2.4 The precise definition of a limit

1. definition of limit 極限定義
2. one-side limit 單邊極限
3. infinite limit 無限極限

ε	epsilon 艾普西龍	δ	delta 黛兒塔

什麼叫靠近 (approach)? 一公分? 一公尺? 一公里?
你問我靠你有多近? 我挨你有幾分? 你去想一想, 你去看一看, ε-δ 我的近。

0.1 Definition of limit

Recall: $\lim_{x\to a} f(x) = L \iff f(x) \to L$ as $x \to a$. 怎麼說明靠近 (approach "$\to$")?
要用 ε-δ 語言: 以 ε & δ 代表距離, 用來描述靠近。

Define: $f(x)$ is defined on (b, c) with $b < a < c$ (except a possibly).

$$\boxed{\lim_{x\to a} f(x) = L}$$

$$\text{if } \boxed{\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \ni 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.}$$

如果對所有 $\varepsilon > 0$, 都存在 $\delta > 0$, 使得只要 $0 < |x-a| < \delta$, 就會 $|f(x)-L| < \varepsilon$。

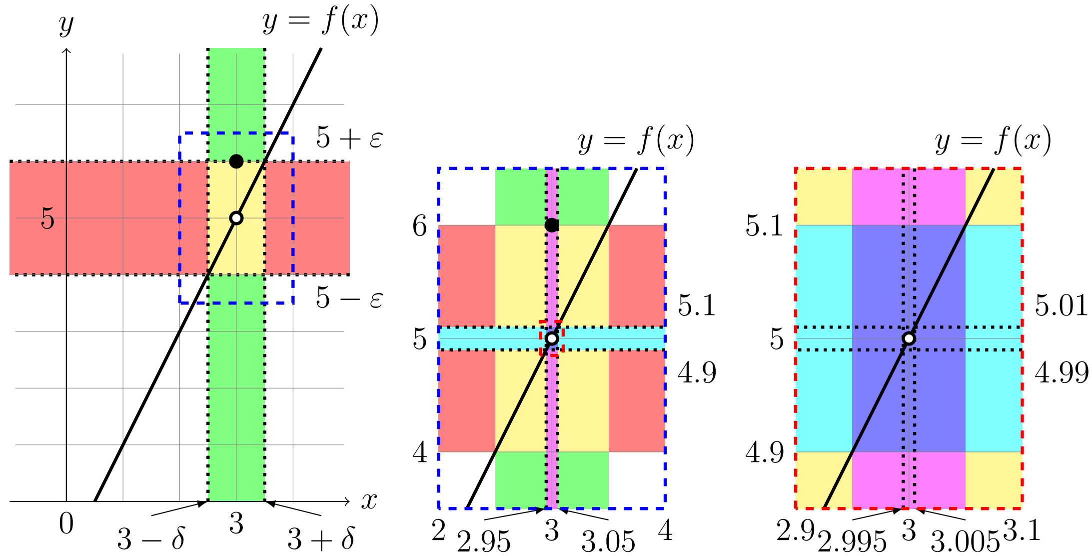
Notation: $\boxed{\forall}$: for all 對所有;
$\boxed{\exists}$: exists 存在;
$\boxed{\ni}$: such that (s.t.) 使得;
$\boxed{\Longrightarrow\ (\Rightarrow, \to)}$: implies 若 (前者爲眞) 則 (後者爲眞)。

$\lim_{x\to a} f(x) = L$ 代表: 不管你要求 $f(x)$ 以任何 (你給的) 距離 ε 靠近 L, 它能保證只要 x 與 a 距離在某個 (一定存在的) δ 以內就有。

反過來 (脚色互換), 要證明 $\lim_{x\to a} f(x) = L$, 就要對任意給定的 ε 找出 δ, 證明只要 x 是以 δ 的距離 (或更小) 靠近 a, $f(x)$ 就會以 (至少有) ε 的距離靠近 L。

Ex: $f(x) = \begin{cases} 2x-1, & x \neq 3 \\ 6, & x = 3 \end{cases}$, $\lim_{x\to 3} f(x) = 5$ (polynomial).

How to prove $f(x) \to 5$ as $x \to 3$?



Case 1. $\varepsilon = 0.1$.

$|f(x) - L| = |(2x-1) - 5| = 2|x-3| < 0.1 \iff |x-3| < 0.05$.

