4.3 How derivatives affect the shape of a graph

微分應用之四: 分析函數圖形。 (★ 授課順序與 §4.4 調換。)

1. the first derivative 一階導數 f'
2. the second derivative 二階導數 f''

0.1 The first derivative

Increasing/Decreasing Test: 增減測試

(a) $f' > 0 \implies f$ is increasing 遞增。(有些書上用 $\nearrow$ 符號表示。)

(b) $f' < 0 \implies f$ is decreasing 遞減。(有些書上用 $\searrow$ 符號表示。)

Proof. (a) ((b) is similar) Say $f'(x) > 0$ on (a, b). $\forall\ x_1, x_2 \in (a, b)$ with $x_1 < x_2$, f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), by the Mean Value Theorem, $\exists\ c \in (x_1, x_2)$, $\ni f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$, $f(x_2) > f(x_1)$. Therefore, f is increasing on (a, b). ■

Note: 遞增/遞減 的區間通常回答開區間 ((a, b)不含端點), 除非另外檢查。

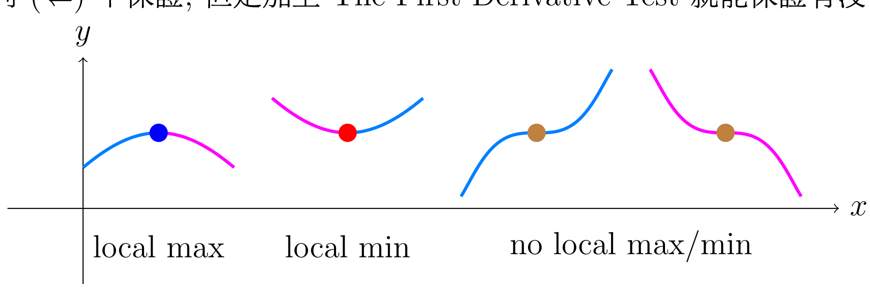
The First Derivative Test: 一階導數測試
c is a ***critical*** number of a ***continuous*** function f

(a) f' changes from positive to negative at $c \implies f$ has local max at c.

(b) f' changes from negative to positive at $c \implies f$ has local min at c.

(c) f' does **NOT** change sign at $c \implies f$ has **NO** local max/min at c.

Recall: critical number c: $f'(c) = 0$ or $f'(c)$ 不存在。
Recall: f has local max/min at $c \implies c$ is a critical number.
反向 ($\Leftarrow$) 不保證, 但是加上 The First Derivative Test 就能保證有沒有。



Example 0.1 *Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing, and its extreme values.*

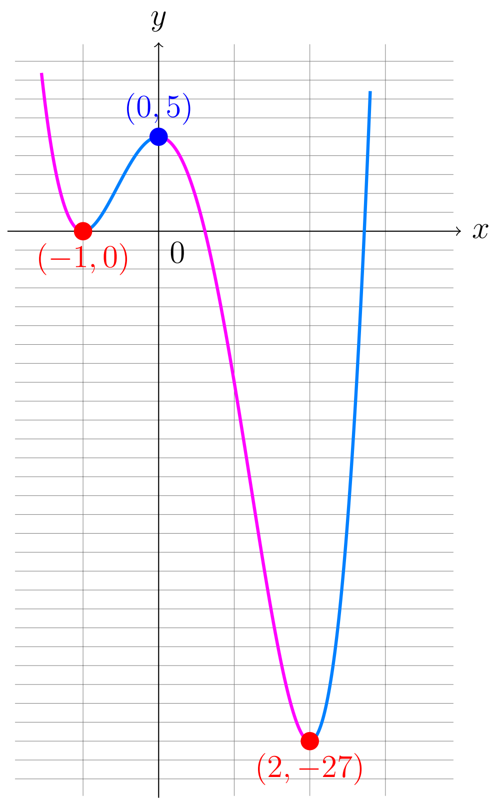
(找臨界值分段) $f'(x) = 12x(x-2)(x+1)$, $f'(x) = 0$ *when* $x = -1, 0, 2$.
(計算 f' *正負判斷增減)*

