11.1 Sequences

1. (infinite) sequence
2. limit of sequence
3. Monotone Convergence Theorem

0.1 Infinite sequence

Define: A ***sequence*** 數列 is a list of numbers written in a definite order:

$$\underbrace{a_1}_{\text{1st term}}, \underbrace{a_2}_{\text{2nd term}}, \ldots, \underbrace{a_n}_{n\text{-th term}}, \ldots$$

Notation: 數列寫法: (不一定要從 1 開始, 沒寫的通常指從 1 而終。)

- $\boxed{\{a_1, a_2, a_3, \ldots\}}$
- $\boxed{\{a_n\}}$ or $\boxed{\{a_n\}_{n=1}^{\infty}}$
- $\boxed{a_n = n \text{ 的公式}}$

Example 0.1

(a) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$, $\quad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, $\quad a_n = \frac{n}{n+1}$.

(b) $\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$,

$\left\{\frac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$, $\quad a_n = \frac{(-1)^n(n+1)}{3^n}$.

(c) $\left\{0, 1, \sqrt{2}, \ldots, \sqrt{n-3}, \ldots\right\}$, $\quad \left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$, $\quad a_n = \sqrt{n-3}$, $n \geq 3$.

(d) $\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, \ldots, \cos\frac{n\pi}{6}, \ldots\right\}$, $\quad \left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$, $\quad a_n = \cos\frac{n\pi}{6}$, $n \geq 0$.

Question: 要寫前幾項? 幾項都不夠, 能寫出第 n 項最清楚。

Example 0.2 *Find a formula for the general term a_n of the sequence*

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

$a_n = (-1)^{n-1}\frac{n+2}{5^n}$. *(答案不是唯一的!)* ■

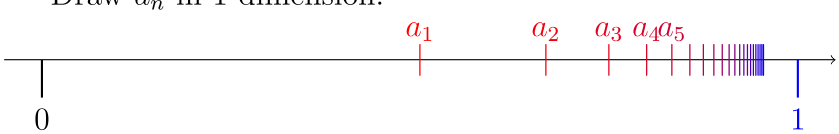
Example 0.3 *(很多數列不見得找得到公式, 甚至沒有公式。)*
(a) π*:* $3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9 \ldots.$
(b) e*:* $2, 7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots.$
(c) 斐波那契 *(*生兎子*)* 數列 ***Fibonacci sequence****:*

$$\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}.$$

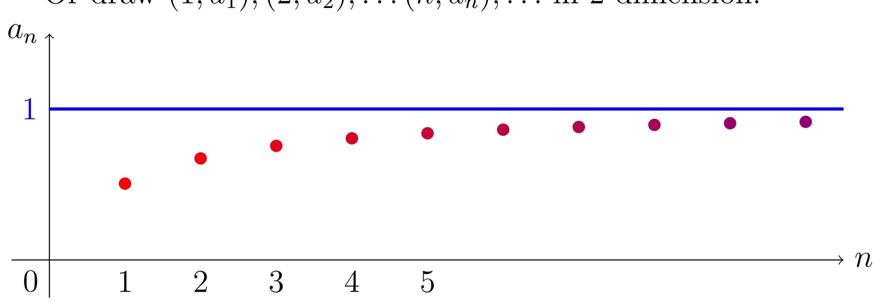
♦ 可以表示成*:* $f_1 = 1,\ f_2 = 1,$ *and* $f_n = f_{n-1} + f_{n-2},\ n \geq 3$ *(*遞迴關係*).*
In fact: $f_n = \dfrac{1}{\sqrt{5}}\Big[\Big(\dfrac{1+\sqrt{5}}{2}\Big)^n - \Big(\dfrac{1-\sqrt{5}}{2}\Big)^n\Big].$

Question: Where does the sequence goes?

Consider $a_n = \dfrac{n}{n+1}$, $\{a_n\} = \Big\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \cdots\Big\}.$
Draw a_n in 1-dimension:



Or draw $(1, a_1), (2, a_2), \ldots (n, a_n), \ldots$ in 2-dimension:



(*Can you find the limit tonight? It's where* a_n*'s are.*)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \iff a_n = \frac{n}{n+1} \to 1 \text{ as } n \to \infty.$$

(*It's enough to make Lin and his students. Believe the very best.*)

0.2 Limit of sequence

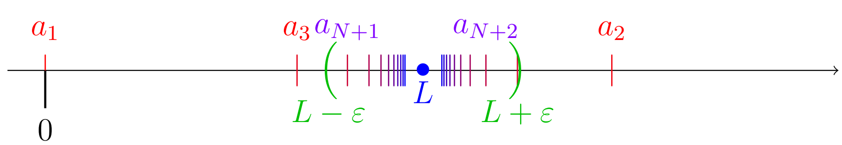
Define: A sequence $\{a_n\}$ has the ***limit*** L, write

