11.2 Series

1. (infinite) series
2. find the sum of series

0.1 Infinite series

Define: A ***series*** 級數 is written in a definite order:

$$\underbrace{a_1}_{\text{1st term}} + \underbrace{a_2}_{\text{2nd term}} + \ldots + \underbrace{a_n}_{n\text{-th term}} + \ldots$$

Notation: 級數寫法: (沒寫通常代表從 $n = 1$ 到 ∞。)

- $\boxed{\boldsymbol{a_1 + a_2 + a_3 + \ldots}}$
- $\boxed{\sum \boldsymbol{a_n}}$ or $\boxed{\sum_{\boldsymbol{n=1}}^{\infty} \boldsymbol{a_n}}$
- $\boxed{\sum_{\boldsymbol{n=1}}^{\infty} (\boldsymbol{n} \text{ 的公式})}$

Question: 無窮多項怎麼加? 不能加, 用極限。

Define: The ***n-th partial sum*** $\boldsymbol{s_n}$ of a series is the sum of its **first $\boldsymbol{n}$ terms**. 一個級數的第 n 個部分和是它的前 n 項和。(不是加到第 n 項。)

級數 $\sum a_n$ 的部分和們: (a_n 從 $n = 1$ 開始, s_n 剛好是加到第 n 項。)

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ &\vdots \\ s_n &= a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i. \end{aligned}$$

從一個序列 $\{a_n\}$ 產生另一個序列 $\{s_n\}$, 用 $\{s_n\}$ 的收發來定義 $\sum a_n$ 的收發。

Define: Given a series $\sum a_n$, let s_n denote its n-th partial sum. If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$, then $\sum a_n$ is called ***convergent*** 收斂 and written $\sum a_n = s$. The number $\boxed{s}$ is called the ***sum*** 和 of the series. If $\{s_n\}$ is divergent, then $\sum a_n$ is called ***divergent*** 發散.
(級數不講發散至無窮 (diverges to ∞)。)

Note: $\sum a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$. Compare $\int_1^{\infty} f(x)\, dx = \lim_{t\to\infty} \int_1^{t} f(x)\, dx$.

Example 0.1 ***Geometric series*** *幾何 (等比) 級數*

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum ar^{n-1},\ a \neq 0,$$

where r is the ***common ratio*** *公比.*

When $r = 1$, $s_n = na \to \pm\infty$ as $n \to \infty$, divergent.
When $r \neq 1$, $s_n = \dfrac{a(1-r^n)}{1-r}$.
If $-1 < r < 1$, then $\lim\limits_{n\to\infty} r^n = 0$ and hence

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n = \frac{a}{1-r},$$ is convergent.

If $r \leq -1$ or $r > 1$, then $\{r^n\}$ is divergent, and so $\{s_n\}$ is divergent. ■

Fact: $\boxed{\sum \boldsymbol{ar^{n-1}} \text{ is } \begin{cases} \text{convergent with the sum } \dfrac{\boldsymbol{a}}{\boldsymbol{1-r}} & \text{if } |\boldsymbol{r}| < \mathbf{1} \\ \text{divergent} & \text{if } |\boldsymbol{r}| \geq \mathbf{1} \end{cases}}$

Example 0.2 *Find the sum of the geometric series* $5 - \dfrac{10}{3} + \dfrac{20}{9} - \dfrac{40}{27} + \ldots$.

$$a = 5,\ r = -\frac{2}{3} \text{ and } |r| < 1,\ \sum 5(-\frac{2}{3})^{n-1} = \frac{5}{1-(-\frac{2}{3})} = 3.$$ ■

Example 0.3 *Is* $\sum 2^{2n}3^{1-n}$ *convergent or divergent?*

$$\sum 2^{2n}3^{1-n} = \sum 4(\frac{4}{3})^{n-1},\ a = 4 \text{ and } |r| = \left|\frac{4}{3}\right| \geq 1,$$ divergent. ■

Example 0.4 *Find the sum of the series* $\sum\limits_{n=0}^{\infty} x^n$, *where* $|x| < 1$.

