3.3 Derivatives of trigonometric functions

1. two limits on trigonometric function 兩個三角函數的極限
   $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ & $\lim\limits_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$

2. derivatives of trigonometric functions 六個三角函數的導函數
   $(\sin)'=\ \cos,\quad (\tan)'=\ \sec^2,\quad (\sec)'=\ \sec\tan,$
   $(\cos)'=-\sin,\quad (\cot)'=-\csc^2,\quad (\csc)'=-\csc\cot\,.$

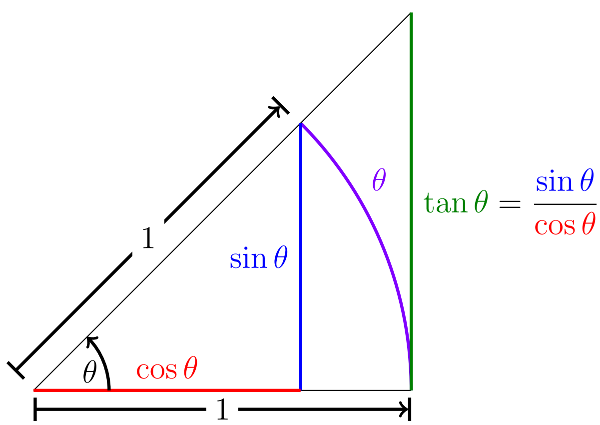
用定義 (極限) 來求三角函數的導函數, $f'(x)=\lim\limits_{\theta\to 0}\dfrac{f(x+\theta)-f(x)}{\theta}$.

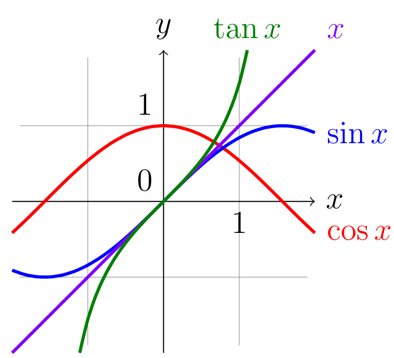
Recall: 合角公式: $\begin{cases}\sin(x+\theta)=\sin x\cos\theta+\cos x\sin\theta,\\ \cos(x+\theta)=\cos x\cos\theta-\sin x\sin\theta.\end{cases}$

Identify: $\sin^2 x+\cos^2 x=1$, $\tan^2 x+1=\sec^2 x$, $\cot^2 x+1=\csc^2 x$.

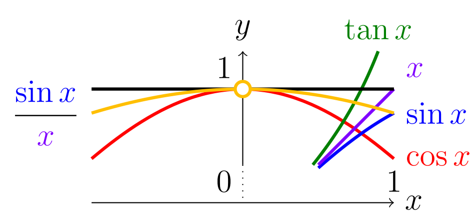
0.1 two limits on trigonometric function

1. $\boxed{\lim\limits_{\theta\to 0}\dfrac{\sin\boldsymbol{\theta}}{\boldsymbol{\theta}}=1}$



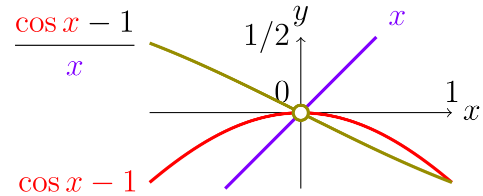


$$\sin\theta<\quad\theta\quad<\tan\theta=\frac{\sin\theta}{\cos\theta}$$
$$\cos\theta<\frac{\sin\theta}{\theta}<\quad 1$$



$\because\lim\limits_{\theta\to 0}\cos\theta=1=\lim\limits_{\theta\to 0}1$. By the Squeeze Theorem, $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$. ■

2. $\boxed{\displaystyle \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0}$



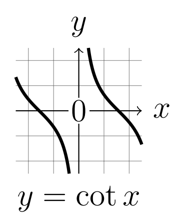
$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\frac{\cos\theta+1}{\cos\theta+1}\right) && (\cos\theta+1\to 2\neq 0)\\
&= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)} && (\sin^2+\cos^2=1)\\
&= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\frac{-\sin\theta}{\cos\theta+1}\right) && (\text{why? try!})\\
&= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta+1},\\
\because \lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta+1} &= \frac{-\lim\limits_{\theta\to 0}\sin\theta}{\lim\limits_{\theta\to 0}\cos\theta+1} = -\frac{0}{1+1} = 0,\\
\therefore \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} &= 1\cdot 0 = 0. && \blacksquare
\end{aligned}$$

Example 0.1 $\displaystyle \lim_{x\to 0}\frac{\sin 7x}{4x} = ?$

Let $\theta = 7x$, *then* $\theta\to 0 \iff x\to 0$.