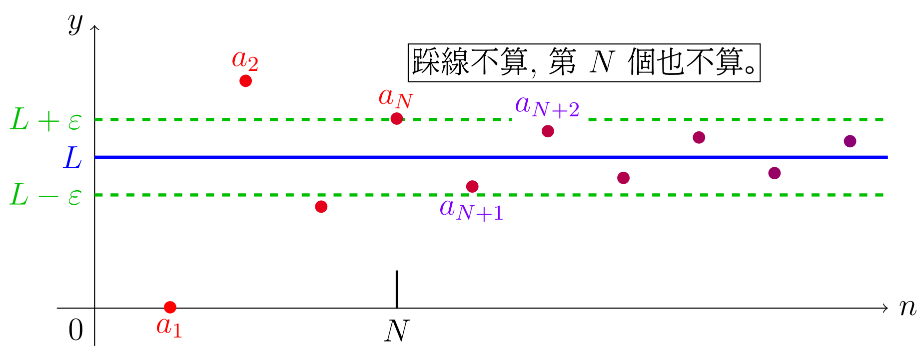
$$\boxed{\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty}$$

a_n approaches L as n sufficiently large. 只要 n 夠大, a_n 就會靠近 L。

$$\text{if } \boxed{\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\ \ni n > N \implies |a_n - L| < \varepsilon}.$$

For $\varepsilon > 0$, there exists an integer N, such that if $n > N$ then $|a_n - L| < \varepsilon$.
對所有 $\varepsilon > 0$, 存在正整數 N, 使得只要 $n > N$, 就會 $|a_n - L| < \varepsilon$。





If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence $\{a_n\}$ ***converges (is convergent)***; otherwise, ***diverges (is divergent)***. 極限存在叫收斂, 否則叫發散。

Define: $\{a_n\}$ ***diverges to*** ∞ 發散至無限。

$$\boxed{\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty \text{ as } n \to \infty}$$

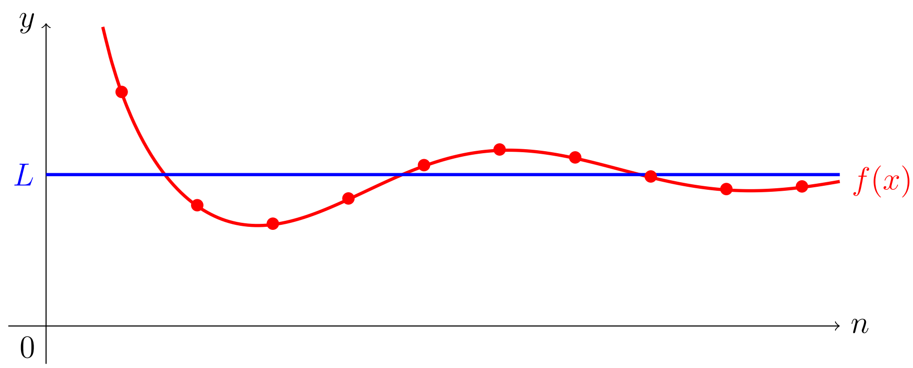
a_n is arbitrarily large as n sufficiently large. 只要 n 夠大, a_n 就會任意大。

$$\text{if } \boxed{\forall\, M > 0,\ \exists\, N \in \mathbb{N},\ \ni n > N \implies a_n > M.}$$

Note: 無限處無限極限, 此時極限不存在, $\{a_n\}$ 發散。
(2∞& → *To infinity and beyond!* — Buzz Lightyear)

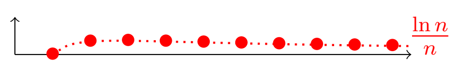
Question: $\lim\limits_{n\to\infty} a_n = ?$ 用 ε, δ 太麻煩了, 介紹五個方法。

Theorem 1 *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$*, then* $\lim_{n\to\infty} a_n = L$.
(有個函數 f 把 a_n 連起來, 所以 a_n 跟著他走。)



Note: 1. $\lim_{x\to\infty} f(x) = \infty \implies \lim_{n\to\infty} a_n = \infty$. (一起奔向宇宙浩瀚無垠。)
2. 反過來不對! $f(x) = \sin \pi x$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0$, but $\lim_{n\to\infty} f(x) \nexists$.
(是 a_n 跟著 $f(x)$ 走, 不是 $f(x)$ 跟著 a_n 走。)
3. 好用的結果: For $r > 0$, $\lim_{x\to\infty} \frac{1}{x^r} = 0 \implies \boxed{\lim_{n\to\infty} \frac{1}{n^r} = 0}$.

