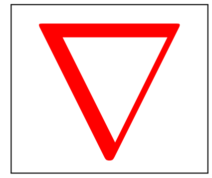
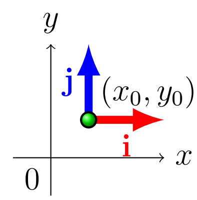
14.6 Directional derivatives and the gradient vector

1. directional derivatives 方向導數 $\boldsymbol{D}_{\mathbf{u}} f = \nabla f \bullet \mathbf{u}$
2. the gradient vector 梯度向量 $\nabla f = \langle f_x, f_y(, f_z)\rangle$
3. $\max\limits_{\mathbf{u}} \boldsymbol{D}_{\mathbf{u}} f$ & tangent plane to level surface



0.1 Directional derivatives

Recall: $z = f(x, y)$, then the partial derivatives

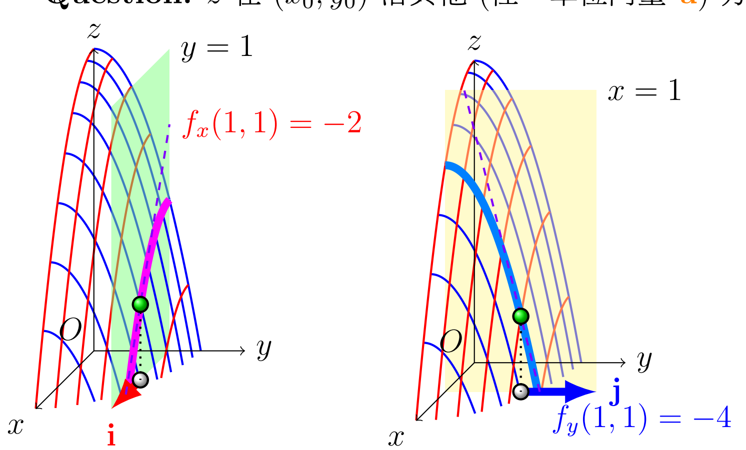


$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

分別代表 z 在 (x_0, y_0) 沿 x-軸 ($\mathbf{i} = \langle 1, 0\rangle$) 與 y-軸 ($\mathbf{j} = \langle 0, 1\rangle$) 方向的變化率。

Question: z 在 (x_0, y_0) 沿其他 (任一單位向量 $\mathbf{u}$) 方向的變化率=?



Define: The ***directional derivative*** 方向導數 of f at (x_0, y_0) in the direction of a **unit vector** **單位向量** $\mathbf{u} = \langle a, b\rangle$ is

$$\boxed{\boldsymbol{D}_{\mathbf{u}} \boldsymbol{f}(\boldsymbol{x_0}, \boldsymbol{y_0}) = \lim_{h \to 0} \frac{\boldsymbol{f}(\boldsymbol{x_0} + \boldsymbol{ha}, \boldsymbol{y_0} + \boldsymbol{hb}) - \boldsymbol{f}(\boldsymbol{x_0}, \boldsymbol{y_0})}{\boldsymbol{h}}}$$

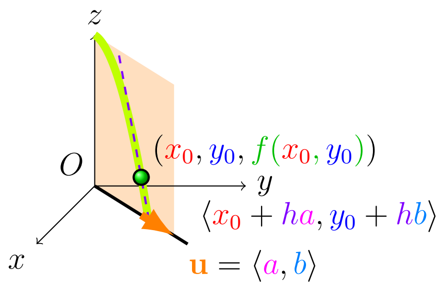
if the limit exists. ($\boldsymbol{D}_{\mathbf{u}} f$ 就是 f 在 $\mathbf{u}$ 方向的變化率 (rate of change)。)

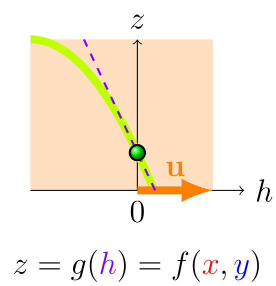
Theorem 1 *If f is a* ***differentiable*** *function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and*

