1179: Probability
Lecture 10 — Expected Value, Variance, and Moments

Ping-Chun Hsieh (謝秉均)

October 15, 2021

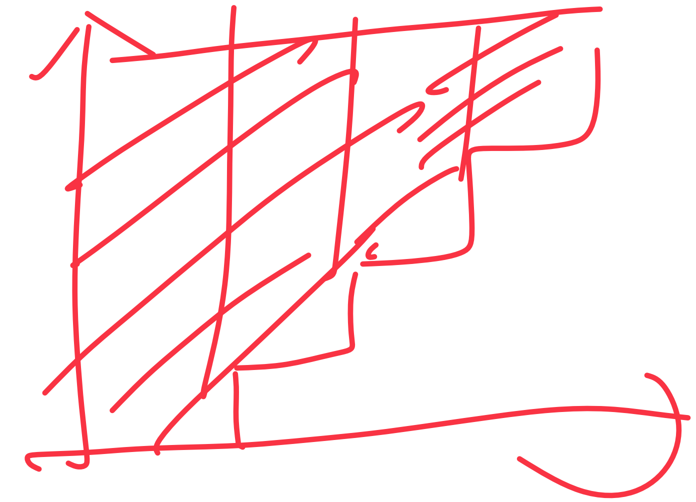
This Lecture

1. Expected Value

2. Variance and Moments

- Reading material: Chapter 4.4-4.5

X $\quad E[X] = \sum_{x \in S} x \cdot P_X(x)$



X

Expected Value

Example: St. Petersburg Paradox

- Example: We are asked to pay 10000 dollars to play a game.
  - We can keep flipping a fair coin until a head is observed.
  - If the 1st head occurs at n-th toss, then we get a prize of 2^n dollars and the game is over.
  - Shall we play this game?

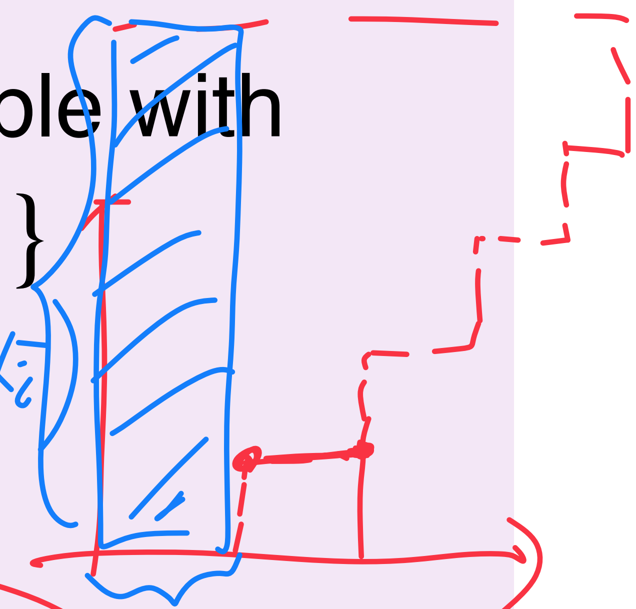
Expected Value of a Discrete Random Variable: An Alternative Expression

Expected Value (or Mean / Expectation):

Let X be a non-negative discrete random variable with

- the set of possible values $S = \{x_1, x_2, x_3 \cdots\}$
- CDF of X is $F_X(t)$

$1-F_X(x_i)$



Denote $x_0 = 0$. The expected value of X is

$$E[X] = \sum_{i=1}^{\infty} \underbrace{(x_i - x_{i-1})}_{x_i - x_{i-1}} \cdot \left(1 - F_X(x_i^-)\right)$$

$F_X(x_i^-) = F_X(x_{i-1})$

- What if $S = \{1,2,3 \cdots\}$?

$$E[X] = \sum_{i=1}^{\infty} 1 \cdot \left(1 - F_X(x_i^-)\right)$$

- How about continuous cases?

Example: Using the Alternative Expression

- Example: Suppose X is a discrete random variable
  - For X, the set of possible values $A = \{2,4,6,8\cdots\}$
  - The CDF of X is $F_X(t) = 1 - \frac{1}{t^2}, t \in A$
  - What is $E[X]$?

