Problem 1

(a) $E[XY] = \int_0^1 \int_{-y}^{y} 1 \, dx\,dy$

$= \int_0^1 \frac{1}{2} x^2 y \Big|_{-y}^{y} dy$

$= 0$

$E[X] = \int_{-1}^{0} x f_X(x) dx + \int_0^1 x f_X(x) dx$

$= \int_{-1}^{0} x \int_{-x}^{1} 1 \, dy\,dx + \int_0^1 x \cdot \int_x^1 1 \, dy\,dx$

$= \int_{-1}^{0} (x^2 + x) \, dx + \int_0^1 (-x^2 + x) \, dx$

$= \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_{-1}^{0} + \left(\frac{-1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^1$

$= 0 - \left(\frac{-1}{3} + \frac{1}{2}\right) + \left(\frac{-1}{3} + \frac{1}{2} - 0\right)$

$= 0$

$E[Y] = \int_0^1 y f_Y(y) dy$

$= \int_0^1 y \int_{-y}^{y} 1 \, dx\,dy$

$= \int_0^1 2y^2 dy$

$= \frac{2}{3} y^3 \Big|_0^1$

$= \frac{2}{3}$ $\Rightarrow E[XY] = E[X] \cdot E[Y]$ #

(b) Suppose $A=(\frac{1}{2},1)$ $B=(0,\frac{1}{2})$

$$P(U\in A, V\in B)=\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} 0 \, dx\,dy$$

$$=0$$

$$P(U\in A)=\int_{\frac{1}{2}}^{1} f_X(x)\, dx$$

$$=\int_{\frac{1}{2}}^{1}\int_x^1 1\, dy\, dx$$

$$=\int_{\frac{1}{2}}^{1} (1-x)\, dx$$

$$=\frac{-1}{2}x^2+x\Big|_{\frac{1}{2}}^{1}$$

$$=(\frac{-1}{2}+1)-(\frac{-1}{8}+\frac{1}{2})$$

$$=\frac{1}{8}$$

$$P(V\in B)=\int_0^{\frac{1}{2}} f_Y(y)\, dy=\int_0^{\frac{1}{2}}\int_{-y}^{y} 1\, dx\,dy$$

$$=\frac{2}{3}y^3\Big|_0^{\frac{1}{2}}$$

$$=\frac{1}{12}$$

$\Rightarrow P(U\in A, V\in B)\neq P(U\in A)\cdot P(V\in B) \Rightarrow X, Y$ are not independent #

Problem 2

(a) $f_{X|Y}(x,y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx} = \dfrac{f_{XY}(x,y)}{\int_0^{1-y} \frac{1}{\frac{1\cdot 1}{2}}\,dx} = \dfrac{f_{XY}(x,y)}{2-2y}$

$$\Rightarrow f_{X|Y}(x,y) = \begin{cases} \frac{1}{1-y} & x+y<1 \\ 0 & x+y\geq 1 \end{cases}$$

(b) $E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x,y)\,dx$

$$= \int_0^{1-y} x \cdot \frac{1}{1-y}\,dx$$

$$= \frac{1}{1-y} \cdot \frac{1}{2}x^2 \Big|_0^{1-y}$$

$$= \frac{1-y}{2}$$

By LIE, $E[X] = E[E[X|Y=y]]$

$$= \int_{-\infty}^{\infty} g(y) \cdot f_Y(y)\,dy$$

$$= \int_0^1 \frac{1-y}{2} \cdot \int_0^{1-y} 2\,dx\,dy$$

$$= \int_0^1 (1-y)^2\,dy$$

$$= \frac{1}{3}y^3 - y^2 + y \Big|_0^1 = \frac{1}{3} \quad \#$$

Problem 3

(a) $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_{-1}^{3} e^{tx} \cdot \frac{1}{4}\,dx = \frac{1}{4t}(e^{3t} - e^{-t})$

$$E[X] = M_X'(0) = \left[\frac{1}{4} t^{-1}(e^{3t} - e^{-t})\right]' \Big|_{t=0}$$

$$= \frac{1}{4}\left(-t^{-2}(e^{3t} - e^{-t}) + t^{-1}(3e^{3t} + e^{-t})\right) \Big|_{t=0}$$

$$\overset{L'H.}{=} \lim_{t \to 0} \frac{1}{4}\left(\frac{-3e^{3t} - e^{-t}}{2t} + 9e^{3t} - e^{-t}\right)$$

$$\overset{L'H.}{=} \lim_{t \to 0} \frac{1}{4}\left(\frac{-9e^{3t} + e^{-t}}{2} + 8\right)$$

$$= 1$$

$$E[X^2] = M_X''(0) = \lim_{t \to 0}\left(\frac{e^{3t} - e^{-t}}{2t^3} - \frac{3e^{3t} + e^{-t}}{2t^2} + \frac{9e^{3t} - e^{-t}}{4t}\right)$$

$$\overset{L'H.}{=} \lim_{t \to 0}\left(\frac{3e^{3t} + e^{-t}}{6t^2} - \frac{9e^{3t} - e^{-t}}{4t} + \frac{9e^{3t} - e^{-t}}{4t}\right)$$

