# 1179: Probability
# Lecture 17 — Joint Distributions

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# This Lecture

1. Joint Distributions of Two Random Variables

2. Joint PMF and Marginal PMF

- Reading material: Chapter 8.1

# Joint CDF of Two Random Variables

# Joint CDF

$F_X(t) \Leftarrow P(X \leq t),\ t \in \mathbb{R}$

**Joint CDF**: Let $X$ and $Y$ be two random variables defined on the same sample space $\Omega$. The joint CDF $F_{XY}(t,u)$ is defined as

$$F_{XY}(t,u) = P(X \leq t, Y \leq u), \quad \forall t, u \in \mathbb{R}$$

and

$= P(\{\omega : X(\omega) \leq t \text{ and } Y(\omega) \leq u\})$

- ✓ $0 \leq F_{XY}(t,u) \leq 1$?
- Suppose $t_1 \leq t_2$ and $u_1 \leq u_2$, then $F_{XY}(t_1,u_1) \leq F_{XY}(t_2,u_2)$?
- What is $F_{XY}(\infty, \infty)$? How about $F_{XY}(-\infty, -\infty)$?

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# Event Probabilities and Joint CDF (I)

CDF of X

$$F_{XY}(t,u) = P(X \le t, Y \le u), \;\; \forall t, u \in \mathbb{R}$$

- $P(X \le t) = ?$ $P(X \le t, Y \le \infty) = F_{XY}(t, \infty)$

CDF of Y

- $P(Y \le u) = ?$ $P(X \le \infty, Y \le u) = F_{XY}(\infty, u)$

# Marginal CDF

**Marginal CDF**: Let $X$ and $Y$ be two random variables defined on the same sample space $\Omega$, and the joint CDF is $F_{XY}(t, u)$. The marginal CDF of $X$ and $Y$ are

