⧫ 4.7 Optimization Problems

微分應用之七: 優化問題, 求最佳解。

Closed Interval Method: $f(a)$, $f(b)$ and critical number c: $f'(c) = 0$ or $\nexists$

The first derivative test: f' change sign at $c \implies f(c)$ local max/min.

The second derivative test: $f'(c) = 0$, $f''(c) \lessgtr 0 \implies f(c)$ local max/min.

Replace variable or use implicit differentiation.

Example 0.1 *A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river. He need no fence along the river. What are the dimensions of the field that has the largest area?*
一農有 1200 m 籬沿直河圍矩形, 如何有最大面積?

Let depth x m and width y m.
Maximize area $A = xy$ under $2x + y = 1200$, $0 \le x \le 600$.

[Sol 1: replace y] ($y = 1200 - 2x$)
$A(x) = xy = -2x^2 + 1200x$, $A'(x) = -4x + 1200$.
$A'(x) = 0$ *when* $x = 300$, $y = 1200 - 2x = 600$.
Extreme values: $A(300) = 180000$ *and* $A(0) = A(600) = 0$ *(邊界).*

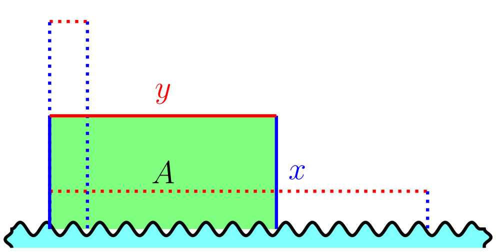
[Sol 2: implicit differentiation] (把 A 與 y 想像成 x 的函數)

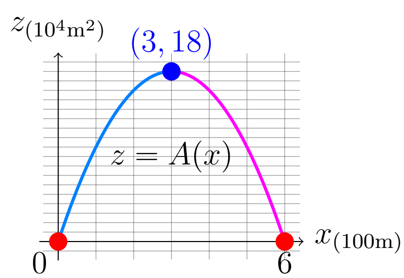
$$A = xy,\ 2x + y = 1200,\ \overset{\frac{d}{dx}}{\Longrightarrow}\ \frac{dA}{dx} = y + x\frac{dy}{dx},\ 2 + \frac{dy}{dx} = 0.$$

$$\text{Let } \frac{dA}{dx} = 0 \text{ (消去} \frac{dy}{dx}\text{)},\ \implies\ y - 2x = 0,\ y = 2x,$$

(代入) $2x + y = 4x = 1200 \implies x = 300$, $y = 2x = 600$.

Ans: 300 m deep and 600 m wide (with area 180,000 m^2). ■





Example 0.2 *A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.*
造 1 *L* 圓罐最少材料 *(*面積*)?*

Let radius r cm and height h cm ($1\ L = 1000\ cc = 1000\ cm^3$).
Minimize area $A = 2\pi r^2 + 2\pi r h$ under $\pi r^2 h = 1000$, $r > 0$.

[Sol 1: replace h] ($h = 1000/\pi r^2$)

$A(r) = 2\pi r^2 + 2\pi r(\frac{1000}{\pi r^2}) = 2\pi r^2 + \frac{2000}{r}$, $A'(r) = 4\pi r - \frac{2000}{r^2}$.

$A'(r) = 0$ *when* $r = \sqrt[3]{\frac{500}{\pi}}$, $h = \frac{1000}{\pi r^2} = 2\sqrt[3]{500/\pi}$.

$A'(r)$ *change sign from* $- \to +$ *at* $\sqrt[3]{500/\pi} \implies$ *local min.*

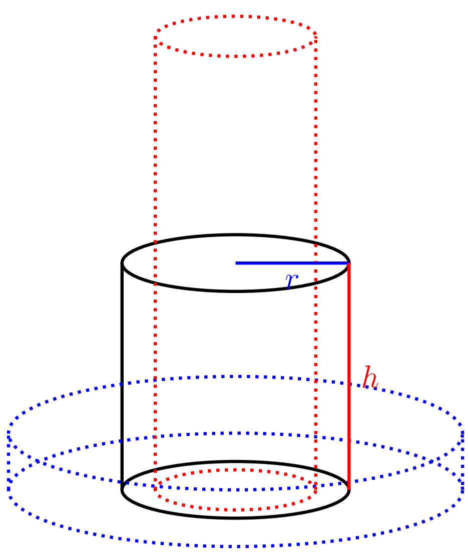
*[Sol 2: implicit differentiation] (*把 A 與 h 想像成 r 的函數*)*