(Adopt the convention $x^0 = 1$ even when $x = 0$)

$$a = x^0 = 1,\ |r| = |x| < 1,\ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$ ■

Fact: $\boxed{\sum_{\boldsymbol{n=0}}^{\infty} \boldsymbol{x^n} = \frac{\mathbf{1}}{\mathbf{1}-\boldsymbol{x}} \text{ for } |\boldsymbol{x}| < \mathbf{1}.}$

1. 你我約定 x 的零次等於一, 也答應永遠都不爲 $x = 0$ 擔心。
2. 說好不寫的是從 $n = 1$ 開始, 從 $n = 0$ 開始的要寫。

Example 0.5 *Recurring/Repeating decimal (循環小數是有理數 $\in \mathbb{Q}$):*
(a) $0.\overline{9} = 1$ *(Exercise 11.2.49); (b)* $2.3\overline{17} = \dfrac{1147}{495}$.

$$(a)\ 0.\overline{9} = 0.999... = \frac{9}{10} + \frac{9}{100} + \ldots = \sum \frac{9}{10}(\frac{1}{10})^{n-1} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{9}{9} = 1.$$

$$(b)\ 2.3\overline{17} = 2.3171717... = 2.3 + \sum \frac{17}{1000}(\frac{1}{100})^{n-1} = 2.3 + \frac{\frac{17}{1000}}{1-\frac{1}{100}}$$

$$= 2.3 + \frac{17}{990} = 2 + \frac{317-3}{990} = 2\frac{157}{495} = \frac{1147}{495}.$$

$$0.a_1a_2\ldots a_s\overline{b_1b_2\ldots b_t} = \frac{a_1a_2\ldots a_sb_1b_2\ldots b_t - a_1a_2\ldots a_s}{\underbrace{99\ldots9}_{t}\underbrace{00\ldots0}_{s}}$$ ■

$(\dfrac{\text{全部} - \text{不循環}}{|\text{全部}|'s\ 9 - |\text{不循環}|'s\ 9}$, 不要背。)

Example 0.6 (前後相消) *Show that the series* $\sum \dfrac{1}{n(n+1)}$ *is convergent and find its sum.*

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n}(\frac{1}{i} - \frac{1}{i+1}) \\ &= (\frac{1}{1} - \cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}) + (\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}) + \ldots + (\cancel{\frac{1}{n}} - \frac{1}{n+1}) \\ &= 1 - \frac{1}{n+1}, \qquad \text{(算出部分和的公式)} \\ \lim_{n\to\infty} s_n &= \lim_{n\to\infty}(1 - \frac{1}{n+1}) = 1 - 0 = 1. \end{aligned}$$

So $\sum \dfrac{1}{n(n+1)}$ *is convergent with the sum* 1. ■

Skill: $s_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n}(b_i - b_{i+1}) = (b_1 - b_2) + \cdots + (b_n - b_{n+1}) = b_1 - b_{n+1}$,
$\sum a_n = \lim_{n\to\infty} s_n = b_1 - \lim_{n\to\infty} b_{n+1}$ if the limit exists.

Example 0.7 *Show that the* ***harmonic series*** 調和級數

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is divergent.

Proof. *Consider* s_{2^n}*. (直接看* s_n *算不出來)*

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2}, \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) \;=\; 1 + \frac{2}{2}, \\
s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) \;=\; 1 + \frac{3}{2}, \\
s_{16} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16} \\
&> 1 + \underbrace{\frac{1}{2}}_{1\text{項}} + \underbrace{(\frac{1}{4} + \frac{1}{4})}_{2\text{項}} + \underbrace{(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8})}_{4\text{項}} + \underbrace{(\frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16})}_{8\text{項}} \\
&= 1 + \frac{4}{2}, \; s_{32} \;>\; 1 + \frac{5}{2}, \; s_{64} \;>\; 1 + \frac{6}{2}, \dots, \;\Longrightarrow\; s_{2^n} \;>\; 1 + \frac{n}{2}.
\end{aligned}$$

$s_{2^n} \to \infty$ as $n \to \infty$, so $\{s_n\}$ is divergent.

