15.9 Change of variables in multiple integrals

0.1 Transformation

Recall: Substitution: Let $x = x(u) = g(u)$, $dx = g'(u)\ du$,

$$\begin{aligned}\int_{a=g(c)}^{b=g(d)} f(x)\ dx &= \int_c^d f(g(u))\color{red}{g'(u)}\ du \qquad \text{(通常都是由右變成左)}\\ \int_a^b f(x)\ dx &= \int_c^d f(x(u))\color{red}{\frac{dx}{du}}\ du\end{aligned}$$

Polar: Let $x = x(r,\theta) = r\cos\theta$, $y = y(r,\theta) = r\sin\theta$, $dA = r\ dr\ d\theta$,

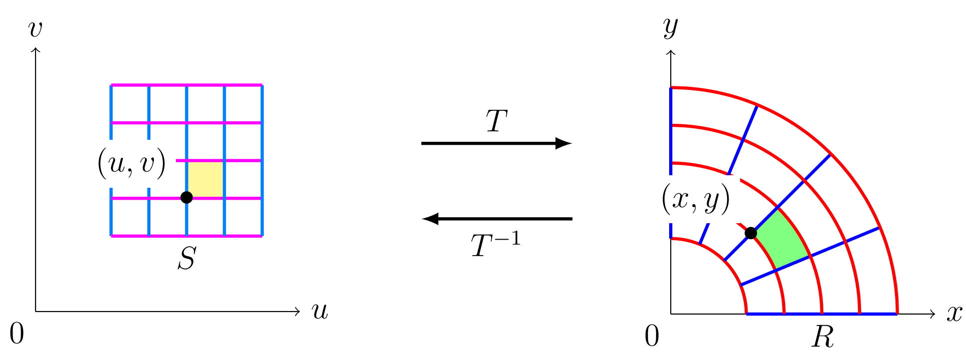
$$\iint\limits_R f(x,y)\ dA = \iint\limits_S f(r\cos\theta, r\sin\theta)\cdot \color{red}{r}\ dr\ d\theta \qquad \text{(極/圓柱座標)}$$

Define: A ***transformation*** 變換 T from uv-plane to xy-plane:

$$T(u,v) = (x,y), \quad x = x(u,v) = g(u,v), \quad y = y(u,v) = h(u,v)$$

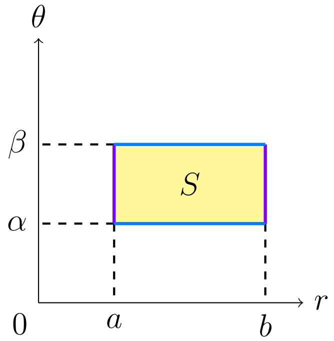
T is called a $\boldsymbol{C^1}$ transformation if g and h have continuous first-order partial derivatives, and is called ***one-to-one*** if no two points (u,v) have the same image $T(u,v) = (x,y)$. If T is one-to-one, then it has an ***inverse transformation*** T^{-1} from xy-plane to uv-plane:

$$T^{-1}(x,y) = (u,v), \quad u = u(x,y) = G(x,y), \quad v = v(x,y) = H(x,y)$$

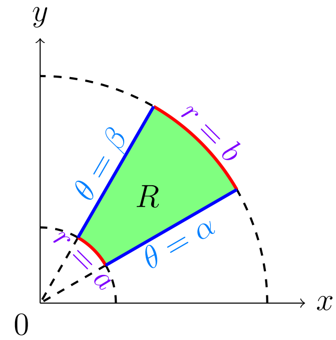


♦ **Extra:** $\boldsymbol{C^k}$ = has continuous k-th derivatives, $\boldsymbol{C^0}$ = continuous.
ex: $x^{k+1}\sin\dfrac{1}{x}$ is C^k but not C^{k+1}; e^x, $\sin x$, $\cos x$, and polynomials are $\boldsymbol{C^\infty}$.

