14.2 Limits and continuity

1. limit
2. continuous
3. functions of more than two variables

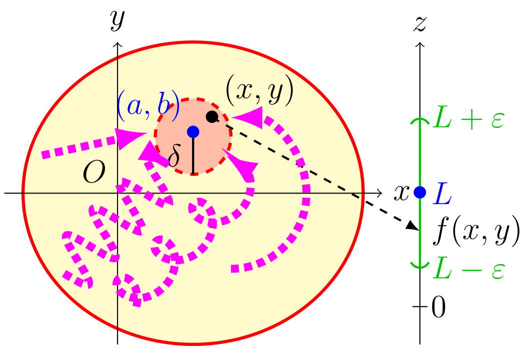
0.1 Limit of functions of two variables

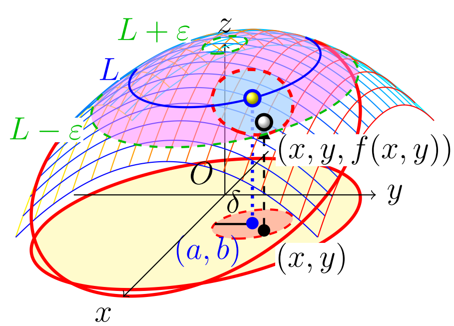
Define: Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). (f 在 (a, b) 附近都有定義, 在 (a, b) 可以無定義。)
Then the ***limit*** of $f(x, y)$ as (x, y) approaches (a, b) is L,

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = L}, \qquad \boxed{\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L},$$

$$\text{or } \boxed{f(x,y) \to L \text{ as } (x,y) \to (a,b)},$$

if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$,
$\ni (x, y) \in D$, $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon$.





Note: 1. 多變數函數要各方向極限都要一樣。(極限存在 — 殊途同歸。)
Recall: $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$.
2. 不見得以 (a, b) 爲圓心半徑 δ 的圓盤裡的點都有定義, 所以要 $(x, y) \in D$.
3. $\lim_{(x,y)\to(a,b)} f(x, y) \neq \lim_{x\to a} \lim_{y\to b} f(x, y)$ or $\lim_{y\to b} \lim_{x\to a} f(x, y)$.
分開是代表有先後順序 (近的先算), 兩者是不一樣的意思。

★ 差異之一: 兩個方向 (直線) v.s. 任何方向 (曲線)。

Theorem 1 **(判斷極限不存在)** *If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$* ***does not exist****. (sketch of proof: $\varepsilon = |L_1 - L_2|/2$.)*
(有兩個路徑的極限不同, 極限就不存在 — 殊途不同歸。)

Skill: 計算沿著 $y = f(x)$ or $x = g(y)$ 路徑的極限: 代入變單變數再求極限。

Example 0.1 *Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.*

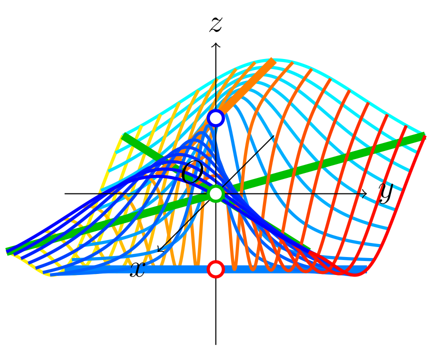
along x-axis ($y = 0$), then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} \overset{y=0}{=} \lim\limits_{x\to 0} \dfrac{x^2 - 0^2}{x^2 + 0^2} = 1.$

$\dfrac{x^2 - y^2}{x^2 + y^2} \to 1$ *as* $(x,y) \to (0,0)$ *along* $y = 0$.

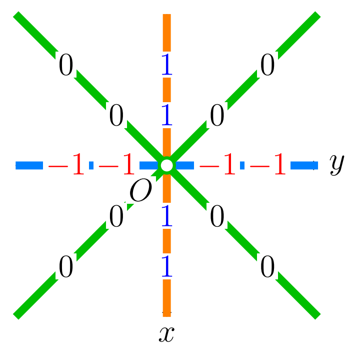
along y-axis ($x = 0$), then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} \overset{x=0}{=} \lim\limits_{y\to 0} \dfrac{0^2 - y^2}{0^2 + y^2} = -1.$

$\dfrac{x^2 - y^2}{x^2 + y^2} \to -1$ *as* $(x,y) \to (0,0)$ *along* $x = 0$.