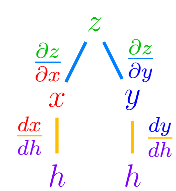
$$\boxed{\boldsymbol{D}_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b}$$

Proof. Let $g(h) = f(x_0 + ha, y_0 + hb)$, then by definition, $g'(0) = \boldsymbol{D}_{\mathbf{u}} f(x_0, y_0)$. Consider $g(h) = f(x, y)$, $x = x_0 + ha$, $y = y_0 + hb$, then by the Chain Rule, $g'(h) = \dfrac{\partial f}{\partial x}\dfrac{dx}{dh} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dh} = f_x(x, y)a + f_y(x, y)b$. When $h = 0$, $x = x_0$, $y = y_0$, $\boldsymbol{D}_{\mathbf{u}} f(x_0, y_0) = g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$. ■

Attention: 如果 f 有偏導數, 只能算 $\mathbf{u} = \mathbf{i}$ 或 $\mathbf{j}$ ($\boldsymbol{D}_{\mathbf{i}} f = f_x$ & $\boldsymbol{D}_{\mathbf{j}} f = f_y$), 其它方向要 (有連續的偏導數 $\Longrightarrow$)可微分, 不然要用定義。

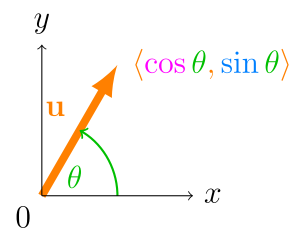






Note: 單位向量常見的表示法:
A unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta\rangle$, where θ is the angle with the positive x-axis, so

$$\boxed{\boldsymbol{D}_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta}.$$



Remark: 方向導數怎麼找: 1. 定義 (求極限) & 2. 定理 (要 f 可微分)。

Example 0.1 *Find the directional derivative $\boldsymbol{D}_{\mathbf{u}} f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ is the unit vector given by angle $\theta = \dfrac{\pi}{6}$. What is $\boldsymbol{D}_{\mathbf{u}} f(1, 2)$?*

$$\boldsymbol{D}_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2} = \frac{3\sqrt{3}}{2}x^2 - \frac{3}{2}x + (4 - \frac{3\sqrt{3}}{2})y.$$

$$\boldsymbol{D}_{\mathbf{u}} f(1, 2) = \frac{3\sqrt{3}}{2}(1)^2 - \frac{3}{2}(1) + (4 - \frac{3\sqrt{3}}{2})(2) = \frac{13 - 3\sqrt{3}}{2}.$$ ■

0.2 The gradient vector

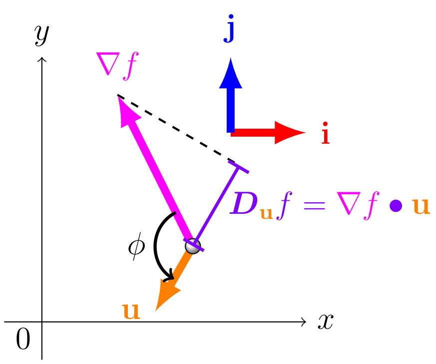
Functions of two variables

Define: If f is a function of two variables x and y, then the ***gradient*** [ˈgrediənt] 梯度 (坡度) of f is the vector function ∇f (or ***grad*** f, 唸作 "del f" or "nabla [ˈnæblə] f") defined by (注意方向, 不要跟 Δ "delta" 搞混。)

$$\boxed{\boldsymbol{\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}}}$$

If f is differentiable and $\mathbf{u} = \langle a, b\rangle$ be a unit vector, then

$$\begin{aligned} \boldsymbol{D}_{\mathbf{u}} f(x,y) &= f_x(x,y)a + f_y(x,y)b \\ &= \langle f_x(x,y), f_y(x,y)\rangle \bullet \langle a, b\rangle \\ &= \nabla f(x,y) \bullet \mathbf{u} \end{aligned}$$



