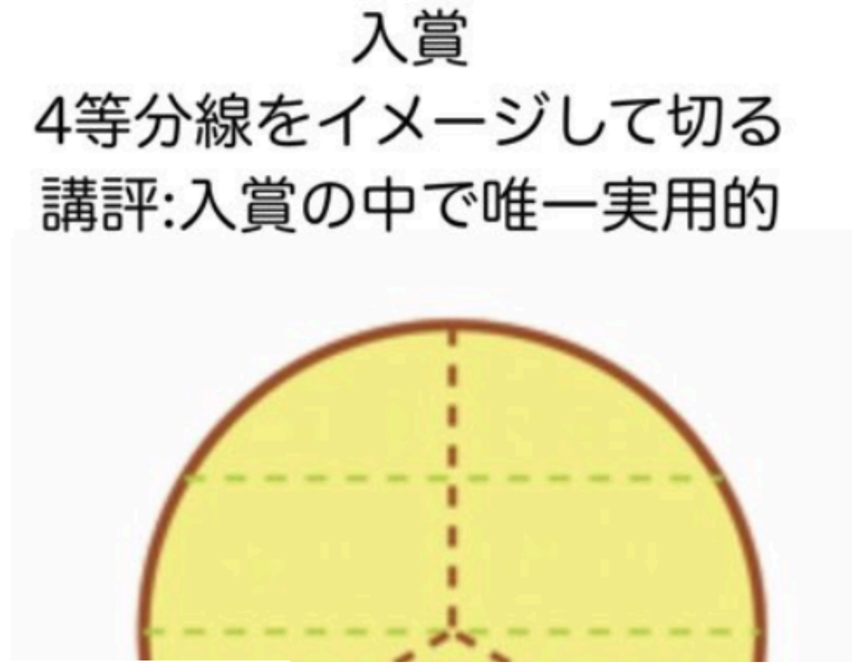
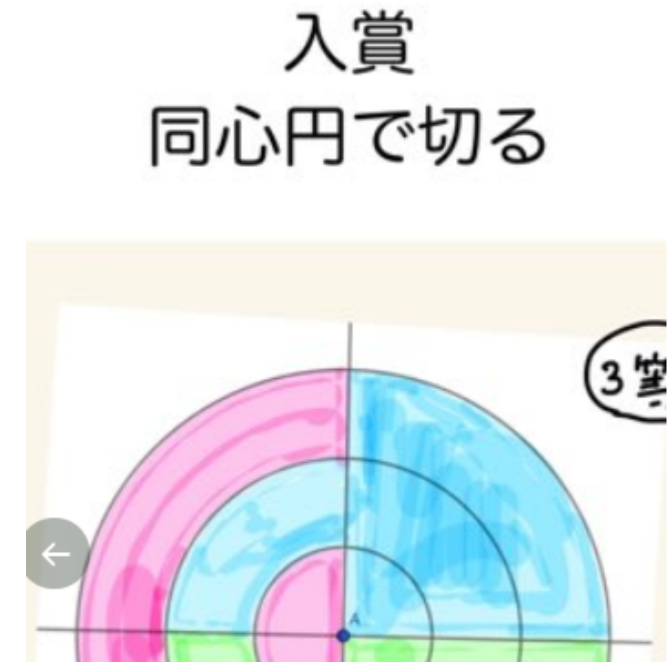
1179: Probability
Lecture 13 — Continuous Random Variables

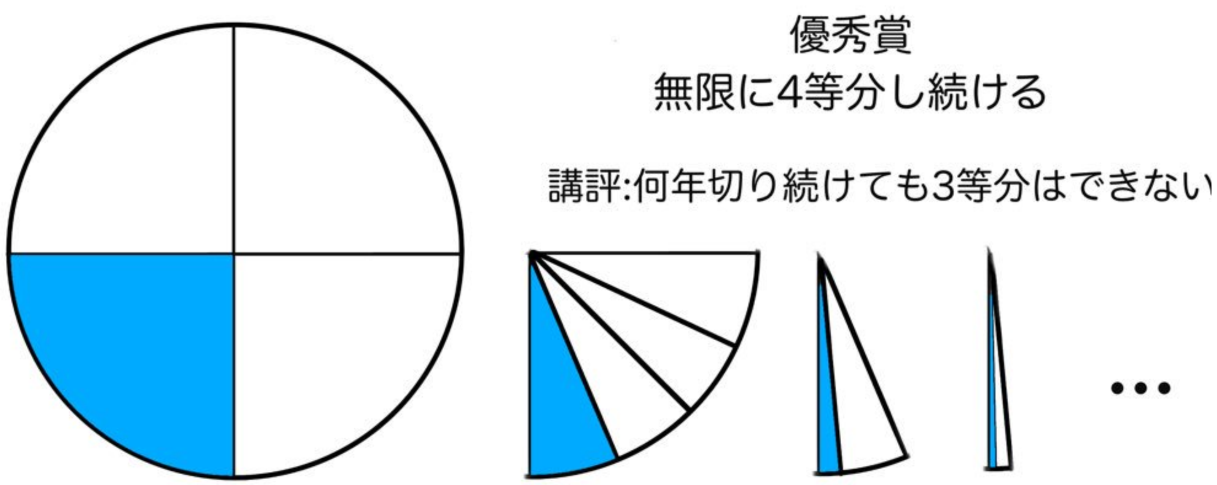
Ping-Chun Hsieh (謝秉均)

October 27, 2021

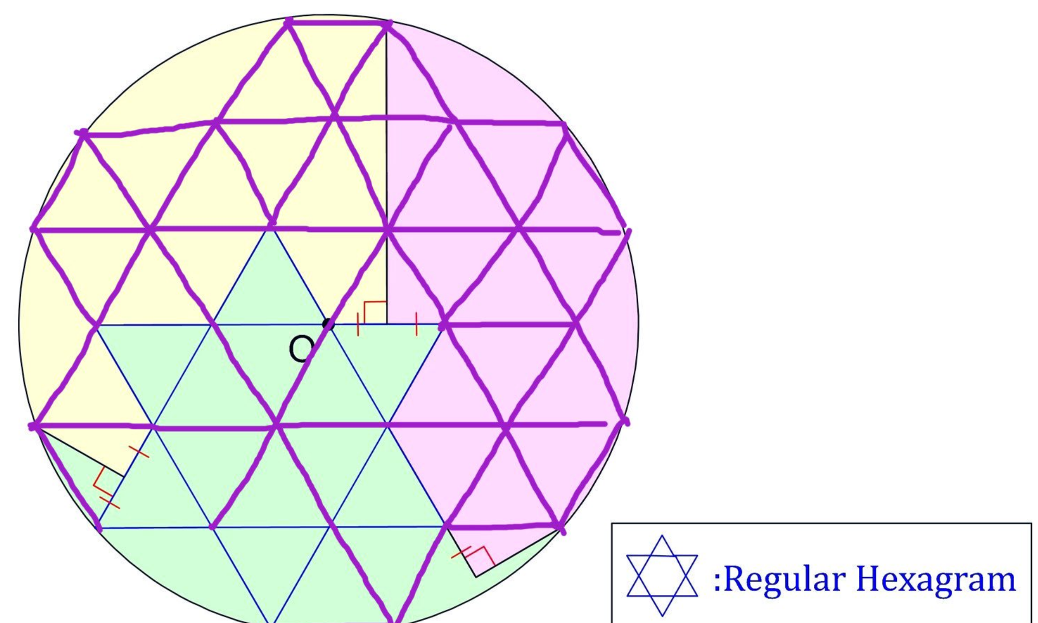
How to Cut a Cake into 3 Equal Pieces?







$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots = \frac{1}{3}$$



https://twitter.com/mathlava/status/1163796284760662017?s=21

Announcements

- HW2 has been posted on E3 (Due: 11/1, 9pm)
- About late HW submissions (starting from HW2):
  - 20% deduction for submissions 0~48 hours past due
  - HW submitted >48 hours after the deadline will not be graded
- Midterm on 11/10 (on Wednesday, in class)
  - 10:10am - 12:10pm
  - Coverage: Lec 1 - Lec 16
  - You are allowed to bring a cheat sheet (A4 size, 2-sided, without any attachments)
  - Locations: EC015 and EC022

EC022

		109550092 邱彥禎		109550094 陳侑澤		109550095 王力得		109550096 杜峯	
	109705071 許晉		109550104 劉彥甫		109550106 涂圓綠		109550108 王俊閔		
		109700006 陳樂燊		109550090 李以恩		109700018 陳韋齊			
	M093567 朱以箴		109700022 吳維誠		109700026 侯翔寓				
		109700045 陳晶		109700048 朱哲毅					
	109704026 顏于斌		109704001 許恒睿						

Blackboard(Left)

	109550098 吳柏橙		109550099 楊竣喆		109550100 陳宇駿		109550103 范劍維		
		109550110 陳尚奇		109550201 林家輝		109550202 白詩愷			
			109550203 黃承豪		109550206 陳品劭		109612019 林伯偉		
				109652022 林家鴻		109652026 王昱宸			
					109652039 林立倫		109652050 趙曼真		
						109705002 李天宰			

Blackboard(Right)

	109550113 黃宥嘉		109550116 楊傑宇		109550122 王宇晨		109550123 林芷瑩		109550124 陳俊佑		109550126 蔣欣穎
		109550130 林念慈		109550132 吳念蓉		109550134 梁詠晴		109550136 邱弘竣		109550138 江涵宇	
	109550140 藍良碩		109550150 呂則諺		109550158 陳建嘉		109550160 張綺恩		109550162 郭子頡		
		109550164 徐聖哲		109550168 林慧旻		109550170 邱奕庭		109550174 孟祥蓉		109550178 黃昱翰	
			109550182 莊婕妤		109550186 李嘉玲		109550187 梁巍濤		109550192 鄭佳楨		
		109550194 龍偉亮		109550198 卜銳凱		109550199 林能財		109550200 許涵羲			

Blackboard(Middle)

