1.4 Exponential functions

1. exponent 指數 a^b
2. exponential function 指數函數 $f(x) = a^x$
3. number e 歐拉數

0.1 Exponent

Define: $\boxed{a^b}$ a 的 b 次方 ("a to the (power of) b", "a to the b-th power", or "the b-th power of a"), a 稱爲底數 (***base***), b 稱爲指數 (***exponent***)。 $2^\heartsuit$
(b 個 a 相加 $(+)$ 等於 a 乘以 b $(a \times b)$; b 個 a 相乘 $(\times)$ 等於 a 的 b 次方 (a^b)。)

	$2 \times n$	$\overbrace{2 + \ldots + 2}^{n}$
$\mathbb{N}$	2×3	$2 + 2 + 2 = 6$
$\mathbb{Z}$	2×0	0(有加跟沒加一樣)
	$2 \times (-3)$	$0 - 2 - 2 - 2 = -6$
$\mathbb{Q}$	$2 \times \frac{1}{3}$	$\frac{2}{3} = 2/3$
$\mathbb{R}$	$2 \times \sqrt{3}$	$\approx 2 \times 1.732 = 3.464$

	2^n	$\overbrace{2 \times \ldots \times 2}^{n}$
$\mathbb{N}$	2^3	$2 \times 2 \times 2 = 8$
$\mathbb{Z}$	2^0	1(有乘跟沒乘一樣)
	2^{-3}	$1 \div 2 \div 2 \div 2 = \frac{1}{8} = \frac{1}{2^3}$
$\mathbb{Q}$	$2^{\frac{1}{3}}$	$\sqrt[3]{2}$
$\mathbb{R}$	$2^{\sqrt{3}}$	$\approx 2^{1.732} \approx 3.322$

$$3 < \pi < 4 \qquad 2^{3} < 2^\pi < 2^{4}$$
$$3.1 < \pi < 3.2 \qquad 2^{3.1} < 2^\pi < 2^{3.2}$$
$$3.14 < \pi < 3.15 \qquad 2^{3.14} < 2^\pi < 2^{3.15}$$
$$3.141 < \pi < 3.142 \qquad 2^{3.141} < 2^\pi < 2^{3.142}$$
$$3.1415 < \pi < 3.1416 \qquad 2^{3.1415} < 2^\pi < 2^{3.1416}$$
$$3.14159 < \pi < 3.14160 \qquad 2^{3.14159} < 2^\pi < 2^{3.14160}$$
$$3.141592 < \pi < 3.141593 \qquad 2^{3.141592} < 2^\pi < 2^{3.141593}$$
$$3.1415926 < \pi < 3.1415927 \qquad 2^{3.1415926} < 2^\pi < 2^{3.1415927}$$

(存在無窮多的有理數逼近 $\pi \implies$ 無窮多的 $2^{\text{有理數}}$ 逼近 2^π。)

♦: **Story of** π: 劉徽: $\pi \approx 3.14$ 稱爲徽率; 祖沖之: $3.1415926 < \pi < 3.141927$; 以 $\dfrac{22}{7}$ 爲約率, $\dfrac{355}{113} \approx 3.1415929$ 爲密率, 稱爲祖率。

Note: $\sqrt{2} \times \sqrt{3} = \sqrt{6} \approx 2.449$, $\sqrt{2}^{\sqrt{3}} \approx 1.414^{1.732} \approx 1.823$.
(Check by yourself: which is large: 2^3 v.s. 3^2? $\frac{1}{2}^{\frac{1}{3}}$ v.s. $\frac{1}{3}^{\frac{1}{2}}$? $\sqrt{2}^{\sqrt{3}}$ v.s. $\sqrt{3}^{\sqrt{2}}$?)

Law of exponents 指數律: $(a > 0,\ b, c \in \mathbb{R})$

1 加: $a^{b+c} = a^b \times a^c$; $\quad (\overbrace{a \times \cdots \times a}^{b}) \times (\overbrace{a \times \cdots \times a}^{c}) = \overbrace{a \times \cdots \times a}^{b+c} = a^{b+c}$

2 減: $a^{b-c} = a^b \div a^c$; $\quad (\overbrace{a \times \cdots \times a}^{b}) \div (\overbrace{a \times \cdots \times a}^{c}) = a^{b-c}$

3 乘: $a^{bc} = (a^b)^c$; $\quad \overbrace{(\underbrace{a \times \cdots \times a}_{b}) \times \cdots \times (\underbrace{a \times \cdots \times a}_{b})}^{c} = a^{bc}$

4 除: $a^{b/c} = \sqrt[c]{a^b} = (\sqrt[c]{a})^b$; $\quad \overbrace{\sqrt[c]{a^b} \times \cdots \times \sqrt[c]{a^b}}^{c} = a^b$

5 分配: $(ab)^c = a^c b^c$; $\quad \overbrace{ab \times \cdots \times ab}^{c} = (\overbrace{a \times \cdots \times a}^{c}) \times (\overbrace{b \times \cdots \times b}^{c}) = a^c b^c$

6 零: $\boxed{\boldsymbol{a^0 := 1}}$, $\boxed{\mathbf{0}^{\boldsymbol{b}} \mathbf{:= 0}}$, $\boxed{\mathbf{0^0}}$: undetermined(未定).

