4.5 Summary of curve sketching

微分應用之五: 畫圖。

如何畫圖?

找幾個點連起來? (X) 點太少, 或是這些點不夠關鍵。

用繪圖軟體畫? (X) 計算不夠精準, 位數不足, 會誤導極值存在, 看不出來。

Guidelines of sketching curve 注意事項

A. **Domain** 定義域。

B. **Intercepts** x-,y-軸交點: $(x, 0)$ with $f(x) = 0$, and $(0, f(0))$.

C. **Symmetry** 對稱性:
f is even 偶函數 if $f(-x) = f(x)$: 對稱 y-軸 $(x = 0)$; ex: $\cos x$;
f is odd 奇函數 if $f(-x) = -f(x)$: 對稱原點 $(0, 0)$; ex: $\sin x$;
f is periodic 週期函數 if $f(x + p) = f(x)$: 複製 $[0, p]$; ex: $\sin x$.

D. **Asymptotes** 漸近線: (離原點越遠跟函數圖形越靠近的線。)
Vertical Asymptote 垂直 : $x = a$ if $\lim\limits_{x\to a/a^+/a^-} f(a) = \infty/-\infty$.
(a 通常不在 domain, 只要看 a^+/a^-.)
Horizontal Asymptote 水平 : $y = L$ if $\lim\limits_{x\to\pm\infty} f(a) = L$. (if defined)
Slant Asymptote 斜: $y = mx + b$ if $\lim\limits_{x\to\pm\infty} [f(x) - (mx + b)] = 0$. (?)

E. **Interval of increasing/decreasing** 遞增/減區間:
Critical number c: $f'(c) = 0$ or does not exist. 以 c 分界考慮
$f'(x) > 0$: increasing, $f'(x) < 0$: decreasing.

F. **Local max/min** 極值: The first/second derivative test:

For critical number c, $f'(x)$: $\begin{cases} + \to - & \text{local max,} \\ - \to + & \text{local min,} \\ \text{no change} & \text{no local max/min.} \end{cases}$

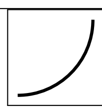
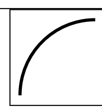
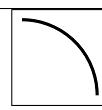
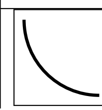
$f'(c) = 0$ & $f''(c) > 0$: local min, $f'(c) = 0$ & $f''(c) < 0$: local max.

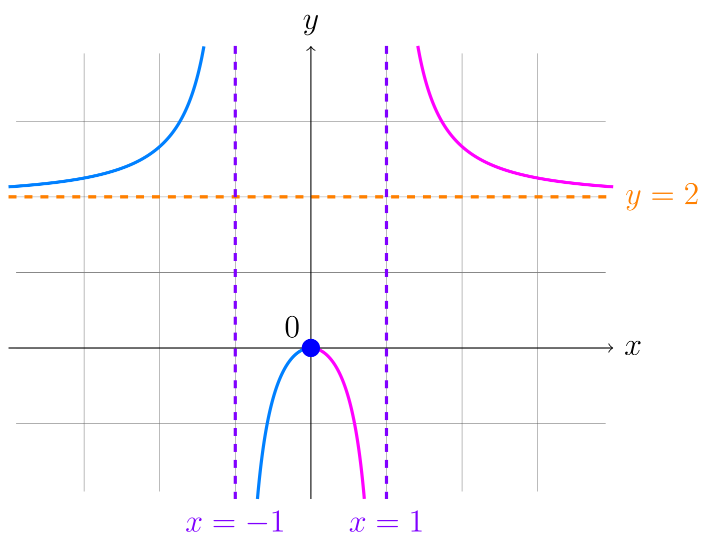
G. **Concavity & inflection point** 凹性與反曲點:
Find $f''(p) = 0$ or does not exist. 以 p 分界考慮
$f''(x) > 0$: Concave Upward, $f''(x) < 0$: Concave Downward;
Inflection point $(p, f(p))$: f is continuous and f'' change sign at p.

H. **Just Sketch It.** ✓

Example 0.1 *Sketch* $y = \dfrac{2x^2}{x^2-1}$.