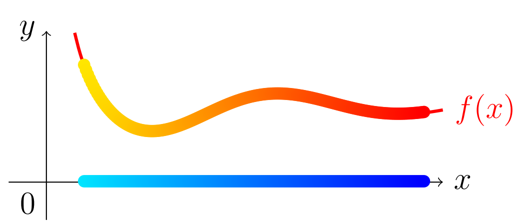
Example 0.4 *Calculate* $\lim_{n\to\infty} \frac{\ln n}{n}$.

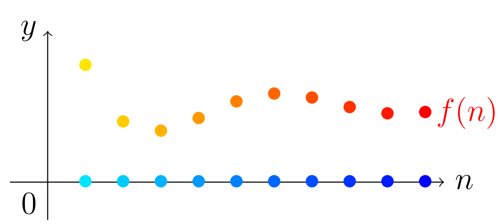


Let $f(x) = \frac{\ln x}{x}$. $\ln x \to \infty$ *as* $x \to \infty$. $(\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}})$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{l'H}{=} \lim_{x\to\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \overset{\text{Thm 1}}{\Longrightarrow} \lim_{n\to\infty} \frac{\ln n}{n} = 0. \qquad \blacksquare$$

Skill: 1. 怎麼找 $f(x)$? 把 n 改成 x。($\sin, \tan, \ln$ 不可以改成 si$x$, ta$x$, lx。)
2. 雖然 $\lim_{x\to\infty} f(x)$ 的 x 是連續的變大, $\lim_{n\to\infty} f(n)$ 的 n 是離散的變大,
但是計算上可以把 n 當變數做 ℓ'Hospital Rule: $\lim_{n\to\infty} \frac{\ln n}{n} \overset{l'H}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$.





Theorem 2 (Limit Laws 極限律: 加減乘除常數倍)
If $\{a_n\}$ *and* $\{b_n\}$ *are convergent sequences,* $\lim\limits_{n\to\infty} a_n = L$ *and* $\lim\limits_{n\to\infty} b_n = M$*, and* c *is a constant, then* (跟函數版本一樣, 要極限存在 *(*收斂*)* 才可以用*!)*

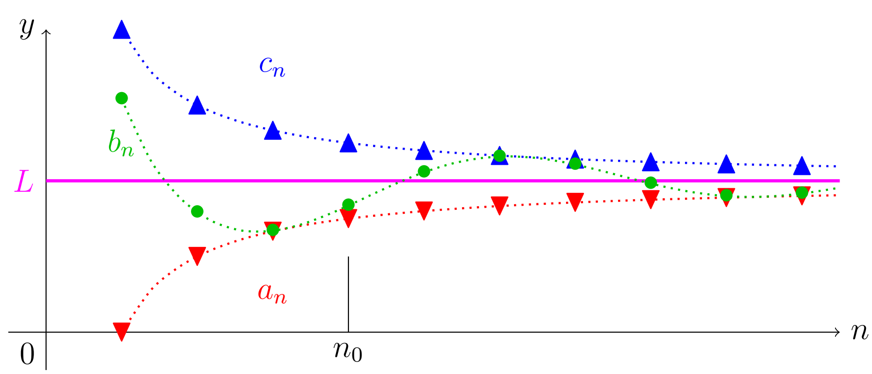
$$\begin{aligned}
\lim_{n\to\infty}(a_n+b_n) &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L+M \\
\lim_{n\to\infty}(a_n-b_n) &= L-M \\
\lim_{n\to\infty} ca_n &= cL \\
\lim_{n\to\infty} c &= c \\
\lim_{n\to\infty}(a_nb_n) &= L\cdot M \\
\lim_{n\to\infty}\frac{a_n}{b_n} &= \frac{L}{M} \ \textit{if } M\neq 0 \qquad \text{(分母不爲零)} \\
\lim_{n\to\infty} a_n^p &= L^p \ \textit{if } p>0 \textit{ and } a_n>0 \qquad \text{(冪次/開根)}
\end{aligned}$$

Example 0.5 *Find* $\lim\limits_{n\to\infty}\dfrac{n}{n+1}=1.$

$$\lim_{n\to\infty}\frac{n}{n+1} \overset{(\div n)}{\underset{(\div n)}{=}} \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}\frac{1}{n}} = \frac{1}{1+0} = 1. \qquad \blacksquare$$

Skill: 直接把 $\lim\limits_{n\to\infty}$ 丟進去算:
極限都存在, 就是答案; 有極限不存在 (或分母爲零), 換別招。

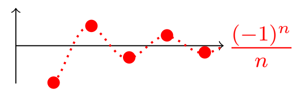
Theorem 3 (Squeeze Theorem 夾擠定理)
If $a_n \le b_n \le c_n$ *for* $n\ge n_0$ *and* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$*, then* $\lim\limits_{n\to\infty} b_n = L$*.*



Note: Prove Theorems 2 & 3: by Theorem 1 & 函數 version.

