Problem 1

(a)

$$E[e^{-tX_i}] = \int_0^\infty e^{-tx} \cdot f_{X_i}(x)\,dx \quad \text{(non-negative)}$$

$$\le \int_0^\infty e^{-tx}\,dx$$

$$= \frac{-1}{t} e^{-tx} \Big|_0^\infty$$

$$= \frac{-1}{t}(0-1)$$

$$= \frac{1}{t} \quad \#$$

(b) $$P\left(\sum_{i=1}^{N} X_i \le \varepsilon N\right) = P\left(\underbrace{-\sum_{i=1}^{N} X_i}_{Z} \ge -\varepsilon N\right) \le e^{-t(-\varepsilon N)} \cdot M_Z(t)$$

$$= e^{t\varepsilon N} \cdot E[e^{-t(X_1+X_2+\cdots+X_N)}]$$

$$= e^{t\varepsilon N} \cdot E[e^{-tX_1}] \cdot E[e^{-tX_2}] \cdot \cdots \cdot E[e^{-tX_N}]$$

$$\le e^{t\varepsilon N} \cdot \left(\frac{1}{t}\right)^N = \frac{e^{t\varepsilon N}}{t^N}$$

To minimize $\frac{e^{t\varepsilon N}}{t^N}$: $\frac{d\left(\frac{e^{t\varepsilon N}}{t^N}\right)}{dt} = \frac{Ne^{t\varepsilon N}(\varepsilon t - 1)}{t^{N+1}} = 0 \Rightarrow t = \frac{1}{\varepsilon}$

$$\Rightarrow P\left(\sum_{i=1}^{N} X_i \le \varepsilon N\right) \le \frac{e^{\frac{1}{\varepsilon}\cdot\varepsilon\cdot N}}{\left(\frac{1}{\varepsilon}\right)^N} = (e\varepsilon)^N \quad \#$$

Problem 2

Suppose $A = \{ w : X_n(w) \overset{a.s}{\not\to} a \}$, $B = \{ w : Y_n(w) \overset{a.s}{\not\to} b \}$

then $\{ w : X_n(w) \cdot Y_n(w) \overset{a.s}{\not\to} a \cdot b \} \subseteq A \cup B = \emptyset$

$\Rightarrow P(\{ w : X_n(w) \cdot Y_n(w) \overset{a.s}{\not\to} ab \}) \leq P(A \cup B) = 0$

By probability axiom $1 = P(\{ w : X_n(w) \cdot Y_n(w) \overset{a.s}{\not\to} ab \}) = 0$

$\Rightarrow X_n \cdot Y_n \xrightarrow{a.s} ab$ #

Problem 3

(a)

$$\lim_{n\to\infty} E[(X_n-C)^2]=0 \Rightarrow \frac{\lim_{n\to\infty} E[(X_n-C)^2]}{\varepsilon^2}=0$$

$$\geq \lim_{n\to\infty} P((X_n-C)^2 \geq \varepsilon^2) \text{ (by Markov's inequality)}$$

By probability axiom 1: $\lim_{n\to\infty} P((X_n-C)^2 \geq \varepsilon^2)=0$

$\Rightarrow \lim_{n\to\infty} P(|X_n-C| \geq \varepsilon)=0 \Rightarrow$ convergence in probability #

(b) construct $X_n=\begin{cases} 0 & \text{w.p.}\left(1-\frac{1}{n}\right) \\ n & \text{w.p.}\ \frac{1}{n} \end{cases}$

$\lim_{n\to\infty} P(|X_n-0| \geq \varepsilon)=\lim_{n\to\infty} P(X_n \geq \varepsilon) \Leftrightarrow \lim_{n\to\infty} P(X_n=n)$ (for $\varepsilon>0$)

$=0 \Rightarrow$ convergence in probability

$\lim_{n\to\infty} E[(X_n-0)^2]=\lim_{n\to\infty} E[X_n^2]=0^2+n^2\left(\frac{1}{n}\right)=n \neq 0$

$\Rightarrow$ doesn't imply convergence in mean square #