5.2 The Definite integral

1. definite integral 定積分
2. evaluating integral 計算積分
3. midpoint rule 中點
4. property of definite integral 定積分性質

$$\boxed{\int_{*}^{*} * \, dx}$$

0.1 Definite integral

A function f defined on $[a,b]$, divide $[a,b]$ into n intervals $[x_{i-1}, x_i]$,
$a = x_0 < x_1 < \cdots < x_n = b$, $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$,
(sample points) $x_i^* \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$.
$\boxed{\sum_{i=1}^{n} f(x_i^*)\Delta x}$ 稱爲 f 在 $[a,b]$ 的 ***Riemann sum*** 黎曼和。

Recall: 如果 f 非負連續, 到 x-軸面積 $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$。

Define: The ***definite integral*** 定積分 of f from a to b is

$$\boxed{\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

if the limit ***exists***, and is independent on the choices of sample points, and we say that f is ***integrable*** 可積分 on $[a,b]$.
定積分就是黎曼和的極限, 可積分就是有定積分, 也就是黎曼和的極限存在。

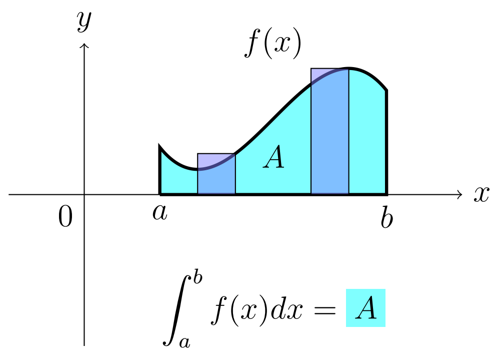
Note 1: 符號解釋: f (對 x) 從 a 到 b (注意順序) 的定積分

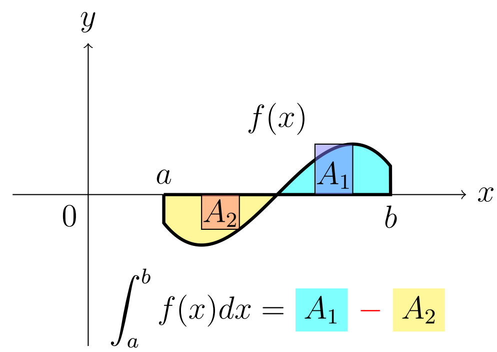
$$\int_a^b f(x)\, dx : \text{積分號} \ {}^{\text{上限}}_{\text{下限}} \ \text{積分域} \ d\text{變數}$$

$\int$: integral sign 積分號 (by Leibniz); $f(x)$: integrand 積分域;
a, b: lower/upper limits of integration 積分的上下限; dx: 表示對 x 積分。
integration (n.u.): 算積分的步驟; integrate (v.t.): 對...積分。

Note 2: 定積分 $\int_a^b f(x)\ dx$ 是一個 (極限) 數字 (與 x 無關),
所以 $\int_a^b f(x)\ dx = \int_a^b f(t)\ dt = \int_a^b f(r)\ dr$, x 換成其他符號 (t, r) 都一樣。

Note 3: 當 $f \geq 0$ on $[a, b]$, 黎曼和就是用長方形估計 f 底下的面積。
如果不是, 則是 ***net area*** 淨面積 $=$ x-軸上方的面積 減 x-軸下方的面積。





$f(x_i) \geq 0$: $f(x_i)\Delta x$ $=$ 長方形面積, $f(x_i) \leq 0$: $f(x_i)\Delta x$ $= -$ 長方形面積。

幾何意義: f 從 a 到 b 的定積分, 就是 $y = f(x)$ 在 $[a, b]$ 到 x-軸的淨面積。

Note 4: 用 ε-δ 語言:

$$\forall \varepsilon > 0,\ \exists N > 0,\ \ni n > N \implies \left| \int_a^b f(x)\ dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon.$$

Note 5: 不一定要把 $[a, b]$ 均分: 只要 $\Delta x_i = x_i - x_{i-1} \to 0$ as $n \to \infty$, 則

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Note 6: (Theorem)
如果 f 連續 或 只有有限個 jump discontinuities, 則 f 可積分 (integrable)。

Note 7: (Theorem)
如果 f 可積分, 黎曼和的樣本點選擇 {左, 右, 中, 大, 小} 都得到一樣的定積分。

Example 0.1 (變成定積分) *Express* $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3+x_i\sin x_i)\Delta x$ *as an integral on the interval* $[0,\pi]$.