Theorem 4 (Absolutely converge to zero 絕對收斂到零)
If $\lim\limits_{n\to\infty} |a_n| = 0$, *then* $\lim\limits_{n\to\infty} a_n = 0$. *(想想看, 反過來對嗎?)*

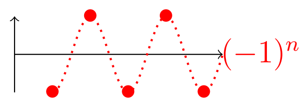
Example 0.6 *Evaluate* $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n}$ *if it exists.*



$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0 \overset{\text{Thm 4}}{\Longrightarrow} \lim_{n\to\infty} \frac{(-1)^n}{n} = 0. \qquad \blacksquare$$

Example 0.7 *Determine whether* $a_n = (-1)^n$ *is convergent or divergent.*

$\lim\limits_{n\to\infty} a_n$ *does not exist,* $\{(-1)^n\}$ *is divergent.* $\blacksquare$

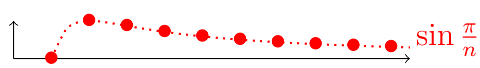


Note: 絕對收斂到零才會收斂 (一樣到零), 不是零可能會發散。
ex: $\lim\limits_{n\to\infty} |(-1)^n| = \lim\limits_{n\to\infty} 1 = 1 \not\Longrightarrow \lim\limits_{n\to\infty} (-1)^n = 1$.

Timing: 常用在 a_n 裡面有 $(-1)^n$ 的情形。

Theorem 5 (Continuity 連續性)
If $\lim\limits_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L, *then* $\lim\limits_{n\to\infty} f(a_n) = f(L)$.
(連續函數可傳遞極限: $\lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n)$, 遇到連續函數把 $\lim\limits_{n\to\infty}$ 往裡傳。*)*

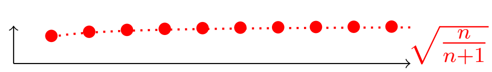
Example 0.8 *Find* $\lim\limits_{n\to\infty} \sin(\dfrac{\pi}{n})$.



$\because \lim\limits_{n\to\infty} \dfrac{\pi}{n} = 0$ *and* $\sin x$ *is continuous at* 0,

$$\lim_{n\to\infty} \sin\frac{\pi}{n} \overset{\text{Thm 5}}{=} \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin 0 = 0. \qquad \blacksquare$$

Example 0.9 *Find* $\lim\limits_{n\to\infty} \sqrt{\dfrac{n}{n+1}}$.



$\because \lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$ *and* $\sqrt{x}$ *is continuous at* 1 *(on* $(0,\infty)$*)*,

$$\lim_{n\to\infty} \sqrt{\frac{n}{n+1}} \overset{\text{Thm 5}}{=} \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \sqrt{1} = 1. \qquad \blacksquare$$

Recall: $\lim\limits_{x\to 0^+} \sqrt{x} = 0$, $\lim\limits_{x\to 0} \sqrt{x}$ does not exist.

Example 0.10 *Discuss the convergence of the sequence* $a_n = \dfrac{n!}{n^n}$.

$$a_1 = 1,\ a_2 = \frac{1\times 2}{2\times 2} = \frac{1}{2},\ a_3 = \frac{1\times 2\times 3}{3\times 3\times 3} = \frac{2}{9},\ a_n = \frac{1\times 2\times\cdots\times n}{n\times n\times\cdots\times n}.$$

$$0 < a_n = \frac{1}{n}\left(\frac{2}{n}\times\frac{3}{n}\times\cdots\times\frac{n}{n}\right) < \frac{1}{n}\cdot 1 = \frac{1}{n}.$$ *(選 <1 沒用, 夾不住。)*

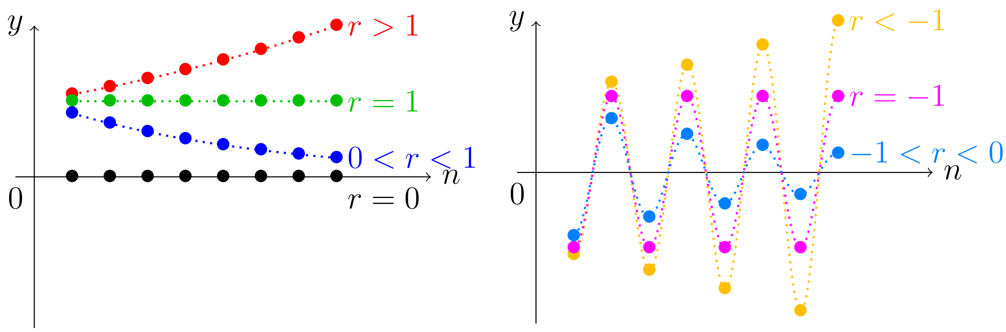
Since $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty}\frac{1}{n} = 0$, *by the Squeeze Theorem,* $\lim\limits_{n\to\infty} a_n = 0$. ■