Interval	$f'(x)$	$f(x)$
$x < -1$	$(- * - * - =)-$	*decreasing on* $(-\infty, -1)$
$-1 < x < 0$	$(- * - * + =)+$	*increasing on* $(-1, 0)$
$0 < x < 2$	$(+ * - * + =)-$	*decreasing on* $(0, 2)$
$x > 2$	$(+ * + * + =)+$	*increasing on* $(2, \infty)$

(判斷極值)

abs. max	*no*
abs. min	$f(2) = -27$
local max	$f(0) = 5$
local min	$f(-1) = 0$ *and* $f(2) = -27$

■



Example 0.2 *Find where $g(x) = x + 2\sin x$, $0 \le x \le 2\pi$ is increasing and decreasing, and its extreme values.*

$g'(x) = 1 + 2\cos x$, $g'(x) = 0$ *when* $x = \frac{2}{3}\pi, \frac{4}{3}\pi$.

Interval	$g'(x)$	$g(x)$
$0 < x < \frac{2}{3}\pi$	$+$	*increasing on* $(0, \frac{2}{3}\pi)$
$\frac{2}{3}\pi < x < \frac{4}{3}\pi$	$-$	*decreasing on* $(\frac{2}{3}\pi, \frac{4}{3}\pi)$
$\frac{4}{3}\pi < x < 2\pi$	$+$	*increasing on* $(\frac{4}{3}\pi, 2\pi)$

(代好算的數字判斷正負)

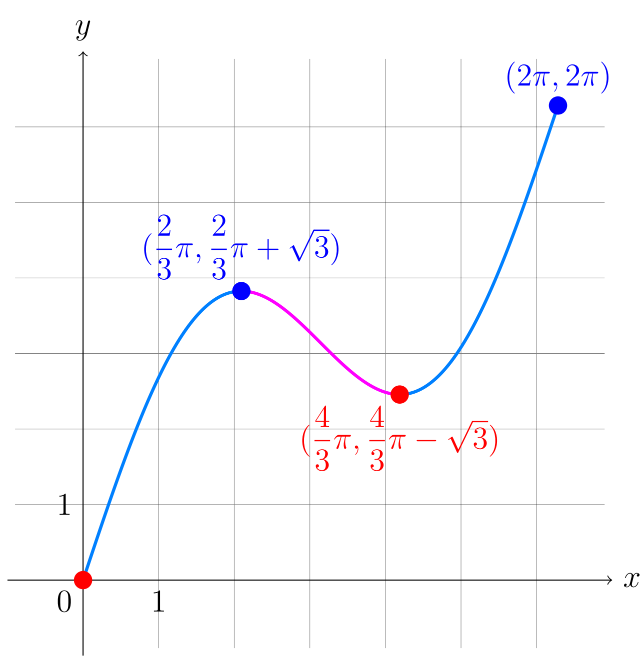
$g'(\frac{\pi}{2}) = 1 + 2(0) = 1$

$g'(\pi) = 1 + 2(-1) = -1$

$g'(\frac{3\pi}{2}) = 1 + 2(0) = 1$

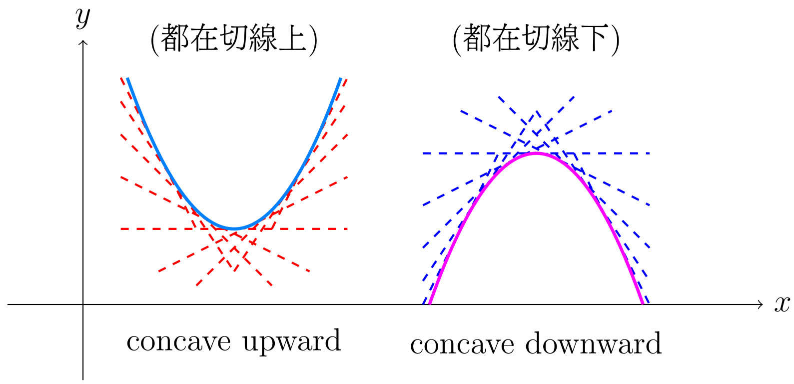
abs. max	$g(2\pi) = 2\pi \approx 6.28$
abs. min	$g(0) = 0$
local max	$g(\frac{2}{3}\pi) = \frac{2}{3}\pi + \sqrt{3} \approx 3.83$
local min	$g(\frac{4}{3}\pi) = \frac{4}{3}\pi - \sqrt{3} \approx 2.46$

$\blacksquare$



0.2 The second derivative

Define:
f is ***concave upward*** on an interval I if f lies above all its tangent lines.
f is ***concave downward*** on an interval I if f lies below all its tangent lines.