Question: What is a transformation? 換座標就是一種變換。



$$\xrightarrow{\quad T \quad} \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$



$$T(r,\theta) = (x,y) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta).$$

$$\begin{array}{cccc}
T: & r\theta\text{-plane} & \longrightarrow & xy\text{-plane} \\
(\text{點}) & (r,\theta) & \mapsto & (x,y) \\
(\text{線}) & r = a & \mapsto & x^2+y^2=a^2 \\
 & \theta = \alpha & \mapsto & y = \tan\alpha \cdot x \\
(\text{區域}) & S & \mapsto & R \\
 & \displaystyle\iint_S g(r,\theta)\ dA & \mapsto & \displaystyle\iint_R f(x,y)\ dA
\end{array}$$

Question: Why transformation? Substitution, 換個好算的積分。

$$\iint_R f(x,y)\ dA \;\underset{T}{\overset{T^{-1}}{\rightleftarrows}}\; \iint_S g(r,\theta)\ dA$$

$$\int_?^? \int_?^? f(x,y)\ dx\ dy \quad = \quad \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\cdot r\ dr\ d\theta$$

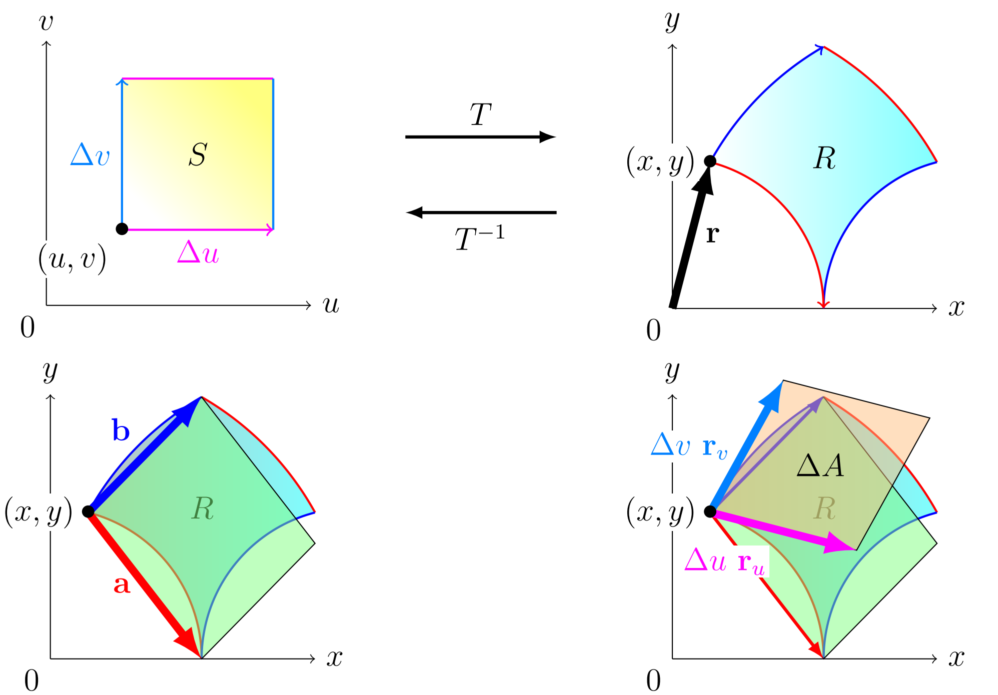
Question: 多乘的 r 哪來的? 面積的變換率。極矩形的面積:

$$\begin{array}{ccccc}
\Delta x\Delta y & = & \Delta A & \approx & r\Delta\theta\cdot\Delta r \\
\downarrow & & \downarrow & & \downarrow \\
dx\ dy & = & dA & = & r\ dr\ d\theta
\end{array}$$

Question: Transformation 怎麼算變換率? Jacobian 雅可比。

♦: Jacobian [dʒɑˋkobiən], 字源於德國數學家卡爾・古斯塔夫・雅各・雅可比 (Carl Gustav Jacob Jacobi) [jɑˋkobi]。

0.2 Jacobian (2D) — 面積的變換率



Let $\mathbf{r}(u,v) = g(u,v)\mathbf{i} + h(u,v)\mathbf{j}$ be the position vector of $(x,y) = T(u,v)$. The area $A(T(S)) = \boxed{A(R)} \approx \boxed{|\mathbf{a}\times\mathbf{b}|}$.

