Data Structures

What are data structures?

A data structure is a way to store and organize data in order to facilitate access (e.g. search) and modifications (e.g. insertion, deletion).

No single data structure works well for all purposes, so it is important to know the strengths and limitations of a data structure.

n elements	search cost	insertion cost
sorted array	O(log n)	O(n)
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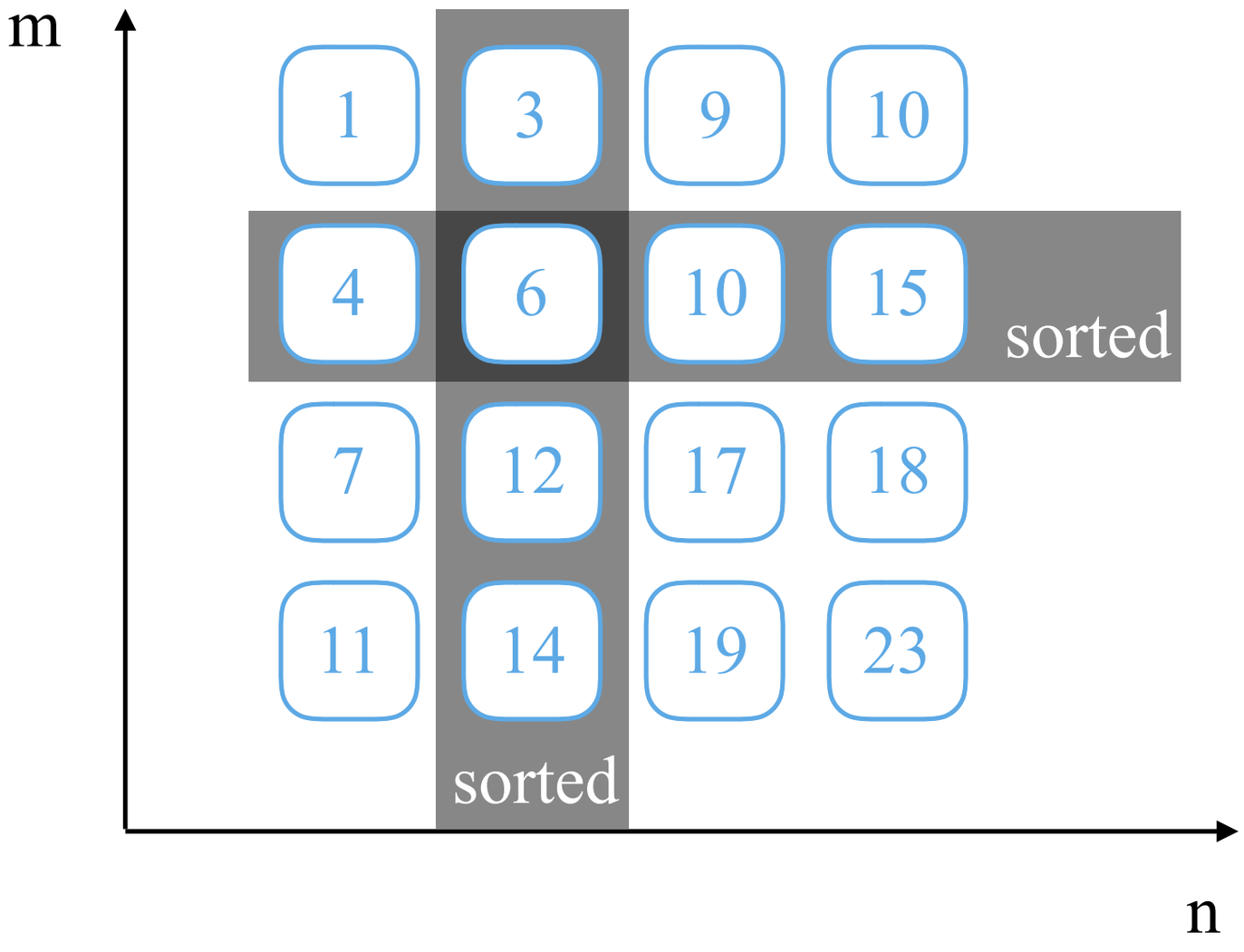
When do unsorted arrays outperform sorted ones?

2D Sorted Arrays

Young Tableau

An m by n Young Tableau is an m by n matrix so that each row and column is sorted in nondecremental order.

Example.

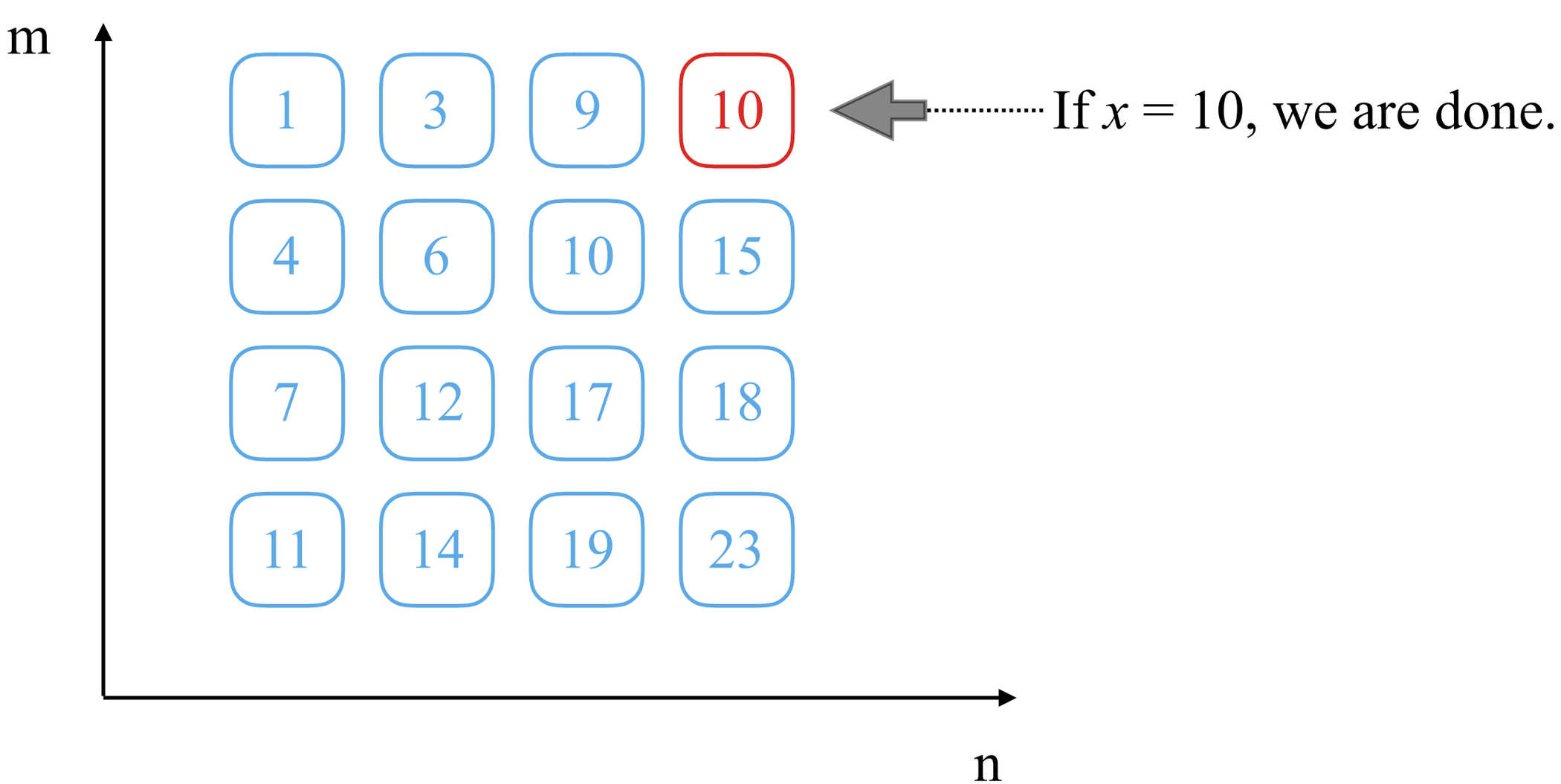


Search on a Young Tableau

Input: a query x.

Output: "Yes," if x is a member in the tableau; "No," otherwise.

Search cost is O(n+m).



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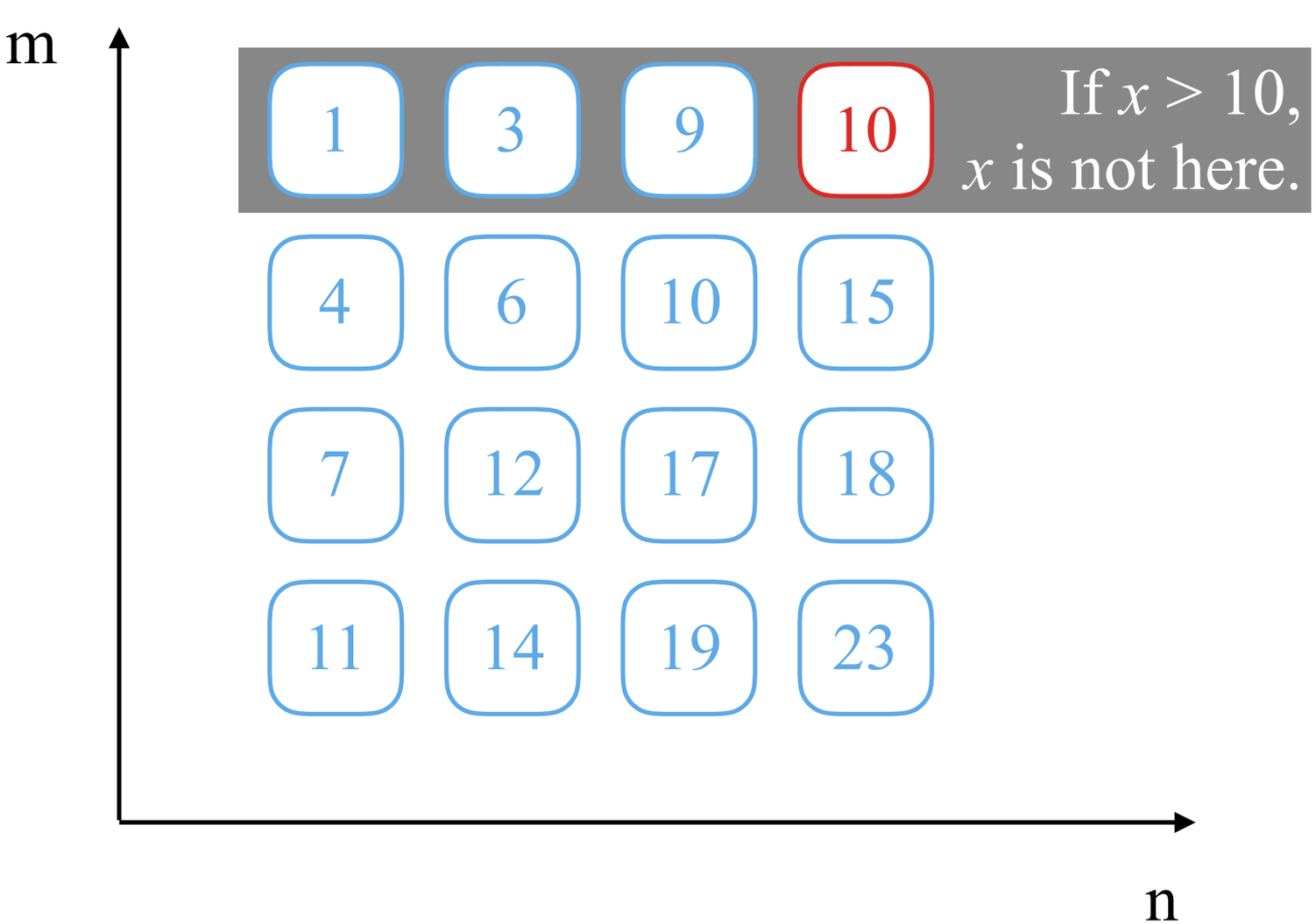

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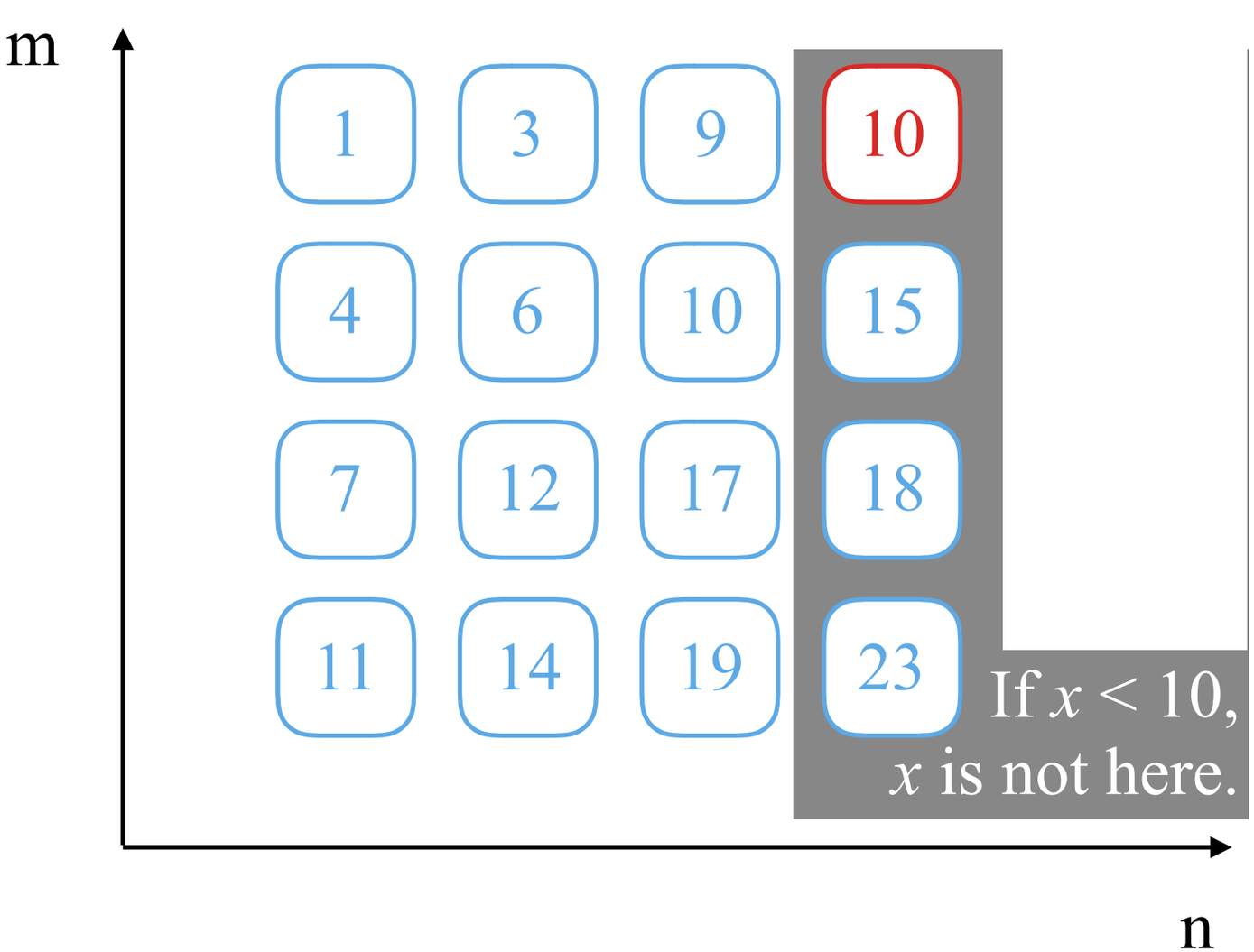