$n! := n\times(n-1)\times\cdots\times 2\times 1$, 唸作 “$n$ factorial [fæk'torɪəl]”, “n 階乘”。

Example 0.11 *For what value of r is the sequence $\{r^n\}$ convergent?*

(a) $\lim\limits_{x\to\infty} a^x = \infty$ *for* $a > 1$, $\lim\limits_{x\to\infty} a^x = 0$ *for* $0 < a < 1$,
So $\lim\limits_{n\to\infty} r^n = \infty$ *if* $r > 1$ *and* $\lim\limits_{n\to\infty} r^n = 0$ *if* $0 < r < 1$.
(b) $\lim\limits_{n\to\infty} 1^n = \lim\limits_{n\to\infty} 1 = 1$, $\lim\limits_{n\to\infty} 0^n = \lim\limits_{n\to\infty} 0 = 0$.
(c) $-1 < r < 0$, $\lim\limits_{n\to\infty} |r^n| = \lim\limits_{n\to\infty} |r|^n = 0$.
(d) $r \le -1$, $\{r^n\}$ *diverges.* ■

$$\textbf{Fact: } \lim_{n\to\infty} r^n = \begin{cases} \mathbf{0} & \text{if } -1 < r < 1 \quad \text{(convergent)} \\ \mathbf{1} & \text{if } r = 1 \quad \text{(convergent)} \\ \boldsymbol{\infty} & \text{if } r > 1 \quad \text{(divergent to infinity)} \\ \nexists & \text{if } r \le -1 \quad \text{(divergent)} \end{cases}$$



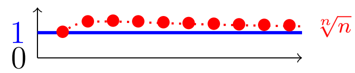
♦: $\{ar^{n-1}\}$ 稱爲等比/幾何 (geometric) 數列, r 稱爲公比 (common ratio)。

Recall: 7 important limits.

1. $\boxed{\lim\limits_{n\to\infty}\frac{\ln n}{n}=0}$.

2. $\boxed{\lim\limits_{n\to\infty} x^n = 0 \text{ for } |x|<1}$.

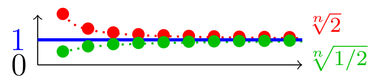
3. $\boxed{\lim\limits_{n\to\infty}\sqrt[n]{n}=1}$.



Proof. $\sqrt[n]{n}=n^{1/n}=e^{\frac{\ln n}{n}}$, $\lim\limits_{n\to\infty}\frac{\ln n}{n}=0$, e^x is continuous at 0,

$$\lim_{n\to\infty}\sqrt[n]{n}=\lim_{n\to\infty}e^{\frac{\ln n}{n}}=e^{\lim\limits_{n\to\infty}\frac{\ln n}{n}}=e^0=1.$$

4. $\boxed{\lim\limits_{n\to\infty}\sqrt[n]{x}=1 \text{ for } x>0}$.



Proof. $\sqrt[n]{x}=x^{1/n}=e^{\frac{1}{n}\ln x}$, $\lim\limits_{n\to\infty}\frac{\ln x}{n}=0$, e^x is continuous at 0,

$$\lim_{n\to\infty}\sqrt[n]{x}=\lim_{n\to\infty}e^{\frac{1}{n}\ln x}=e^{\lim\limits_{n\to\infty}\frac{\ln x}{n}}=e^0=1.$$

5. $\boxed{\lim\limits_{n\to\infty}(1+\frac{x}{n})^n=e^x \text{ for } x}$. $(1^\infty\to 0\bullet\infty\to\frac{0}{0})$

Proof. $\lim\limits_{n\to\infty}n\ln(1+\frac{x}{n})=\lim\limits_{n\to\infty}\frac{\ln(1+\frac{x}{n})}{1/n}\overset{l'H}{=}\lim\limits_{n\to\infty}\frac{\frac{-x/n^2}{1+x/n}}{-1/n^2}=\lim\limits_{n\to\infty}\frac{x}{1+\frac{x}{n}}=x$,

e^x is continuous on $\mathbb{R}$, $\lim\limits_{n\to\infty}(1+\frac{x}{n})^n=\lim\limits_{n\to\infty}e^{n\ln(1+x/n)}=e^{\lim\limits_{n\to\infty}n\ln(1+x/n)}=e^x$.

6. $\boxed{\lim\limits_{n\to\infty}\frac{x^n}{n!}=0 \text{ for } x}$.

