11.5 Alternating series

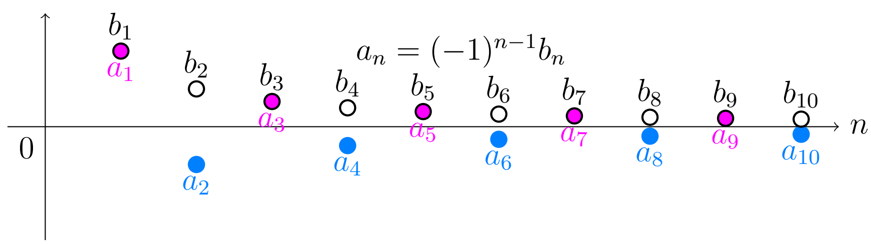
1. alternating series test $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, $b_n \searrow 0$.

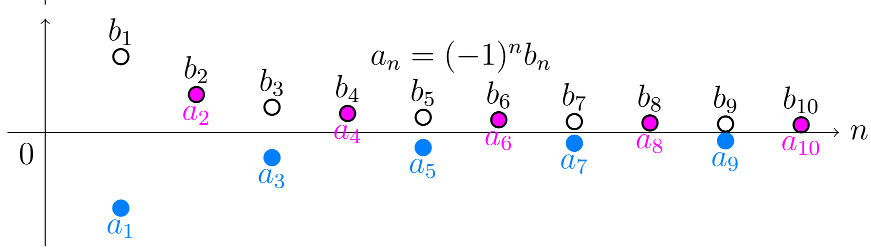
2. estimate sums $|R_n| \le b_{n+1}$

$$
\begin{array}{l}
1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\
1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots \\
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \\
1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \cdots
\end{array}
$$

Define: An ***alternating*** [ˋɔltɚˏnetɪŋ] series 交錯級數 is a series whose terms are alternately positive and negative. (正負交錯)

$\sum a_n$ with $a_n = (-1)^{n-1}b_n$ or $a_n = (-1)^n b_n$, where $b_n = |a_n|$.





$$
\begin{array}{rlllllll}
\sum a_n & = & a_1 & +a_2 & +a_3 & +a_4 & +\cdots+ & a_n & +\cdots \\
 & = & b_1 & -b_2 & +b_3 & -b_4 & +\cdots+ & (-1)^{n-1}b_n & +\cdots \\
 & \text{or} & & & & & & & \\
 & = & -b_1 & +b_2 & -b_3 & +b_4 & +\cdots+ & (-1)^n b_n & +\cdots
\end{array}
$$

Note: $(-1)^{\text{奇數}} = -1$, $(-1)^{\text{偶數}} = 1$.

0.1 Alternating series test

Theorem 1 (Alternating Series Test, Leibniz Test)
If $b_n \geq b_{n+1} > 0$ for n and $\lim\limits_{n\to\infty} b_n = 0$, (遞減到零, 方便記為: $b_n \searrow 0$)
then the alternating series $\sum(-1)^{n-1}b_n$ (also $\sum(-1)^n b_n$) is convergent.

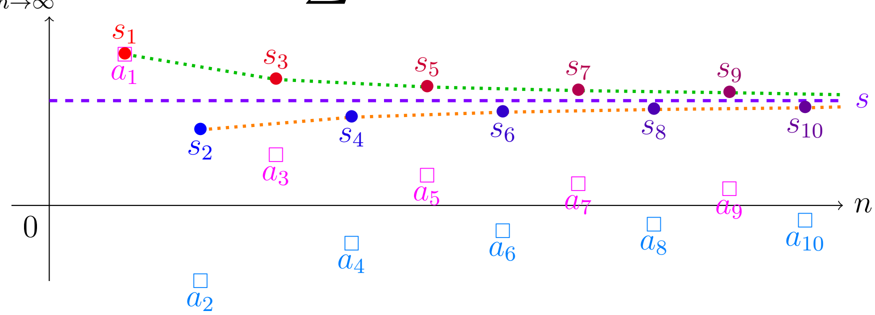
Proof. (證明 $\sum(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \ldots$ 就好)

