5.5 The substitution rule

1. for indefinite integral (變數變換) 不定積分
2. for definite integral (變數變換) 定積分
3. symmetry 對稱性

0.1 The substitution rule for indefinite integral

Recall: Chain rule: Let $y = f(u)$ and $u = u(x)$, then

$$\frac{df(u(x))}{dx} = \frac{df(u)}{du}\frac{du(x)}{dx}[= f'(u)u'(x)] \qquad \left(\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}\right)$$

Theorem 1 *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\boxed{\int \boldsymbol{f(g(x))g'(x)\ dx} = \int \boldsymbol{f(u)\ du}}$$

Proof. Let F be an antiderivative of f, $F'(u) = f(u)$.
(代入 $u = g(x)$, 用 Chain Rule 對 x 微分。)

$$\frac{d}{dx}[F(g(x))] = \frac{d}{du}F(u)\frac{d}{dx}g(x) = F'(u)g'(x) = f(g(x))g'(x).$$

$\Longrightarrow$ $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.
$\therefore \displaystyle\int f(g(x))g'(x)\ dx = F(g(x)) + C = F(u) + C = \int f(u)\ du.$ ∎

Skill: 把積分裡的 dx 與 du 當成微分(differential)(其實不是) 來幫忙換:

$$u = u(x), \ \boxed{\boldsymbol{du = u'(x)\ dx}}, \text{(變數名與函數名一樣方便使用。)}$$

$$\begin{array}{ccccc} \int & f(u(x)) & u'(x)\ dx & x \text{ 的函數對 } x \text{ 積分} \\ \downarrow & \downarrow & \downarrow & \text{換成} \\ \int & f(u) & du & u \text{ 的函數對 } u \text{ 積分} \end{array}$$

Timing: 合成函數的積分。 **Goal:** 換成簡單的函數來積分。
Question: 怎麼選擇適當的 $u = u(x)$? $\displaystyle\int$經驗 d作業

Attention: 換的時候要把 x 都換成 u 的函數, 最後要把 u 換回 x 的函數。

Example 0.1 $\displaystyle\int 2x\sqrt{1+x^2}\ dx = ?$

Let $u = \overset{\text{可省略}}{\boxed{u(x) =}}\, 1+x^2$, *then* $du = \overset{\text{可省略}}{\boxed{u'(x)\ dx = (1+x^2)'\ dx =}}\, 2x\ dx.$

$$\therefore \int 2x\sqrt{1+x^2}\ dx \overset{\text{換成}u}{=} \int \sqrt{u}\ du = \frac{2}{3}u^{3/2} + C \overset{\text{換回}x}{=} \frac{2}{3}(1+x^2)^{3/2} + C. \quad \blacksquare$$

Example 0.2 $\displaystyle\int x^3\cos(x^4+2)\ dx = ?$

Let $u = x^4+2$, *then* $du = 4x^3\ dx$, $x^3\ dx = \dfrac{1}{4}\ du.$

$$\therefore \int x^3\cos(x^4+2)\ dx = \int \frac{1}{4}\cos u\ du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C. \quad \blacksquare$$

Example 0.3 $\displaystyle\int \sqrt{2x+1}\ dx = ?$

[Sol 1] Let $u = 2x+1$, *then* $du = 2\ dx$, $dx = \dfrac{1}{2}\ du.$

$$\therefore \int \sqrt{2x+1}\ dx = \int \frac{1}{2}\sqrt{u}\ du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C.$$

[Sol 2] Let $v = \sqrt{2x+1}$, *then* $dv = \dfrac{1}{\sqrt{2x+1}}\ dx = \dfrac{1}{v}\ dx$, $dx = v\ dv.$

$$\therefore \int \sqrt{2x+1}\ dx = \int v \cdot v\ dv = \frac{1}{3}v^3 + C = \frac{1}{3}(2x+1)^{3/2} + C. \quad \blacksquare$$

Example 0.4 $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\ dx = ?$

Let $u = 1-4x^2$, *then* $du = -8x\ dx$, $x\ dx = -\dfrac{1}{8}\ du.$

$$\therefore \int \frac{x}{\sqrt{1-4x^2}}\ dx = \int -\frac{1}{8}u^{-1/2}\ du = -\frac{1}{4}u^{1/2} + C = -\frac{1}{4}\sqrt{1-4x^2} + C. \quad \blacksquare$$

Example 0.5 $\displaystyle\int e^{5x}\ dx = ?$

Let $u = 5x$, *then* $du = 5\ dx$, $dx = \dfrac{1}{5}\ du.$

$$\therefore \int e^{5x}\ dx = \int \frac{1}{5}e^u\ du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C. \quad \blacksquare$$

Skill: 怎麼檢查對不對? 一樣, 用微分! (這時候一定會用上連鎖律。)

Example 0.6 *(換乾淨)* $\int \sqrt{1+x^2}x^5\ dx = ?$

Let $u = 1+x^2$*, then* $du = 2x\ dx$, $x\ dx = \dfrac{1}{2}\ du$.