$$\overset{L'H.}{=} \lim_{t \to 0} \frac{9e^{3t} - e^{-t}}{12t}$$

$$\overset{L'H.}{=} \lim_{t \to 0} \frac{27e^{3t} + e^{-t}}{12}$$

$$= \frac{7}{3}$$

$Var[X] = E[X^2] - E^2[X] = \frac{7}{3} - 1 = \frac{4}{3}$ #

(b) $M_Y(t) = E[e^{tY}] = \sum_{y \in Y} e^{ty} \cdot P_Y(y)$

$= \sum_{y=1}^{\infty} e^{ty} \cdot \frac{6}{\pi^2 y^2}$

$= \frac{6}{\pi^2} \sum_{y=1}^{\infty} \frac{e^{ty}}{y^2}$

$\lim_{y \to \infty} \frac{e^{ty}}{y^2} \overset{\text{L.H.}}{=} \lim_{y \to \infty} \frac{te^{ty}}{2y} \overset{\text{L.H.}}{=} \lim_{y \to \infty} \frac{t^2 e^{ty}}{2} \longrightarrow \infty$ for all $t \in (0, \infty)$

$\Rightarrow$ there exists no interval of the form $(-\delta, \delta)$ such that $M_Y(t)$ exists #

4. (a) $X \sim B(5, \frac{1}{3})$

$$P_X(k) = \begin{cases} C^5_k (\frac{1}{3})^k (\frac{2}{3})^{5-k} & , k = 0, 1, 2, 3, 4, 5 \\ 0 & , \text{otherwise} \end{cases}$$ #

(b) $X \sim \text{poisson}(5)$

$$P_X(k) = \begin{cases} \frac{e^{-5T} \cdot (5T)^k}{k!} & , k = 0, 1, 2, 3 \cdots \\ 0 & , \text{otherwise} \end{cases}$$ #

5. $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \Delta_1 & 0 \\ \Delta_2\rho & \Delta_2\sqrt{1-\rho^2} \end{bmatrix}}_{A} \begin{bmatrix} Z \\ W \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \det(A) = \Delta_1\Delta_2\sqrt{1-\rho^2}$$

$$\Rightarrow \begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} \dfrac{x_1-\mu_1}{\Delta_1} \\ \dfrac{-\rho(x_1-\mu_1)}{\Delta_1\sqrt{1-\rho^2}} + \dfrac{x_2-\mu_2}{\Delta_2\sqrt{1-\rho^2}} \end{bmatrix}$$

$$f_{x_1,x_2}(x,y) = \frac{1}{|\det(A)|} f_{Z,W}(z,w)$$

$$= \frac{1}{\Delta_1\Delta_2\sqrt{1-\rho^2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x_1-\mu_1}{\Delta_1}\right)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\left(\frac{-\rho(x_1-\mu_1)}{\Delta_1\sqrt{1-\rho^2}} + \frac{x_2-\mu_2}{\Delta_2\sqrt{1-\rho^2}}\right)^2}{2}}$$

$$= \frac{1}{2\pi\Delta_1\Delta_2\sqrt{1-\rho^2}} e^{\frac{-1}{2}\left(\frac{(x_1-\mu_1)^2}{\Delta_1^2} + \left(\frac{\rho^2(x_1-\mu_1)^2}{\Delta_1^2(1-\rho^2)} + \frac{(x_2-\mu_2)^2}{\Delta_2^2(1-\rho^2)} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\Delta_1\Delta_2(1-\rho^2)}\right)\right)}$$

$$= \frac{1}{2\pi\Delta_1\Delta_2\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left(\frac{(1-\cancel{\rho^2})(x_1-\mu_1)^2}{\Delta_1^2} + \cancel{\frac{\rho^2(x_1-\mu_1)^2}{\Delta_1^2}} + \frac{(x_2-\mu_2)^2}{\Delta_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\Delta_1\Delta_2(1-\rho^2)}\right)}$$

$$= \frac{1}{2\pi\Delta_1\Delta_2\sqrt{1-\rho^2}} \cdot e^{-\frac{\frac{(x_1-\mu_1)^2}{\Delta_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\Delta_1\Delta_2} + \frac{(x_2-\mu_2)^2}{\Delta_2^2}}{2(1-\rho^2)}} \quad \#$$

6. define $\left(\int_S |f(\omega)|^p\right)^{\frac{1}{p}} = ||f||_p$

$\Rightarrow$ prove that $||fg||_1 \le ||f||_p ||g||_q$

by young's inequality : $|f(s)g(s)| \le \frac{f(s)^p}{p} + \frac{g(s)^q}{q}$

integral on both side : $||fg||_1 \le \frac{||f||_p^p}{p} + \frac{||g||_q^q}{q}$

also $||f||_p = ||g||_q = 1$

$\Rightarrow ||fg||_1 \le \frac{1}{p} + \frac{1}{q} = 1 = ||f||_p ||g||_q$

which can expand to $E[|XY|] \le E[|X|^p]^{\frac{1}{p}} E[|Y|^q]^{\frac{1}{q}}$ #