11.11 Application of Taylor polynomials

1. Approximating function
2. Application to Physics

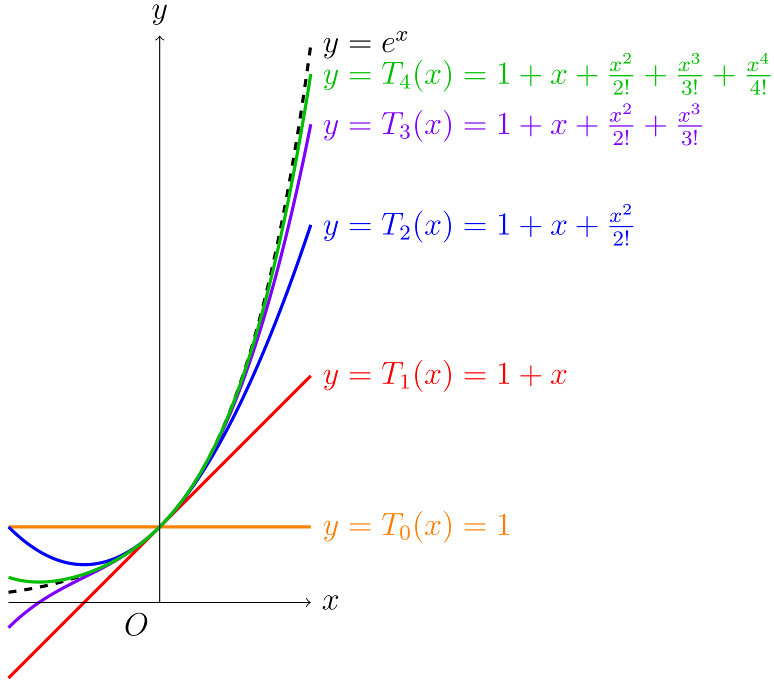
0.1 Approximating function

Suppose $f(x)$ is the sum of its Taylor series at a,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Then $\lim_{n\to\infty} T_n(x) = f(x)$ and hence $f(x) \approx T_n(x)$.

$T_1(x) = f(a) + f'(a)(x-a)$: linear approximation (一次) 線性逼近。



Note: 估計誤差的方法:

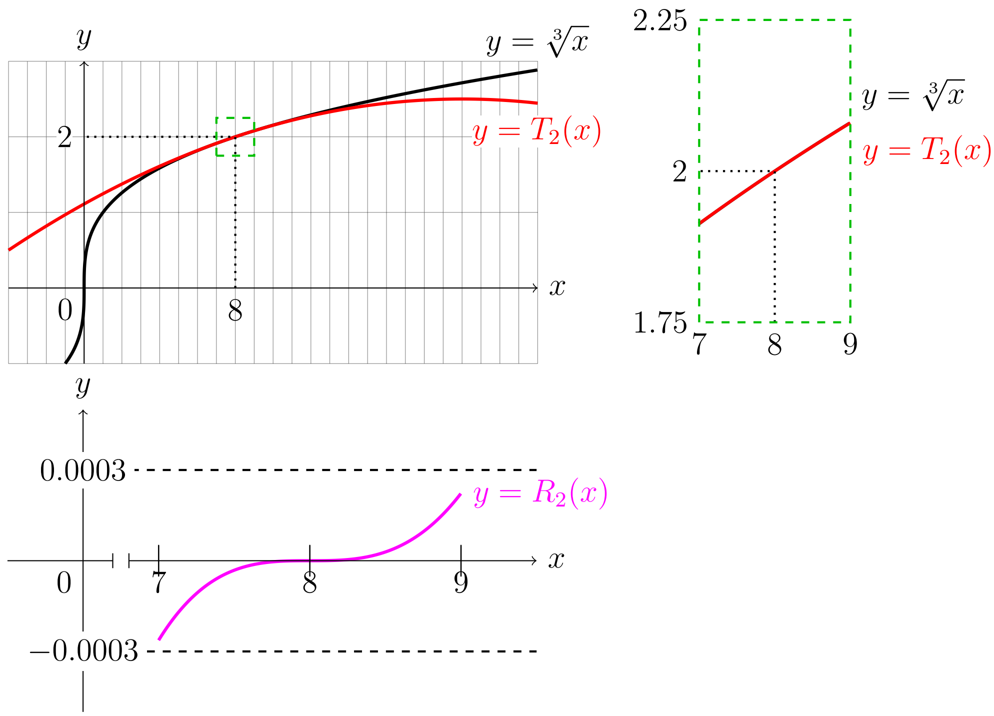
1. 畫 $R_n(x) = f(x) - T_n(x)$ 看圖;
2. 如果交錯, 用 Alternating Series Estimation Theorem, $\dfrac{|f^{(n+1)}(a)|}{(n+1)!}|x-a|^{n+1}$;
3. 用 Taylor's Inequality, $|R_n(x)| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$, $|f^{(n+1)}(x)| < M$.

Example 0.1 *(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.*
(b) How accurate is this approximation when $7 \le x \le 9$?

$$f'(x) = \frac{1}{3}x^{-2/3},\ f''(x) = -\frac{2}{9}x^{-5/3},\ f'''(x) = \frac{10}{27}x^{-8/3}\ \text{(估計誤差用)}.$$

$$\begin{aligned} \sqrt[3]{x} \ \approx\ T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 \end{aligned}$$

$\because$ *$x < 8$ is not alternating, use Taylor's inequality.*
For $7 \le x \le 9$, $|f'''(x)| \le |f'''(7)| = \frac{10}{27} \cdot 7^{-8/3} < 0.0021$.
Choose $M = 0.0021$, $|R_2(x)| \le \frac{M}{3!}|x-8|^3 \le \frac{0.0021}{6} \cdot 1^3 < 0.0004$.
(用畫圖軟體的會發現 $|R_2(x)| < 0.0003$.) ■



Example 0.2 *(a) What is the maximum error possible in using the approximation* $\sin x \approx x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ *when* $-0.3 \le x \le 0.3$*? Use this to find* $\sin 12°$ *correct to six decimal places.*
(b) For what value of x *is this approximation accurate to within 0.00005?*

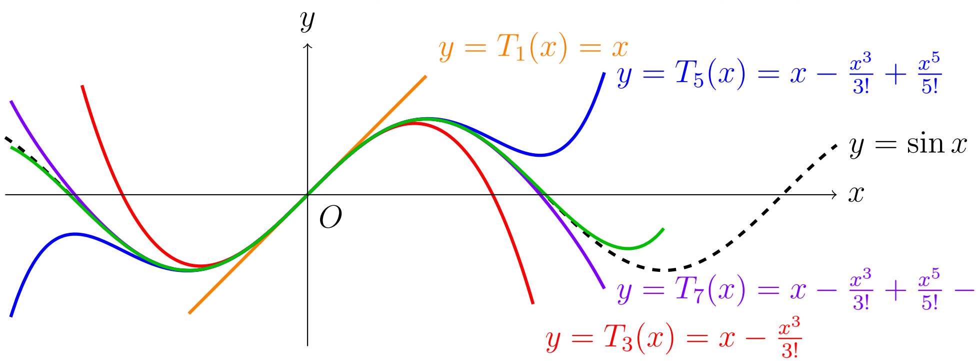
(a) By the ASET, error $\le \left|\dfrac{x^7}{7!}\right| = \dfrac{(0.3)^7}{7!} \approx 4.3 \times 10^{-8}$.

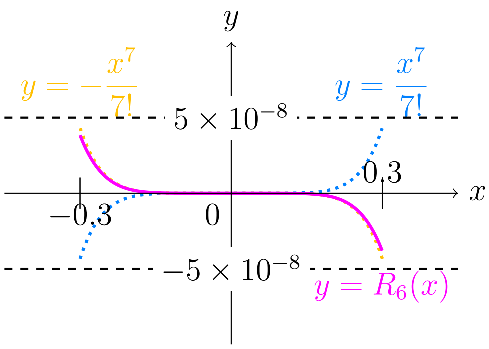
$\sin 12° = \sin\dfrac{\pi}{15} \approx \dfrac{\pi}{15} - \dfrac{1}{3!}(\dfrac{\pi}{15})^3 + \dfrac{1}{5!}(\dfrac{\pi}{15})^5 \approx 0.20791169 \approx 0.207912.$

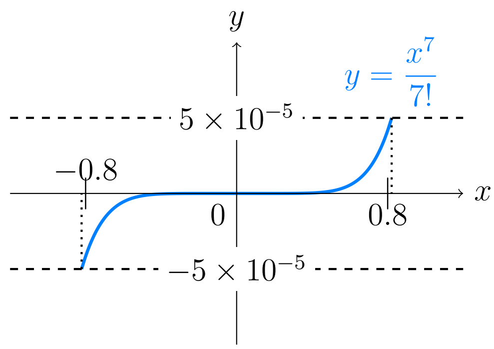
(b) $\dfrac{|x^7|}{7!} < 0.00005$, $|x| < (0.252)^{1/7} \approx 0.82$. ■

Note: 1. 如果用 Taylor Inequality, $R_6(x) = \sin x - \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}\right)$, $|(\sin x)^{(7)}| \le 1$, $|R_6(x)| \le |x|^7/7!$, 跟 ASET 一樣準。

2. $\because \sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $T_1(x) = T_2(x) = x$, $T_3(x) = T_4(x) = x - \dfrac{x^3}{3!}$, $T_5(x) = T_6(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$, $T_7(x) = T_8(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$.







0.2 Application to physics

Einstein's Theory 愛因斯坦理論

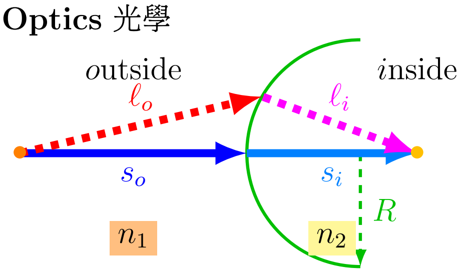
$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

m_0 靜止質量, m 速度 v 的質量, c 光速。

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x)^n = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots \quad (|x| < 1)$$

$$\begin{aligned}
K &= mc^2 - m_0c^2 \qquad \text{(kinetic energy 動能 等於 總能量差)} \\
&= m_0c^2 \left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right] \qquad \left(\left|\frac{v^2}{c^2}\right| < 1\right) \\
&= m_0c^2 \left\{ \left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \dots \right] - 1 \right\} \\
&= m_0\cancel{c^2} \left(\frac{1}{2}\frac{v^2}{\cancel{c^2}} + \underline{\frac{3}{8}\frac{v^4}{c^4} + \dots} \right) \qquad (\underline{\text{忽略不計}}) \\
&\approx \frac{1}{2}m_0v^2 \qquad \text{(一階逼近, 得到 } \textbf{\textit{Newtonian physics}} \text{ 牛頓物理。)}
\end{aligned}$$

Optics 光學



Eugene Hecht 由 Fermat's principle: 光線走最快路線, 推得:

$$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o} \right).$$

R: 球面半徑, n_1, n_2: indexes of refraction 折射係數。

$\because \sin x = x - \frac{x^3}{3!} + \dots$, $\cos x = 1 - \frac{x^2}{2!} + \dots$.

When $\theta \approx 0$, $\sin\theta \approx \theta$, $\cos\theta \approx 1$, $\ell_o \approx s_o$, $\ell_i \approx s_i$.

一階逼近, 得到 ***Gaussian optics*** 高斯光學 (***first-order optics***):

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}.$$