EC015

			0516107 吳健銘		0516114 洪江金		0613148 吳茂華		0616247 張育茹		0616249 劉南宏	
	0616317 游宗穎		0710871 陳治銓		0710873 田隆生			0711064 徐詠祺		0711078 簡佑任		0711088 張文彥
0711529 陳冠儒		0716250 葉書德		0716306 柯立恩			0716309 劉峻豪		0716312 葉佳翰		0716332 林威俊	
	0811209 簡右群		0812249 楊謹安		0813131 黃暄富			0816076 陳沛妤		0816108 周子同		0816132 蔡欣龍
0816138 楊卓敏		0816164 鍾凡		10501125 陳祺侑			10811017 張皓丞		109511097 江廷威		109550002 王君豪	
	109550003 陳茂祥		109550004 紀政良		109550006 陳虹蓓			109550008 王禹博		109550010 何銘鵬		109550012 鄧閔璟
109550014 黃姿清		109550015 林承佑		109550018 郭昀			109550020 胡景竑		109550022 吳文心		109550024 廖兆琪	
	109550026 楊詠翔		109550028 陳吉遠		109550030 李霂菱			109550032 楊秉宇		109550034 黃迺絜		109550036 張嘉文
109550038 廖柏任		109550039 楊富翔		109550040 高鈺鴻			109550050 王承勳		109550054 楊弘詣		109550056 黃韋傑	
	109550058 謝旻辰		109550060 陳星宇		109550062 胡正一			109550064 陳姵帆		109550066 張瑋倫		109550070 林季囷
109550072 何嘉娓		109550074 吳秉澍		109550076 林睿騰			109550080 蔡琮偉		109550082 廖宗瑋		109550088 林哲安	

Blackboard

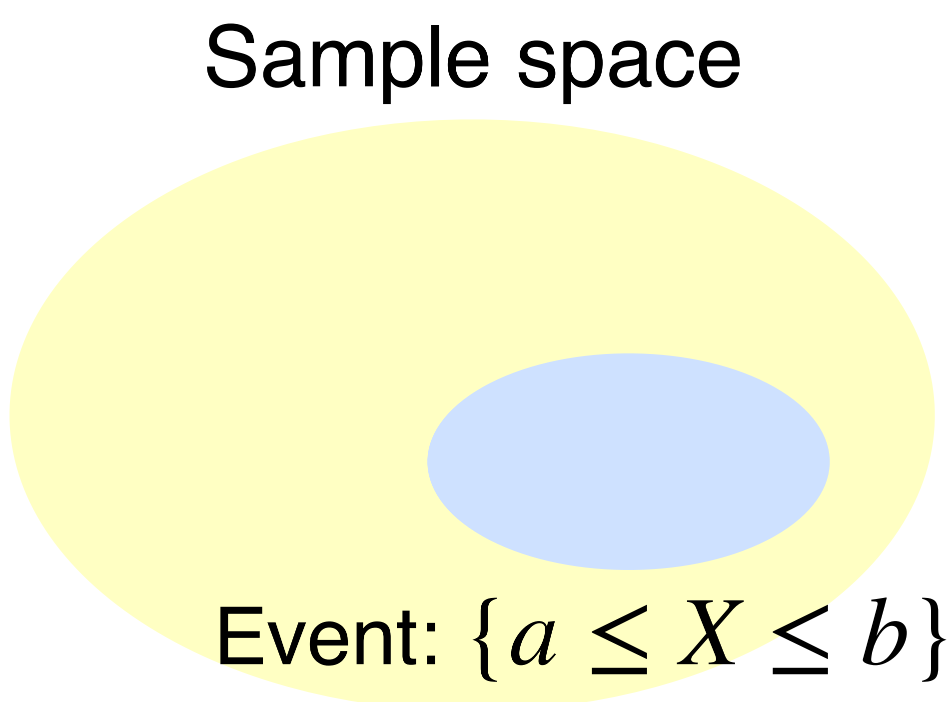
This Lecture

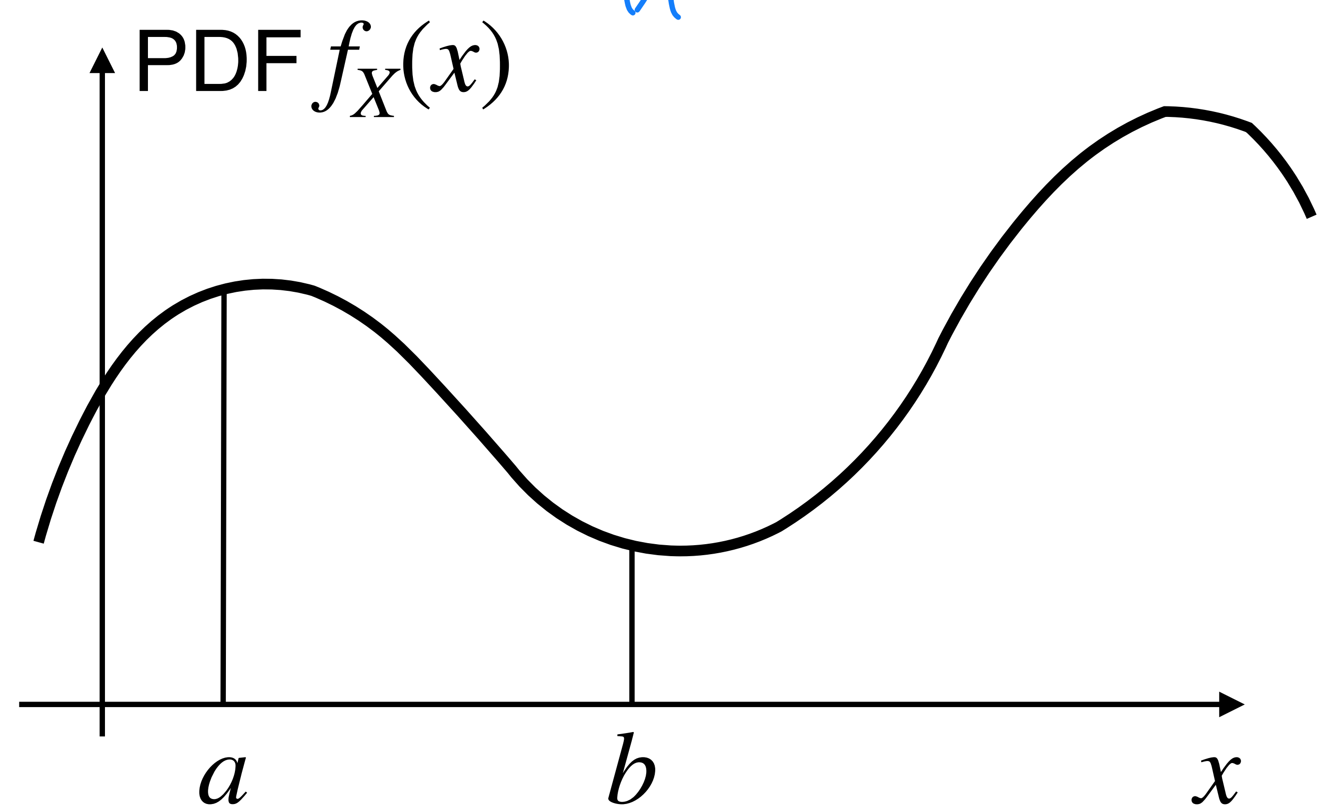
1. Continuous Random Variables

2. Special Continuous Random Variables

- Reading material: Chapter 6.1 and 7.1~7.3

Review: Probability Density Function (PDF)





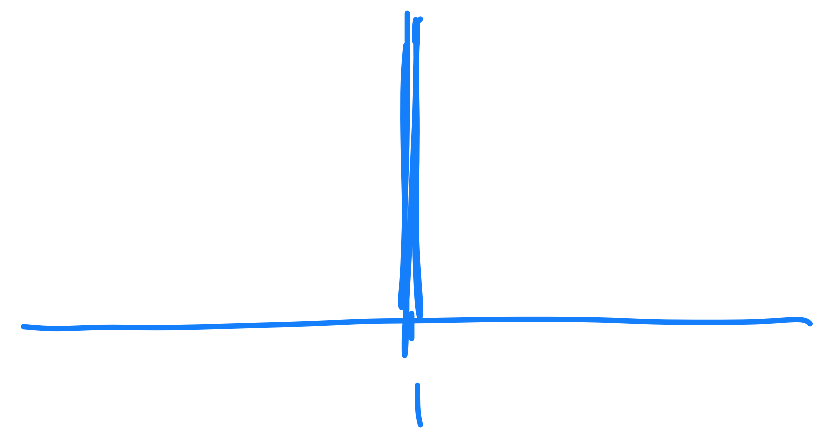
Probability Density Function (PDF):
Let X be a random variable. Then, $f_X(x)$ is the PDF of X if for every subset B of the real line, ~~we have~~

$$P(X \in B) = \int_B f_X(x)dx$$

X : discrete random variable

$$P(X=k) = \begin{cases} 1, & \text{if } k=1 \\ 0, & \text{otherwise} \end{cases}$$

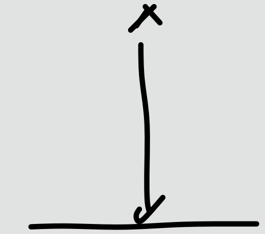
PMF



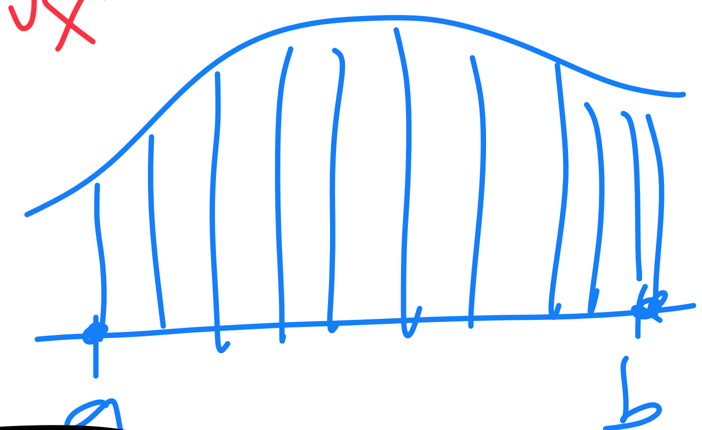
PDF f_X

$$1 = P(X=1) = \int_{\{1\}} \underbrace{f_X(x)}_{\substack{\| \\ \text{Delta}}} dx$$

Express Other Quantities Using PDF



1. $P(X \in \mathbb{R}) =$ $1 = \int_{-\infty}^{+\infty} f_X(x)\,dx \equiv \int_{\mathbb{R}} f_X(x)\,dx$



$F_X(t)$ $\parallel$

2. $P(X \leq t) =$ $\int_{-\infty}^{t} f_X(x)\,dx$

3. $P(a \leq X \leq b) =$ $\int_a^b f_X(x)\,dx$

$\int_{x \in \mathbb{R} \setminus \mathbb{Q}} 1 \cdot dx$

4. $P(a \leq X < b) =$ $\underbrace{\int_a^b f_X(x)\,dx - \underbrace{P(X=b)}_{=0}}_{=} = \int_a^b f_X(x)\,dx$