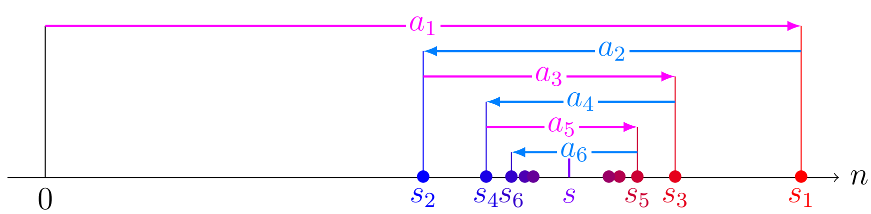
考慮偶數項的部分和 $\{s_{2n}\}$: $\because b_n \geq b_{n+1}$,

$s_2 \ = b_1 - b_2 \geq 0,$
$s_4 \ = b_1 - b_2 + b_3 - b_4 = s_2 + b_3 - b_4 \geq s_2,$
$s_{2n} = b_1 - b_2 + b_3 - b_4 + \cdots + b_{2n-1} - b_{2n} = s_{2n-2} + b_{2n-1} - b_{2n} \geq s_{2n-2},$
$\Longrightarrow \ \{s_{2n}\}$ is non-decreasing (increasing);
$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \ldots - (b_{2n-2} - b_{2n-1}) - b_{2n} < b_1,$
$\Longrightarrow \ \{s_{2n}\}$ is bounded (by b_1).
By the Monotone Convergence Theorem, $\{s_{2n}\}$ converges. Say $\lim\limits_{n\to\infty} s_{2n} = s$.

考慮奇數項的部分和 $\{s_{2n+1}\}$: $\because \lim\limits_{n\to\infty} b_n = 0$,

$\Longrightarrow \ \lim\limits_{n\to\infty} s_{2n+1} = \lim\limits_{n\to\infty}(s_{2n} + b_{2n+1}) = \lim\limits_{n\to\infty} s_{2n} + \lim\limits_{n\to\infty} b_{2n+1} = s + 0 = s.$
$\therefore \lim\limits_{n\to\infty} s_n = s$ and hence $\sum(-1)^{n-1}b_n$ converges (to s). ∎





Note: $b_n \searrow 0 \implies \sum(-1)^n b_n$ converges. 三個條件: 交錯, 遞減, 到零, 才會保證收斂。(缺少任何一個都有反例, 試著造造看。)

Example 0.1 *The **alternating harmonic series** (交錯調和級數)*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum \frac{(-1)^{n-1}}{n}$$

$b_n = \dfrac{1}{n} > \dfrac{1}{n+1} = b_{n+1}$, *and* $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{1}{n} = 0$. *(* $b_n \searrow 0$ *)*

So $\sum \dfrac{(-1)^{n-1}}{n}$ *converges by the Alternating Series Test.* ■

> **Recall:** harmonic series $\sum \dfrac{1}{n}$ diverges.
>
> **Fact:** $\sum \dfrac{(-1)^{n-1}}{n} = \ln 2$. (Exercise 11.5.36)

♦ **Proof.** (a) $s_{2n} = \sum_{i=1}^{2n} \dfrac{1}{i} - 2\sum_{i=1}^{n} \dfrac{1}{2i} = h_{2n} - h_n$,

(b) (Exercise 11.3.44) $t_n = h_n - \ln n \searrow \gamma > 0$ (Euler's constant) by MCT.