Let $f(x) = \dfrac{2x^2}{x^2-1}$.
A. Domain $\{x \neq \pm 1\} = (-\infty,-1) \cup (-1,1) \cup (1,\infty)$.
B. Intercept $(0,0)$.
C. $f(-x) = f(x)$ *even. (左右對稱)*
D. $\lim\limits_{x\to 1^+} f(x) = \infty$ *or* $\lim\limits_{x\to 1^-} f(x) = -\infty$*, V.A.:* $x = 1$.
$\lim\limits_{x\to -1^+} f(x) = -\infty$ *or* $\lim\limits_{x\to -1^-} f(x) = \infty$*, V.A.:* $x = -1$.
$\lim\limits_{x\to \pm\infty} f(x) = 2$*. H.A:* $y = 2$.
E–G.
$f' = \dfrac{-4x}{(x^2-1)^2}$, $f' = 0$ *when* $x = 0$, $\nexists$ *when* $x = \pm 1$ *(not in domain).*
$f'' = \dfrac{4(3x^2+1)}{(x^2-1)^3} \neq 0$, $\nexists$ *when* $x = \pm 1$ *(not in domain).*

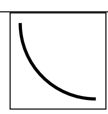
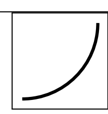
	<-1	-1	$-1<x<0$	0	$0<x<1$	1	$1<$
f'	$+$	$\nexists$	$+$	0	$-$	$\nexists$	$-$
f''	$+$	$\nexists$		$-$		$\nexists$	$+$
		*no*		*max*		*no*	

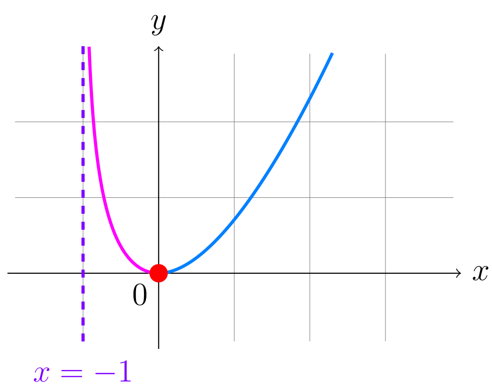


Skill: 增減以臨界值作分界, 每段中代入好算的數字判斷 f' 的正負。

Example 0.2 *Sketch* $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

A. Domain $\{x > -1\} = (-1, \infty)$.
B. Intercept $(0, 0)$.
C. No symmetry.
D. $\lim\limits_{x \to -1^+} f(x) = \infty$. *V.A.:* $x = -1$.
$\lim\limits_{x \to \infty} f(x) = \infty$. *H.A: none.*
E–G.
$f' = \dfrac{x(3x+4)}{2(x+1)^{3/2}}$, $f' = 0$ *when* $x = 0$*(,* $x = -\dfrac{4}{3} \notin (-1, \infty)$*).*
$f'' = \dfrac{3x^2+8x+8}{4(x+1)^{5/2}} > 0$. *(Both does not exist when* $x = -1 \notin (-1, \infty)$*).*
*(*分子用判別式 $b^2-4ac = 8^2-4\times3\times8 < 0$ 或 $3x^2+8x+8 = x^2+2(x+4)^2 \geq 0$.*)*

	$-1 < x < 0$	0	$0 <$
f'	$-$	0	$+$
f''	$+$		
		*min*	



Note: 知道定義域的好處: 沒有圖就不用畫到那邊。
Skill: 臨界值不在定義域的不用看!
Attention: 要在定義域的才算臨界值, 不要數錯!

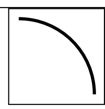
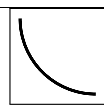
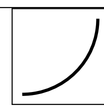
Example 0.3 *Sketch* $f(x) = xe^x$.

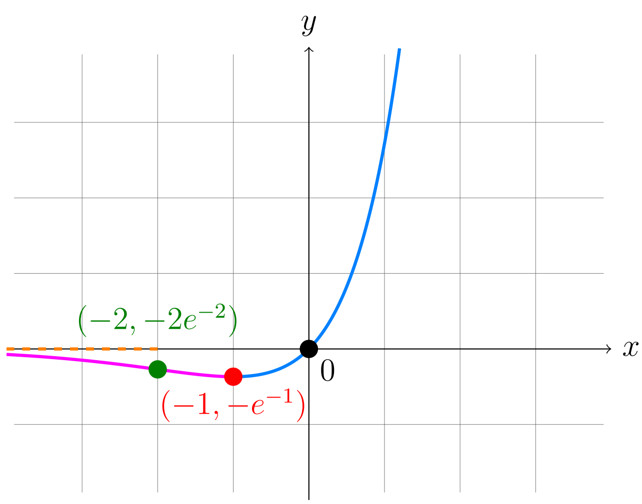
A. Domain $\mathbb{R}$.
B. Intercept $(0,0)$.
C. No symmetry.
D. $\lim\limits_{x\to\infty} f(x) = \infty$, *(*不可以寫 $\cancel{= \infty \cdot e^{\infty} = \infty \cdot \infty}$, 直接寫 $= \infty$.*)*

