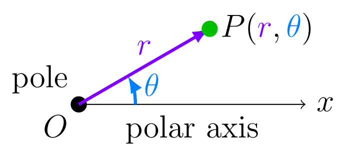
10.3 Polar coordinates

1. polar coordinates 極座標 $x = r\cos\theta$, $y = r\sin\theta$.
2. polar curve 極曲線
3. symmetry 對稱性
4. tangent 切線 $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ if $\frac{dx}{d\theta} \neq 0$.

0.1 Polar coordinate

以前的 (x, y) 叫 Cartesian coordinates system 卡氏 (直角) 坐標系,
Newton 引進 polar coordinates system 極座標系:

Define:



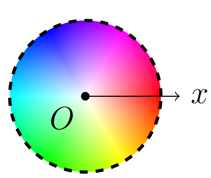
- ***pole (origin)*** 極 (原點): O.
- ***polar axis*** 極軸: $\overrightarrow{Ox}$, x-axis 正向。
- ***polar coordinates*** 極座標 $\boxed{(\boldsymbol{r}, \boldsymbol{\theta})}$, where $r = |\overrightarrow{OP}|$ and $\theta = \angle xOP$.

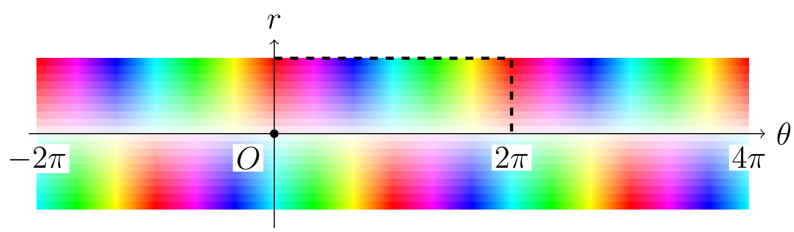
Note:

1. θ counterclockwise $+$ 逆時針為正, clockwise $-$ 順時針為負。
2. 表示法不唯一: $(\boldsymbol{r}, \boldsymbol{\theta} + \boldsymbol{2\pi}) = (\boldsymbol{r}, \boldsymbol{\theta})$ for $r > 0$, $O(0, \theta)$ for any θ. (差一圈)
3. 延伸定義 $(\boldsymbol{-r}, \boldsymbol{\theta}) = (\boldsymbol{r}, \boldsymbol{\theta} + \boldsymbol{\pi})$. (差半圈)
4. 座標轉換:

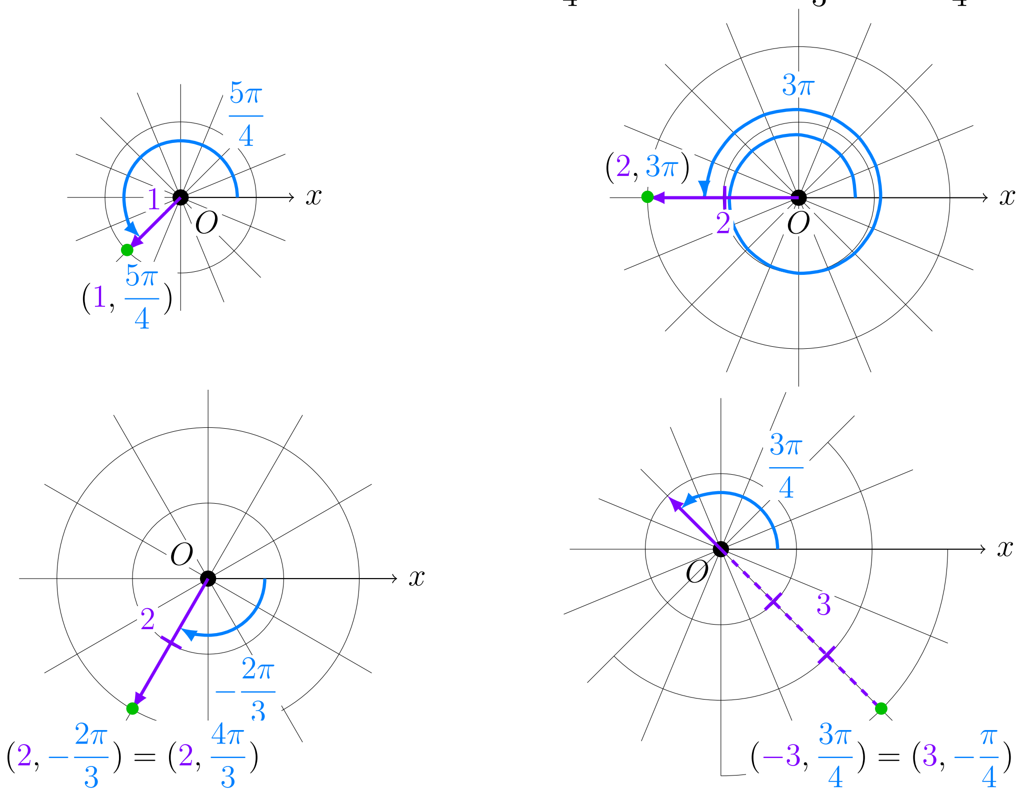
$$\text{Polar} \to \text{Cartesian: } \boxed{\boldsymbol{x = r\cos\theta,\ y = r\sin\theta}}. \text{ (唯一解)}$$