Therefore, the harmonic series $\sum \frac{1}{n}$ *is divergent.* ■

Fact: $\boxed{\sum \frac{\mathbf{1}}{\boldsymbol{n}} \text{ is divergent.}}$

♦ First proof by Nicole Oresme in 1350s, there are 45+ proofs.

Honsberger: $s_9 > \frac{9}{10}$, $s_{99} > \frac{9}{10} + \frac{90}{100} = 2\frac{9}{10}$, $s_{10^n-1} > n\frac{9}{10} \to \infty$ as $n \to \infty$.

Leonard Gillman:
$$\begin{aligned}
\sum \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \\
&> \underbrace{\frac{1}{2} + \frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{6} + \frac{1}{6}} + \cdots \\
&= \quad 1 \quad + \quad \frac{1}{2} \quad + \quad \frac{1}{3} \quad + \cdots = \sum \frac{1}{n}. (\to\leftarrow)
\end{aligned}$$

0.2 Find the sum of series

Theorem 1 *(級數收斂單項歸零)*
If the series $\sum a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} s_{n-1} = s$ since $n-1 \to \infty$ as $n \to \infty$.
Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$. ■

Attention: 反過來不對! $\lim_{n\to\infty} a_n = 0 \not\Longrightarrow \sum a_n$ converges, ex: $\sum \frac{1}{n}$.

Theorem 2 (Test for Divergence) *(單項不歸零則級數發散)*
If $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum a_n$ *is divergent.*

Example 0.8 $\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$*(單項不歸零)*, $\sum \frac{n^2}{5n^2+4}$ *diverges*. ■

Theorem 3 *If* $\sum a_n$ *and* $\sum b_n$ *are convergent series,* $\sum a_n = s$ *and* $\sum b_n = t$, *and* c *is a constant, then so are* $\sum ca_n$, $\sum(a_n + b_n)$ *and* $\sum(a_n - b_n)$, *and*

$$\begin{aligned}
\sum ca_n &= c\sum a_n = cs, \\
\sum(a_n+b_n) &= \sum a_n + \sum b_n = s+t, \\
\sum(a_n-b_n) &= \sum a_n - \sum b_n = s-t.
\end{aligned}$$

Note: 加減常數倍, 沒有乘除! Compare with $\lim_{x\to\infty} f(x)$ & $\int_a^\infty f(x)\,dx$.

Example 0.9 *Find the sum of the series* $\sum \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

$$\sum \frac{1}{n(n+1)} = 1 \text{ and } \sum \frac{1}{2^n} = \sum \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1,$$

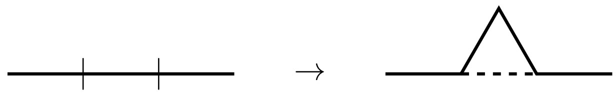
so $\sum \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3\sum \frac{1}{n(n+1)} + \sum \frac{1}{2^n} = 3\cdot 1 + 1 = 4.$ ■

Note: 有限項不影響級數的收斂或發散!