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} \;\; \left(\boldsymbol{\infty \cdot 0 \to \frac{\infty}{\infty}}\right)$$

$$\overset{l'H}{=} \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^x = 0. \;\; H.A\text{: } y = 0.$$

E–G.
$f' = e^x(1+x)$, $f' = 0$ *when* $x = -1$.
$f'' = e^x(2+x)$, $f'' = 0$ *when* $x = -2$.

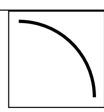
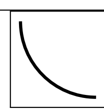
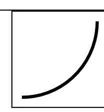
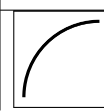
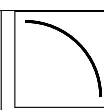
	< -2	-2	$-2 < x < -1$	-1	$-1 <$
f'		$-$		0	$+$
f''	$-$	0		$+$	
		*IP*		*min*	

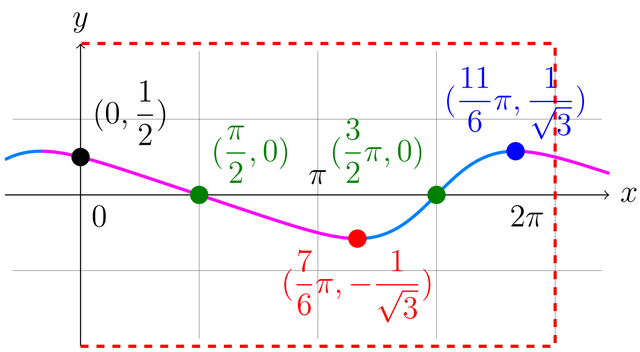


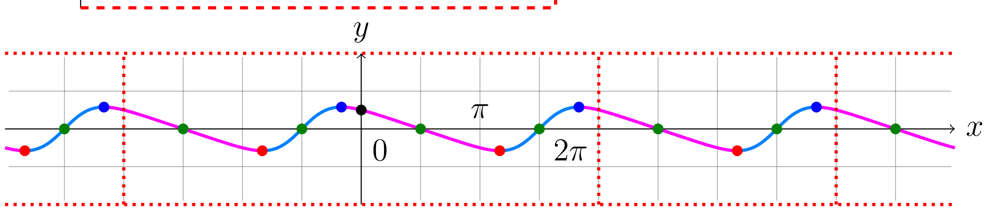
Skill: 凹性找 $f''(x) = 0$ or $\nexists$ 的地方 (f' 的臨界值) 做分界。
Note: 漸近線剛好是座標軸 ($x = 0$ or $y = 0$) 可以省略標示 (虛線)。

Example 0.4 *Sketch* $f(x) = \dfrac{\cos x}{2+\sin x}$.

A. Domain $\mathbb{R}$.
B. Intercept $((\frac{1}{2}+n)\pi, 0)$, $(0, \frac{1}{2})$.
C. Periodic with period 2π. *Draw* $[0, 2\pi)$ *and repeat it.*
D. No asymptote.
E–G.
$f' = -\dfrac{1+2\sin x}{(2+\sin x)^2}$, $f' = 0$ *when* $x = \frac{7}{6}\pi, \frac{11}{6}\pi$. *(*只看 $[0, 2\pi)$*.)*
$f'' = \dfrac{-2\cos x(1-\sin x)}{(2+\sin x)^3}$, $f'' = 0$ *when* $x = \frac{\pi}{2}, \frac{3}{2}\pi$. *(*只看 $[0, 2\pi)$*.)*

	$0 \wr \frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2} \wr \frac{7}{6}\pi$	$\frac{7}{6}\pi$	$\frac{7}{6}\pi \wr \frac{3}{2}\pi$	$\frac{3}{2}\pi$	$\frac{3}{2}\pi \wr \frac{11}{6}\pi$	$\frac{11}{6}\pi$	$\frac{11}{6}\pi \wr 2\pi$
f'	$-$			0	$+$			0	$-$
f''	$-$	0	$+$			0	$-$		
		*IP*		*min*		*IP*		*max*	