$\mathbf{a} = \mathbf{r}(u+\Delta u, v) - \mathbf{r}(u,v) \approx \Delta u\ \mathbf{r}_u, \qquad \mathbf{b} = \mathbf{r}(u, v+\Delta v) - \mathbf{r}(u,v) \approx \Delta v\ \mathbf{r}_v.$

$$\boxed{A(R)} \approx \boxed{|\mathbf{a}\times\mathbf{b}|} \approx \boxed{|(\Delta u\ \mathbf{r}_u)\times(\Delta v\ \mathbf{r}_v)|} = |\mathbf{r}_u \times \mathbf{r}_v|\Delta u\Delta v$$

$$\begin{pmatrix} \mathbf{r}_u = x_u\mathbf{i} + y_u\mathbf{j}, \\ \mathbf{r}_v = x_v\mathbf{i} + y_v\mathbf{j}. \end{pmatrix} = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} \right| \Delta u\Delta v = \left| \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} \right| \Delta u\Delta v$$

$$= \left| \begin{vmatrix} \dfrac{\partial x}{\partial u} & \boxed{\dfrac{\partial x}{\partial v}} \\ \boxed{\dfrac{\partial y}{\partial u}} & \dfrac{\partial y}{\partial v} \end{vmatrix} \right| \boxed{A(S)}. \qquad (\boxed{\cdot}\ \text{換了順序})$$

Note: 裡面的 $|\cdot|$ 是行列式, 外面的 $|\cdot|$ 是絕對值。

Define: The ***Jacobian (determinant)*** 雅可比 (行列式) of a transformation $T(u,v)=(x,y)$ is

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

$$\begin{aligned}\Longrightarrow\ \Delta A(=\Delta x\Delta y) &\approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v\\ \iint_R f(x,y)\,dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta A\\ &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i,v_j),h(u_i,v_j))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v\\ &\approx \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv\end{aligned}$$

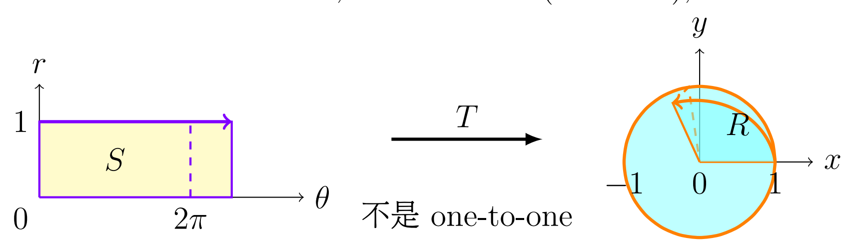
Theorem 1 *Suppose that T is a C^1 transformation whose* **Jacobian is nonzero** *and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose f is continuous on R and that R and S are type I or type II plane region. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then*

$$\boxed{\iint_R f(x,y)\ dA=\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv}$$

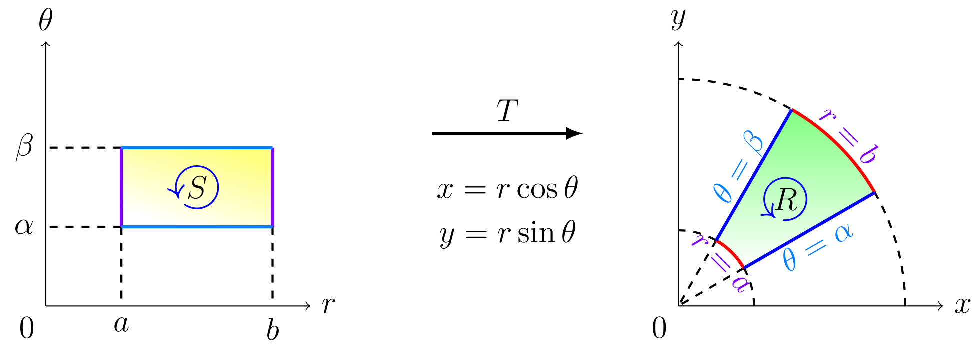
Note: 1. $x\to g(u,v)$, $y\to h(u,v)$, $dA\to\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$

2. Jacobian 是個行列式, 有可能是負的。要乘的是 Jacobian 的絕對值。

3. R and S 要能 one-to-one, 就是不能重疊 (除了邊界), 例如:



Example 0.1 *Double Integrals in Polar Coordinates.*



$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r > 0$$

(except perhaps on the boundary: $r = 0$).

$$\begin{aligned} \iint\limits_R f(x,y)\ dx\ dy &= \iint\limits_S f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\ dr\ d\theta \\ &= \iint\limits_S f(r\cos\theta, r\sin\theta)\cdot \boxed{r}\ dr\ d\theta. \end{aligned}$$ ■

