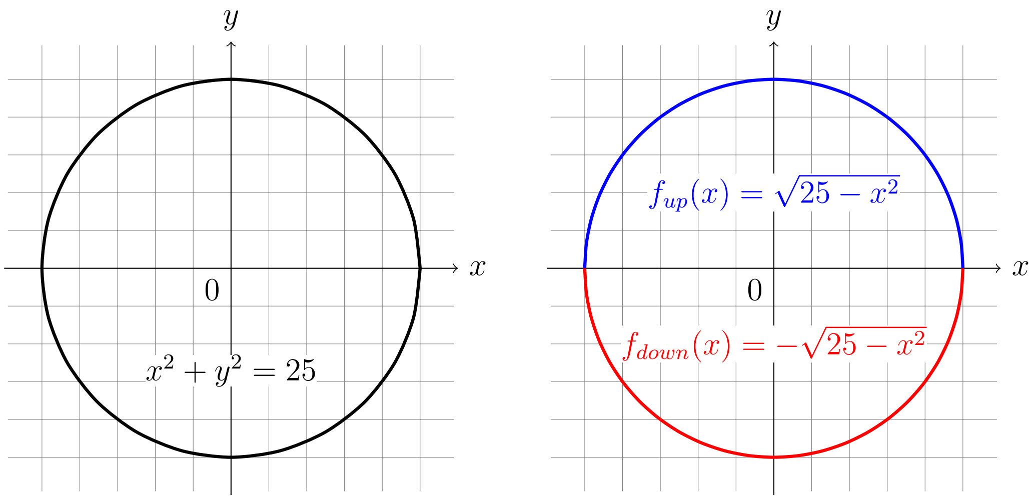
3.5 Implicit differentiation

1. implicit differentiation 隱微分
2. differentiation of inverse trigonometric function 反三角函數的微分

0.1 Implicit differentiation

一個函數 f 畫在圖上 $y = f(x)$ 可以求導數求切線.
如果一個圖沒辦法表示成一個函數 (一對多) 該怎麼求切線?

Example 0.1 $x^2 + y^2 = 25$ *find tangent line.*



How? $y = \pm\sqrt{25 - x^2}$. *(把 y 變成 x 的函數, 結果有兩個。)*
Let $f_{up}(x) = \sqrt{25 - x^2}$ *and* $f_{down}(x) = -\sqrt{25 - x^2}$ *on* $[-5, 5]$,
then $f'_{up}(x) = \dfrac{-x}{\sqrt{25 - x^2}}$ *and* $f'_{down}(x) = \dfrac{x}{\sqrt{25 - x^2}}$ *on* $(-5, 5)$.

Tangent line at $(x_0, y_0)(\neq (\pm 5, 0))$*:*

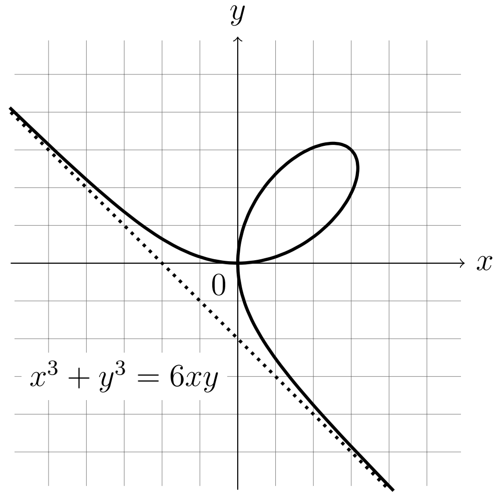
$(x_0, y_0) = (x_{up}, y_{up})$ *at* $\frown$	$(x_0, y_0) = (x_{down}, y_{down})$ *at* $\smile$
$y = \dfrac{-x_{up}}{\sqrt{25 - {x_{up}}^2}}(x - x_{up}) + y_{up}$	$y = \dfrac{x_{down}}{\sqrt{25 - {x_{down}}^2}}(x - x_{down}) + y_{down}$
$= \dfrac{-x_{up}}{y_{up}}(x - x_{up}) + y_{up}$	$= \dfrac{x_{down}}{-y_{down}}(x - x_{down}) + y_{down}$

