5.4 Indefinite integrals and the Net Change Theorem

1. indefinite integral 不定積分 $\int f(x)\ dx$
2. the Net Change Theorem 淨變化定理 $\int v(t)\ dt$ v.s. $\int |v(t)|\ dt$

0.1 Indefinite integral

TFTC (2) 告訴我們積分與反導數的關係: 積 a 到 b = 反導代 b 減 a。
我們叫這個“f 的反導數”爲“f 的不定積分”。

Define: The ***indefinite integral*** 不定積分 of a continuous f (on its domain)

$$\boxed{\int f(x)\ dx = F(x) \iff F'(x) = f(x)}.$$

Note: 定積分 $\displaystyle\int_a^b f(x)\ dx$ 是一個數字 (極限值 $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$);
不定積分 $\displaystyle\int f(x)\ dx$ 是一個函數, 寫成“函數+C”的形式, 也就是 f 最一般的反導數。

定積分與不定積分的關係: (TFTC)

$$\boxed{\int_a^b f(x)\ dx = \left[\int f(x)\ dx\right]_a^b}.$$

Example 0.1

$$\int x^2\ dx = \frac{x^3}{3} + C,\ \int \sec^2 x\ dx = \tan x + C,\ \int \frac{1}{x^2}\ dx = -\frac{1}{x} + C.$$

Note: 雖然用 $F(x)+C$ 來代表 $\displaystyle\int f(x)\ dx$, 但是 C 只在同一個區間有效;
寫成不同區間要用不同的常數。

$$\int \frac{1}{x^2}\ dx = -\frac{1}{x} + C = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x > 0, \\ -\dfrac{1}{x} + C_2 & \text{if } x < 0. \end{cases}$$

Table 1: Table of indefinite integrals:

$\int cf(x)\ dx = c\int f(x)\ dx$	常數倍
$\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$	加減
$\int c\ dx = cx + C$	常數
$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	(逆冪次律)
$\int \frac{1}{x}\ dx = \ln\|x\| + C$	(注意絕對值)
$\int e^x\ dx = e^x + C,\ \int a^x\ dx = \frac{a^x}{\ln a} + C \quad (a > 0)$	指數
$\int \ln x\ dx = x\ln x - x + C$	對數
$\int \log_a x\ dx = x\log_a x - \frac{x}{\ln a} + C \quad (a > 0)$	
$\int \sin x\ dx = -\cos x + C,\ \int \cos x\ dx = \sin x + C$	三角
$\int \sec^2 x\ dx = \tan x + C,\ \int \csc^2 x\ dx = -\cot x + C$	
$\int \sec x \tan x\ dx = \sec x + C$	
$\int \csc x \cot x\ dx = -\csc x + C$	
$\int \tan x\ dx = \ln\|\sec x\| + C$	
$\int \sec x\ dx = \ln\|\sec x + \tan x\| + C$	
$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + C = -\cos^{-1} x + C$	反三角
$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + C = -\cot^{-1} x + C$	
$\int \frac{1}{x\sqrt{x^2-1}}\ dx = \sec^{-1} x + C = -\csc^{-1} x + C$	
$\int \sinh x\ dx = \cosh x + C,\ \int \cosh x\ dx = \sinh x + C$	雙曲三角

Example 0.2 $\int (10x^4 - 2\sec^2 x)\ dx = ?$

$$\int (10\,x^4 - 2\,\sec^2 x)\ dx = 10\int x^4\ dx - 2\int \sec^2 x\ dx$$

$= 10\,\dfrac{x^5}{5} - 2\tan x + C = 2x^5 - 2\tan x + C$. *(共用一個 C 就好。)* ■

Example 0.3 $\int \dfrac{\cos\theta}{\sin^2\theta}\ d\theta = ?$

$$\int \frac{\cos\theta}{\sin^2\theta}\ d\theta = \int \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}\ d\theta = \int \csc\theta\cot\theta\ d\theta = -\csc\theta + C.$$ ■

(不太容易想到, 也可以用變數變換 §5.5。)

Example 0.4 *(Recall §5.2 Ex 0.2)* $\int_0^3 (x^3 - 6x)\ dx = ?$

$\int (x^3 - 6x)\ dx = \dfrac{x^4}{4} - 3x^2 + C$, *(找不定積分)*

$\int_0^3 (x^3 - 6x)\ dx = \dfrac{x^4}{4} - 3x^2\Big]_0^3 = \left[\dfrac{3^4}{4} - 3(3)^2\right] - \left[\dfrac{0^4}{4} - 3(0)^2\right]$

$= \dfrac{81}{4} - 27 - 0 + 0 = -6.75$. *(用 TFTC (2) 比用 $\lim\sum$ 好算多了。)* ■

Example 0.5 $\int_0^2 \left(2x^3 - 6x + \dfrac{3}{x^2+1}\right)\ dx = ?$

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)\ dx = \frac{x^4}{2} - 3x^2 + 3\tan^{-1} x\Big]_0^2$$

$= \left[\dfrac{2^4}{2} - 3(2)^2 + 3\tan^{-1} 2\right] - \left[\dfrac{0^4}{2} - 3(0)^2 + 3\tan^{-1} 0\right]$

$= 8 - 12 + 3\tan^{-1} 2 - 0 + 0 - 0 = -4 + 3\tan^{-1} 2 (\approx -0.67855)$. ■

Example 0.6 $\int_1^9 \dfrac{2t^2 + t^2\sqrt{t} - 1}{t^2}\ dt = ?$

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\ dt \overset{\text{簡化}}{=} \int_1^9 (2 + t^{1/2} - t^{-2})\ dt = 2t + \frac{2}{3}t^{3/2} + t^{-1}\Big]_1^9$$

$= \left[2(9) + \dfrac{2}{3}(9)^{3/2} + 9^{-1}\right] - \left[2(1) + \dfrac{2}{3}(1)^{3/2} + 1^{-1}\right]$

$= 18 + 18 + \dfrac{1}{9} - 2 - \dfrac{2}{3} - 1 = 32\dfrac{4}{9}$. ■

Skill: 怎麼檢查反導數找得對不對? 微分!

