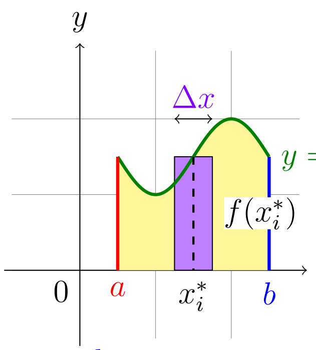
6.1 Areas between curves

應用突入: 面積篇　　　　　　　我左看, 右看, 上看, 下看, 原來每個積分都很簡單。

英語教室: region [ˋridʒən] 區域, area [ˋɛrɪə] 面積, curve [kɝv] 曲線。

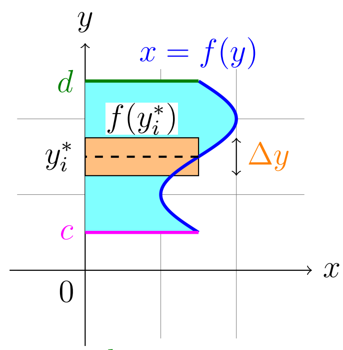
1. 無交錯 $\int f - g \ \ dx$ & $\int f - g \ \ dy$
2. 有交錯 $\int |f - g| \ dx$ & $\int |f - g| \ dy$

Recall: 單一函數 f 的情形: 面積就是近似長方形面積和的極限。



$\Delta x = \dfrac{b-a}{n}$, $x_i^* \in [x_{i-1}, x_i]$.

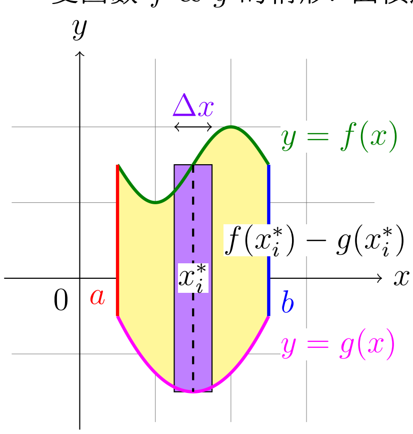
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\ dx.$$



$\Delta y = \dfrac{d-c}{n}$, $y_i^* \in [y_{i-1}, y_i]$.

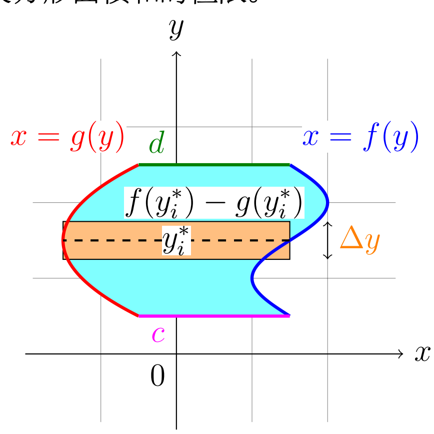
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(y_i^*)\Delta y = \int_c^d f(y)\ dy.$$

雙函數 f & g 的情形: 面積還是近似長方形面積和的極限。



$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

$$= \int_a^b \quad ? \quad ? \qquad ? \ dx.$$



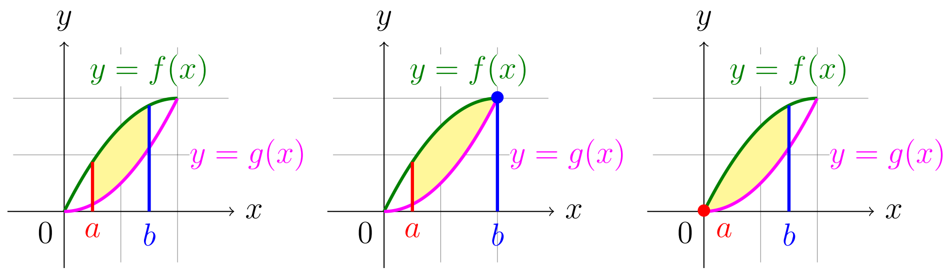
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(y_i^*) - g(y_i^*)]\Delta y$$

$$= \int_c^d \quad ? \quad ? \qquad ? \ dy.$$

0.1 (無交錯)

