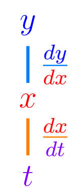
14.5 The chain rule

1. chain rules
2. implicit differentiation

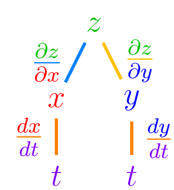
0.1 Chain rule



Recall: Chain Rule Theorem 連鎖律: If $y = f(x)$ is a differentiable function of x, where $x = g(t)$ is a differentiable function of t, then $y = f(g(t))$ is a differentiable function of t, and $\frac{dy}{dt} = \frac{df}{dx}\frac{dg}{dt}$.

Theorem 1 (Chain Rule (Case 1))
If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are differentiable function of t, then z is a differentiable function of t and,

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$



t: independent variables 獨立變數;
*x, y: **intermediate** variables 中介變數;*
z: dependent variable 相依變數。

Proof. Any $\Delta t \neq 0$ produces $\Delta x = g(t + \Delta t) - g(t)$, $\Delta y = h(t + \Delta t) - h(t)$, and $\Delta z = f(g(t + \Delta t)), h(t + \Delta t)) - f(g(t), h(t))$ (changes in t, x, y, and z).
$\because$ f is differentiable, $\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ (by definition).

When $\Delta t \to 0$, $\because$ g and h are differentiable and hence continuous, $\Delta x \to 0$ and $\Delta y \to 0$, $\implies$ $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. By definition of derivative,

$$\begin{aligned}
\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y\right) \\
&= \frac{\partial f}{\partial x}\lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t} + \lim_{\Delta t \to 0}\varepsilon_1 \cdot \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0}\varepsilon_2 \cdot \lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.
\end{aligned}$$

■

Note: 通常會用 $\dfrac{\partial z}{\partial x}$ 代替 $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial z}{\partial y}$ 代替 $\dfrac{\partial f}{\partial y}$。(符號省著點用)

Note: 比較:

$$\begin{array}{lcccccc} \text{全微分} & dz & = & \dfrac{\partial z}{\partial x}\,dx & + & \dfrac{\partial z}{\partial y}\,dy \\ \updownarrow & & & \updownarrow & & \updownarrow \\ \text{連鎖律} & \dfrac{dz}{dt} & = & \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} & + & \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} \end{array}$$

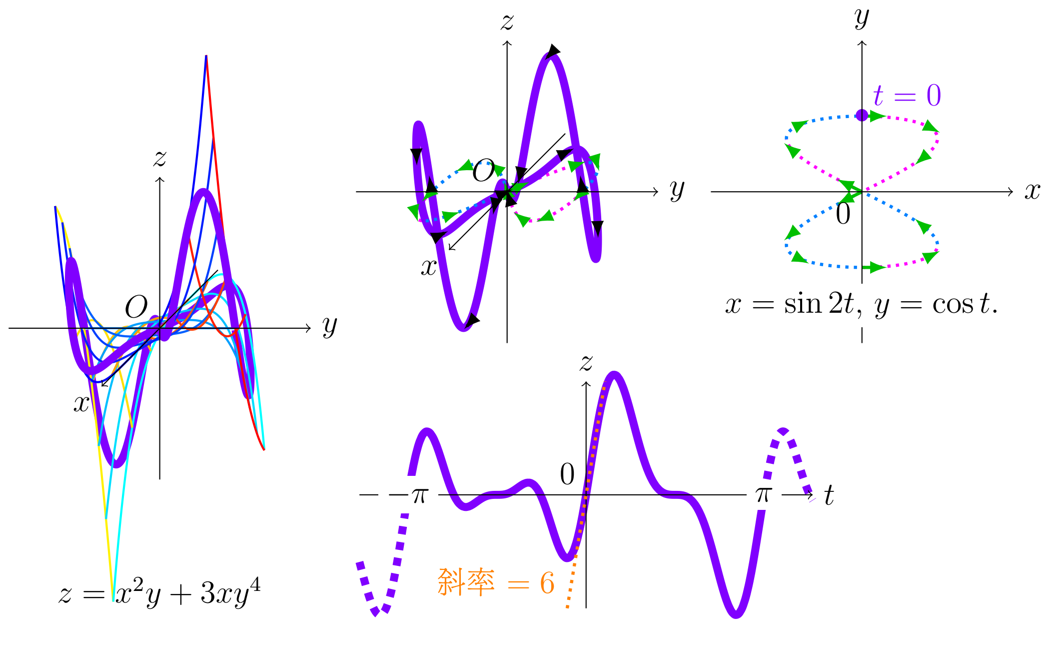
Attention: 每個中介變數都要微分, 單變數用 $\dfrac{d}{d}$, 多變數用 $\dfrac{\partial}{\partial}$。

