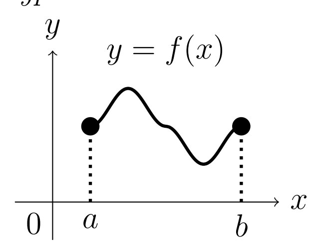
4.2 The Mean Value Theorem

微分應用之二: 瞬間即平均。

1. Rolle's Theorem 羅爾定理
2. Mean Value Theorem 均值定理

0.1 Rolle's Theorem

Theorem 1 (Rolle's Theorem)
Let f be a function that satisfies the following three hypothesis:
1. f is ***continuous*** *on $[a, b]$,* 閉連續
2. f is ***differentiable*** *on (a, b),* 開可微
3. $\boldsymbol{f(a) = f(b)}$. 頭尾同
*Then $\exists$ $\boldsymbol{c \in (a, b)}$, $\ni$(*某處水平*)*



$$\boxed{\boldsymbol{f'(c) = 0}}.$$

Proof.
Case 1. $f(x) = k$ is a constant function. $\Longrightarrow$ $f'(c) = 0, \forall\, c \in (a, b)$.
Case 2. $f(x) > f(a)$ for some $x \in (a, b)$. Hypothesis 1 + Extreme Value Theorem $\Longrightarrow$ f has max in (a, b) ($f(a) = f(b)$ are not). Hypothesis 2 + Fermat's Theorem $\Longrightarrow$ $\exists\, c \in (a, b) \ni f'(c) = 0$.
Case 3. $f(x) < f(a)$ for some $x \in (a, b)$. Similarly, f has min in (a, b) and $\exists\, c \in (a, b) \ni f'(c) = 0$. ■

Example 0.1 經過同一點時, 期間會有速率爲零。

Proof. *Let $s(t)$ be position function, then $v(t) = s'(t)$ is the velocity function. By Rolle's Theorem, $s(a) = s(b)$, then $\exists\, c \in (a, b) \ni v(c) = 0$.* ■

Example 0.2 $x^3 + x - 1$ *has exactly one real root.*

Proof. $\because$ *$f(x) = x^3 + x - 1$ is continuous and differentiable on $\mathbb{R}$.*
*(*勘根定理證明有根*) $f(0)f(1) = (-1) \cdot 1 < 0$, $\exists\, c \in (0, 1) \ni f(c) = 0$.*
*(*證明只有一根*) Suppose there are two roots a, b, i.e. $f(a) = f(b) = 0$.*
By Rolle's Theorem, $\exists\, c \in (a, b) \ni f'(c) = 0$.
But $f'(x) = 3x^2 + 1 > 0$ for all x, a contradiction.
Therefore, f has exactly one root. ■

0.2 Mean Value Theorem

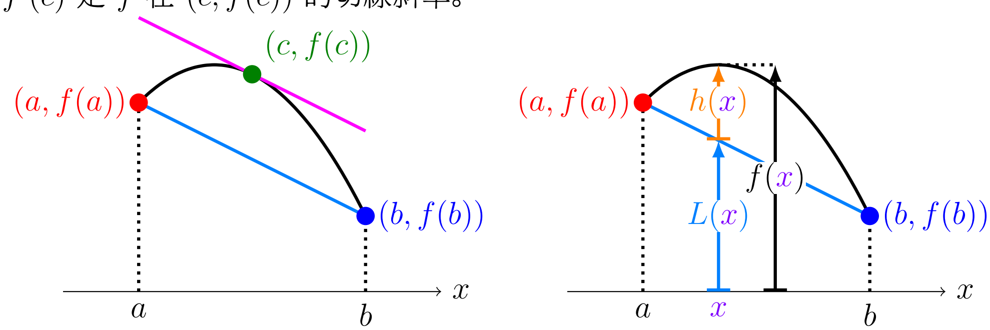
Theorem 2 (Mean Value Theorem) *Let f be a function that satisfies the following two hypothesis.*
1. f is ***continuous*** *on $[a, b]$,* 閉連續
2. f is ***differentiable*** *on (a, b),* 開可微
Then $\exists\ \boldsymbol{c} \in (\boldsymbol{a}, \boldsymbol{b}) \ni$