$2 \cdot i$

$2\times 1,\ 2\times 2,$

$E[X] = \sum x \cdot P_X(x)$

$E[X] = \sum_{i=1}^{\infty} (x_i - x_{i-1}) \cdot (1 - F_X(x_{i-1}))$

$\left(X_{i-1} = 2(i-1) \right)$

$$E[X] = \sum_{i=1}^{\infty} \underbrace{(X_i - X_{i-1})}_{=2} \cdot \underbrace{\left(1 - F_X(X_{i-1})\right)}_{=\begin{cases} 1 - \left(1 - \frac{1}{(2(i-1))^2}\right), & i \geq 2 \\ 0, & i = 1 \end{cases}}$$

$$= \sum_{i=2}^{\infty} 2 \cdot \frac{1}{(2(i-1))^2} = \frac{1}{2} \boxed{\sum_{i=2}^{\infty} \frac{1}{(i-1)^2}} = \frac{1}{2} \cdot \frac{\pi^2}{6}$$

$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \right)$

Basel Problem

A Property of Expected Value

Theorem (Expectation of a Function of r.v.):

?

1. Let X be a discrete random variable with
   - the set of possible values S
   - PMF of X is $p_X(x)$
2. Let $g(\cdot)$ be a real-valued function

$g(\cdot): \mathbb{R}^1 \to \mathbb{R}^1$

$Y = g(X)$

Y is still a random variable

The expectation of $g(X)$ is

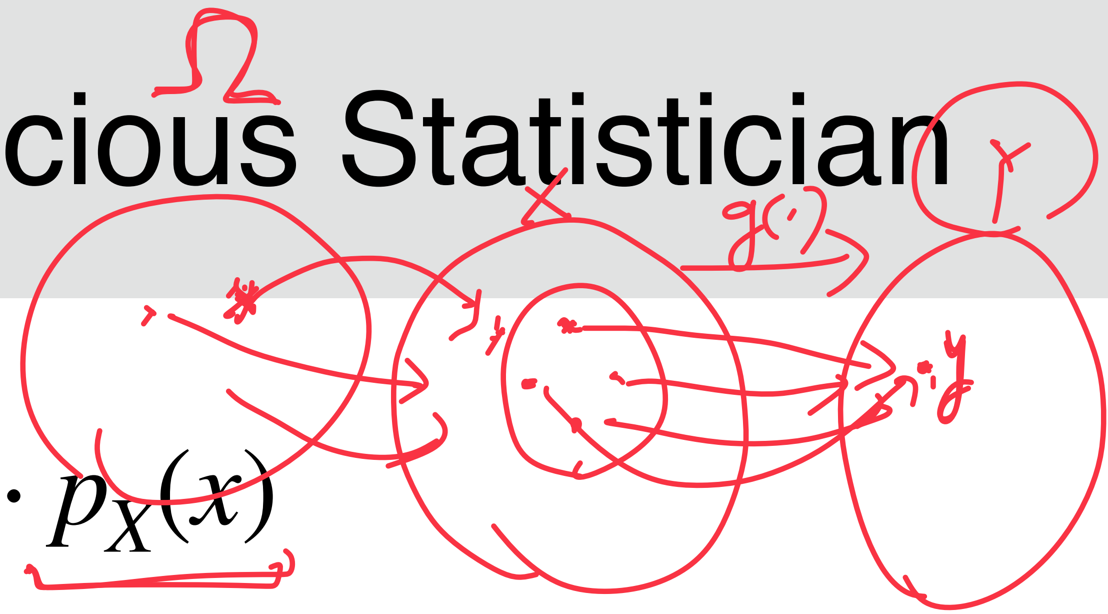
Y

$$E[g(X)] = \sum_{x \in S} g(x) \cdot p_X(x)$$

$$E[Y] = \sum_{\text{all possible } y} y \cdot P_Y(y)$$

- Is this intuitive? Do we need a proof?
- Also called Law of the unconscious statistician (LOTUS)

Proof of Law of the Unconscious Statistician



$$E[g(X)] := \sum_{x\in S} g(x)\cdot p_X(x)$$

Define $Y = g(X)$, Suppose $P_Y(y)$ is the PMF of Y.

$$E[g(X)] = E[Y] = \sum_{\text{all possible } y} y\cdot P_Y(y)$$

$$= \sum_{\text{all possible } y} y\cdot \sum_{x:\, g(x)=y} P_X(x)$$

$$\sum_{\substack{\text{all possible } y\\ x:\, g(x)=y}} y\cdot P_X(x)$$

$$= \sum_{\text{all possible } x} g(x)\cdot P_X(x)$$

Linearity of Expected Values (I)

$E[2X] = 2 \cdot E[X]$

Linearity Property (I):

Let X be a discrete random variable and α, β be real numbers. Then, we have

$$E[\alpha X + \beta] = \alpha \cdot E[X] + \beta$$

$g(X) = \alpha X + \beta$

- How to show this?