Skill: 求(偏)導數可以先算出中介變數的值再一起代入。

Example 0.1 *If $z = x^2y + 3xy^4$, $x = \sin 2t$, $y = \cos t$, find $\dfrac{dz}{dt}$ when $t = 0$.*

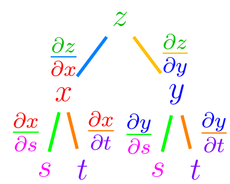
$$\begin{aligned} \left.\frac{dz}{dt}\right|_{t=0} &= \left[\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right]_{t=0} \\ &= \left[(2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)\right]_{t=0} \\ &\quad (\textit{When } t = 0 \implies x = \sin 0 = 0,\ y = \cos 0 = 1.) \\ &= \left[(2(0)(1) + 3(1)^4)(2\cos(2\cdot 0)) + ((0)^2 + 12(0)(1)^3)(-\sin 0)\right] \\ &= 6. \end{aligned}$$

■



Theorem 2 (Chain Rule (Case 2))
$z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$ are differentiable, then

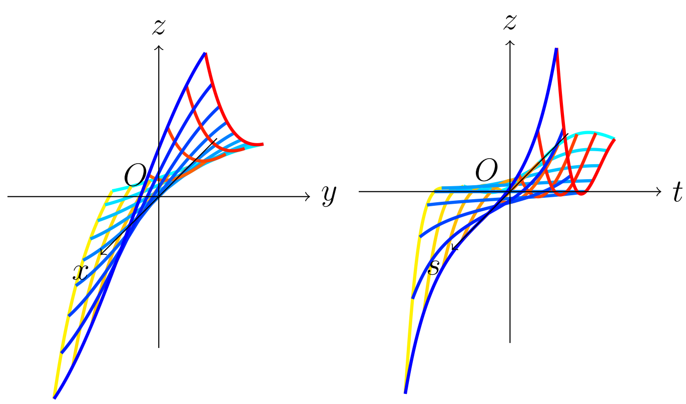
$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}}, \quad \boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}.$$



Example 0.2 *If $z = e^x \sin y$, $x = st^2$, $y = s^2 t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.*

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= (e^x \sin y)(t^2) + (e^x \cos y)(2st) = t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t). \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= (e^x \sin y)(2st) + (e^x \cos y)(s^2) = 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t).
\end{aligned}$$

(可以先代入變成 $z = e^x \sin y = e^{st^2} \sin(s^2 t)$ 再偏微分。) ■



$z = e^x \sin y$ $\qquad$ $z = e^{st^2} \sin(s^2 t)$

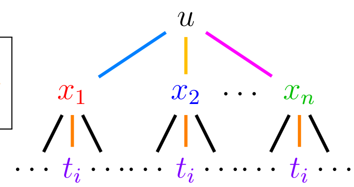
Note: 先代入再(偏)微分做的事都一樣, 只是容易漏微。

Attention: 偏微分時, 被當成常數的變數必須是同一層的變數。

Attention: 求(偏)導函數時, 微完要把中介變數 (x, y) 都換成(最下層的)獨立變數 (s, t) 的函數。

Theorem 3 (Chain Rule (General Version))
$u = f(x_1, x_2, \ldots, x_n)$, $x_j = g(t_1, t_2, \ldots, t_m)$, $j = 1, 2, \ldots, n$, are differentiable, then

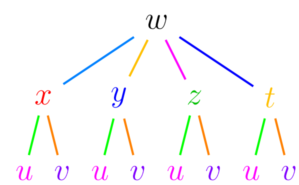
$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}}$$



for $i = 1, 2, \ldots, m$.