極限不同, 所以不存在。 ■



$$z = \frac{x^2 - y^2}{x^2 + y^2}$$

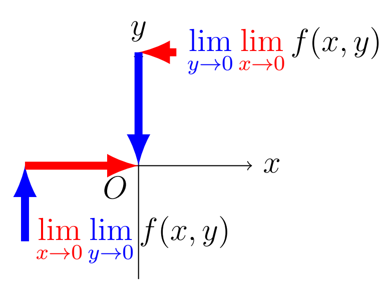


Note:

$$\lim_{x\to 0}\lim_{y\to 0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - 0}{x^2 + 0} = \lim_{x\to 0} 1 = 1;$$

$$\lim_{y\to 0}\lim_{x\to 0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{0 - y^2}{0 + y^2} = \lim_{y\to 0} -1 = -1.$$

與 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ 不一樣。



Example 0.2 *Does* $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ *exist?*

along x*-axis* $(y=0)$*, then* $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = 0$.

$\frac{xy}{x^2+y^2} \to 0$ *as* $(x,y)\to(0,0)$ *along* $y=0$.

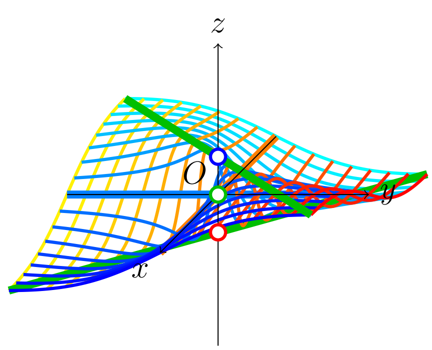
along y*-axis* $(x=0)$*, then* $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = 0$.

$\frac{xy}{x^2+y^2} \to 0$ *as* $(x,y)\to(0,0)$ *along* $x=0$. *(*極限一樣沒用, 再找一條。*)*

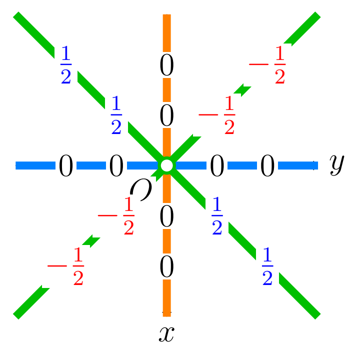
along $x=y$*, then* $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$.

$\frac{xy}{x^2+y^2} \to \frac{1}{2}$ *as* $(x,y)\to(0,0)$ *along* $x=y$.

極限不同, 所以不存在。 ■



$z = \frac{xy}{x^2+y^2}$



Note:

$\lim_{x\to 0}\lim_{y\to 0} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0} = \lim_{x\to 0} 0 = 0;$

$\lim_{y\to 0}\lim_{x\to 0} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0\cdot y}{0+y^2} = \lim_{y\to 0} 0 = 0.$

但與 $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ 還是不一樣。

Example 0.3 *(只看直線還不夠) Does* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ *exist?*

along $y = mx$, *then* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} = 0.$

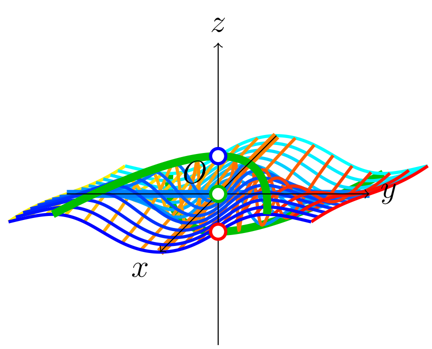
$\dfrac{xy^2}{x^2+y^4} \to 0$ *as* $(x,y)\to(0,0)$ *along* $y = mx$.

*(*沿過原點的直線 *(*含 y-軸*)* 極限都是 0。*)*

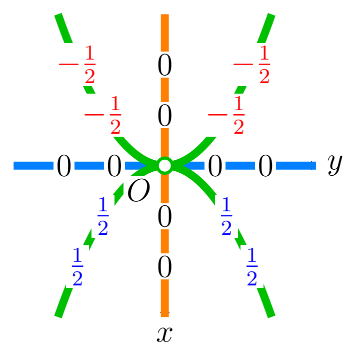
along $x = y^2$, *then* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}.$

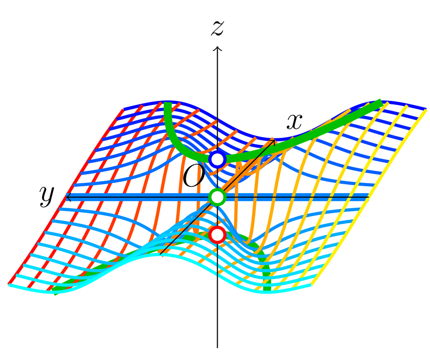
$\dfrac{xy^2}{x^2+y^4} \to \dfrac{1}{2}$ *as* $(x,y)\to(0,0)$ *along* $x = y^2$.