幾何意義:
$\because \nabla f \bullet \mathbf{u} = |\nabla f||\mathbf{u}|\cos\phi = |\nabla f|\cos\phi$,
方向導數 $\boldsymbol{D}_{\mathbf{u}} f$ 就是梯度向量 ∇f 在單位向量 $\mathbf{u}$ 方向的投影量 ($\mathrm{proj}_{\mathbf{u}}\nabla f$)。

Example 0.2 *If $f(x,y) = \sin x + e^{xy}$, then*
$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle \cos x + ye^{xy}, xe^{xy}\rangle$.
$\nabla f(0,1) = \langle \cos 0 + 1e^{0\cdot 1}, 0e^{0\cdot 1}\rangle = \langle 2, 0\rangle$. ■

Example 0.3 *Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.*

1. 先算 ∇f: $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$;
2. 代入 $(2,-1)$: $\nabla f(2,-1) = 2(2)(-1)^3\mathbf{i} + (3(2)^2(-1)^2 - 4)\mathbf{j} = -4\mathbf{i} + 8\mathbf{j}$;
3. 算單位向量: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{2}{\sqrt{29}}\mathbf{i} + \dfrac{5}{\sqrt{29}}\mathbf{j}$;
4. 算方向導數: $\boldsymbol{D}_{\mathbf{u}} f(2,-1) = \nabla f(2,-1) \bullet \mathbf{u} = \dfrac{-4\cdot 2 + 8\cdot 5}{\sqrt{29}} = \dfrac{32}{\sqrt{29}}$. ■

Functions of three variables

Let $f(x,y,z)$ be a function of three variables of x, y and z.

$$\boxed{\begin{aligned}\nabla f(x,y,z) &= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \\ &= f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}\end{aligned}}$$

If f is differentiable and $\mathbf{u} = \langle a, b, c\rangle$ be a unit vector, then

$$\begin{aligned}\boldsymbol{D}_{\mathbf{u}} f(x,y,z) &= \lim_{h\to 0}\frac{f(x+ha, y+hb, z+hc) - f(x,y,z)}{h} \\ &= f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c \\ &= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \bullet \langle a, b, c\rangle \\ &= \nabla f(x,y,z) \bullet \mathbf{u}\end{aligned}$$

Example 0.4 *If $f(x,y,z) = x\sin yz$, find the gradient of f and the directional derivative of f at $(1,3,0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.*

1. 先算 ∇f: $\nabla f(x,y,z) = \langle \sin yz, xz\cos yz, xy\cos yz\rangle$;

2. 代入 $(1,3,0)$:

$\nabla f(1,3,0) = \langle \sin(3\cdot 0), (1)(0)\cos(3\cdot 0), (1)(3)\cos(3\cdot 0)\rangle = \langle 0,0,3\rangle$;

3. 算單位向量: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{-1}{\sqrt{6}}\right\rangle$;

4. 算方向導數: $\boldsymbol{D}_{\mathbf{u}} f(1,3,0) = \nabla f(1,3,0) \bullet \mathbf{u}$

$= \dfrac{0(1)+0(2)+3(-1)}{\sqrt{6}} = \dfrac{-3}{\sqrt{6}} (= -\dfrac{\sqrt{6}}{2} = -\sqrt{\dfrac{3}{2}})$. ■

Functions of more variables

If f is a differentiable function and $\mathbf{u}$ be a unit vector, then

$$\boldsymbol{D}_{\mathbf{u}} f(\mathbf{x}) = \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{u}) - f(\mathbf{x})}{h} = \nabla f(\mathbf{x}) \bullet \mathbf{u}$$

