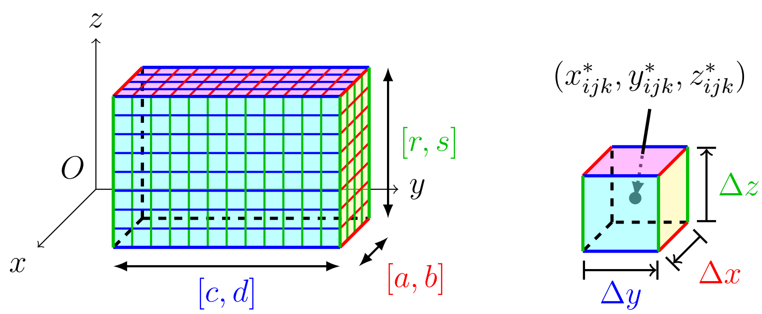
# 15.6 Triple integrals

1. Triple integral over a box (§15.1)
2. Fubini's Theorem (iterated integral) (§15.1)
3. Triple integral over a general bounded region (§15.2 + 3)
4. Application (§15.4)

## 0.1 Triple Integral over a box




**Define:** The ***triple integral*** of $f$ over the box $B$ is the limit of the ***triple Riemann sum*** 三重積分是三重黎曼和的極限。

$$\boxed{\iiint\limits_{B} f(x,y,z)\,dV = \lim_{\ell,m,n\to\infty}\sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V}$$

where $B = \{(x,y,z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\} = [a,b]\times[c,d]\times[r,s]$.
把 $[a,b]\times[c,d]\times[r,s]$ 分成 $\ell\times m\times n$ 等分, 樣本點 (sample point) $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ 在每個小盒子 $B_{ijk}$, 體積是 $\Delta V = \Delta x\Delta y\Delta z = \dfrac{b-a}{\ell}\dfrac{d-c}{m}\dfrac{s-r}{n}$.
當 $f$ 連續, 或是有界並且只有有限多的平面不連續
$\Longrightarrow$ 三重黎曼和極限存在 $\Longleftrightarrow$ $f$ 在 $B$ 上可積分(integrable)。

**Note:** $\iiint\limits_{?} ?\ dV$ 是固定寫法, $V$ 是指體積 (Volume), $B$ 是指方盒 (Box)。

## 0.2 Fubini's Theorem

**Theorem 1 (Fubini's Theorem for Triple Integral)**
*If $f$ is* **continuous** *on the rectangular box* $B = [a, b] \times [c, d] \times [r, s]$*, then*

$$\boxed{\iiint_B \boldsymbol{f(x, y, z)\ dV} = \int_a^b \int_c^d \int_r^s \boldsymbol{f(x, y, z)\ dz\ dy\ dx}}$$

**Note:** 迭代積分 (iterated integral) 還有其他五種 (換 $dx$, $dy$, $dz$ 順序):

$$\begin{aligned}
&\iiint_B f(x, y, z)\ dV \\
&= \int_a^b \int_c^d \int_r^s f(x, y, z)\ dz\ dy\ dx = \int_a^b \int_r^s \int_c^d f(x, y, z)\ dy\ dz\ dx \\
&= \int_r^s \int_a^b \int_c^d f(x, y, z)\ dy\ dx\ dz = \int_c^d \int_a^b \int_r^s f(x, y, z)\ dz\ dx\ dy \\
&= \int_c^d \int_r^s \int_a^b f(x, y, z)\ dx\ dz\ dy = \int_r^s \int_c^d \int_a^b f(x, y, z)\ dx\ dy\ dz
\end{aligned}$$

**Attention:** 由內而外積分, 偏積分把其他變數當常數, 注意順序, (TFTC) 多變數代入時註明變數。

$$a \le x \le b,\ c \le y \le d,\ r \le z \le s$$

$$\int_a^b \int_c^d \int_r^s f(x, y, z)\ dz\ dy\ dx$$

**Note:** If $f(x, y, z) = g(x)h(y)u(z)$ and $B = [a, b] \times [c, d] \times [r, s]$, then (函數分開 (天時), 在(矩形) 盒子裡(地利) $\Longrightarrow$ 可以分開積 (人和)。)

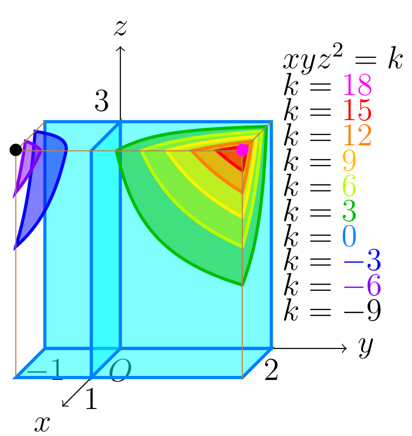
$$\boxed{\iiint_B \boldsymbol{f(x, y, z)\ dV} = \int_a^b \boldsymbol{g(x)\ dx} \int_c^d \boldsymbol{h(y)\ dy} \int_r^s \boldsymbol{u(z)\ dz}}$$

**Example 0.1** *Evaluate the triple integral* $\displaystyle\iiint_B xyz^2\ dV$, *where* $B$ *is the rectangular box given by* $B = \{(x,y,z) : 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$.

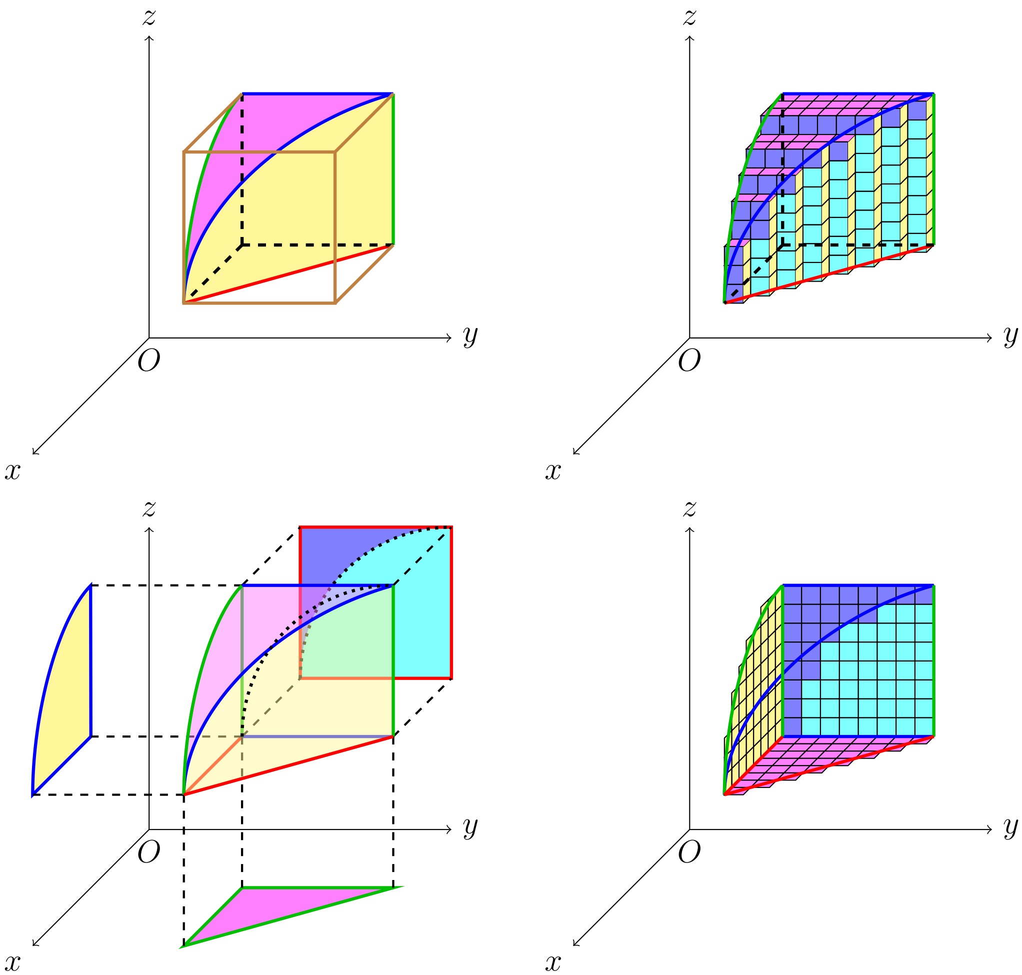
$$\begin{aligned}
\iiint_B xyz^2\ dV &= \int_0^1 \int_{-1}^2 \int_0^3 xyz^2\ dz\ dy\ dx \\
\text{(對 } z \text{ 偏積分, } x,y \text{ 當常數。)} &= \int_0^1 \int_{-1}^2 \left[xy\frac{z^3}{3}\right]_{z=0}^{z=3} dy\ dx \\
\text{(積完 } z \text{ 就沒 } z\text{, 三重變雙重。)} &= \int_0^1 \int_{-1}^2 9xy\ dy\ dx \\
\text{(對 } y \text{ 偏積分, } x \text{ 當常數。)} &= \int_0^1 \left[9x\frac{y^2}{2}\right]_{y=-1}^{y=2} dx \\
\text{(積完 } y \text{ 就沒 } y\text{, 雙重變定積。)} &= \int_0^1 \frac{27}{2}x\ dx = \left[\frac{27}{2}\frac{x^2}{2}\right]_0^1 = \frac{27}{4}. \quad \blacksquare
\end{aligned}$$

$$\begin{aligned}
\iiint_B xyz^2\ dV &= \int_0^1 x\ dx \int_{-1}^2 y\ dy \int_0^3 z^2\ dz \qquad \text{(可以分開)} \\
&= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_{-1}^2 \left[\frac{z^3}{3}\right]_0^3 = \frac{1}{2}\cdot\frac{3}{2}\cdot 9 = \frac{27}{4}.
\end{aligned}$$