Proof. let $k=\lceil |x|\rceil$, ($\lfloor a\rfloor$: 'floor of a', $\lceil a\rceil$: 'ceiling of a'.) then for $n>k$,

$$0<\left|\frac{x^n}{n!}\right|=\frac{|x|^n}{n!}<\frac{k^n}{n!}=\frac{k^k}{k!}\left(\frac{k}{k+1}\frac{k}{k+2}\cdots\frac{k}{n-1}\right)\frac{k}{n}<\frac{k^k}{k!}\cdot 1\cdot\frac{k}{n}=\frac{k^{k+1}}{n\cdot k!},$$

$\lim\limits_{n\to\infty}0=0=\lim\limits_{n\to\infty}\frac{k^{k+1}}{n\cdot k!}$, $\implies\lim\limits_{n\to\infty}\left|\frac{x^n}{n!}\right|\overset{S.T.}{=}0$, and so $\lim\limits_{n\to\infty}\frac{x^n}{n!}=0$.

7. $\boxed{\lim\limits_{n\to\infty}\frac{1}{n^x}=0 \text{ for } x>0}$.

Proof. $\exists$ odd positive integer $p\geq\frac{1}{x}$, $0<\frac{1}{n}<1$, $0<\frac{1}{n^x}=(\frac{1}{n})^x\leq(\frac{1}{n})^{\frac{1}{p}}$.

$\lim\limits_{n\to\infty}\frac{1}{n}=0$, and $x^{\frac{1}{p}}$ is continuous at 0, (所以要奇數; 偶數也可以, 但要說明。)

by continuity $\lim\limits_{n\to\infty}(\frac{1}{n})^{\frac{1}{p}}=0^{\frac{1}{p}}=0=\lim\limits_{n\to\infty}0$, $\implies\lim\limits_{n\to\infty}\frac{1}{n^x}\overset{S.T.}{=}0$.

0.3 Monotone Convergence Theorem

Define: A sequence $\{a_n\}$ is

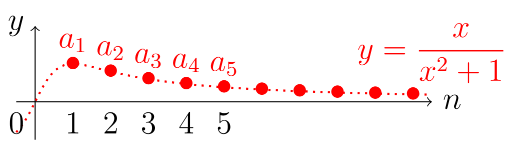
- ***monotonic*** 單調 if it is either:
  - ***increasing*** 遞增 if $a_n < a_{n+1}\ \forall\ n \geq 1$.
  - ***non-decreasing*** 非遞減 if $a_n \leq a_{n+1}\ \forall\ n \geq 1$.
  - ***decreasing*** 遞減 if $a_n > a_{n+1}\ \forall\ n \geq 1$.
  - ***non-increasing*** 非遞增 if $a_n \geq a_{n+1}\ \forall\ n \geq 1$.
- ***bounded*** 有界 if it is both:
  - ***bounded above*** 上有界 if $\exists\ M \ni a_n \leq M\ \forall\ n \geq 1$.
  - ***bounded below*** 下有界 if $\exists\ m \ni a_n \geq m\ \forall\ n \geq 1$.

Example 0.12 $\left\{\dfrac{3}{n+5}\right\}$ *is decreasing.* $\because\ \dfrac{3}{n+5} > \dfrac{3}{(n+1)+5}$. ■

Example 0.13 $a_n = \dfrac{n}{n^2+1}$ *is decreasing.*

$$\because\ \frac{n}{n^2+1} > \frac{(n+1)}{(n+1)^2+1} \iff n[(n+1)^2+1] > (n+1)(n^2+1) \iff$$
$\cancel{n^3} + \cancel{2}n^2 + \cancel{2}n > \cancel{n^3} + \cancel{n^2} + \cancel{n} + 1,\ n^2 + n > 1.$

[function sol] Consider $f(x) = \dfrac{x}{x^2+1}$, $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ *when* $x^2 > 1$*. So* f *is decreasing on* $(-\infty, -1) \cup (1, \infty)$.
$a_1 = \dfrac{1}{2} > \dfrac{2}{5} = a_2$*, and* $a_n = f(n) > f(n+1) = a_{n+1}$ *for* $n > 1$. ■



Example 0.14 $a_n = n$ *is bounded below* $(a_n > 0)$*, not bounded above.*
$a_n = \dfrac{n}{n+1}$ *is bounded* $0 < a_n < 1$.

Note: 有界不保證收斂 Bounded $\not\Longrightarrow$ convergent, ex: $a_n = (-1)^n$.
單調不保證收斂 Monotonic $\not\Longrightarrow$ convergent, ex: $a_n = n$.

有界不收斂, 單調不收斂, 那有界加上單調呢?