$$\text{Cartesian} \to \text{Polar: } r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}. \text{ (解不唯一)}$$

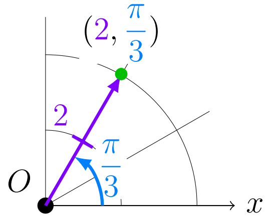




Example 0.1 *Plot polar coordinates* $(1, \frac{5\pi}{4})$, $(2, 3\pi)$, $(2, -\frac{2\pi}{3})$, $(-3, \frac{3\pi}{4})$.



Example 0.2 *Convert polar coordinates* $(2, \pi/3)$ *to Cartesian coordinates.*

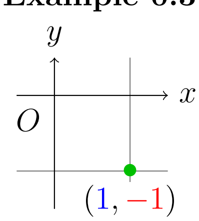


$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1,$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}.$$

$(x, y) = (1, \sqrt{3})$. ■

Example 0.3 *Represent Cartesian coordinates* $(1, -1)$ *by polar coordinates.*



$r^2 = x^2 + y^2 = (1)^2 + (-1)^2 = 2$, $r = \pm\sqrt{2}$.

$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$, $\theta = \frac{3\pi}{4}$ *or* $\frac{7\pi}{4}$,

$\because (1, -1)$ *in the fourth quadrant,*

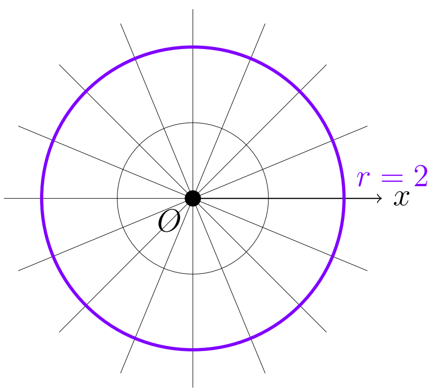
$(r, \theta) = (\sqrt{2}, \frac{7\pi}{4}) = (-\sqrt{2}, \frac{3\pi}{4})$. ■

0.2 Polar curve

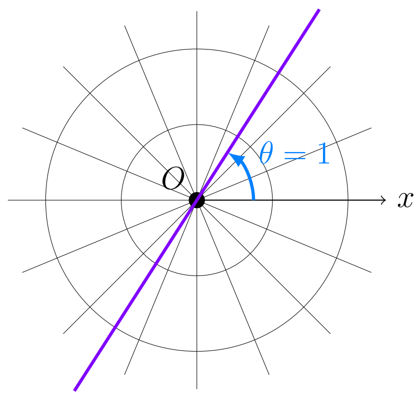
Cartesian curve: $\{(x, y) : y = f(x) \text{ or } F(x, y) = 0\}$.
The ***graph*** of a ***polar equation*** 極方程式 $r = f(\theta)$ or $F(r, \theta) = 0$ is

$$\{(r, \theta) : r = f(\theta) \text{ or } F(r, \theta) = 0\}$$

Example 0.4 $r = 2$.