(c) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} (h_{2n} - h_n) = \lim_{n\to\infty} [(t_{2n} + \ln 2n) - (t_n + \ln n)]$

$= \gamma - \gamma + \ln 2 = \ln 2$. (§11.9 將介紹用冪級數來得到 $\sum \dfrac{(-1)^{n-1}}{n} = \ln 2$.) ■

Example 0.2 *The series* $\sum \dfrac{(-1)^n 3n}{4n-1}$ *is alternating.*

> $\lim_{n\to\infty} \dfrac{3n}{4n-1} = \dfrac{3}{4} \neq 0$. *(* b_n 沒有 $\searrow 0$, 不能用 *Alternating Series Test!)*

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{(-1)^n 3n}{4n-1}$ *does not exist.*

So $\sum \dfrac{(-1)^n 3n}{4n-1}$ *diverges by the Test for Divergence.* ■

Example 0.3 *Test the series* $\sum \dfrac{(-1)^{n+1} n^2}{n^3+1}$ *for convergence or divergence.*

> $b_n = \dfrac{n^2}{n^3+1}$ *(直接看遞減不容易)*

Let $f(x) = \dfrac{x^2}{x^3+1}$, *then* $f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2} < 0$ *when* $x > \sqrt[3]{2} \approx 1.26$.

So $b_n = f(n) > f(n+1) = b_{n+1}$ *for* $n \geq 2$. *(有限項不影響)*

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{n^2}{n^3+1} = \lim_{n\to\infty} \dfrac{1/n}{1+1/n^3} = \dfrac{0}{1+0} = 0$.

So $\sum (-1)^{n+1} \dfrac{n^2}{n^3+1}$ *converges by the Alternating Series Test.* ■

0.2 Estimate sums

Theorem 2 (Alternating Series Estimation Theorem)

$$\boxed{|R_n| \le b_{n+1}} \quad \textit{(第一個被忽略的項。)}$$

Proof. For $b_1 - b_2 + b_3 - b_4 + \cdots = s$, $s_{2n+1} \searrow s$, $s_{2n} \nearrow s$.
$s_n = b_1 - b_2 + \cdots + (-1)^{n-1} b_n$, $R_n = (-1)^n b_{n+1} + (-1)^{n+1} b_{n+2} + \cdots$.

$$|R_n| = |s - s_n| = \left\{ \begin{array}{ll} s - s_n < s_{n+1} - s_n & \text{if } n \text{ even} \\ s_n - s < s_n - s_{n+1} & \text{if } n \text{ odd} \end{array} \right\} = |s_{n+1} - s_n| = b_{n+1}. \quad \blacksquare$$

Attention: 這個估計法只有交錯級數能用。

Example 0.4 *Find the sum of the series* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ *correct to three decimal places.* 小數三位, $0! := 1$

$$b_n = \frac{1}{n!} > \frac{1}{n!}\frac{1}{n+1} = \frac{1}{(n+1)!} = b_{n+1},$$

and $0 < \dfrac{1}{n!} < \dfrac{1}{n} \to 0$ *as* $n \to \infty$, $\lim\limits_{n\to\infty} b_n = 0$, *by the Squeeze Theorem.*
So $\displaystyle\sum \frac{(-1)^n}{n!}$ *converges by the Alternating Series Test.*

$$\begin{aligned} s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \\ &= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdots \end{aligned}$$

$b_6 = \dfrac{1}{720} \approx 0.0014$, $b_7 = \dfrac{1}{5040} \approx 0.0002$,
and $s_6 = 1 - 1 + \dfrac{1}{2} - \dfrac{1}{6} + \dfrac{1}{24} - \dfrac{1}{120} + \dfrac{1}{720} \approx 0.3680\overline{5}$,
So $s \approx 0.368$. *(誤差* < 0.0002 *不會影響小數第三位。)* $\blacksquare$

(書上這裡的 b_n & s_n 是從 $n = 0$ 算起, s_6 指的是前 7 項的部分和。因爲書上不是用前 n 項和的 (沒有明確的) 定義, 我覺得這樣寫並不恰當, 容易搞混。另外, 第 8 項是 $-\dfrac{1}{5040}$, 或許擔心會影響小數第三位; 事實上, $s_7 = 1 - 1 + \dfrac{1}{2} - \dfrac{1}{6} + \dfrac{1}{24} - \dfrac{1}{120} + \dfrac{1}{720} - \dfrac{1}{5040} \approx 0.367\overline{8571542}$, 四捨五入後還是 0.368.)