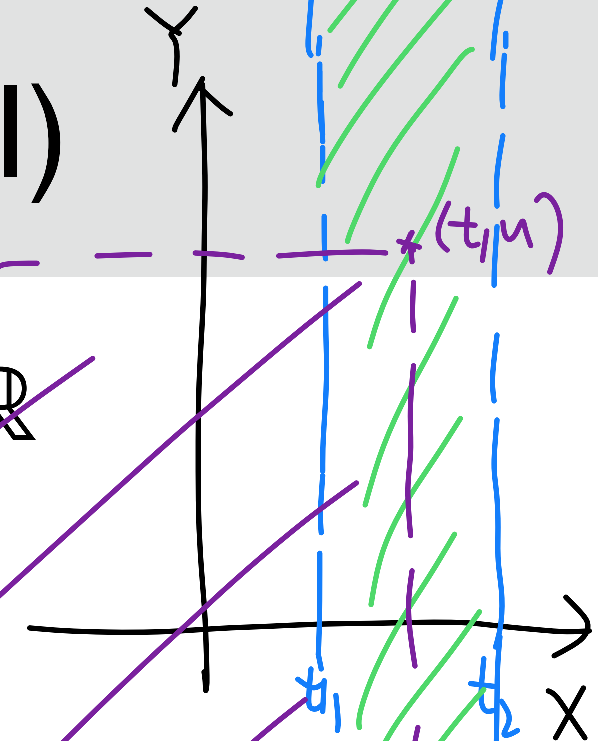
marginal CDF of X

$$F_X(t) = P(X \leq t) = F_{XY}(t, \infty)$$

$$F_Y(t) = P(Y \leq t) = F_{XY}(\infty, t)$$

marginal CDF of Y

# Event Probabilities and Joint CDF (II)



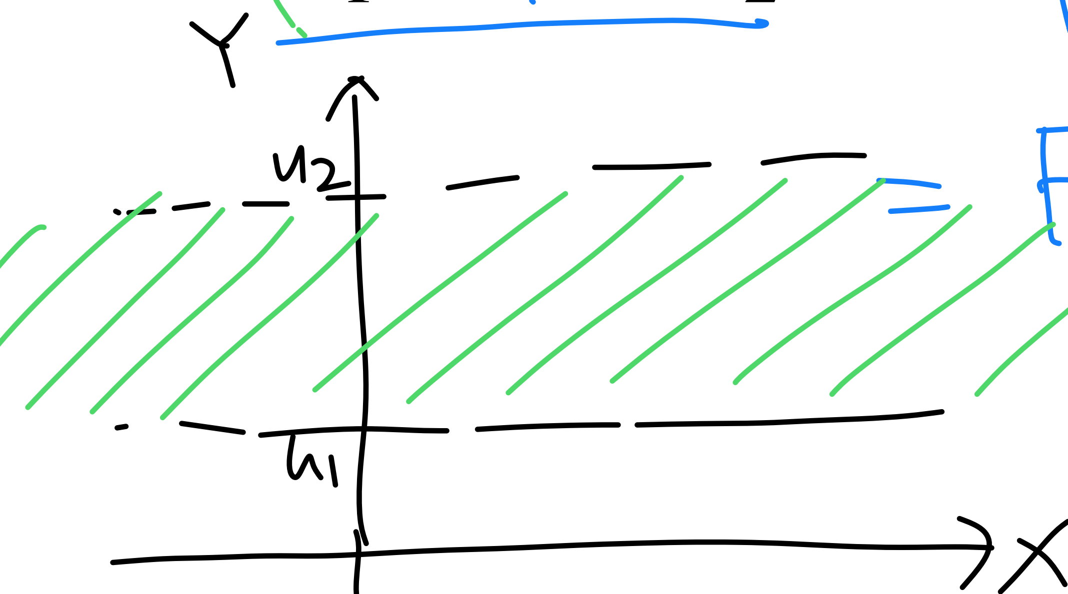

$$F_{XY}(t,u) = P(X \le t, Y \le u), \ \forall t,u \in \mathbb{R}$$

- $P(t_1 < X \le t_2) = ?$ $\quad P(X \le t_2) - P(X \le t_1)$

$$= F_X(t_2) - F_X(t_1) = F_{XY}(t_2, \infty) - F_{XY}(t_1, \infty)$$

- $P(u_1 < Y \le u_2) = ?$ $\quad P(Y \le u_2) - P(Y \le u_1)$

$$= F_{XY}(\infty, u_2) - F_{XY}(\infty, u_1)$$



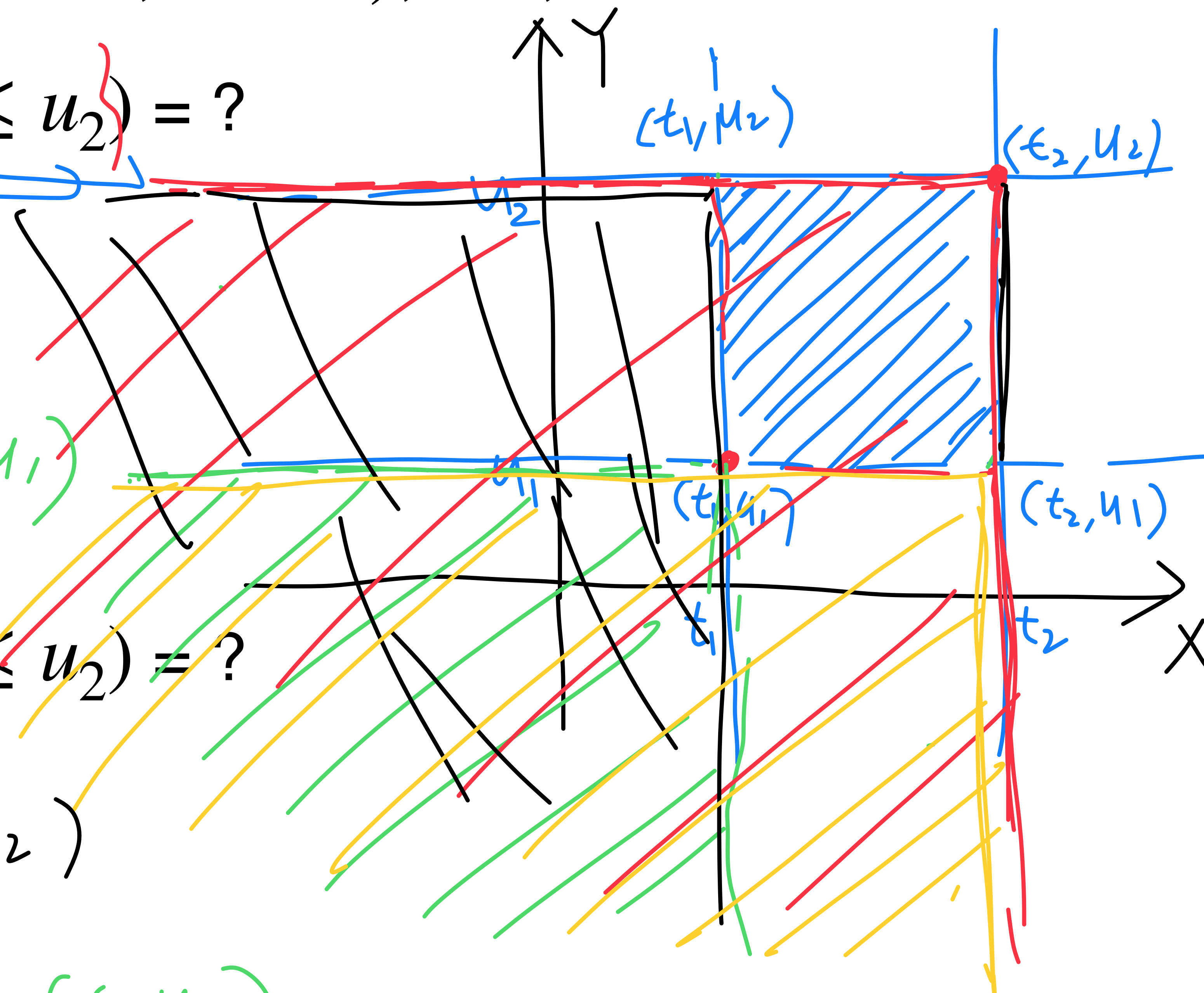
# Event Probabilities and Joint CDF (III)