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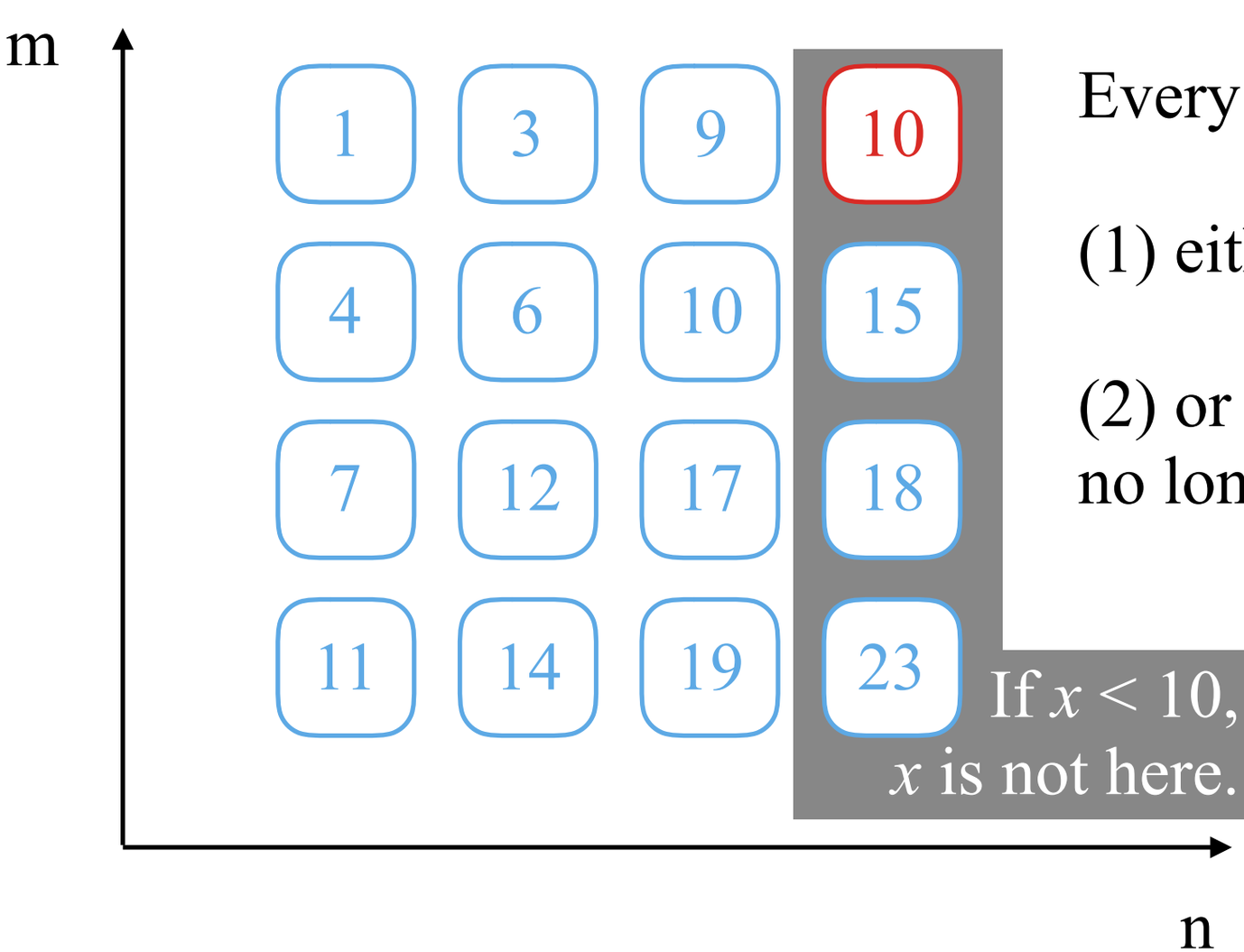

Every time a comparison is made,

(1) either x is found,

(2) or a row (column) of candidates are no longer possible.

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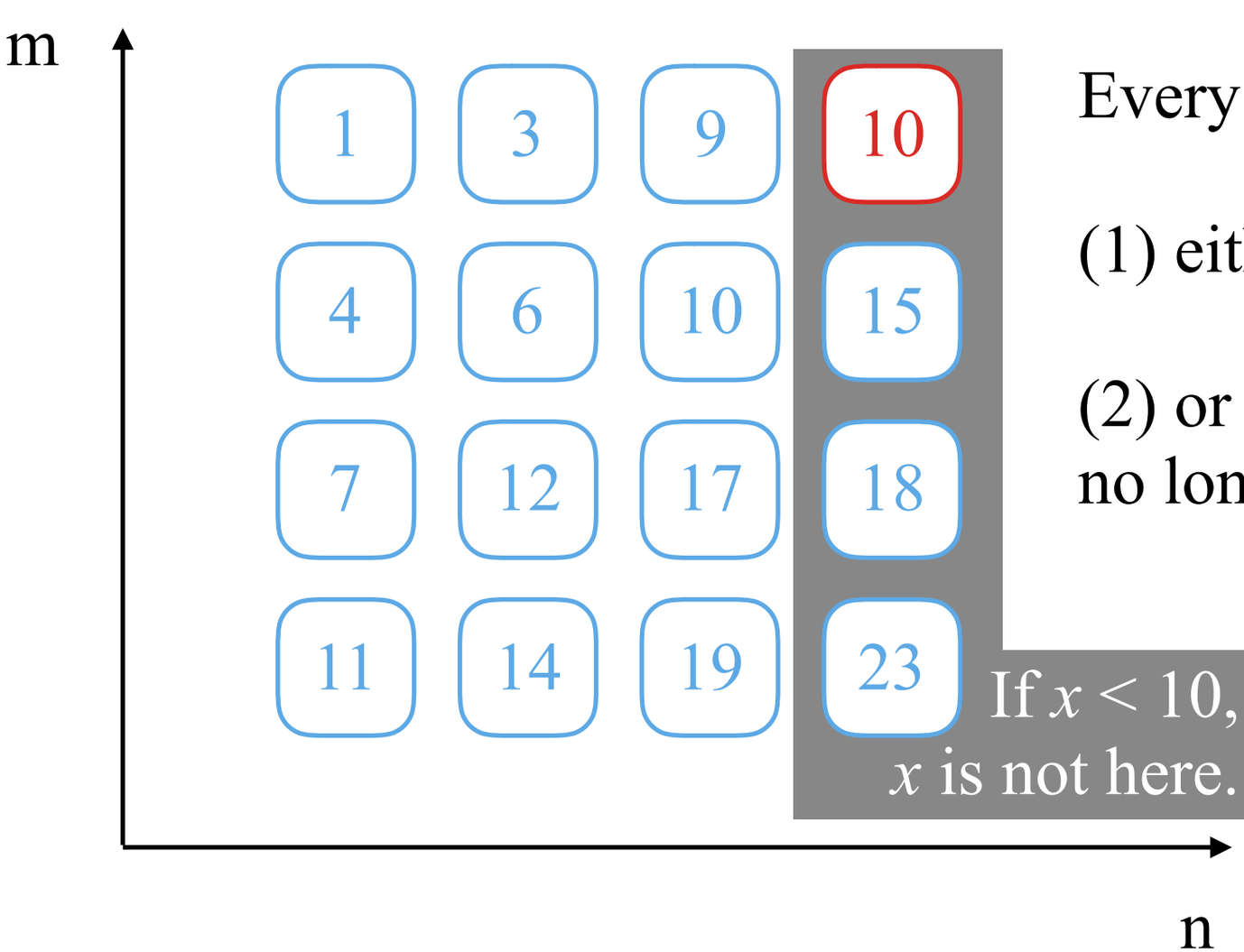

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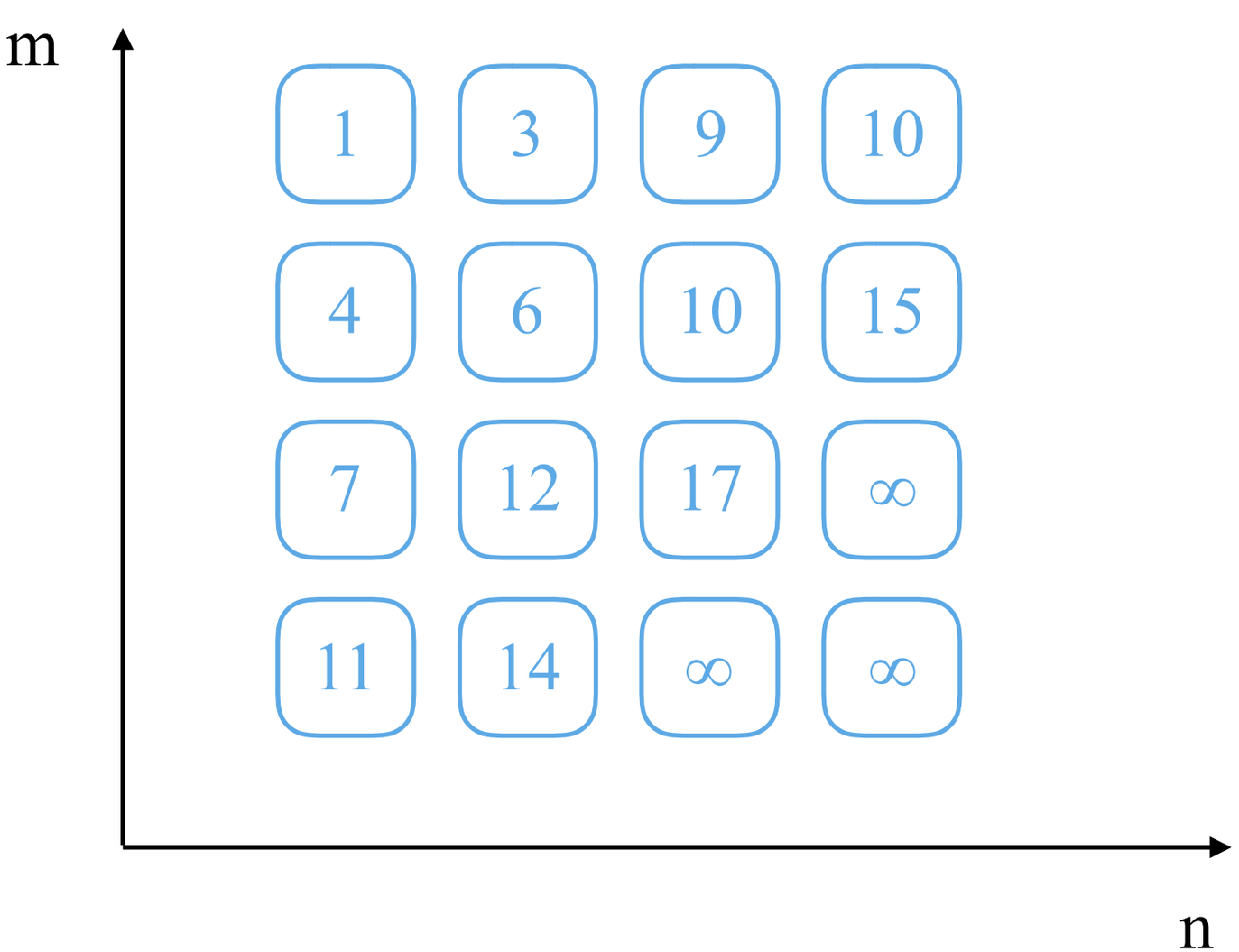
There are O(n+m) rows and columns, yielding the bound.

Insertion on a Young Tableau

Input: a newly-added element x.

No output. Rearrange data to satisfy the requirements of Young Tableau.

Insertion cost is O(n+m).



∞ denotes an empty slot.

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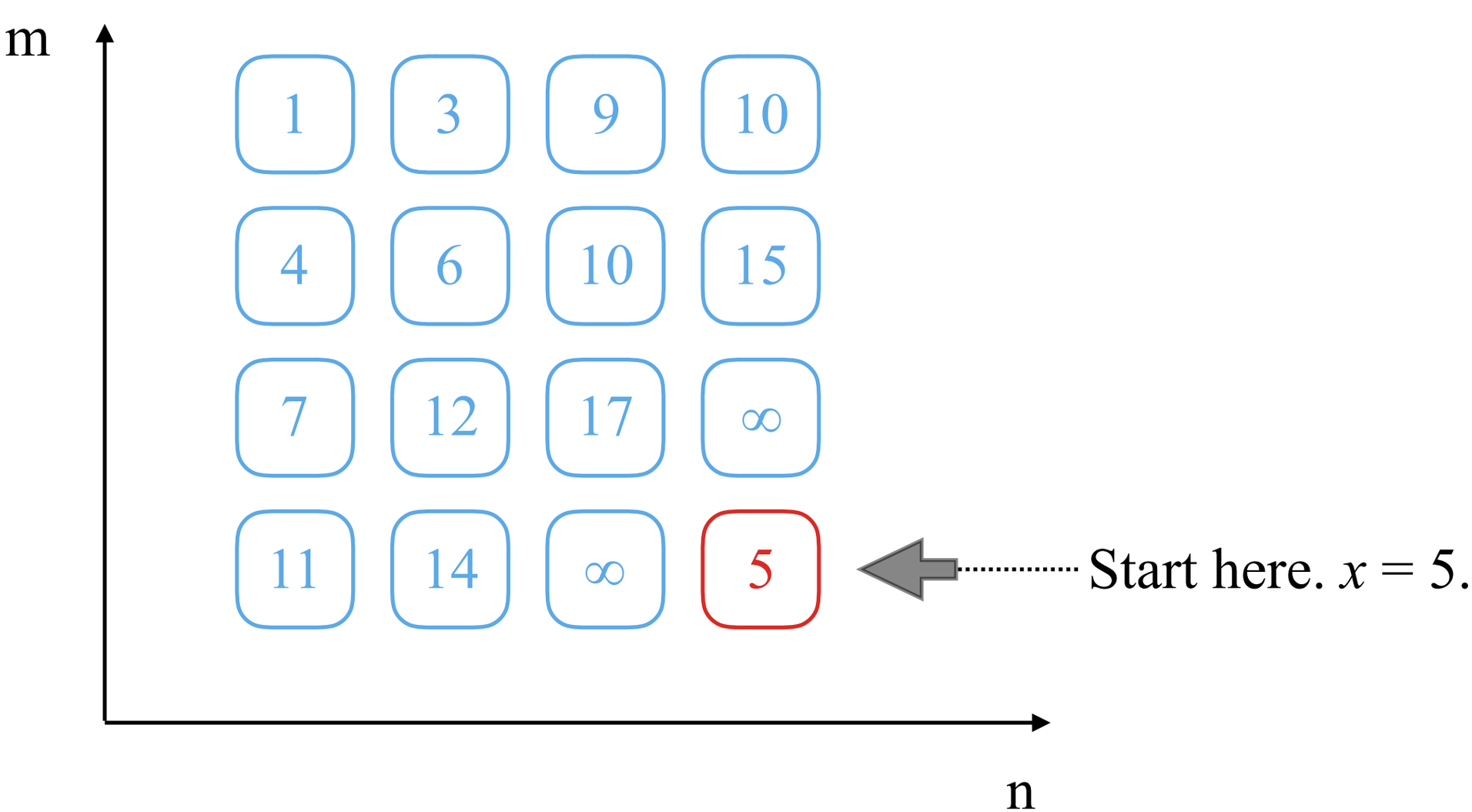