Example 0.3 *Chain Rule for $w = f(x, y, z, t)$, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, $t = t(u, v)$.*

$$\begin{aligned}
\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}, \\
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}.
\end{aligned}$$

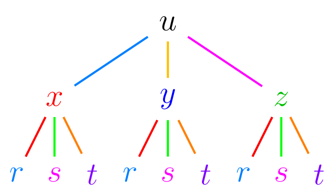


■

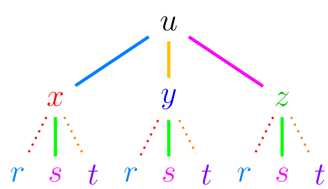
Skill: 公式怎麼背? 不用背! 畫出樹狀圖, 路上的相乘, 要的路相加。

Example 0.4 *If $u = x^4 y + y^2 z^3$, $x = rse^t$, $y = rs^2 e^{-t}$, $z = r^2 s \sin t$, find $\dfrac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.*

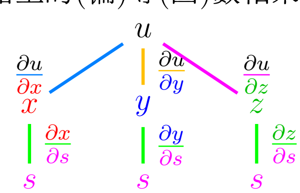
劃出樹狀圖



找出底部是 s 的路



路上的(偏)導(函)數相乘



$$\begin{aligned}
\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} \qquad \textit{(再把這些路上的乘積相加)} \\
&= (4x^3 y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2 z^2)(r^2 \sin t) \\
&\quad \textit{(When } r = 2,\ s = 1,\ t = 0,\ \implies x = 2,\ y = 2,\ z = 0.\textit{)} \\
&= 4(2)^3(2)(2)e^0 + ((2)^4 + 2(2)(0)^3)2(2)(1)e^{-0} + 3(2)^2(0)^2(2)^2 \sin 0 \\
&= 192.
\end{aligned}$$

■

Skill: 先寫出公式再分別計算, 可以避免漏微。

Example 0.5 *If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$, and f is differentiable, show that*

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

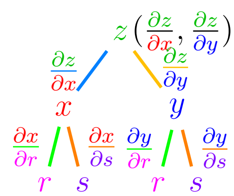
Proof. *Let $x = s^2 - t^2$ and $y = t^2 - s^2$, then $g(s,t) = f(x,y)$. (自設變數)*

$$\begin{aligned}
\frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s),\\
\frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t),\\
t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} &= \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0.
\end{aligned}$$

■

Example 0.6 *If $z = f(x,y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$, $y = 2rs$, find (a) $\frac{\partial z}{\partial r}$ and (b) $\frac{\partial^2 z}{\partial r^2}$.*

$$\text{(a)}\quad \begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\\
&= \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s) = 2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y};
\end{aligned}$$



$$\text{(b)}\quad \begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right) \qquad \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\ \text{跟}\ z\ \text{的角色一樣。}\right)\\
&= \left[2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\right] + \left[2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\right] \qquad (\text{偏微}\ r,\ \text{把}\ s\ \text{當常數。})\\
&= 2\frac{\partial z}{\partial x} + 2r\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\right]\\
&\qquad + 2s\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r}\right]\\
&= 2\frac{\partial z}{\partial x} + 2r\frac{\partial^2 z}{\partial x^2}(2r) + 2r\frac{\partial^2 z}{\partial y \partial x}(2s) + 2s\frac{\partial^2 z}{\partial x \partial y}(2r) + 2s\frac{\partial^2 z}{\partial y^2}(2s)\\
&= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}.
\end{aligned}$$

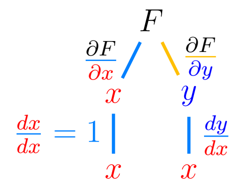
By Clairaut's Theorem, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ since they are continuous. ■

0.2 Implicit Differentiation

Suppose $F(x,y) = 0$ (or constant C) defines y implicitly as a differential function of x, i.e. $y = y(x)$, where $F(x, y(x)) = 0$.

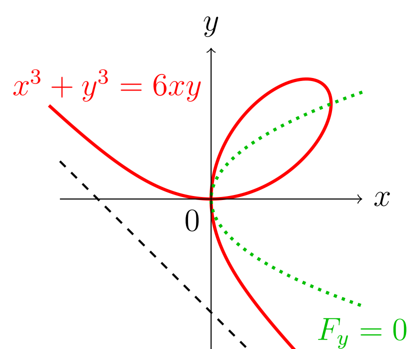
If F is differentiable, then by Chain Rule $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$. $(\frac{dx}{dx} = 1)$

$$\boxed{\boldsymbol{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \qquad \text{if } \frac{\partial F}{\partial y} \neq 0.}}$$



Skill: 不要背公式! 會背錯。想像 y 是 x 的函數做 (偏) 微分。

Example 0.7 *Find y' if $x^3 + y^3 = 6xy$.*



Folium of Descartes
笛卡爾的葉形線

Consider $F(x,y) = x^3 + y^3 - 6xy = 0$, then $F_x = 3x^2 - 6y$ and $F_y = 3y^2 - 6x$,

$$\begin{aligned} y' &= \left(\frac{dy}{dx} = \right) - \frac{F_x}{F_y} \\ &= -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}. \end{aligned}$$

(When $F_y = 3y^2 - 6x \neq 0$.)

