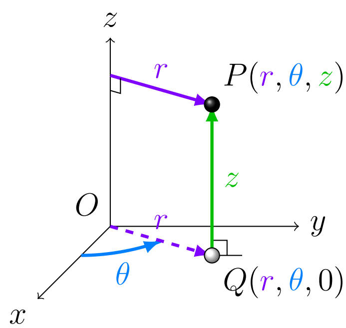
15.7 Triple integrals in cylindrical coordinates

1. cylindrical coordinates
2. triple integrals in cylindrical coordinates

$$x \to r\cos\theta,\ y \to r\sin\theta,\ dV \to r\ dz\ dr\ d\theta.$$

0.1 Cylindrical coordinates

Cylindrical ([sɪˋlɪndrɪkl]) ***coordinate system*** 圓柱坐標系: $P(r, \theta, z)$.



r: P 到 z-軸的距離;
θ: OP 在 xy-平面的投影 OQ 與正 x-軸夾角;
z: P 到 xy-平面的距離。

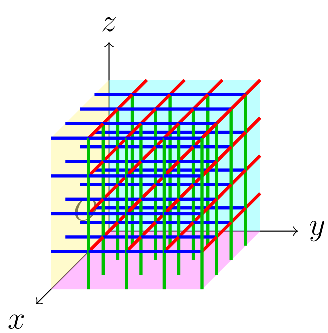
Note: 圓柱座標就是極座標加 z-軸, (與極座標一樣) 表示法不是唯一:

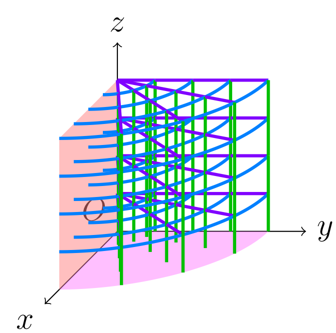
$$(r, \theta, z) = (r, \theta + 2\pi, z) = (-r, \theta + \pi, z).$$

座標變換: cylindrical coordinate $\longleftrightarrow$ Cartesian (rectangular) coordinate:

$$\boxed{\boldsymbol{x = r\cos\theta \quad y = r\sin\theta \quad z = z}}$$

$$\boxed{x^2 + y^2 = r^2 \quad \tan\theta = \frac{y}{x} \quad z = z}$$





Example 0.1 *(a) Plot the point with cylindrical coordinates* $(2, \frac{2\pi}{3}, 1)$ *and find its rectangular coordinates.*
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

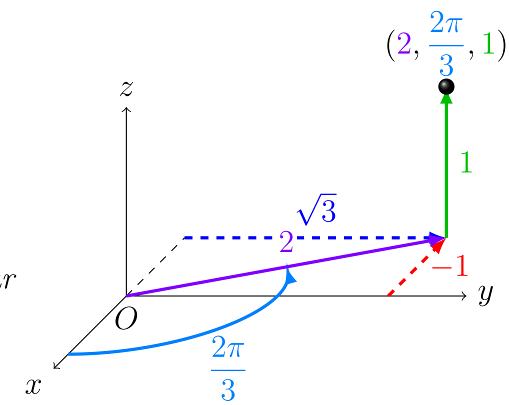
(a)

$x = r\cos\theta = 2\cos\frac{2\pi}{3} = 2(-\frac{1}{2}) = -1;$

$y = r\sin\theta = 2\sin\frac{2\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3};$

$z = 1.$

The point is $(-1, \sqrt{3}, 1)$ *in rectangular coordinates.*



(b)

$r^2 = x^2 + y^2 = 3^2 + (-3)^2 = 18,$

$r = \pm 3\sqrt{2};$

$\tan\theta = \frac{y}{x} = \frac{-3}{3} = -1,$

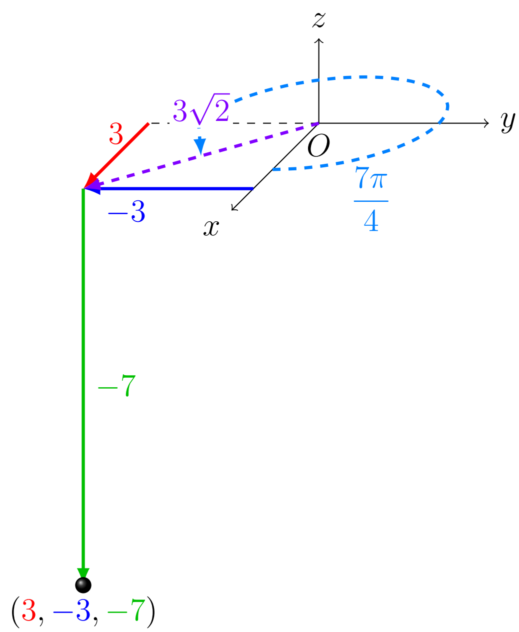
$\theta = \left(2n + \frac{7}{4}\right)\pi$ *or* $\left(2n + \frac{3}{4}\right)\pi,$