極限不同, 所以不存在。 ■

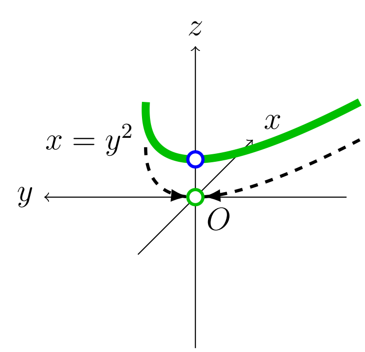


$z = \dfrac{xy^2}{x^2+y^4}$





$z = \dfrac{xy^2}{x^2+y^4}$



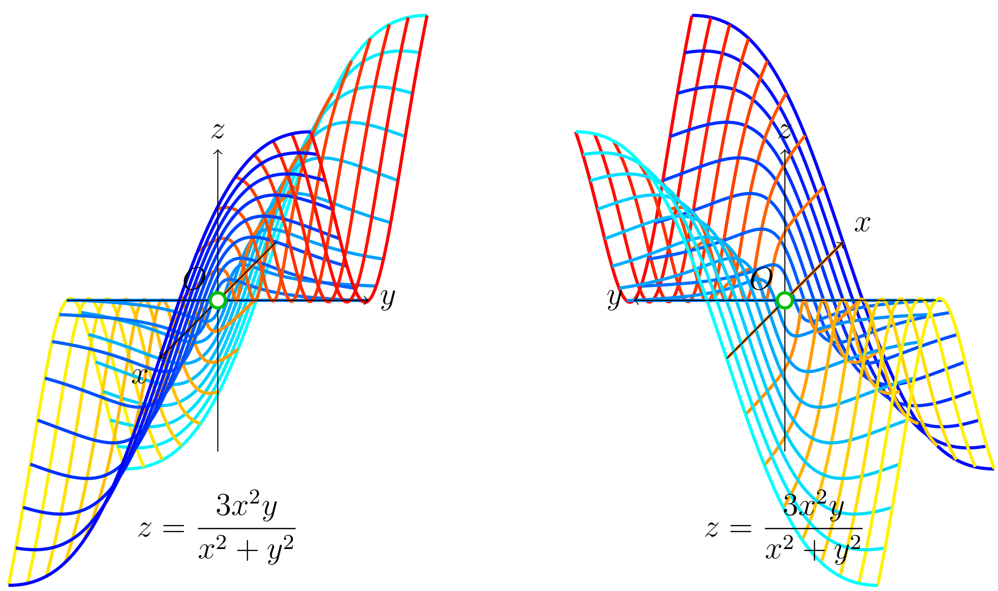
Example 0.4 *(有極限怎麼找) Find* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ *if it exists.*

Try: $\dfrac{3x^2y}{x^2+y^2} \to 0$ *as* $(x,y)\to(0,0)$ *along* $x=0,\ y=0,\ x=y^2,\ y=x^2$.
So if the limit exists, it should be 0. *(殊途同歸, 找條好算的路徑算極限。)*

Step 1. Guess δ*: find* $\delta \ni 0 < \sqrt{x^2+y^2} < \delta \implies \left|\dfrac{3x^2y}{x^2+y^2}(-0)\right| < \varepsilon$.
Since $x^2 \le x^2+y^2$ *and* $y^2 \le x^2+y^2$,
$\left|\dfrac{3x^2y}{x^2+y^2}\right| \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2} < \varepsilon,\ \sqrt{x^2+y^2} < \dfrac{\varepsilon}{3} = \delta.$

Step 2. Prove it works:
$\forall\ \varepsilon > 0$, *choose* $\delta = \dfrac{\varepsilon}{3}$, *if* $0 < \sqrt{x^2+y^2} < \delta$, *then*
$\left|\dfrac{3x^2y}{x^2+y^2}\right| \le 3\sqrt{x^2+y^2} < 3\delta = 3\dfrac{\varepsilon}{3} = \varepsilon.$

By the definition of limit, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$ ■



Note: 求極限的方法:

1. $\lim\limits_{(x,y)\to(a,b)} x = a$, $\lim\limits_{(x,y)\to(a,b)} y = b$, $\lim\limits_{(x,y)\to(a,b)} c = c$ (常數).

2. **Limit Laws** 極限律 (各極限要存在):
加減乘除常數倍 $+, -, \times, \div$(分母$\neq 0$), c.

