

5.3 The Fundamental Theorem of Calculus

Theorem 1 Suppose f is **continuous on $[a, b]$** . 閉連續

$$(1) \quad g(x) = \int_a^x f(t) dt \implies g'(x) = f(x)$$

$$(2) \quad F'(x) = f(x) \implies \int_a^b f(x) dx = F(b) - F(a)$$

前言：定積分是黎曼和的極限，如果函數長得很複雜，極限就會很難算。
有沒有一個好的方法可以用來算定積分？

0.1 TFTC (1)

Theorem 2 (The Fundamental Theorem of Calculus, Part 1)

If f is **continuous on $[a, b]$** 閉連續, then

$$g(x) = \int_a^x f(t) dt$$

(1) g is **continuous on $[a, b]$** 閉連續 and

(2) **differentiable on (a, b)** 開可微, and

(3)

$$g'(x) = f(x)$$

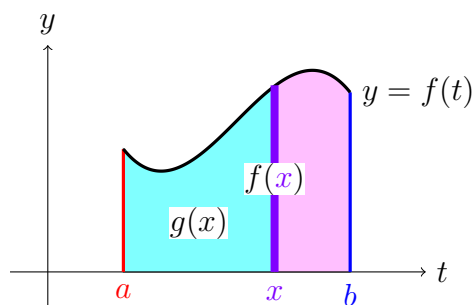
$\because f$ 連續 (on $[a, x]$) 所以可積,

$\therefore g$ 是可以定義的 (黎曼和極限存在)。

當 f 都是正的 (> 0 , 或非負 ≥ 0),

$g(x)$ 代表 f 以下從 a 到 x 的面積。

$g'(x)$ 代表在 x 下一個瞬間增加的面積
(改變率), 就是 $f(x)$ 的長度。



Leibniz notation: 先積一遍, 再微不變。

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Note: f 要閉連續, x 要一致, t 也要一致。

Proof of TFTC (1).

(1) “ g is differentiable on (a, b) , and hence is continuous on (a, b) .”

For $x \in (a, b)$,

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \quad (\text{導數定義}) \\ &= \lim_{h \rightarrow 0} \left\{ \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right] \right\} \\ &= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(t) dt}{h} \quad (\text{性質 7. 分段}) \end{aligned}$$

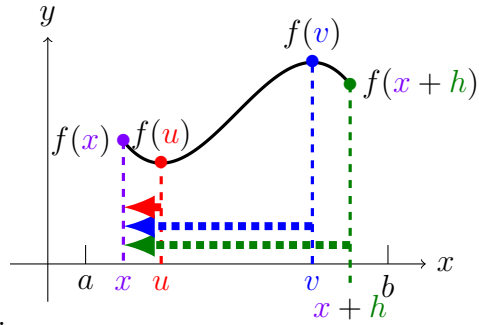
假設 $h > 0$ 而且 $x+h \leq b$ 。

$\because f$ 在 $[x, x+h] \subseteq [a, b]$ 連續, by the Extreme Value Theorem,

$\exists u, v \in [x, x+h], \exists f$ 有最小值 $f(u)$ 與最大值 $f(v)$ 。

根據定積分的性質 (§5.2, 10. 上下界),

$$\begin{aligned} f(u)h &\leq \int_x^{x+h} f(t) dt \leq f(v)h, \\ f(u) &\leq \frac{\int_x^{x+h} f(t) dt}{h} \leq f(v). \end{aligned}$$



只要 $h \rightarrow 0$ ($x+h \rightarrow x$), 會有 $u, v \rightarrow x$;

因為 f 在 x 連續 $\implies f(u), f(v) \rightarrow f(x)$.

By the Squeeze Theorem, $\lim_{h \rightarrow 0^+} \frac{\int_x^{x+h} f(t) dt}{h} \stackrel{S.T.}{=} f(x)$.

對 $h < 0$ ($x+h \geq a$) 類似。

Therefore, g is differentiable on (a, b) and $g'(x) = f(x)$.

一舉三得

(2) “ $g(x)$ is continuous from the right at a and from the left at b .”

For $x \in (a, b)$, 因為 f 在 $[a, x]$ 連續, by the Extreme Value Theorem,

$\exists u, v \in [a, x] \ni f(u)(x-a) \leq \int_a^x f(t) dt \leq f(v)(x-a)$.

只要 $x \rightarrow a^+$, 會有 $u, v \rightarrow a^+$; 因為 f 在 a 右連續 $\implies f(u), f(v) \rightarrow f(a)$.