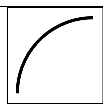
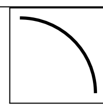


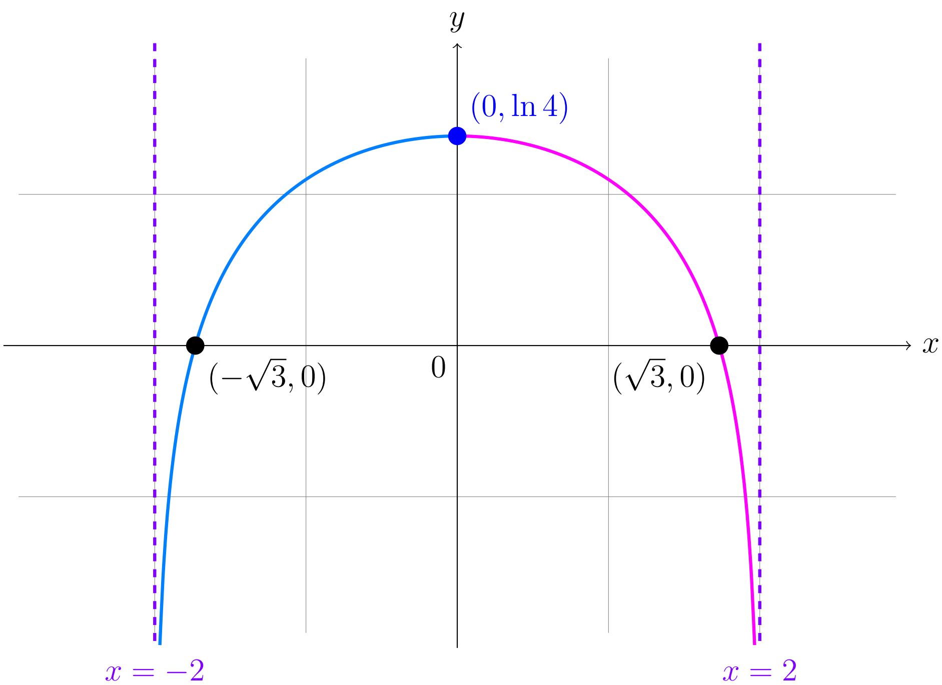


Skill: 看出週期 (通常是三角的) 函數畫一段就夠了。

Example 0.5 *Sketch* $f(x)=\ln(4-x^2)$.

A. Domain $(-2,2)$.
B. Intercept $(0,\ln 4),(\pm\sqrt{3},0)$.
C. $f(-x)=f(x)$, *even. (左右對稱)*
D. $\lim\limits_{x\to 2^-} f(x)=-\infty$, $\lim\limits_{x\to -2^+} f(x)=-\infty$, *V.A.:* $x=2$, $x=-2$.
E–G.
$f'=\dfrac{-2x}{4-x^2}$, $f'=0$ *when* $x=0$. *(*$\nexists$ *when* $x=\pm 2\notin(-2,2)$.*)*
$f''=\dfrac{-8-2x^2}{(4-x^2)^2}<0$. *(*$\nexists$ *when* $x=\pm 2\notin(-2,2)$.*)*

	$-2\sim 0$	0	$0\sim 2$
f'	$+$	0	$-$
f''		$-$	
		*max*	



Example 0.6 *Sketch* $f(x) = \dfrac{x^3}{x^2+1}$.

A. Domain $\mathbb{R}$.

B. Intercept $(0,0)$.

C. $f(-x) = -f(x)$*, odd. (旋轉對稱)*

D. $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$*, no asymptote.*

Skill: 當 $x \to \pm\infty$ 很大/小, $+1$ 影響不大, $\dfrac{x^3}{x^2\cancel{+1}} \approx x$.