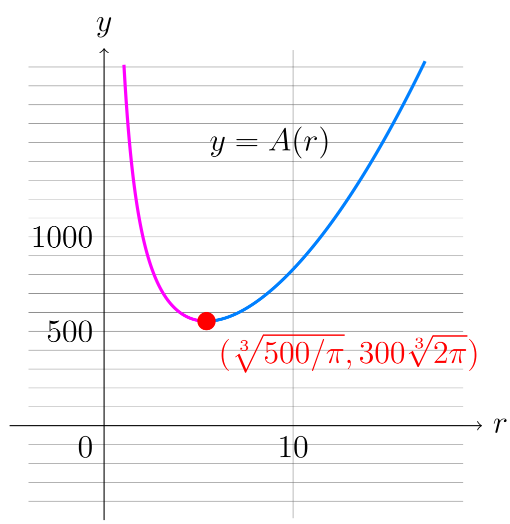
$\overset{\frac{d}{dr}}{\Longrightarrow} \frac{dA}{dr} = 2\pi(2r + h + r\frac{dh}{dr})$, $\pi r(2h + r\frac{dh}{dr}) = 0$.

Let $\frac{dA}{dr} = 0$ *(*消去 $\frac{dh}{dr}$*),* $\implies 2r - h = 0$, $h = 2r$,

*(*代入*)* $\pi r^2 h = \pi r^2(2r) = 1000$, $r = \sqrt[3]{500/\pi}$, $h = 2r = 2\sqrt[3]{500/\pi}$.

Ans: radius $\sqrt[3]{500/\pi}(\approx 5.4)$ cm and height $2\sqrt[3]{500/\pi}(\approx 10.8)$ cm. ■
(with area $300\sqrt[3]{2\pi} \approx 553.6$ cm ^{2}.)





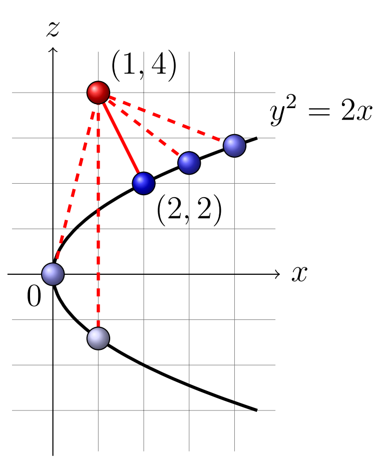
Example 0.3 *Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.*
找 $y^2 = 2x$ 上離 $(1, 4)$ 最近的點。

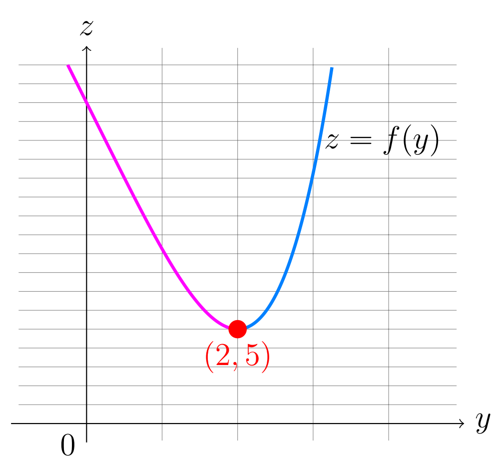
The point (x, y) and the distance $d = \sqrt{(x-1)^2 + (y-4)^2}$. Since minimize d^2 also minimize d, minimize $f = d^2 = (x-1)^2 + (y-4)^2$ under $y^2 = 2x$.