0.2 The Net Change Theorem

Recall: f is continuous on $[a, b]$, $\int_a^b f(x)\ dx = F(b) - F(a)$, $F'(x) = f(x)$.

F 是 f 的反導數, 代入得到: $\int_a^b F'(x)\ dx = F(b) - F(a)$.

$F'(x)$ 代表 rate of change 改變率, $F(b) - F(a)$ 代表 net change 淨改變。

Theorem 1 (Net Change Theorem 淨改變定理)
The integral of a rate of change is the net change:

$$\int_a^b F'(x)\ dx = F(b) - F(a)$$

Example 0.7 $V(t)$ 是水量, $V'(t)$ 是注水率。

$$\int_{t_1}^{t_2} V'(t)\ dt = V(t_2) - V(t_1).$$

Example 0.8 $n(t)$ 是人口, $n'(t)$ 是人口成長(=出生−死亡)率。

$$\int_{t_1}^{t_2} n'(t)\ dt = n(t_2) - n(t_1).$$

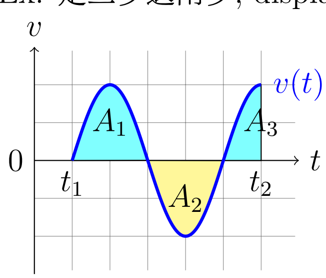
Example 0.9 $s(t)$ 是位置, $s'(t) = v(t)$ 是速率。

$$\int_{t_1}^{t_2} v(t)\ dt = s(t_2) - s(t_1).$$

$\int_{t_1}^{t_2} v(t)\ \ dt = s(t_2) - s(t_1)$: the net change of position = displacement.

$\int_{t_1}^{t_2} |v(t)|\ dt$: the total distance traveled. (加上絕對值)

Ex: 走三步退兩步, displacement: $+3 - 2 = 1$, distance: $+3 + 2 = 5$.



$$\int_{t_1}^{t_2} v(t)\ \ dt \ = \ A_1 - A_2 + A_3$$

$$\int_{t_1}^{t_2} |v(t)|\ dt \ = \ A_1 + A_2 + A_3$$

Example 0.10 *A particle 粒子 moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).*
(a) Find the displacement of the particle during the time period $1 \le t \le 4$.
(b) Find the distance traveled during this time period.

(a) $\displaystyle\int_1^4 v(t)\ dt = \int_1^4 (t^2 - t - 6)\ dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4$
$\displaystyle = \left[\frac{(4)^3}{3} - \frac{(4)^2}{2} - 6(4)\right] - \left[\frac{(1)^3}{3} - \frac{(1)^2}{2} - 6(1)\right] = -\frac{9}{2} = -4.5.$

(b) (去掉絕對值, 要先知道 $v(t)$ 什麼時候是正的/負的 — 解 $v(t) = 0$。)
$v(t) = t^2 - t - 6 = (t-3)(t+2) = 0$ *when* $t = -2, 3$,
$v(t) > 0$ *when* $t > 3$ *or* $t < -2$, *and* $v(t) < 0$ *when* $-2 < t < 3$,
$\therefore$ 把 $[1, 4]$ 分成 $[1, 3] \cup [3, 4]$ 兩段。

分段與去絕對值同時完成

$$\int_1^4 |v(t)|\ dt = \int_1^3 -v(t)\ dt + \int_3^4 v(t)\ dt$$

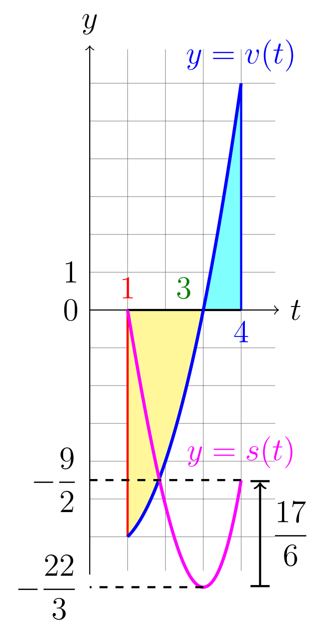
反導數同樣只差一個負號

$$= \left[-\left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \left\{\left[-\left(\frac{(3)^3}{3} - \frac{(3)^2}{2} - 6(3)\right)\right] - \left[-\left(\frac{(1)^3}{3} - \frac{(1)^2}{2} - 6(1)\right)\right]\right\}$$

$$+ \left\{\left[\frac{(4)^3}{3} - \frac{(4)^2}{2} - 6(4)\right] - \left[\frac{(3)^3}{3} - \frac{(3)^2}{2} - 6(3)\right]\right\}$$

$$= \boxed{\frac{22}{3}} + \boxed{\frac{17}{6}} = \frac{61}{6}\ (\approx\ 10.17).$$



$$\left(\boxed{\frac{17}{6}} - \boxed{\frac{22}{3}} = -\frac{9}{2}\right)$$

Ans: Displacement 位移 -4.5 *m; traveled distance* 旅行距離 $\dfrac{61}{6}$ *m.* ■

胡適: 大膽假設, 小心求證。— 大膽列式, 小心計算。