Theorem 6 (Completeness Axiom of Real Numbers 實數的完備性公設)
*Every nonempty set of real numbers that has an upper bound has a **least upper bound (lub)**. (有上界就有最小上界: 描述實數沒有洞。)*

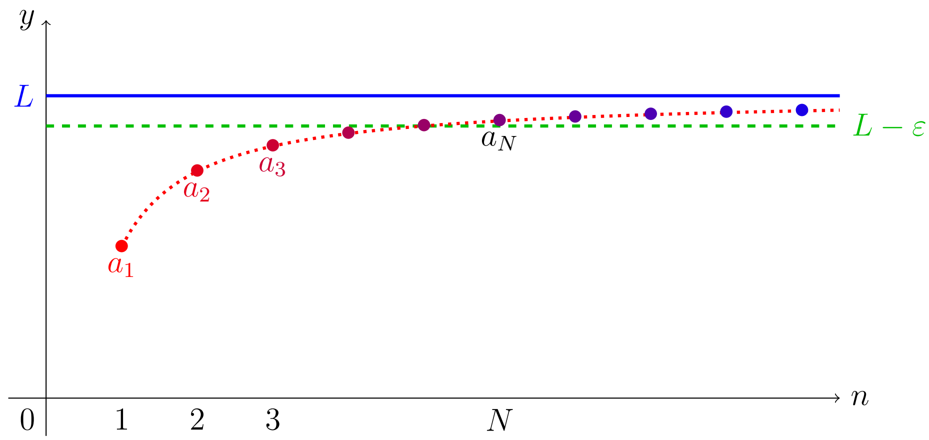
Note: Equivalently, 有下界就有最大下界 (***greatest lower bound, glb***).

Theorem 7 (Monotone Convergence Theorem 單調收斂定理)
Every bounded, monotonic sequence is convergent. (有界單調會收斂。)

Proof. Suppose $\{a_n\}$ is increasing. By the Completeness Axiom of Real Numbers, $\{a_n\}$ has a least upper bound L.

Goal: $\lim\limits_{n\to\infty} a_n = L \iff \forall\, \varepsilon > 0, \exists\, N \in \mathbb{N}, \ni n > N \implies |a_n - L| < \varepsilon$.
$\forall\, \varepsilon > 0$, $L - \varepsilon$ is not an upper bound, so $\exists\, N > 0 \ni a_N > L - \varepsilon$.
For $n > N$, since $\{a_n\}$ is increasing, $(L \geq) a_n > a_N > L - \varepsilon$, $|a_n - L| < \varepsilon$.
By the definition, $\lim\limits_{n\to\infty} a_n = L$.

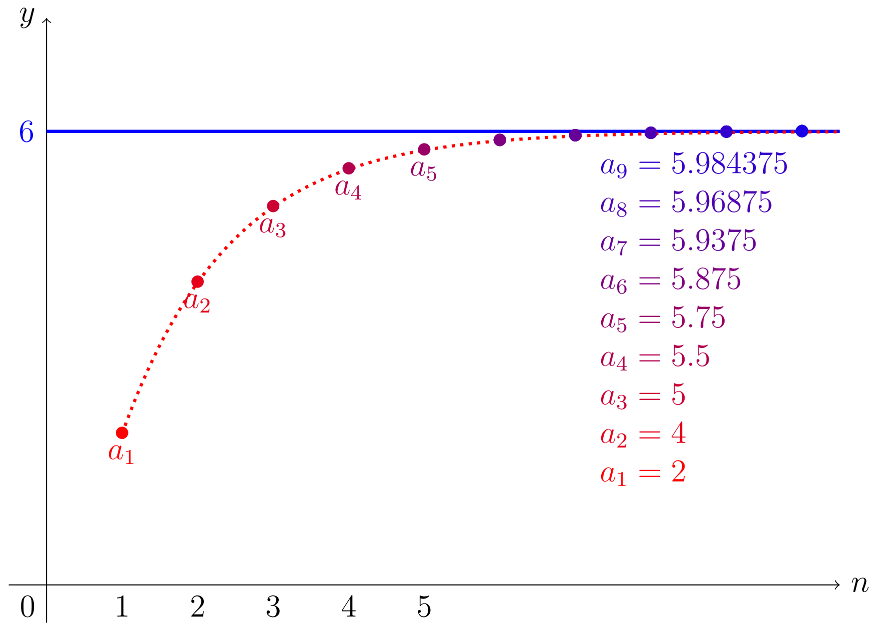
Similarly when $\{a_n\}$ is decreasing (use the greatest lower bound). ■



Note: 其實可以分成: 1. 遞增有上界會收斂; 2. 遞減有下界會收斂。

Question: Why MCT(Monotone Convergence Theorem)?
有些數列不是用 n 的公式表示, 就不能使用前面學過的五種方法求極限, 但是如果知道他收斂, 就可以取 $\lim\limits_{n\to\infty}$。

Example 0.15 *Show that* $\{a_n\}$ *defined by* $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$ *for* $n \geq 1$ *is convergent and find* $\lim\limits_{n\to\infty} a_n$. *(♦: 後項用前幾項表示稱爲遞迴關係。)*



Proof. *(a)* $\{a_n\}$ *is increasing and (b)* $\{a_n\}$ *is bounded above by* 6.