(課本順序是 $\displaystyle\int_0^3 \int_{-1}^2 \int_0^1 xyz^2\ dx\ dy\ dz$; 算算看其他四種順序答案是否一樣。)



## 0.3 Triple integral over a general bounded region




$$\iiint\limits_E f(x,y,z)\ dV = \iiint\limits_B F(x,y,z)\ dV$$

where $E$ is a region bounded by some surfaces inside a box $B$, and

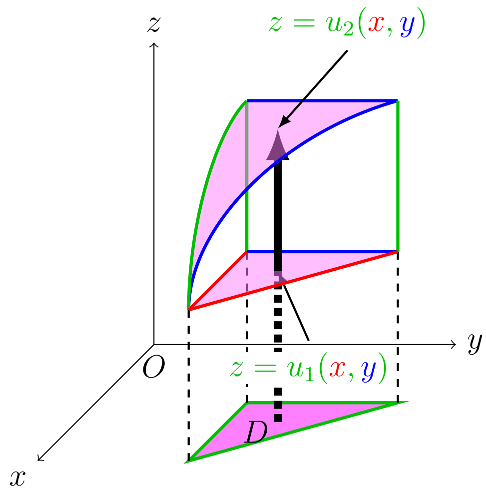
$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in E \\ 0 & \text{if } (x,y,z) \in B \setminus E. \end{cases}$$

**Question:** 怎麼從三重積分變成迭代積分? 積分的順序與上下界怎麼寫?

$$\iiint\limits_E f(x,y,z)\ dV = \int_?^? \int_?^? \int_?^? f(x,y,z)\ d?\ d?\ d?$$

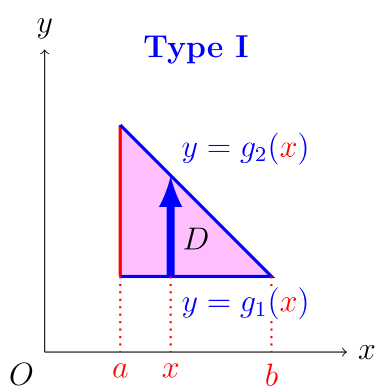
### 0.3.1 Type 1

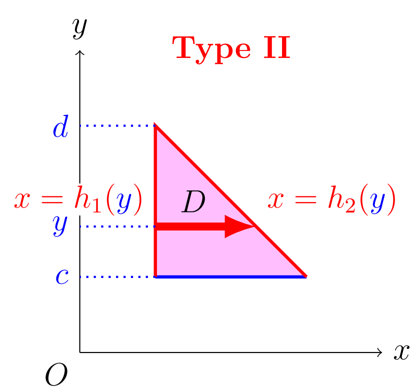
$E = \{(x, y, z) : (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$,
where $D$ is the projection of $E$ on to the $xy$-plane.
($E$ 是 $D$ 上 $z$ 介於兩個 $x, y$ 的函數之間, $D$ 是在 $z = 0$ 的投影。)




(先對 $z$ 從下面 ($u_1$) 到上面 ($u_2$) 偏積分, 積完變成一個 $x, y$ 的函數在 $D$ 上的雙重積分。再根據 $D$ 的形狀決定先對 $y$ (**Type I**) 或是先對 $x$ (**Type II**) 偏積分。)

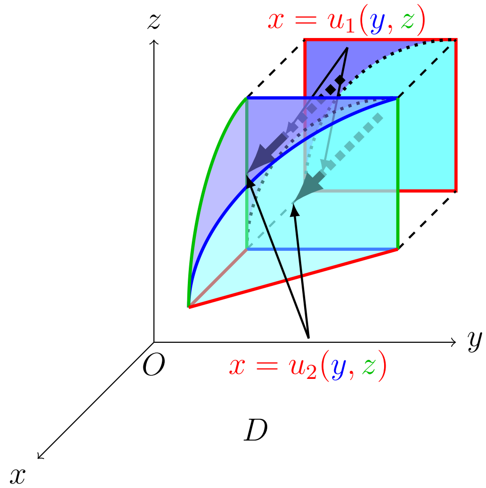
$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \iint_D \left[ \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA \\
(\textbf{Type I}) &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx \\
(\textbf{Type II}) &= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dx\, dy
\end{aligned}$$



### 0.3.2 Type 2

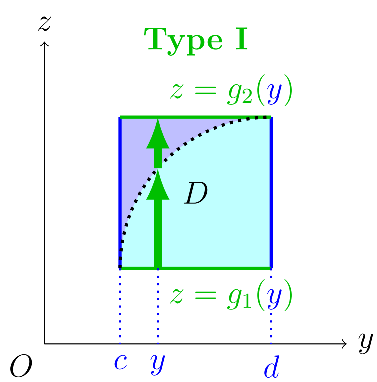
$E = \{(x,y,z) : (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$,
where $D$ is the projection of $E$ on to the $yz$-plane.
($E$ 是 $D$ 上 $x$ 介於兩個 $y,z$ 的函數之間, $D$ 是在 $x=0$ 的投影。)

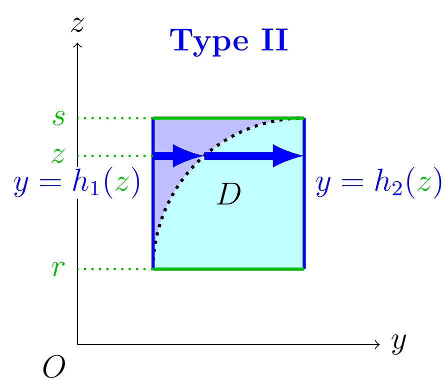



(先對 $x$ 從後面 ($u_1$) 到前面 ($u_2$) 偏積分, 積完變成一個 $y,z$ 的函數在 $D$ 上的雙重積分。再根據 $D$ 的形狀決定先對 $z$ (**Type I**) 或是先對 $y$ (**Type II**) 偏積分。)

**Note:** 這時候 $dA$ 是 $dz\,dy$ 或 $dy\,dz$.

$$\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA \\
(\textbf{Type I}) &= \int_c^d \int_{g_1(y)}^{g_2(y)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dz\,dy \\
(\textbf{Type II}) &= \int_r^s \int_{h_1(z)}^{h_2(z)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dy\,dz
\end{aligned}$$

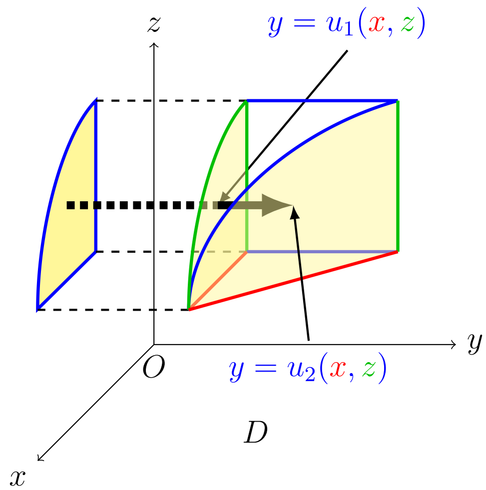



(這個例圖要分成兩個積分, 因爲 $x$ 的上下界不完全一樣。)

### 0.3.3 Type 3

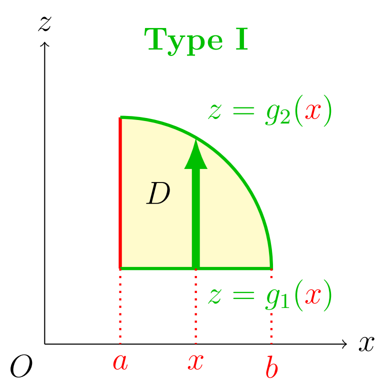
$E = \{(x, y, z) : (x, z) \in D,\ u_1(x, z) \leq y \leq u_2(x, z)\}$,
where $D$ is the projection of $E$ on to the $xz$-plane.
($E$ 是 $D$ 上 $y$ 介於兩個 $x, z$ 的函數之間, $D$ 是在 $y = 0$ 的投影。)

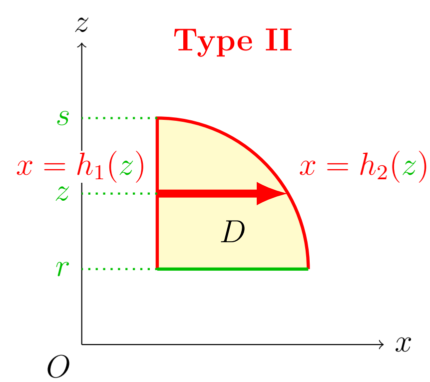



(先對 $y$ 從左面 ($u_1$) 到右面 ($u_2$) 偏積分, 積完變成一個 $x, z$ 的函數在 $D$ 上的雙重積分。再根據 $D$ 的形狀決定先對 $z$ (**Type I**) 或是先對 $x$ (**Type II**) 偏積分。)