所以只要 x 以 0.05 的距離靠近 3, $f(x)$ 就會以 0.1 的距離靠近 5。

Case 2. $\varepsilon = 0.01$.

$|f(x) - L| = |(2x-1) - 5| = 2|x-3| < 0.01 \iff |x-3| < 0.005$.

所以只要 x 以 0.005 的距離靠近 3, $f(x)$ 就會以 0.01 的距離靠近 5。

Case 3. $\varepsilon > 0$.

$|f(x) - L| = |(2x-1) - 5| = 2|x-3| < \varepsilon \iff |x-3| < \dfrac{\varepsilon}{2}$.

所以$\overset{\boxed{\text{as}}}{\text{只要}}$ x 以 $\delta \overset{(\leq)}{=} \dfrac{\varepsilon}{2}$ (或更小) 的距離$\overset{\boxed{\to 3}}{\text{靠近}}$ 3, $f(x)$ 就會以 (至少有) ε 的距離$\overset{\boxed{\to 5}}{\text{靠近}}$ 5. $\therefore$ By the definition of limit, $\lim_{x\to 3} f(x) = 5$.

Note: 因果不要搞反了: $|f(x) - L| < \varepsilon \not\Longrightarrow 0 < |x-a| < \delta$.

$f(x)$ 靠近 L 的地方可能很多, 可能 x 離 a 很遠, 但是 $f(x)$ 還是很靠近 L。

0.2 One-side limit

Define: $f(x)$ is defined on (b, a) (resp. (a, c)).

$$\boxed{\lim_{\substack{x\to a^- \\ a^+}} f(x) = L}$$

$$\text{if } \boxed{\forall\, \varepsilon > 0, \exists\, \delta > 0, \ni \begin{array}{c} a-\delta < x < a \\ a < x < a+\delta \end{array} \implies |f(x) - L| < \varepsilon.}$$

(Prove by definition "$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$".)

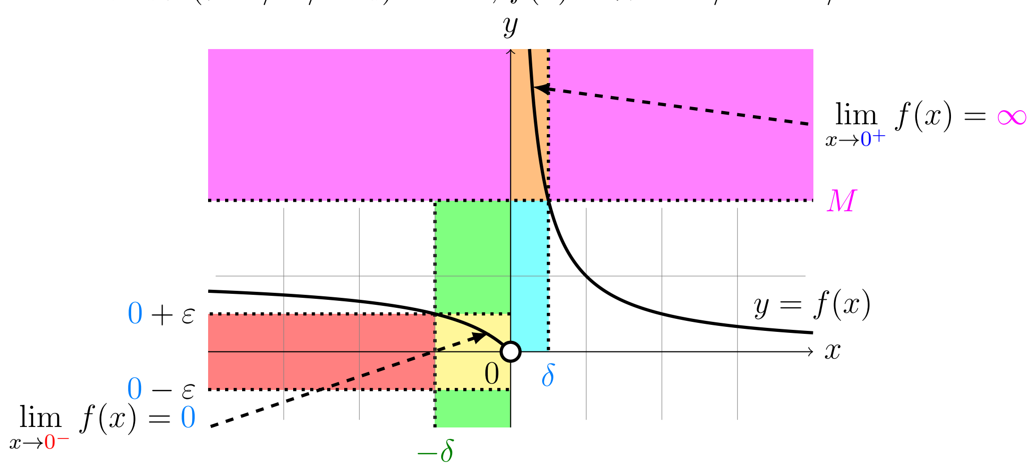
0.3 Infinite limit

Define: $f(x)$ is defined on $(b, a) \cup (a, c)$ (resp. (b, a) or (a, c)).

$$\boxed{\lim_{\substack{x\to a \\ a^- \\ a^+}} f(x) = \begin{array}{r} \infty \\ -\infty \end{array}}$$

$$\text{if } \boxed{\forall \begin{array}{l} M > 0, \\ N < 0 \end{array} \exists\, \delta > 0, \ni \begin{array}{c} 0 < |x-a| < \delta \\ a-\delta < x < a \\ a < x < a+\delta \end{array} \implies \begin{array}{l} f(x) > M. \\ f(x) < N \end{array}}$$

怎麼描述任意大/小? 任何 (至少比零)大/小的 M/N, 都能找到 δ, 保證只要 x 以 δ 的距離 (從兩/左/右邊) 靠近 a, $f(x)$ 就會比 M/N 還大/小。



How to prove limit by the definition (find δ): (標準流程)

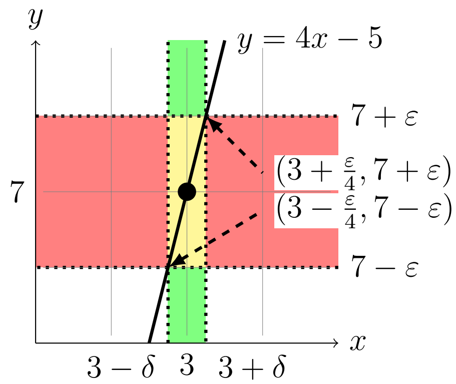
Step 1. Guessing a value for δ ($\delta = \delta(\varepsilon)$). (說明 δ 是怎麼找到的。)

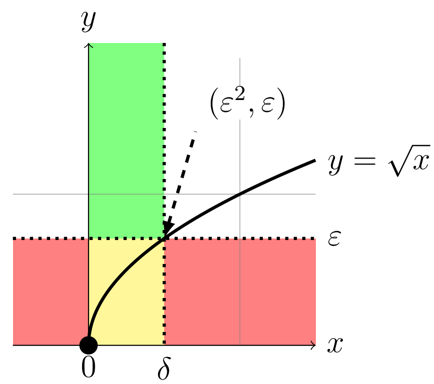
Step 2. Showing this δ works. (驗證符合定義的描述。)

Example 0.1 *Prove* $\lim\limits_{x\to 3}(4x-5) = 7$.

Prove: "$\forall\ \varepsilon > 0,\ \exists\ \delta > 0,\ \ni\ 0 < |x-3| < \delta \implies |(4x-5)-7| < \varepsilon$."
1. (Guess) $|(4x-5)-7| < \varepsilon \iff 4|x-3| < \varepsilon \iff |x-3| < \varepsilon/4$,
*(*比較 $0 < |x-3| < \delta$*) guess* $\delta = \varepsilon/4$.
2. (Show) Given $\varepsilon > 0$*, choose* $\delta = \varepsilon/4$.
If $0 < |x-3| < \delta$*, then* $|(4x-5)-7| = 4|x-3| < 4\cdot\delta = 4\cdot\varepsilon/4 = \varepsilon$.
Therefore, by the definition (of the limit), $\lim\limits_{x\to 3}(4x-5) = 7$. ■

Skill 1: 用 $|f(x)-L| < \varepsilon$ 推出 $|x-a| < \delta(\varepsilon)$, 猜 $\delta = \delta(\varepsilon)$.





Example 0.2 *Prove* $\lim\limits_{x\to 0^+}\sqrt{x} = 0$.

Prove: "$\forall\ \varepsilon > 0,\ \exists\ \delta > 0,\ \ni\ 0 < x < \delta \implies |\sqrt{x}-0| < \varepsilon$."
1. $|\sqrt{x}-0| = \sqrt{x} < \varepsilon \iff x < \varepsilon^2$*, guess* $\delta = \varepsilon^2$.
2. Given $\varepsilon > 0$*, choose* $\delta = \varepsilon^2$.
If $0 < x < \delta$*, then* $|\sqrt{x}-0| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = |\varepsilon| = \varepsilon$.
Therefore, by the definition (of the right-hand limit), $\lim\limits_{x\to 0^+}\sqrt{x} = 0$. ■

Attention: $\lim\limits_{x\to 0}\sqrt{x} \neq 0$. (Can you explain why?)

Example 0.3 *Prove* $\lim\limits_{x\to a} c = c$. *(Choose* $\delta = 1$*.)*

Example 0.4 *Prove* $\lim\limits_{x\to a} x = a$. *(Choose* $\delta = \varepsilon$*.)*

Example 0.5 *Prove* $\lim_{x\to 3} x^2 = 9$.

1. $|x^2-9| = |x+3||x-3|$. *(*$|x-3|$ 很靠近零, 但是 $|x+3|$ 呢*?)*

idea: *If* $|x+3| < C$ *for some* $C > 0$, *then let* $|x-3| < \dfrac{\varepsilon}{C}$ *and hence* $|x^2-9| < C\cdot\dfrac{\varepsilon}{C} = \varepsilon$.

try: *When* $|x-3| < 1$, $|x+3| < 7$*; so let* $C = 7$ *and guess* $\delta = \min\left\{1, \dfrac{\varepsilon}{7}\right\}$.

2. Given $\varepsilon > 0$, *choose* $\delta = \min\left\{1, \dfrac{\varepsilon}{7}\right\}$. *(*選最小才能保證 $<, <$ 都成立。*)*

If $0 < |x-3| < \delta$, *then* $0 < |x-3| < 1 \Rightarrow |x+3| < 7$, *and* $0 < |x-3| < \dfrac{\varepsilon}{7}$,

so $|x^2-9| = |x+3||x-3| < 7\cdot\dfrac{\varepsilon}{7} = \varepsilon$.

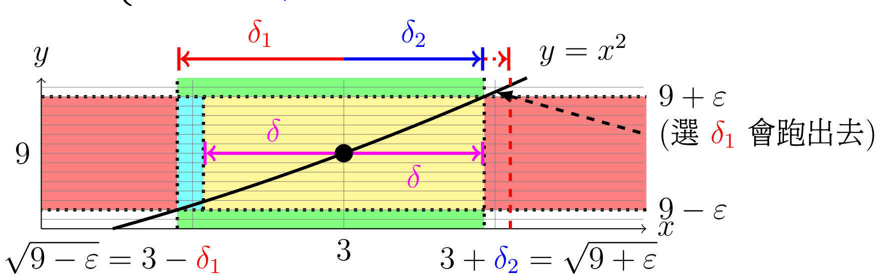
Therefore, by the definition, $\lim_{x\to 3} x^2 = 9$. ■

Skill 2: δ 可以嘗試一些數字 (like 1) 夾住其他乘積項, 再讓 δ 取最小值。

[Another method]: (用 **Skill 1**)

$$
\begin{array}{llcccl}
\because & & & |x^2-9| & < & \varepsilon \\
\Longleftrightarrow & -\varepsilon & < & x^2-9 & < & \varepsilon \\
\Longleftrightarrow & 9-\varepsilon & < & x^2 & < & 9+\varepsilon \\
\Longrightarrow & \sqrt{9-\varepsilon} & < & x & < & \sqrt{9+\varepsilon} \quad (\text{when } \varepsilon < 9) \\
\Longleftrightarrow & \sqrt{9-\varepsilon}-3 & < & x-3 & < & \sqrt{9+\varepsilon}-3 \quad (\sqrt{9-\varepsilon}-3 < 0) \\
\Longrightarrow & 3-\sqrt{9-\varepsilon} & > & |x-3| & < & \sqrt{9+\varepsilon}-3
\end{array}
$$

$$
\text{Choose } \delta = \begin{cases} \min\{3-\sqrt{9-\varepsilon}, \sqrt{9+\varepsilon}-3\} & \text{when } \varepsilon < 9, \\ \sqrt{9+\varepsilon}-3 & \text{when } \varepsilon \geq 9. \end{cases}
$$



從點 (a, L) 沿著 $y = f(x)$ 找第一次跑出 $y = L+\varepsilon$ 與 $y = L-\varepsilon$ 包圍的 x (解 $|f(x)-L| = \varepsilon$), 選擇 $\delta = \min\{|x-a|\}$ (要取最小, 這也是最大可能的 δ), 但是有時候不好算。

♡考: 已知 $\lim_{x\to a} f(x) = L$, 給定 ε, 找最大/可用的 δ。(100,101,102會考)

Example 0.6 *Prove limit law: (addition)*

$$\boxed{\lim_{x\to a} f(x) = L \,\&\, \lim_{x\to a} g(x) = M \implies \lim_{x\to a}[f(x)+g(x)] = L+M.}$$

Proof. *Given* $\varepsilon > 0$. $|[f(x)+g(x)]-(L+M)| = |(f(x)-L)+(g(x)-M)| \le |f(x)-L|+|g(x)-M|$. $(\because |a+b| \le |a|+|b|.)$

$\because \lim_{x\to a} f(x) = L$, $\exists\ \delta_1 > 0$, $\ni 0 < |x-a| < \delta_1 \implies |f(x)-L| < \frac{\varepsilon}{2}$.

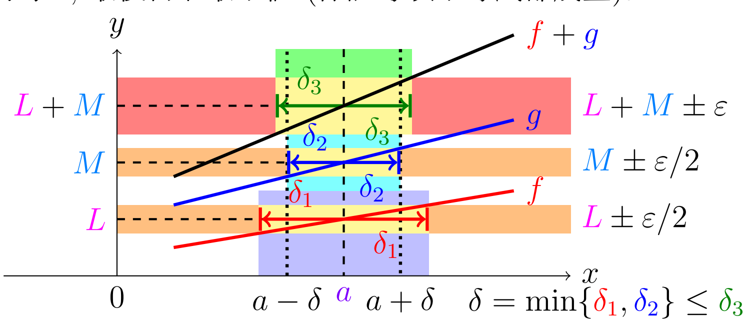
$\because \lim_{x\to a} g(x) = M$, $\exists\ \delta_2 > 0$, $\ni 0 < |x-a| < \delta_2 \implies |g(x)-M| < \frac{\varepsilon}{2}$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

If $0 < |x-a| < \delta$, *then* $0 < |x-a| < \delta_1$ *and* $0 < |x-a| < \delta_2$,
and so $|f(x)-L| < \frac{\varepsilon}{2}$ *and* $|g(x)-M| < \frac{\varepsilon}{2}$,
$\implies |[f(x)+g(x)]-(L+M)| \le |f(x)-L|+|g(x)-M| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$

Therefore, by the definition, $\lim_{x\to a}[f(x)+g(x)] = L+M$. ■

Skill 3: 用 triangle inequality 三角不等式 ($|a+b| \le |a|+|b|$, $|a+b+c| \le |a|+|b|+|c|$, ...) 分成總和爲 ε 的多項 ($\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$, $\frac{\varepsilon}{3}+\frac{2\varepsilon}{3}$, $\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$, ...), 找出個別的 δ, 最後再取最小値 (保證每項不等式都成立)。



Example 0.7 (Extended) *(continue)* $\implies \lim_{x\to a} f(x)g(x) = LM$.

Proof. $|fg-LM| = |(fg-Lg)+(Lg-LM)| \le |f-L||g|+|L||g-M|$.

1. $\exists\ \delta_1 > 0$, $\ni 0 < |x-a| < \delta_1 \implies |g-M| < 1 \iff |g| < |M|+1$;
2. $\exists\ \delta_2 > 0$, $\ni 0 < |x-a| < \delta_2 \implies |f-L| < \frac{\varepsilon}{2(|M|+1)}$;
3. $\exists\ \delta_3 > 0$, $\ni 0 < |x-a| < \delta_3 \implies |g-M| < \frac{\varepsilon}{2(|L|+1)}$. *(*避開 $L=0$*)*

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. *If* $0 < |x-a| < \delta$, *then (*略*) and*
$|fg-LM| < \frac{\varepsilon}{2(|M|+1)} \cdot (|M|+1) + |L| \cdot \frac{\varepsilon}{2(|L|+1)} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$ ■

Example 0.8 *(infinite limit) Prove* $\lim_{x\to 0}\frac{1}{x^2}=\infty$.

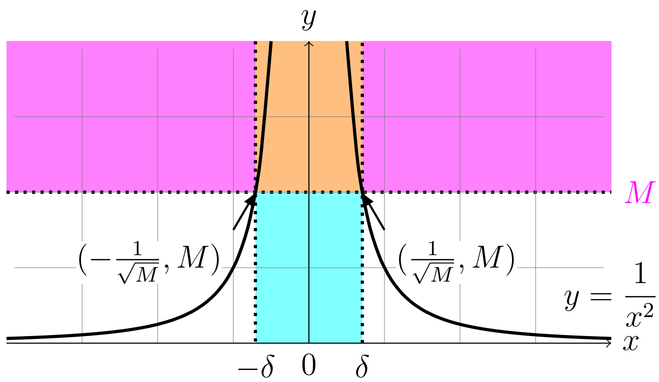
Prove: "$\forall\ M>0,\ \exists\ \delta>0,\ \ni 0<|x-0|<\delta \implies \frac{1}{x^2}>M$."

1. $\frac{1}{x^2}>M \iff |x|<\frac{1}{\sqrt{M}}$, *guess* $\delta=\frac{1}{\sqrt{M}}$.

2. *Given* $M>0$, *choose* $\delta=\frac{1}{\sqrt{M}}$.

If $0<|x-0|<\delta$, *then* $\frac{1}{x^2}>\frac{1}{\delta^2}=\frac{1}{(\frac{1}{\sqrt{M}})^2}=M$.

Therefore, by the definition, $\lim_{x\to 0}\frac{1}{x^2}=\infty$. ($\frac{1}{x^2}\to\infty$ *as* $x\to 0$.) ■



Remind: $\lim_{x\to a} f(x) = L$ or $f(x)\to L$ as $x\to a$

$\lim_{x\to a} f(x)=$	L	or	$f(x)\to$ L	as $x\to a$
a^-	∞		∞	a^-
a^+	$-\infty$		$-\infty$	a^+

if $\forall\ \varepsilon>0$, $\exists\ \delta>0,\ \ni 0<|x-a|<\delta \implies |f(x)-L|<\varepsilon$.

$M>0$	$a-\delta<x<a$	$f(x)>M$
$N<0$	$a<x<a+\delta$	$f(x)<N$

When proving

- limit: $0\ \boxed{<}\ |x-a|<\delta$ 避開 $x=a$ 的情形。
- one-side limit: $a-\delta<x<a$ & $a<x<a+\delta$ 左右邊不同。
- infinite limit: $f(x)>M$ & $f(x)<N$ 沒有絕對值。

Remark: 計算極限的方法: 極限律, 左右極限, 夾擠定理, 都可用 ε-δ 證明。
(Try to prove by ε-δ: limit laws, left/right-hand limits, Squeeze Theorem.)

⧫ Additional: Proof of left/right-hand limits

"$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$"

Proof. $(\Rightarrow)$ $\forall\ \varepsilon > 0$,

$\because \lim_{x\to a} f(x) = L$, $\exists\ \delta > 0$, $\ni 0 < |x-a| < \delta \implies |f(x)-L| < \varepsilon$.

If $a-\delta < x < a$, then $0 < a-x = |x-a| < \delta \implies |f(x)-L| < \varepsilon$.

$\therefore$ by the definition, $\lim_{x\to a^-} f(x) = L$.

If $a < x < a+\delta$, then $0 < x-a = |x-a| < \delta \implies |f(x)-L| < \varepsilon$.

$\therefore$ by the definition, $\lim_{x\to a^+} f(x) = L$.

$(\Leftarrow)$ $\forall\ \varepsilon > 0$,

$\because \lim_{x\to a^-} f(x) = L$, $\exists\ \delta_1 > 0$, $\ni a-\delta_1 < x < a \implies |f(x)-L| < \varepsilon$;

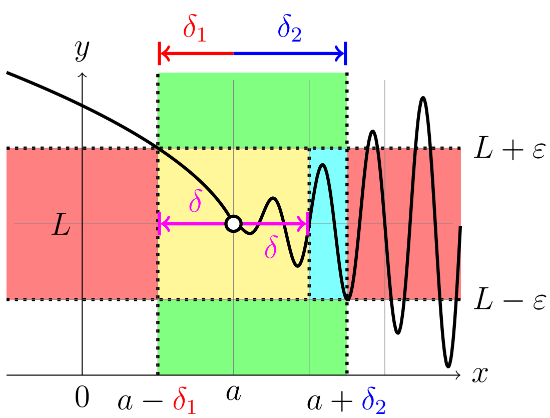
$\because \lim_{x\to a^+} f(x) = L$, $\exists\ \delta_2 > 0$, $\ni a < x < a+\delta_2 \implies |f(x)-L| < \varepsilon$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

If $0 < |x-a| < \delta$, then $\begin{Bmatrix} \text{either } -\delta < x-a < 0,\ a-\delta_1 < a-\delta < x < a \\ \text{or}\quad 0 < x-a < \delta,\ a < x < a+\delta < a+\delta_2 \end{Bmatrix}$

$\implies |f(x)-L| < \varepsilon$.

$\therefore$ by the definition, $\lim_{x\to a} f(x) = L$. ■



⧫ Additional: Proof of limit "does not exist"

"$\lim_{x\to a} f(x) \neq L$, $\forall\ L \in \mathbb{R}$"

$\forall\ L \in \mathbb{R}$, $\exists\ \varepsilon > 0$, $\ni \forall\ \delta > 0$, $\exists\ x$ with $0 < |x-a| < \delta$ and $|f(x)-L| \geq \varepsilon$.