5. $P(a < X < b) =$ $\int_a^b f_X(x)\,dx - \underbrace{P(X=a)}_{=0} - \underbrace{P(X=b)}_{=0}$

Lebesgue integral

Riemann integral

How to Check if a PDF is Valid?

$$\int_{[0,1]\cup[2,3]} f\,dx = \int f + \int$$

- Recall: 3 Axioms of Probability

1. $P(X \in \mathbb{R}) = 1$ $\Rightarrow$ $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

2. $P(X \in A) \geq 0$, for all A $\Rightarrow$ $\boxed{\int_A f_X(x)\,dx \geq 0, \text{ for all } A}$

Sufficient condition:

$f_X(x) \geq 0$, for all $x \in \mathbb{R}$

This holds by the def. of integration

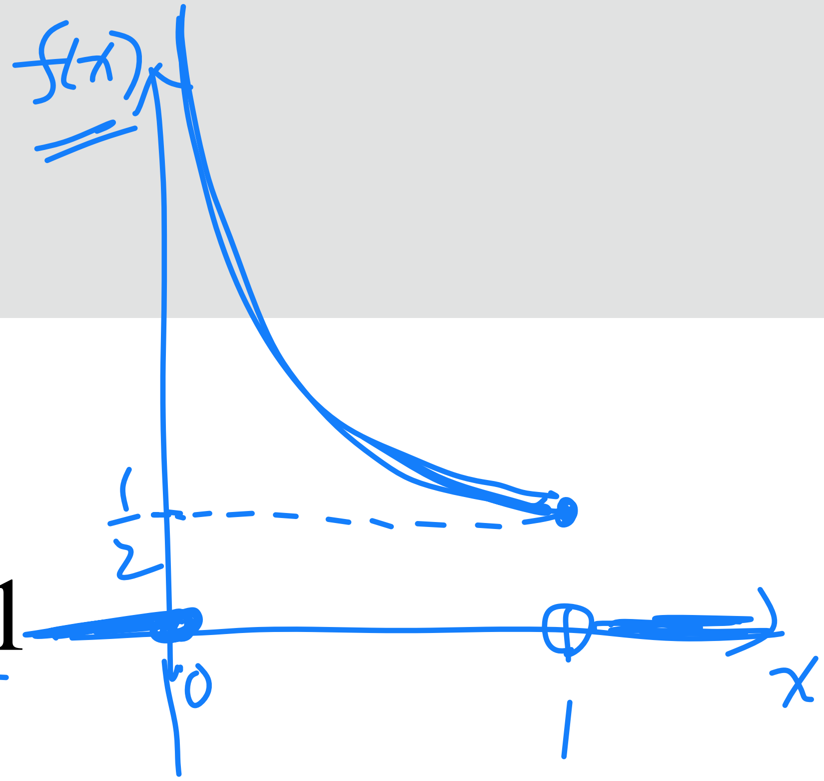
3. Let $A_1, A_2, \cdots$ be mutually exclusive sets of real numbers, then

$$P(X \in \bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} P(X \in A_i) \Rightarrow \boxed{\int_{\bigcup_{i\geq 1} A_i} f_X(x)\,dx = \sum_{i\geq 1} \int_{A_i} f_X(x)\,dx}$$

Example: From PDF to CDF

- Example: Consider the following PDF

$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & ,0 < x \le 1 \\ 0 & \text{, otherwise} \end{cases}$$



- Is $f(x)$ a valid PDF of some random variable?

① Nonnegativity: ✓ (Since $f(x) \geq 0$ for all $x \in \mathbb{R}$)

② Total probability = 1: $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx$

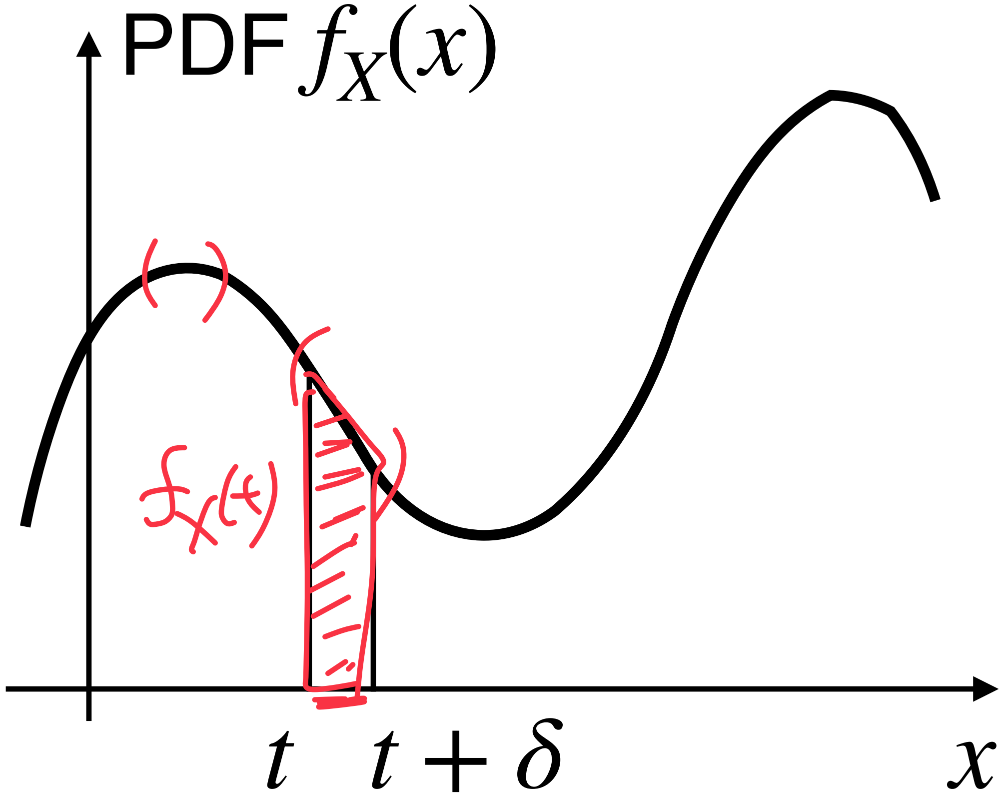
$= \sqrt{x}\Big|_0^1 = \sqrt{1} - \sqrt{0}$
$= 1$

From CDF to PDF

$$\lim_{\delta \to 0} \frac{P(t < X \le t+\delta)}{\delta} \qquad F_X'(t)$$

- CDF: $F_X(t) = P(X \le t) = \int_{-\infty}^{t} f_X(x)dx$

$$= \lim_{\delta \to 0} \frac{F_X(t+\delta) - F_X(t)}{\delta} = f_X(t)$$



- Suppose PDF is continuous
- $F_X(t+\delta) - F_X(t) = ?$

$$= P(t < X \le t+\delta) = \int_t^{t+\delta} f_X(x)dx = f_X(c)\cdot\delta$$

Mean value theorem of integration

for some $c \in [t, t+\delta]$

From CDF to PDF (Formally)

Derivative of CDF is PDF:
Let X be a random variable with a CDF $F_X(\cdot)$ and a PDF $f_X(\cdot)$. If $f_X(\cdot)$ is continuous at x_0, then

$$F'_X(x_0) = f_X(x_0)$$

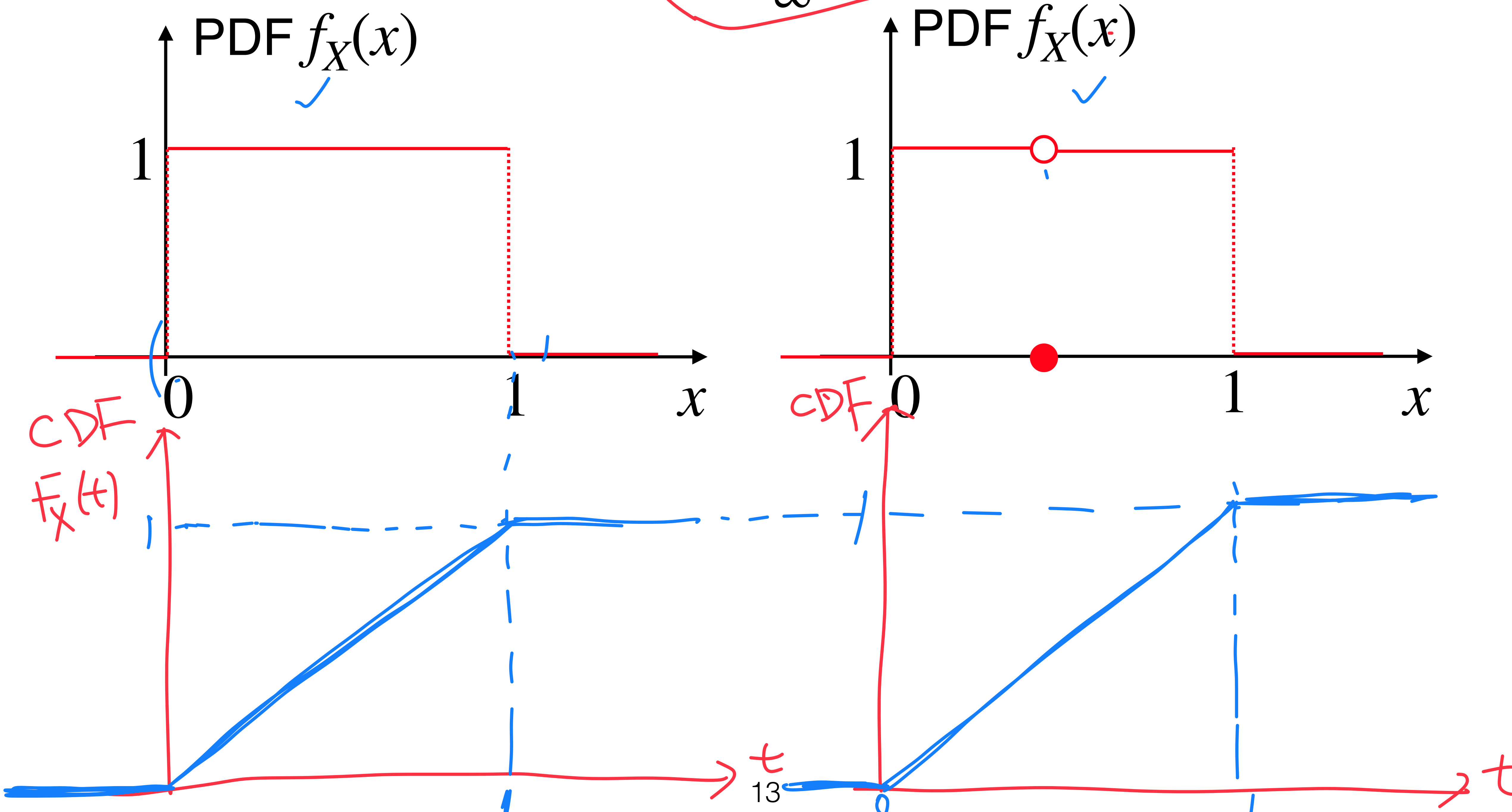
- Any similar results in calculus?