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}.$$

Equivalently,

$$\boxed{f(b) - f(a) = f'(c)(b - a)}.$$

Note: $\dfrac{f(b) - f(a)}{b - a}$ 是從 $(a, f(a))$ 到 $(b, f(b))$ 的割線斜率。
$f'(c)$ 是 f 在 $(c, f(c))$ 的切線斜率。



Proof. Let $y = L(x) = f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a)$ be the secant line function through $(a, f(a))$ and $(b, f(b))$, and let

$$h(x) = f(x) - L(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

$\because f$ is continuous on $[a, b]$ and differentiable on (a, b), so are L and h. (*1.&2.*)
$\because h(a) = f(a) - f(a) - \dfrac{f(b) - f(a)}{b - a}(a - a) = 0,$
and $h(b) = f(b) - f(a) - \dfrac{f(b) - f(a)}{b - a}(b - a) = 0, \therefore h(a) = h(b).$ ...... (*3.*)

By Rolle's Theorem, $\exists\ c \in (a, b) \ni h'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a} = 0,$
$\Longrightarrow\ f'(c) = \dfrac{f(b) - f(a)}{b - a}$ or $f(b) - f(a) = f'(c)(b - a).$ ∎

Example 0.3 *A car traveled 180 km in 2 hours, then velocity 90 km/h at least once.*

Example 0.4 *$f(0) = -3$, $f'(x) \le 5$ for all x, how large can $f(2)$ be?*

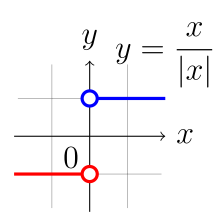
Proof. $\because f$ *is differentiable (and hence continuous) for all x.*
By the Mean Value Theorem, $\exists\, c \in (0,2)$, $\ni f(2) - f(0) = f'(c)(2-0)$.
$\implies f(2) = 2f'(c) + f(0) \le 2 \cdot 5 - 3 = 7.$ ∎

Theorem 3 *$f'(x) = 0$ for all $x \in (a,b)$ then f is **constant** on (a,b).*
開可微零導數是常數

Proof. $\forall\, x_1, x_2 \in (a,b)$, $x_1 < x_2$, f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$.
By the Mean Value Theorem, $\exists\, c \in (x_1, x_2)$,
$\ni f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0, \implies f(x_1) = f(x_2)$.
Therefore, f is constant on (a,b). ∎

Corollary 4 *$f'(x) = g'(x)$ for all $x \in (a,b)$ then $f - g$ is constant on (a,b); that is $f(x) = g(x) + c$ where c is a constant.* 同導數差常數

Note: 要 (a,b), 不可斷。
Ex: $f(x) = \dfrac{x}{|x|}$ on $D = \{x \ne 0\}$, $f'(x) = 0$ on D, but $f(x)$ is not constant. If choose $D = (0,\infty)$ or $D = (-\infty, 0)$ then f is constant.



Example 0.5 *Prove identity $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$.*

Proof. *Let $f(x) = \tan^{-1} x + \cot^{-1} x$.*
Then $f'(x) = \dfrac{1}{1+x^2} + \dfrac{-1}{1+x^2} = 0$, so $f(x)$ is constant.
Therefore, $f(x) = f(1) = \tan^{-1} 1 + \cot^{-1} 1 = \dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2}$. ∎

Additional: $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, $\sec^{-1} x + \csc^{-1} x = \dfrac{\pi}{2}$.
$\left((\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}},\ (\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2-1}}.\right)$

Quiz 4.2

National Chiao Tung University Campus Run is about 4.5 km. Suppose that you finish it in one hour, and your position function (from the beginning) is continuous (on a closed interval) and differentiable (on an open interval). Prove that your velocity reaches 1.25 m/s at least once during the running.

交大校園路跑約 4.5 km. 假設你一小時跑完, 而且位置函數是閉連續開可微. 證明途中你必定曾經達到速率 1.25 m/s.