**Note:** 這時候 $dA$ 是 $dz\ dx$ 或 $dx\ dz$.

$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\right] dA \\
(\textbf{Type I}) &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\, dz\, dx \\
(\textbf{Type II}) &= \int_r^s \int_{h_1(z)}^{h_2(z)} \int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\, dx\, dz
\end{aligned}$$




(如果 $D$ 要分成多個, 三重積分也要分成多個積分。)

**Note:** 偏積分一次之後, 除了 **Type I** or **Type II**, 還可以換成極座標:

$$\iiint\limits_E f(x,y,z)\ dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\right]\ dA$$

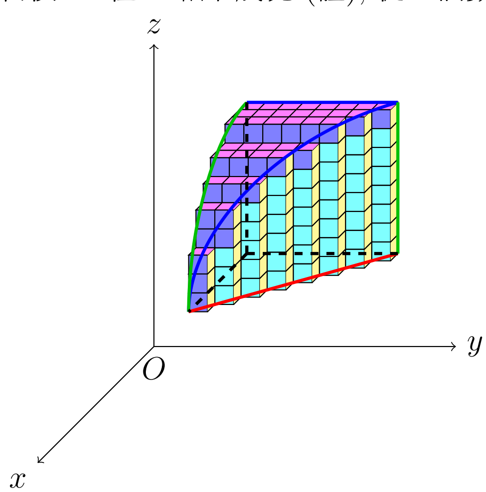
$$= \iint\limits_D g(x,y)\ dA = \int_\alpha^\beta \int_\gamma^\delta g(r\cos\theta, r\sin\theta)\cdot r\ dr\ d\theta. \qquad (\text{勿忘乘 } r!)$$

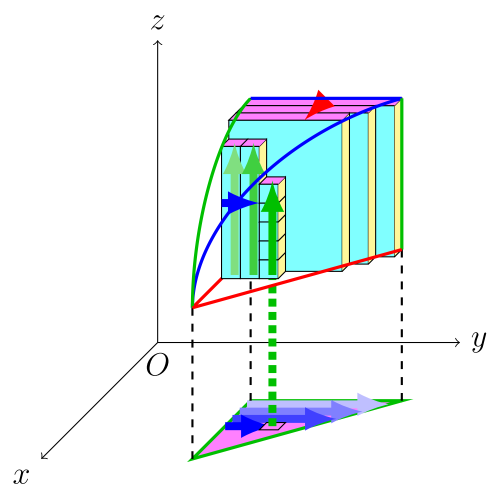
**Attention:** 越外層上下界的變數越少, 裡層上下界寫成外層變數的函數。上下界不是單一表示的函數 $(f = \{\vdots)$ 要分成多個積分相加。

**Skill:** 1. 先積誰就投影到誰 $= 0$, 再由投影形狀決定再積誰。
2. 找投影的邊界函數: 投影到誰 $= 0$, 就代入邊界函數消去誰。
3. 從包圍曲面找上下界: 把要積的變數寫成其他變數的函數。
4. 從上下界找包圍曲面: 對應的變數 $=$ 對應的上下界。

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dy\ dx \iff \begin{array}{l} x = b,\ y = g_2(x),\ z = u_2(x,y), \\ x = a,\ y = g_1(x),\ z = u_1(x,y). \end{array}$$

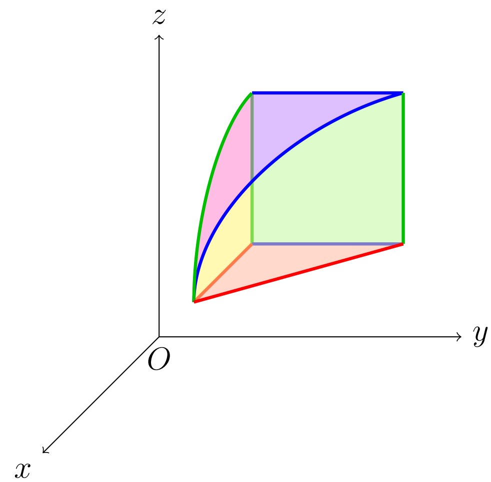
可以想像成把 $E$ 切成很多小盒子, 要先積誰就看從哪個方向先累積。
先積 $z$: 往 $z$-軸連成條 (線), 從一個 $x, y$ 的函數累積到另一個 $x, y$ 的函數;
再積 $y$: 往 $y$-軸織成片 (面), 從一個 $x$ 的函數累積到另一個 $x$ 的函數;
終積 $x$: 往 $x$-軸堆成塊 (體), 從一個數字累積到另一個數字。



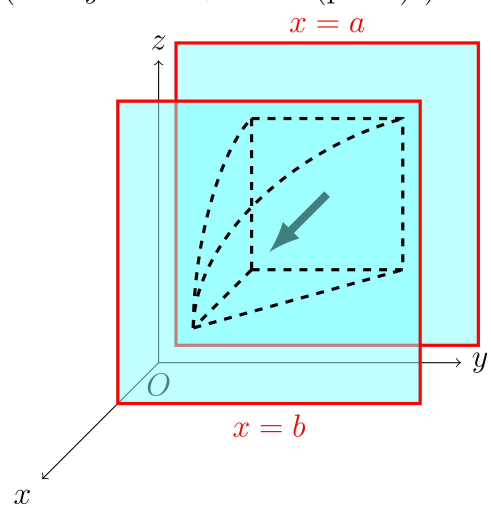



**Question:** How to change types? **Answer:** 畫圖, 重新投影。

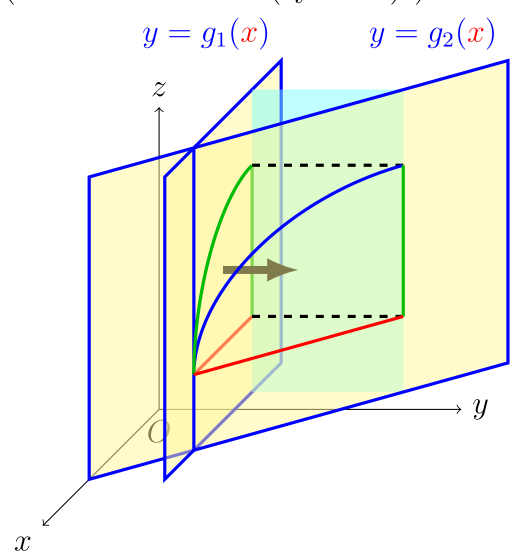
$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$



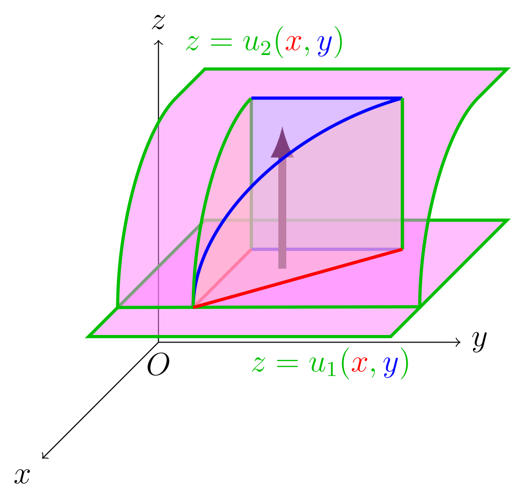

1. 最外層: $x = a$ & $x = b$
(平行 $yz$-平面的兩平面 (plane)。)




2. 中間層: $y = g_1(x)$ & $y = g_2(x)$
(平行 $z$-軸的兩柱面 (cylinder)。)

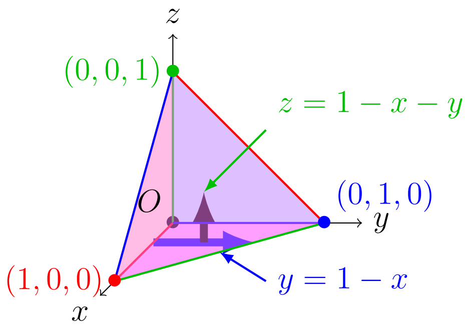


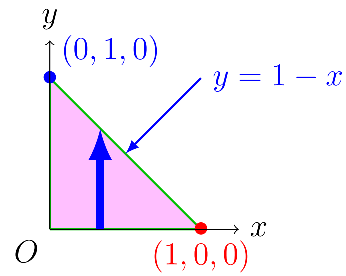

3. 最內層: $z = u_1(x,y)$ & $z = u_2(x,y)$
(兩曲面 (surface)。)




**Observation:** 其實三重積分能列式的迭代積分只有 兩平面+ 兩柱面+ 兩曲面 包住的區域 (雙重積分的是 兩 (垂直/水平) 線+ 兩曲線), 但是任何區域都可以切成這些形狀。

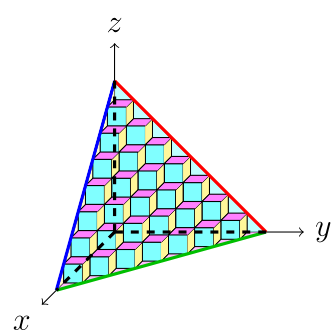
**Example 0.2** *Evaluate* $\displaystyle\iiint_E z\,dV$, *where* $E$ *is the solid tetrahedron bounded by the four planes* $x=0$, $y=0$, $z=0$, *and* $x+y+z=1$.