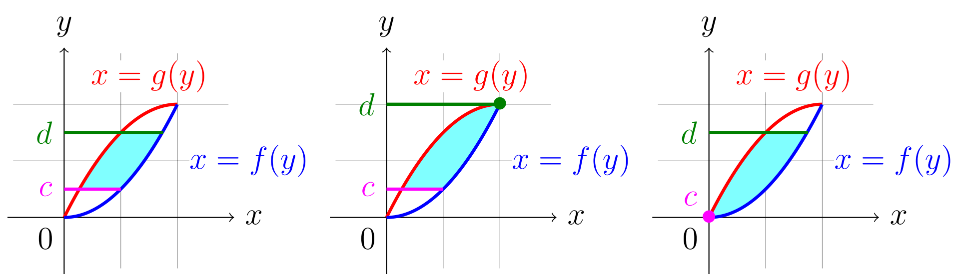
Theorem 1 (上 f 下 g 左 a 右 b)
*The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continues and $f(x) \geq g(x)$ for all x in $[a, b]$, is (*上下兩函數, 左右兩垂直線。*)*

$$\boxed{\boldsymbol{A = \int_a^b [f(x) - g(x)]\, dx}} \left(\int_{左}^{右} 上 - 下 \, dx \right)$$



Theorem 2 (上 d 下 c 左 g 右 f)
*The area A of the region bounded by the curves $x = f(y)$, $x = g(y)$, and the lines $y = c$, $y = d$, where f and g are continues and $f(y) \geq g(y)$ for all y in $[c, d]$, is (*上下兩水平線, 左右兩函數。*)*

$$\boxed{\boldsymbol{A = \int_c^d [f(y) - g(y)]\, dy}} \left(\int_{下}^{上} 右 - 左 \, dy \right)$$

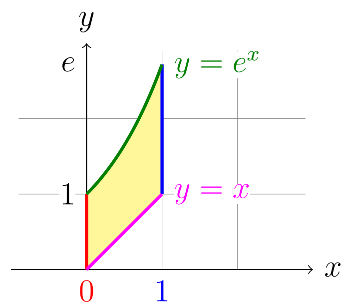


Note: 會畫圖 (§4.3 + 4.5) 很重要, 能知道誰是上上下下左左右右 BABA。

Example 0.1 *Find the area of the region bounded(界限) by $y = e^x$, $y = x$, $x = 0$, $x = 1$.*

$$
\begin{aligned}
A &= \int_0^1 (e^x - x)\, dx \quad &&\text{(列式 } \textit{formulate}\text{)}\\
&= \left[e^x - \frac{x^2}{2}\right]_0^1 &&\text{(}\textit{TFTC}\text{)}\\
&= \left[e - \frac{1}{2}\right] - [1 - 0] &&\text{(計算 } \textit{calculate}\text{)}\\
&= e - 1.5.
\end{aligned}
$$

■



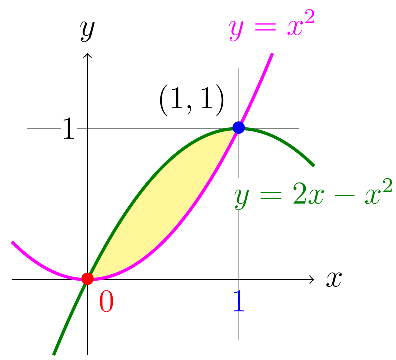
Example 0.2 *Find the area of the region enclosed(包圍) by the parabolas $y = x^2$ and $y = 2x - x^2$.*

Solve $x^2 = y = 2x - x^2$, $2x(x-1) = 0$,

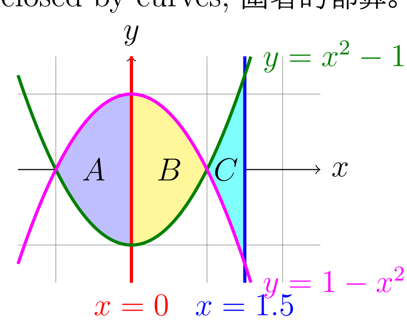
$(x, y) = (0, 0), (1, 1)$. *(解交點找範圍)*

$$
\begin{aligned}
A &= \int_0^1 [(2x - x^2) - (x^2)]\, dx\\
&= \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \left[1 - \frac{2}{3}\right] - [0 - 0]\\
&= \frac{1}{3}.
\end{aligned}
$$