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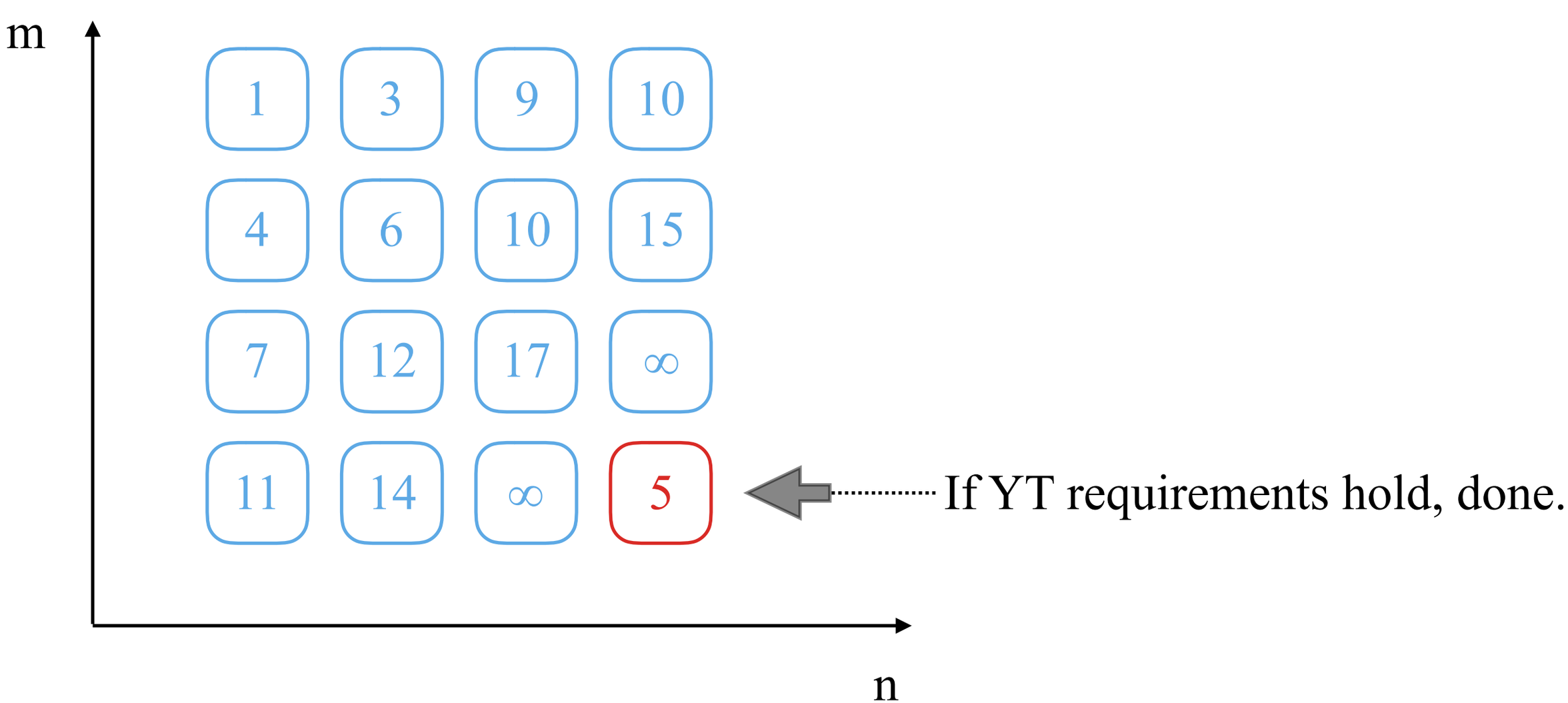

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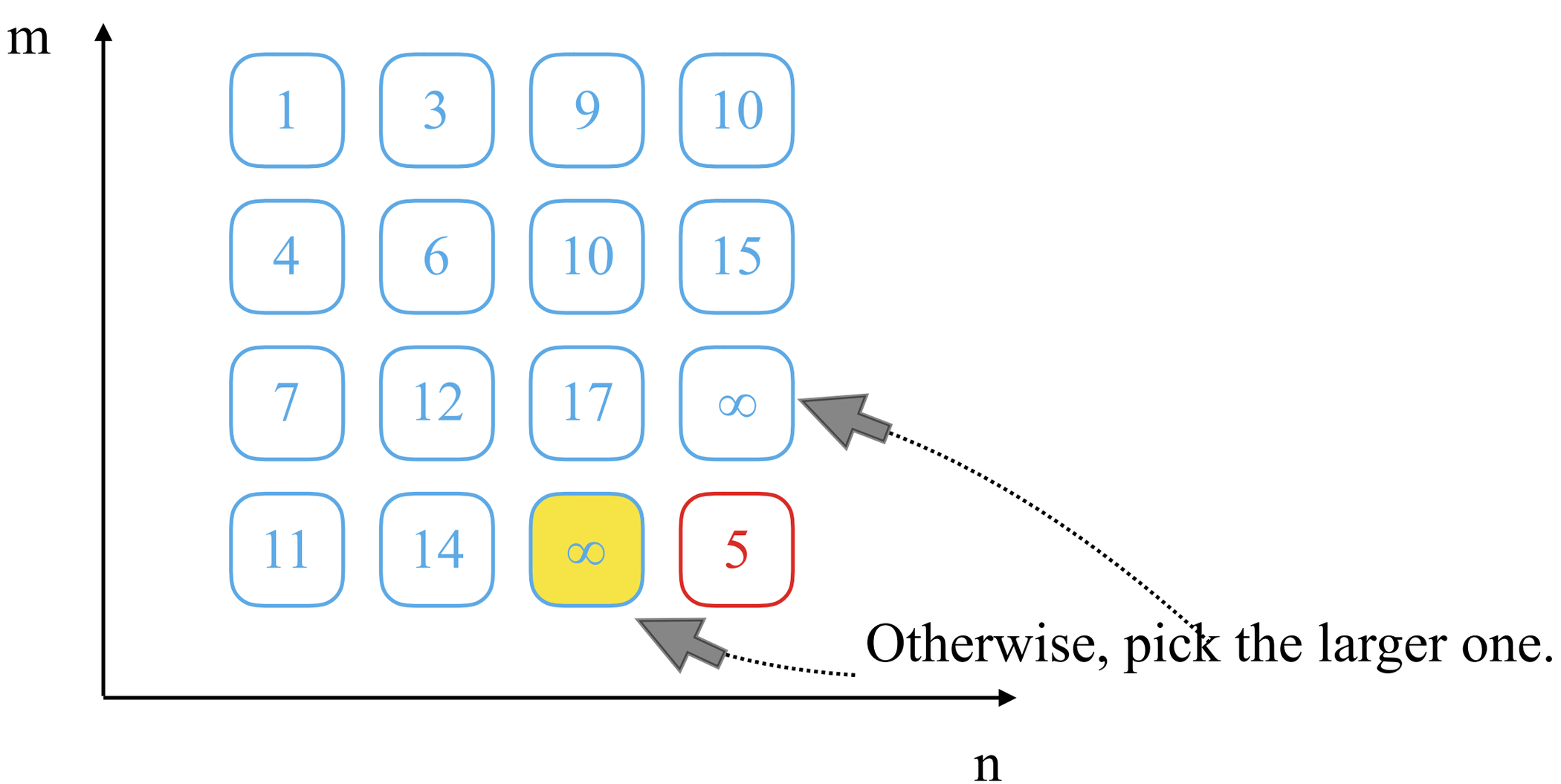

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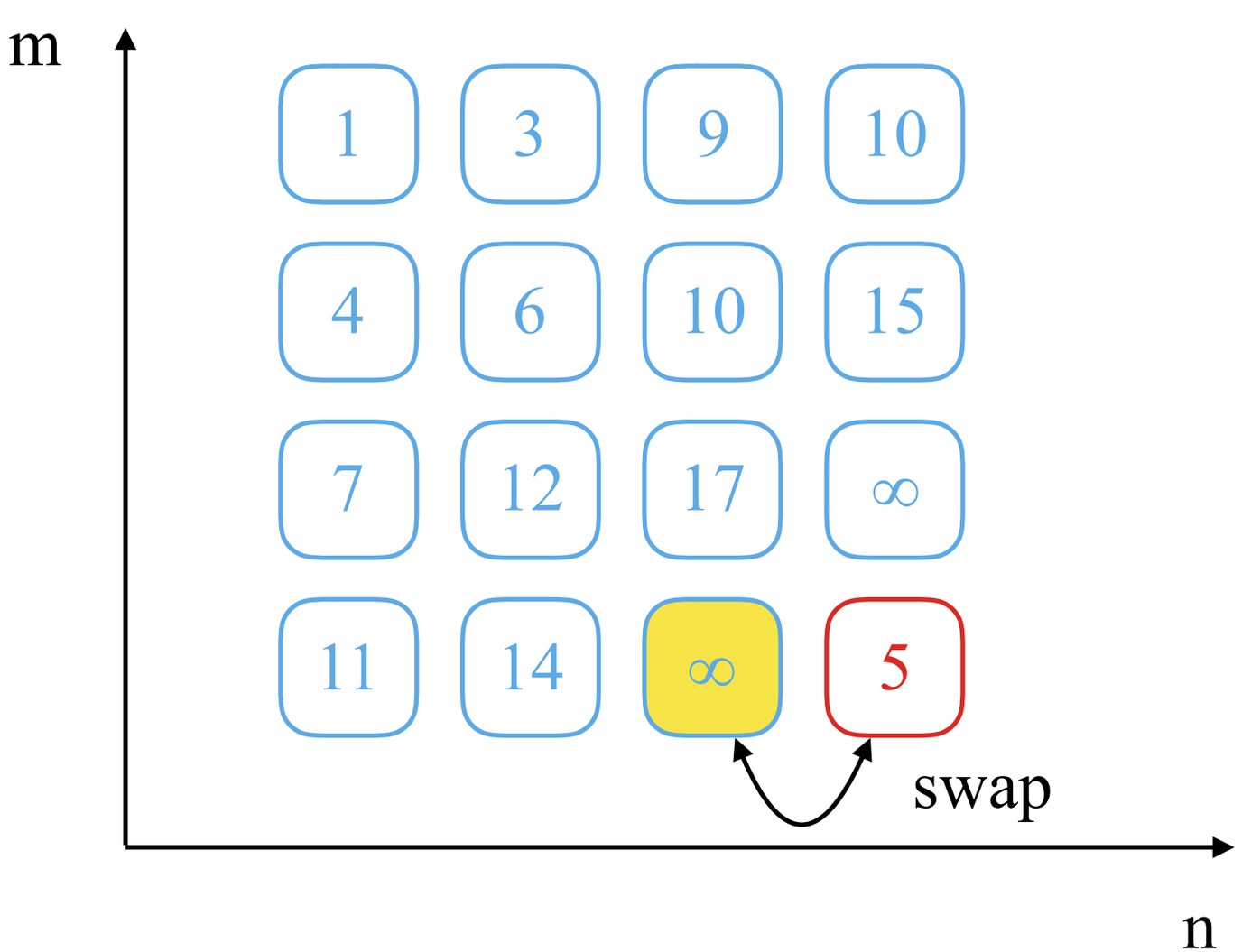

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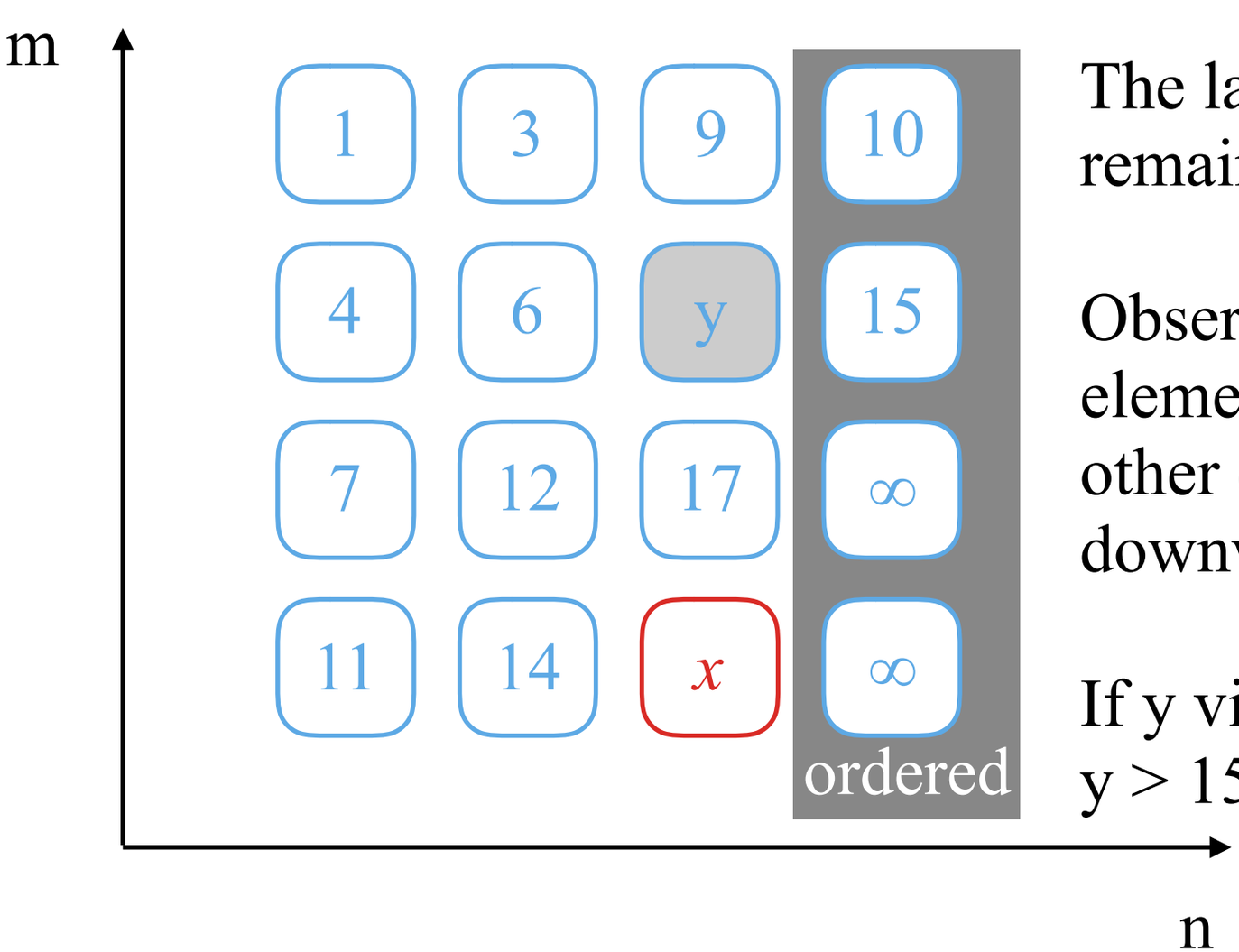

The last column is ordered now, and remains ordered in the subsequent steps.

Observe: (1) x is smaller than any element in the last column, and (2) other elements can move only downward and rightward.

If y violates the requirements, then $y > 15$, which is impossible.