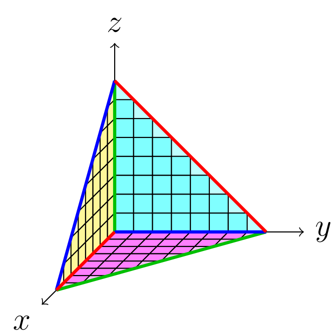


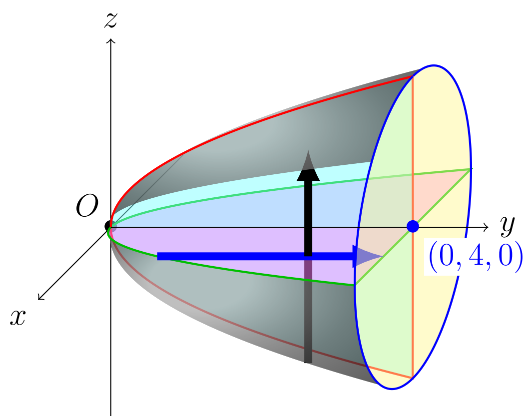

投影到 *xy-plane* $(z=0)$:
*z-axis:* 看往正 $z$ 軸的黑粗箭頭的頭尾的 $z$ 座標, 把邊界函數變成 $z=?(x,y)$,
$\Longrightarrow$ 在 $z=0$ 與 $z=1-x-y$(解 $x+y+z=1$) 之間;
*y-axis:* 看往正 $y$ 軸的粉紅箭頭的頭尾的 $y$ 座標, 把投影邊界函數變成 $y=?(x)$,
$\Longrightarrow$ 在 $y=0$ 與 $y=1-x$(解 $0=z=1-x-y$) 之間;
*x-axis:* 看極端的 $x$ 座標, $\Longrightarrow$ 在 $x=0$ 與 $x=1$(解 $0=y=1-x$) 之間。
$E=\{(x,y,z): 0\le x\le 1,\ 0\le y\le 1-x,\ 0\le z\le 1-x-y\}$.
(描述 $E$ 的 $x,y,z$ 範圍就是積分對應的上下界。)

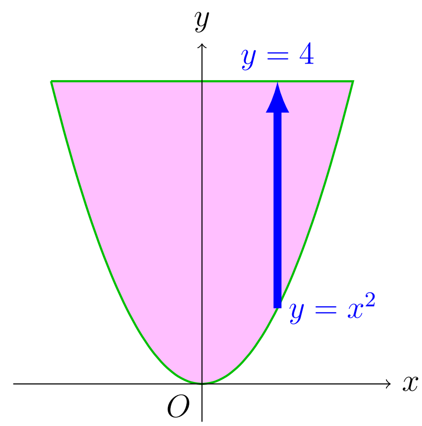
$$\begin{aligned}
\iiint_E z\,dV &= \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx \\
&= \int_0^1\int_0^{1-x}\left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\,dx \\
&= \int_0^1\int_0^{1-x}\frac{(1-x-y)^2}{2}\,dy\,dx \\
\binom{u=1-x-y,}{du=-dy,} &= \int_0^1\left[-\frac{(1-x-y)^3}{6}\right]_{y=0}^{y=1-x} dx \\
&= \int_0^1 \frac{(1-x)^3}{6}\,dx \\
\binom{v=1-x,}{dv=-dx,} &= \left[-\frac{(1-x)^4}{24}\right]_0^1 \\
&= \frac{1}{24}.
\end{aligned}$$
■



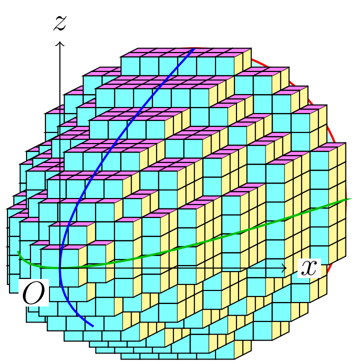
**Example 0.3** *Evaluate* $\displaystyle\iiint\limits_E \sqrt{x^2+z^2}\ dV$*, where* $E$ *is the region bounded by the paraboloid* $y=x^2+z^2$ *and the plane* $y=4$*.*

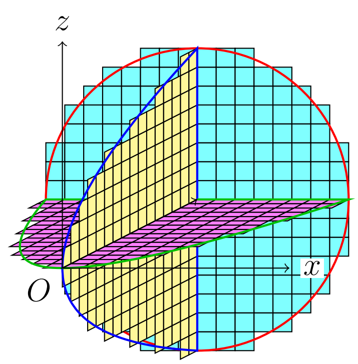


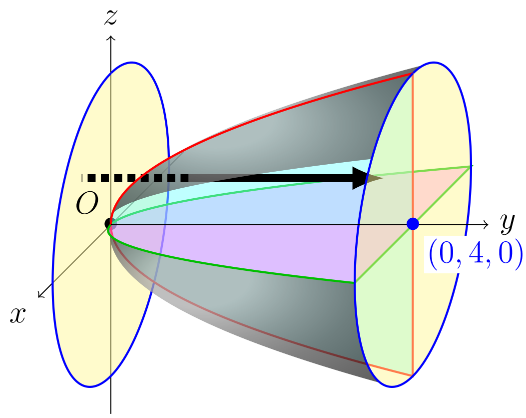



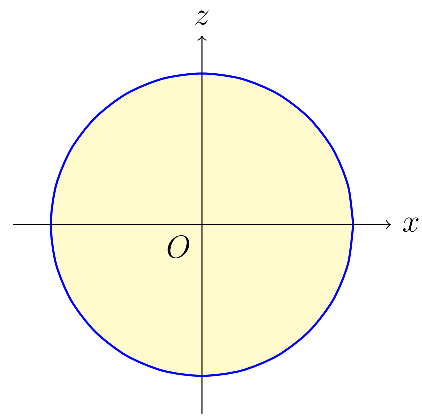

投影到 *xy-plane* $(z=0)$*:*
*z-axis:* 在 $z=-\sqrt{y-x^2}$ 與 $z=\sqrt{y-x^2}$(解 $y=x^2+z^2$) 之間;
*y-axis:* 在 $y=x^2$(解 $z=0$, $y=x^2+z^2$) 與 $y=4$ 之間;
*x-axis:* 在 $x=-2$ 與 $x=2$(解 $4=y=x^2$) 之間。
$E=\{(x,y,z): -2\le x\le 2,\ x^2\le y\le 4,\ -\sqrt{y-x^2}\le z\le\sqrt{y-x^2}\}$.

$$\iiint\limits_E \sqrt{x^2+z^2}\ dV=\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}}\sqrt{x^2+z^2}\ dz\ dy\ dx=\cdots\ (\text{很難算}).$$



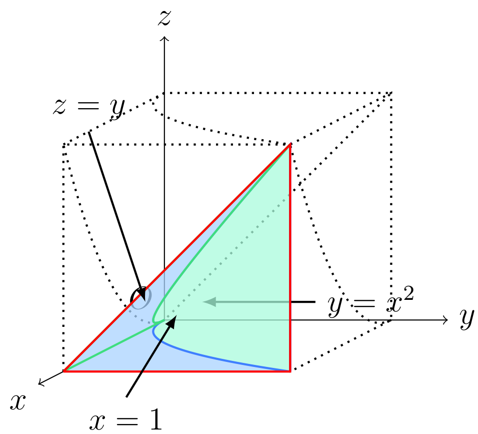


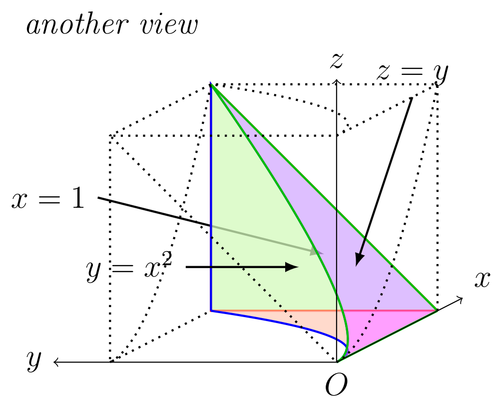




換個方向, 投影到 *xz-plane* ($y=0$):
*y-axis:* 在 $y=x^2+z^2$ 與 $y=4$ 之間;
*xz-plane:* 在 $x^2+z^2\le 4$ (解 $4=y=x^2+z^2$). 換極座標:
$D=\{(x,z): x^2+z^2\le 4\}=\{(r,\theta): 0\le r\le 2,\ 0\le\theta\le 2\pi\}$,
$E=\{(x,y,z):(x,z)\in D,\ x^2+z^2\le y\le 4\}$.

$$
\begin{aligned}
\iiint_E \sqrt{x^2+z^2}\,dV &= \iint_D \left[\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}\,dy\right]dA \\
&= \iint_D \left[y\sqrt{x^2+z^2}\right]_{y=x^2+z^2}^{y=4} dA \\
&= \iint_D (4-x^2-z^2)\sqrt{x^2+z^2}\,dA \\
&\qquad \text{(用極座標: } x=r\cos\theta,\ z=r\sin\theta,\ \text{別忘乘 } r\text{。)} \\
&= \int_0^{2\pi}\int_0^2 (4-r^2)r\cdot r\,dr\,d\theta \\
\text{(可以分開)} &= \int_0^{2\pi} d\theta \int_0^2 (4r^2-r^4)\,dr \\
&= \Big[\theta\Big]_0^{2\pi}\left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 \\
&= 2\pi\cdot\frac{64}{15} = \frac{128}{15}\pi. \qquad \blacksquare
\end{aligned}
$$

**Example 0.4** *Express the iterated integral* $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\ dz\ dy\ dx$ *as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to* $x$*, then* $z$*, and then* $y$*.*