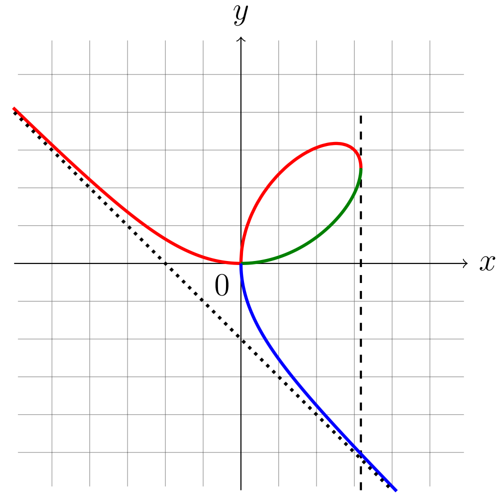
$$\Longrightarrow y = -\frac{x_0}{y_0}(x - x_0) + y_0,\ x_0 x + y_0 y = 25.$$

(tangent line at $(\pm 5, 0)$*?)*

考題一定有陷阱, 開平方根有正有負, 考試前請詳閱課本講義勤做練習考古題。

Example 0.2 $x^3 + y^3 = 6xy$: *the **folium of Descartes**(笛卡兒的葉形線)*





How? 分三段? (hard) 變函數? (harder) 算導數? (hardest)

♦: $y = \sqrt[3]{\dfrac{\sqrt{x^6 - 32x^3} - x^3}{2}} - \sqrt[3]{\dfrac{\sqrt{x^6 - 32x^3} + x^3}{2}},$

$\sqrt[3]{\dfrac{\sqrt{x^6 - 32x^3} - x^3}{2}}\omega - \sqrt[3]{\dfrac{\sqrt{x^6 - 32x^3} + x^3}{2}}\omega^2,$

$\sqrt[3]{\dfrac{\sqrt{x^6 - 32x^3} - x^3}{2}}\omega^2 - \sqrt[3]{\dfrac{\sqrt{x^6 - 32x^3} + x^3}{2}}\omega, \quad \omega = \dfrac{-1 + \sqrt{3}i}{2}.$

How to solve: the tangent line of $F(x, y) = 0$ at (x_0, y_0)?

隱普利系特 地佛連喜耶遜
Implicit Differentiation [ɪmˋplɪsɪt dɪˏfərɛnʃɪˋeʃən] 隱微分 **:**

Step 1. Differentiating with respect to x ($\dfrac{d}{dx}$) both sides of $F(x, y) = 0$.
等式兩邊對 x 微分。

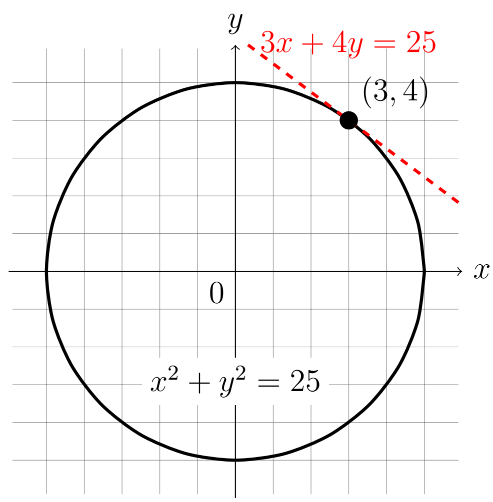
Step 2. Imaging $y = y(x)$ and applying the Chain Rule to solve $\dfrac{dy}{dx} = G(x, y)$.
把 $y = y(x)$ 當作 x 的函數, 用連鎖律求出 y' 寫成一個 x, y 的函數。