♦ **Question:** Why and when Jacobian 有正有負?
變換前 (S) 後 (R) 邊界繞的方向是否一致:

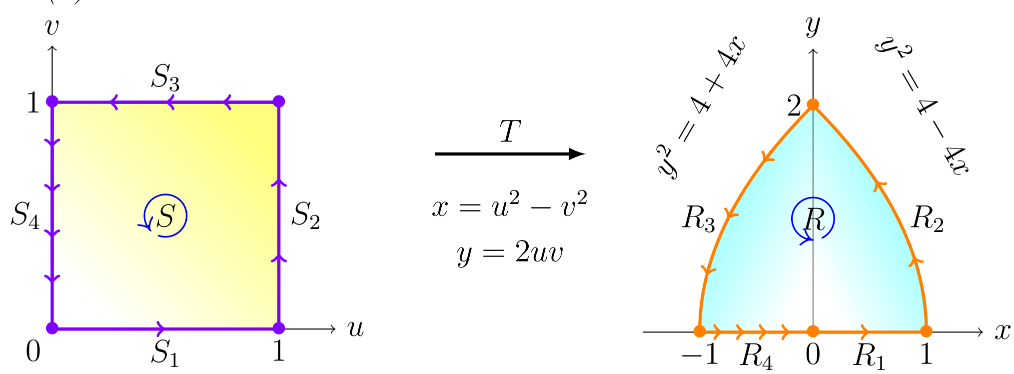
$$\circlearrowleft \xrightarrow{+} \circlearrowleft, \quad \circlearrowright \xrightarrow{+} \circlearrowright, \quad \circlearrowleft \xleftrightarrow{-} \circlearrowright.$$

Skill: $\displaystyle\iint\limits_R f(x,y)\ dA = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$:
Step 1. 找 u, v; Step 2. 逆算 x, y; Step 3. Jacobian.

Question: 怎麼找 u and v? 由 R 的邊界或是要積分的函數 f:
Ex: R 由 $G(x,y) = a$, $G(x,y) = b$, $H(x,y) = c$, and $H(x,y) = d$ 包圍,
或是 $f(x,y) = F(G(x,y), H(x,y))$ 且 F 長得比 f 簡單.
$\Longrightarrow$ Let $u = G(x,y)$ and $v = H(x,y)$, 逆算 $x = g(u,v)$ and $y = h(u,v)$.

Example 0.2 *(a) A transformation is defined by the equations* $x = u^2 - v^2$ $y = 2uv$. *Find the image of the square* $S = \{(u, v) : 0 \le u \le 1, 0 \le v \le 1\}$.
(b) Use (a) to evaluate $\displaystyle\iint_R y\, dA$, *where* R *is the region bounded by the* x*-axis and the parabolas* $y^2 = 4 - 4x$ *and* $y^2 = 4 + 4x$, $y \ge 0$.

(a)



i	S_i	R_i
1	$\{(u,v) : v = 0,\ 0 \le u \le 1\}$	$\{(x,y) : y = 0,\ 0 \le x \le 1\}$
2	$\{(u,v) : u = 1,\ 0 \le v \le 1\}$	$\{(x,y) : x = 1 - \frac{y^2}{4},\ 0 \le y \le 2\}$
3	$\{(u,v) : v = 1,\ 0 \le u \le 1\}$	$\{(x,y) : x = \frac{y^2}{4} - 1,\ 0 \le y \le 2\}$
4	$\{(u,v) : u = 0,\ 0 \le v \le 1\}$	$\{(x,y) : y = 0,\ -1 \le x \le 0\}$

(b) $T(S) = R$ *and* $S = [0,1] \times [0,1]$, $x = u^2 - v^2$, $y = 2uv$.