Compare $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$, *we have* $f(x)=x^3+x\sin x$,

$$\begin{array}{ccccc} & \lim_{n\to\infty}\sum_{i=1}^{n} & (x_i{}^3+x_i\sin x_i) & \Delta x \\ & \updownarrow & \updownarrow & \updownarrow \\ = & \int_0^{\pi} & (x^3+x\sin x) & dx. \end{array}$$

$\blacksquare$

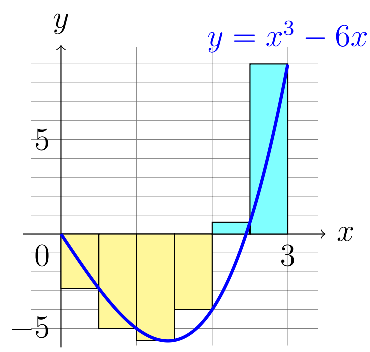
Skill: Find Δx & x_i. (*Try yourself*: $\lim_{n\to\infty}\sum_{i=1}^{n}((\frac{i\pi}{n})^3+\frac{i\pi}{n}\sin\frac{i\pi}{n})\frac{\pi}{n}$.)

0.2 Evaluating integral

$$\begin{array}{rrcll}
 & \sum_{i=1}^{n} i & = & 1+2+\cdots+n & = \frac{n(n+1)}{2}; \\
 & \sum_{i=1}^{n} i^2 & = & 1^2+2^2+\cdots+n^2 & = \frac{n(n+1)(2n+1)}{6}; \\
 & \sum_{i=1}^{n} i^3 & = & 1^3+2^3+\cdots+n^3 & = \left[\frac{n(n+1)}{2}\right]^2; \\
 & \sum_{i=1}^{n} c & = & \overbrace{c+c+\cdots+c}^{n} & = cn; \\
\text{(常數倍)} & \sum_{i=1}^{n} ca_i & = & ca_1+ca_2+\cdots+ca_n & \\
 & & = & c(a_1+a_2+\cdots+a_n) & = c\sum_{i=1}^{n} a_i; \\
\text{(加)} & \sum_{i=1}^{n}(a_i+b_i) & = & \sum_{i=1}^{n} a_i+\sum_{i=1}^{n} b_i; & \\
\text{(減)} & \sum_{i=1}^{n}(a_i-b_i) & = & \sum_{i=1}^{n} a_i-\sum_{i=1}^{n} b_i. &
\end{array}$$

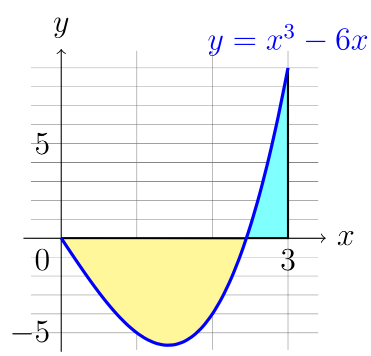
Example 0.2 *(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking right endpoint, $a = 0, b = 3, n = 6$. (b) Evaluate $\int_0^3 (x^3 - 6x)\, dx$.*

(a) $[0, 3]$ 分成 *6* 段*:* $x_0 = a = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = b = 3,$ *and* $\Delta x = \frac{3-0}{6} = \frac{1}{2}$.



$$\begin{aligned}
R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x \\
&= f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x \\
&\quad + f(2)\Delta x + f(2.5)\Delta x + f(3)\Delta x \\
&= \frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) \\
&= -3.9375.
\end{aligned}$$

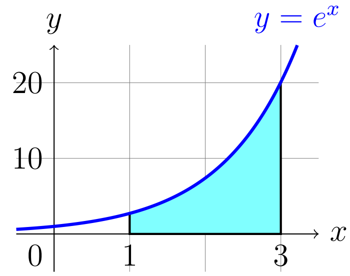
(b) $x_i = \frac{3i}{n}$ *and* $\Delta x = \frac{3}{n}$.



$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\frac{3}{n} \\
&= \frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i \\
&= \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2} \\
&= \frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right).
\end{aligned}$$

$$\begin{aligned}
\int_0^3 (x^3 - 6x)\, dx &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right] \\
&= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75. \qquad \blacksquare
\end{aligned}$$

Example 0.3 *(a) Set up an express for $\int_1^3 e^x\ dx$ as a limit of sums.*
(b) Evaluate the expression.