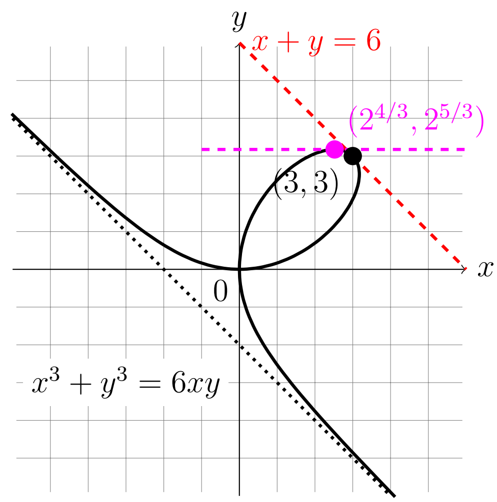
Step 3. $G(x_0, y_0) = \dfrac{dy}{dx}\Big|_{x=x_0, y=y_0}$ is the slope of tangent line at (x_0, y_0), and the equation of the tangent line is $y = G(x_0, y_0)(x - x_0) + y_0$.
代入 $x = x_0, y = y_0$ 解 y' 得到切線斜率，寫出切線方程式。

Skill: 如果只求 y': 對 x 微分完就代 x_0, y_0 (**Step 1+3**), 解 y' 的一次方程式。

Example 0.3 *Tangent line of $x^2+y^2=25$ at $(3,4)=$?*

1. $\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$, $2x+2y\frac{dy}{dx}=0$. $\quad\left[6+8y'=0, y'=-\frac{3}{4}\right]$
2. $\frac{dy}{dx}=-\frac{x}{y}$. $\quad$ (↑ 兩邊微分 ; ← 導函數由 x, y 表示; ↓ 代入得導數.)
3. $\left.\frac{dy}{dx}\right|_{x=3,y=4}=-\frac{3}{4}$, *and* $y=-\frac{3}{4}(x-3)+4$ (*or* $3x+4y=25$). ■





Example 0.4 *(a) Tangent line of $x^3+y^3=6xy$ at $(3,3)=$?*
(b) Point whose horizontal tangent line in the first quadrant =?

(a) 1. $\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(6xy)$, $3x^2+3y^2y'=6y+6xy'$.

2. $y'=\frac{2y-x^2}{y^2-2x}$. $\quad\left[27+27y'=18+18y', y'=-1\right]$

3. $\left.y'\right|_{x=3,y=3}=\frac{2\cdot 3-3^2}{3^2-2\cdot 3}=-1$, *and* $y=-(x-3)+3$ (*or* $x+y=6$).

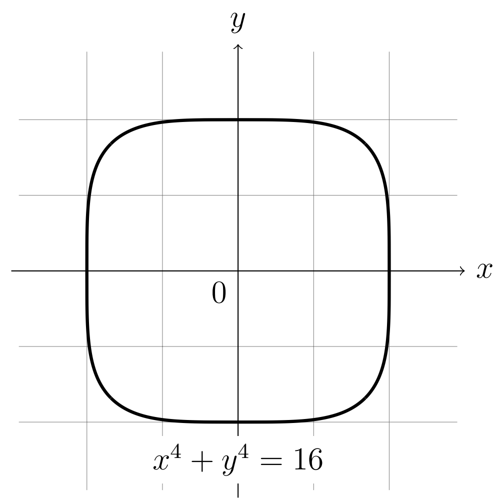
(b) $y'=0 \implies 2y-x^2=0$, $y=\frac{x^2}{2}$,

(代入 $x^3+y^3=6xy$) $x^3+(\frac{x^2}{2})^3=6x\frac{x^2}{2}$, $x^3(x^3-2^4)=0$, $x=0, 2^{4/3}$.

$x\neq 0$ ($\because$ *first quadrant*) $\implies x=2^{4/3}$, $y=\frac{(2^{4/3})^2}{2}=2^{2\times 4/3-1}=2^{5/3}$.

(& 分母 $y^2-2x=2^{10/3}-2^{1+4/3}\neq 0$.) ■

Example 0.5 $x^4 + y^4 = 16,\ y'' = ?$



$$4x^3 + 4y^3 y' = 0,\ y' = -\frac{x^3}{y^3}.$$

$$\begin{aligned}
y'' &= (y')' = \left(-\frac{x^3}{y^3}\right)' \\
&= -\frac{(x^3)' y^3 - x^3 (y^3)'}{(y^3)^2} \\
&= -\frac{3x^2 y^3 - 3x^3 y^2 y'}{y^6} \\
&= -\frac{3x^2 y^3 - 3x^3 y^2 (-\frac{x^3}{y^3})}{y^6} \qquad \textit{(代入 } y' = -\frac{x^3}{y^3}\textit{)} \\
&= -\frac{3x^2 y^4 + 3x^6}{y^7} \qquad \textit{(這個答案也是對的)} \\
&= -\frac{3x^2 (x^4 + y^4)}{y^7} \qquad \textit{(代入 } x^4 + y^4 = 16 \textit{ 簡化)} \\
&= -\frac{48x^2}{y^7}. \qquad \blacksquare
\end{aligned}$$