Fundamental Theorem of Calculus

From CDF to PDF: Why Continuity is Required?

- CDF: $F_X(t) = P(X \le t) = \int_{-\infty}^{t} f_X(x)dx$

"identifiability"

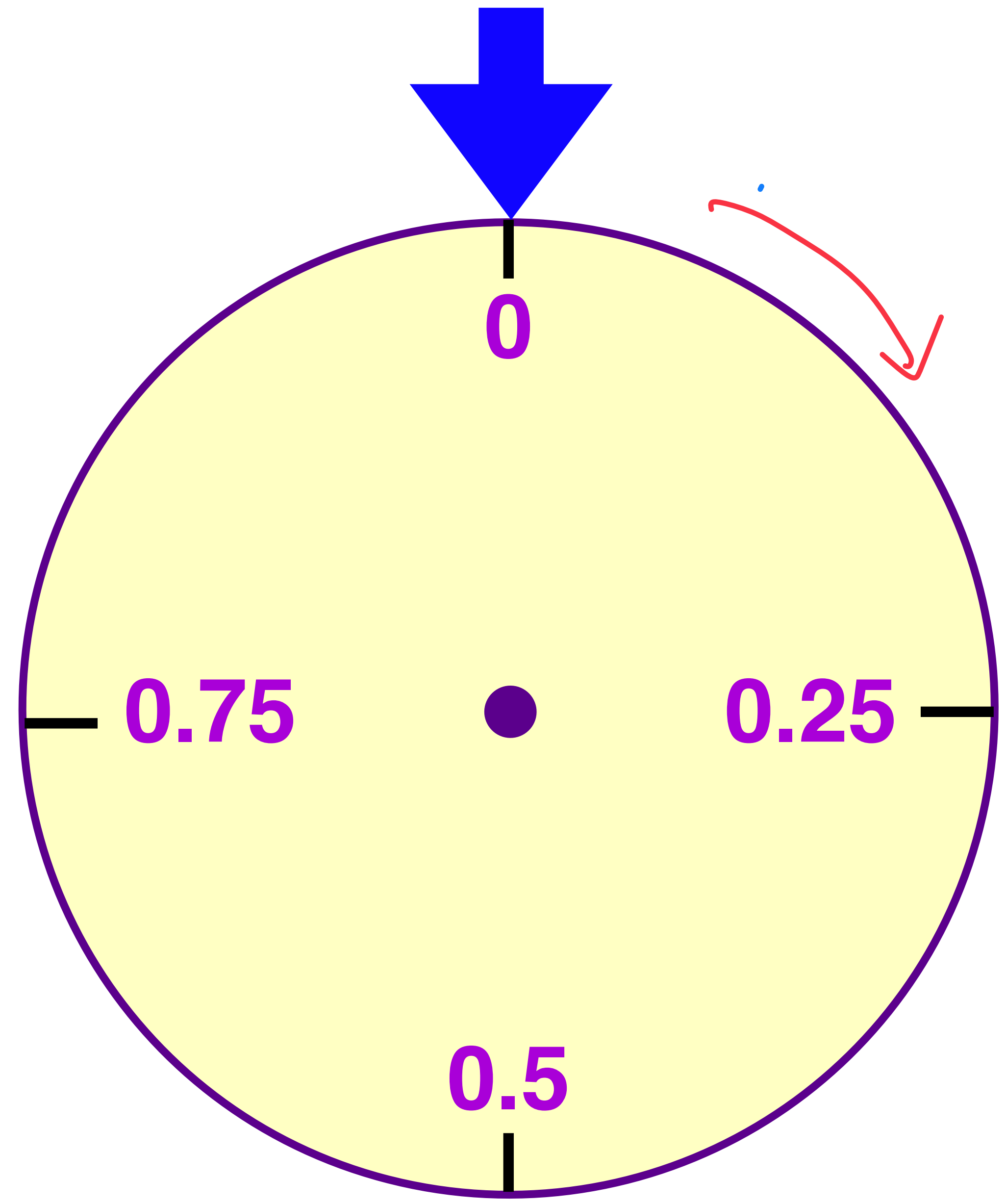


Special Continuous Random Variables

1. (Continuous) Uniform Random Variables

- Example: A bus arrive at a random time between 9:15am and 9:30am

- Example: Play wheel of fortune



- What are the common features?
  - "Principle of indifference"

1. Uniform Random Variables (Formally)

$X \sim Unif(a,b)$

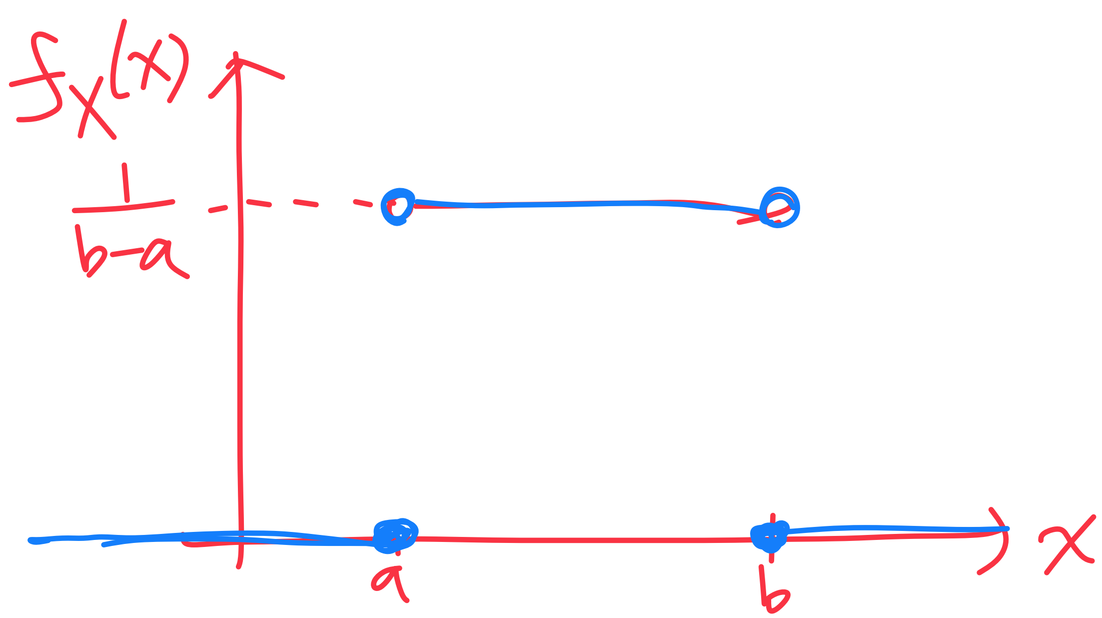
Uniform Random Variables: A random variable X is uniform with parameters a, b $(a < b)$ if its PDF is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{, if } a < x < b \\ 0 & \text{, otherwise} \end{cases}$$

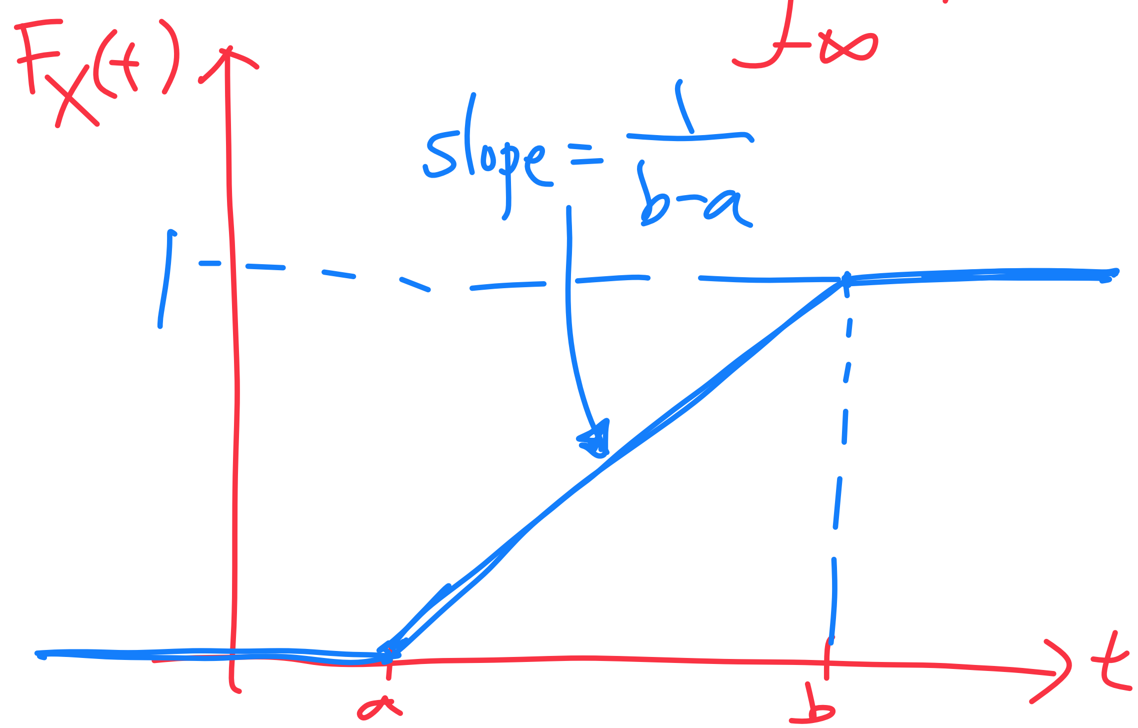
$F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(x)\,dx$

- How to plot the PDF and CDF of X?