$\int \sqrt{1+x^2}x^4 \cdot x\ dx = \int \sqrt{u}x^4 \cdot \frac{1}{2}\ du$ (**Wrong!** 要把 x 換光。)

$x^4 = (x^2)^2 = (u-1)^2$, $x^5\ dx = (x^2)^2 x\ dx = \dfrac{1}{2}(u-1)^2\ du$.

$\therefore \int \sqrt{1+x^2}x^5\ dx = \int u^{1/2} \cdot \frac{1}{2}(u-1)^2\ du = \int \frac{1}{2}u^{5/2} - u^{3/2} + \frac{1}{2}u^{1/2}\ du$

$= \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C = \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$

(or $= \sqrt{1+x^2}\left(\frac{1}{7}x^6 + \frac{1}{35}x^4 - \frac{4}{105}x^2 + \frac{8}{105}\right) + C$*).* ■

Example 0.7 $\int \tan x\ dx = ?$

$\tan x = \dfrac{\sin x}{\cos x}$. *Let* $u = \cos x$*, then* $du = -\sin x\ dx$, $\sin x\ dx = -\ du$.

$\therefore \int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx = \int -\frac{du}{u} = -\ln|u| + C \qquad \left(\int \frac{dx}{x} = \ln|x| + C\right)$

$= -\ln|\cos x| + C$ *(ok, but)* $= \ln|\cos x|^{-1} + C = \ln|\sec x| + C$. *(好記)* ■

加入你的不定積分表: $\boxed{\int \tan x\ dx = \ln|\sec x| + C}$

Example 0.8 *(Extra)* $\int \sec x\ dx = ?$ *(用變數變換比較繁瑣)*

$(\ln|\sec x + \tan x|)' = \dfrac{(\sec x + \tan x)'}{\sec x + \tan x} = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= \dfrac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$. $(\sec x + \tan x = 0?)$

(By the Chain Rule & $(\ln|x|)' = \dfrac{1}{x}$, 因爲都有 $\sec x$, *domain* 一樣, 是反導數。)

$\therefore \int \sec x\ dx = \ln|\sec x + \tan x| + C$. ■

加入你的不定積分表: $\boxed{\int \sec x\ dx = \ln|\sec x + \tan x| + C}$

0.2 The substitution rule for definite integral

Theorem 2 *If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then*

$$\boxed{\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du}$$

Proof. Let F be an antiderivative of f, $F'(u) = f(u)$,
$\Longrightarrow\ [F(g(x))]' = F'(g(x))g'(x) = f(g(x))g'(x)$.

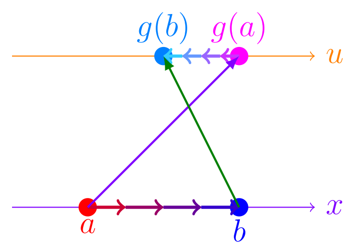
By TFTC, $\displaystyle\int_a^b f(g(x))g'(x)\ dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$,

$$\|$$

and $\displaystyle\int_{g(a)}^{g(b)} f(u)\ du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$. ■

Remark: 定積分的時候, <u>上下界</u>要跟著換:
(想像: 變數變換就像用不同的單位計算; 台幣換美金, 數字也要變, 總價值不變。)
when $u = g(x)$,

$$\begin{array}{ccccccccc} g(x), & g'(x)\ dx, & x & & a & & b. \\ \updownarrow & \updownarrow & \updownarrow & 從 & \updownarrow & 到 & \updownarrow \\ u, & du, & u & & g(a) & & g(b). \end{array}$$



(不一定會 $g(a) \le g(b)$, 有可能大小反過來。)

Solve: 兩種方法
1. 用不定積分算出來反導數 $F(g(x))$, 再把 x 代 b 減代 a。
2. 用定積分算出 $F(u)$ (不要代入 $u = g(x)$), u 代 $g(b)$ 減代 $g(a)$。

有時候 f & g 很複雜, 變回 $F(g(x))$ 代 b 減代 a 計算會變得很複雜, 不如直接 $F(u)$ 代 $g(b)$ 減代 $g(a)$ 計算會簡單些, 答案都是一樣。

Example 0.9 $\displaystyle\int_0^4 \sqrt{2x+1}\ dx =?$

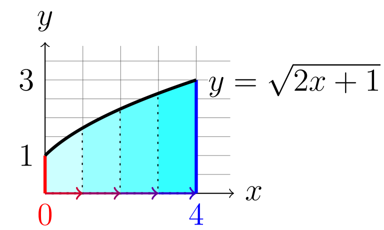
[Sol 1] (先反導再代) $\displaystyle\int \sqrt{2x+1}\ dx \overset{過程略}{=} \frac{1}{3}(2x+1)^{3/2} + C,$