Ex:

$$
\begin{aligned}
2^{3+5} &= 2^3 \times 2^5 = 2^8, \\
2^{3-5} &= 2^3 \div 2^5 = 2^{-2} = \frac{1}{2^2}, \\
2^{3\times 5} &= (2^3)^5 = 2^{15}, \\
2^{3/5} &= \sqrt[5]{2^3} = \sqrt[5]{2}^{\,3}, \\
(2 \times 3)^5 &= 2^5 \times 3^5 = 2^5 \cdot 3^5 = 2^5 3^5.
\end{aligned}
$$

Note: (方根唸法) For $0 < a \in \mathbb{R}$,
$\sqrt{a}(= \sqrt[2]{a})$: a 的平方根 (“square root of a”);
$\sqrt[3]{a}$: a 的立方根 (“cube root of a”);
$\sqrt[n]{a}$: a 的 n 次方根 (“the n-th root of a”).

Extend: For $0 < a \in \mathbb{R}$ and $n \in \mathbb{Z}$, $(-a)^n = \begin{cases} a^n & \text{if } n \text{ is even;} \\ -a^n & \text{if } n \text{ is odd.} \end{cases}$

Ex: $(-1)^0 = (-1)^2 = 1$, $(-1)^1 = (-1)^{-1} = -1$.

0.2 Exponential function

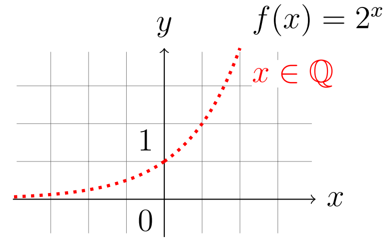
$f(x) = x^n$, $n \in \mathbb{N}$ ***power function*** 冪次函數 ($x^{1/2} = ?$ $x^{\sqrt{2}} = ?$ $x^x = ?$)
Define: The ***exponential function of base*** $a (> 0)$ 指數函數 (以 a 爲底)

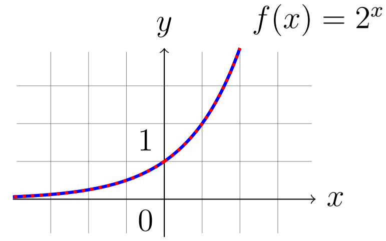
$$\boxed{\boldsymbol{f(x) = a^x}}$$ ←變數在指數
← 底數大於零

Question: Why $a > 0$?
Answer: $(-1)^{1/2} = \sqrt{-1} := i$, 0^{-1} undefined, $\therefore$ 只考慮正底數 $(a > 0)$。

Question: How to define a^x?

1. $x = n \in \mathbb{N}$, $a^n := \overbrace{a \times \ldots \times a}^{n}$.
2. $x = 0$, $a^0 := 1$.
3. $x = -n$, $a^{-n} := \dfrac{1}{a^n}$.
4. $x = \dfrac{p}{q} \in \mathbb{Q}$, $a^{p/q} := \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.
5. $x \in \mathbb{R} \setminus \mathbb{Q}$, $x \approx r \in \mathbb{Q}$, $a^x :\approx a^r$.
6. Domain 定義域: $(-\infty, \infty) = \mathbb{R}$.
   Range 值域: $(0, \infty)$.





(Practice by yourself: find $f(x) = 2^x$ for $x = 3, 0, -3, \dfrac{1}{3}, \sqrt{3}$.)

Note: Intervals 區間表示: ("(,)"不含端點小括號, "[,]"包含端點中括號。)

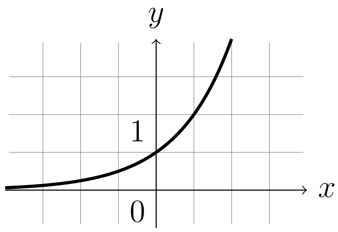
$$(a, b) = \{x : a < x < b\}, \quad (a, b] = \{x : a < x \leq b\},$$
$$[a, b) = \{x : a \leq x < b\}, \quad [a, b] = \{x : a \leq x \leq b\},$$

$\infty(-\infty)$: 無限大 (小) 不是可確定的數, 必用小括號 "$\ldots, \infty)$" & "$(-\infty, \ldots$".