Insertion on a Young Tableau

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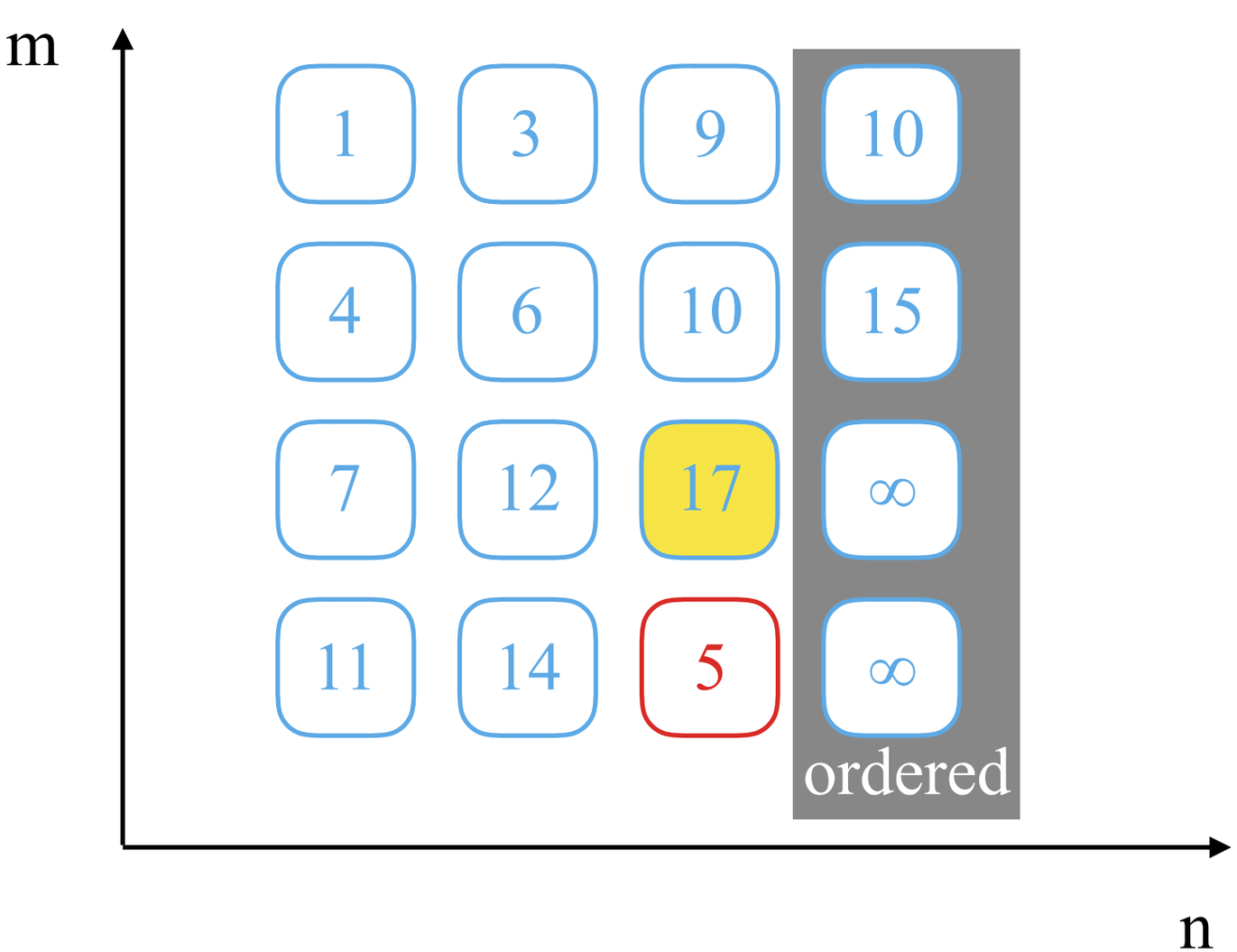

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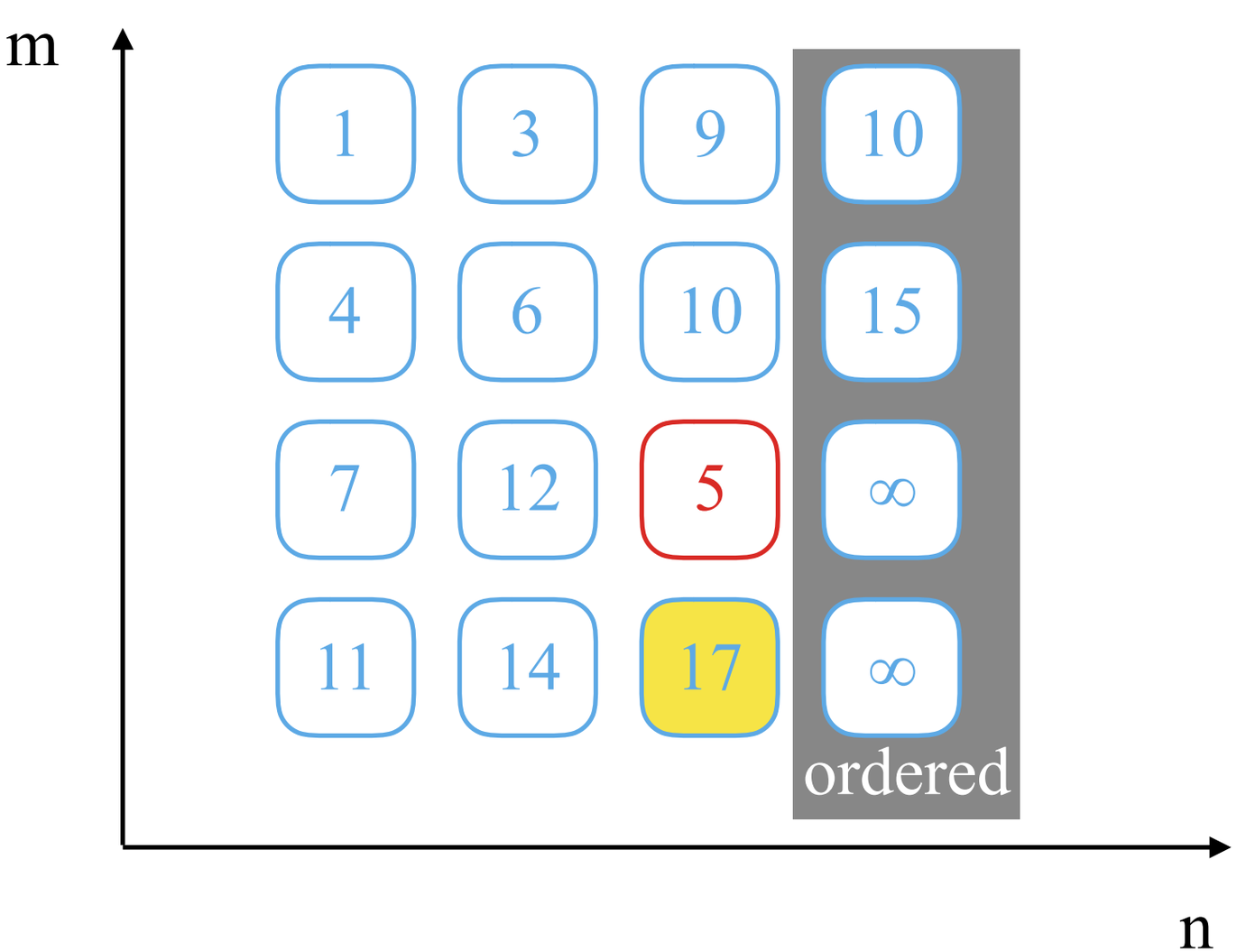

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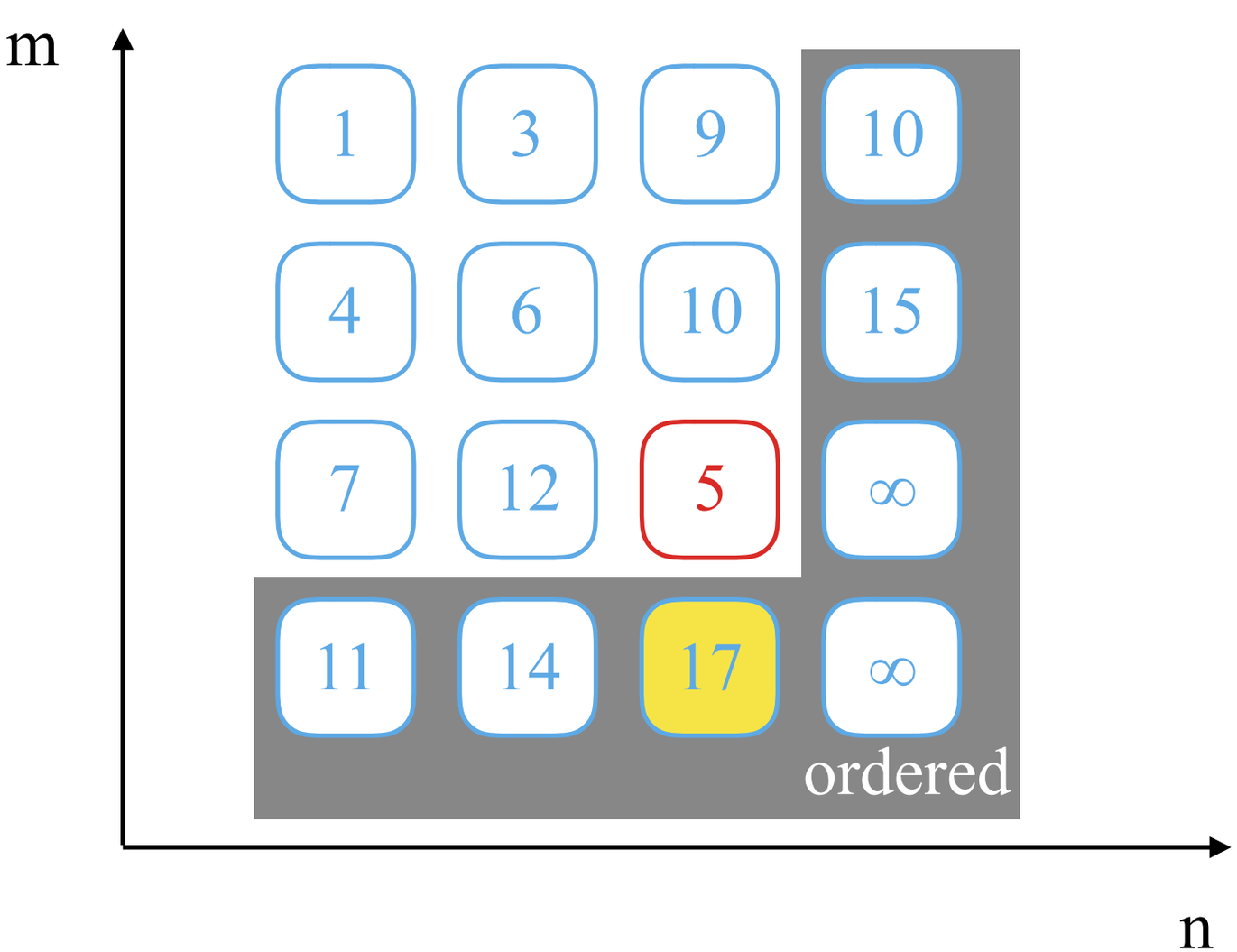

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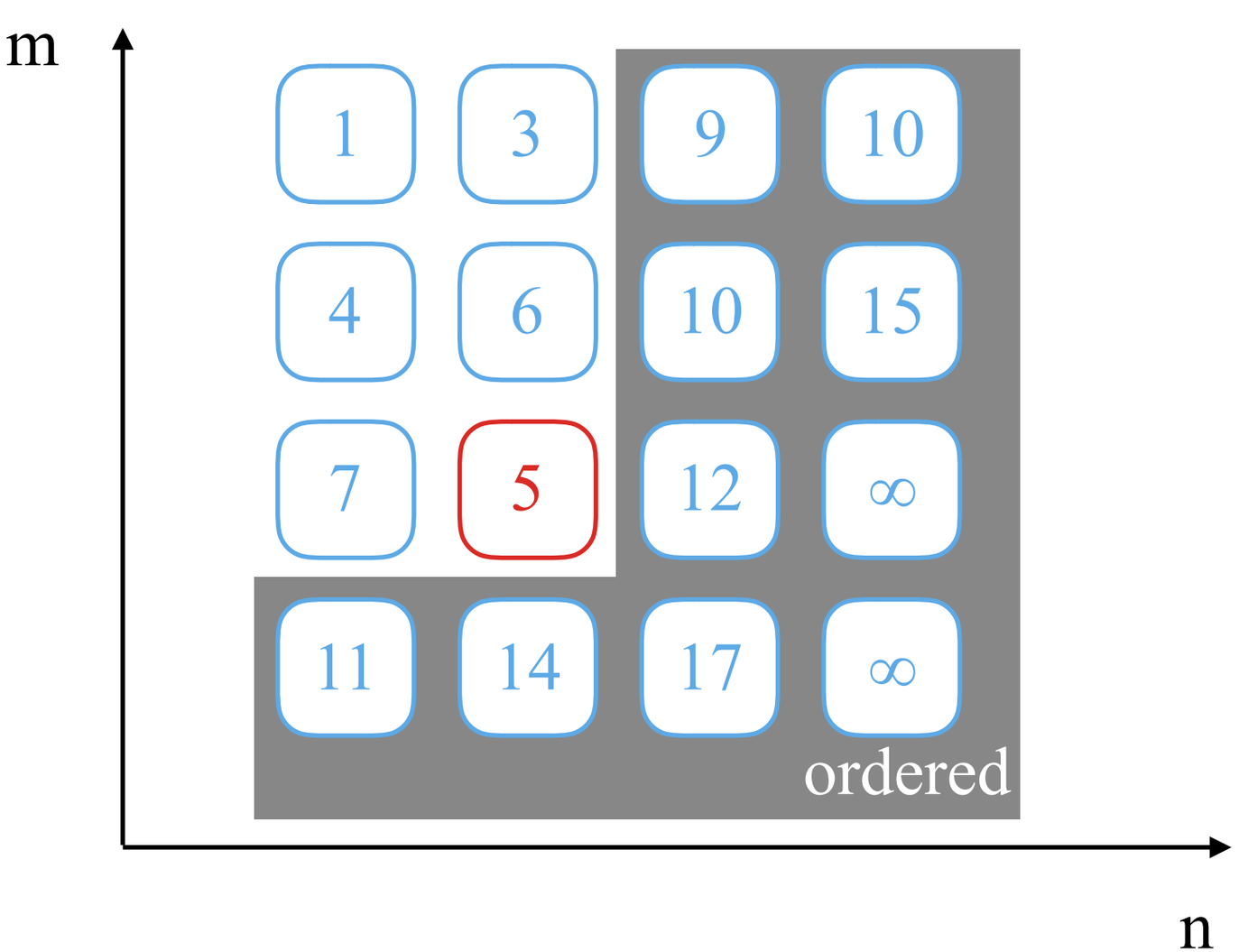

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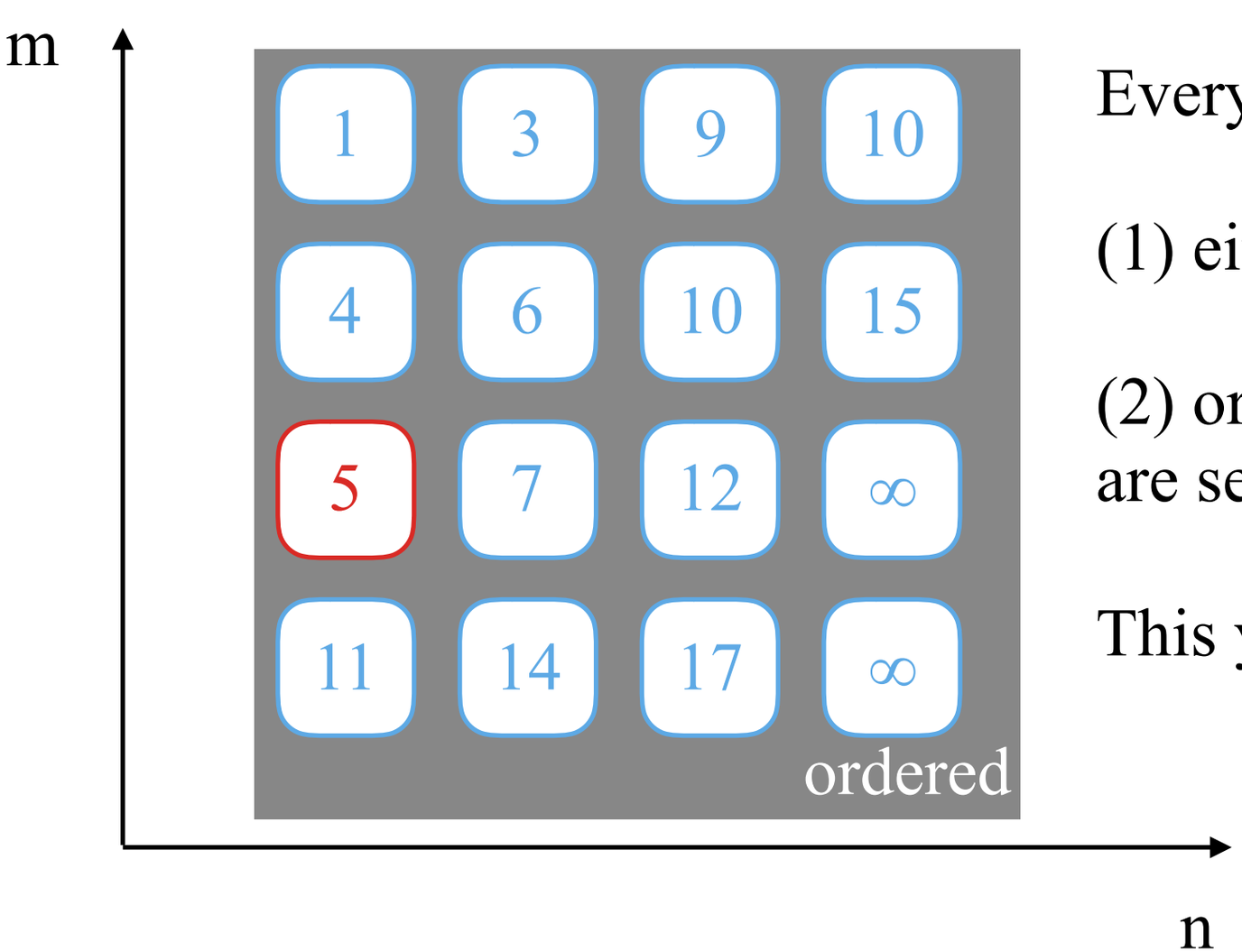


Every time a comparison is made,

(1) either x is settled,

(2) or a row (column) of elements are settled.