By the Squeeze Theorem,

$$\lim_{x \rightarrow a^+} g(x) = \lim_{x \rightarrow a^+} \int_a^x f(t) dt \stackrel{S.T.}{=} 0 = \int_a^a f(t) dt = g(a).$$

對 $x \rightarrow b^-$ 類似: $\lim_{x \rightarrow b^-} g(x) = \lim_{x \rightarrow b^-} \int_a^x f(t) dt \stackrel{S.T.}{=} \int_a^b f(t) dt = g(b)$.

Therefore, g is continuous on $[a, b]$. ■

Example 0.1 Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2} dt$.

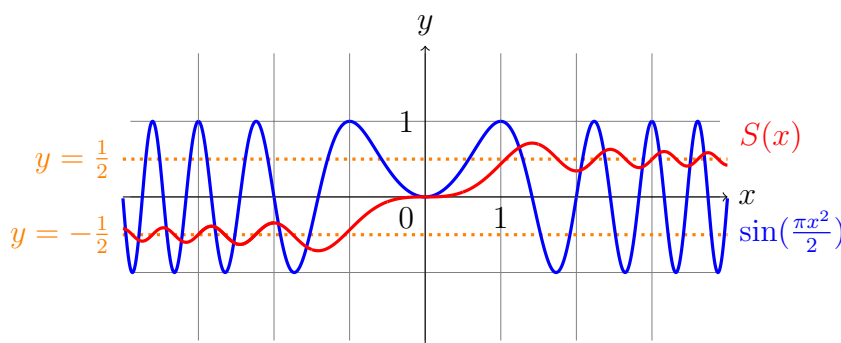
$\because \sqrt{1+x^2}$ is continuous on \mathbb{R} , by T FTC part 1, $g'(x) = \sqrt{1+x^2}$. ■

Example 0.2 Find the derivative of Fresnel (Sine) Function

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt.$$

French physicist Augustin Fresnel [frei`nɛl] 菲涅耳: theory of diffraction of light wave 光波繞射理論. (Other is $C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt$.)

$\because \sin\left(\frac{\pi x^2}{2}\right)$ is continuous on \mathbb{R} , by T FTC part 1, $S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$. ■



Example 0.3 (Chain Rule) Find $\frac{d}{dx} \int_1^{x^4} \sec t dt$.

Let $u = x^4$,

$$\begin{aligned} \frac{d}{dx} \int_1^{x^4} \sec t dt &= \frac{d}{dx} \int_1^u \sec t dt && (\text{對 } u \text{ 的函數微 } x) \\ &= \frac{d}{du} \left[\int_1^u \sec t dt \right] \cdot \frac{du}{dx} && (\text{前項用 T FTC}) \\ &= \sec u \cdot (x^4)' && (\text{別忘代回 } u = x^4) \\ &= 4x^3 \sec(x^4). \end{aligned}$$

($4x^3$ 放 $\sec x^4$ 前面可以省略 “ \cdot ”, 放後面不可省 $\sec x^4 \cdot 4x^3$ 。) ■

Skill: 不一致, 就要設新變數, 再用 Chain Rule 讓他一致。

0.2 T FTC (2)

Theorem 3 (The Fundamental Theorem of Calculus, Part 2)

If f is **continuous on** $[a, b]$ 閉連續, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is any **antiderivative** of f , i.e. $F' = f$.

算積分雖可以用 Riemann sum 逼近, 但太複雜, 極限也不好求。

但是, 用 T FTC (2) 只要找反導數代上界減代下界。

Leibniz notation: 先微再積, 代上減下。

$$\int_a^b \frac{d}{dx} F(x) dx = F(b) - F(a)$$

Attentin: 不是隨便的 F 都可以, F 要是個閉連續函數的反導數。

Note: 其他寫法: (不推薦單邊中括號的寫法。)

$$F(b) - F(a) = \left[F(x) \right]_a^b = \left[F(x) \right]_a^b = \left[F(x) \right]_a^b.$$

Proof of T FTC (2).

Let $g(x) = \int_a^x f(t) dt$. By T FTC part 1, $g'(x) = f(x)$ (g 是 f 的反導數).

So $F(x) = g(x) + C$, is continuous on $[a, b]$ and differentiable on (a, b) as g .
Therefore,

$$\begin{aligned} F(b) - F(a) &= (g(b) + C) - (g(a) + C) \\ &= \int_a^b f(t) dt - \int_a^a f(t) dt \\ &= \int_a^b f(x) dx. \quad (\text{定積分 } t \text{ 換成 } x \text{ 都一樣。}) \end{aligned}$$

(C 可以是任意常數, 但是計算時是同一個。)