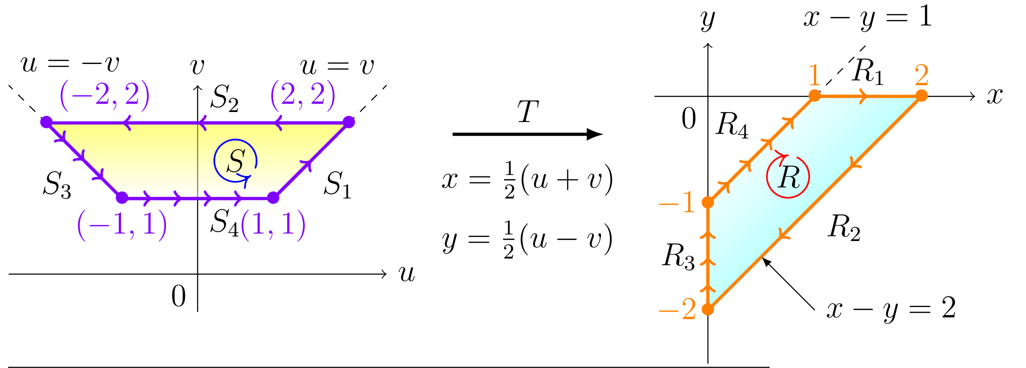
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0 \text{ (邊界不算)}$$

$$\iint_R y\, dA = \iint_S 2uv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = \int_0^1 \int_0^1 2uv(4u^2 + 4v^2)\, du\, dv$$

$$= \int_0^1 \left[2u^4 v + 4u^2 v^3 \right]_{u=0}^{u=1} dv = \int_0^1 (2v + 4v^3)\, dv = \left[v^2 + v^4 \right]_0^1 = 2. \qquad \blacksquare$$

Example 0.3 *Evaluate* $\displaystyle\iint_R e^{(x+y)/(x-y)}\, dA$, *where* R *is the trapezoidal region with vertices* $(1,0)$, $(2,0)$, $(0,-2)$, *and* $(0,-1)$.
R *is bounded by* $y=0$, $x-y=2$, $x=0$, *and* $x-y=1$.
Consider transformation T^{-1} *from xy-plane to uv-plane:*

$$u = x+y, \quad v = x-y$$



i	S_i		R_i	
1	$u=v$,	$1\le v\le 2$	$y=0$,	$1\le x\le 2$
2	$v=2$,	$-2\le u\le 2$	$x-y=2$,	$-2\le y\le 0$
3	$u=-v$,	$-2\le v\le -1$	$x=0$,	$-2\le y\le -1$
4	$v=1$,	$-1\le u\le 1$	$x-y=1$,	$-1\le y\le 0$

$S=\{(u,v): 1\le v\le 2,\ -v\le u\le v\}$.
Then the transformation T *from uv-plane to xy-plane is:*

$$x=\frac{1}{2}(u+v), \quad y=\frac{1}{2}(u-v)$$

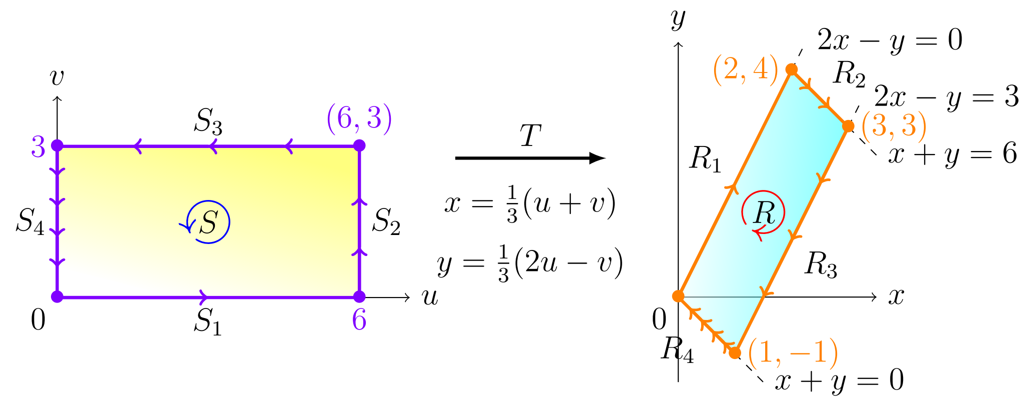
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{2} \neq 0 \qquad \left(\begin{array}{l} A(S) = 3, \\ A(R) = 1.5. \end{array} \right)$$

$$\iint_R e^{(x+y)/(x-y)}\, dA = \iint_S e^{u/v} \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\, du\, dv = \int_1^2 \int_{-v}^{v} e^{u/v} \left|-\frac{1}{2}\right|\, du\, dv$$

$$= \int_1^2 \left[\frac{v}{2} e^{u/v}\right]_{u=-v}^{u=v} dv = \int_1^2 \frac{v}{2}(e-e^{-1})\, dv = \frac{e-e^{-1}}{2}\left[\frac{v^2}{2}\right]_1^2 = \frac{3}{4}(e-e^{-1}). \quad \blacksquare$$