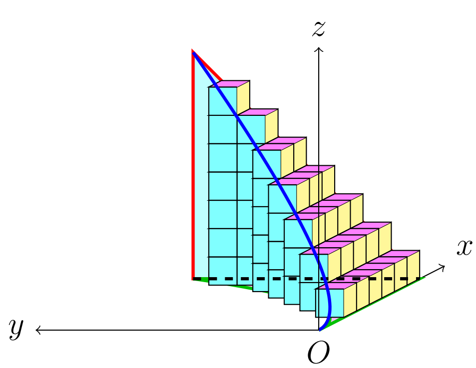

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\ dz\ dy\ dx = \iiint_E f(x,y,z)\ dV,$$

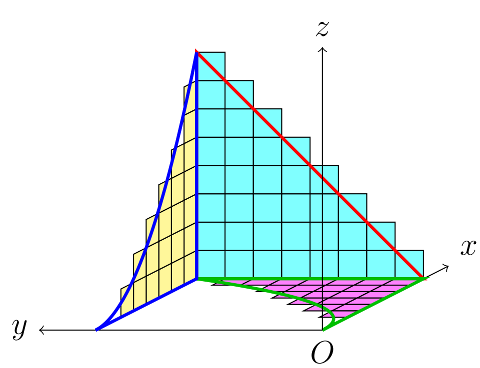
*where* $E = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}$.
*(由迭代積分的式子反推) E is bounded by: (包圍 E 的曲面)*

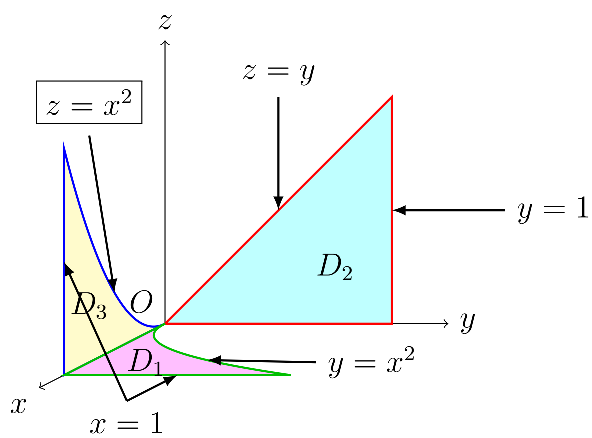
$$x = 0,\ x = 1,\ y = 0,\ y = x^2,\ z = 0,\ \text{and}\ z = y.$$

目標: 找出新積分順序的上下界:

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\ dz\ dy\ dx = \int_?^? \int_?^? \int_?^? f(x,y,z)\ dx\ dz\ dy.$$

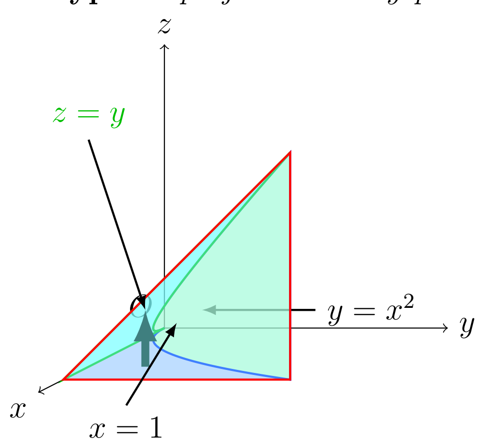


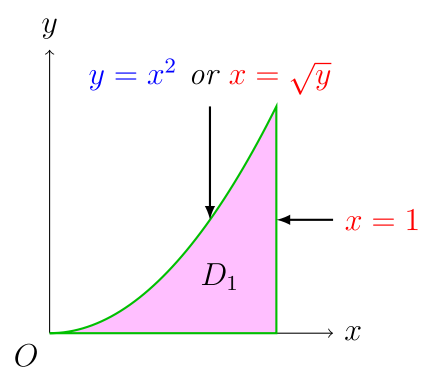






**Note:** $\boxed{z = x^2}$ 是由 $z = y$ 與 $y = x^2$ 解出來的。在 *xz-plane*, 把 $y$ 消去。

**Type 1**: *projection on xy-plane* $(z = 0)$: $0 \le z \le y$.



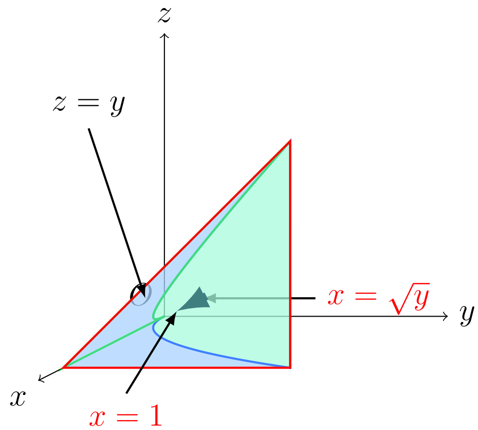



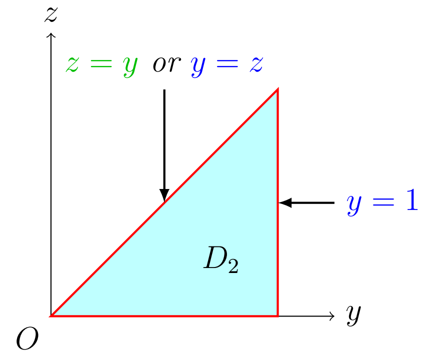

$D_1 = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x^2\} = \{(x, y) : 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$,

$$\iiint_E f\ dV = \underline{\int_0^1 \int_0^{x^2} \int_0^y f\ dz\ dy\ dx} = \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f\ dz\ dx\ dy.$$

(爲了方便把 $f(x, y, z)$ 省略爲 $f$。)

**Type 2**: *projection on yz-plane* $(x=0)$: (解 $y=x^2$) $\sqrt{y}\le x\le 1$.

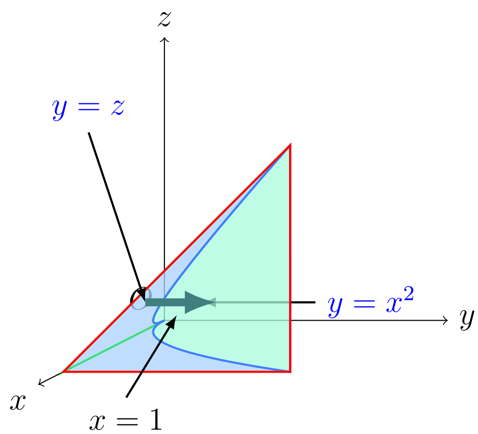


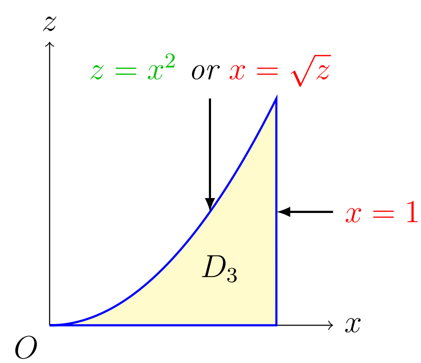




$D_2=\{(y,z): 0\le y\le 1,\ 0\le z\le y\}=\{(y,z): 0\le z\le 1,\ z\le y\le 1\}$,

$$\iiint_E f\ dV = \underline{\int_0^1\int_0^y\int_{\sqrt{y}}^1 f\ dx\ dz\ dy} = \int_0^1\int_z^1\int_{\sqrt{y}}^1 f\ dx\ dy\ dz.$$

**Type 3**: *projection on xz-plane* $(y=0)$: $z\le y\le x^2$.




$D_3=\{(x,z): 0\le z\le 1,\ \sqrt{z}\le x\le 1\}=\{(x,z): 0\le x\le 1,\ 0\le z\le x^2\}$,

$$\iiint_E f\ dV = \int_0^1\int_{\sqrt{z}}^1\int_z^{x^2} f\ dy\ dx\ dz = \int_0^1\int_0^{x^2}\int_z^{x^2} f\ dy\ dz\ dx.$$

*Answer:* $\underline{\int_0^1\int_0^{x^2}\int_0^{y} f(x,y,z)\ dz\ dy\ dx = \int_0^1\int_0^{y}\int_{\sqrt{y}}^{1} f(x,y,z)\ dx\ dz\ dy}$. ■

## 0.4 Application

$$
\begin{aligned}
f(x) \geq 0 &\implies \int_a^b f(x)\, dx\text{: } y = f(x) \text{ 到 } x\text{-axis 從 } a \text{ 到 } b \text{ 的}\overset{\text{area}}{\text{面積}}\text{。} \\
f(x,y) \geq 0 &\implies \iint\limits_D f(x,y)\, dA\text{: } z = f(x,y) \text{ 到 } xy\text{-plane 在 } D \text{ 上的}\overset{\text{volume}}{\text{體積}}\text{。} \\
f(x,y,z) \geq 0 &\implies \iiint\limits_E f(x,y,z)\, dV\text{: 四維物體的 }\overset{\text{hypervolume}}{\text{超體積}}\text{ 。}
\end{aligned}
$$

一個三維的立體 (solid) 佔有區域 (region) $E$:

♡ **volume** 體積 $V(E) = \iiint\limits_E \, dV$. $(f(x,y,z) = 1)$

♡ **mass** 質量 $m = \iiint\limits_E \rho(x,y,z)\, dV$, where $\rho(x,y,z)$ is the density.