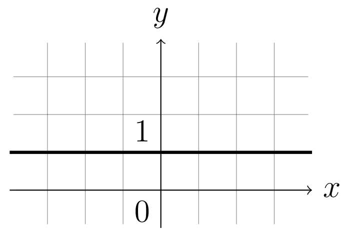
區間的聯集 (union)$\boxed{\bigcup}$與交集 (intersection)$\boxed{\bigcap}$:
$(a, b) \cup (c, d) = \{x : a < x < b \text{ or } c < x < d\}$, $(-\infty, 0) \cup (0, \infty) = \{x \neq 0\}$.
$(a, b) \cap (c, d) = \{x : a < x < b \text{ and } c < x < d\}$, $(-\infty, 0) \cap (0, \infty) = \emptyset$.

我找不到, 我到不了, 你所謂的, 無限大或小。

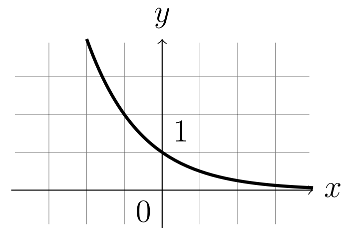
Three types of exponential functions:



$f(x) = a^x$, $a > 1$

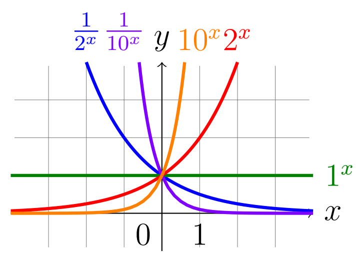


$f(x) = a^x$, $a = 1$



$f(x) = a^x$, $0 < a < 1$

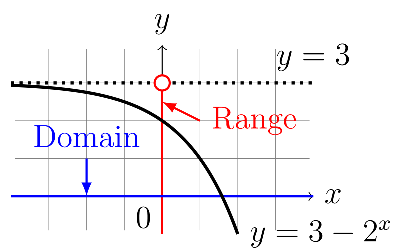
Note: $\boxed{\left(\frac{\mathbf{1}}{\boldsymbol{a}}\right)^{\boldsymbol{x}} = \frac{\mathbf{1}}{\boldsymbol{a}^{\boldsymbol{x}}} = \boldsymbol{a}^{-\boldsymbol{x}}}$,
$y = a^{-x}$ 與 $y = a^x$ 以 $x = 0$ (y-軸) 對稱.



Example 0.1 *Range of* $f(x) = 3 - 2^x = ?$

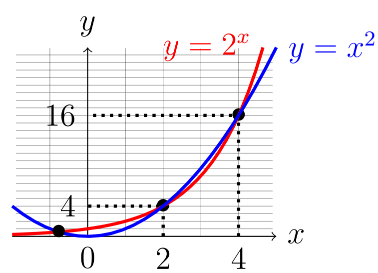
$$\begin{aligned} 2^x &: \mathbb{R} \to (0, \infty) \quad \text{(所有正數)} \\ -2^x &: \mathbb{R} \to (-\infty, 0) \quad \text{(所有負數)} \\ 3 - 2^x &: \mathbb{R} \to (-\infty, 3) \quad \text{(所有比 3 小的數)} \end{aligned}$$

Range: $(-\infty, 3)$ *(or* $\{x \in \mathbb{R} : x < 3\}$*).* ■

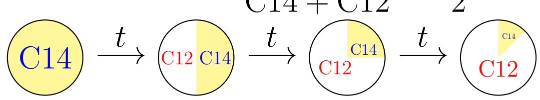


Example 0.2 *How many intersection of* 2^x *&* x^2*?*

$2^4 = 16 = 4^2$, $2^2 = 4 = 2^2$.
(只能從圖看出第三點)
Three: $x = 4$, $x = 2$, $x \approx -0.7666$. ■
(Try yourself: 怎麼說明沒有第四點?)



Application: Estimate population of human(人口), bacterial(細菌), radiometric dating(放射性定年) C14(C12:5730yr), U235(Pb207) or U238(Pb206).
radiometric dating: (C14) 半衰期 t: $\frac{\text{C14}}{\text{C14} + \text{C12}} = (\frac{1}{2})^{x/t}$, $x = t \lg \frac{\text{C14} + \text{C12}}{\text{C14}}$;



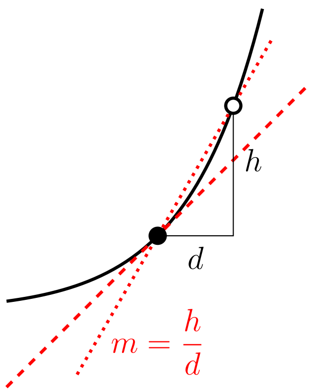
population: 起初 N 個, 每 t 時間變成 a 倍 (增加 $a - 1$ 倍): $P(x) = Na^{x/t}$.