Conclusion: f 可微分 $\Longrightarrow$ 方向導數 等於 梯度 內積 單位向量。

(當成(向量)函數時, 常省略 “(x,y)”, “(x,y,z)”, “$(\mathbf{x})$”。)

$$\boxed{\boldsymbol{D}_{\mathbf{u}} f = \nabla f \bullet \mathbf{u}}$$

0.3 Maximum directional derivative and tangent plane to level surface

Directional derivative 方向導數, 該方向的變化量。

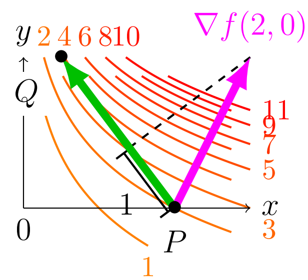
Question: 那個方向改變最大? **Answer:** 梯度向量的方向。

Question: 有多大? **Answer:** 梯度向量那麼大。

Theorem 2 *Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $\boldsymbol{D}_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.*

Proof. $\because \cos\theta \le 1$, $\boldsymbol{D}_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta \le |\nabla f|$, and "=" occurs when $\theta = 0$ (the same direction). (何時最小?) ■

Example 0.5 *(a) If $f(x,y) = xe^y$, find the rate of change of f at $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.*
(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?



(a) 1. 先算 ∇f: $\nabla f(x,y) = \langle e^y, xe^y\rangle$;
2. 代入 $(2,0)$: $\nabla f(2,0) = \langle e^0, 2e^0\rangle = \langle 1, 2\rangle$;
3. 算單位向量:
$\mathbf{v} = \overrightarrow{PQ} = \langle -1.5, 2\rangle$, $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$;
4. 算方向導數: $\boldsymbol{D}_{\mathbf{u}}f(2,0) = \nabla f(2,0) \bullet \mathbf{u} = 1(-\frac{3}{5}) + 2(\frac{4}{5}) = 1$.
(從 P 往 Q 走 1 單位, 高度上升約 1 單位。)
(b) $\nabla f(2,0) = \langle 1, 2\rangle$, value is $|\nabla f(2,0)| = \sqrt{5}$. ■

Example 0.6 *Suppose the temperature at (x,y,z) is $T(x,y,z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius, and x, y, z in meters. In what direction does the temperature increase fastest at $(1,1,-2)$? What is this maximum rate of change?*

1. 先算 ∇T: $\nabla T(x,y,z) = \langle -160x/(1+x^2+2y^2+3z^2)^2, -320y/(1+x^2+2y^2+3z^2)^2, -480z/(1+x^2+2y^2+3z^2)^2\rangle$,

2. 代入 $(1,1,-2)$: $\nabla T(1,1,-2) = \dfrac{160}{256}\langle -1,-2,6\rangle = \dfrac{5}{8}\langle -1,-2,6\rangle$,

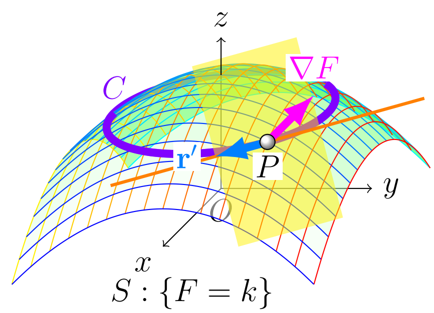
$|\nabla T(1,1,-2)| = \dfrac{5}{8}\sqrt{41} \approx 4°\text{C/m}$. ■

Question: 等高$\left\{\begin{matrix}\text{線}\\ \text{面}\end{matrix}\right.$ $(F=k)$ 的切$\left\{\begin{matrix}\text{線}\\ \text{平面}\end{matrix}\right.$的法向量? **Answer:** 梯度向量。

A level surface S with $F(x,y,z)=k$, and a point $P(x_0,y_0,z_0)$ on S. Let $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$ be a curve C on S through P.