This yields a time bound O(n+m).

Summary

n elements	search cost	insertion cost
sorted array	O(log n)	O(n)
Young tableau	$O(n^{1/2})$	$O(n^{1/2})$
unsorted array	O(n)	O(1)

Heaps

Heaps

An array A is a max heap if for every i in [1, n], element A[i] has value less than or equal to its parent A[parent(i)] where

$$\text{parent(i)} = \lfloor i/2 \rfloor.$$

Min heaps are ordered in the opposite way; that is, A[i] ≥ A[parent(i)] for every i in [1, n].

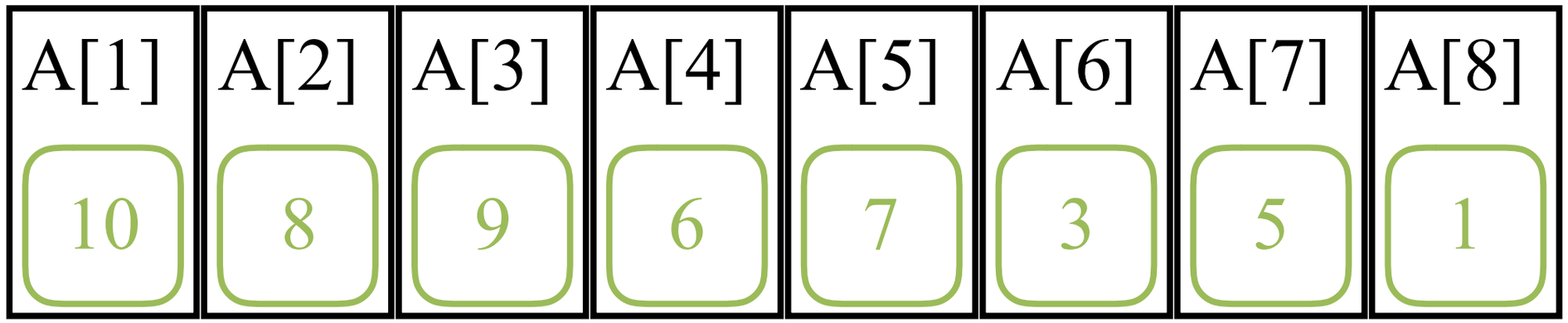
Example.

array A

A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]
10	8	9	6	7	3	5	1

View heaps as binary trees

array A



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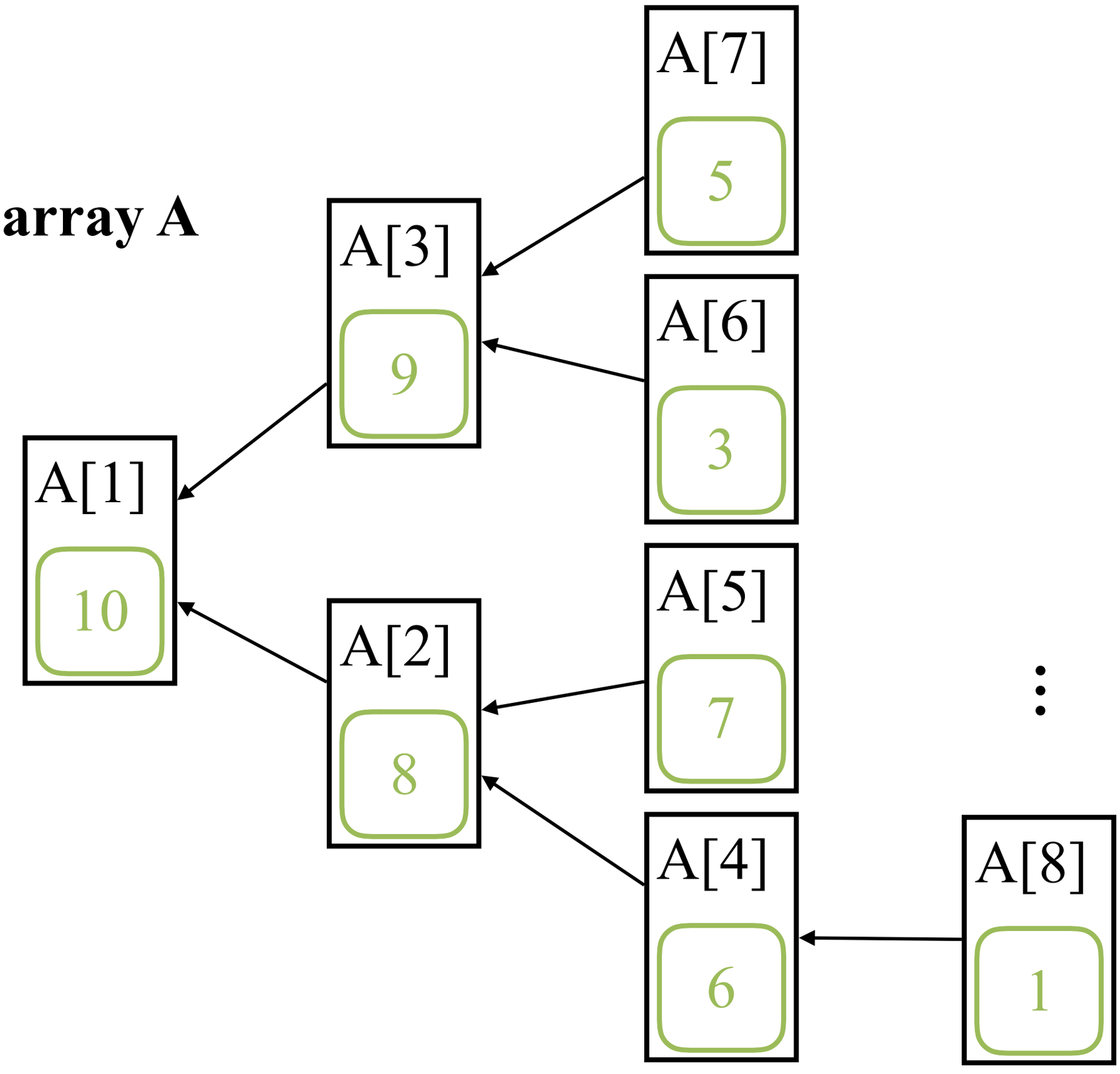

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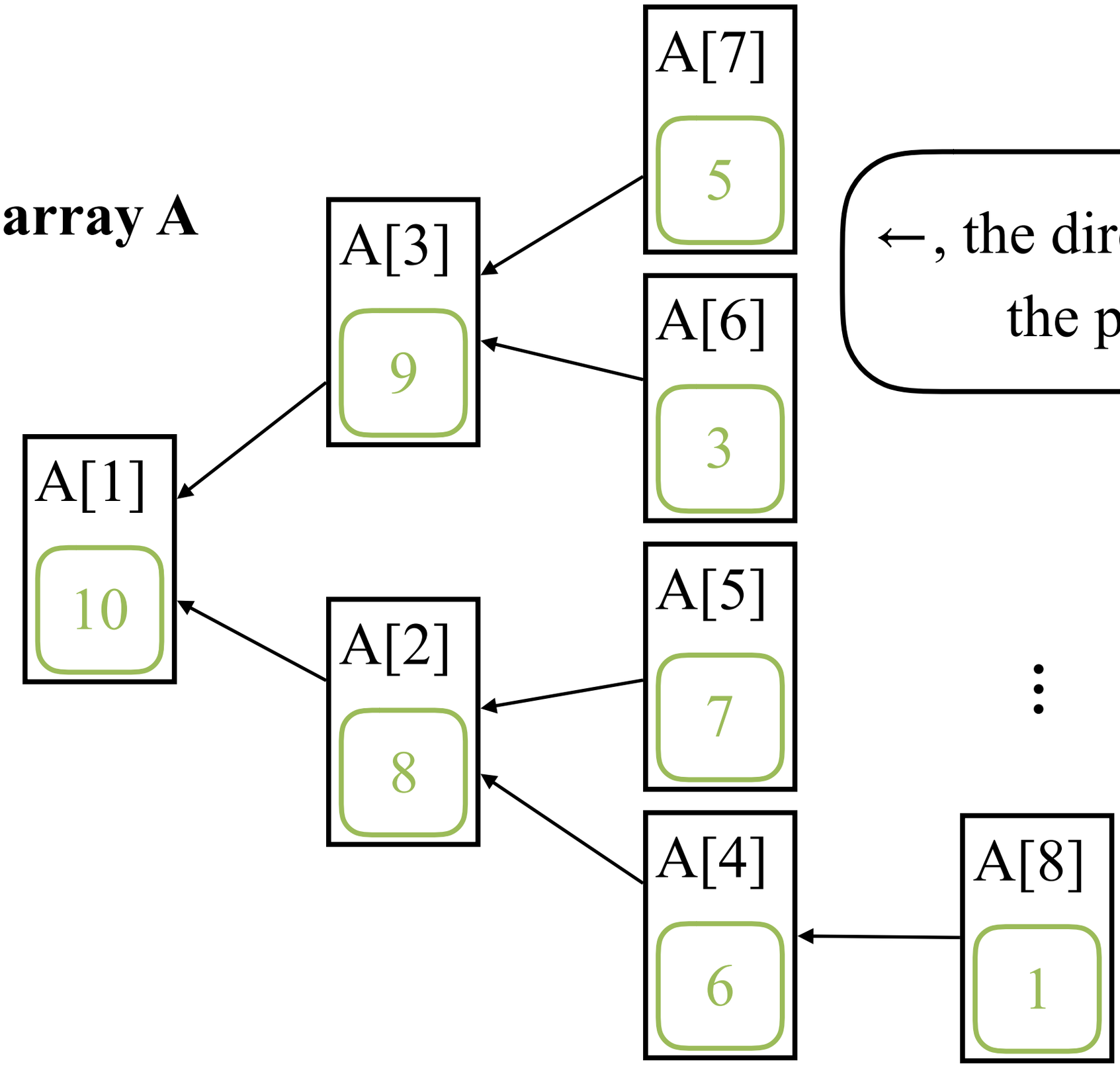
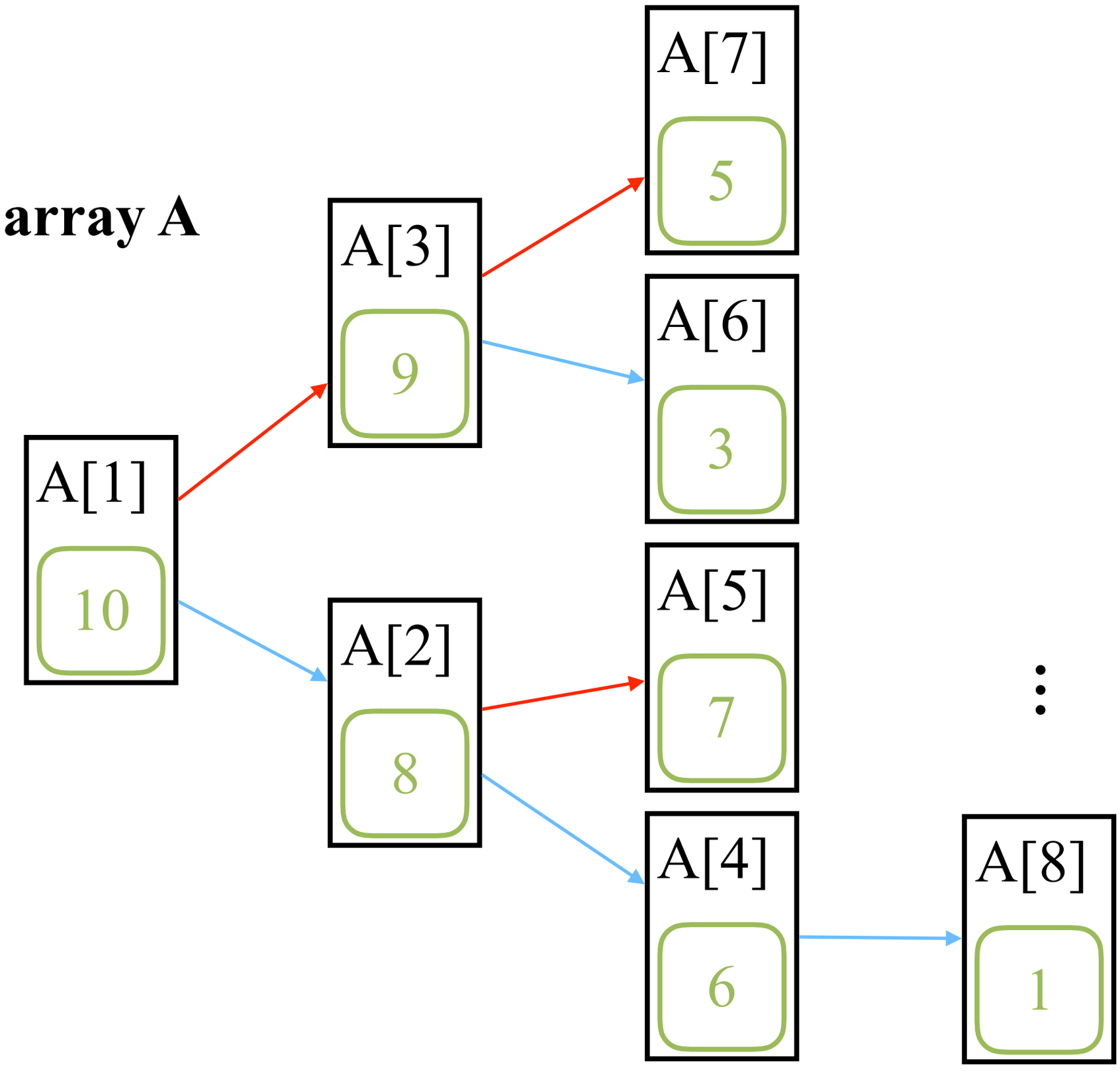


←, the direction pointed to the parent node.