PDF:



CDF:



Example: Uniform Distribution

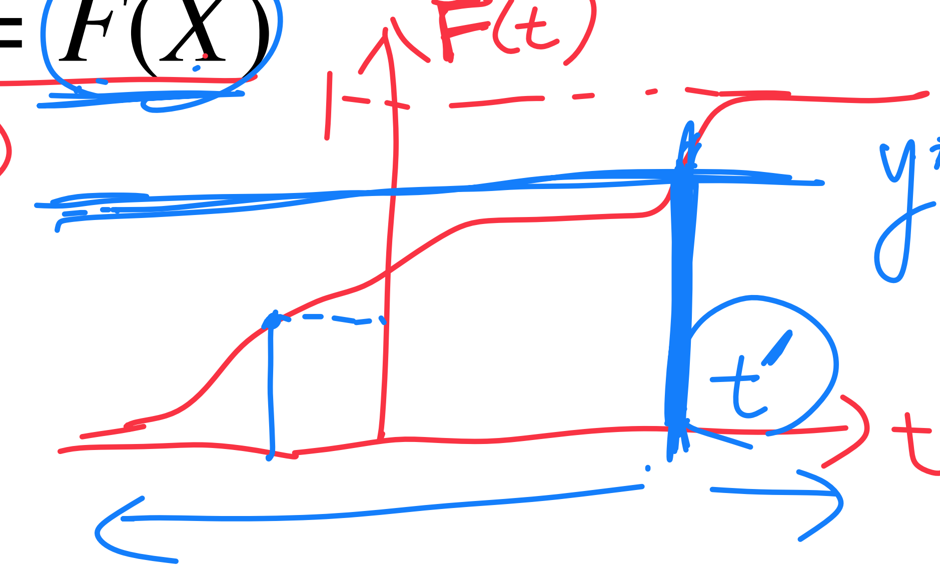
- Example: Let X be a random variable with CDF $F(t)$.
  - Define another random variable $Y = F(X)$
  - What type of random variable is Y?

ω

$F(t') = y^*$

$P(X \le t') = F(t') = y^*$

$F(X) \le y^*$ if $X \le t'$



$P(Y \le 0.2) = \underbrace{P(F(X) \le 0.2)}$

$= P(\{\omega : F(X(\omega)) \le 0.2\}) = 0.2$

$Y \sim Unif(0,1)$

For any $t^* \in [0,1]$:

$P(Y \le t^*) = t^*$

For $t^* > 1$:

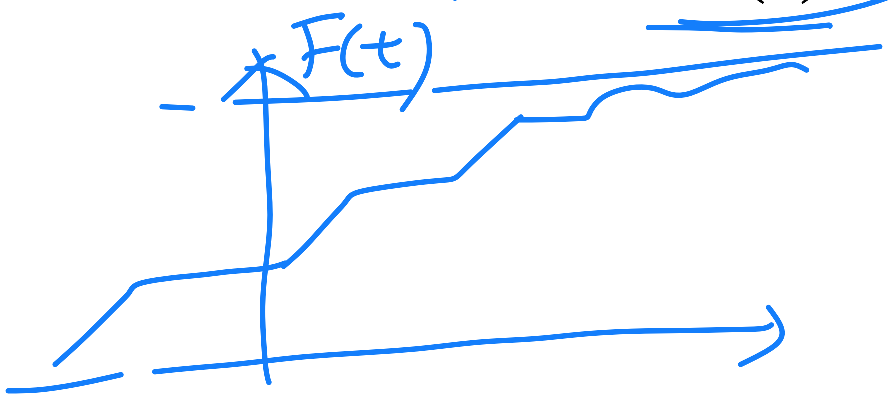
$P(Y \le t^*) = 1$

For $t^* < 0$:

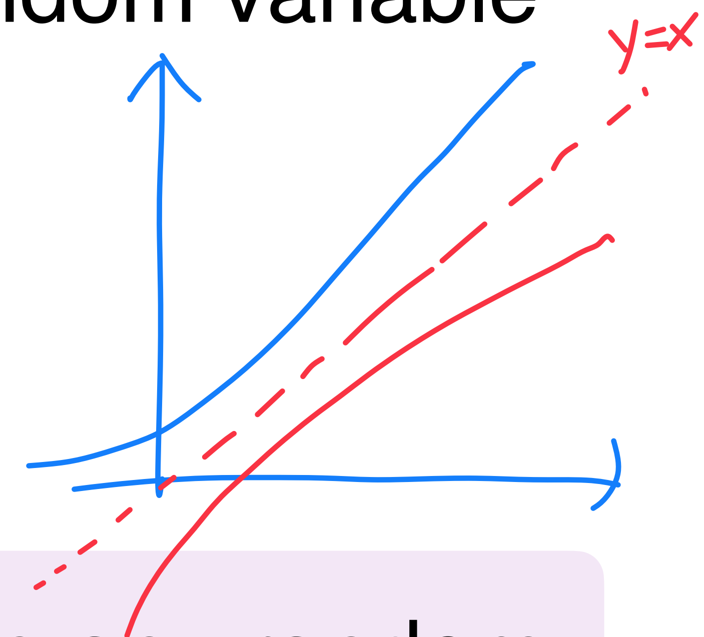
$P(Y \le t^*) = 0$

Why Are Uniform Random Variables Useful?

- Question: How to generate a customized random variable X with CDF $F(t)$?



np.random.uniform
geometric
binomial



Inverse Transform Sampling (ITS): Generate any random variable with CDF $F(t)$ from a uniform random variable

1. Generate a random variable $U \sim \text{Unif}(0,1)$

2. Let $X = F^{-1}(U)$, where $F^{-1}(u) := \inf\{z : F(z) \geq u\}$

Proof: Inverse Transform Sampling

$P(X \le t) = F(t)$

Inverse Transform Sampling: Generate <u>any</u> random variable with CDF $F(t)$ from a uniform random variable

1. Generate a random variable $U \sim$ Unif$(0,1)$
2. Let $X = F^{-1}(U)$, where $F^{-1}(u) := \inf\{z : F(z) \geq u\}$

F is non-decreasing

- $\underbrace{P(F^{-1}(U) \leq x)}$ $= P(F(F^{-1}(U)) \le F(x))$

$= P(U \le F(x)) = F(x)$

$P(\{X \le 1\})$

$= P(\{2X \le 2\})$

The CDF of X is indeed $F(x)$

Example: Inverse Transform Sampling

- Example: Generate a random variable X with CDF

$$F_X(t) = \begin{cases} 1 - \exp(-t^2) & , t \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

Normal Random Variables

2. Standard Normal Random Variables

- **Motivation**: Consider $X \sim \text{Binomial}(n, \frac{1}{2})$

np

$$\text{PMF} = P(X=k) = \begin{cases} C^n_k \cdot \left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{2}\right)^{n-k}, & k=0,1,\cdots,n \\ 0, & \text{otherwise} \end{cases}$$

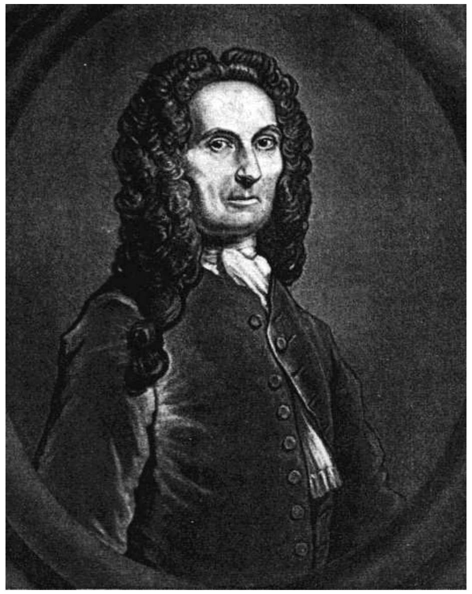
- Define $Y = \dfrac{X - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}}$. What is the CDF of Y vs n?

$n \to \infty$

$$E[X] = \frac{1}{2} \cdot n$$

$$Var[X] = n \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right) = \frac{1}{4} \cdot n$$