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \lim_{x\to 0}\left(\frac{7}{4}\frac{\sin 7x}{7x}\right) = \frac{7}{4}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{7}{4}\cdot 1 = \frac{7}{4}. \qquad \blacksquare$$

Example 0.2 $\displaystyle \lim_{x\to 0} x\cot x = ?$



$\because \displaystyle\lim_{x\to 0^{\pm}}\cot x = \pm\infty$ *does not exist,* 不能用極限律乘法。

$\because \displaystyle\lim_{x\to 0}\cos x = 1$ *and* $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1\neq 0$, 可以用極限律除法。

$$\therefore \lim_{x\to 0} x\cot x = \lim_{x\to 0}\frac{x\cos x}{\sin x} = \lim_{x\to 0}\frac{\cos x}{\frac{\sin x}{x}} = \frac{\lim\limits_{x\to 0}\cos x}{\lim\limits_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1. \qquad \blacksquare$$

Skill: 化成已知的極限:

$\displaystyle\lim_{x\to 0}\frac{e^x-1}{x} = 1$; $\displaystyle\lim_{x\to -\infty} e^x = \lim_{x\to 0^-} e^{1/x} = \lim_{x\to\infty} e^{-x} = 0$; $\displaystyle\lim_{x\to\pm\infty}\frac{1}{x^r} = 0,\ r\in\mathbb{Q}^+$;

$\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$; $\displaystyle\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$; $\displaystyle\lim_{x\to 0} x\sin\frac{1}{x} = 0$; $\displaystyle\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

0.2 Derivatives of trigonometric functions

$$\boxed{\begin{array}{llllll}(\sin x)'= & \cos x, & (\tan x)'= & \sec^2 x, & (\sec x)'= & \sec x\tan x,\\ (\cos x)'= & -\sin x, & (\cot x)'= & -\csc^2 x, & (\csc x)'= & -\csc x\cot x.\end{array}}$$

1. $\dfrac{d}{dx}\sin x = \cos x.$ .... $\boxed{\color{blue}{(\sin x)'} = \color{red}{\cos x}}$

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\theta\to 0}\frac{\sin(x+\theta)-\sin x}{\theta}\\
&= \lim_{\theta\to 0}\frac{\sin x\cos\theta+\cos x\sin\theta-\sin x}{\theta}\\
&= \lim_{\theta\to 0}\left(\sin x\frac{\cos\theta-1}{\theta}+\cos x\frac{\sin\theta}{\theta}\right)\\
&= \sin x\,\color{red}{\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}}+\cos x\,\color{blue}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}\\
&= \sin x\cdot\color{red}{0}+\cos x\cdot\color{blue}{1}\\
&= \cos x. \qquad\qquad (\text{sine 導數到了})
\end{aligned}$$

2. $\dfrac{d}{dx}\cos x = -\sin x.$ .... $\boxed{\color{red}{(\cos x)'} = \color{red}{-}\color{blue}{\sin x}}$

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{\theta\to 0}\frac{\cos(x+\theta)-\cos x}{\theta}\\
&= \lim_{\theta\to 0}\frac{\cos x\cos\theta-\sin x\sin\theta-\cos x}{\theta}\\
&= \lim_{\theta\to 0}(\cos x\frac{\cos\theta-1}{\theta}-\sin x\frac{\sin\theta}{\theta})\\
&= \cos x\,\color{red}{\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}}-\sin x\,\color{blue}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}\\
&= \cos x\cdot\color{red}{0}-\sin x\cdot\color{blue}{1}\\
&= -\sin x.
\end{aligned}$$

3. $\dfrac{d}{dx}\tan x = \sec^2 x$. .... $\boxed{(\tan x)' = \sec^2 x}$

Apply Quotient Rule on $\tan x = \dfrac{\sin x}{\cos x}$.