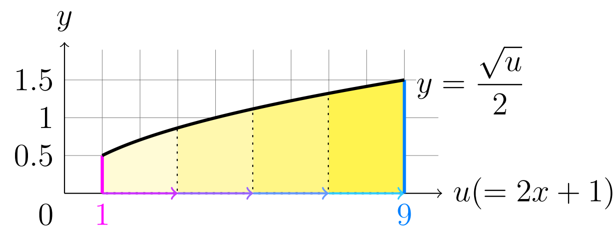
$$\therefore \int_0^4 \sqrt{2x+1}\ dx = \frac{1}{3}(2x+1)^{3/2}\Big|_0^4 = \frac{1}{3}(2\cdot 4+1)^{3/2} - \frac{1}{3}(2\cdot 0+1)^{3/2} = \frac{26}{3}.$$

Skill: 反導數有加一加二加山加海家豪佳俊, 找誰? 找嘉玲 *(加零)*。

[Sol 2] (上下一起換) Let $u = 2x+1$, *then* $du = 2\ dx$, $dx = \frac{1}{2}\ du$,
when $x = 0$, $u = 2\cdot 0 + 1 = 1$, *when* $x = 4$, $u = 2\cdot 4+1 = 9$. *(上下界的變換)*

$$\therefore \int_0^4 \sqrt{2x+1}\ dx = \int_1^9 \frac{1}{2}\sqrt{u}\ du = \frac{1}{3}u^{3/2}\Big|_1^9 = \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{26}{3}. \qquad \blacksquare$$

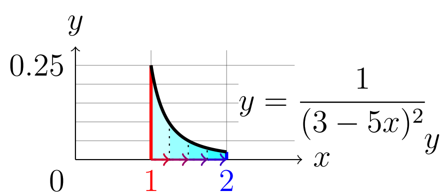


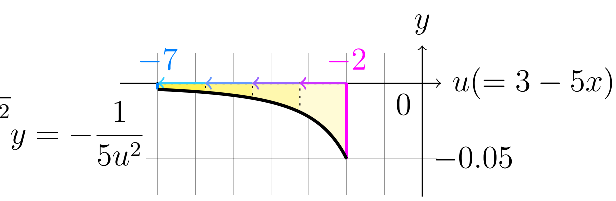


Example 0.10 $\int_1^2 \frac{dx}{(3-5x)^2} = ?$

Let $u = 3-5x$, *then* $du = -5\ dx$, $dx = -\frac{1}{5}\ du$,
when $x = 1$, $u = -2$, *when* $x = 2$, $u = -7$.

$$\therefore \int_1^2 \frac{dx}{(3-5x)^2} = \int_{-2}^{-7} -\frac{1}{5u^2}\ du = \frac{1}{5u}\Big|_{-2}^{-7} = \frac{-1}{35} - \frac{-1}{10} = \frac{1}{14}. \qquad \blacksquare$$

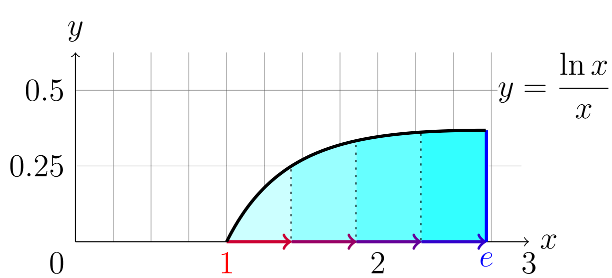


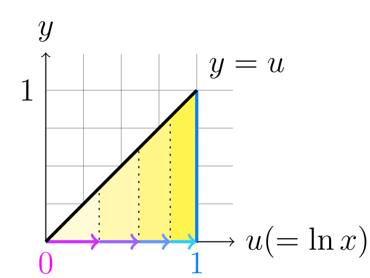


Example 0.11 $\int_1^e \frac{\ln x}{x}\ dx = ?$

Let $u = \ln x$, *then* $du = \frac{1}{x}\ dx$, *when* $x = 1$, $u = 0$, *when* $x = e$, $u = 1$.

$$\therefore \int_1^e \frac{\ln x}{x}\ dx = \int_0^1 u\ du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}. \qquad \blacksquare$$





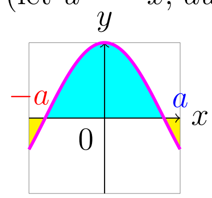
0.3 Symmetry

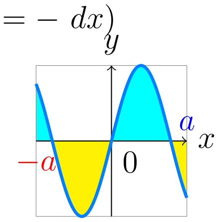
Theorem 3 *Suppose f is continuous on $[-a, a]$.*
(a) If f is even $[f(-x) = f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$.
(b) If f is odd $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\ dx = 0$.