By LOTUS: $E[\alpha X + \beta] = E[g(X)] = \sum_{\text{all } x} g(x) \cdot P_X(x)$

$$= \sum_{\text{all } x} (\alpha x + \beta) \cdot P_X(x) = \underbrace{\boxed{\sum_{\text{all } x} \alpha \cdot x \cdot P_X(x)}}_{= \alpha \cdot E[X]} + \underbrace{\boxed{\sum_{\text{all } x} \beta \cdot P_X(x)}}_{= \beta \cdot 1}$$

Linearity of Expected Values (II)

Linearity Property (II):

Let X be a discrete random variable and $g(\cdot), h(\cdot)$ be real ~~numbers~~ functions. Then, we have

$$E[g(X) + h(X)] = E[g(X)] + E[h(X)]$$

- How to show this?

By LOTUS:

$$E[g(X)+h(X)] = \sum_{\text{all } x} \big(g(x)+h(x)\big) \, p_X(x)$$

Conditional Expectation

- Example: Roll a fair 6-sided die once
  - Define $X =$ the number that we observe
  - Given that $X \geq 4$, what is the expected value of X?

Conditional Expectation:

Let X be a discrete random variable with the set of possible values $S = \{x_1, x_2, x_3 \cdots\}$. Let A be an event. The expected value of X conditioned on A

$$E[X|A] := \sum_{x \in S} x \cdot P(X = x | A)$$

Example: Taiwan Receipt Lottery

- Example: Suppose we have a receipt at hand
  - Define X = the prize we get
  - What is $E[X]$?
  - Given that the last digit is 7, what is the expected value of X?

109年 7-8月 統一發票開獎		
特別獎	13362795	與左欄號碼相同者獎金1000萬元
特獎	27580166	與左欄號碼相同者獎金200萬元
頭獎	53227282 35082085 37175928	頭獎 與頭獎號碼完全相同者獎金20萬元 二獎 與頭獎末7碼相同者各得獎金4萬元 三獎 與頭獎末6碼相同者各得獎金1萬元 四獎 與頭獎末5碼相同者各得獎金4000元 五獎 與頭獎末4碼相同者各得獎金1000元 六獎 與頭獎末3碼相同者各得獎金200元
增開六獎	987 614	末3碼與增開六獎號碼相同者各得獎金200元

正確資訊請以財政部提供為準 中央社祝您幸運中獎

Variance and Moments

Moments and Others

LOTUS:

$$E[g(X)] := \sum_{x \in S} g(x) \cdot p(x)$$

- Example: $g(X) = X^2$ $\Rightarrow E[X^2]$ ······ 2nd moment of X
- Example: $g(X) = X^n$ $\Rightarrow E[X^n]$ ···· nth moment of X
- Example: $g(X) = (X - \mu_X)^2$ $\Rightarrow E[(X-\mu_X)^2]$ ···· 2nd central moment of X (Variance)
- Example: $g(X) = (X - \mu_X)^n$ $\Rightarrow E[(X-\mu_X)^n]$ ··· nth central moment of X
- Example: $g(X) = e^{tX}$ $\Rightarrow E[e^{tX}]$ ···· moment generating function of X (MGF)

Variance

Variance (2nd central moment):
Let X be a discrete random variable with the set of possible values S and PMF $p_X(x)$. The variance of X is

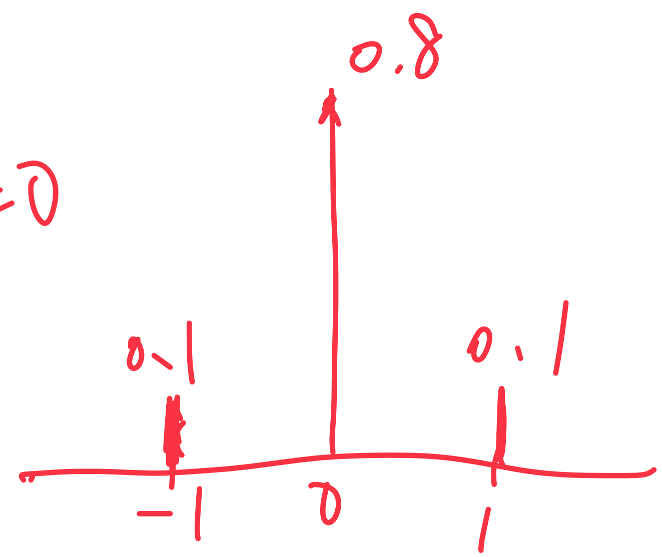
$$\text{Var}[X] := E[(X - \mu_X)^2] = \sum_{x \in S} (x - \mu_X)^2 \cdot p_X(x)$$

$\|$ $E[X]$

- Sometimes we use the notation: $\sigma_X^2 \equiv \text{Var}[X]$
- Variance captures the variability of a random variable