View heaps as binary trees



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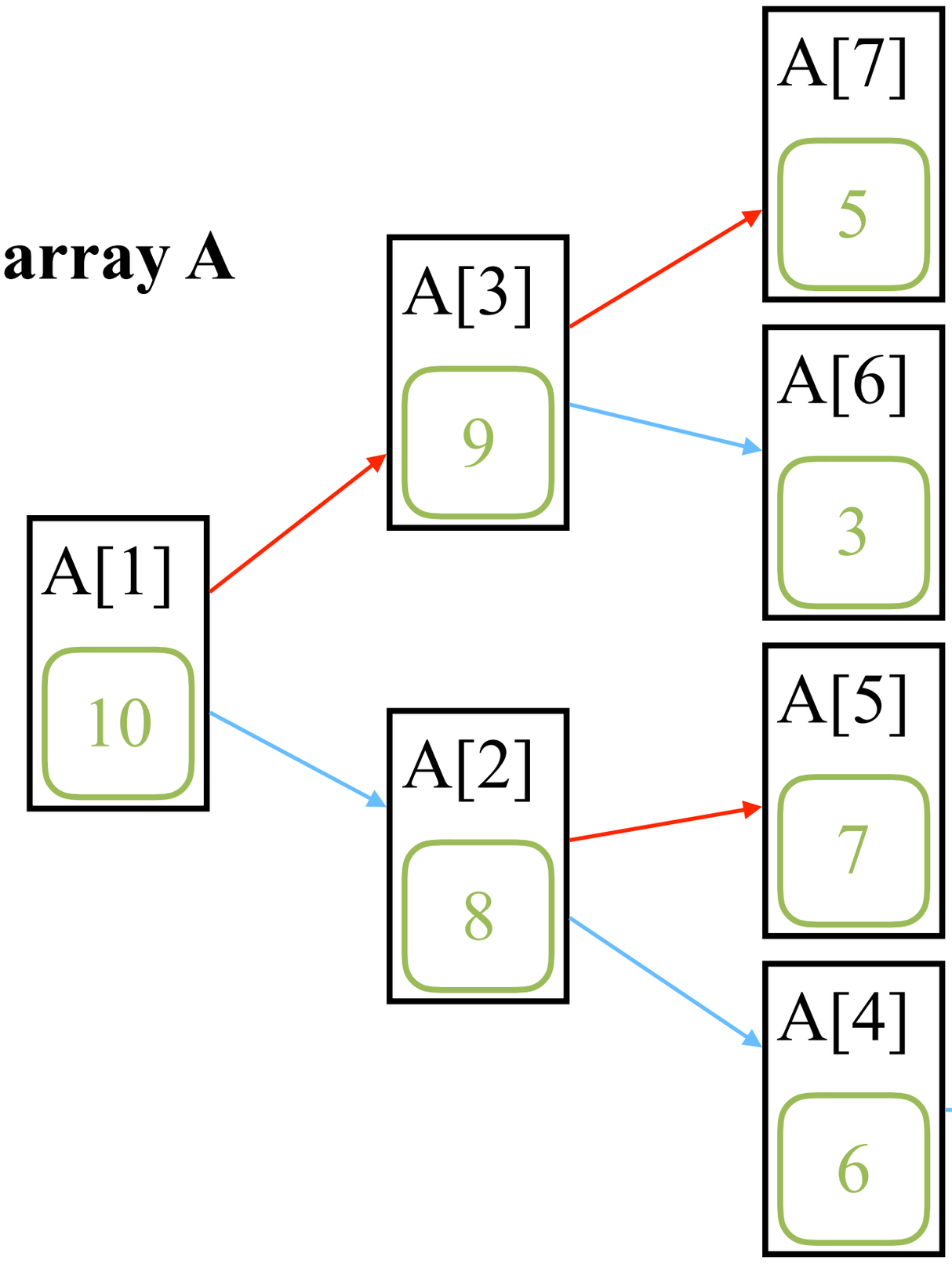


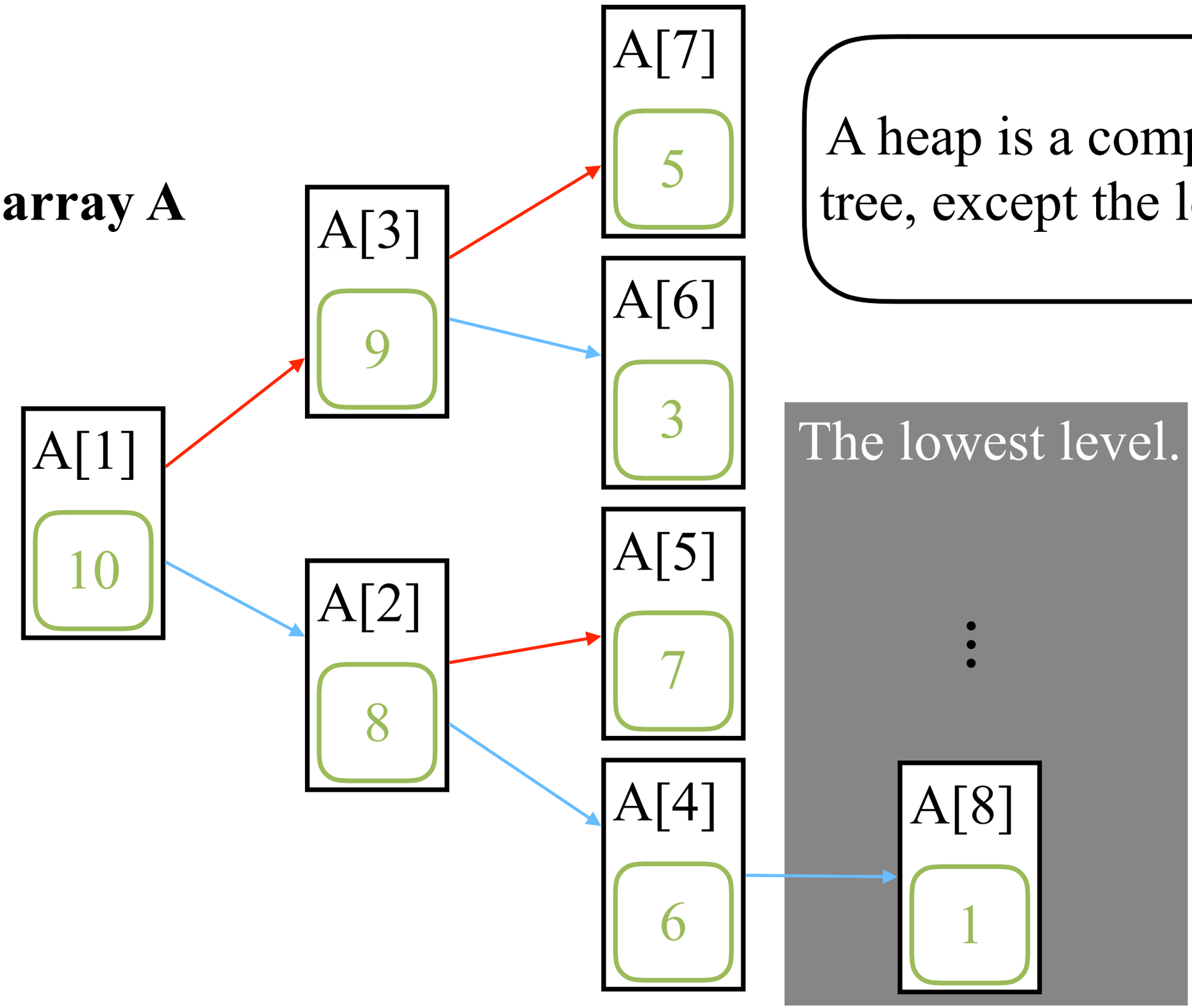
← , the direction pointed to the left child.
Left(i) = 2*i.

← , the direction pointed to the right child.
Right(i) = 2*i+1.

⋮

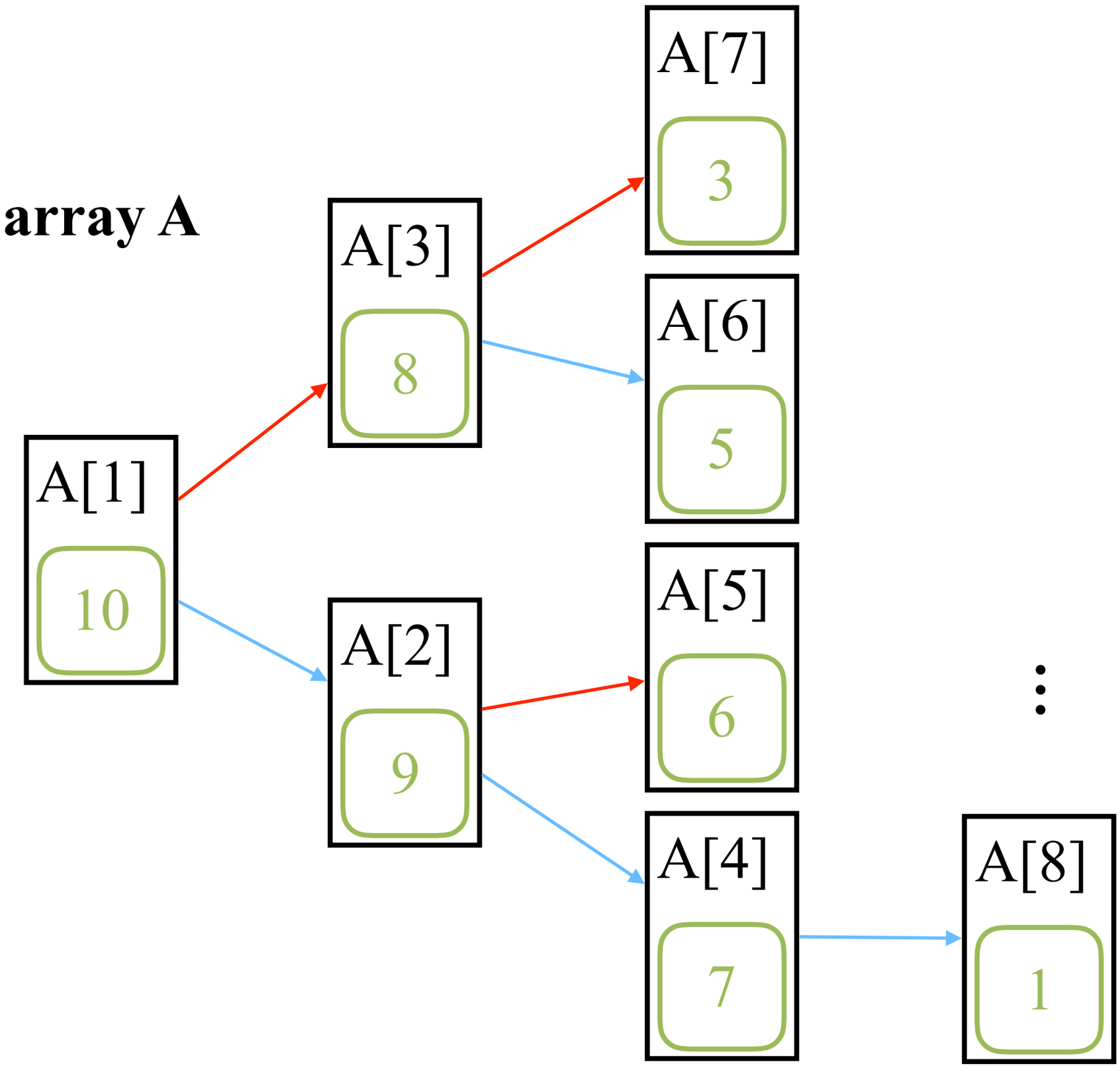
View heaps as binary trees

array A



A heap is a complete binary tree, except the lowest level.

Is sorted array a heap?



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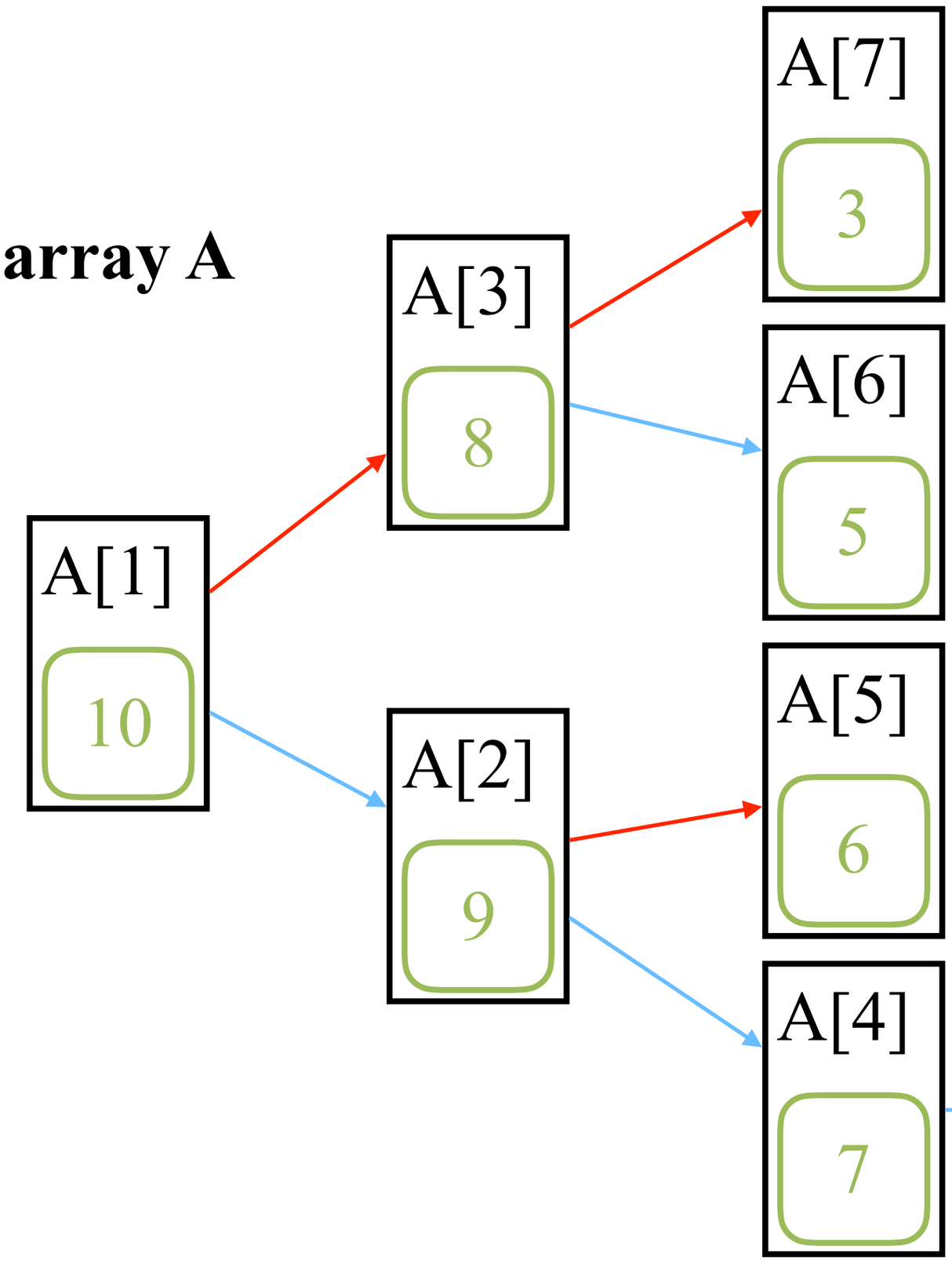

Yes. Why do we need heaps?

Construction Time

Given n elements, constructing a heap of the n elements needs $O(n)$ time.

However, sorting the n elements requires $\Omega(n \log n)$ time.

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sort must be a heap

heap may not be sorted

That is why heaps are not subsumed
by sorted arrays.

Heapification

Max-Heapify(A, i)

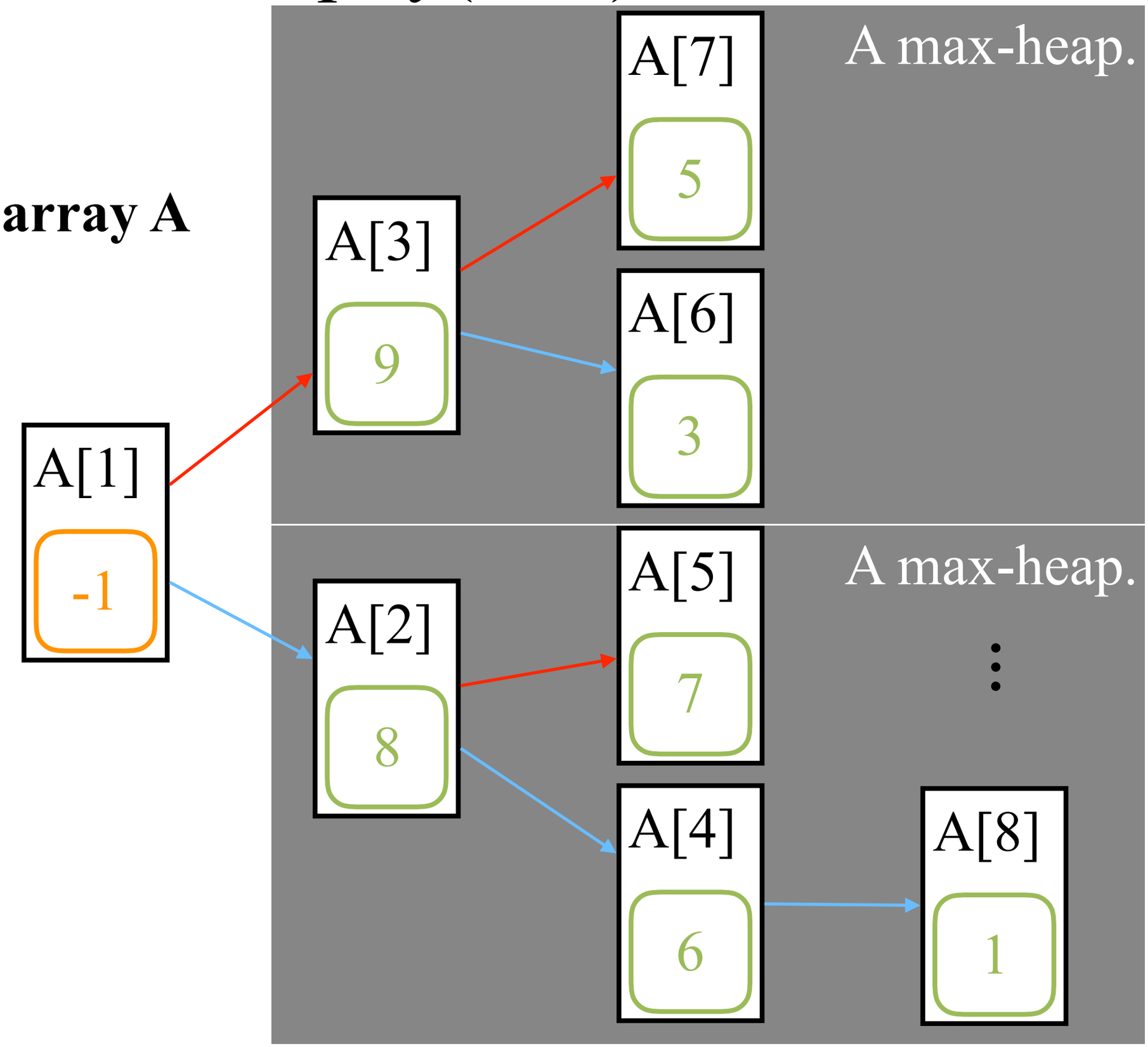
convert the subtree rooted at node i into a max heap assuming that

(1) the subtree rooted at node Left(i) is a max heap, and

(2) the subtree rooted at node Right(i) is a max heap.