Skill: 求 y'' at $x = x_0$ 時, 有時候不見得 $y' = y'(x, y)$ 代進去會好算, 這時候就要算出 $y'(x_0, y_0)$, 把它帶入解 y'' 的式子中的 y'.

Example 0.6 (Extended) $e^{x+y} = x$ *at* $x = 1,\ y' = ?\ y'' = ?$

(對 $e^{x+y} = x$ 微分) $\quad (1 + y')e^{x+y} = 1, \cdots\cdots (*)$

$\because e^{x+y} = x = 1, \therefore \quad (1 + y') \cdot 1 = 1$

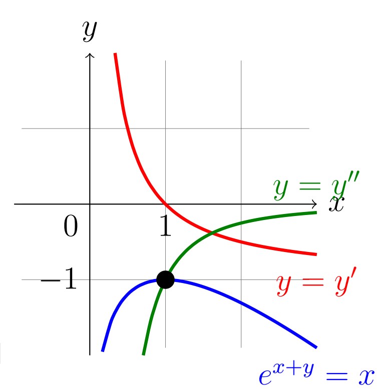
(這時的 y' 是代入 $x = 1$ 的狀態)

$\Longrightarrow y' = 0;$

(對 $(*)$ 再微分) $\quad y'' e^{x+y} + (1 + y')^2 e^{x+y} = 0,$

代 $\begin{cases} e^{x+y} = x = 1 \\ \textit{and } y' = 0 \end{cases}, \quad y'' \cdot 1 + (1 + 0)^2 \cdot 1 = 0,$

$\Longrightarrow y'' = -1. \qquad \blacksquare$



0.2 Differentiation of inverse trigonometric function

Apply implicit differentiation. (公式記憶法: 正餘一樣, 餘有負號。)

$$\boxed{\begin{array}{lll} (\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}, & (\tan^{-1}x)' = \dfrac{1}{1+x^2}, & (\sec^{-1}x)' = \dfrac{1}{x\sqrt{x^2-1}} \\ (\cos^{-1}x)' = \dfrac{-1}{\sqrt{1-x^2}}, & (\cot^{-1}x)' = \dfrac{-1}{1+x^2}, & (\csc^{-1}x)' = \dfrac{-1}{x\sqrt{x^2-1}} \end{array}}$$

1. $\boxed{\mathbf{(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}}}$ $y = \sin^{-1}x \iff \sin y = x \ \&\ y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

   $\frac{d}{dx}\sin y = \frac{d}{dx}x$, $\cos y\frac{dy}{dx} = 1$. $\because y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos y = \sqrt{1-\sin^2 y} \geq 0$.

   $\therefore \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$.

2. $\boxed{\mathbf{(\cos^{-1}x)' = \dfrac{-1}{\sqrt{1-x^2}}}}$ $y = \cos^{-1}x \iff \cos y = x \ \&\ y \in [0, \pi]$

   $\frac{d}{dx}\cos y = \frac{d}{dx}x$, $-\sin y\frac{dy}{dx} = 1$. $\because y \in [0, \pi]$, $\sin y = \sqrt{1-\cos^2 y} \geq 0$.

   $\therefore \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$.

3. $\boxed{\mathbf{(\tan^{-1}x)' = \dfrac{1}{1+x^2}}}$ $y = \tan^{-1}x \iff \tan y = x \ \&\ y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

   $\frac{d}{dx}\tan y = \frac{d}{dx}x$, $\sec^2 y\frac{dy}{dx} = 1$. $\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$.