3. The **Squeeze Theorem** 夾擠定理: (夾得好&夾得緊)
$g \le f \le h$ (near (a,b)), $\lim\limits_{(x,y)\to(a,b)} g = \lim\limits_{(x,y)\to(a,b)} h = L \implies \lim\limits_{(x,y)\to(a,b)} f = L$.

Proof. [Another proof of $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0$]:

$\because 0 \le \dfrac{x^2}{x^2+y^2} \le 1$ for $(x,y) \neq (0,0)$, $\implies -|y| \le \dfrac{x^2}{x^2+y^2} \cdot y \le |y|$,
and $\lim\limits_{(x,y)\to(0,0)} -|y| = \lim\limits_{y\to 0} -|y| = 0 = \lim\limits_{y\to 0} |y| = \lim\limits_{(x,y)\to(0,0)} |y|$.

$\therefore$ By the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2} = 0$, and by the limit laws,
$\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 3 \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2} = 3 \cdot 0 = 0$. ■

Skill: (不保證100%正確) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^? + y^? + x^?y^?}{x^? + y^?}$:

如果分子有項次數 $\le$ 分母次數 $(\frac{\text{小等}}{\text{大}})$, 很可能沒極限 $\implies$ 找路徑;

如果分子每項次數 $>$ 分母次數 $(\frac{\text{大}}{\text{小}})$, 很可能有極限 $\implies$ 求極限。

Ex: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ & $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ 不存在 (上$2 = 2$下);
$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ 不存在 (上$3 < 4$下); $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$ 存在 (上$3 > 2$下)。

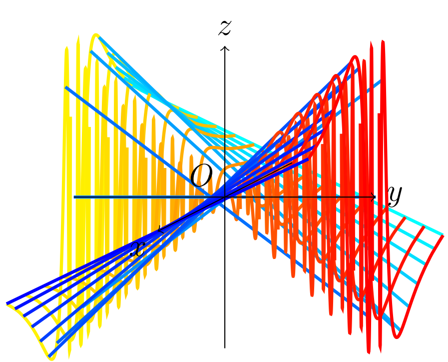
Skill: $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ with $f(a,b)$ undefined:
不存在: 找路徑: $x = 0$, $y = 0$, $x = \pm y$, $y = g(x)$ 或 $x = h(y)$, 讓 $f(x, g(x))$ 或 $f(h(y), y)$ 可以約分, 得到不同的極限。
存在: 證明, $\forall\, \varepsilon > 0$, choose $\delta > 0$, $\ni$ ..., 或 the Squeeze Theorem.
(極限=? 因爲任何路徑都一樣, 沿著好算的路徑如: $x = 0$ or $y = 0$.)

Note: 選擇的路徑要通過求極限的點。

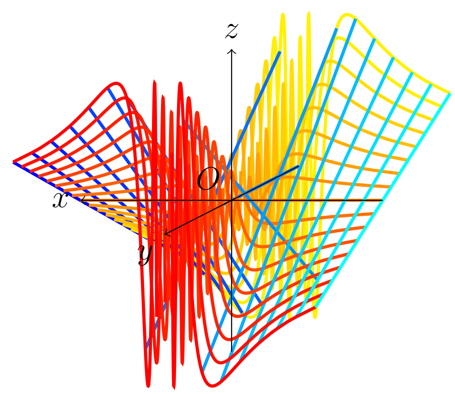
Example 0.5 (extended) *Find* $\lim_{(x,y)\to(0,0)} f(x,y)$ *where*

$$f(x,y) = \begin{cases} y\sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

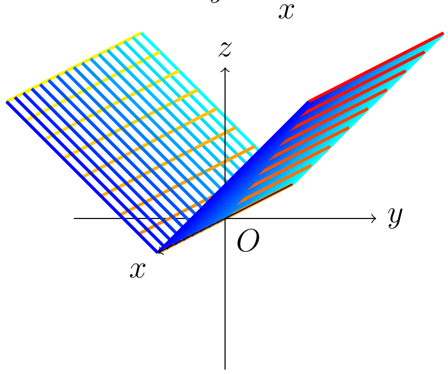
Consider $g(x,y) = -|y|$ *and* $h(x,y) = |y|$.
$\because -1 \le \sin\dfrac{1}{x} \le 1$, $g(x,y)(= -|y|) \le y\sin\dfrac{1}{x} \le (|y| =) h(x,y)$,
and $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{y\to 0} -|y| = 0$, $\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{y\to 0} |y| = 0$,
$\therefore$ *By the Squeeze Theorem,* $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. ■