$n \in \mathbb{Z};$

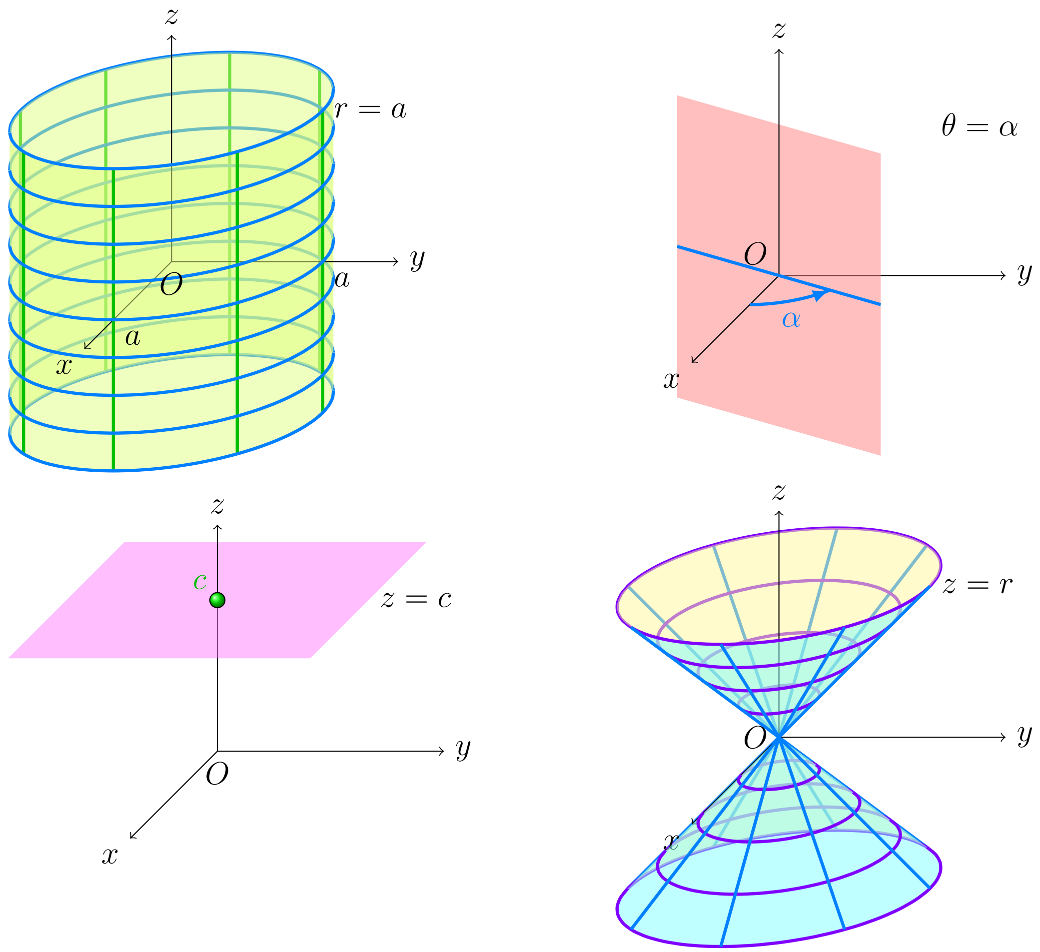
(當 $r > 0$, $(x, y) = (r\cos\theta, r\sin\theta)$, 前者 $(+, -)$, 後者 $(-, +)$.)

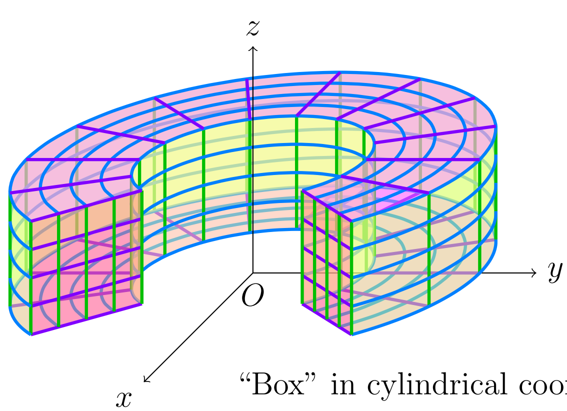
$z = -7.$

The point is $\left(3\sqrt{2}, \left(2n + \frac{7}{4}\right)\pi, -7\right)$ *or* $\left(-3\sqrt{2}, \left(2n+\frac{3}{4}\right)\pi, -7\right)$ *in cylindrical coordinates.* ■



