13.2 Derivatives and integrals of vector functions

1. derivative and unit tangent vector
2. integral and antiderivative

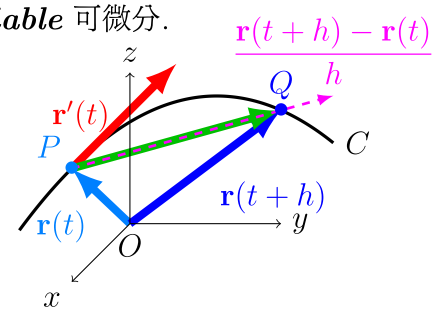
0.1 Derivative and unit tangent vector

Define: The ***derivative*** 導 (函) 數 $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined as

$$\boxed{\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}}$$

if the limit exists, and $\mathbf{r}$ is called ***differentiable*** 可微分.

(幾何意義) If points P & Q have position vector $\mathbf{r}(t)$ & $\mathbf{r}(t+h)$, then secant vector $\overrightarrow{PQ} = \mathbf{r}(t+h) - \mathbf{r}(t)$, and $\frac{1}{h}\overrightarrow{PQ}$ has the same direction. If $\lim_{Q\to P} \frac{1}{h}\overrightarrow{PQ}$ exists, call the tangent vector at P. (切向=lim 割向)



Define: $\mathbf{r}'(t)$ is called the ***tangent vector*** 切向量 to the curve C defined by $\mathbf{r}$ at the point, provided $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$ (導數存在但不是零向量), and the ***tangent line*** 切線 to C at P through P and parallel $\mathbf{r}'(t)$.

Define: The ***unit tangent vector*** 單位切向量

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}$$

Theorem 1 *If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, h are differentiable functions, then (分量函數微分, 又是看分量函數。)*

$$\boxed{\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}}$$

Define: The ***second derivative*** 二階導數 of $\mathbf{r}$ is $\boxed{\mathbf{r}''} = (\mathbf{r}')'$, the third one is $\mathbf{r}'''$, the n-th derivative $n(\geq 4)$階導數 is $\boxed{\mathbf{r}^{(n)}} = (\mathbf{r}^{(n-1)})'$.

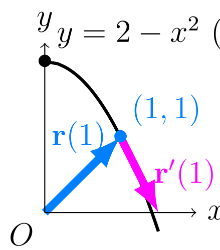
Example 0.1 *(a) Find the derivative of* $\mathbf{r}(t) = (1+t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$.
(b) Find the unit tangent vector at $t = 0$.

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + (1-t)e^{-t}\mathbf{j} + 2\cos 2t\mathbf{k}$, $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$,

(b) $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \dfrac{1}{\sqrt{5}}\mathbf{j} + \dfrac{2}{\sqrt{5}}\mathbf{k}$. *(先算* $\mathbf{T}(t)$ *再代* $t = 0$ *也可以, but...)* ■

Example 0.2 *Find* $\mathbf{r}'(t)$ *for* $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2-t)\mathbf{j}$ *and sketch* $\mathbf{r}(1)$ *and* $\mathbf{r}'(1)$.

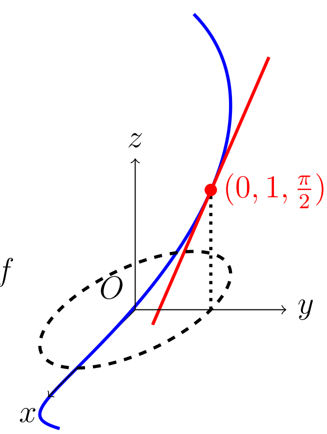
$$\mathbf{r}(1) = \mathbf{i} + \mathbf{j}(= \langle 1, 1\rangle),\ \mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j},\ \mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}(= \langle \frac{1}{2}, -1\rangle). \quad ■$$



Example 0.3 *Find parametric equations for the tangent line to the helix with parametric equations* $x = 2\cos t$, $y = \sin t$, $z = t$ *at* $(0, 1, \frac{\pi}{2})$.

The vector equation of the helix is

$$\mathbf{r}(t) = \langle 2\cos t, \sin t, t\rangle.$$



Solve $\mathbf{r}(t) = \langle 0, 1, \frac{\pi}{2}\rangle \implies t = \frac{\pi}{2}$.
(給點座標找 t *有時不好算。)*
$\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1\rangle$, $\mathbf{r}'(\frac{\pi}{2}) = \langle -2, 0, 1\rangle$.
The tangent line through $(0, 1, \frac{\pi}{2})$ *parallel* $\langle -2, 0, 1\rangle$ *of parametric equations:*

$$x = (0 - 2s =) - 2s,\ y = (1 + 0s =)1,\ z = \frac{\pi}{2} + s.$$

(Why s*? 因爲* t *用過了。)* ■

Theorem 2 (Differentiation Rules) *Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are differentiable vector functions,* c *is a scalar, and* f *is a differentiable function. Then*

1. $\dfrac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$ *(addition)*

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$ *(constant multiplication)*

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$ *(function multiplication)*

4. $\dfrac{d}{dt}[\mathbf{u}(t)\bullet\mathbf{v}(t)] = \mathbf{u}'(t)\bullet\mathbf{v}(t)+\mathbf{u}(t)\bullet\mathbf{v}'(t)$ *(dot product)*

5. $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$ *(cross produce)*

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ *(Chain Rule)*

Proof. of dot product formula.