By the Chain Rule,

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0.$$



$$\begin{aligned}\because \nabla F &= \langle F_x,F_y,F_z\rangle,\\ \mathbf{r}' &= \langle x',y',z'\rangle,\\ \nabla F\bullet\mathbf{r}' &= F_x x'+F_y y'+F_z z' \;=\; 0.\end{aligned}$$

When $t=t_0$, $x=x(t_0)=x_0$, $y=y(t_0)=y_0$, $z=z(t_0)=z_0$.

$$\nabla F(x_0,y_0,z_0)\bullet\mathbf{r}'(t_0)=0$$

The gradient vector at P is perpendicular to the tangent vector to any curve C on S through P. (在 P 的梯度向量垂直任何通過 P 的曲線切向量。)

If $\nabla F(x_0,y_0,z_0)\neq\mathbf{0}$, then the ***tangent plane*** 切平面 ***to*** S ***at*** P is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

and the ***normal line*** 法線 ***to*** S ***at*** P is

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

Note: 簡單記法: Let $\mathbf{x}=\langle x,y,z\rangle$, $\mathbf{x_0}=\langle x_0,y_0,z_0\rangle$.

$\boxed{\text{切平面: } \nabla F\bullet(\mathbf{x}-\mathbf{x_0})=0}$ $\boxed{\text{法線: } \nabla F=t(\mathbf{x}-\mathbf{x_0})}$

When $z=f(x,y)$, let $\boldsymbol{F(x,y,z)=f(x,y)-z}$, then the tangent plane to the surface of $z=f$ at (x_0,y_0) is also the one of $F=0$ at $(x_0,y_0,z_0=f(x_0,y_0))$ is (雙變數函數 看成 三變數函數 的等高面)

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+(-1)(z-z_0)=0$$

(注意: $F_z=-1$) 跟之前 (14,4) 的切平面公式一樣。

Conclusion: 梯度向量垂直等高面 [3D]/線 [2D]。

Example 0.7 *Find the equations of the tangent plane and normal line at the point* $(-2, 1, -3)$ *to the ellipsoid* $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$.

The ellipsoid is the level surface with $k = 3$ *of* $F(x, y, z) = \dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9}$.

$\nabla F = \langle F_x, F_y, F_z \rangle = \left\langle \dfrac{x}{2}, 2y, \dfrac{2z}{9} \right\rangle$,

$\nabla F(-2, 1, -3) = \left\langle \dfrac{(-2)}{2}, 2(1), \dfrac{2(-3)}{9} \right\rangle = \left\langle -1, 2, -\dfrac{2}{3} \right\rangle$;

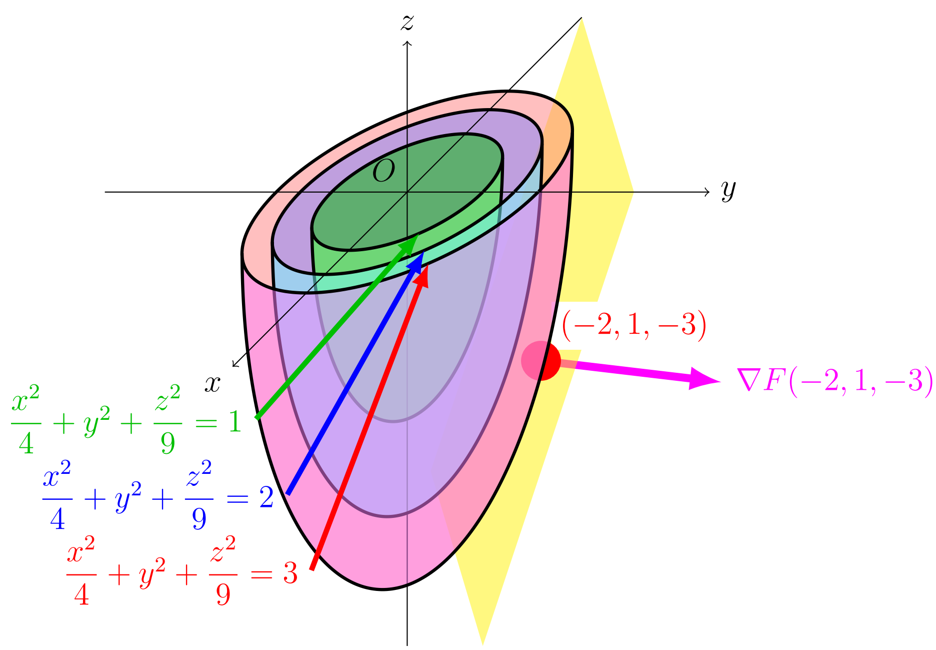
tangent plane: $(-1)(x - (-2)) + 2(y - 1) + (-\dfrac{2}{3})(z - (-3)) = 0$.