Example 0.4 (Extra 1) $\displaystyle\iint_R (x+y)^2\, dA$, *where R is bounded by $x+y=0$, $x+y=6$, $2x-y=0$, and $2x-y=3$.*
Consider transformation T^{-1} from xy-plane to uv-plane:

$$u = x+y, \quad v = 2x-y$$



i	S_i		R_i	
1	$v=0,$	$0 \le u \le 6$	$2x-y=0,$	$0 \le x \le 2$
2	$u=6,$	$0 \le v \le 3$	$x+y=6,$	$2 \le x \le 3$
3	$v=3,$	$0 \le u \le 6$	$2x-y=3,$	$1 \le x \le 3$
4	$u=0,$	$0 \le v \le 3$	$x+y=0,$	$0 \le x \le 1$

$S = \{(u,v) : 0 \le u \le 6,\ 0 \le v \le 3\} = [0,6] \times [0,3]$.
Then the transformation T from uv-plane to xy-plane is:

$$x = \frac{1}{3}(u+v), \quad y = \frac{1}{3}(2u-v)$$

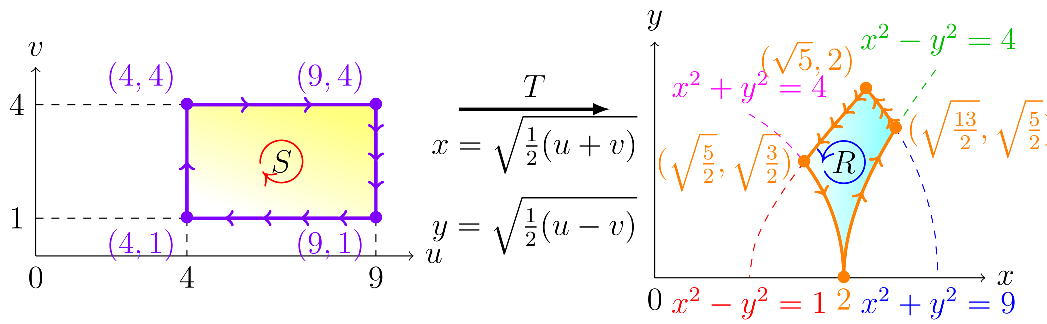
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{3} & \dfrac{1}{3} \\ \dfrac{2}{3} & -\dfrac{1}{3} \end{vmatrix} = -\frac{1}{3} \neq 0 \qquad \left(\begin{array}{l} A(S) = 18, \\ A(R) = 6. \end{array} \right)$$

$$\iint_R (x+y)^2\, dA = \iint_S u^2 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = \int_0^3 \int_0^6 u^2 \left| -\frac{1}{3} \right| du\, dv$$

$$= \frac{1}{3} \int_0^6 u^2\, du \int_0^3 dv = \frac{1}{3} \left[\frac{u^3}{3} \right]_0^6 \cdot \Big[v \Big]_0^3 = \frac{1}{3} \cdot 72 \cdot 3 = 72. \qquad \blacksquare$$

Example 0.5 (Extra 2) $\displaystyle\iint_R xy\ dA$, *where R is bounded by $x^2+y^2=4$, $x^2+y^2=9$, $x^2-y^2=1$, and $x^2-y^2=4$ in the first quadrant.*
Consider transformation T^{-1} from xy-plane to uv-plane:

$$u = x^2+y^2, \quad v = x^2-y^2$$



i	S_i		R_i	
1	$u=4,$	$1\le v\le 4$	$x^2+y^2=4,$	$\sqrt{5/2}\le x\le 2$
2	$v=4,$	$4\le u\le 9$	$x^2-y^2=4,$	$2\le x\le\sqrt{13/2}$
3	$u=9,$	$1\le v\le 4$	$x^2+y^2=9,$	$\sqrt{5}\le x\le\sqrt{13/2}$
4	$v=1,$	$4\le u\le 9$	$x^2-y^2=1,$	$\sqrt{5/2}\le x\le\sqrt{5}$