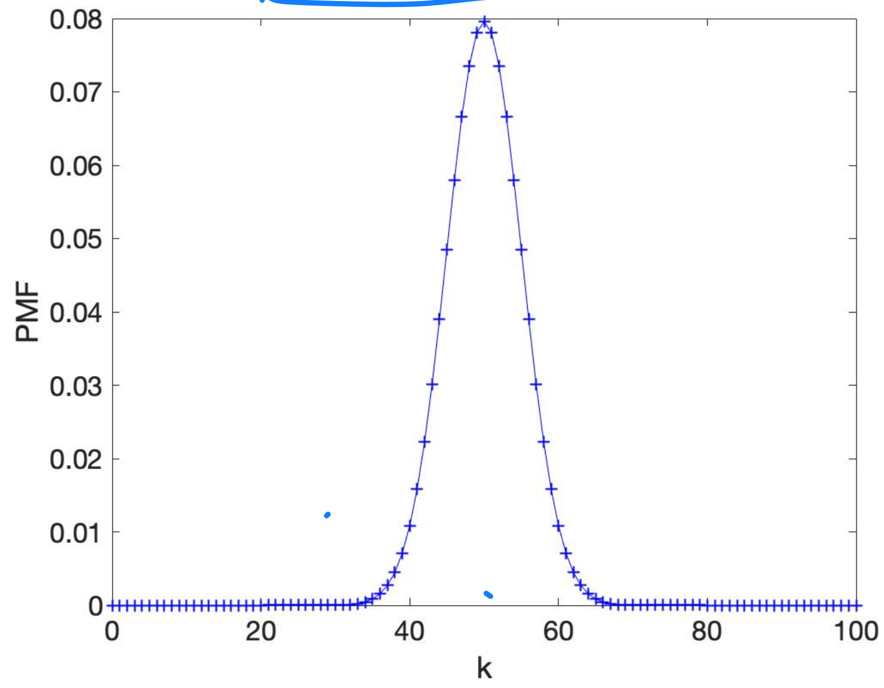
$$Y = \frac{X - E[X]}{\sqrt{Var[X]}}$$



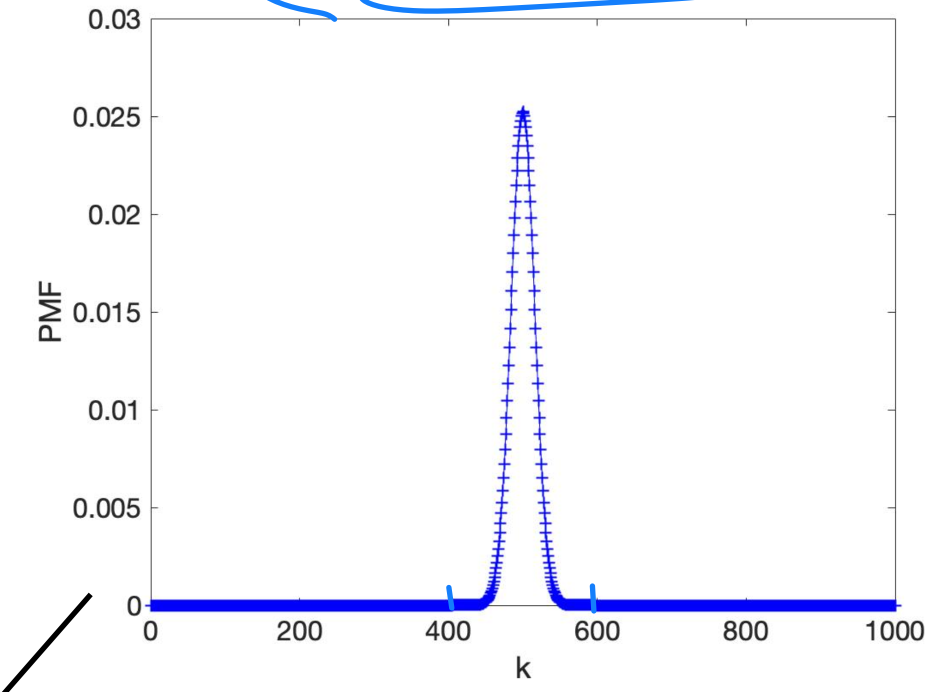
Abraham de Moivre

Plotting $X \sim \text{Binomial}(n, 1/2)$

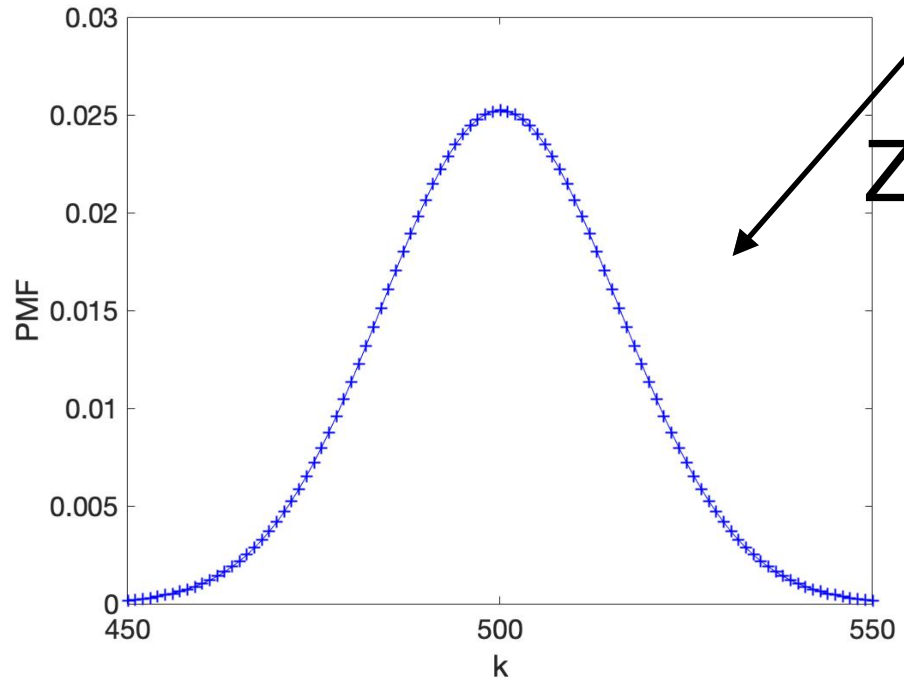
$n = 100$



$n = 1000$

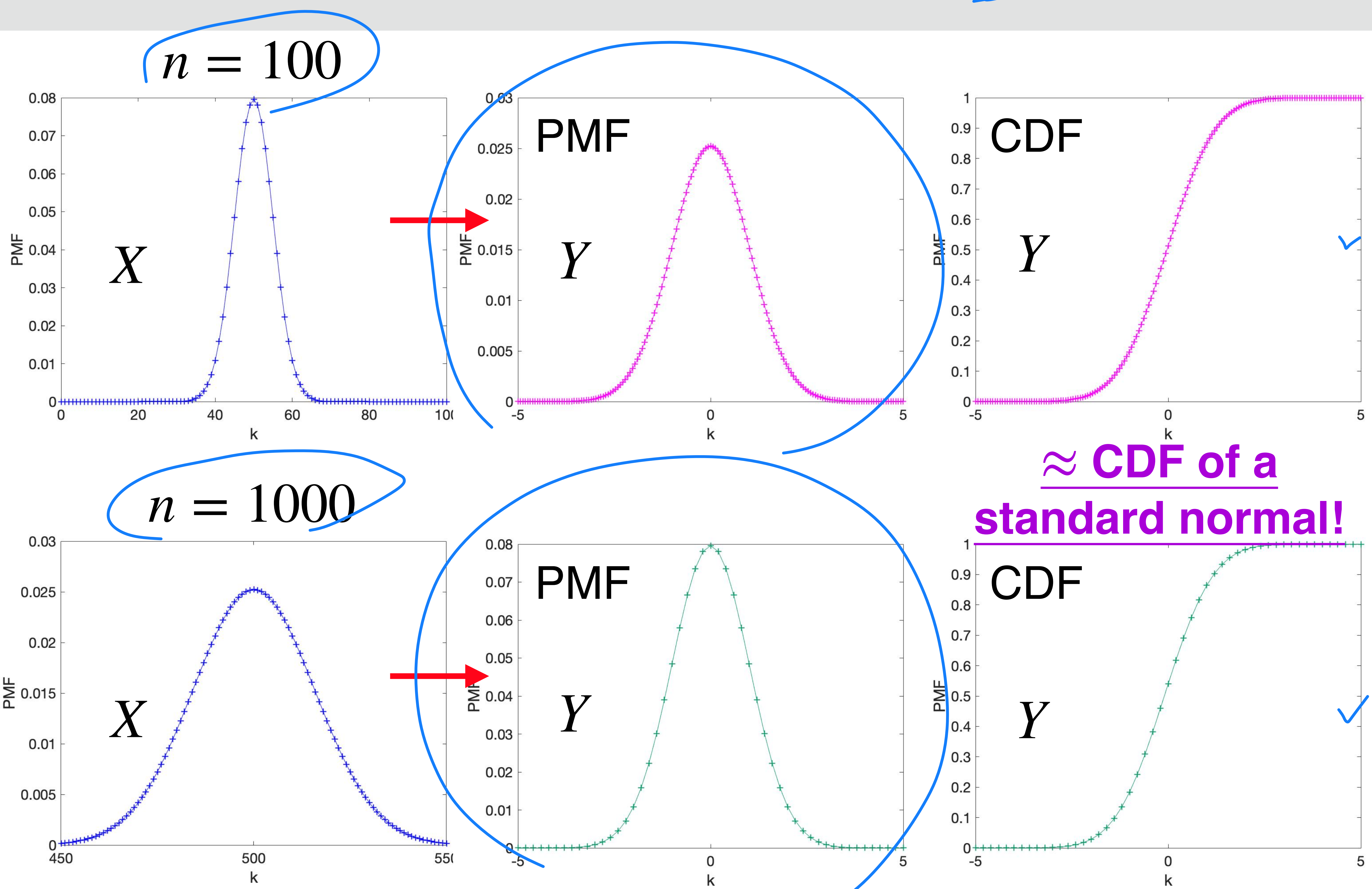


$n = 1000$



Zoom in to the range
$k \in [450, 550]$

Plotting $Y = (X - 0.5n)/(0.5\sqrt{n})$



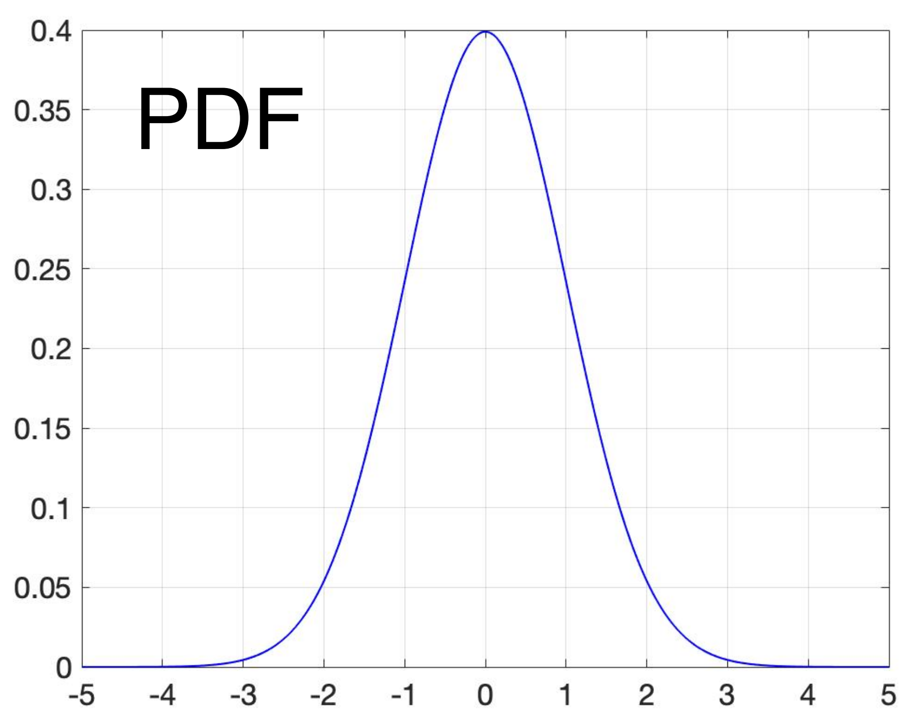
≈ CDF of a standard normal!