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

♦ 4. $\dfrac{d}{dx}\cot x = -\csc^2 x$. .... $\boxed{(\cot x)' = -\csc^2 x}$

Apply Quotient Rule on $\cot x = \dfrac{\cos x}{\sin x}$.

$$\begin{aligned}\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{(\cos x)'\sin x - \cos x(\sin x)'}{\sin^2 x}\\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.\end{aligned}$$

5. $\dfrac{d}{dx}\sec x = \sec x\tan x$. .... $\boxed{(\sec x)' = \sec x\tan x}$

Apply Quotient Rule on $\sec x = \dfrac{1}{\cos x}$.

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x} = \frac{(1)'\cos x - 1(\cos x)'}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x\tan x.\end{aligned}$$

♦ 6. $\dfrac{d}{dx}\csc x = -\csc x\cot x$. .... $\boxed{(\csc x)' = -\csc x\cot x.}$

Apply Quotient Rule on $\csc x = \dfrac{1}{\sin x}$.

$$\begin{aligned}\frac{d}{dx}\csc x &= \frac{d}{dx}\frac{1}{\sin x} = \frac{(1)'\sin x - 1(\sin x)'}{\sin^2 x}\\ &= \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x}\frac{\cos x}{\sin x} = -\csc x\cot x.\end{aligned}$$

Example 0.3 $(x^2\sin x)' = ?$

$$(x^2\sin x)' = (x^2)'\sin x + x^2(\sin x)' = 2x\sin x + x^2\cos x. \qquad \blacksquare$$

Example 0.4 $y = \dfrac{\sec x}{1+\tan x}$ 水平切線處 $x = ?$

Let $f(x) = \dfrac{\sec x}{1+\tan x}$, *solve* $f'(x) = 0$.

$$f'(x) = \left(\frac{\sec x}{1+\tan x}\right)' = \frac{(\sec x)'(1+\tan x) - \sec x(1+\tan x)'}{(1+\tan x)^2}$$

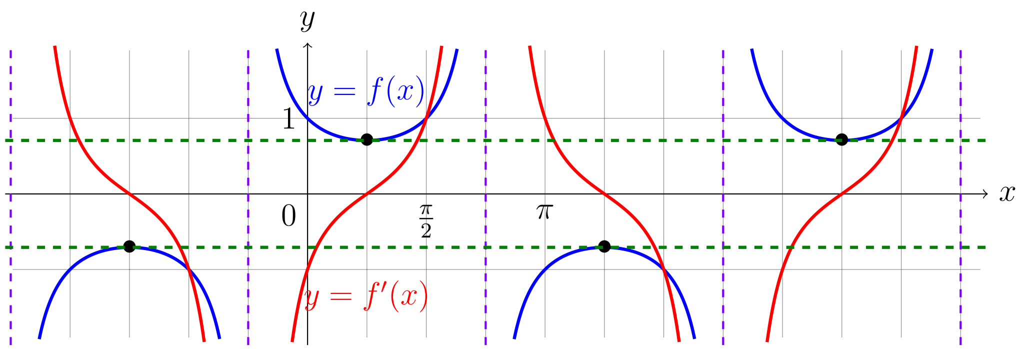
$$= \frac{\sec x\tan x(1+\tan x) - \sec^3 x}{(1+\tan x)^2} = \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}.$$

(★: 微分後不要乘開, 養成因式分解的好習慣。)

$$f'(x) = 0 \iff \tan x = 1(\because |\sec x| \geq 1) \iff x = (n+\frac{1}{4})\pi,\ n \in \mathbb{Z}. \qquad \blacksquare$$

Note: $(a, f(a))$ 有水平切線 $\iff$ 切線斜率爲零 $\iff$ $f'(a) = 0$。



Remark: 出現頻率: (由上而下, 由高而低。)

$(\sin x)'$	$=$	$\cos x$
$(\cos x)'$	$=$	$-\sin x$
$\lim\limits_{x\to 0}\dfrac{\sin x}{x}$	$=$	1
$(\tan x)'$	$=$	$\sec^2 x$
$(\sec x)'$	$=$	$\sec x\tan x$
$\lim\limits_{x\to 0}\dfrac{\cos x - 1}{x}$	$=$	0
$(\cot x)'$	$=$	$-\csc^2 x$
$(\csc x)'$	$=$	$-\csc x\cot x$

公式記憶法:

$(\sin x)' = \cos x,\qquad (\cos x)' = -\sin x,$

$(\tan x)' = \sec^2 x,\qquad (\cot x)' = -\csc^2 x,$

$(\sec x)' = \sec x\tan x,\qquad (\csc x)' = -\csc x\cot x.$

$\leftrightarrow$正餘交換 $\uparrow$餘有負號