Max-Heapify(A, 1)

array A



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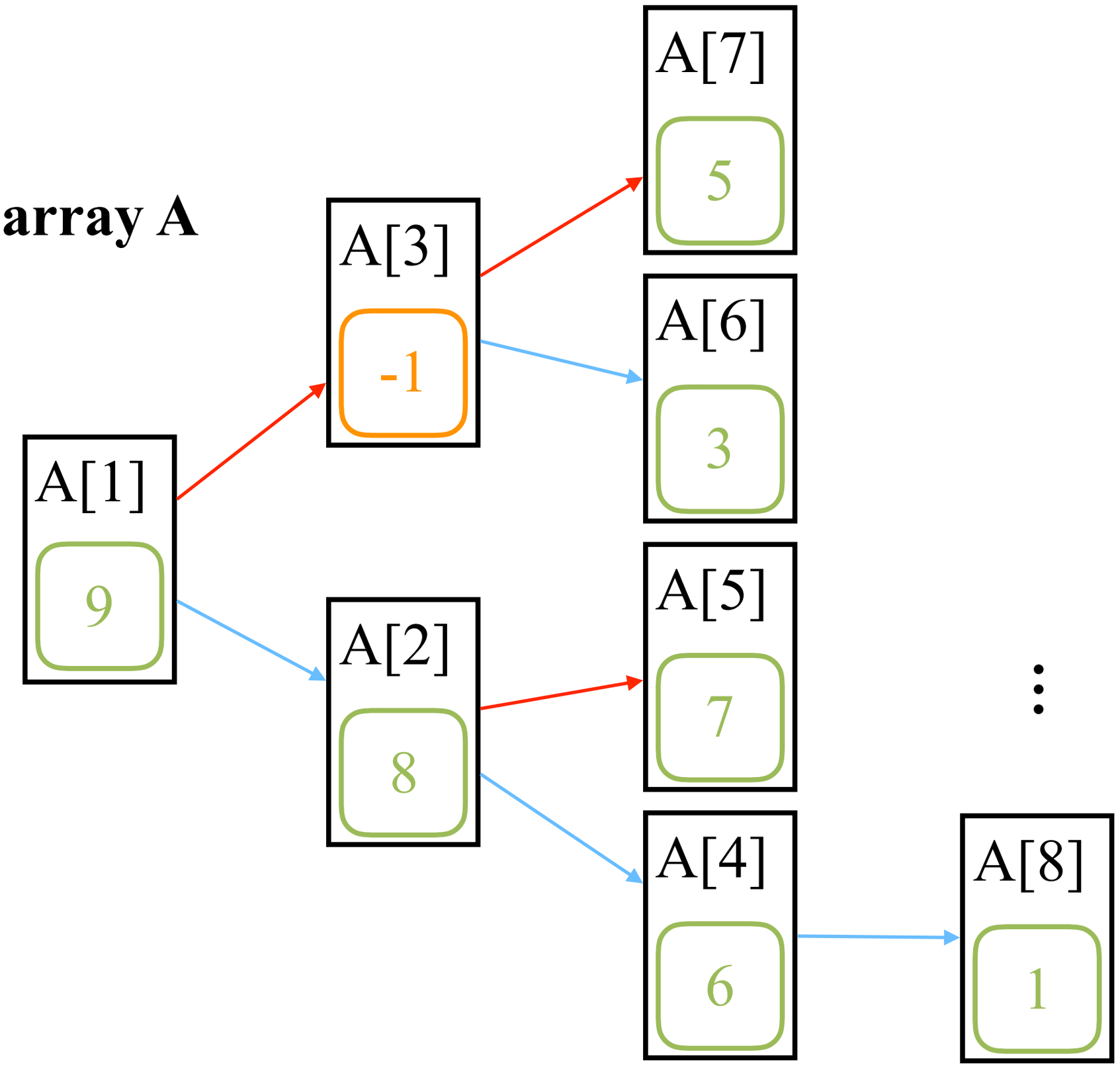

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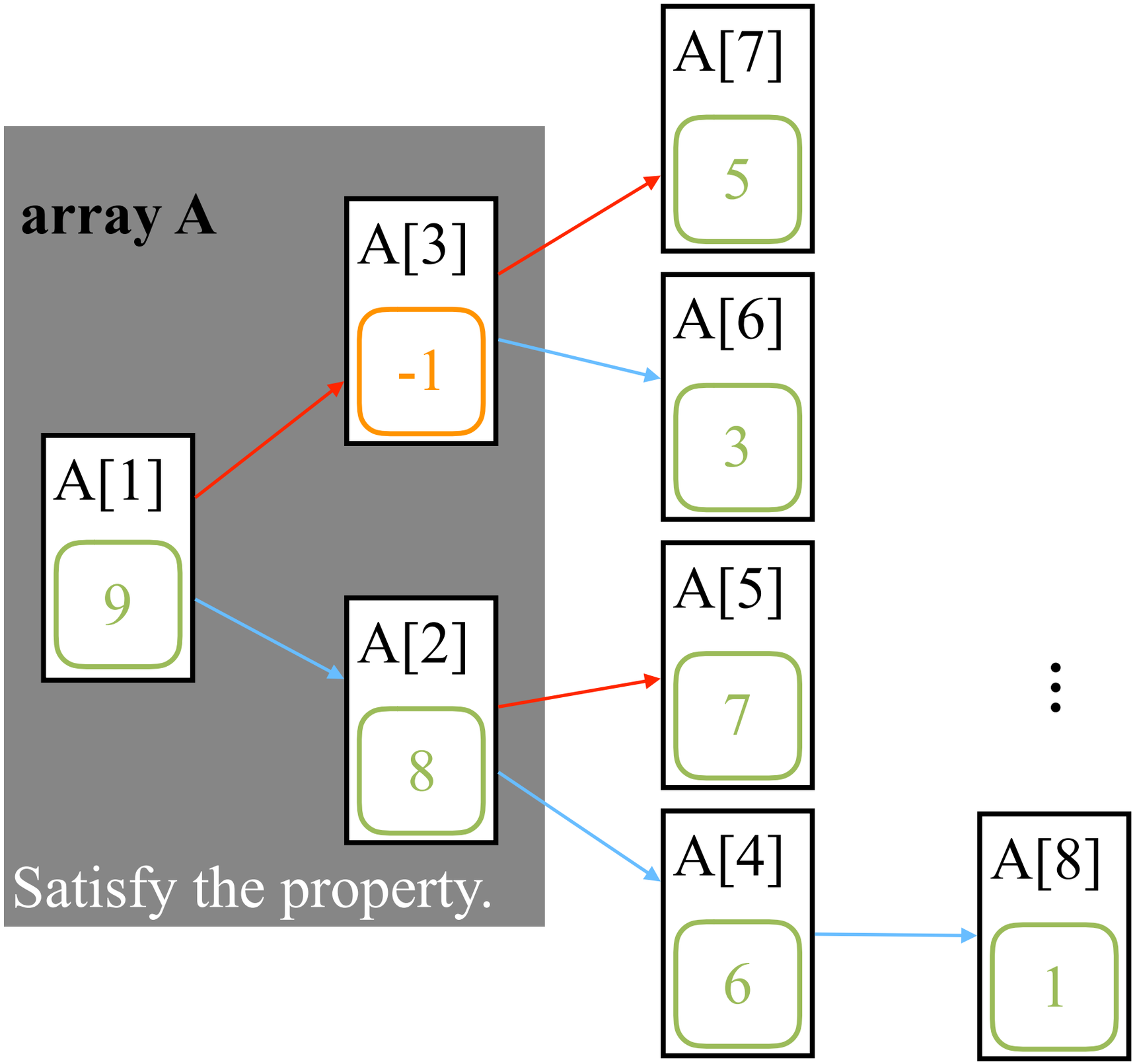

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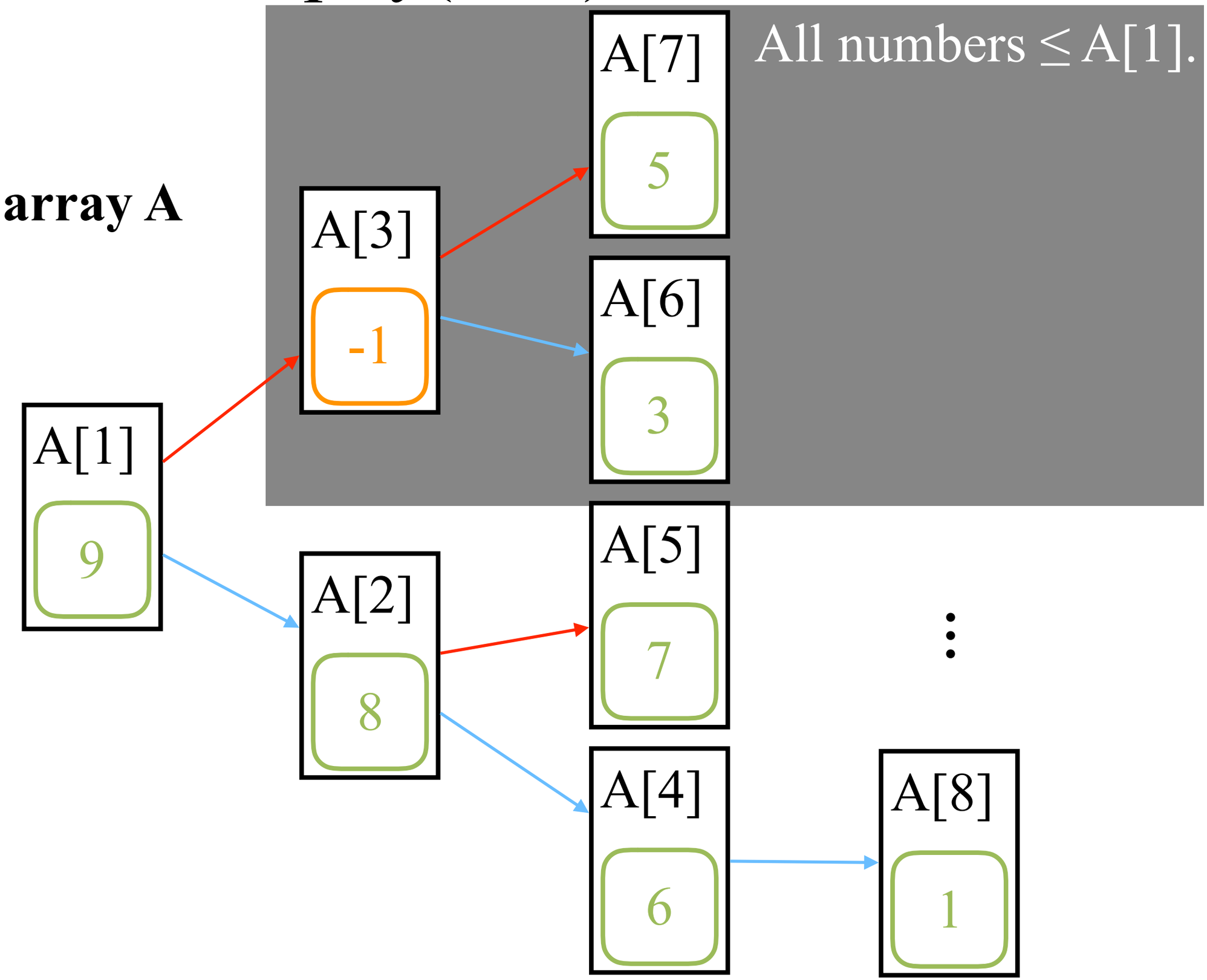