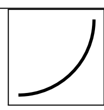
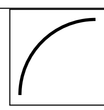
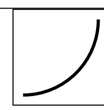
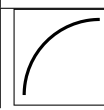
$$\lim_{x\to\pm\infty}[f(x)-x] = \lim_{x\to\pm\infty}\left(\frac{x^3}{x^2+1}-x\right) = \lim_{x\to\pm\infty}\frac{-x}{x^2+1}\ \left(\frac{\infty}{\infty}\right)$$

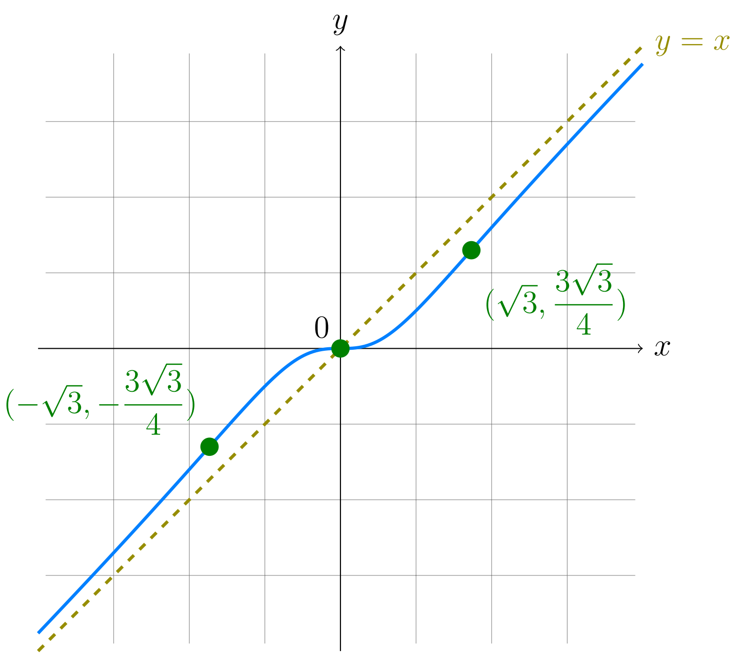
$$\overset{l'H}{=} \lim_{x\to\pm\infty}\frac{-1}{2x} = 0, \text{ Slant asymptote: } y = x.$$

E–G.

$f' = \dfrac{x^2(x^2+3)}{(x^2+1)^2}$, $f' = 0$ *when* $x = 0$.

$f'' = \dfrac{2x(3-x^2)}{(x^2+1)^3}$, $f'' = 0$ *when* $x = 0, \pm\sqrt{3}$.

	$< -\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3} < x < 0$	0	$0 < x < \sqrt{3}$	$\sqrt{3}$	$\sqrt{3} <$
f'			$+$	0		$+$	
f''	$+$	0	$-$	0	$+$	0	$-$
		IP		IP		IP	



Note: 何時有斜漸進線? 如果是有理函數 $\dfrac{f(x)}{g(x)}$, f 的次數比 g 的次數恰多 1。

Skill: 有理函數得到斜漸進線? 用長除法 $\dfrac{f(x)}{g(x)} = \boxed{mx+b} + \dfrac{r(x)}{g(x)}$.

$\Longrightarrow$ S.A.: $y = mx + b$.

Ex: $\dfrac{x^3}{x^2+1} = x + \dfrac{-x}{x^2+1}$.

$$
\begin{array}{rllll}
 & \boxed{x} & & & \\
\cline{2-5}
x^2+1) & x^3 & +0x^2 & +0x & +0 \\
-) & \boxed{x^3} & & +x & \\
\cline{2-5}
 & & & -x &
\end{array}
$$

(乘以能消去最高次的)
(由高往低排, 缺項補零)
(次數比分母小就停止)

Note: 有理函數以外很難猜, ex: $x - \tan^{-1} x$ (Exercise 4.5.71),
要驗證: $\lim\limits_{x\to\pm\infty} |f(x) - (mx+b)| = 0$.

Do some practice: Exercise 4.5.61 $\sim$ 68 (rational function).

Exercise 4.5.69.(exponential function) $1 + \frac{1}{2}x + e^{-x}$. (S.A.: $y = 1 + \frac{1}{2}x$.)

Exercise 4.5.70.(exponential function) $1 - x + e^{1+x/3}$. (S.A.: $y = 1 - x$.)
(Hint: $\lim\limits_{x\to-\infty} e^x = 0$.)

Exercise 4.5.72.(root function) $\sqrt{x^2+4x}$. (S.A.: $y = x+2$, $y = -x-2$.)
(Hint: $\sqrt{x^2+4x} = \sqrt{(x+2)^2 - 4} \approx \sqrt{(x+2)^2} = |x+2|$.)