[Sol 1: replace x] ($x = y^2/2$)
$f(y) = (x-1)^2 + (y-4)^2 = (\frac{y^2}{2} - 1)^2 + (y-4)^2$,
$f'(y) = 2(\frac{y^2}{2} - 1)y + 2(y-4) = y^3 - 8$.
$f'(y) = 0$ *when* $y = 2$, $x = \frac{y^2}{2} = 2$.
$f'(y)$ *change sign from* $- \to +$ *at* $2 \implies$ *local min.*

[Sol 2: implicit differentiation] (可以同時 (a) 對 x 或 (b) 對 y 隱微分)
(a) $\frac{df}{dx} = 2(x-1) + 2(y-4)\frac{dy}{dx}$, $2y\frac{dy}{dx} = 2$. $\frac{df}{dx} = 0 \implies 2x - \frac{8}{y} \overset{*}{=} y^2 - \frac{8}{y} = 0$,
(b) $\frac{df}{dy} = 2(x-1)\frac{dx}{dy} + 2(y-4)$, $2y = 2\frac{dx}{dy}$. $\frac{df}{dy} = 0 \implies 2xy - 8 \overset{*}{=} y^3 - 8 = 0$,
($*$ 代入 $2x = y^2$) $\implies$ $y = 2$, $x = 2$.

Ans: <u>$(2, 2)$ *(with distance $\sqrt{5}$)*</u>. *(注意! 距離是 $\sqrt{5}$ 不是 5.)* ■





Attention: 隱微分要對同一個變數微分, 而且 $\frac{dy}{dx} \neq 1 \div \frac{dx}{dy}$.

Example 0.4 *A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?*
如何從 A 以 6 km/h 過 3 km 河並以 8 km/h 跑至下游 8 km 的 B 最快?

Let x *be distance from C to D.* $time = \dfrac{distance}{rate}$.

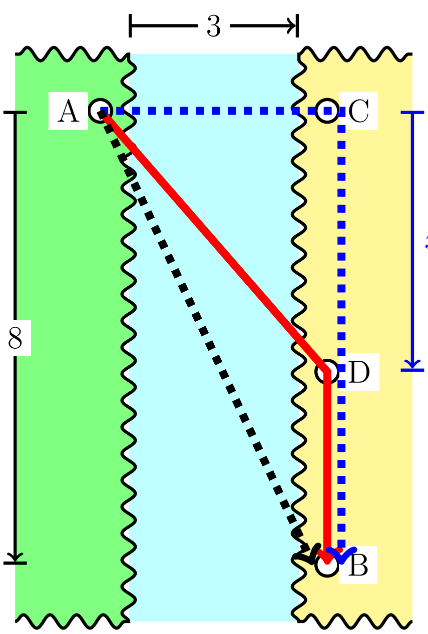
Minimize $T(x) = \dfrac{\sqrt{x^2+3^2}}{6} + \dfrac{8-x}{8}$ *under* $0 \le x \le 8$.

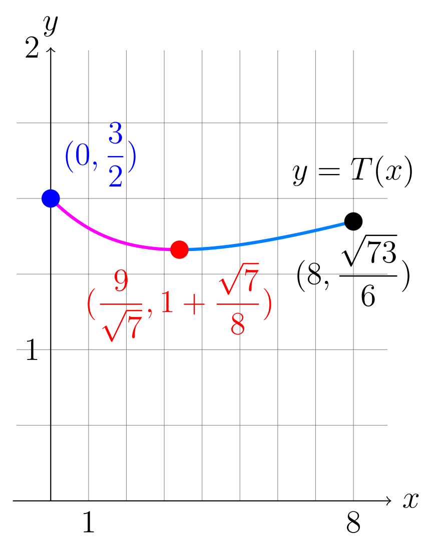
$T'(x) = \dfrac{x}{6\sqrt{x^2+3^2}} - \dfrac{1}{8}$. $T'(x) = 0$ *when* $x = \dfrac{9}{\sqrt{7}}$.

$T(\dfrac{9}{\sqrt{7}}) = 1 + \dfrac{\sqrt{7}}{8} \approx 1.33$, $T(0) = \dfrac{3}{2} = 1.5$, $T(8) = \dfrac{\sqrt{73}}{6} \approx 1.42$.