$$F_{XY}(t,u) = P(X \le t, Y \le u), \ \forall t, u \in \mathbb{R}$$

- $P(t_1 < X \le t_2, u_1 < Y \le u_2) = ?$

$= F_{XY}(t_2, u_2) - F_{XY}(t_1, u_2)$

$- F_{XY}(t_2, u_1) + F_{XY}(t_1, u_1)$




- $P(t_1 < X < t_2, u_1 < Y \le u_2) = ?$

$= F_{XY}(\bar{t}_2, u_2) - F_{XY}(t_1, u_2)$

$- F_{XY}(\bar{t}_2, u_1) + F_{XY}(t_1, u_1)$

# Example: A Random Point in a Unit Square

- Example: Suppose a point $(X, Y)$ is selected randomly from the unit square $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.
  - What is the joint CDF of $X$ and $Y$, i.e. $F_{XY}(t, u)$?

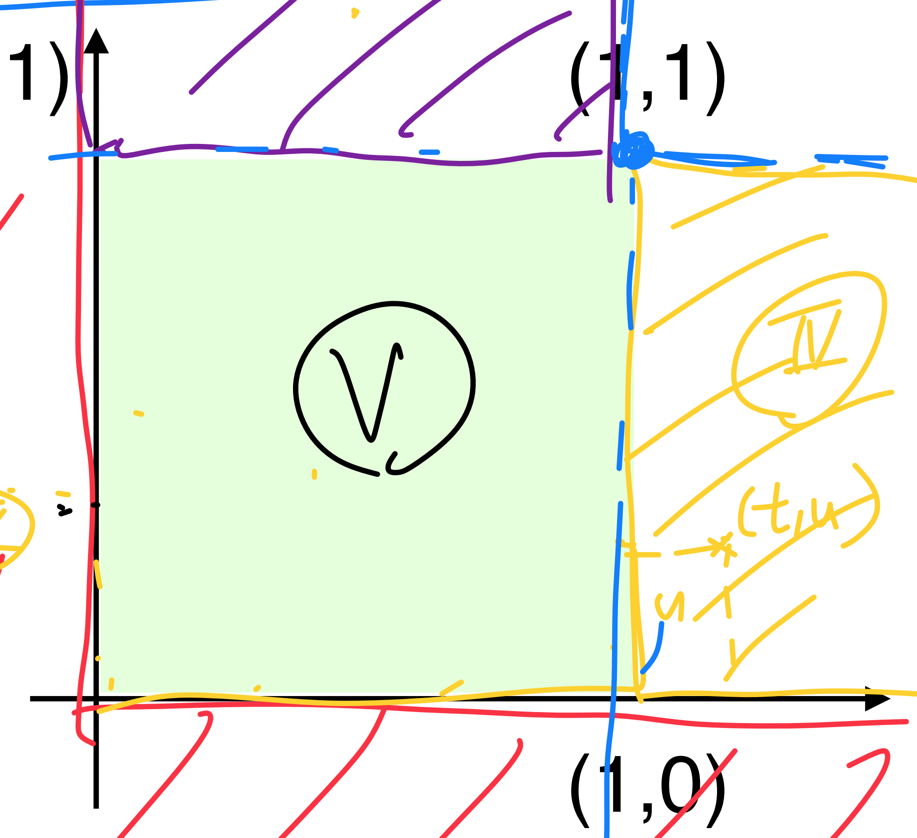
$$F_{XY}(t,u) = \begin{cases} 1, & t>1 \text{ and } u>1 \quad \text{I} \\ 0, & t<0 \text{ or } u<0 \quad \text{II} \\ u, & t>1 \text{ and } u\in[0,1] \quad \text{IV} \\ t, & t\in[0,1] \text{ and } u>1 \quad \text{IV} \\ t\cdot u, & t\in[0,1] \text{ and } u\in[0,1] \quad \text{V} \end{cases}$$