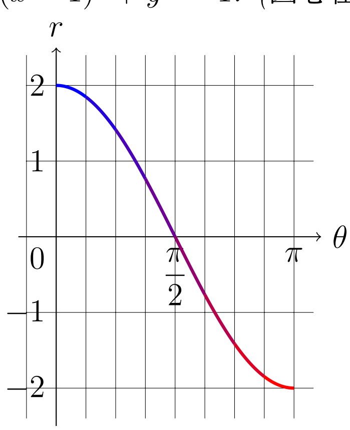
Example 0.5 $\theta = 1$.



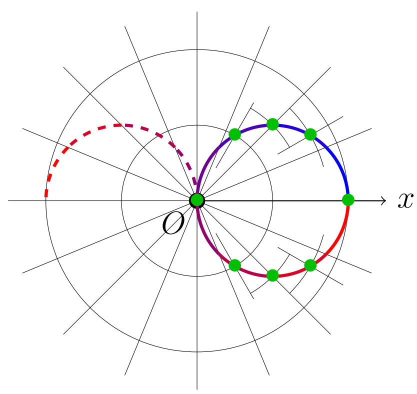
Note: 1. $r = a$ 是半徑 a 圓心 O 的圓。2. $\theta = t$ 是夾極軸 t 的直線。

Example 0.6 $r = 2\cos\theta$, *and find a Cartesian equation.*

$x = r\cos\theta$, $x^2 + y^2 = r^2 = r \cdot 2\cos\theta = 2x$
$\Longrightarrow (x-1)^2 + y^2 = 1$. *(圓心在 $(1, 0)$ 的單位圓。)* ■

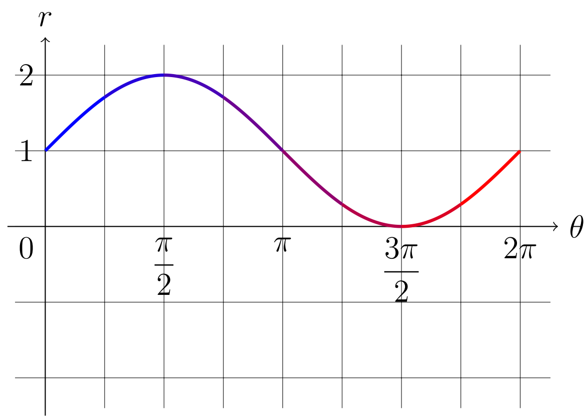


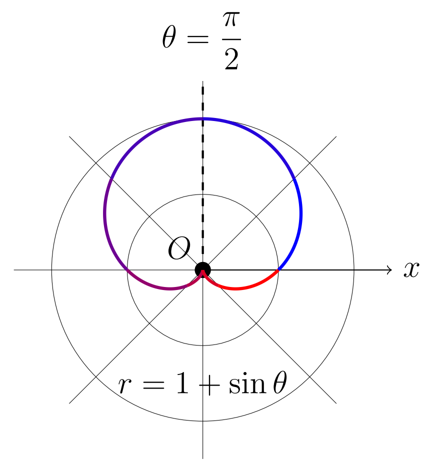
($0 \le \theta \le \pi$ 就已經一圈。)



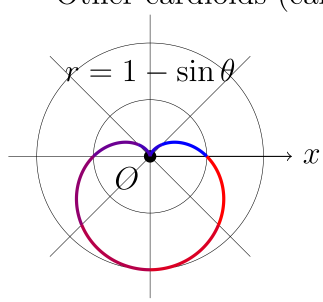
θ	r
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