- **moment** 力矩 about $yz$-, $xz$-, and $xy$-planes are

$$
\begin{aligned}
M_{yz} &= \iiint\limits_E x\rho(x,y,z)\, dV, \\
M_{xz} &= \iiint\limits_E y\rho(x,y,z)\, dV, \\
M_{xy} &= \iiint\limits_E z\rho(x,y,z)\, dV.
\end{aligned}
$$

- **center of mass** 質心 (質量中心, 權重是密度 $(\rho)$ 的加權平均座標。)

$$
(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right).
$$

- **centroid** 形心 (形狀中心, 等於均質 (constant $\rho(x,y,z)$) 時的質心。)

$$
\left( \frac{\iiint_E x\, dV}{\iiint_E \, dV}, \frac{\iiint_E y\, dV}{\iiint_E \, dV}, \frac{\iiint_E z\, dV}{\iiint_E \, dV} \right).
$$

- **moment of inertia 轉動慣量** about $x$-, $y$-, and $z$-axes, and the origin $O$ are

$$
\begin{aligned}
I_x &= \iiint_E (y^2 + z^2)\rho(x, y, z)\ dV, \\
I_y &= \iiint_E (x^2 + z^2)\rho(x, y, z)\ dV, \\
I_z &= \iiint_E (x^2 + y^2)\rho(x, y, z)\ dV, \\
I_O &= \iiint_E (x^2 + y^2 + z^2)\rho(x, y, z)\ dV.
\end{aligned}
$$

- **electric charge 電荷**

$$Q = \iiint_E \sigma(x, y, z)\ dV,$$

where $\sigma(x, y, z)$ is the charge density.

- **probability 機率**

$$P((X, Y, Z) \in E) = \iiint_E f(x, y, z)\ dV.$$

where $X$, $Y$, and $Z$ are random variables, $f(x, y, z)$ is their joint density function, satisfying

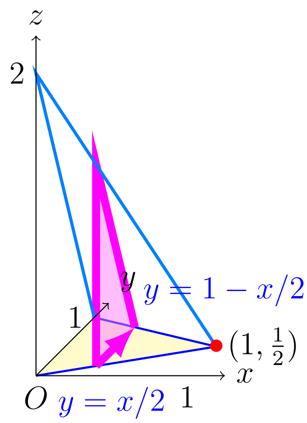
$$f(x, y, z) \geq 0, \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\ dz\ dy\ dx = 1.$$

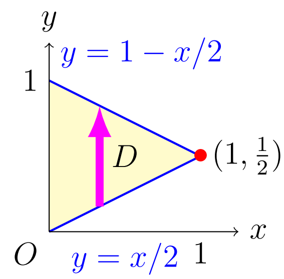
In particular,

$$P(a \leq X \leq b, c \leq Y \leq d, r \leq Z \leq s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\ dz\ dy\ dx.$$

**Example 0.5** *Use a triple integral to find the volume of the tetrahedron $T$ bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, and $z=0$.*

原本: 看成 $z=f(x,y)=(2-x-2y)$ 在 $D$ 上到 *xy-plane* 的體積。

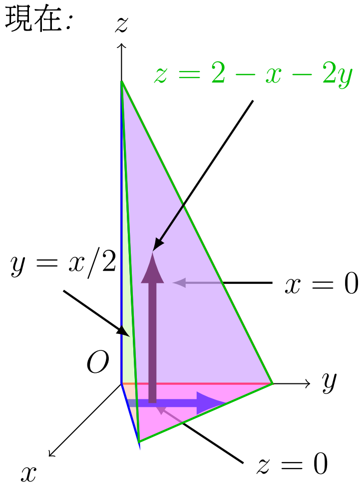


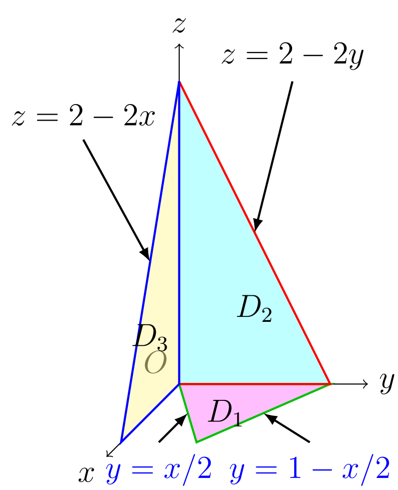




$$V=\iint_D f(x,y)\,dA=\int_0^1\int_{x/2}^{1-x/2}(2-x-2y)\,dy\,dx$$

$$=\int_0^1\Big[2y-xy-y^2\Big]_{y=x/2}^{y=1-x/2}dx=\int_0^1\left(x^2-2x+1\right)dx=\left[\frac{x^3}{3}-x^2+x\right]_0^1=\frac{1}{3}.$$

現在:

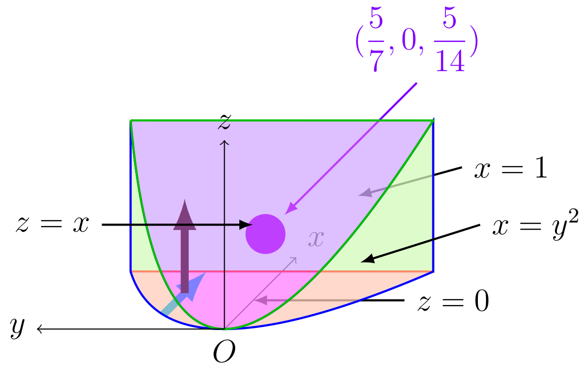


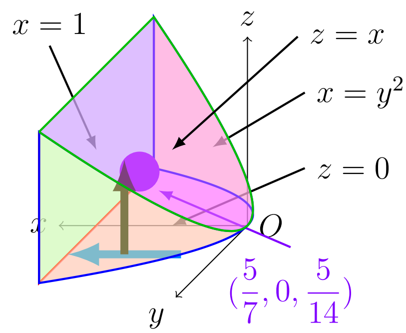



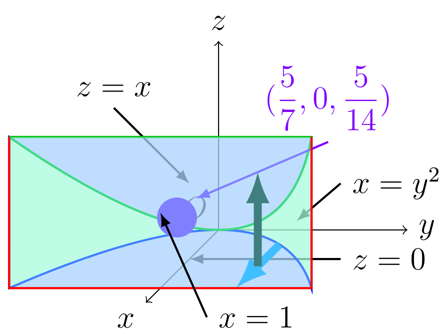

$$V(E)=\iiint_E dV=\int_0^1\int_{x/2}^{1-x/2}\int_0^{2-x-2y}dz\,dy\,dx \qquad (\textbf{Type 1 + Type I})$$

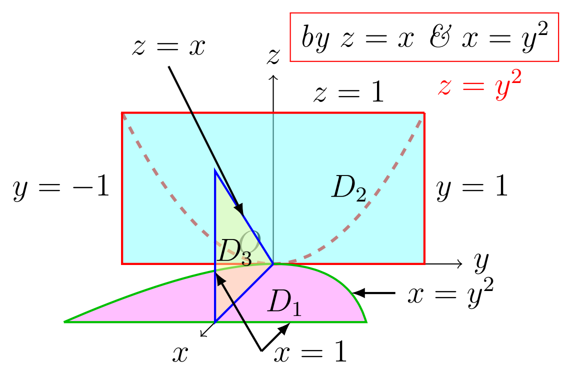
$$=\int_0^1\int_{x/2}^{1-x/2}(2-x-2y)\,dy\,dx=\cdots=\frac{1}{3}.$$ *(Try other types yourself.)* ■

**Example 0.6** *Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the plane $z = 0$, $z = x$, and $x = 1$.*




**Type 1**: *projection on xy-plane ($D_1$)*

$$\begin{aligned} E &= \{(x,y,z) : 0 \le x \le 1,\ -\sqrt{x} \le y \le \sqrt{x},\ 0 \le z \le x\} \\ &= \{(x,y,z) : -1 \le y \le 1,\ y^2 \le x \le 1,\ 0 \le z \le x\}. \qquad (\checkmark\text{上下界最好算}) \end{aligned}$$

**Type 2**: *projection on yz-plane ($D_2$) (要分兩塊或三塊)*

$$\begin{aligned} E &= \{(x,y,z) : -1 \le y \le 1,\ 0 \le z \le y^2,\ y^2 \le x \le 1\} \\ &\cup \{(x,y,z) : -1 \le y \le 1,\ y^2 \le z \le 1,\ z \le x \le 1\}. \\ &= \{(x,y,z) : 0 \le z \le 1,\ -1 \le y \le -\sqrt{z},\ y^2 \le x \le 1\} \\ &\cup \{(x,y,z) : 0 \le z \le 1,\ -\sqrt{z} \le y \le \sqrt{z},\ z \le x \le 1\} \\ &\cup \{(x,y,z) : 0 \le z \le 1,\ \sqrt{z} \le y \le 1,\ y^2 \le x \le 1\}. \end{aligned}$$

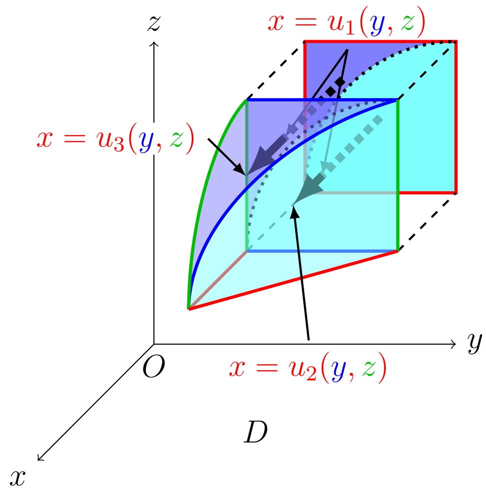
**Type 3**: *projection on xz-plane ($D_3$)*