(a) $f(x) = e^x$, $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$,
$x_i = 1 + i\Delta x = 1 + \dfrac{2i}{n}$.

$$\int_1^3 e^x\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n} e^{1+\frac{2i}{n}}\frac{2}{n} = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e^{1+\frac{2i}{n}}.$$

(b) ($a + ar + \cdots + ar^{n-1} = \dfrac{ar^n - a}{r-1}$, $t = \dfrac{2}{n} \to 0^+ \iff n \to \infty$.)

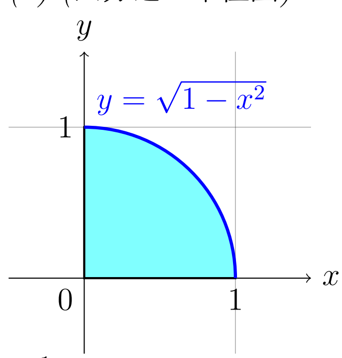
$$\sum_{i=1}^{n} e^{1+\frac{2i}{n}} = e^{\frac{n+2}{n}} + e^{\frac{n+4}{n}} + \cdots + e^{\frac{3n}{n}} = \frac{e^{\frac{3n+2}{n}} - e^{\frac{n+2}{n}}}{e^{\frac{2}{n}} - 1} = (e^3 - e)\frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}} - 1},$$

$$\int_1^3 e^x\ dx = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e^{1+\frac{2i}{n}} = \lim_{n\to\infty}(e^3 - e)\frac{\frac{2}{n}e^{\frac{2}{n}}}{e^{\frac{2}{n}} - 1} = (e^3 - e)\lim_{t\to 0^+}\frac{te^t}{e^t - 1}\ \left(\frac{\mathbf{0}}{\mathbf{0}}\right)$$

$$\overset{l'H}{=} (e^3 - e)\lim_{t\to 0^+}\frac{e^t + te^t}{e^t} = (e^3 - e)\frac{1 + 0\cdot 1}{1} = e^3 - e. \qquad \blacksquare$$

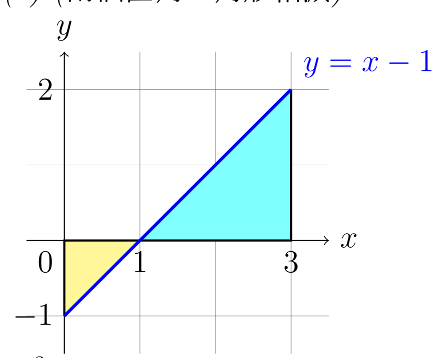
Example 0.4 *Evaluate integrals by areas. (有時候會有較簡單的算法。)*
(a) $\int_0^1 \sqrt{1-x^2}\ dx$. *(b)* $\int_0^3 (x-1)\ dx$.

(a) (四分之一單位圓)



$$\int_0^1 \sqrt{1-x^2}\ dx = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}.$$

(b) (兩個直角三角形相減)



$$\int_0^3 (x-1)\ dx = \frac{2\cdot 2}{2} - \frac{1\cdot 1}{2} = \frac{3}{2}. \qquad \blacksquare$$

0.3 Midpoint rule

黎曼和的樣本點可以選 {左, 右, 中, 大, 小}, 什麼叫選 midpoint 中點?

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b, \quad \Delta x = x_i - x_{i-1} = \frac{b-a}{n},$$

$$\boxed{\boldsymbol{\bar{x_i}} = \frac{x_i + x_{i-1}}{2}} \in [x_{i-1}, x_i].$$

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x_i})\Delta x \approx \sum_{i=1}^{n} f(\bar{x_i})\Delta x$$

Example 0.5 *Use midpoint rule with $n = 5$ to approximate* $\int_1^2 \frac{1}{x}\, dx$.

$\Delta x = \frac{2-1}{5} = \frac{1}{5}$, $\bar{x_1} = 1.1, \bar{x_2} = 1.3, \bar{x_3} = 1.5, \bar{x_4} = 1.7, \bar{x_5} = 1.9$.

$$\begin{aligned}\int_1^2 \frac{1}{x}\, dx &\approx \frac{1}{5}[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \\ &= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right) \\ &\approx 0.691908.\end{aligned}$$