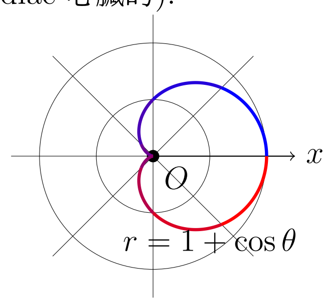
Example 0.7 $r = 1 + \sin\theta$. *(**cardioid** [ˈkɑrdɪˌɔɪd], [卡底歐乙的] 心臟線)*

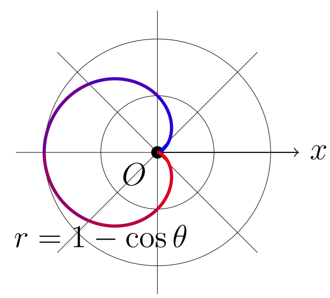




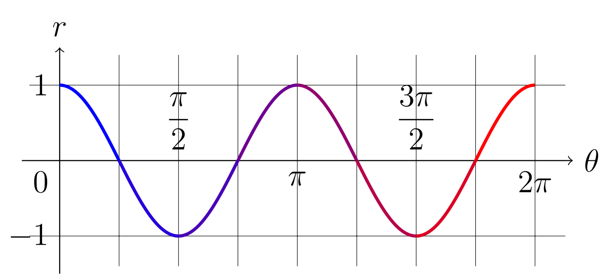
Other cardioids (cardiac 心臟的):

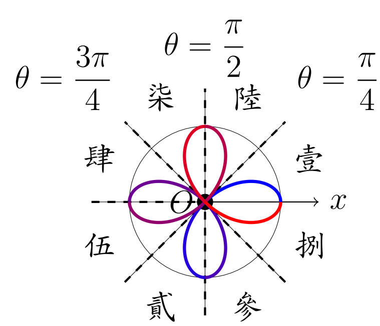






Example 0.8 $r = \cos 2\theta$. *(**four-leaved rose**)*





Additionally:

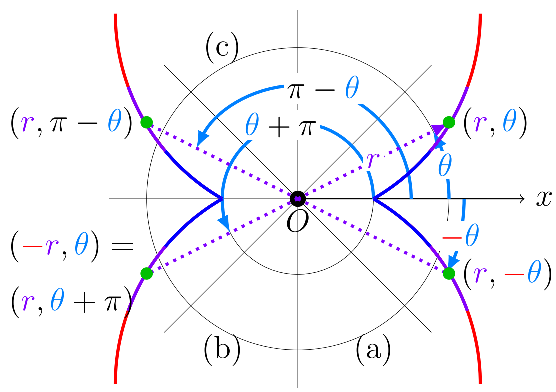
1. $r = a\sin\theta$, $r = a\cos\theta$ 也都是圓。(半徑=? 圓心=?)
2. $r = a\sec\theta$, $r = a\csc\theta$ 也是直線。(什麼線?)

0.3 Symmetry

(a) If $F(r, -\theta) = F(r, \theta)$, then the curve is symmetric about the polar axis. Ex: $\cos(-\theta) = \cos\theta$.

(b) If $F(-r, \theta) = F(r, \theta)$ or $F(r, \theta + \pi) = F(r, \theta)$, then the curve is symmetric about the pole. Ex: $\tan(\theta + \pi) = \tan\theta$.

(c) If $F(r, \pi - \theta) = F(r, \theta)$, then the curve is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$ (y-axis). Ex: $\sin(\pi - \theta) = \sin\theta$.



0.4 Tangent

The polar curve $r = f(\theta)$ is also the curve of parametric equations with parameter θ: (當成參數 θ 的參數方程/曲線)

$$\boxed{\boldsymbol{x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.}}$$

$$\boxed{\boldsymbol{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}, \qquad \text{if } \frac{dx}{d\theta} \neq 0}}$$

Note: if $\dfrac{dy}{d\theta} = 0 \neq \dfrac{dx}{d\theta}$: horizontal tangent 水平切線;
if $\dfrac{dy}{d\theta} \neq 0 = \dfrac{dx}{d\theta}$: vertical tangent 垂直切線;
if $\dfrac{dy}{d\theta} = 0 = \dfrac{dx}{d\theta}$ 什麼都有可能。