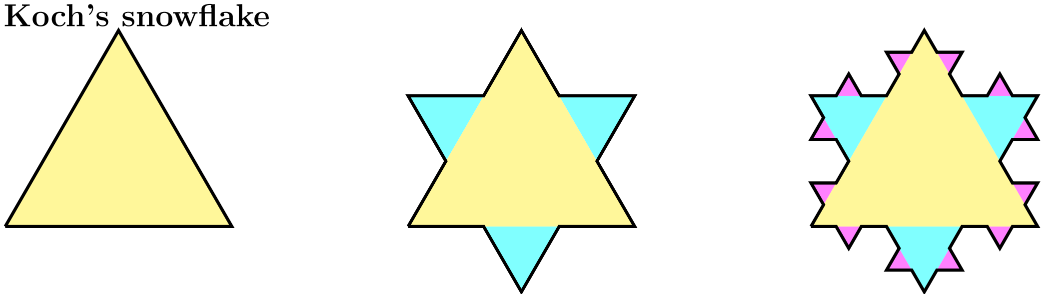
If $\sum_{N+1}^{\infty} a_n$ is convergent/divergent, so is $\sum a_n = \sum_{1}^{N} a_n + \sum_{N+1}^{\infty} a_n$.

♦ Additional: Fractals 碎形

von Koch curve



Koch's snowflake



Length: $L_1 = 3$, $L_2 = 3 \cdot (\frac{4}{3})^1$, $L_3 = 3 \cdot (\frac{4}{3})^2$, $\cdots$, $L_n = 3(\frac{4}{3})^{n-1}$.

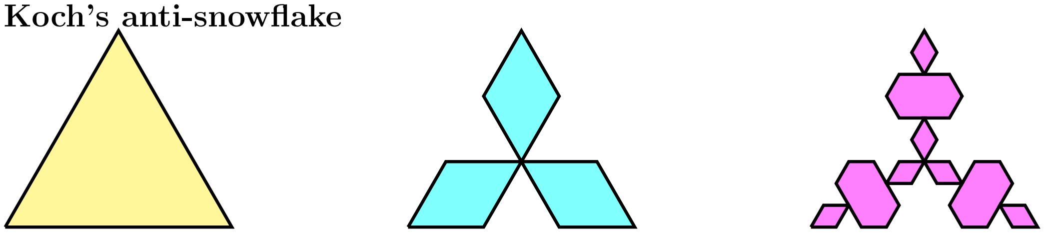
Total length $L = \lim_{n\to\infty} L_n = \infty$. ($|r| = |\frac{4}{3}| > 1$)

Increased Area: $A_1 = \triangle$, $A_2 = (\frac{1}{9})^1 \cdot 3 \cdot 4^0 \triangle$, $A_3 = (\frac{1}{9})^2 \cdot 3 \cdot 4^1 \triangle$, $\cdots$,

$A_n = (\frac{1}{9})^{n-1} \cdot 3 \cdot 4^{n-2} \triangle = \frac{1}{3}(\frac{4}{9})^{n-2} \triangle$.

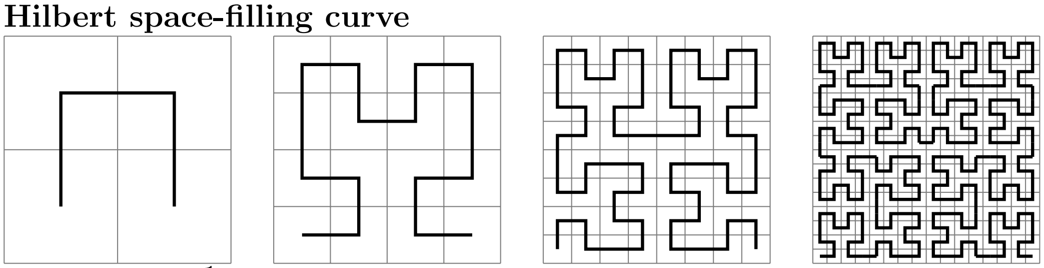
Total area $A = \sum_{n=1}^{\infty} A_n = (1 + \frac{1/3}{1 - 4/9})\triangle = \frac{8}{5}\triangle$. ($|r| = |\frac{4}{9}| < 1$)

Koch's anti-snowflake



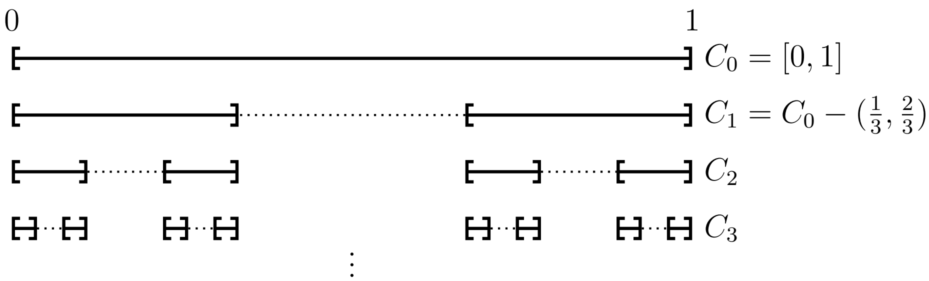
Total length $\lim_{n\to\infty} L_n = \infty$, total area $\sum_{n=1}^{\infty} A_n = (1 - \frac{1/3}{1 - 4/9})\triangle = \frac{2}{5}\triangle$.

Hilbert space-filling curve



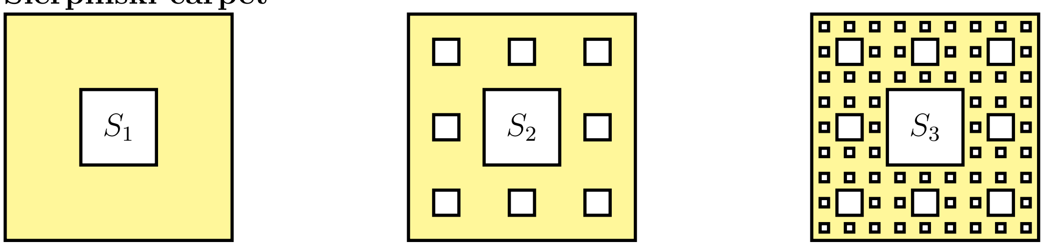
Length: $2^n - \frac{1}{2^n}$.

Cantor set (Exercise 11.2.89)

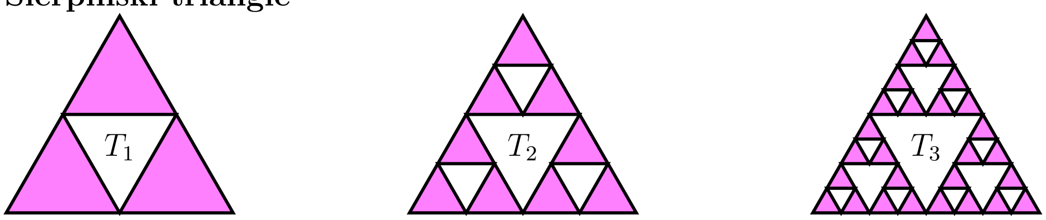


Cantor set $\mathcal{C} = \bigcap_{n=0}^{\infty} C_n = \lim_{n\to\infty} C_n$, infinite many points, but zero length.

Sierpinski carpet

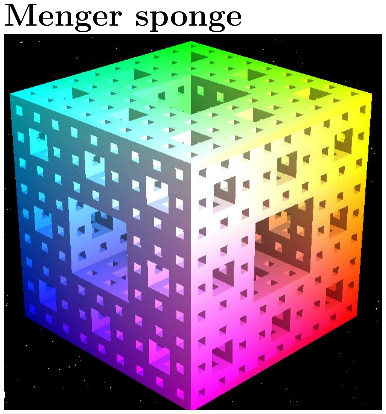


Sierpinski triangle

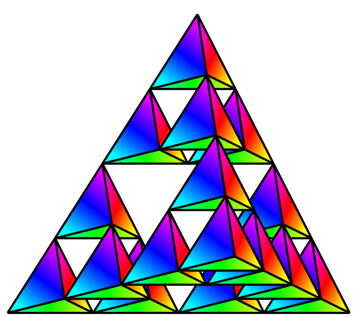


$S = \lim_{n\to\infty} S_n$ has zero area. $T = \lim_{n\to\infty} T_n$ has zero area.

Menger sponge



Seirpinski tetrahedron



Hausdorff dimension:
$\mathcal{C}$: $\log_3 2 \approx 0.63$.
S : $\log_3 8 \approx 1.89$.
T : $\log_2 3 \approx 1.58$.
MS : $\log_3 20 \approx 2.73$.
ST : $\log_2 4 = 2$.

♦ Additional: $1+1+1+1+\cdots=-1/2$?

在某一年裡有兩位物理學家分別在巴賽隆納演講不同主題, 但是他們在介紹時都說了一句令人難忘的話:
「大家都知道, $1+1+1+1+\cdots=-1/2$。」
或許意味著「如果你不知道, 那你繼續聽下去也沒用。」

黎曼 ζ 函數 Riemann zeta function: $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$.

格蘭迪級數 Grandi's series: $\sum_{n=0}^{\infty}(-1)^n=1-1+1-1+\cdots$.

- $\mathbf{1-1+1-1+\cdots=1/2}$.
  Let $S=1-1+1-1+\cdots$,
  $1-S=1-(1-1+1-1+\cdots)=1-1+1-1+\cdots=S$,
  $S=1/2$.

- $\mathbf{1-2+3-4+\cdots=1/4}$.
  Let $T=1-2+3-4+\cdots$,
  $T+T=(1-2+3-4+\cdots)+(1-2+3-4+\cdots)$
  $=1-(2-1)+(3-2)-(4-3)+\cdots=1-1+1-1+\cdots=S=1/2$,
  $T=1/4$.

  歐拉 Euler 在 1749 給出:

- $\boldsymbol{\zeta(0)=\sum_{n=1}^{\infty}1=1+1+1+1+\cdots=-1/2}$.
  $\zeta(0)-2\zeta(0)=(1+1+1+1+\cdots)-(2+2+2+2+\cdots)$
  $=1+(1-2)+1+(1-2)+\cdots=1-1+1-1+\cdots=S=1/2$,
  $\zeta(0)=-1/2$.

- $\boldsymbol{\zeta(-1)=\sum_{n=1}^{\infty}n=1+2+3+4+\cdots=-1/12}$.
  $\zeta(-1)-4\zeta(-1)=(1+2+3+4+\cdots)-(4+8+12+16+\cdots)$
  $=1+(2-4)+3+(4-8)+\cdots=1-2+3-4+\cdots=T=1/4$,
  $\zeta(-1)=-1/12$.

(想想看, 到底是哪裡出了問題? 如果你不知道, 那你還是得繼續聽下去。)

⧫ Additional: Euclid's theorem: Infinitely many primes.

質數/素數 (prime [number]): 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, ...

Proof.

- (Euclid)

  如果有限多個質數 $p_1, p_2, \ldots, p_n$, 則 $\displaystyle\prod_{i=1}^{n} p_i + 1$ 是新的質數, 矛盾! ■

- (Euler)

  根據數論基本定理 (Fundamental Theorem of Arithmetic):
  每個正整數 n 有唯一的質因數分解 (unique prime factorization).

  Assume $n = 2^k 3^\ell 5^m \cdots$, then

$$\begin{aligned}
\prod_{\text{prime } p} \frac{1}{1-1/p} &= \prod_{\text{prime } p} \sum_{i=0}^{\infty} \frac{1}{p^i} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k} \times \sum_{\ell=0}^{\infty} \frac{1}{3^\ell} \times \sum_{m=0}^{\infty} \frac{1}{5^m} \times \cdots \\
&= \sum_{k,\ell,m,\cdots \geq 0} \frac{1}{2^k 3^\ell 5^m \cdots} \\
&= \sum_{n=1}^{\infty} \frac{1}{n}.
\end{aligned}$$

  (如果有限多個質數, 則調和級數收斂。)
  因爲調和級數發散, 所以質數有無限多個。 ■