3.1 Derivatives of polynomials and exponential functions

1. derivative of polynomials 多項式函數的導函數
   $(c)' = 0$, $x' = 1$, $(x^n)' = nx^{n-1}$, $(cf)' = cf'$, $(f \pm g)' = f' \pm g'$.

2. derivative of exponential functions 指數函數的導函數
   $(e^x)' = e^x$, $(a^x)' = a^x \ln a$, $a > 0$.

3. normal line 法線 沒有白雪的痕跡, 也不隨時間退後。

Recall:

- Definition of limit:

  $\lim_{x \to a} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0, \ni\ 0 < |x-a| < \delta \implies |f(x)-L| < \varepsilon$.

- limit laws: 加減乘除常數倍, c & x, power 冪次 (^{n}), root 開根 ($\sqrt[n]{\ }, > 0$ when n even.)

- derivative: $\dfrac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ if the limit exists.

0.1 Derivative of polynomials

Polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, $a_i \in \mathbb{R}$, $a_n \neq 0$.
$f'(x) =$? 5 steps: c, x, x^n, cf, $f \pm g$.

Step 1. $\dfrac{d}{dx} c = 0$. 常數函數 (constant function) .... $\boxed{\boldsymbol{(c)' = 0}}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

Step 2. $\dfrac{d}{dx} x = 1$. 恆等函數 (identity function) .... $\boxed{\boldsymbol{(x)' = 1}}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} 1 = 1.$$

Step 3. $\frac{d}{dx}x^n = nx^{n-1}$. 冪次函數 (power function) .... $\boxed{\boldsymbol{(x^n)' = nx^{n-1}}}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n - x^n}{h}$$
$$= \lim_{h\to 0}\left[nx^{n-1} + h\underbrace{\left(\frac{n(n-1)}{2}x^{n-2} + \ldots + nxh^{n-3} + h^{n-2}\right)}_{(*)}\right]$$
$$= nx^{n-1}.$$

Note: 目前只證明 $n \in \mathbb{N}$, 實際上 $n \in \mathbb{R}$ 都對。(see §3.6 Power Rule).
negative integer: (Exercise 3.1.65)
$\frac{d}{dx}x^{-1} = \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} = -x^{-2} = -1x^{-1-1}$.
rational number: (§2.8 Example 0.3)
$\frac{d}{dx}x^{\frac{1}{2}} = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1}$.

Example 0.1 *(Exercise 3.2.64(c))* $\frac{d}{dx}x^{-n} = -nx^{-n-1}$.

$$(x^{-n})' = \lim_{h\to 0}\frac{(x+h)^{-n} - x^{-n}}{h} = \lim_{h\to 0}\frac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h}$$
$$= \lim_{h\to 0}\frac{-[(x+h)^n - x^n]}{hx^n(x+h)^n} \qquad \textit{(通分)}$$
$$= \lim_{h\to 0}\frac{-nx^{n-1}h + (*)h^2}{hx^n(x+h)^n} \qquad \textit{(乘開, $*$ 是個 x 與 h 的多項式)}$$
$$= \lim_{h\to 0}\left[\frac{-n}{x(x+h)^n} + h\frac{*}{x^n(x+h)^n}\right] \qquad \textit{($x \neq 0$, 極限存在)}$$
$$= \frac{-n}{x^{n+1}} = -nx^{-n-1}.$$
■

(已經證明 $n \in \mathbb{Z} \cup \{\frac{1}{2}\}$, $(x^n)' = nx^{n-1}$.)

Step 4. c is a constant, f is differentiable, (常數倍)

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x). \quad \cdots\cdots\cdots\cdots \quad \boxed{(\boldsymbol{cf})' = \boldsymbol{cf'}}$$

$$\lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0}\left(c\frac{f(x+h)-f(x)}{h}\right)$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x).$$ ($\because$ 極限常數倍 & f' 極限的存在。)

Step 5. f and g are differentiable, (加減)

$$\frac{d}{dx}[f(x)\pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \quad \cdots\cdots\cdots \quad \boxed{(\boldsymbol{f}\pm\boldsymbol{g})' = \boldsymbol{f'}\pm\boldsymbol{g'}}$$

$$\lim_{h\to 0}\frac{[f(x+h)\pm g(x+h)]-[f(x)\pm g(x)]}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} \pm \frac{g(x+h)-g(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$= f'(x)\pm g'(x)$. ($\because$ 極限加減 (或是 $f+(-1)g$) & f', g' 兩極限的存在。)

By **Steps 1$\sim$5**,

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \\ f'(x) &= \frac{d}{dx}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\right) \\ &= na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1 . \end{aligned}$$ (不要背!)