Example 0.9 *(a) For the cardioid $r = 1+\sin\theta$, find the slope of the tangent line when $\theta = \pi/3$.*
(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

$$r = 1+\sin\theta,\ \frac{dr}{d\theta} = \cos\theta,$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}.$$

$$(a)\ \frac{dy}{dx}\Big|_{\theta=\pi/3} = \frac{\cos\frac{\pi}{3}(1+2\sin\frac{\pi}{3})}{(1+\sin\frac{\pi}{3})(1-2\sin\frac{\pi}{3})} = \frac{\frac{1}{2}(1+2\cdot\frac{\sqrt{3}}{2})}{(1+\frac{\sqrt{3}}{2})(1-2\cdot\frac{\sqrt{3}}{2})} = -1.$$

[Quick sol] 如果只是要算切線, 計算:

$$\frac{dy}{d\theta}\Big|_{\theta=\pi/3} = \frac{1}{2}(1+\sqrt{3}),\ \frac{dx}{d\theta}\Big|_{\theta=\pi/3} = -\frac{1}{2}(1+\sqrt{3}),\ \textit{then}\ \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = -1.$$

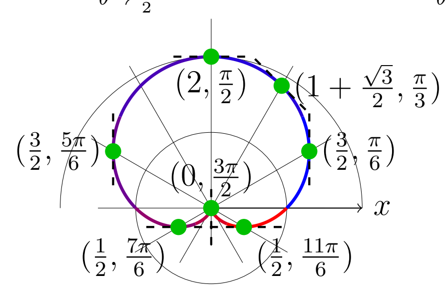
$$(b)\ \frac{dy}{d\theta} = \cos\theta(1+2\sin\theta) = 0\ \textit{when}\ \theta = \frac{\pi}{2}, \boxed{\frac{3\pi}{2}}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

$$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0\ \textit{when}\ \theta = \boxed{\frac{3\pi}{2}}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

When $\theta = \frac{3\pi}{2}$, $\frac{dy}{d\theta}$ 與 $\frac{dx}{d\theta}$ 同時爲 0 $(\frac{\mathbf{0}}{\mathbf{0}})$, 要看看是不是端點。*(*不是*)*
用 *ℓ'Hospital's rule* 求極限:

$$\lim_{\theta\to\frac{3\pi}{2}^-}\frac{dy}{dx} = \Big(\lim_{\theta\to\frac{3\pi}{2}^-}\frac{1+2\sin\theta}{1-2\sin\theta}\Big)\Big(\lim_{\theta\to\frac{3\pi}{2}^-}\frac{\cos\theta}{1+\sin\theta}\Big) = -\frac{1}{3}\lim_{\theta\to\frac{3\pi}{2}^-}\frac{\cos\theta}{1+\sin\theta}$$