■



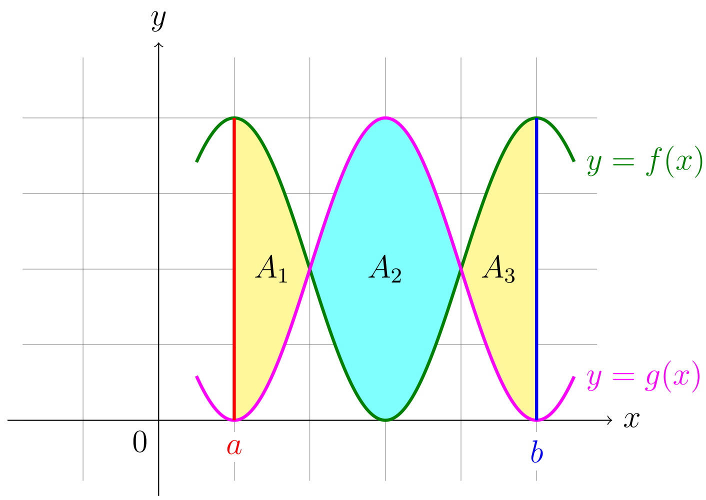
Note: "bounded" 與 "enclosed" 的差別:
bounded by curves $y = \cdots$ (上下), $x = \cdots$ (左右), 超過範圍的不算;
enclosed by curves, 圍著的都算。



Curves:	$y = x^2 - 1$,
	$y = 1 - x^2$,
	$x = 0$,
	$x = 1.5$.
Bounded:	$B + C$;
Enclosed:	$A + B + C$.

0.2 (有交錯)

有時候 $f(x) \geq g(x)$, 有時候 $f(x) \leq g(x)$.



$$\int_a^b f(x) - g(x) \ dx = A_1 - A_2 + A_3 \qquad \text{(無絕對值)}$$

$$\int_a^b |f(x) - g(x)| \ dx = A_1 + A_2 + A_3 \qquad \text{(有絕對值)}$$

Theorem 3 *The area between curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is*

$$\boxed{\boldsymbol{A = \int_a^b |f(x) - g(x)| \ dx}}$$

Theorem 4 *The area between curves $x = f(y)$ and $x = g(y)$ and between $y = c$ and $y = d$ is*

$$\boxed{\boldsymbol{A = \int_c^d |f(y) - g(y)| \ dy}}$$

Skill:

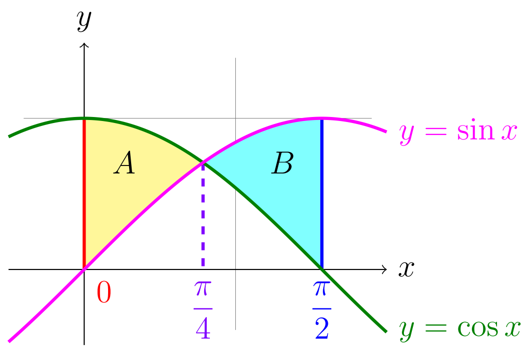
1. 畫出大概的圖形, 找出分段點, 消去絕對值, 變成無交錯版本算面積。
2. 注意上下左右, 減錯會差很大。

Example 0.3 *Find the area of the region bounded by the curves* $y = \sin x$, $y = \cos x$, $x = 0$ *and* $x = \pi/2$.

$\sin x = y = \cos x$ *when* $x = \frac{\pi}{4}$, *(找分段點, 解絕對值內*$= 0$.*)*

$\cos x \geq \sin x$ *when* $0 \leq x \leq \frac{\pi}{4}$, *and* $\cos x \leq \sin x$ *when* $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$.

$$
\begin{aligned}
&\int_0^{\pi/2} |\cos x - \sin x| \; dx \\
&= \int_0^{\pi/4} (\cos x - \sin x) \; dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \; dx \\
&= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\
&= \Big[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\Big] - [\sin 0 + \cos 0] + \Big[-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\Big] - \Big[-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\Big] \\
&= \Big[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\Big] - [0 + 1] + [-0 - 1] - \Big[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\Big] \\
&= 2\sqrt{2} - 2.
\end{aligned}
$$
■