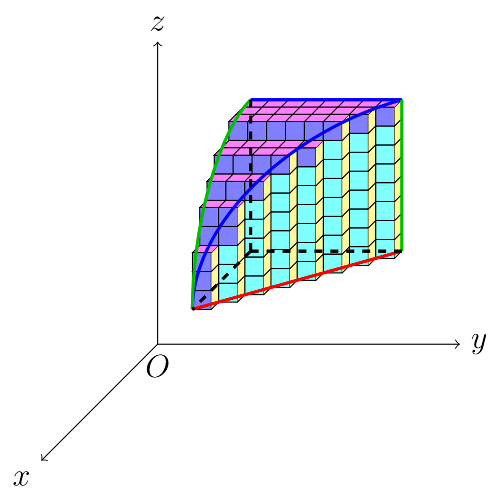
$$\begin{aligned} E &= \{(x,y,z) : 0 \le x \le 1,\ 0 \le z \le x,\ -\sqrt{x} \le y \le \sqrt{x}\} \\ &= \{(x,y,z) : 0 \le z \le 1,\ z \le x \le 1,\ -\sqrt{x} \le y \le \sqrt{x}\}. \end{aligned}$$

$$
\begin{aligned}
E &= \{(x,y,z) : -1 \le y \le 1,\ y^2 \le x \le 1,\ 0 \le z \le x\},\ \rho(x,y,z) = \rho. \\
m &= \iiint_E \rho \, dV \;=\; \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} \rho \, dz \, dx \, dy \;=\; \int_{-1}^{1}\int_{y^2}^{1} \Big[\rho z\Big]_{z=0}^{z=x} dx \, dy \\
&= \rho \int_{-1}^{1}\int_{y^2}^{1} x \, dx \, dy \;=\; \rho \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy \;=\; \frac{\rho}{2}\int_{-1}^{1} (1-y^4) \, dy \\
&= \rho \int_{0}^{1} (1-y^4) \, dy \;=\; \rho\Big[y - \frac{y^5}{5}\Big]_0^1 \;=\; \frac{4\rho}{5}. \quad \textit{($1-y^4$ is even)} \\
M_{yz} &= \iiint_E x\rho \, dV \;=\; \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} \rho x \, dz \, dx \, dy \;=\; \int_{-1}^{1}\int_{y^2}^{1} \Big[\rho x z\Big]_{z=0}^{z=x} dx \, dy \\
&= \rho \int_{-1}^{1}\int_{y^2}^{1} x^2 \, dx \, dy \;=\; \rho \int_{-1}^{1} \left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy \;=\; \frac{\rho}{3}\int_{-1}^{1} (1-y^6) \, dy \\
&= \frac{2\rho}{3}\int_{0}^{1} (1-y^6) \, dy \;=\; \frac{2\rho}{3}\Big[y - \frac{y^7}{7}\Big]_0^1 \;=\; \frac{4\rho}{7}, \quad \textit{($1-y^6$ is even)} \\
\bar{x} &= \frac{M_{yz}}{m} \;=\; \frac{4\rho}{7} \div \frac{4\rho}{5} \;=\; \frac{5}{7}. \\
\bar{y} &= 0. \quad \because E \textit{ and } \rho \textit{ are symmetric about } xz\textit{-plane} \implies M_{xz} = 0. \\
M_{xy} &= \iiint_E z\rho \, dV \;=\; \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} \rho z \, dz \, dx \, dy \;=\; \int_{-1}^{1}\int_{y^2}^{1} \left[\rho\frac{z^2}{2}\right]_{z=0}^{z=x} dx \, dy \\
&= \frac{\rho}{2} \int_{-1}^{1}\int_{y^2}^{1} x^2 \, dx \, dy \;=\; \frac{\rho}{2} \int_{-1}^{1} \left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy \;=\; \frac{\rho}{6}\int_{-1}^{1} (1-y^6) \, dy \\
&= \frac{\rho}{3}\int_{0}^{1} (1-y^6) \, dy \;=\; \rho\Big[y - \frac{y^7}{7}\Big]_0^1 \;=\; \frac{2\rho}{7}, \quad \textit{($1-y^6$ is even)} \\
\bar{z} &= \frac{M_{xy}}{m} \;=\; \frac{2\rho}{7} \div \frac{4\rho}{5} \;=\; \frac{5}{14}.
\end{aligned}
$$

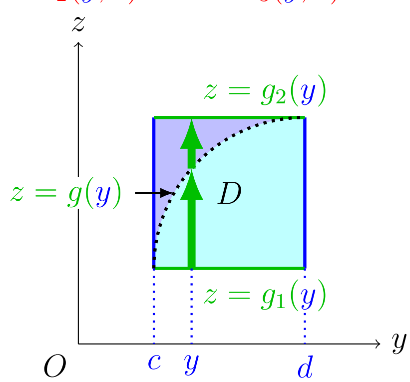
*Therefore the center of mass (also centroid) is* $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{5}{7}, 0, \frac{5}{14}\right)$. ■

## ♦ Additional: 0.3.2 type 2



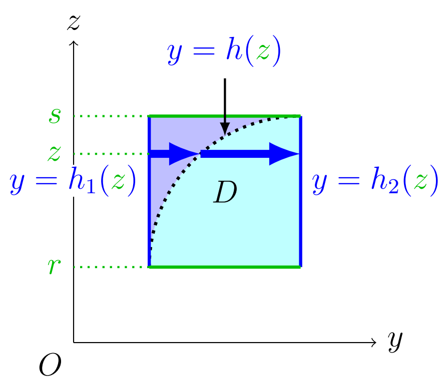

由 $u_2(y,z) = x = u_3(y,z)$ 解出 $z = g(y)$ & $y = h(z)$.




$$\iiint_E f(x,y,z)\ dV$$

$$= \int_c^d \int_{g_1(y)}^{\boxed{g(y)}} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dx\ dz\ dy$$

$$+ \int_c^d \int_{\boxed{g(y)}}^{g_2(y)} \int_{u_1(x,y)}^{u_3(x,y)} f(x,y,z)\ dx\ dz\ dy.$$



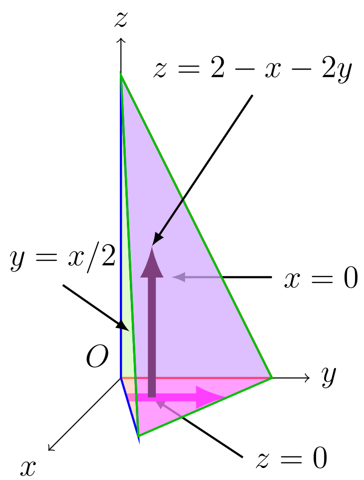

$$\iiint_E f(x,y,z)\ dV$$

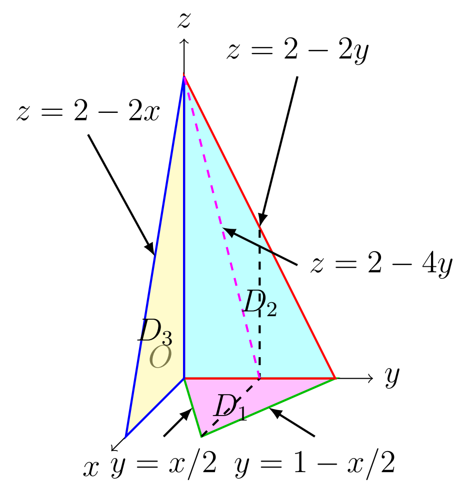
$$= \int_r^s \int_{h_1(z)}^{\boxed{h(z)}} \int_{u_1(x,y)}^{u_3(x,y)} f(x,y,z)\ dx\ dy\ dz$$

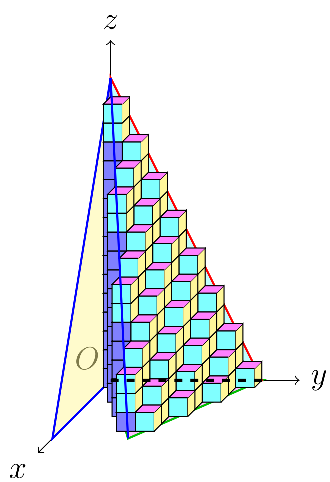
$$+ \int_r^s \int_{\boxed{h(z)}}^{h_2(z)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dx\ dy\ dz.$$

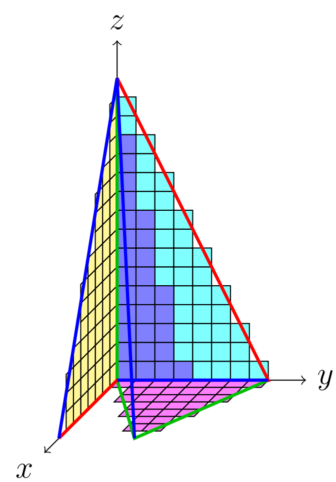
## ⧫ Additional: Example 0.5 all types

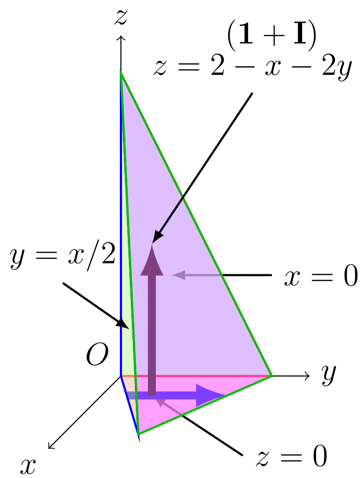
**Example 0.5** *Use a triple integral to find the volume of the tetrahedron $T$ bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.*