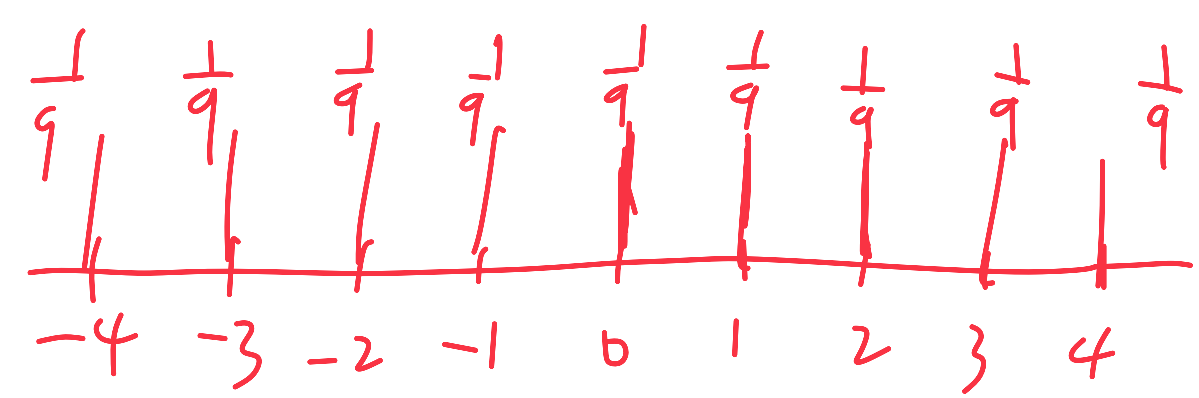
X_1

$E[X_1]=0$



X_2

$E[X_2]=0$



Variance: An Alternative Explanation

- Example: Suppose we are given a random variable X
  - We need to output a prediction of X (denoted by z)
  - Penalty of prediction is $(X - z)^2$
  - What is the minimum expected penalty?

Another Way for Calculating Variance

$\alpha X + 0$

Theorem:
Let X be a random variable. Then, we have

$$\underbrace{\mathrm{Var}[X]}\, := \underbrace{E[X^2]}_{平方的平均} - \underbrace{(E[X])^2}_{平均的平方}$$

$$\boxed{\mu_X \equiv E[X]}$$

- How to show this?

$$\mathrm{Var}[X] = E\left[(X-\mu_X)^2\right]$$

$$= E\left[X^2 - 2\cdot\mu_X X + \mu_X^2\right]$$

$$= \underbrace{E[X^2]}_{} + \underbrace{E[-2\cdot\mu_X X]}_{= -2\mu_X\cdot \underbrace{E[X]}_{=\mu_X}} + \underbrace{E[\mu_X^2]}_{=\mu_X^2} = E[X^2] - \mu_X^2$$

Properties of Variance

1. Var$(X + c) =$ Var(X)?

2. Var$(aX) = a \cdot$ Var(X)?

3. Var$(\,|X|\,) =$ Var(X)?

4. $E(X^2) \geq (E(X))^2$?

Existence of Moments

- Example: Suppose X is a random variable with PMF $p_X(x)$

$$p_X(k) = \begin{cases} \dfrac{1}{2k(k+1)} & , k = 1,2,3,\cdots \\ \dfrac{1}{2k(k-1)} & , k = -1, -2, -3,\cdots \end{cases}$$

- Does $E[X]$ exist?

Rearrangement of Series

- Example: Consider a series $\{a_n\}$: $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \cdots$
  - What is $\sum_{n=1}^{\infty} a_n$?

- Example: Rearrange $\{a_n\}$ as $\{b_n\}$:

  $$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{M}, -1, \frac{1}{M+1}, \cdots, \frac{1}{2M}, -\frac{1}{2} \cdots,$$

  - What is $\sum_{n=1}^{\infty} b_n$?

Riemann Rearrangement Theorem

Riemann Rearrangement Theorem:
Let $\{a_n\}$ be a sequence of numbers. If $\{a_n\}$ satisfies that

1. $\sum_{n=1}^{\infty} a_n$ converges
2. $\sum_{n=1}^{\infty} |a_n| = \infty$

Then, for any $B \in \mathbb{R} \cup \{\infty\}$, there exists a <u>rearrangement</u> $\{b_n\}$ of $\{a_n\}$ such that $\sum_{n=1}^{\infty} b_n = B$

Existence of Moments (Formally)

Existence of Moments:
Let X be a random variable. Then, the n-th moment of X (i.e. $E[X^n]$) is said to exist if $E[\,|X^n|\,] < \infty$

1-Minute Summary

1. Expected Value

- Definition / alternative expression

2. Variance and Moments

- Definition / alternative explanation using penalty / properties
- Existence of moments