Proof. $\displaystyle\int_{-a}^{0} f(x)\ dx = -\int_{0}^{-a} f(x)\ dx$ (上下界互換差負號)

$$\overset{變數變換}{=} \int_{0}^{a} f(-u)\ du \overset{u\to x}{=} \int_{0}^{a} f(-x)\ dx \qquad (\text{let } u = -x,\ du = -\ dx)$$

$$= \begin{cases} \displaystyle\int_{0}^{a} f(x)\ dx & \text{if } f \text{ is even;} \\ \displaystyle-\int_{0}^{a} f(x)\ dx & \text{if } f \text{ is odd.} \end{cases}$$





$$\therefore \int_{-a}^{a} f(x)\ dx = \int_{-a}^{0} f(x)\ dx + \int_{0}^{a} f(x)\ dx$$

$$= \begin{cases} \displaystyle\int_{0}^{a} f(x)\ dx + \int_{0}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx & \text{if } f \text{ is even;} \\ \displaystyle\cancel{-\int_{0}^{a} f(x)\ dx} + \cancel{\int_{0}^{a} f(x)\ dx} = 0 & \text{if } f \text{ is odd.} \end{cases} \qquad \blacksquare$$

Example 0.12 $\displaystyle\int_{-2}^{2} (x^6 + 1)\ dx = ?$

$\because$ $x^6 + 1$ *is even, (*如果用 $\displaystyle\left[\frac{x^7}{7} + x\right]_{-2}^{2}$ 也可以, 只是容易算錯。*)*

$$\therefore \int_{-2}^{2} (x^6 + 1)\ dx = 2\int_{0}^{2} (x^6 + 1)\ dx = 2\left[\frac{x^7}{7} + x\right]_{0}^{2} = \frac{284}{7}. \qquad \blacksquare$$

Example 0.13 $\displaystyle\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\ dx = ?$

$\because$ $\displaystyle\frac{\tan(-x)}{1 + (-x)^2 + (-x)^4} = -\frac{\tan x}{1 + x^2 + x^4}$ *is odd, (*看出來就不用算。*)*

$$\therefore \int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\ dx = 0. \qquad \blacksquare$$

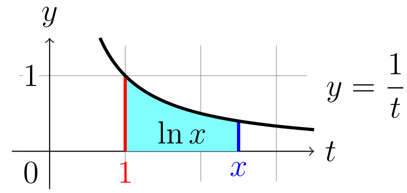
Timing: 使用對稱性時機: 1. 是否爲奇/偶函數; 2. 範圍($[-a, a]$)對稱 y-軸。

♦ Additional: Logarithm defined as an integral

課本上是用極限定義 e $\left(\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1\right)$, 再定義 $\ln x$ 爲 e^x 的反函數。
歷史上是用積分定義 $\ln x$, 再定義 e $(\ln e = 1)$ 以及定義 e^x 爲 $\ln x$ 的反函數。

Define: The ***natural logarithmic function*** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt$$



[由定義可證明 導數公式 與 對數律:]

- By TFTC, $\implies$ $(\ln x)' = \dfrac{1}{x}$.
- Let $u = t/x$, $x\,du = dt$, $\displaystyle\int_x^{xy}\frac{1}{t}\,dt = \int_1^y \frac{x}{xu}\,du = \int_1^y \frac{1}{u}\,du = \ln y$,
  $\displaystyle\implies \ln(xy) = \int_1^{xy}\frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^{xy}\frac{1}{t}\,dt = \ln x + \ln y.$
- $\displaystyle 0 = \int_1^1 \frac{1}{t}\,dt = \ln 1 = \ln(\frac{1}{y}y) = \ln\frac{1}{y} + \ln y$, $\displaystyle\ln\frac{1}{y} = -\ln y$,
  $\displaystyle\implies \ln\frac{x}{y} = \ln(x\frac{1}{y}) = \ln x + \ln\frac{1}{y} = \ln x - \ln y.$
- Let $u = t^{1/r}$, $\displaystyle du = \frac{1}{r}t^{1/r-1}\,dt = \frac{1}{r}\frac{u}{t}\,dt$, $\displaystyle\frac{1}{t}\,dt = \frac{r}{u}\,du$,
  $\displaystyle\implies \ln x^r = \int_1^{x^r}\frac{1}{t}\,dt = \int_1^x \frac{r}{u}\,du = r\int_1^x \frac{1}{u}\,du = r\ln x.$

Define: e is the solution to $\ln x = 1$. e^x is the inverse function of $\ln x$.