Together prove them by the Mathematical Induction 數學歸納法 *on* n:

Claim: "$a_n < a_{n+1}$ *and* $a_n \leq 6$ *for* $n \in \mathbb{N}$*".*

Step 1. For $n = 1$, $a_1 = 2 < 4 = a_2 \leq 6$.

Step 2. Suppose true for $n = k$.

Step 3. Prove true for $n = k + 1$*: by induction hypothesis(*歸納法假設*),*

$$a_k < a_{k+1} \implies a_{k+1} = \frac{1}{2}(a_k + 6) < \frac{1}{2}(a_{k+1} + 6) = a_{k+2}, \text{ and}$$

$$a_k \leq 6 \implies a_{k+1} = \frac{1}{2}(a_k + 6) \leq \frac{1}{2}(6 + 6) = 6.$$

$\therefore a_n < a_{n+1}$ *and* $a_n \leq 6$ *for* $n \in \mathbb{N}$ □

(c) Find L. *(*注意! 要有收斂才能用 *Limit Laws* 極限律。*)*

By the Monotone Convergence Theorem, $\{a_n\}$ *converges and let* $\lim\limits_{n\to\infty} a_n = L$.

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(L + 6),\ L = 6. \qquad \blacksquare$$

Usage: 1. 先假設收斂算極限 L, 當作上/下界。

2. 檢查: 單調 & 有界, 則 $\{a_n\}$ 收斂 by MCT & $\lim\limits_{n\to\infty} a_n = L$。

Example 0.16 (extend) $a_1 = 2,\ a_{n+1} = \dfrac{1}{a_n},\ \lim\limits_{n\to\infty} a_n = ?.$

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L},\ L^2 = 1,\ L = \pm 1.\ \textit{(}\textbf{Wrong!}\textit{)}$$

(如果有收斂, 可能是 1, 也可能是 -1, 根據 a_1 不同而定。)

$a_1 = 2,\ a_2 = \dfrac{1}{2},\ a_3 = 2,\ a_4 = \dfrac{1}{2},\ \ldots.$ 答案是*: does not exist!.* ■

Example 0.17 (extend) $a_1 = 1,\ a_{n+1} = 1 + \dfrac{1}{1 + a_n}.$ *(Exercise 11.1.92)*

Let $x = \lim\limits_{n\to\infty} a_n$, *solve* $x = 1 + \dfrac{1}{1+x}$, $x = \pm\sqrt{2}$.

When $a_n > \sqrt{2} \implies 0 < a_{n+1} = 1 + \dfrac{1}{1+a_n} < 1 + \dfrac{1}{1+\sqrt{2}} = \sqrt{2}$

$\implies a_n > a_{n+2} > \sqrt{2} \implies a_{n+1} < a_{n+3} < \sqrt{2}$. *(隔項遞增/減)*

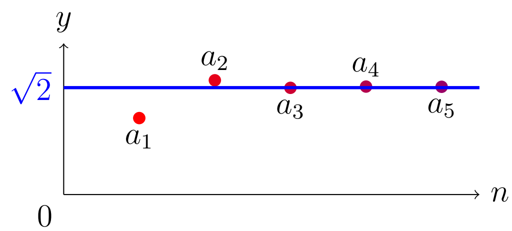
$\implies \{a_{2n}\}$ *&* $\{a_{2n+1}\}$ *are bounded and monotonic.*

By the MCT $\{a_{2n}\}$ *&* $\{a_{2n+1}\}$ *are convergent with* $\lim\limits_{n\to\infty} a_{2n} = \sqrt{2} = \lim\limits_{n\to\infty} a_{2n+1}$.

$\therefore \lim\limits_{n\to\infty} a_n = \sqrt{2}$. *(Pythagoras' constant 畢達哥拉斯常數)* ■

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$ *(連分數)*

$1, \dfrac{3}{2}, \dfrac{7}{5}, \dfrac{17}{12}, \dfrac{41}{29}, \cdots$



Example 0.18 (extend) $a_1 = 1,\ a_{n+1} = 1 + \dfrac{1}{a_n}.$

Let $x = \lim\limits_{n\to\infty} a_n$, $x = 1 + \dfrac{1}{x}$, $x = \dfrac{1 \pm \sqrt{5}}{2}$ *(*$\because a_1 = 1,\ a_n > 0$, 負不合*).*

Similarly, $\lim\limits_{n\to\infty} a_n = \dfrac{1+\sqrt{5}}{2}$. *(Golden ratio 黃金比例)* ■

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

$1, 2, \dfrac{3}{2}, \dfrac{5}{3}, \dfrac{8}{5}, \cdots$