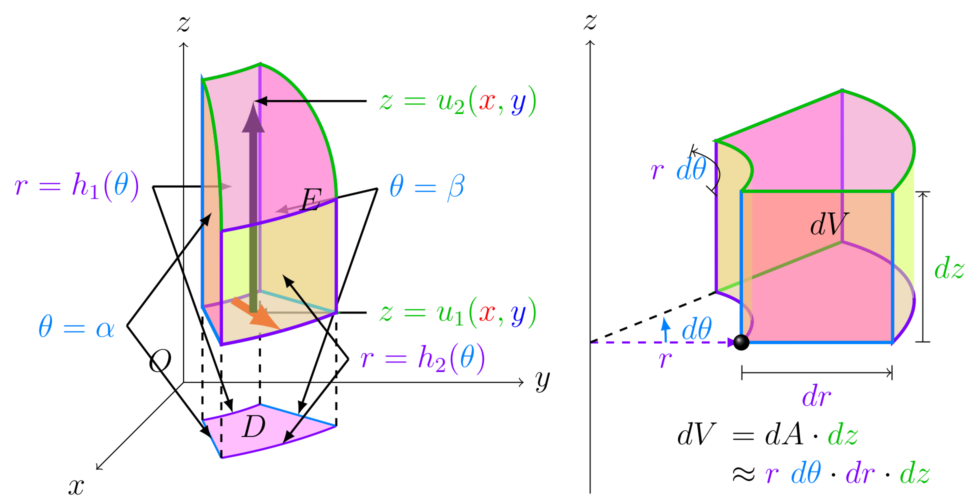
Example 0.2 *Graphs: (a) $r = a$; (b) $\theta = \alpha$; (c) $z = c$; (d) $z = r$.*





"Box" in cylindrical coordinates, baumkuchen[German].

0.2 Triple integrals in cylindrical coordinates



$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$,
$D = \{(r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$.

$$\boxed{\begin{aligned}&\iiint\limits_E f(x,y,z)\ dV \\ &= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\cdot r\ dz\ dr\ d\theta\end{aligned}}$$

Note: $x \to r\cos\theta$, $y \to r\sin\theta$, z 不變,

$$\bigstar\boxed{dV \to \boldsymbol{r}\ dz\ dr\ d\theta}\bigstar$$

Note: If $E =$“box” and $f(r\cos\theta, r\sin\theta, z)\cdot r = g(\theta)h(r)u(z)$, 可以分開

$$\int_\alpha^\beta \int_a^b \int_c^d f(r\cos\theta, r\sin\theta, z)\cdot r\ dz\ dr\ d\theta = \int_\alpha^\beta g(\theta)\ d\theta \int_a^b h(r)\ dr \int_c^d u(z)\ dz$$

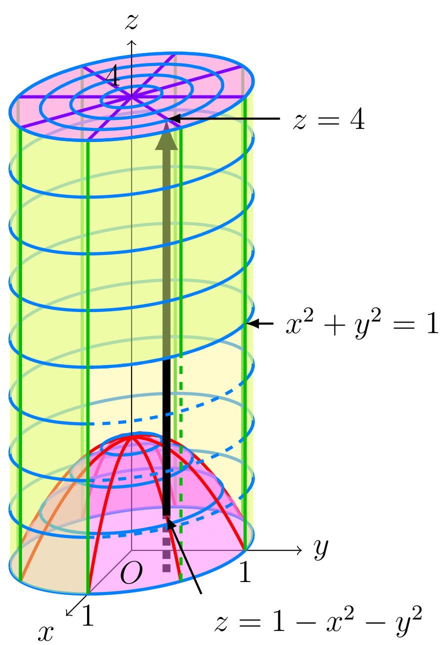
Note: If 偏積完 z 之後, $D =$“polar rectangle” and