Fact: 閉區間 $[a, b]$ 上連續函數的反導數也連續, 而且可微分 (on (a, b))。 ■

Example 0.4 $\int_1^3 e^x dx = ?$

$\because e^x$ is an antiderivative of e^x ($e^x + C$ is the most general one),

$$\therefore \int_1^3 e^x dx = e^x \Big|_1^3 = e^3 - e^1 = e^3 - e. \quad \blacksquare$$

Example 0.5 Find the area under $y = x^2$ from 0 to 1. ($\int_0^1 x^2 dx = ?$)

$$\because \frac{x^3}{3} \text{ is an antiderivative of } x^2, \therefore \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}. \quad \blacksquare$$

Example 0.6 $\int_3^6 \frac{dx}{x} (= \int_3^6 \frac{1}{x} dx) = ?$

$\because \ln x$ is an antiderivative of $\frac{1}{x}$ for $x > 0$,

$$\therefore \int_3^6 \frac{1}{x} dx = \ln x \Big|_3^6 = \ln 6 - \ln 3 = \ln \frac{6}{3} = \ln 2. \quad \blacksquare$$

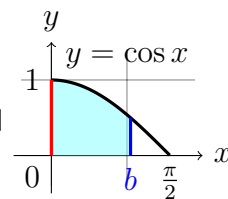
Note: $\int_a^b \frac{dx}{x} = \int_a^b \frac{1}{x} dx$ 是習慣的寫法, 不可以約掉 x ; $\ln |x| + C$ is the

most general one for $x \neq 0$, 因為 $[3, 6] \subseteq (0, \infty)$, 這裡只要 $\ln x$ 就好。

Example 0.7 Find the area of cosine curve from 0 to b , where $0 \leq b \leq \frac{\pi}{2}$.

$\because \sin x$ is an antiderivative of $\cos x$,

$$\therefore \int_0^b \cos x dx = \sin x \Big|_0^b = \sin b - \sin 0 = \sin b. \quad \blacksquare$$



Example 0.8 What wrong with the calculation?

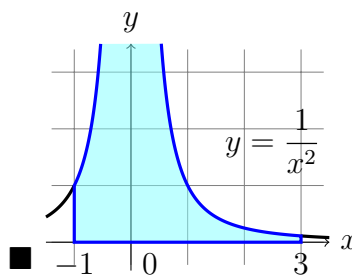
$$\int_{-1}^3 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}$$

應該是正的。

$\because \frac{1}{x^2}$ is **NOT** continuous on $[-1, 3]$,

\therefore TFTC does not hold.

In fact, $\int_{-1}^3 \frac{1}{x^2} dx$ does not exist.



0.3 T FTC

Theorem 4 Suppose f is **continuous on** $[a, b]$. 閉連續

$$g(x) = \int_a^x f(t) dt \implies g'(x) = f(x)$$

$$F'(x) = f(x) \implies \int_a^b f(x) dx = F(b) - F(a)$$

Recall: Leibniz notation:

先積一遍, 再微不變。(f 要在 $[a, b]$ 連續)

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

先微再積, 代上減下。(F' 要在 $[a, b]$ 連續)

$$\int_a^b \frac{d}{dx} F(x) dx = F(b) - F(a)$$

Additional: $f(x)$ 閉連續 $\implies g(x) = \int_a^x f(t) dt$ 是閉連續開可微。而且 $f(x)$ 的反導數 $F(x) = g(x) + C$, 所以也是閉連續開可微。

但是反過來, 一個閉連續開可微的函數 $G(x)$, 它的導函數 $G'(x)$ **不一定**是閉連續; 就是說, $G(x)$ 不一定是個閉連續函數的反導數。所以 (不適用 T FTC)

$$\int_a^b \frac{d}{dx} G(x) dx \not\asymp G(b) - G(a)$$

例如: $G(x) = \sqrt{x}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$, but

$G'(x) = \frac{1}{2\sqrt{x}}$ is continuous on $(0, 1]$, **不能用** T FTC: $\int_0^1 \frac{dx}{2\sqrt{x}} \not\asymp \sqrt{x} \Big|_0^1$.

(此例要用 §7.8 瑕積分:

$$\int_0^1 \frac{dx}{2\sqrt{x}} = \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{2\sqrt{x}} = \lim_{t \rightarrow 0^+} \sqrt{x} \Big|_t^1 = \sqrt{1} - \lim_{t \rightarrow 0^+} \sqrt{t} = 1 - 0 = 1.)$$

後言: Barrow (Newton's teacher) 發現:

求切線 (微分) 與 求面積 (積分) 互為逆程序 (inverse process)。

Newton & Leibniz 發展出一套有系統的方法, 並建立學說 (極限, 黎曼和, ... 等)。積分可以用 $\lim \sum$ 來算, 但是用反導數會更好算; 我們將會介紹其他的技巧。(§ 7)