$S=\{(u,v): 4\le u\le 9,\ 1\le v\le 4\}=[4,9]\times[1,4]$.
Then the transformation T from uv-plane to xy-plane is:

$$x=\sqrt{\frac{1}{2}(u+v)}, \quad y=\sqrt{\frac{1}{2}(u-v)}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{2}\sqrt{u+v}} & \frac{1}{2\sqrt{2}\sqrt{u+v}} \\ \frac{1}{2\sqrt{2}\sqrt{u-v}} & \frac{-1}{2\sqrt{2}\sqrt{u-v}} \end{vmatrix} = \frac{-1}{4\sqrt{u^2-v^2}} \neq 0$$

$$\iint_R xy\ dA = \iint_S \sqrt{\frac{1}{2}(u+v)}\sqrt{\frac{1}{2}(u-v)}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$$

$$= \int_1^4\int_4^9 \frac{\sqrt{u^2-v^2}}{2}\left|\frac{-1}{4\sqrt{u^2-v^2}}\right|\ du\ dv = \int_1^4\int_4^9 \frac{1}{8}\ du\ dv \qquad (\text{就是 } \frac{1}{8}A(S))$$

$$= \frac{1}{8}(9-4)(4-1) = \frac{15}{8}.$$

■

0.3 Jacobian (3D) — 體積的變換率

A transformation T maps a region S in uvw-space onto a region R in xyz-space:

$$T(u,v,w)=(x,y,z),\quad x=g(u,v,w),\quad y=h(u,v,w),\quad z=k(u,v,w)$$

The ***Jacobian (determinant)*** of T is: $\quad \big(\Delta V \approx |\mathbf{r}_u \bullet (\mathbf{r}_v\times\mathbf{r}_w)|\Delta u\Delta v\Delta w\big)$

$$\boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w}\\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w}\end{vmatrix}}$$

$$\boxed{\begin{aligned}&\iiint\limits_R f(x,y,z)\,dV\\ &=\iiint\limits_S f(g(u,v,w),h(u,v,w),k(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,du\,dv\,dw\end{aligned}}$$

Attention: $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$, 小心不要算反了 $\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}$。

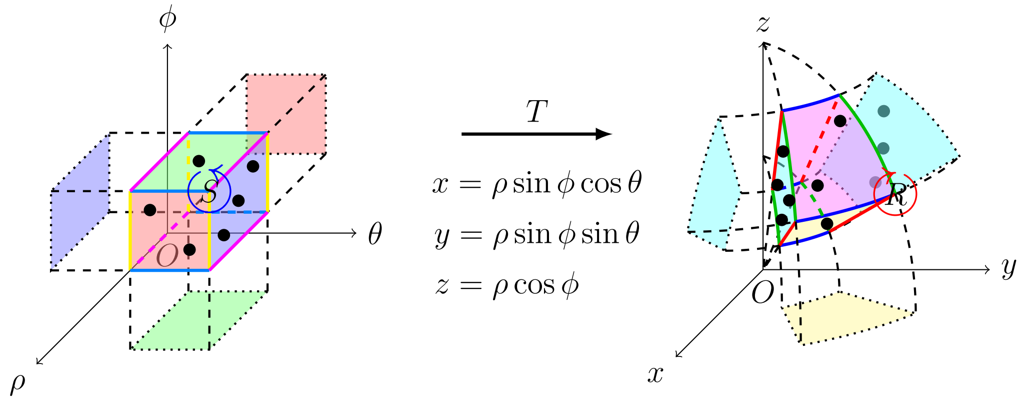
怎麼記?

$$\begin{array}{rlcll} T(u)=x: & dx & \to & \dfrac{dx}{du} & du\\ T(u,v)=(x,y): & dx\ dy & \to & \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| & du\ dv\\ T(u,v,w)=(x,y,z): & dx\ dy\ dz & \to & \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| & du\ dv\ dw\end{array}$$

Note: Jacobian 可以推廣到從 n-維空間 (變數) 到 n-維空間 (變數) 的變換:

$$\frac{\partial(x_1,\cdots,x_n)}{\partial(t_1,\cdots,t_n)}=\begin{vmatrix}\dfrac{\partial x_1}{\partial t_1} & \cdots & \dfrac{\partial x_1}{\partial t_n}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial x_n}{\partial t_1} & \cdots & \dfrac{\partial x_n}{\partial t_n}\end{vmatrix}$$