$F_{XY}(1,1)$

$(-1,-1)$

I

$t\geq 1$
$u=1$



# Joint PMF and Marginal PMF

# Joint PMF of 2 Discrete Random Variables

$P(X=x) = p(x)$

**Joint PMF**: Let $X$ and $Y$ be two discrete random variables defined on the same sample space $\Omega$. The joint PMF $p_{XY}(x, y)$ is defined as

$$p_{XY}(x, y) = P(X = x, Y = y)$$

$S_X$ and $S_Y$ can be different

- Let the sets of possible values of $X$ and $Y$ be $S_X$ and $S_Y$
  - PMF of X: $P(X = x) = P(X=x, Y \in S_Y) = \sum_{y \in S_Y} p_{XY}(x,y)$
  - $P(Y = y) = P(X \in S_X, Y=y) = \sum_{x \in S_X} p_{XY}(x,y)$ PMF of Y

# Marginal PMF

**Marginal PMF**: Let $X$ and $Y$ be two discrete random variables defined on the same sample space $\Omega$, and the joint PMF is $p_{XY}(x, y)$. The marginal PMF of $X$ and $Y$ are

$$P(X = x) = \sum_{y \in S_Y} p_{XY}(x, y)$$

$$P(Y = y) = \sum_{x \in S_X} p_{XY}(x, y)$$

where $S_X$ and $S_Y$ are the sets of possible values of $X$ and $Y$

# Example: From Joint PMF to Marginals

- Example: Let the joint PMF of $X$ and $Y$ be

$$p_{XY}(x, y) = \begin{cases} \frac{1}{25}(x^2 + y^2) & \text{, if } x = 1,2,\ y = 0,1,2 \\ 0 & \text{, otherwise} \end{cases}$$

- What is the marginal PMF of $X$ and $Y$?

# Example: Bernoulli and Poisson

- Example: Consider $X \sim$ Bernoulli$(p)$, $Z_0 \sim$ Poisson$(\lambda_0, T)$ and $Z_1 \sim$ Poisson$(\lambda_1, T)$ (all are independent). Suppose $Y = Z_0$ if $X = 0$, and $Y = Z_1$ if $X = 1$.
  - What is the joint PMF of $X$ and $Y$?
  - What is the marginal PMF of $Y$?