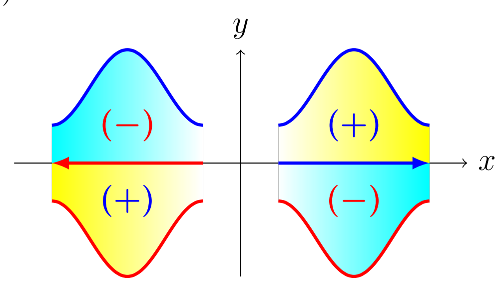
(其實 A 跟 B 一樣大, 可以這樣算:

$$
\begin{aligned}
&A + B = 2A \\
&= 2\int_0^{\pi/4} (\cos x - \sin x) \; dx \\
&= 2(\sqrt{2} - 1) = 2\sqrt{2} - 2.)
\end{aligned}
$$

Note: 積分方向 (上下界的大小) 與函數正負 (在 x-軸的上下) 不同, 得到正負不同, 但是絕對值都是面積 (沒有負的)。

$(\rightarrow)$ $\int_{左}^{右}$ 正 $dx =$ 正, $\int_{左}^{右}$ 負 $dx =$ 負,

$(\leftarrow)$ $\int_{右}^{左}$ 負 $dx =$ 正, $\int_{右}^{左}$ 正 $dx =$ 負.



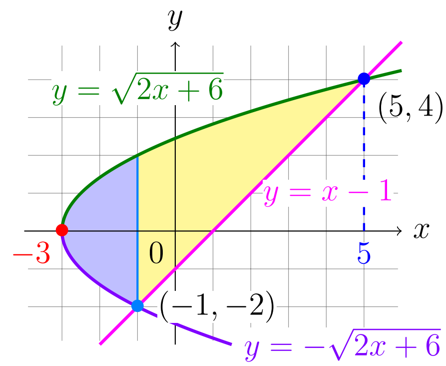
Example 0.4 *Find the area of the region enclosed by the curves $y = x - 1$ and the parabola $y^2 = 2x + 6$.*

[Sol 1]

解交點找範圍 *Solve* $(x-1)^2 = y^2 = 2x+6$,
$x = -1, 5$, $(x, y) = (-1, -2), (5, 4)$.

$\int_{-1}^{5} [\sqrt{2x+6} - (x-1)]\ dx = ...$

*(***Wrong!*** 沒畫圖會看不到 $x = -3$.)*



$\sqrt{2x+6} \geq -\sqrt{2x+6}$ *on* $[-3, -1]$,
and $\sqrt{2x+6} \geq x - 1$ *on* $[-1, 5]$.

$$A = \int_{-3}^{-1} [\sqrt{2x+6} - (-\sqrt{2x+6})]\ dx + \int_{-1}^{5} [\sqrt{2x+6} - (x-1)]\ dx$$

$$= \left[\frac{2}{3}(2x+6)^{3/2}\right]_{-3}^{-1} + \left[\frac{1}{3}(2x+6)^{3/2} - \frac{x^2}{2} + x\right]_{-1}^{5}$$

$$= \left[\frac{2}{3}(2(-1)+6)^{3/2}\right] - \left[\frac{2}{3}(2(-3)+6)^{3/2}\right]$$

$$+ \left[\frac{1}{3}(2(5)+6)^{3/2} - \frac{(5)^2}{2} + (5)\right] - \left[\frac{1}{3}(2(-1)+6)^{3/2} - \frac{(-1)^2}{2} + (-1)\right]$$

$$= \frac{16}{3} + \frac{64}{3} - \frac{25}{2} + 5 - \frac{8}{3} + \frac{1}{2} + 1 = 18.$$

[Another 計算技巧]

Let $u = \sqrt{2x+6}$, *then* $du = \frac{1}{u}\ dx$, $dx = u\ du$,
when $x = -3, -1, 5$, $u = 0, 2, 4$, *respectively.*

$$A = \int_{-3}^{-1} [\sqrt{2x+6} - (-\sqrt{2x+6})]\ dx + \int_{-1}^{5} [\sqrt{2x+6} - (x-1)]\ dx$$

$$= \int_{-3}^{5} \sqrt{2x+6}\ dx - \int_{-3}^{-1} -\sqrt{2x+6}\ dx - \int_{-1}^{5} (x-1)\ dx \qquad \text{(定積分性質)}$$