Concavity Test: 凹性測試

(a)	$f'' > 0 \implies f$ concave upward 凹向上。
(b)	$f'' < 0 \implies f$ concave downward 凹向下。

Note: 凹向上/下減 的區間通常回答開區間 ((a, b)不含端點)。

Define: $(p, f(p))$ is an ***inflection point*** 反曲點 if f ***continuous*** at p and f changes from CU to CD or from CD to CU at $(p, f(p))$. (連續換凹)

Note: 反曲點要連續。Ex: $f(x) = \dfrac{1}{x}$ at 0.

Note: 反曲點要用點座標寫 $(\cdots, \cdots)$。

The Second Derivative Test 二階導數測試
f'' is ***continuous*** near c

(a)	$f'(c) = 0$ and $f''(c) > 0 \implies f$ has local min at c.
(b)	$f'(c) = 0$ and $f''(c) < 0 \implies f$ has local max at c.

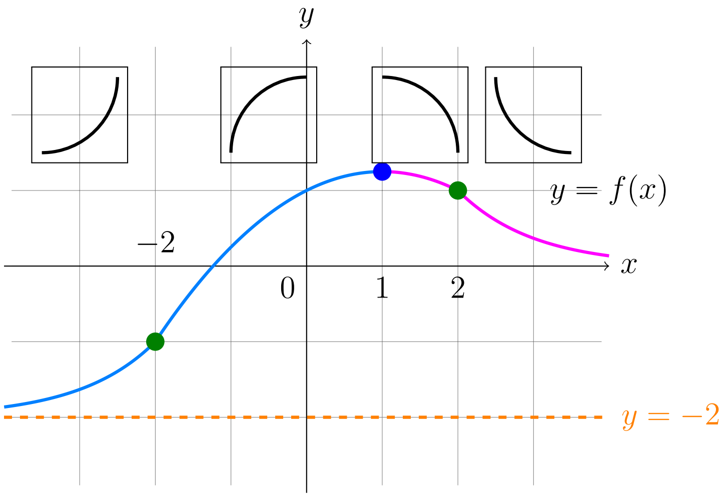
Note: 二階導數測試只能針對 $f'(c) = 0$ 的臨界值, $f'(c)$ 不存在的不能用。

Example 0.3 *Sketch f:*

1. $f'(x) > 0$ *on* $(-\infty, 1)$, $f'(x) < 0$ *on* $(1, \infty)$.

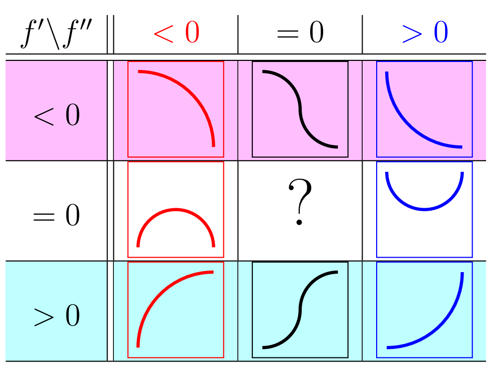
2. $f''(x) > 0$ *on* $(-\infty, -2)$ *and* $(2, \infty)$, $f''(x) < 0$ *on* $(-2, 2)$.

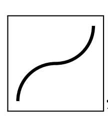
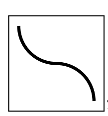
3. $\lim_{x\to-\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 0$.