Let $\mathbf{u} = \langle f_1, f_2, f_3\rangle$ and $\mathbf{v} = \langle g_1, g_2, g_3\rangle$. ("$(t)$" omitted.) Then

$$\begin{aligned}
\mathbf{u}\bullet\mathbf{v} &= f_1g_1+f_2g_2+f_3g_3 = \sum_{i=1}^{3} f_ig_i.\\
(\mathbf{u}\bullet\mathbf{v})' &= (\sum_{i=1}^{3} f_ig_i)' = \sum_{i=1}^{3}(f_ig_i)'\\
&= \sum_{i=1}^{3}(f_i'g_i+f_ig_i') = \sum_{i=1}^{3} f_i'g_i + \sum_{i=1}^{3} f_ig_i'\\
&= \mathbf{u}'\bullet\mathbf{v}+\mathbf{u}\bullet\mathbf{v}'.
\end{aligned}$$ ■

Example 0.4 *Show that if* $|\mathbf{r}(t)| = c$ *a constant, then* $\mathbf{r}'(t)$ *is orthogonal to* $\mathbf{r}(t)$ *for all* t.

$$0 = \frac{d}{dt}c^2 = \frac{d}{dt}[\mathbf{r}(t)\bullet\mathbf{r}(t)] = \mathbf{r}'(t)\bullet\mathbf{r}(t)+\mathbf{r}(t)\bullet\mathbf{r}'(t) = 2\mathbf{r}'(t)\bullet\mathbf{r}(t).$$ ■

Recall: 內積 $0 \iff$ 垂直, 外積 $\mathbf{0} \iff$ 平行。
幾何上來說, $|\mathbf{r}| = c \iff$ 球面上的曲線, 切向量 *(tangent vector)* $\mathbf{r}'$ 總是跟位置向量 *(position vector)* $\mathbf{r}$ 垂直 $(\mathbf{r}\perp\mathbf{r}')$。

0.2 Integral and antiderivative

Define: The ***definite integral*** 定積分 of a continuous vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ on $[a, b]$ is (分量函數積分, 還是看分量函數。)

$$
\begin{aligned}
\int_a^b \mathbf{r}(t)\, dt &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\Delta t \qquad \text{(黎曼和的極限)} \\
&= \lim_{n\to\infty} \left[\Big(\sum_{i=1}^{n} f(t_i^*)\Delta t \Big)\mathbf{i} + \Big(\sum_{i=1}^{n} g(t_i^*)\Delta t \Big)\mathbf{j} + \Big(\sum_{i=1}^{n} h(t_i^*)\Delta t \Big)\mathbf{k} \right] \\
&= \Big(\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\Delta t \Big)\mathbf{i} + \Big(\lim_{n\to\infty} \sum_{i=1}^{n} g(t_i^*)\Delta t \Big)\mathbf{j} + \Big(\lim_{n\to\infty} \sum_{i=1}^{n} h(t_i^*)\Delta t \Big)\mathbf{k},
\end{aligned}
$$

$$
\boxed{\int_a^b \mathbf{r}(t)\, dt = \Big(\int_a^b f(t)\, dt \Big)\mathbf{i} + \Big(\int_a^b g(t)\, dt \Big)\mathbf{j} + \Big(\int_a^b h(t)\, dt \Big)\mathbf{k}}.
$$

Define: $\mathbf{R}(t)$ is an ***antiderivative*** 反導數 of $\mathbf{r}(t)$ if $\boxed{\mathbf{R}'(t) = \mathbf{r}(t)}$.

Define: $\mathbf{R}(t) + \mathbf{C}$ is the ***most general antiderivative*** 最一般反導數 of $\mathbf{r}(t)$, where $\mathbf{C}$ is an ***arbitrary constant vector*** 任意常數向量.

Define: The ***indefinite integral*** 不定積分 of $\mathbf{r}(t)$ is

$$
\boxed{\int \mathbf{r}(t)\, dt = \mathbf{R}(t) + \mathbf{C}}.
$$

Theorem 3 (The Fundamental Theorem of Calculus for vector functions)

For a continuous vector function $\mathbf{r}(t)$ on $[a, b]$,

$$
\boxed{\frac{d}{dt}\int_a^t \mathbf{r}(s)\, ds = \mathbf{r}(t)}, \qquad \boxed{\int_a^b \mathbf{r}(t)\, dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)}.
$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$.

Example 0.5 $\mathbf{r}(t) = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$,

$$
\begin{aligned}
\int \mathbf{r}(t)\, dt &= \Big(\int 2\cos t\, dt \Big)\mathbf{i} + \Big(\int \sin t\, dt \Big)\mathbf{j} + \Big(\int 2t\, dt \Big)\mathbf{k} \\
&= 2\sin t\mathbf{i} - \cos t\mathbf{j} + t^2\mathbf{k} + \mathbf{C}. \\
\int_0^{\pi/2} \mathbf{r}(t)\, dt &= \Big[2\sin t\mathbf{i} - \cos t\mathbf{j} + t^2\mathbf{k} \Big]_0^{\pi/2} = 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}.
\end{aligned}
$$

■