♦ 4. $\boxed{\mathbf{(\sec^{-1}x)' = \dfrac{1}{x\sqrt{x^2-1}}}}$ $y = \sec^{-1}x \iff \sec y = x$

   $\&\ y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$. $\frac{d}{dx}\sec y = \frac{d}{dx}x$, $\sec y\tan y\frac{dy}{dx} = 1$.

   $\because y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$, $\tan y = \sqrt{\sec^2 y - 1} \geq 0$.

   $\therefore \frac{dy}{dx} = \frac{1}{\sec y\tan y} = \frac{1}{\sec y\sqrt{\sec^2 y-1}} = \frac{1}{x\sqrt{x^2-1}}$.

   (開平方根取正, 這就是爲什麼反三角函數要限制三角函數在這些地方。)

   ♦: 有的書上因爲 $\sec^{-1}x$ 值域不同, 會是 $(\sec^{-1}x)' = \frac{1}{|x|\sqrt{x^2-1}}$。

Example 0.7 *Differentiate (a)* $y = \dfrac{1}{\sin^{-1} x}$. *(b)* $f(x) = x \arctan \sqrt{x}$.

(a) $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin^{-1} x)^{-1} = (-1)(\sin^{-1} x)^{-2} \dfrac{d}{dx}(\sin^{-1} x)$
$= (-1)(\sin^{-1} x)^{-2} \dfrac{1}{\sqrt{1-x^2}} = \dfrac{-1}{(\sin^{-1} x)^2 \sqrt{1-x^2}}$.

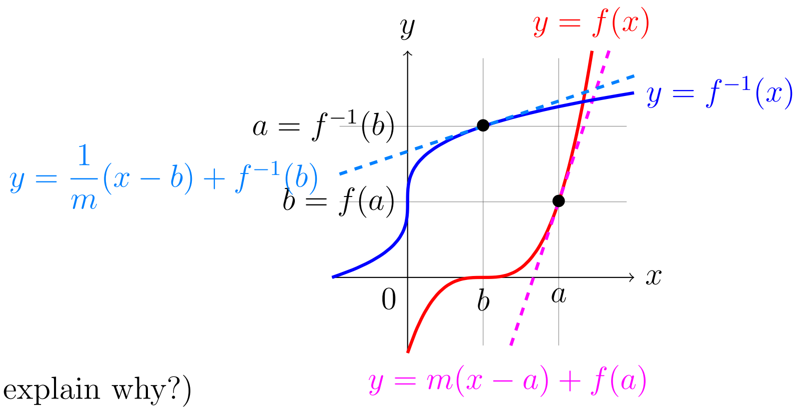
(b) $f'(x) = (x)' \arctan \sqrt{x} + x(\arctan \sqrt{x})' = \arctan \sqrt{x} + x \dfrac{1}{1+(\sqrt{x})^2}(\sqrt{x})'$
$= \arctan \sqrt{x} + x \dfrac{1}{1+(\sqrt{x})^2} \dfrac{1}{2\sqrt{x}} = \arctan \sqrt{x} + \dfrac{\sqrt{x}}{2(1+x)}$. ■

Additional: Derivative of inverse function

Remark: 1. f is continuous and one-to-one $\iff$ f^{-1} is continuous and one-to-one, but f is differentiable $\not\Longrightarrow$ f^{-1} is differentiable.
Ex: $f(x) = x^3$ is differentiable at $x = 0$, but $f^{-1}(x) = \sqrt[3]{x}$ is not.
2. How to solve $\dfrac{d}{dx} f^{-1}$? (Exercise 3.5.77, 101,104 會考考過。)

$$\begin{aligned} f(f^{-1}(x)) &= x && (\text{兩邊 } \tfrac{d}{dx}) \\ f'(f^{-1}(x)) \frac{d}{dx} f^{-1}(x) &= 1 && (\text{Chain rule}) \\ \frac{d}{dx} f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))}. && (\heartsuit) \end{aligned}$$

Observation: $y = f^{-1}(x)$ 在 $x = b$ $(= f(a))$ 的切線斜率 $(\dfrac{1}{m})$, 是 $y = f(x)$ 在 $x = f^{-1}(b)$ $(= a)$ 的切線斜率 (m) 的倒數。



(Can you explain why?)