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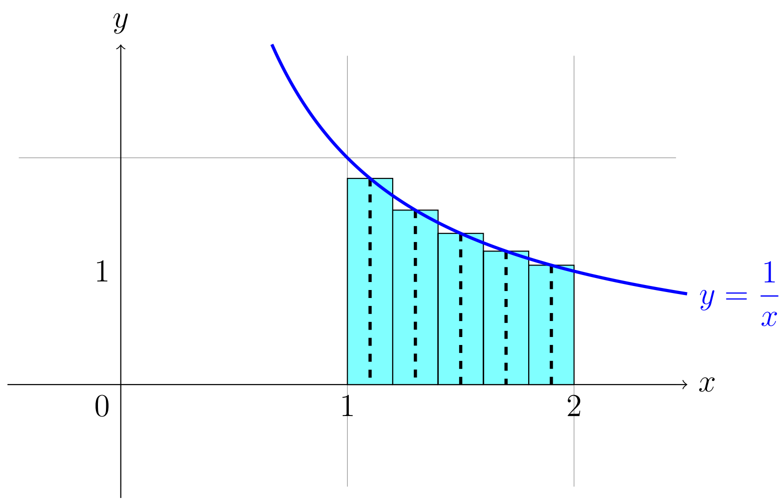


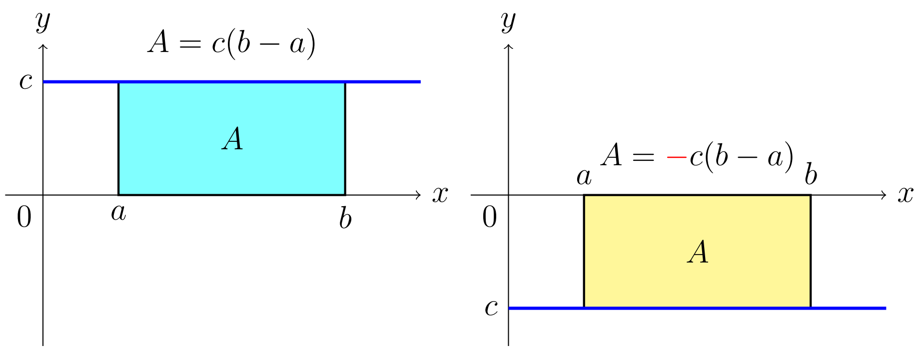
大多的情形下, 挑中點來估計會比挑左右來得準一點。

0.4 Property of definite integral

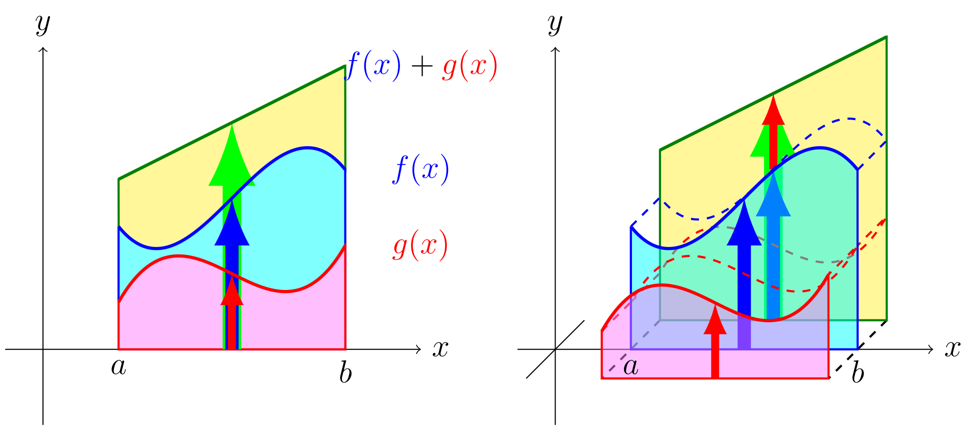
1. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$ $(b \to a)$ 換方向 $(a \to b)$ 差負號。

2. $\displaystyle\int_a^a f(x)\,dx = 0.$ $(a \to a)$ 直線無面積。

3. $\displaystyle\int_a^b c\,dx = c(b-a).$ $\displaystyle\left|\int_a^b \text{常數函數}\,dx\right| =$ 矩形面積。



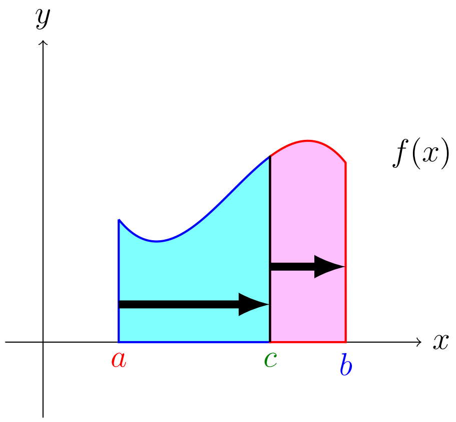
4. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$ 加 (要一樣 $a \to b$)。