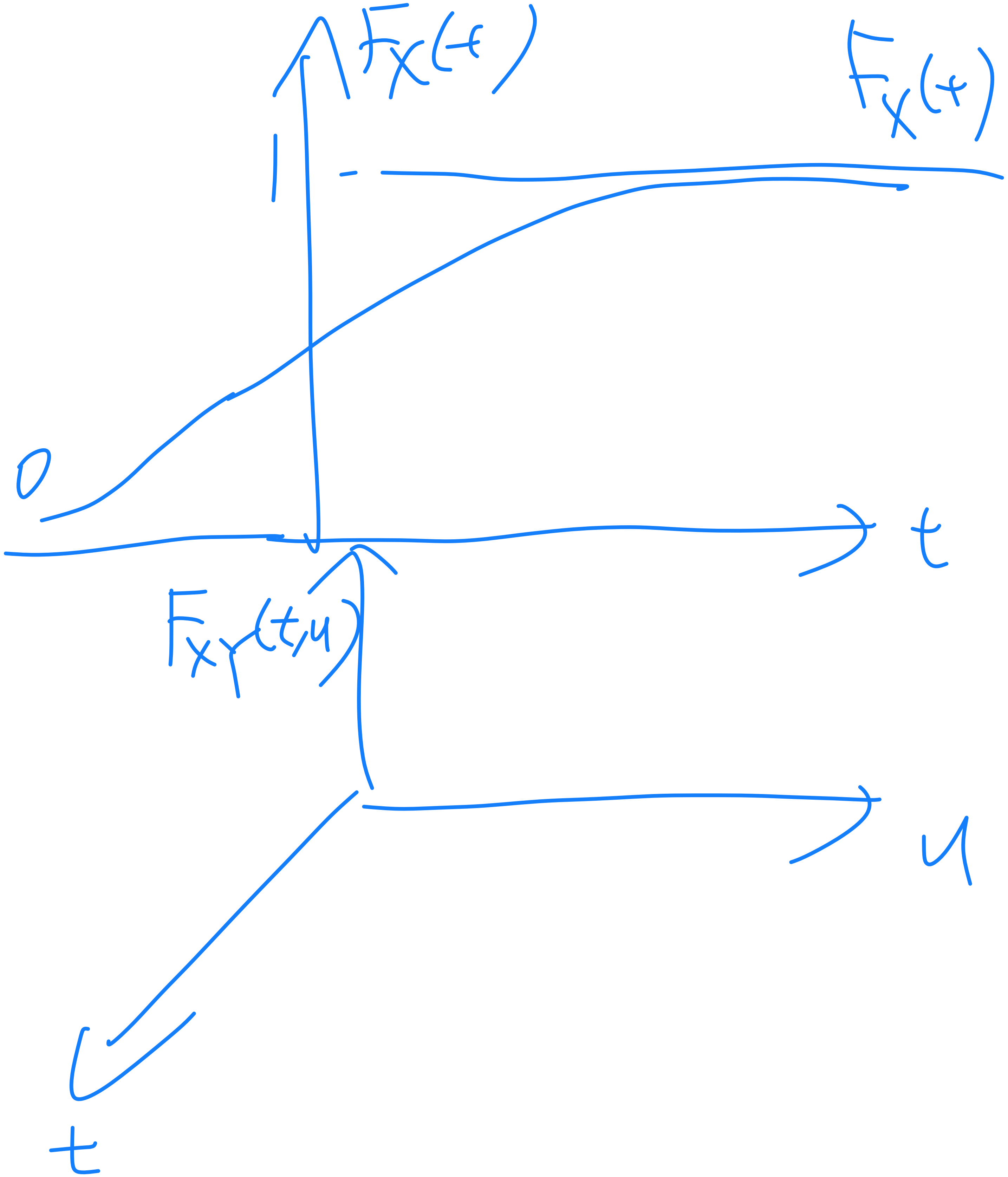
# Revisit: Two Independent Geometric Random Variables

- Example: Consider $X_1 \sim$ Geometric$(p)$, $X_2 \sim$ Geometric$(p)$, and $X_1, X_2$ are independent.
  - What is the joint PMF of $X_1$ and $X_2$?
  - What is the PMF of $X = \min(X_1, X_2)$?

# 1-Minute Summary

$\sum$

| | | |
|---|---|---|
| Joint PMF $P(X=x, Y=y)$ | ?? | Marginal PMF $P(X=x), P(Y=y)$ |
| ?? ?? | | |
| Joint CDF $F_{XY}(x,y)$ | Plug in "$\infty$" ?? | Marginal CDF $F_X(x), F_Y(y)$ |
| Joint PDF $f_{XY}(x,y)$ | | Marginal PDF $f_X(x), f_Y(y)$ |


F_X(t)
F_X(t)
1
0
t
F_XY(t,u)
u
t