(linear(線性) equation: $3x - 6y + 2z + 18 = 0$.*)*

normal line: $\dfrac{x - (-2)}{-1} = \dfrac{y - 1}{2} = \dfrac{z - (-3)}{-2/3}$.

(symmetric(對稱) eq.: $\dfrac{x + 2}{-1} = \dfrac{y - 1}{2} = \dfrac{z + 3}{-2/3}$.*)*

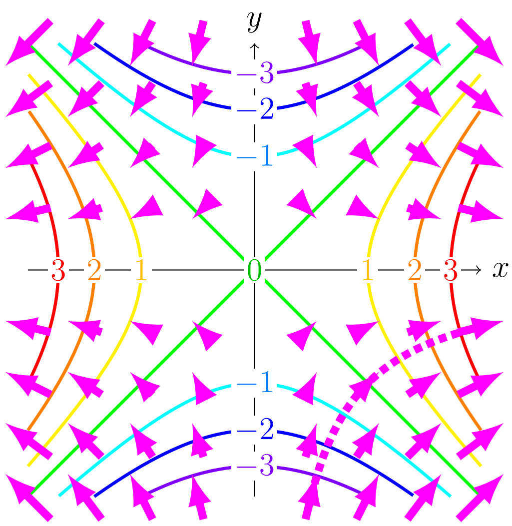
(parametric(參數) eq.: $x = -2 + (-1)t$, $y = 1 + 2t$, $z = -3 + (-\dfrac{2}{3})t$.*)* ■



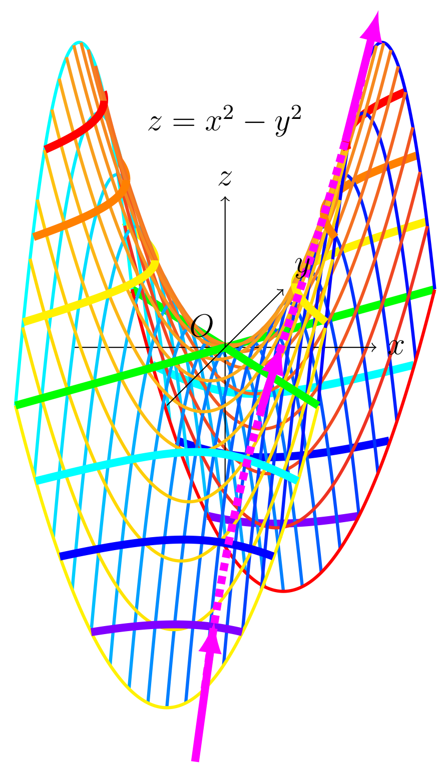
Applications

The method of steepest ascent/descent 最陡上升/下降法: 沿著梯度向量的方向找, 常用在實驗上快速找尋最佳解。

Gradient vector field 梯度向量場:



steepest ascent curve
最陡路線



Great circle (orthodrome 大圓, or Reimannian circle 黎曼圓): 球面上最短距離是走大圓。