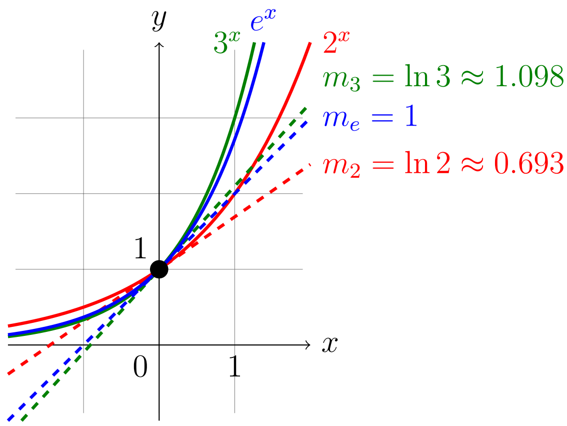
0.3 Number e

Define: ***Euler's number*** 歐拉數 $\boxed{e}$ ≈ 2.718281828,
是 $y = a^x$ 在 $x = 0$ 切線斜率 (slope) 等於 1 的底數 a 值。
In 1727 Leonhard Eular named "e" for "exponential".

Let m_a be the slope of the tangent line of a^x at $x = 0$.

a	1	2	e	3
m_a	0	0.693...	1	1.098...





Note: 找到 e 有什麼好處? $y = e^{rx}$ 在 $(0, 1)$ 的切線斜率就是 r。

Define: The ***natural exponential function*** 自然指數函數
("e to the (power of) x", or "natural exponential of x")

$$\boxed{f(x) = e^x}$$

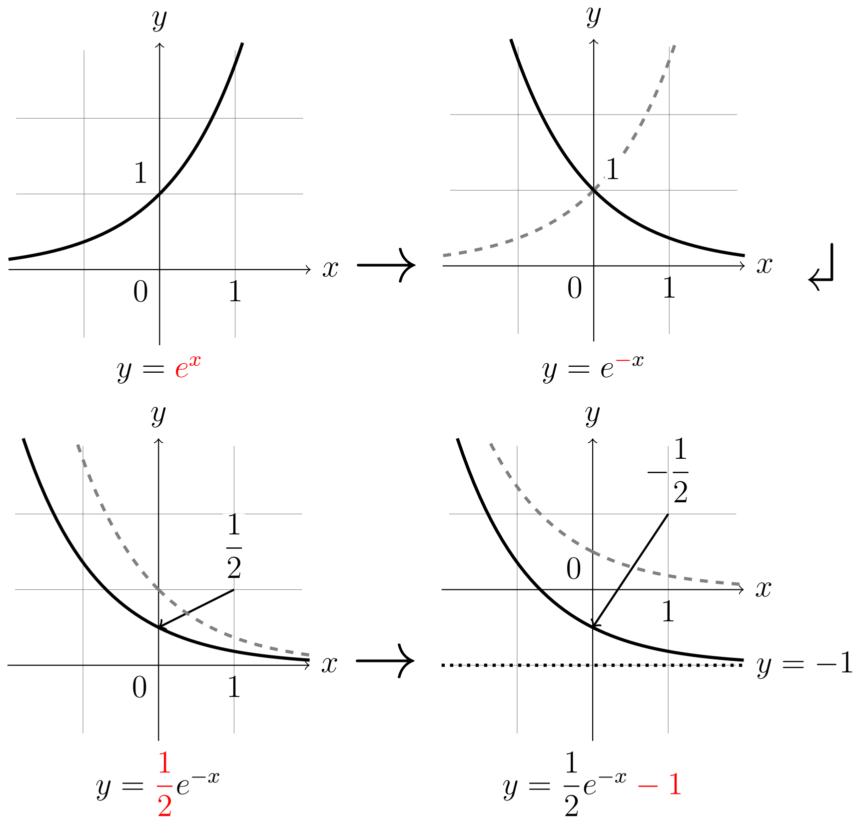
(Find it out by yourself: What is the difference between e^x, ex and x^e?)

Additional: (See §11)

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!},\ e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

♦: Euler's formula: $e^{ix} = \cos x + i\sin x$, Euler's identity: $e^{i\pi} + 1 = 0$.

Example 0.3 *Draw* $y = \dfrac{1}{2}e^{-x} - 1$. *(Domain? Range?)*



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(Domain: $(-\infty, \infty) = \mathbb{R}$*; range:* $(-1, \infty)$*.)*