[Another sol]

$$\begin{aligned} & x^3 + y^3 = 6xy \\ (\frac{d}{dx}:)\ & 3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

(if $y^2 - 2x \neq 0$). ■

Suppose $F(x,y,z) = 0$ and $z = f(x,y)$, $F(x,y,z(x,y)) = 0$

If F and f are differentiable, then by Chain Rule

$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$. $(\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0)$ $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$.

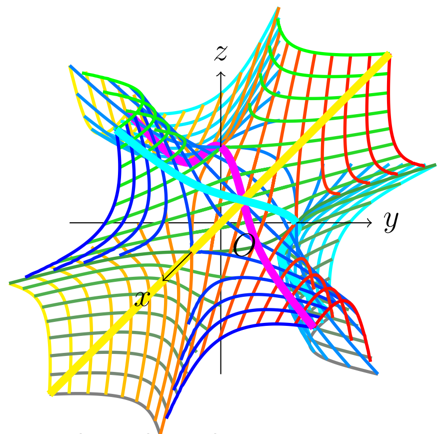
$$\boxed{\boldsymbol{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \ \&\ \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z} \qquad \text{if } \frac{\partial F}{\partial z} \neq 0}}$$

Skill: 不要背公式! 會背錯。想像 z 是 x 與 y 的函數做偏微分。

Example 0.8 *Find* $\dfrac{\partial z}{\partial x}$ *and* $\dfrac{\partial z}{\partial y}$ *if* $x^3+y^3+z^3+6xyz=1$.

Consider $F(x,y,z)=x^3+y^3+z^3+6xyz-1=0$, *then* $F_x=3x^2+6yz$, $F_y=3y^2+6xz$, *and* $F_z=3z^2+6xy$,

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z}\\ &= -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy};\\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z}\\ &= -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}.\end{aligned}$$



$x^3+y^3+z^3+6xyz=1$

(When $F_z=z^2+2xy\neq 0$.*)*

[Another sol]

$$\begin{array}{lcccccl} & x^3 & +\ y^3\ + & z^3 & + & 6xyz & =1\\ (\frac{\partial}{\partial x}:) & 3x^2 & +\ 0\ + & 3z^2\frac{\partial z}{\partial x} & + & 6yz+6xy\frac{\partial z}{\partial x} & =0\end{array}$$

$\Longrightarrow \dfrac{\partial z}{\partial x}=-\dfrac{x^2+2yz}{z^2+2xy}$, $\dfrac{\partial z}{\partial y}=-\dfrac{y^2+2xz}{z^2+2xy}$ *(if* $z^2+2xy\neq 0$*).* ■

Note: 不用搬過去變成 $F(x,y,z)=0$, 直接對等式兩邊做(偏)微分再解。

♦ Additional:

Advanced Calculus: Implicit & Inverse Function Theorems.

Theorem 4 ($\mathbb{R}^2\to\mathbb{R}$ **version)** *If* $F(x,y)$ *is defined on a disk* D *containing* (a,b), $F(a,b)=0$, $F_y(a,b)\neq 0$, F_x *and* F_y *are continuous on* D, *then* $F(x,y)=0$ *defines* $y=f(x)$ *(differentiable) near* (a,b) *with* $y'=-\dfrac{F_x}{F_y}$.

Theorem 5 *If* $f(x)$ *is smooth* (f' *is continuous) near* $x=a$ *and* $f'(a)\neq 0$, *then there exists a (differentiable) function* g *with* $g(f(x))=x$ ($g=f^{-1}$) *near* $f(a)$ *with* $g'(y)=\dfrac{1}{f'(g(y))}$.
(Consider $F(x,y)=f(x)-y$ *and apply Implicit Function Theorem.)*