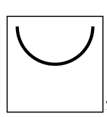
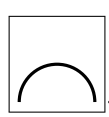


(Think yourself: $f'(1) = 0$ *or does not exist? and* $f''(\pm 2)$*?)*

◆ **Additional:**



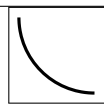
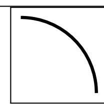
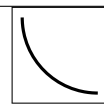
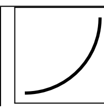
?: 可能有反曲無極值: x^3 , $-x^3$ ,

可能有極值無反曲: x^4 , $-x^4$ .

(二階導數測不到, 要用一階測。)

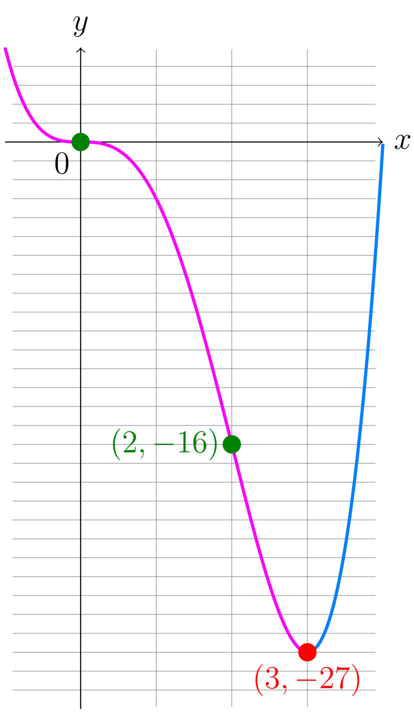
Example 0.4 *Discuss $y = x^4 - 4x^3$ w.r.t. concavity, inflection point, local max/min and Sketch y.*

Let $f(x) = x^4 - 4x^3$.
$f'(x) = 4x^2(x-3)$, $f'(x) = 0$ *when* $x = 0, 3$.
$f''(x) = 12x(x-2)$, $f''(x) = 0$ *when* $x = 0, 2$.

	<0	0	$0<x<2$	2	$2<x<3$	3	$3<$
f'	$-$	0	$-$			0	$+$
f''	$+$	0	$-$	0	$+$		
		*IP*		*IP*		*min*	

	$f(x)$
local max	*no*
local min	$f(3) = -27$
CU	$(-\infty, 0)$ *and* $(2, \infty)$
CD	$(0, 2)$
IP	$(0, 0)$ *and* $(2, -16)$

■



Example 0.5 *Sketch* $f(x) = x^{2/3}(6-x)^{1/3}$.

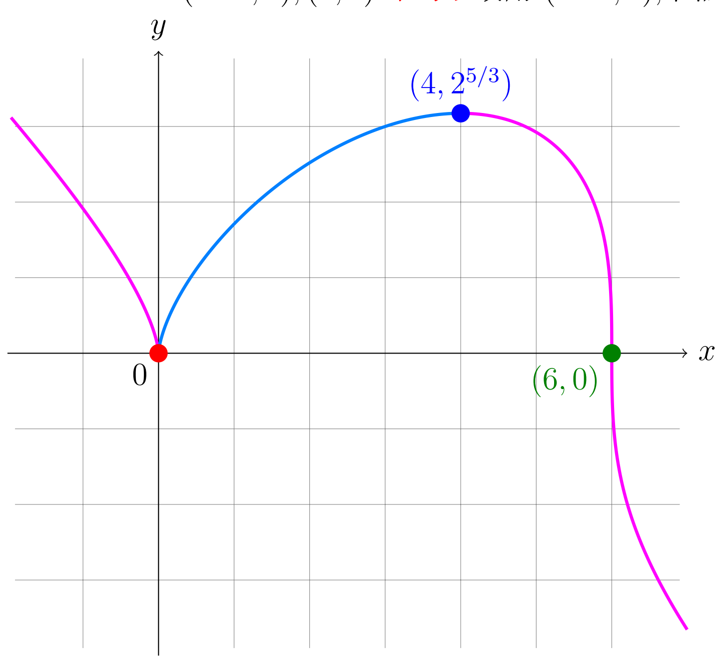
$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}},$$
$$f'(x) = 0 \text{ when } x = 4,\ f'(x) \text{ does not exist when } x = 0, 6.$$
$$f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}},\ f''(x) \text{ does not exist when } x = 0, 6.$$