Example 0.2 $\dfrac{d}{dx}(x^8+12x^5-4x^4+10x^3-6x+5) =?$

$$\begin{aligned} &(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)' \\ &= 8x^{8-1} + 12\times 5x^{5-1} - 4\times 4x^{4-1} + 10\times 3x^{3-1} - 6\times 1 + 0 \\ &= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6 \end{aligned}$$

$8x^7+60x^4-16x^3+30x^2-6.$ ■

0.2 Derivative of exponential functions

$f(x) = a^x$, $a > 0$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h\to 0} \left(a^x \frac{a^h-1}{h}\right) = a^x \lim_{h\to 0} \frac{a^h-1}{h} = f(x)f'(0).$$

f 在 0 的切線斜率: $f'(0) = \lim_{h\to 0} \frac{a^h-1}{h} =?$ $(\ln a.)$

Case 1. $a = e$. **Recall:** e 是定義爲 a^x 在 0 切線斜率是 1 的底。

$$\therefore \boxed{\lim_{h\to 0} \frac{e^h-1}{h} = 1} \text{ and so } \frac{d}{dx}e^x = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x. \quad \ldots \boxed{\boldsymbol{(e^x)' = e^x}}$$

Case 2. $a \neq e$. (if $a = 1$ 是常數 $1^x = 1$, 所以考慮 $a \neq 1$。)

$$a^h = e^{\ln a^h} = (e^{\ln a})^h = e^{h\ln a},$$

$$\lim_{h\to 0} \frac{a^h-1}{h} = \lim_{h\to 0} \frac{e^{h\ln a}-1}{h} \qquad \text{(把底從 } a \text{ 換成 } e\text{)}$$

$$= \lim_{h\to 0} (\ln a \frac{e^{h\ln a}-1}{h\ln a}) \qquad \text{(把分母從 } h \text{ 換成 } h\ln a \text{ 跟 } e \text{ 的指數一致)}$$

$$= \ln a \lim_{h\ln a\to 0} \frac{e^{h\ln a}-1}{h\ln a} = \ln a \lim_{t\to 0} \frac{e^t-1}{t} = \ln a,$$

(令 $t = h\ln a$, 則 $h \to 0 \iff t = h\ln a \to 0$.)

$$\therefore \frac{d}{dx}a^x = a^x \ln a,\ a > 0. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \boxed{\boldsymbol{(a^x)' = a^x \ln a}}$$

When $a = e$, $\ln e = 1$, $(e^x)' = e^x = e^x \ln e$;
when $a = 1$, $\ln 1 = 0$, $(1^x)' = 0 = 1^x \ln 1$. 公式都是對的

Example 0.3 *Find the equation of the tangent line of $y = 2^x$ at $x = 2$.*

Let $f(x) = 2^x$, then $f'(x) = 2^x \ln 2$. ($\ln 2 \approx 0.693$)
切線: $y = f'(2)(x-2) + f(2) = 2^2 \ln 2(x-2) + 2^2 = 4\ln 2(x-2) + 4.$ ■

Attention: x^n 是冪次函數, 導數是 nx^{n-1};
a^x 是指數函數, 導數是 $a^x \ln a$, $\boxed{\text{不是}\cancel{xa^{x-1}}!}$ $\boxed{\text{不是}\cancel{xa^{x-1}}!}$ $\boxed{\text{不是}\cancel{xa^{x-1}}!}$

0.3 Normal line

Recall: $y=f(x)$ 在 a 可微分, 在 $(a,f(a))$ (或 $x=a$) 的 ***tangent line*** 切線

$$y=f'(a)(x-a)+f(a).$$

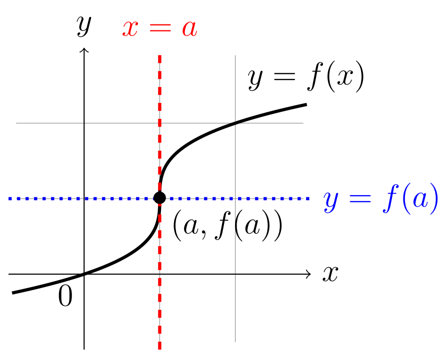
Define: 跟切線垂直在 $(a,f(a))$ 的線叫 ***normal line*** 法線 (垂線)

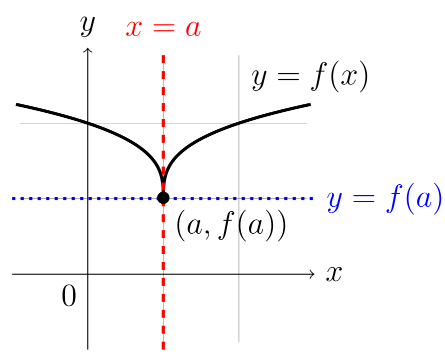
$$\boxed{\boldsymbol{y=\frac{-1}{f'(a)}(x-a)+f(a)}},$$

if $f'(a)$ exists and $f'(a)\neq 0$. (The slope of the normal line is the negative reciprocal 負倒數 of the slope of the tangent line.)

Note: 兩線垂直 $\iff$ 斜率乘積 $=-1$。

Note: 如果 f 在 a 不可微分, 但是 f 在 a 連續, 而且 $\lim\limits_{x\to a^{\pm}}|f'(x)|=\infty$, 則 f 在 $(a,f(a))$ 有垂直切線 (vertical tangent line) $x=a$ 與法線 $y=f(a)$。





Note: 如果 $f'(a)=0$, 切線是水平的 $y=f(a)$, 而法線就是 $x=a$。