$T(x)$ *has absolute min at* $\dfrac{9}{\sqrt{7}}$.

Ans: land at $\dfrac{9}{\sqrt{7}}$ *km downstream (with time* $1 + \dfrac{\sqrt{7}}{8}$ *hour).* ■





Example 0.5 *Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.*
半徑 r 的半圓內最大內接矩形面積。

[Sol 1] (雙變數) Let $P(x, y)$ be the inscribed point in the first quadrant. Maximize area $A = 2xy$ under $x^2 + y^2 = r^2$, $x \geq 0$, $y \geq 0$.

(a) replace y: $A(x) = 2x\sqrt{r^2 - x^2}$, $0 \leq x \leq r$.
$A'(x) = 2\sqrt{r^2 - x^2} - \dfrac{2x^2}{\sqrt{r^2 - x^2}} = \dfrac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$. $A'(x) = 0$ *when* $x = \dfrac{r}{\sqrt{2}}$.
$A(\dfrac{r}{\sqrt{2}}) = 2\dfrac{r}{\sqrt{2}}\sqrt{r^2 - \left(\dfrac{r}{\sqrt{2}}\right)^2} = r^2$, $A(0) = A(r) = 0$.
$A(x)$ has absolute max at $\dfrac{r}{\sqrt{2}}$.

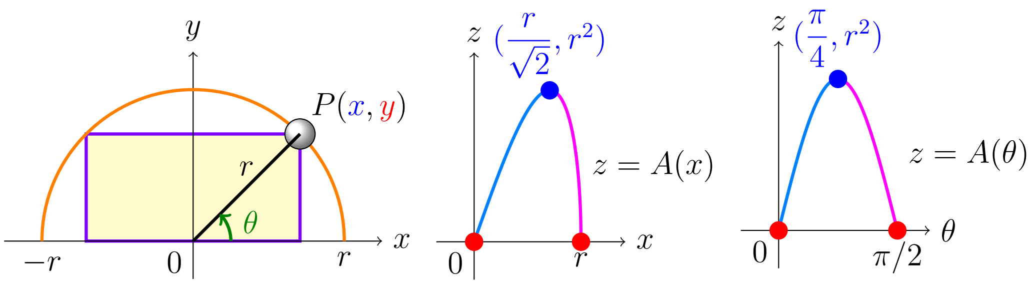
(b) implicit $\dfrac{d}{dx}$: $A' = 2y + 2xy' \overset{Let}{=} 0$, $2x + 2yy' = 0$. $\Longrightarrow$ $x^2 = y^2$,
(代入) $x^2 + y^2 = 2x^2 = r^2$, $x = y = \dfrac{r}{\sqrt{2}}$ *(負不合)*, $A = 2xy = r^2$.

[Sol 2] (單變數) Let θ be the angle between PO and x-axis.

Maximize are $A(\theta) = 2(r\cos\theta)(r\sin\theta) = r^2 \sin 2\theta$ *under* $0 \leq \theta \leq \dfrac{\pi}{2}$.

$A'(\theta) = 2r^2 \cos 2\theta$. $A'(\theta) = 0$ *when* $2\theta = \dfrac{\pi}{2}$, $\theta = \dfrac{\pi}{4}$.
$A(\dfrac{\pi}{4}) = r^2 \sin\dfrac{\pi}{2} = r^2$.

Ans: <u>*Area* r^2</u>. ■



♦ 4.8 Newton's method (optional)

微分應用之八: Newton-Raphson method 牛頓-拉弗森法, 用來求函數近似解。

Let $y = f(x)$.
1. Guess x_1. 過 $(x_1, f(x_1))$ 的切線爲

$$y = f'(x_1)(x - x_1) + f(x_1),$$

交 x-軸於

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

2. When x_n is found, and $f'(x_n) \neq 0$, let

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. $x_n \to a$ as $n \to \infty$, $\implies$ a is a root.

Note: $\{x_n\}$ 可能會不收斂 (not converge)/發散 (diverge), 就重選別的 x_1。