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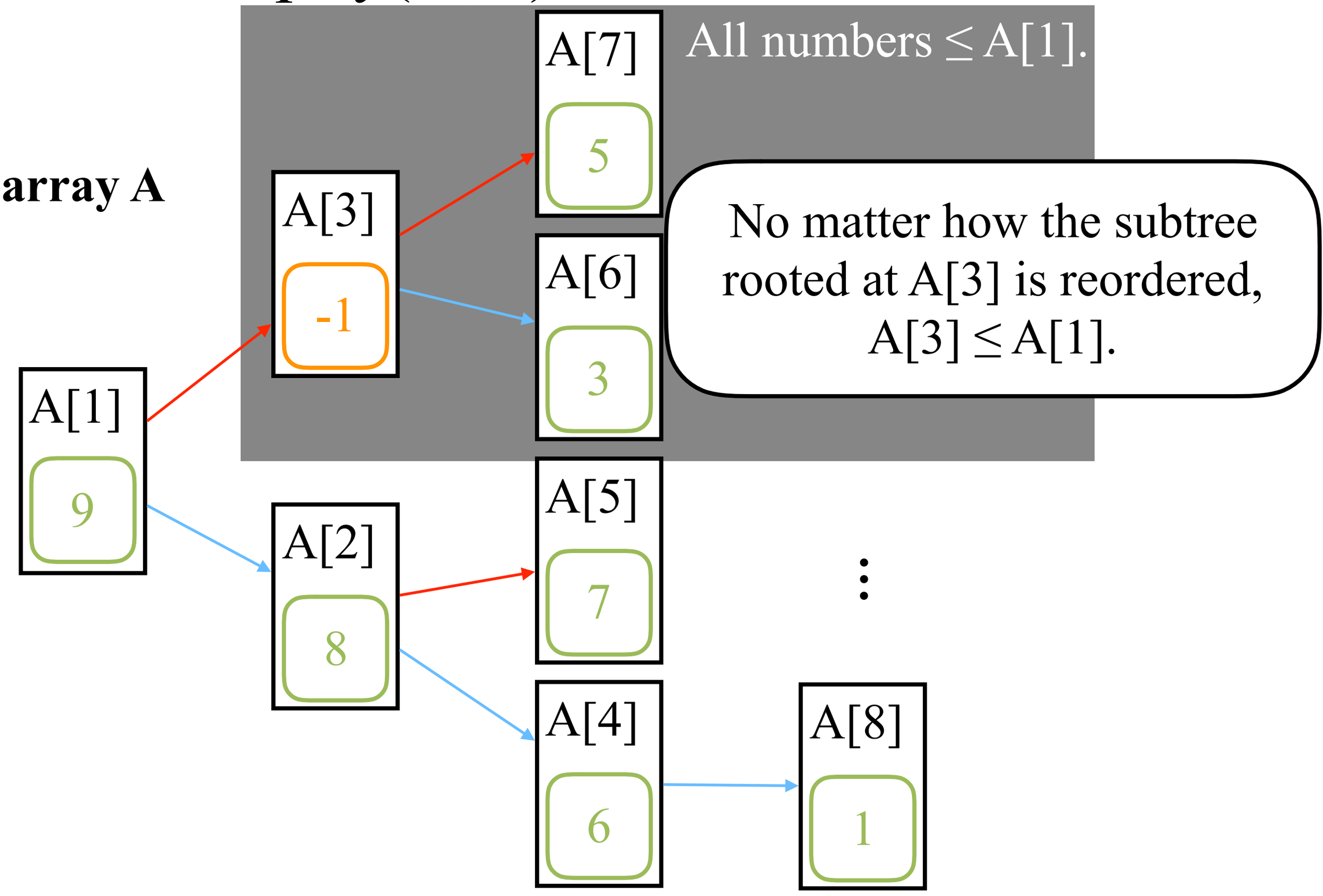
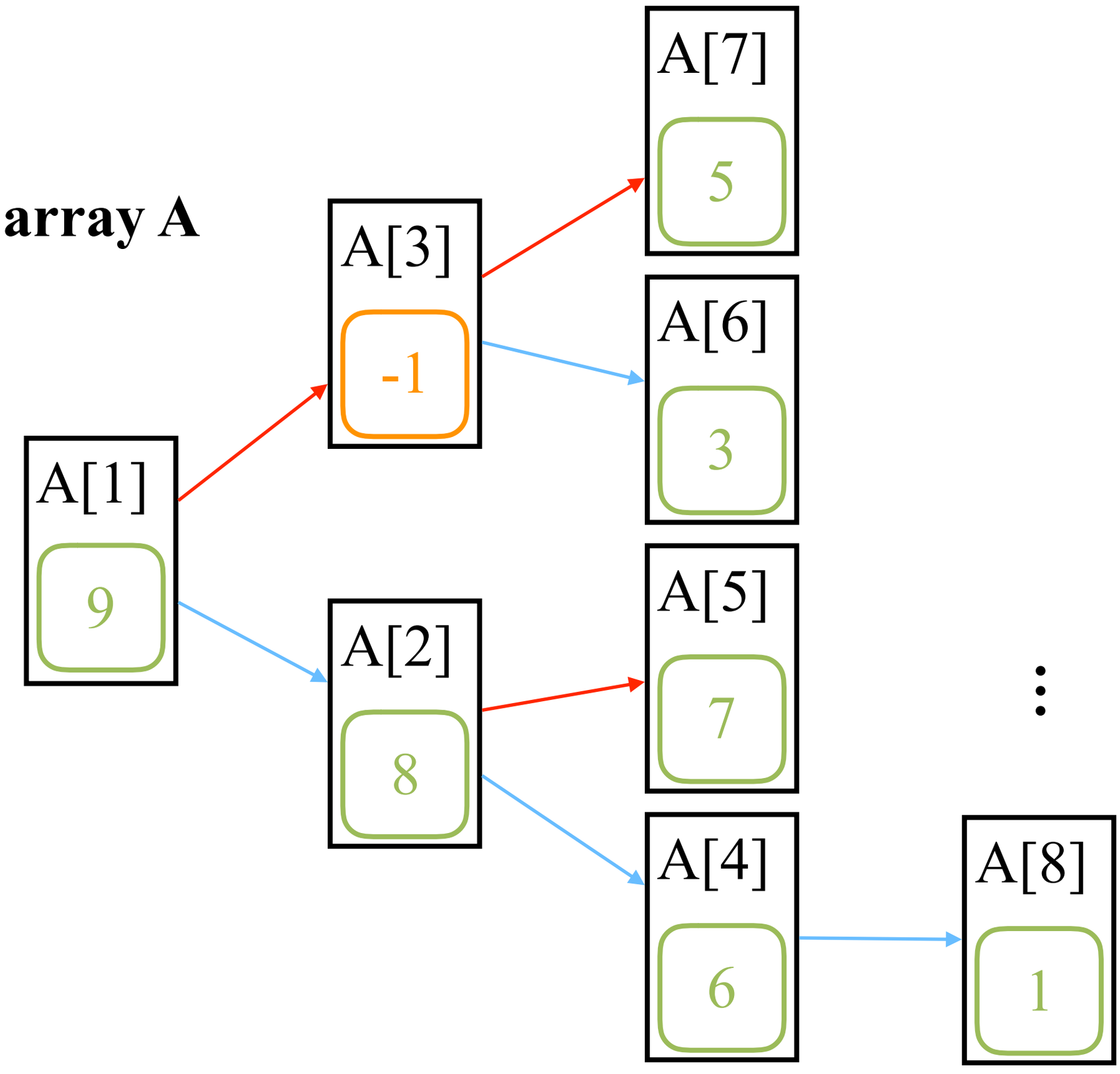
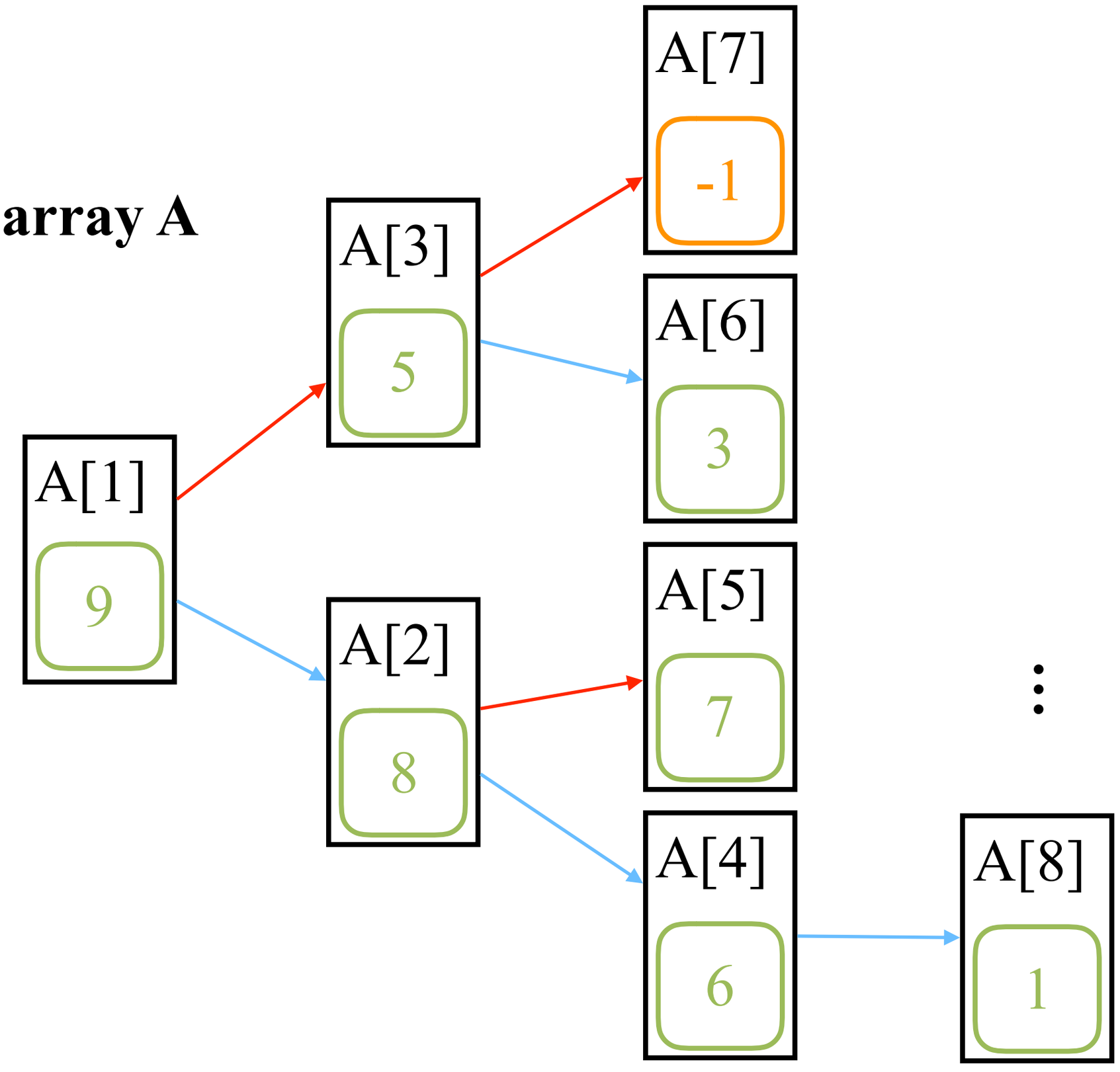


Max-Heapify(A, 3)



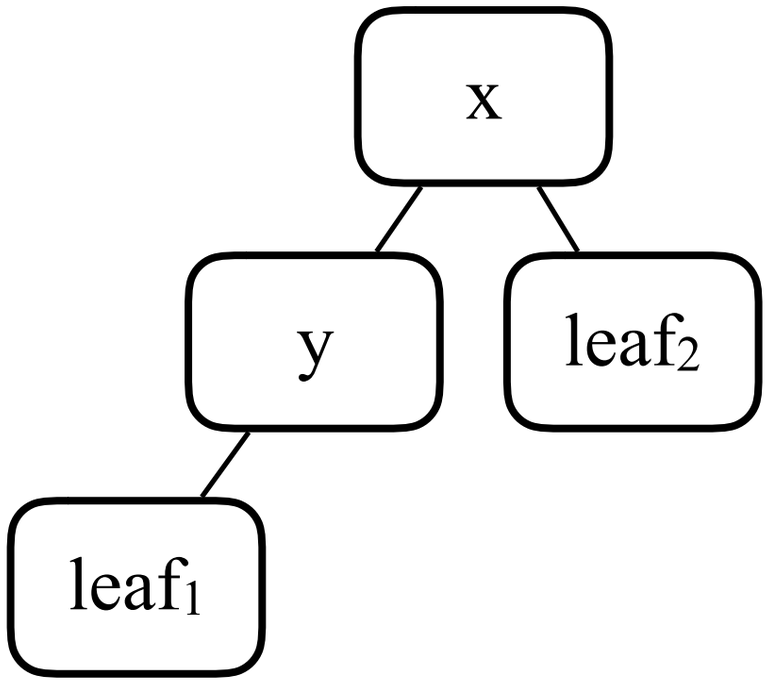
Max-Heapify(A, 3)

array A



Height

Define *height* of a node *v* in a tree to be the number of edges on a longest simple path from *v* to its descendant.



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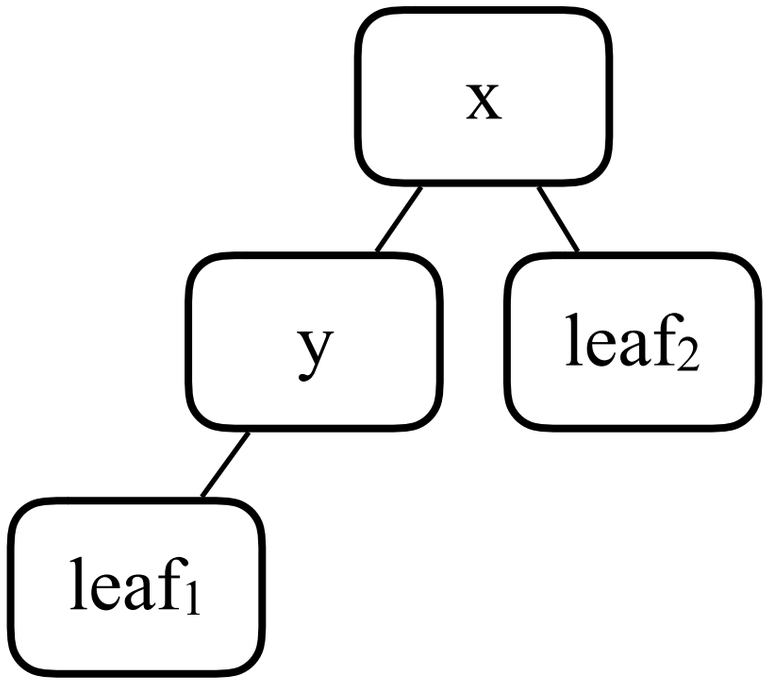

height(x) = 2,
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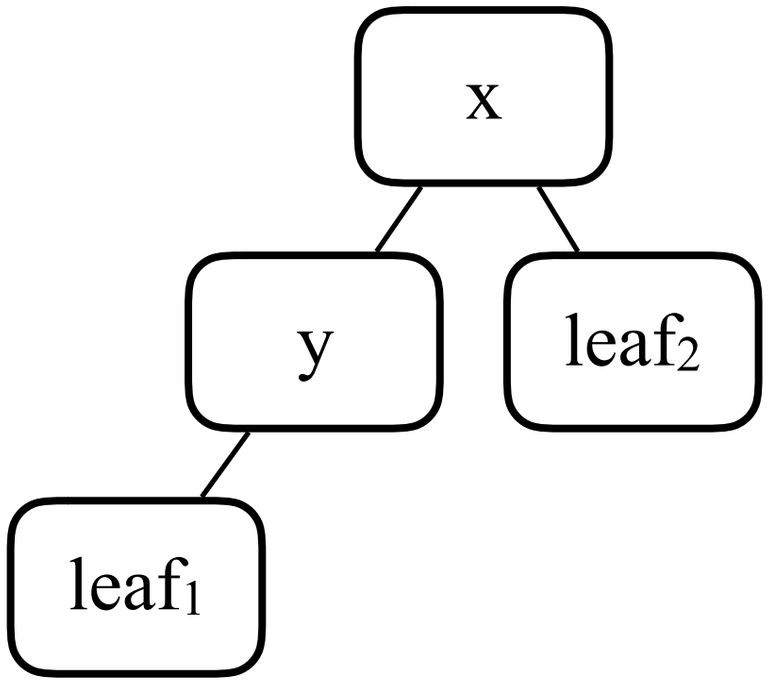

height(x) = 2,
height(y) = 1,
height($leaf_1$) = 0,
height($leaf_2$) = 0.

Heapify(A, i) takes O(height(i)) time, and
height(i) = O(log n). (Why?)

Build a Heap

Build-Map-Heap(A, i)

Convert the subtree rooted at node i into a max heap without the assumption that heapification uses.

Build a Heap

```
Build-Map-Heap(A, i){

    Build-Max-Heap(A, Left(i));
    Build-Max-Heap(A, Right(i));

    Max-Heapify(A, i);
}
```

--- Runtime ---

One may guess that $T(n) = 2T(n/2) + O(\log n)$.

By Master Theorem, the guess implies that $T(n) = O(n)$.

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It is simply a guess rather than a proof because the above algorithm does not necessarily split a problem into two subproblems evenly.

A (More) Rigorous Proof

Observe that $T(n) \leq T(n+1)$ for every n. Thus

$$T(n) \leq T(n')$$

where n' is the smallest power of 2 no less than n.

We can write $T(n) \leq T(n') = 2T(n'/2) + O(\log n')$.

By Master Theorem, we have $T(n) = O(n')$.

Because $n' < 2n$, $O(n') = O(n)$.

Summary

Build a heap of n elements needs O(n) time, which is asymptotically faster than sorting n elements.

Build a Young Tableau of n elements (naively) needs O(n log n) time, which is no faster than sorting n elements.

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Why sorted arrays do not subsume Young Tableau?

Extract-Max

Extract-Max(A, n)

Remove the maximum from an n-element array A and keep the rest of A as a max heap.

Extract-Max

```
Extract-Max(A, int& n){

    int ret = A[1]; A[1] = ∞;

    Max-Heapify(A, 1);

     n --;

   return ret;
}
```

--- Runtime ---

O(log n).

HeapSort

Input: an array A of n integers.

Output: the same array with the n integers ordered nondecrementally.

```
HeapSort(A, n){
    while(n > 1){
        k = Extract-Max(A, n);
        A[n+1] = k;
    }
}
```

--- Runtime ---

There are O(n) loop iterations, and each needs O(log n) time. In total, the running time is O(n log n), which is asymptotically optimal in the comparison-based model.

Summary

n elements	search cost	insertion cost	extract-max
sorted array	O(log n)	O(n)	O(n)
Young tableau	$O(n^{1/2})$	$O(n^{1/2})$	?
unsorted array	O(n)	O(1)	O(n)
heap	?	O(log n)	O(log n)