5. $\displaystyle\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$ 減 (要一樣 $a \to b$)。

6. $\displaystyle\int_a^b [cf(x)]\,dx = c\int_a^b f(x)\,dx.$ 常數倍 (要一樣 $a \to b$)。

7. $\int_{a}^{c} f(x)\ dx + \int_{c}^{b} f(x)\ dx = \int_{a}^{b} f(x)\ dx.$ 分段積分。

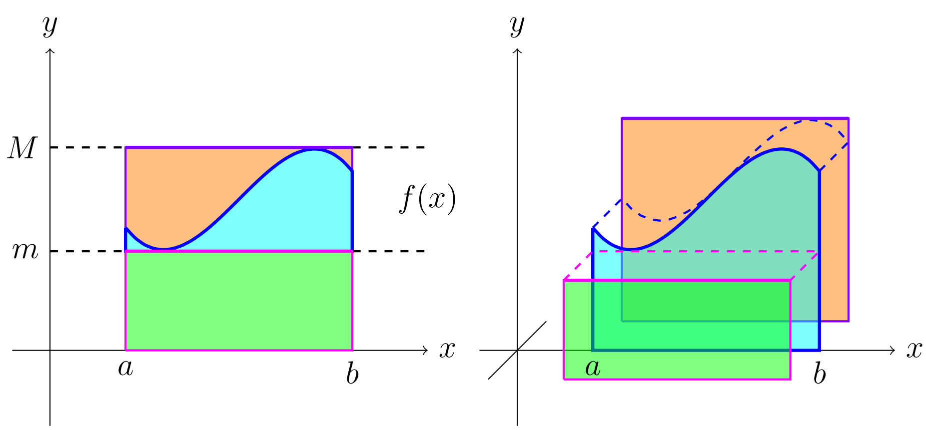


$((a \to c) + (c \to b) = a \to b)$ c 可以不在 a, b 中間。

8. $f(x) \geq 0$ for $a \leq x \leq b \implies \int_{a}^{b} f(x)\ dx \geq 0.$ 正的面積正。

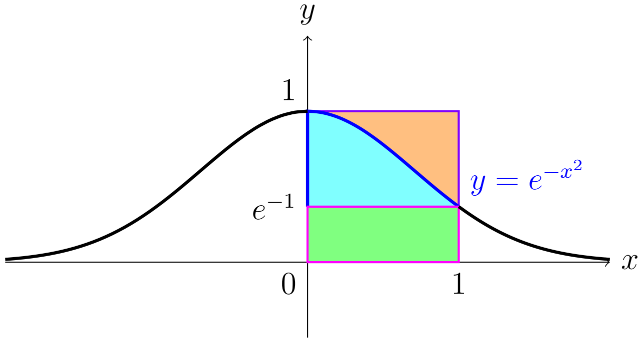
9. $f(x) \geq g(x)$ for $a \leq x \leq b \implies \int_{a}^{b} f(x)\ dx \geq \int_{a}^{b} g(x)\ dx.$大的面積大。

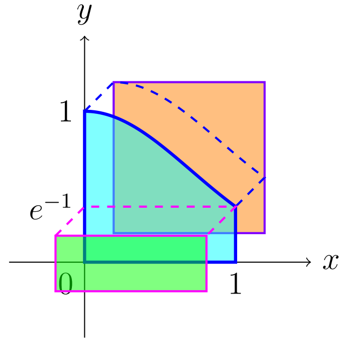
10. $m \leq f(x) \leq M$ for $a \leq x \leq b \implies m(b-a) \leq \int_{a}^{b} f(x)\ dx \leq M(b-a).$



Skill: 利用面積來聯想定積分的性質。

Example 0.6 *Estimate* $\int_0^1 e^{-x^2}\,dx$.





$m = e^{-1} = e^{-1^2} \le e^{-x^2} \le e^{-0^2} = 1 = M$ *for* $x \in [0,1]$,

$0.3679 \approx \boxed{e^{-1}} = e^{-1}(1-0) \le \boxed{\int_0^1 e^{-x^2}\,dx} \le 1(1-0) = \boxed{1}$. ∎

⧫: The **Gaussian function** 高斯函數: e^{-x^2}.

⧫: The **Gaussian/Euler-Poisson integral** 高斯/歐拉-帕松積分:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

(§7.8 improper integral & **Proof.** Exercise 15.3.40)

⧫: The **(Gauss) error function** (高斯) 誤差函數:

$$\operatorname{erf}(x) := \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

$$\int_0^1 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\operatorname{erf}(1) \approx 0.746824.$$