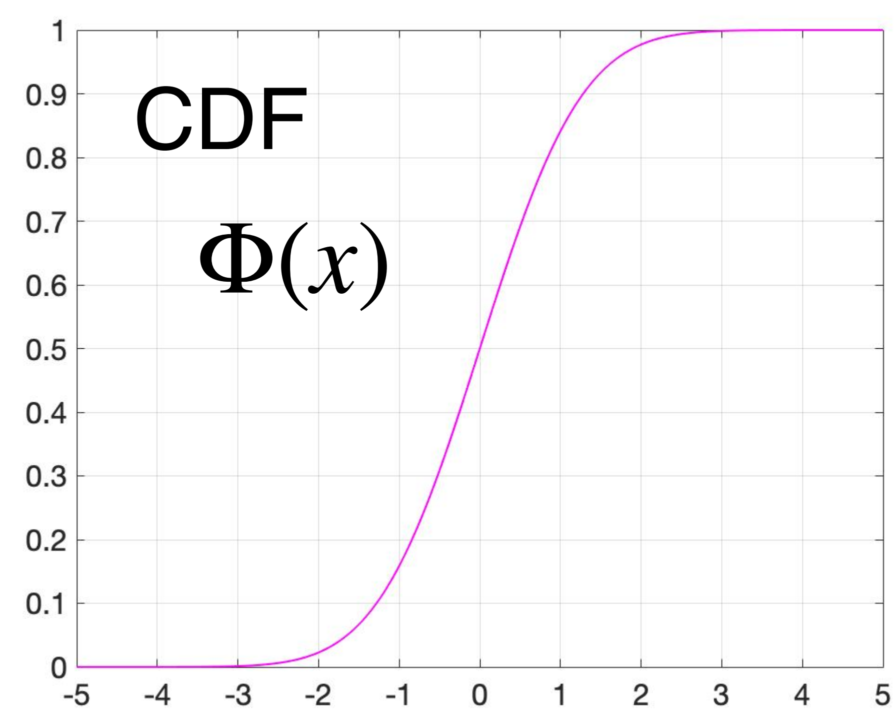
2. Standard Normal Random Variables (Formally)

Standard Normal Random Variables: A random variable X is called standard normal if its PDF is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right), \text{ for all } x \in \mathbb{R}$$

- How to plot the PDF and CDF?





$$E[X] = 0$$

$$\text{Var}[X] = 1$$

2. CDF of Standard Normal (Formally)

- As standard normal is widely applicable, we use a special notation $\Phi(\,\cdot\,)$ for its CDF

CDF of Standard Normal: The CDF of a standard normal random variable X is

$$\Phi(t) := P(X \leq t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx$$

- Question: How to plot $\Phi(t)$?
  - $\Phi(\infty)$ = ? $\Phi(0)$ = ?

Why is Normal Distribution Useful?

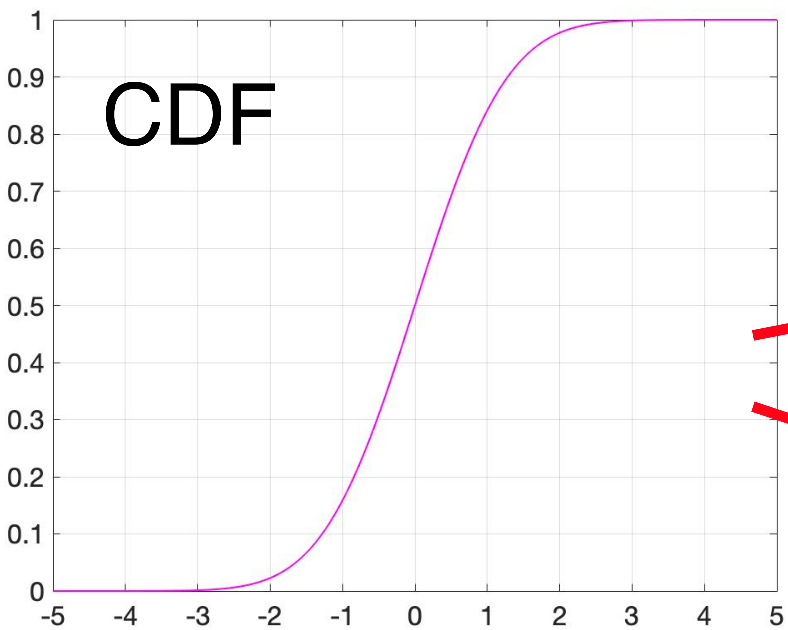
1. **Central Limit Theorem**:

2. **Gaussian Process and Black-Box Optimization**:

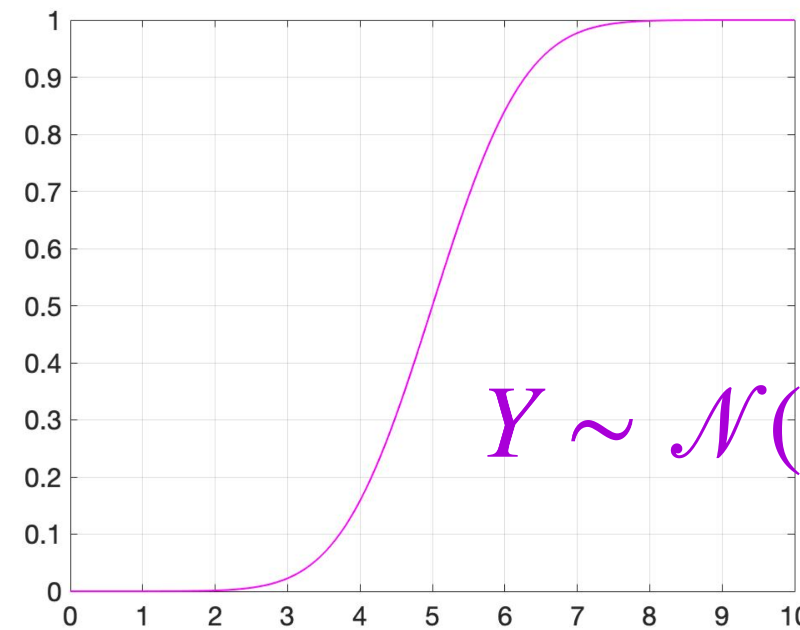
From Standard Normal to Normal: CDF

- X is standard normal

$$F_Y(t) = \Phi(t-5)$$

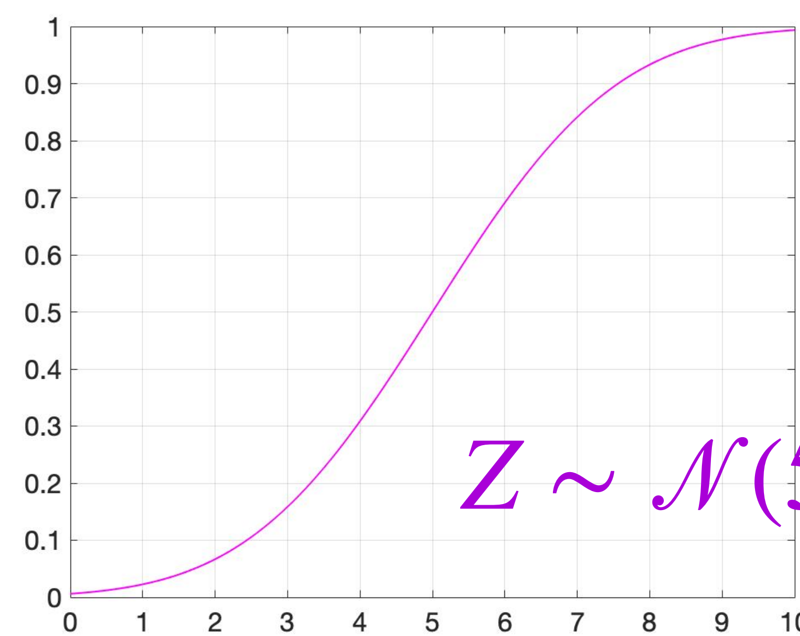


$$Y = X + 5$$



$$Y \sim \mathcal{N}(5,1)$$

$$Z = 2X + 5$$



$$Z \sim \mathcal{N}(5,2^2)$$

$$F_X(t) = \Phi(t)$$

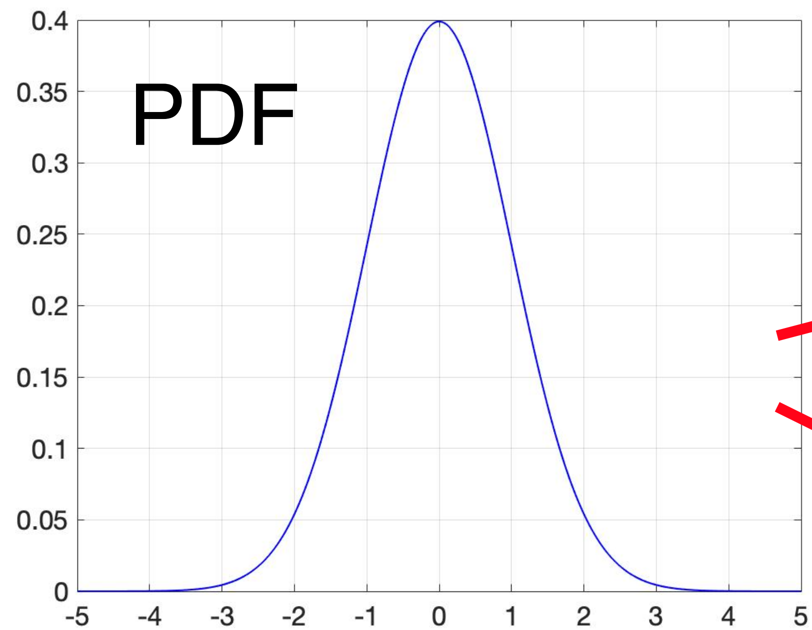
$$X \sim \mathcal{N}(0,1)$$

$$F_Z(t) = \Phi\left(\frac{t-5}{2}\right)$$

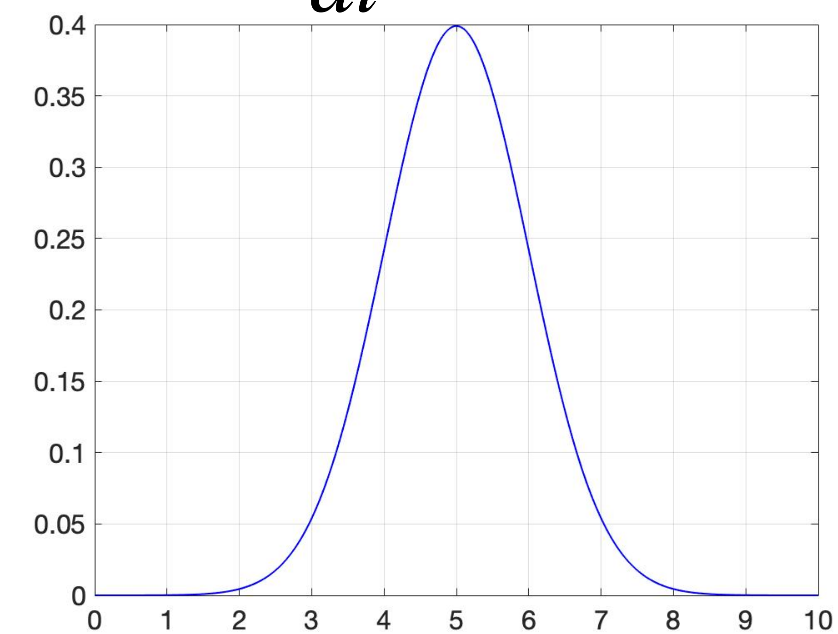
From Standard Normal to Normal: PDF

- X is standard normal

$$f_X(t) = \frac{d\Phi(t-5)}{dt} = \Phi'(t-5)$$

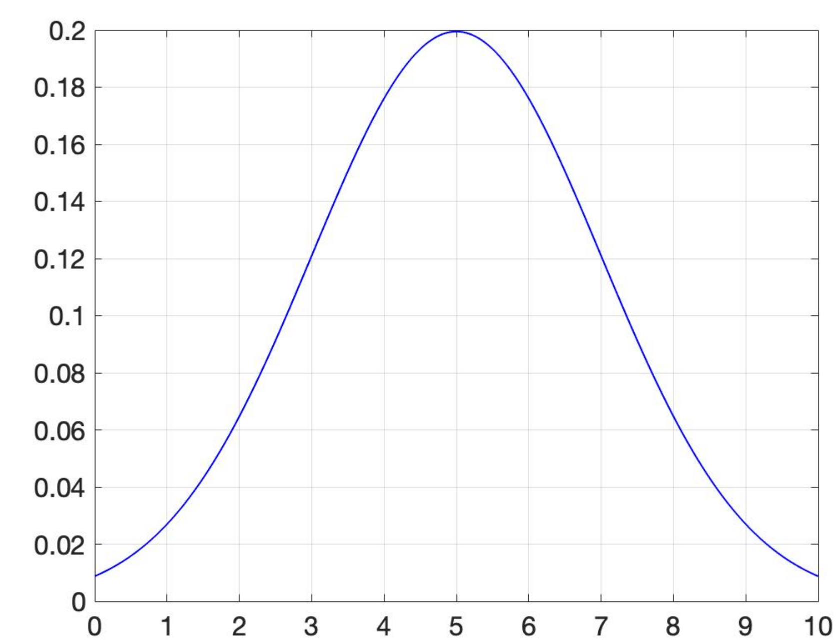


$$Y = X + 5$$



$$f_X(t) = \Phi'(t)$$

$$Z = 2X + 5$$

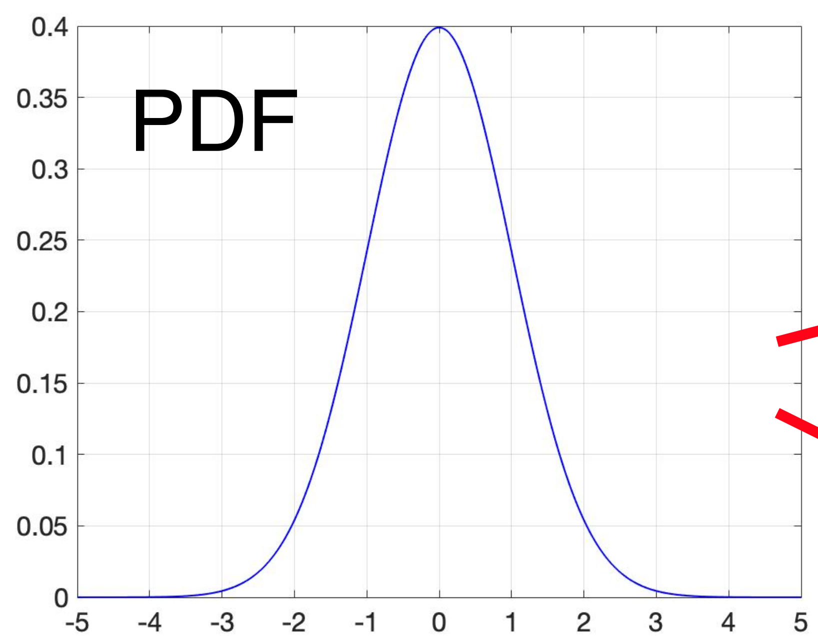


$$f_X(t) = \frac{d\Phi(\frac{t-5}{2})}{dt} = \frac{1}{2}\Phi'(\frac{t-5}{2})$$

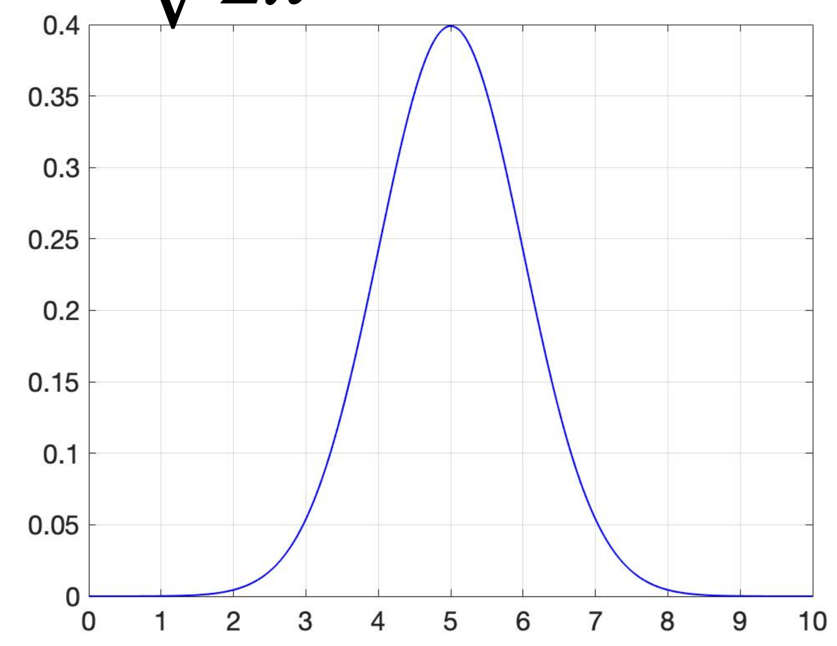
From Standard Normal to Normal: PDF

- X is standard normal

$$f_Y(t) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(t-5)^2}{2}\right)$$

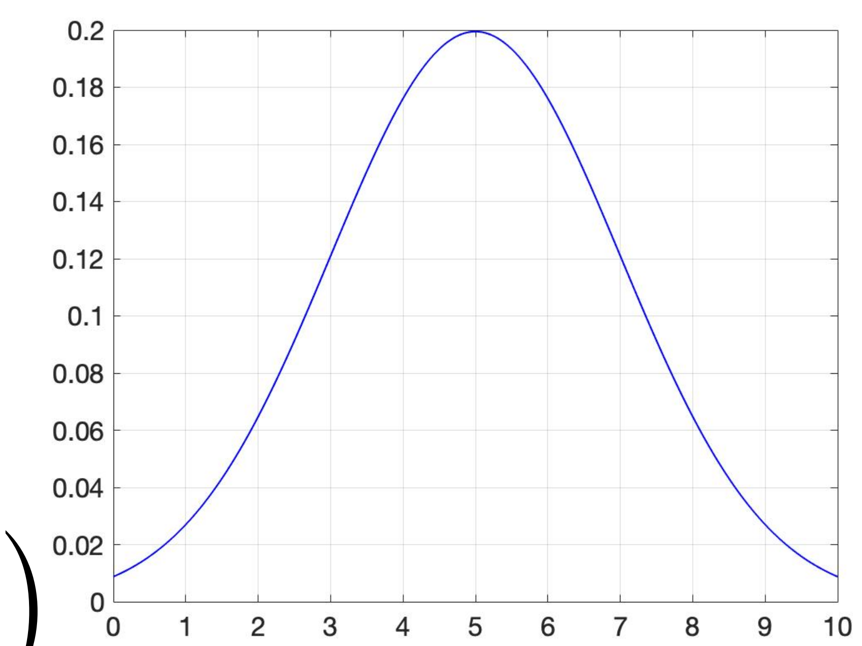


$$Y = X + 5$$



$$f_X(t) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right)$$

$$Z = 2X + 5$$



$$f_Z(t) = \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{-(t-5)^2}{2 \cdot 2^2}\right)$$

A General Recipe for Linear Transformation

- X is a continuous random variable
  - CDF: $F_X(t)$
  - PDF: $f_X(t) = \dfrac{dF_X(t)}{dt}$

- Consider $Y = aX + b,\ a, b \in \mathbb{R}, a \neq 0$
  - CDF $F_Y(t)$?
  - PDF $f_Y(t)$?
  - If $X \sim \mathcal{N}(0,1)$, then $F_Y(t)$ = ?

Normal Random Variables

Normal Random Variables: A random variable X is called normal with parameters μ, σ if its PDF is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

- Notation: $X \sim \mathcal{N}(\mu, \sigma^2)$
- How to plot the PDF?

Example: Normal Distribution

- Example: Let $X \sim \mathcal{N}(-2,5)$
  - What is $P(|X| < 4)$?

Exponential Random Variables

Recall: Geometric Random Variables

- Suppose $X \sim \text{Geometric}(p)$
  - What is the PMF of X?
  - Memoryless property?

- Question: Is there a continuous counterpart of a geometric random variable?

3. Exponential Random Variables

Exponential Random Variables: A random variable X is exponential with parameters $\lambda > 0$ if its PDF is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{, if } x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

- How to plot the PDF of Exp($\lambda = 1$)?

3. Exponential Random Variables

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{, if } x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

- What is the CDF of X?

Memoryless Property

- Suppose $X \sim \text{Exp}(\lambda)$
  - What is $P(X > s + t \,|\, X > t)$?

Example: Nokia 3310

- Example: Suppose the lifetime of a Nokia 3310 is an exponential random variable with mean = 10 years.
  - Suppose a Nokia 3310 was bought 15 years ago.
  - $P(\text{it will last another 5-10 years})$?

Exponential Distribution: A Good Model for Occurrence of Events

- Communication networks: Inter-arrival time between two data packets

- Survival analysis: User’s lifetime (App, social network…)

- Reliability modeling: Amount of time until the hardware on AWS EC2 fails

1-Minute Summary

1. Continuous Random Variables

- Probability density function
- PDF vs CDF

2. Special Continuous Random Variables

- Uniform and Inverse Transform Sampling (ITS)
- Standard Normal
- Exponential and Memoryless Property