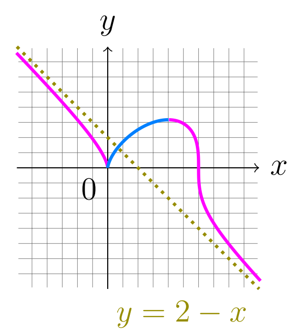
	<0	0	$0<x<4$	4	$4<x<6$	6	$6<$
f'	$-$	$\nexists$	$+$	0	$-$	$\nexists$	$-$
f''	$-$	$\nexists$		$-$		$\nexists$	$+$
		min		*max*		*IP*	

	$f(x)$
local max	$f(4) = 2^{5/3}$
local min	$f(0) = 0$
CU	$(6, \infty)$
CD	$(-\infty, 0), (0, 6)$
IP	$(6, 0)$

■

Attention: $(-\infty, 0), (0, 6)$ 不可以改用 $(-\infty, 6)$, 因爲 $x = 0$ 時不對。





Additional: $y = x^{2/3}(6-x)^{1/3}$ 有斜漸近線 (slant asymptote), 你會找嗎?

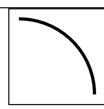
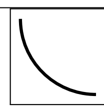
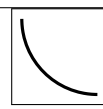
Example 0.6 *(Need asymptote.) Sketch $f(x) = e^{1/x}$ with asymptote.*

$$f'(x) = -\frac{e^{1/x}}{x^2} < 0 \text{ for } x \neq 0,\ f'(x) \text{ does not exist when } x = 0.$$

$$f''(x) = \frac{e^{1/x}(2x+1)}{x^4},\ f'' = 0 \text{ when } x = -\frac{1}{2}, \text{ does not exist when } x = 0.$$

$$\lim_{x\to 0^+} f(x) = \infty,\ \lim_{x\to 0^-} f(x) = 0,\ \Longrightarrow\ v.a.\ x = 0;$$

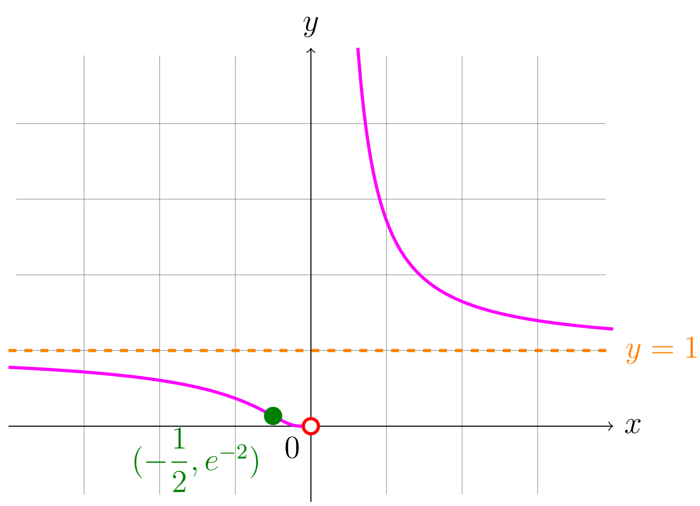
$$\lim_{x\to \infty} f(x) = 1,\ \lim_{x\to -\infty} f(x) = 1,\ \Longrightarrow\ h.a.\ y = 1.$$

	$< -\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2} < x < 0$	0	$0 <$
f'	$-$			$\nexists$	$-$
f''	$-$	0	$+$	$\nexists$	$+$
		*IP*		*no*	

*(0 不在 domain,
不是臨界值,
沒有極值,
也沒有反曲點。)*

	f(x)
local max	*no*
local min	*no*
CU	$(-\frac{1}{2}, 0) \cup (0, \infty)$
CD	$(-\infty, -\frac{1}{2})$
IP	$(-\frac{1}{2}, e^{-2})$

($\cup$: union 聯集。) ■



Note: 沒有漸進線, 不知道怎麼畫。