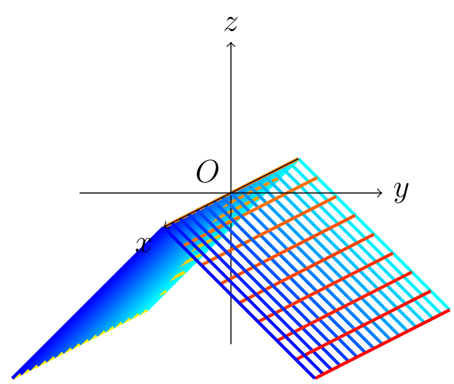
$z = y\sin\dfrac{1}{x}$



$z = y\sin\dfrac{1}{x}$



$z = |y|$



$z = -|y|$

0.2 Continuity of functions of two variables

Define: A function f of two variables is ***continuous*** 連續 at (a,b) if

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}.(\text{極限等於函數值})$$

A ***polynomial function of two variables*** 雙變數多項式: $p(x,y) = \sum_{\substack{m\geq 0\\ n\geq 0}} c_{m,n}x^n y^m$ is continuous everywhere ($\mathbb{R}^2$).

A ***rational function*** 有理函數 is a ratio of polynomials, and is continuous on its domain (分母$\neq 0$).

A continuous function of a continuous function (***composed function*** 合成函數) is a continuous function. (連續函數的連續函數也是連續函數。) $h(x,y) = (f\circ g)(x,y) = f(g(x,y))$ is continuous if $f(x)$ and $g(x,y)$ are continuous.

Example 0.6 *Evaluate* $\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y)$.

$$\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 2^3 - 1^3 2^2 + 3\cdot 1 + 2\cdot 2 = 11. \qquad \blacksquare$$

Example 0.7 *Where is* $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ *continuous?*

f is not defined at $(0,0)$ a rational function, and continuous on its domain $D = \mathbb{R}^2 \setminus \{(0,0)\}$. $\blacksquare$

Example 0.8 *Let* $g(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

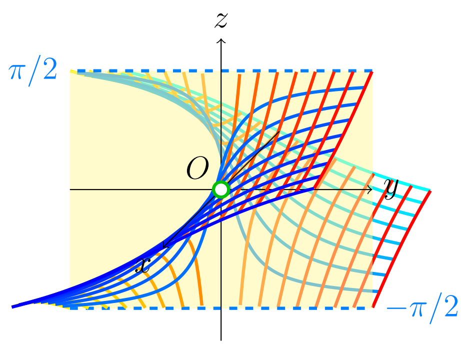
g is defined at $(0,0)$, but $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist. $\blacksquare$

Example 0.9 *Let* $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

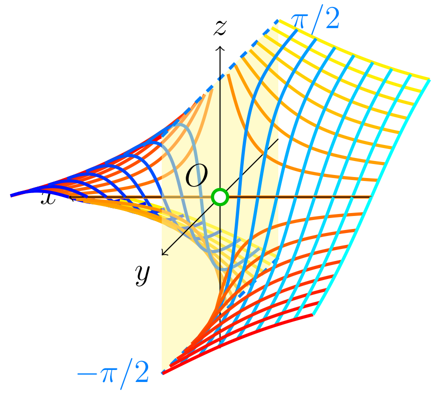
f is continuous for $(x,y)\neq(0,0)$, and $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, so it continuous on $\mathbb{R}^2$. $\blacksquare$

Example 0.10 *Where is* $\arctan(y/x)$ *continuous?*

y/x *is continuous except* $x = 0$, *and* $\arctan t$ *is continuous everywhere, so* $\arctan(y/x)$ *is continuous except* $x = 0$. ■



$$z = \arctan\frac{y}{x}$$



$$z = \arctan\frac{y}{x}$$

0.3 Functions of more than two variables

Define: $f : D \subseteq \mathbb{R}^3 \to R \subseteq \mathbb{R}$,

$$\boxed{\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L} \text{ or } \boxed{f(x,y,z) \to L \text{ as } (x,y,z) \to (a,b,c)},$$

if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, $\ni (x,y,z) \in D$, $0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta \implies |f(x,y,z) - L| < \varepsilon$.

And f is ***continuous*** at (a,b,c) if

$$\boxed{\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)}.$$

Define: If f is defined on $D \subseteq \mathbb{R}^n$, then the ***limit*** of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is L,

$$\boxed{\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L} \text{ or } \boxed{f(\mathbf{x}) \to L \text{ as } \mathbf{x} \to \mathbf{a}},$$

if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, $\ni \mathbf{x} \in D$, $0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |f(\mathbf{x}) - L| < \varepsilon$.

And f is ***continuous*** at $\mathbf{a}$ if

$$\boxed{\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})}.$$