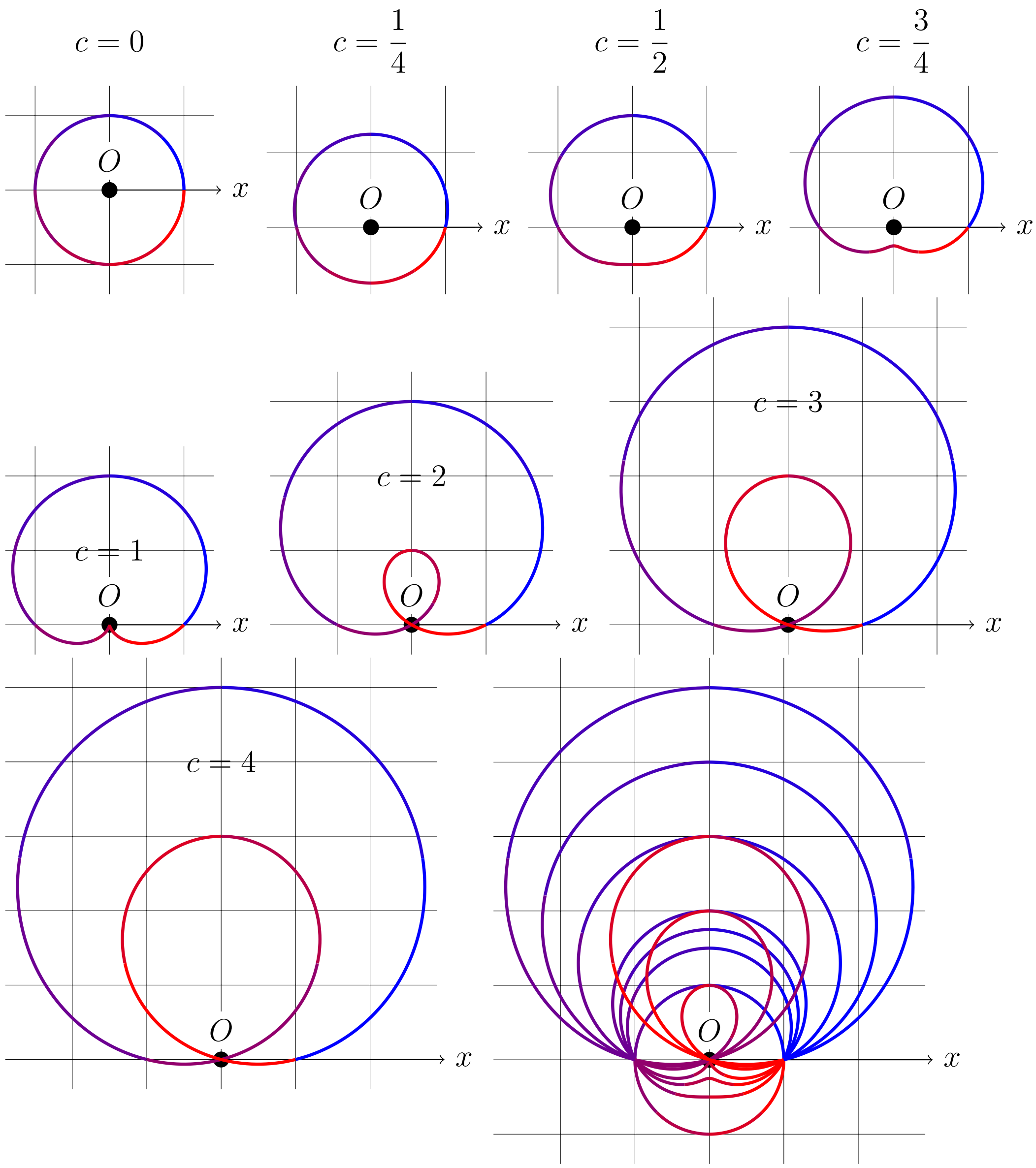
$$\overset{l'H}{=} -\frac{1}{3}\lim_{\theta\to\frac{3\pi}{2}^-}\frac{-\sin\theta}{\cos\theta} = \infty,\ \textit{also}\ \lim_{\theta\to\frac{3\pi}{2}^+}\frac{dy}{dx} = -\infty,\ \therefore\ \textit{vertical tangent.}$$



(r 座標用 $1+\sin\theta$ 算。*)*
Horizontal tangents at point:
$(2, \frac{\pi}{2}), (\frac{1}{2}, \frac{7\pi}{6}), (\frac{1}{2}, \frac{11\pi}{6})$; *(*極座標*)*
and vertical tangents at point:
$(0, \frac{3\pi}{2})\cancel{=(0,0)}, (\frac{3}{2}, \frac{\pi}{6}), (\frac{3}{2}, \frac{5\pi}{6})$. ■

Attention: $0 \neq 1+\sin 0$, 雖然是同一點, 但最好別寫 $(0, 0)$。

Limaçons of Pascal: $r = 1 + c \sin\theta$
帕斯卡的 ***limaçon*** [ˈlɪmɑˌsɑn], [哩麻送] 蝸線 (snail in French)。



♦ Additional: Sketch polar curve