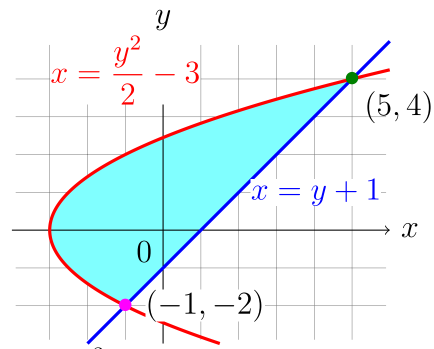
$$= \int_{0}^{4} u^2\ du + \int_{0}^{2} u^2\ du - \int_{-1}^{5} (x-1)\ dx \qquad \text{(變數變換, 不一定要全換)}$$

$$= \left[\frac{u^3}{3}\right]_0^4 + \left[\frac{u^3}{3}\right]_0^2 - \left[\frac{x^2}{2} - x\right]_{-1}^{5} = \frac{64}{3} + \frac{8}{3} - \frac{25}{2} + 5 + \frac{1}{2} + 1 = 18.$$

[Sol 2] (換個角度)

$y+1 = x = \dfrac{y^2}{2} - 3$ *when* $y = -2, 4$,

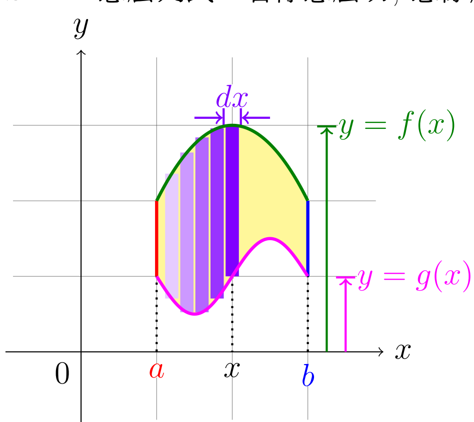
and $y+1 \geq \dfrac{y^2}{2} - 3$ *on* $[-2, 4]$.

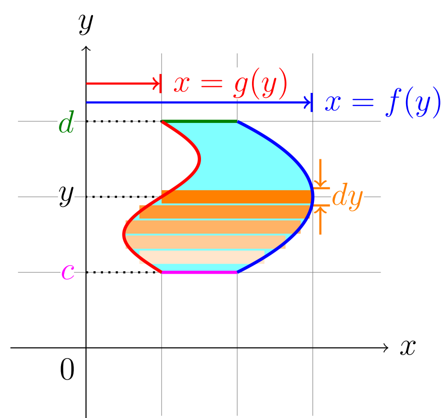


$$A = \int_{-2}^{4} [(y+1) - (\frac{y^2}{2} - 3)]\ dy$$
$$= \int_{-2}^{4} (4 + y - \frac{y^2}{2})\ dy = \left[4y + \frac{y^2}{2} - \frac{y^3}{6}\right]_{-2}^{4}$$
$$= \left[4(4) + \frac{(4)^2}{2} - \frac{(4)^3}{6}\right] - \left[4(-2) + \frac{(-2)^2}{2} - \frac{(-2)^3}{6}\right]$$
$$= 16 + 8 - \frac{32}{3} + 8 - 2 - \frac{4}{3} = 18. \qquad \blacksquare$$

Note: 有時候用 $\displaystyle\int dy$ 比 $\displaystyle\int dx$ 好算.

Skill: 怎麼列式? 看你怎麼切, 想像成長方條面積的累積.





$$A = \int_a^b \overset{\text{height}}{[f(x) - g(x)]}\ \overset{\text{width}}{dx} \qquad A = \int_c^d \overset{\text{width}}{[f(y) - g(y)]}\ \overset{\text{height}}{dy}$$

平淡　　　　　　　浪漫　　　　　　浪漫　　著情感
區域之中製造一些些黎曼, 絲絲點點黎曼累積成積分。