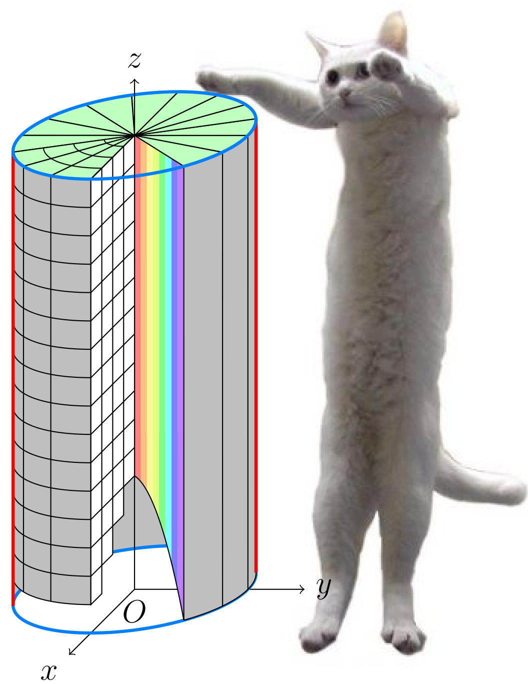
$$\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\cdot r\ dz = g(\theta)h(r),$$ 可以分開

$$\int_\alpha^\beta \int_a^b \int_c^d f(r\cos\theta, r\sin\theta, z)\cdot r\ dz\ dr\ d\theta = \int_\alpha^\beta g(\theta)\ d\theta \int_a^b h(r)\ dr$$

Timing: 函數有 x^2+y^2, y^2+z^2, x^2+z^2, 或是投影是極矩形。

Example 0.3 *A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E. (密度與點到圓柱軸的距離成正比。求 E 質量。)*

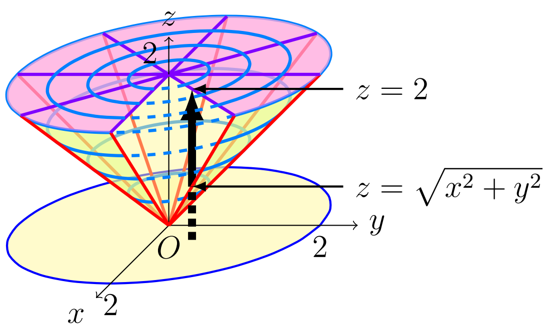


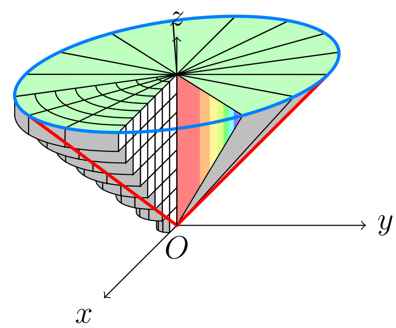


$E=\{(r,\theta,z): 0\le\theta\le 2\pi,\ 0\le r\le 1,\ 1-r^2\le z\le 4\}$.
密度與到 z 軸距離成正比: $f(x,y,z)=K\sqrt{x^2+y^2}=Kr$.

$$\begin{aligned}
m &= \iiint\limits_E f(x,y,z)\,dV = \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 Kr\cdot r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \Big[Kr^2z\Big]_{z=1-r^2}^{z=4}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 Kr^2[4-(1-r^2)]\,dr\,d\theta \\
&= K\int_0^{2\pi}d\theta\int_0^1\left(3r^2+r^4\right)dr \qquad \text{(可以分開)} \\
&= K\Big[\theta\Big]_0^{2\pi}\left[r^3+\frac{r^5}{5}\right]_0^1 = K\cdot 2\pi\cdot\frac{6}{5} = \frac{12\pi K}{5}. \qquad \blacksquare
\end{aligned}$$

Example 0.4 *Evaluate* $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\ dz\ dy\ dx.$





$$E = \{(x,y,z) : -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ \sqrt{x^2+y^2} \le z \le 2\}$$
$$= \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ r \le z \le 2\}.$$

$$\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\ dz\ dy\ dx \ &= \ \iiint_E (x^2+y^2)\ dV \\
&= \ \int_0^{2\pi}\int_0^2\int_r^2 r^2\cdot r\ dz\ dr\ d\theta \ = \ \int_0^{2\pi}\int_0^2 \Big[r^3 z\Big]_{z=r}^{z=2}\ dr\ d\theta \\
&= \ \int_0^{2\pi} d\theta \int_0^2 \left(2r^3 - r^4\right)\ dr \qquad (\text{可以分開}) \\
&= \ \Big[\theta\Big]_0^{2\pi}\left[\frac{r^4}{2} - \frac{r^5}{5}\right]_0^2 \ = \ 2\pi\cdot\left(8 - \frac{32}{5}\right) \ = \ \frac{16\pi}{5}.
\end{aligned}$$
■

Note: 圓柱座標系迭代積分 $\displaystyle\int_0^{2\pi}\int_0^2\int_r^2 r^2\cdot r\ dz\ dr\ d\theta$ 的過程:

$$\int * \ dz \to \int * \ dr \to \int * \ d\theta$$