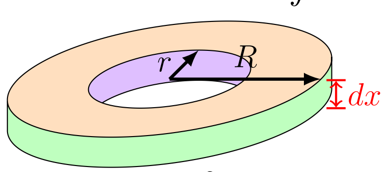
♦ List of Formulas: Area, Volume of Revolution, Arc Length, and Surface Area of Revolution.



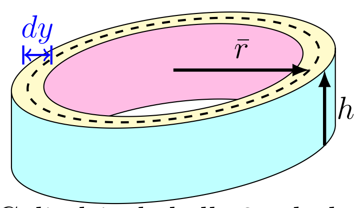


$$ds = \sqrt{(dx)^2 + (dy)^2}$$

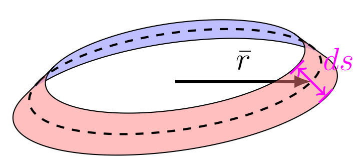
	Cartesian equation $y = f(x)$, $dy = f'(x)\ dx$ $x = g(y)$, $dx = g'(y)\ dy$	parametric equations $\begin{cases} x = f(t),\ dx = f'(t)\ dt \\ y = g(t),\ dy = g'(t)\ dt \end{cases}$
$A = \int \boldsymbol{y\ dx}$	$= \int f(x)\ dx$	$= \int g(t) \cdot f'(t)\ dt$
$= \int \boldsymbol{x\ dy}$	$= \int g(y)\ dy$	$= \int f(t) \cdot g'(t)\ dt$
$V = \int \boldsymbol{\pi R^2\ dx}$	$= \int \pi[f(x)]^2\ dx$	$= \int \pi[g(t)]^2 \cdot f'(t)\ dt$
$= \int \boldsymbol{2\pi\bar{r}h\ dy}$	$= \int 2\pi y g(y)\ dy$	$= \int 2\pi g(t) f(t) \cdot g'(t)\ dt$
$L = \int \boldsymbol{ds}$	$= \int \sqrt{1 + [f'(x)]^2}\ dx$	$= \int \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt$
	$= \int \sqrt{1 + [g'(y)]^2}\ dy$	
$S = \int \boldsymbol{2\pi\bar{r}\ ds}$	$= \int 2\pi y\sqrt{1 + [g'(y)]^2}\ dy$	$= \int 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt$
	$= \int 2\pi f(x)\sqrt{1 + [f'(x)]^2}\ dx$	



Disk: $\pi R^2\ dx$
Washer: $\pi(R^2 - r^2)\ dx$



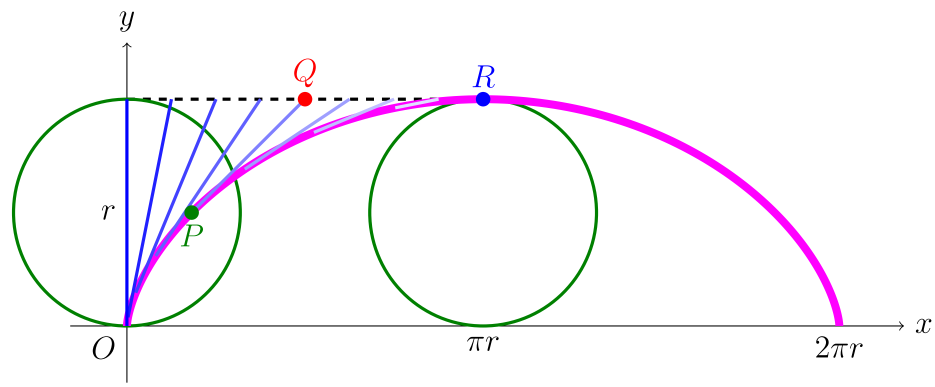
Cylindrical shell: $2\pi\bar{r}h\ dy$



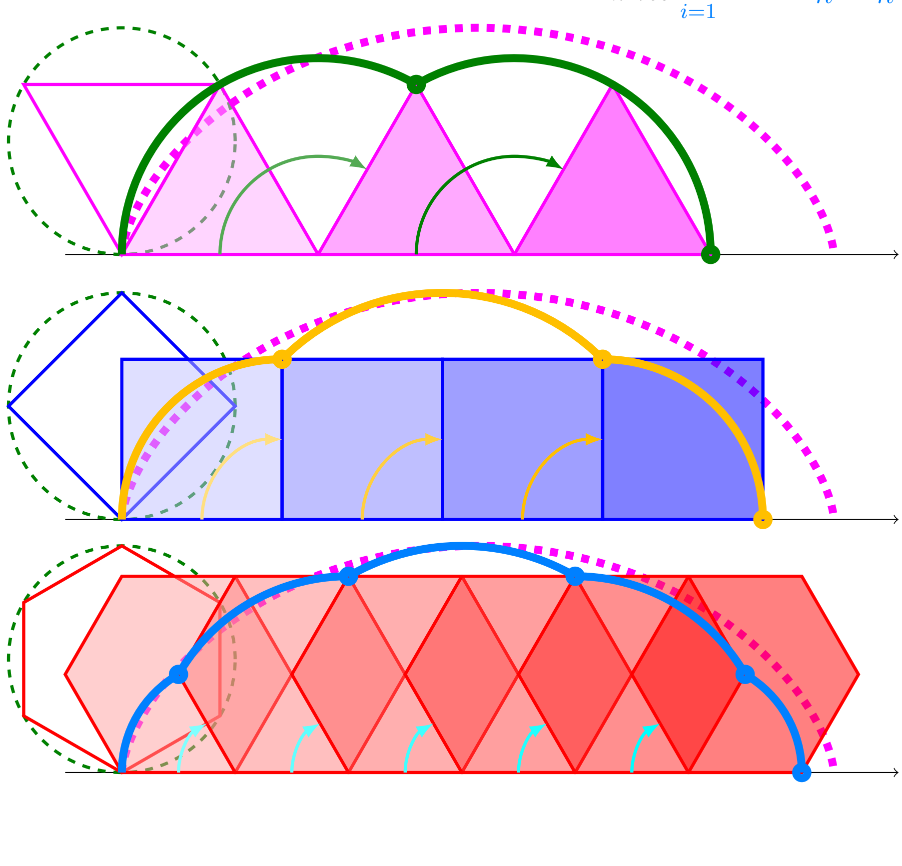
Band: $2\pi\bar{r}\ ds$

⧫ Additional: Geometric proof of area and arc length of one arch of cycloid

1658, Wren's proof: $2\overline{PQ} = \overset{\frown}{PR}$, when $P \to O$, $L = 2\,\overset{\frown}{OR} = 4 \times 2r = 8r$.



1638, Descartes: Rotate a polygon and $L = \lim_{n\to\infty} \sum_{i=1}^{n-1} 2r \sin \frac{i\pi}{n} \cdot \frac{2\pi}{n} = 8r$.



1634, Roberval's proof: