# 11.3 Integral test and estimates of sums

1. integral test $\int_1^\infty f(x)\ dx \iff \sum a_n$.
2. estimate sums $\int_{n+1}^\infty f(x)\ dx \le R_n = s - s_n \le \int_n^\infty f(x)\ dx$

如果能把部分和 $\{s_n\}$ 寫成 $n$ 的公式: $\sum a_n = \lim_{n\to\infty} s_n$.

ex: $\sum \frac{1}{n(n+1)} = 1$, $\sum ar^{n-1} = \frac{1}{1-r}$ for $|r| < 1$.

或是已知的加減常數倍: $\sum (a_n + cb_n) = \sum a_n + c \sum b_n$.

大多的級數和很難算, 但可以:

1. Test 檢驗: 判斷收斂/發散 — 有收斂再試著去算。
2. Estimate 估計 $\sum a_n$.

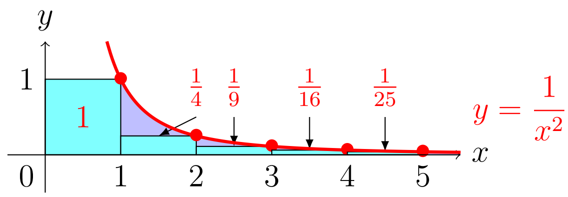
Series with positive term ($a_n > 0$) or non-negative term ($a_n \ge 0$)
正項或非負級數有五種檢驗法:

1. Integral Test 積分 § 11.3
2. Comparison Test 比較 § 11.4
3. Limit Comparison Test 極限比較 § 11.4
4. Root Test 開根 § 11.6
5. Ratio Test 比值 § 11.6

有負項的呢? Alternating 交錯 § 11.5 & Absolute 絕對 § 11.6

## 0.1 Integral test

$\sum \frac{1}{n^2} =?$ Consider $f(x) = \frac{1}{x^2}$:



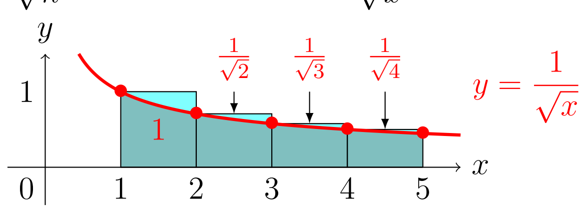

$$
\begin{aligned}
s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots + \frac{1}{n^2} \\
&< \frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\, dx = 2. \qquad \left(\int_1^\infty \frac{dx}{x^2} \text{ is convergent}\right) \\
&\quad \left\{1 + \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = 1 + \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = 1 + \lim_{t\to\infty}(1 - \frac{1}{t})\right\}
\end{aligned}
$$

$\{s_n\}$ monotonic (increasing) and bounded (by 2).

By the Monotone Convergence Theorem, $\sum \frac{1}{n^2} (< 2)$ is convergent.

♦ **Fact:** 1735, Euler: $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

$\sum \frac{1}{\sqrt{n}} =?$ Consider $f(x) = \frac{1}{\sqrt{x}}$:

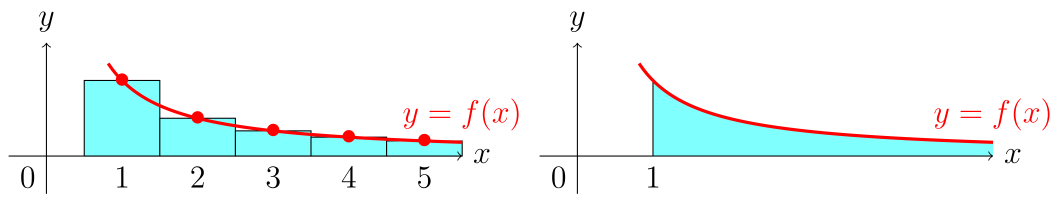



$$
\begin{aligned}
s_n &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} \\
&> \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx = \Big[2\sqrt{x}\Big]_1^{n+1} = 2\sqrt{n+1} - 2.
\end{aligned}
$$

Since $2\sqrt{n+1} - 2 \to \infty$ as $n \to \infty$, ( $\Longrightarrow \int_1^\infty \frac{1}{\sqrt{x}}\, dx$ is divergent)

$s_n \to \infty$ as $n \to \infty$, $\Longrightarrow \sum \frac{1}{\sqrt{n}}$ is divergent.

**Theorem 1 (Integral Test)**
*Suppose $f$ is a* ***continuous****,* ***positive****,* ***decreasing*** *function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum a_n$ is convergent if and only if the improper integral $\displaystyle\int_1^\infty f(x)\,dx$ is convergent.*



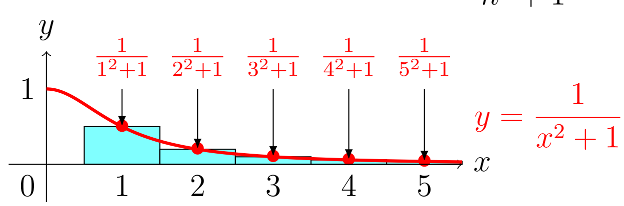

(你收斂就是我收斂, 我收斂就是你收斂, 啊哈! 發散啦!)

**Note:** 1. 不用從 1 而終, ex: $\displaystyle\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \longleftrightarrow \int_4^\infty \frac{1}{(x-3)^2}\,dx.$

2. 要連續, 要正, 不一定要全遞減 — 只要 ***ultimately decreasing*** 終極遞減。
($\exists N \in \mathbb{N} \ni f \searrow \forall\, x \geq N \longleftrightarrow \displaystyle\sum_{n=N}^{\infty} a_n$: 考慮會遞減的部分就好)

3. 只保證收斂/發散一樣, <u>不保證值</u>一樣。$\displaystyle\sum a_n \neq \int_1^\infty f(x)\,dx.$

**Example 0.1** *Test the series $\displaystyle\sum \frac{1}{n^2+1}$ for convergence or divergence.*




*(找函數)* $\displaystyle\frac{1}{x^2+1}$ *continuous, positive, decreasing on* $[1, \infty)$. *(*$\overset{\text{廉政地檢}}{\text{連正遞減}}$*)*

*(瑕積分)* $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1} x \Big|_1^t$
$\displaystyle= \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$

*(做判斷) By the Integral Test,* $\displaystyle\sum \frac{1}{n^2+1}$ *is convergent.* ■

♦ **Fact:** $\displaystyle\sum \frac{1}{n^2+1} = \frac{\pi}{2}\coth\pi - \frac{1}{2} = \frac{\pi}{2}\frac{e^{2\pi}+1}{e^{2\pi}-1} - \frac{1}{2}.$

**Example 0.2** *For what value of $p$ is the series $\sum \frac{1}{n^p}$ convergent?*

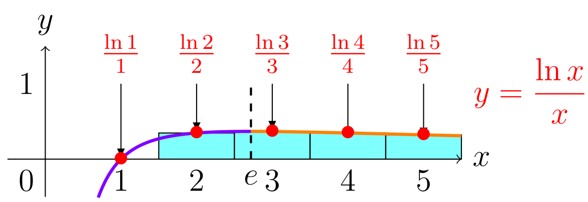
*If $p<0$, $\lim\limits_{n\to\infty}\frac{1}{n^p}=\infty\neq 0$; if $p=0$, $\lim\limits_{n\to\infty}\frac{1}{n^p}=1\neq 0$.*
$\Longrightarrow$ *divergent by the Test for Divergence ($\lim\limits_{n\to\infty}a_n\neq 0$).*
*For $p>0$, $\frac{1}{x^p}$ is continuous, positive, decreasing on $[1,\infty)$.*
*Since $\int_1^\infty \frac{1}{x^p}\,dx$ converges $\iff$ $p>1$. By the Integral Test,*
*$\sum \frac{1}{n^p}$ is convergent for $p>1$ and divergent for $0<p\le 1$.* ■

> **Fact:** The ***p*-series (*p*-級數)**:
>
> $$\sum \frac{\mathbf{1}}{\boldsymbol{n^p}} \text{ is } \begin{cases} \text{convergent} & \text{if } \boldsymbol{p>1}; \\ \text{divergent} & \text{if } \boldsymbol{p\le 1}. \end{cases}$$
>
> When $p=1$, it is the harmonic series $\sum \frac{1}{n}$.

**Example 0.3** $\sum \frac{1}{n^3}$ *converges ($3>1$), $\sum \frac{1}{\sqrt[3]{n}}$ diverges ($1/3\le 1$).*

**Example 0.4** *Determine whether the series $\sum \frac{\ln n}{n}$ converges or diverges.*

*(找函數) $f(x)=\frac{\ln x}{x}$ is continues and positive for $x>1$. (遞減?)*
$f'(x)=\dfrac{\frac{1}{x}\cdot x-\ln x\cdot 1}{x^2}=\dfrac{1-\ln x}{x^2}<0$ *when $x>e\approx 2.71828$.*
*(瑕積分)* $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx=\lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx=\lim_{t\to\infty}\frac{(\ln x)^2}{2}\Big|_1^t=\lim_{t\to\infty}\frac{(\ln t)^2}{2}=\infty.$
*(做判斷) By the Integral Test, $\sum \frac{\ln n}{n}$ diverges. (有限項不影響)* ■




$$\begin{array}{ccc} \displaystyle\int_1^\infty f(x)\,dx & & \displaystyle\sum a_n \\ \Updownarrow & & \Updownarrow \\ \displaystyle\int_3^\infty f(x)\,dx & \overset{I.T.}{\Longleftrightarrow} & \displaystyle\sum_{n=3}^\infty a_n \end{array}$$

**Skill:** 直接看遞減不容易, 可以看導數: $f'(x)<0$ for $x>a$ for some $a$.

## 0.2 Estimate sums

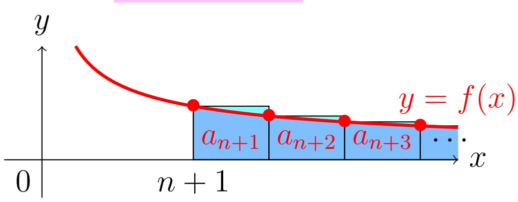
**Define:** For convergent series $\sum a_n$, let $s = \sum a_n$ and $s_n$ be the $n$-th partial sum, then the ($n$-th) ***remainder*** 剩餘項 $R_n = s - s_n$ is the error using $s_n$ to approximate $s$. (收斂級數用剩餘項當作以部分和逼近總和的誤差。)

$$\sum a_n = \overbrace{a_1 + a_2 + \cdots + a_n}^{s_n} + \overbrace{a_{n+1} + \cdots}^{R_n} = s.$$

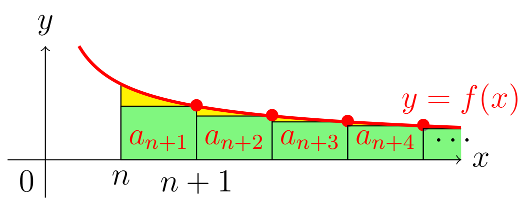
**Theorem 2 (Remainder Estimate for the Integral Test)**
*$f$ is continuous, positive, decreasing for $x \geq n$ with $f(k) = a_k$, and $\sum a_n$ is convergent. If $R_n = s - s_n$, then (比上不足, 比下有餘。)*

$$\boxed{\int_{n+1}^{\infty} f(x)\ dx \leq R_n \leq \int_{n}^{\infty} f(x)\ dx}$$

**Proof.** $\displaystyle\int_{n+1}^{\infty} f(x)\ dx \leq R_n = a_{n+1} + a_{n+2} + \ldots \leq \int_{n}^{\infty} f(x)\ dx.$ ■




(往右畫矩形, 包含住 $y = f(x)$.)




(往左畫矩形, 包含在 $y = f(x)$.)

$$\int_{n+1}^{\infty} f(x)\ dx \leq R_n = s - s_n \leq \int_{n}^{\infty} f(x)\ dx$$

$$\boxed{s_n + \int_{n+1}^{\infty} f(x)\ dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\ dx}$$

**Example 0.5** *(a) Approximate the sum of the series* $\sum \frac{1}{n^3}$ *by using the sum of the first 10 terms. Estimate the error involved in this approximation.*
*(b) How many terms are required to ensure that the sum is accurate to within 0.0005?*
*(c) Estimate the sum with* $n = 10$.

*(a)* $s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{10^3} \approx 1.1975.$ $\frac{1}{x^3}$ *satisfies the condition of the Integral Test.* $R_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{200} = 0.005.$
*(b)* $R_n \le \int_{n}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2} < 0.0005,\ n > \sqrt{1000} \approx 31.6,$ *choose* $n = 32.$
*(c)* $s_{10} + \int_{11}^{\infty} \frac{1}{x^3}\, dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\, dx,\ 1.201664 \le s \le 1.202532,$
$\sum \frac{1}{n^3} \approx \overset{(1.201664+1.202532)/2}{(1.202098 \approx)} 1.2021$ *with error* $< \overset{1.202532-1.202098}{(0.000434 <)} 0.0005.$ ■

♦ **Proof of the Integral Test.**

Let $s_n = \sum_{i=1}^{n} a_i$.

($\Rightarrow$) If $\int_1^{\infty} f(x)\, dx$ is convergent, left $M = a_1 + \int_1^{\infty} f(x)\, dx$.
$\because f > 0,\ s_{n+1} = s_n + a_{n+1} = s_n + f(n+1) > s_n$ (increasing);
$\because f \searrow,\ s_n \le a_1 + \int_1^n f(x)\, dx \le a_1 + \int_1^{\infty} f(x)\, dx = M$ (bounded above).
$\therefore$ By the Monotone Convergence Theorem, $\{s_n\}$ and hence $\sum a_n$ converges.

($\Leftarrow$) If $\int_1^{\infty} f(x)\, dx$ is divergent, $s_n \ge \int_1^{n+1} f(x)\, dx \to \infty$ as $n \to \infty$,
$\lim_{n\to\infty} s_n = \infty$, $\sum a_n$ diverges. ■