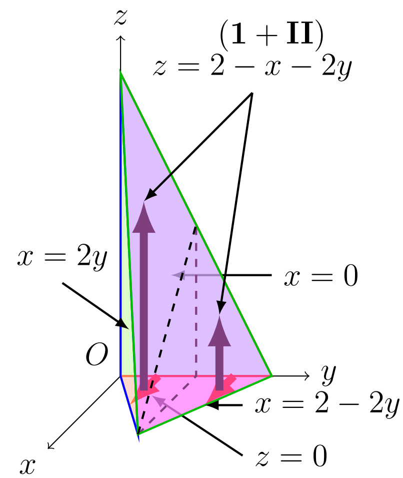


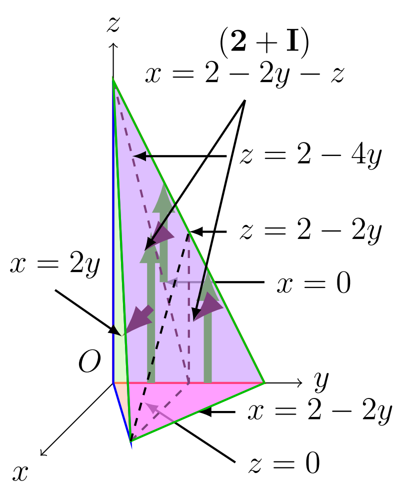



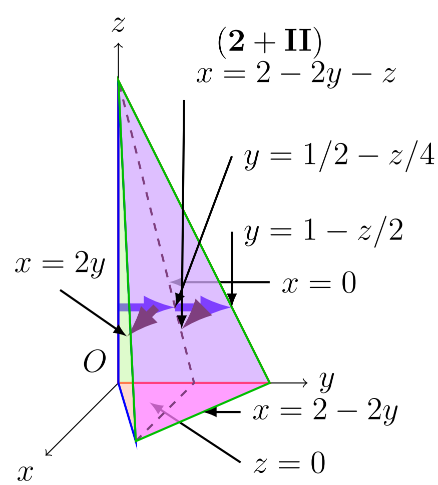



$$V(E) = \iiint\limits_E dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\ dy\ dx \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{1}+\mathbf{I})$$

$$= \int_0^{1/2} \int_0^{2y} \int_0^{2-x-2y} dz\ dx\ dy + \int_{1/2}^1 \int_0^{2-2y} \int_0^{2-x-2y} dz\ dx\ dy \ldots\ldots (\mathbf{1}+\mathbf{II})$$

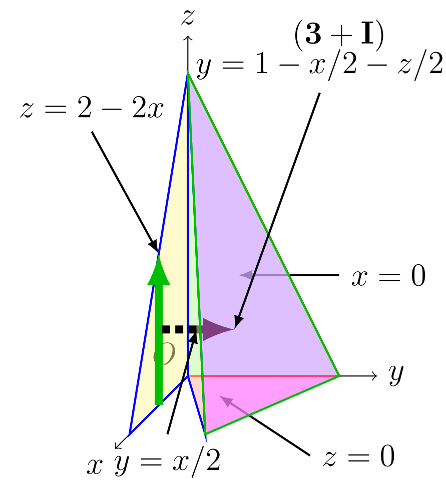


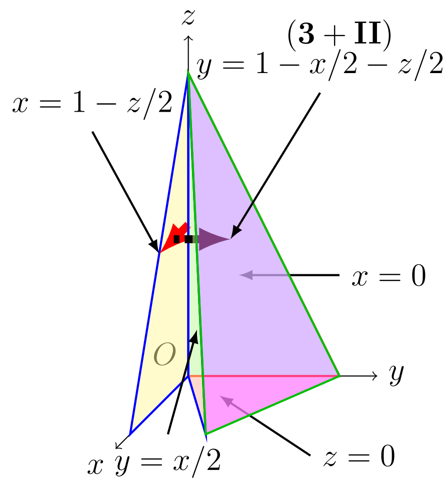




$$V(E) = \iiint\limits_E dV = \int_0^{1/2} \int_0^{2-4y} \int_0^{2y} dx\ dz\ dy$$

$$+ \int_0^{1/2} \int_{2-4y}^{2-2y} \int_0^{2-2y-z} dx\ dz\ dy + \int_{1/2}^1 \int_0^{2-2y} \int_0^{2-2y-z} dx\ dz\ dy \ldots\ldots (\mathbf{2}+\mathbf{I})$$

$$= \int_0^2 \int_0^{1/2-z/4} \int_0^{2y} dx\ dy\ dz + \int_0^2 \int_{1/2-z/4}^{1-z/2} \int_0^{2-2y-z} dx\ dy\ dz \ldots\ldots\ldots (\mathbf{2}+\mathbf{II})$$



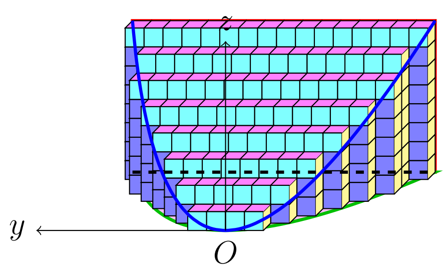



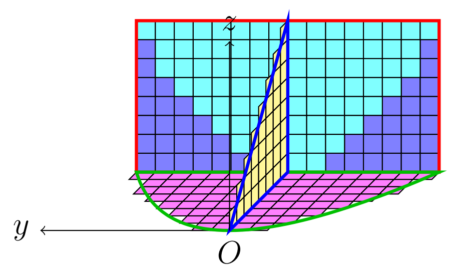

$$V(E) = \iiint\limits_E dV = \int_0^1 \int_0^{2-2x} \int_{x/2}^{1-x/2-z/2} dy\, dz\, dx \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{3+I})$$

$$= \int_0^2 \int_0^{1-z/2} \int_{x/2}^{1-x/2-z/2} dy\, dx\, dz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{3+II})$$

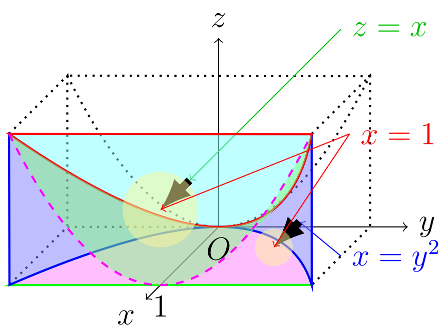
## ♦ Additional: Example 0.6 type 3

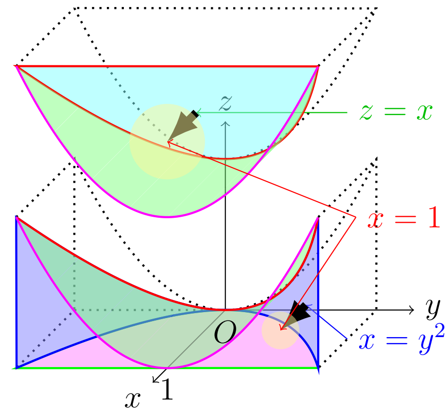
**Example 0.6** *Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the plane $z = 0$, $z = x$, and $x = 1$.*



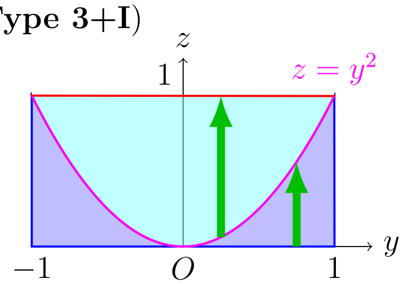



(**Type 3**)




(**Type 3+I**)

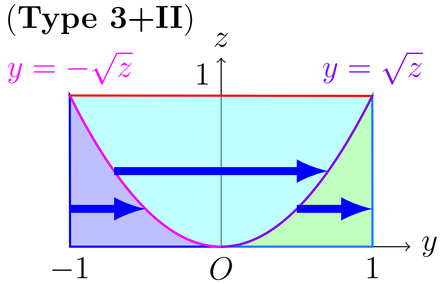



(上) $$\int_{-1}^{1}\int_{y^2}^{1}\int_{z}^{1} f(x,y,z)\, dx\, dz\, dy$$

$$+$$

(下) $$\int_{-1}^{1}\int_{0}^{y^2}\int_{y^2}^{1} f(x,y,z)\, dx\, dz\, dy$$

(**Type 3+II**)




(左) $$\int_{0}^{1}\int_{-1}^{-\sqrt{z}}\int_{y^2}^{1} f(x,y,z)\, dx\, dy\, dz$$

$$+$$

(中) $$\int_{0}^{1}\int_{-\sqrt{z}}^{\sqrt{z}}\int_{z}^{1} f(x,y,z)\, dx\, dy\, dz$$

$$+$$

(右) $$\int_{0}^{1}\int_{\sqrt{z}}^{1}\int_{y^2}^{1} f(x,y,z)\, dx\, dy\, dz$$