Example 0.6 *Triple Integrals in Spherical Coordinates.*



$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\boxed{\rho\sin\phi}\sin\theta & \boxed{\rho}\cos\phi\cos\theta \\ \sin\phi\sin\theta & \boxed{\rho\sin\phi}\cos\theta & \boxed{\rho}\cos\phi\sin\theta \\ \cos\phi & 0 & -\boxed{\rho}\sin\phi \end{vmatrix}$$

$$= \boxed{\rho^2\sin\phi} \begin{vmatrix} \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{vmatrix} \qquad (\boxed{\cdot}\ 提出)$$

$$= \rho^2\sin\phi(-\sin^2\phi\cos^2\theta - \cos^2\phi\sin^2\theta - \sin^2\phi\sin^2\theta - \cos^2\phi\cos^2\theta)$$

$$= -\rho^2\sin\phi(\sin^2\phi + \cos^2\phi)(\sin^2\theta + \cos^2\theta)$$

$$= -\rho^2\sin\phi \neq 0 \ \textit{(except perhaps on the boundary: } \rho = 0 \textit{ or } \phi = 0 \textit{ or } \phi = \pi\textit{)}.$$

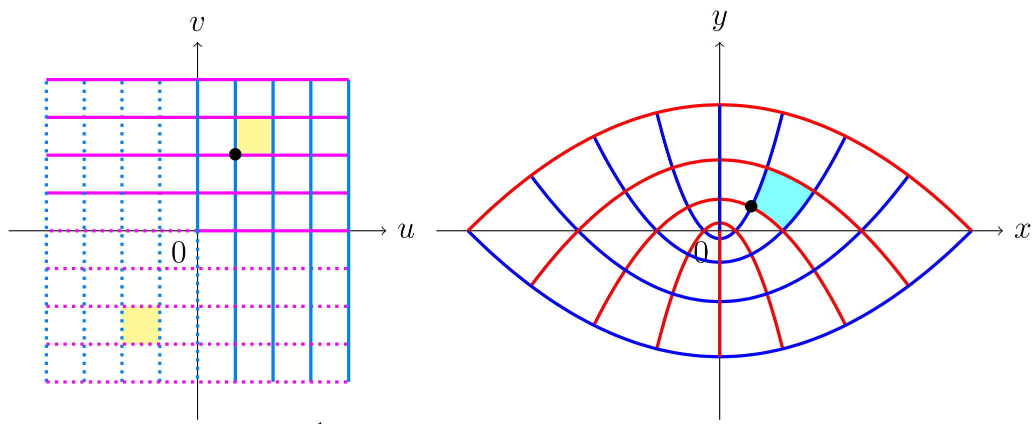
$$\iiint_R f(x,y,z)\, dx\, dy\, dz$$

$$= \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| d\rho\, d\theta\, d\phi$$

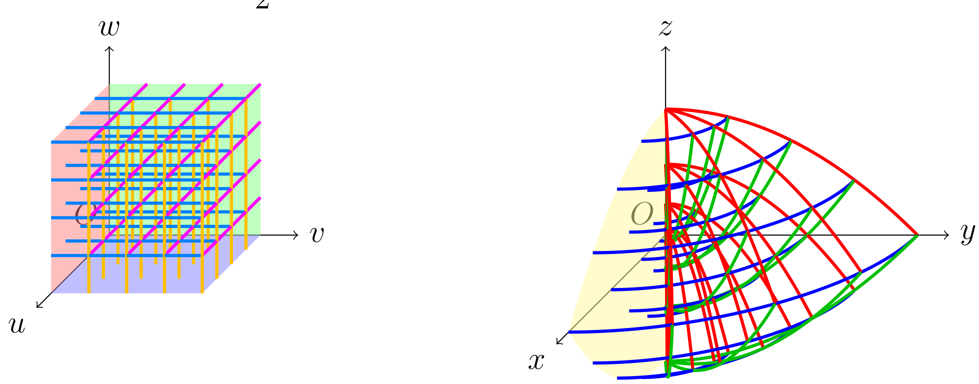
$$= \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot \boxed{\rho^2\sin\phi}\, d\rho\, d\theta\, d\phi. \qquad \blacksquare$$

♦ **Question:** When Jacobian 有正有負? Right/Left-Hand Rule.

Additional: Parabolic Coordinate System



(2D): $x = uv,\ y = \dfrac{1}{2}(v^2 - u^2).$



(3D): $x = uv\cos w,\ y = uv\sin w,\ z = \dfrac{1}{2}(v^2 - u^2).$