14.1 Functions of several variables

認識多變數函數

1. functions of two variables
2. functions of more than two variables

0.1 Function of two variables

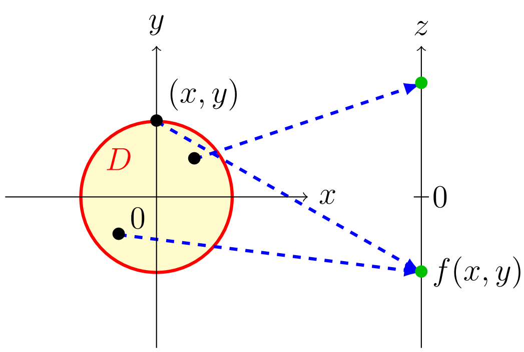
三角形面積 $A = \frac{1}{2}bh$, 可以看成面積是底 (b) 與高 (h) 的函數 $A(b,h) = \frac{1}{2}bh$。
圓柱體積 $V = \pi r^2 h$, 可以看成體積是半徑 (r) 與高 (h) 的函數 $V(r,h) = \pi r^2 h$。

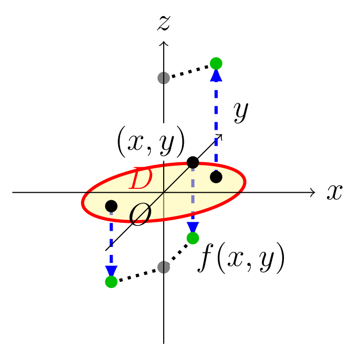
verbally (文字描述)

Define: A ***function of two variables*** 雙變數函數 f is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by $f(x,y)$.

$$f : D \subseteq \mathbb{R}^2 \to R \subseteq \mathbb{R}$$

where, D is the domain 定義域 and $R = \{f(x,y) : (x,y) \in D\}$ is the range 值域 of f.



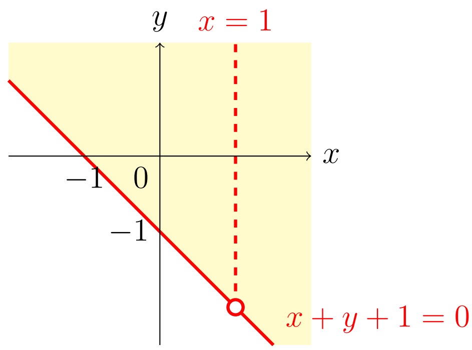


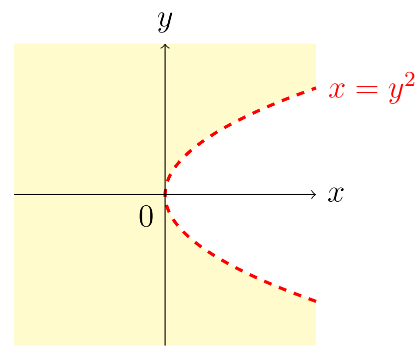
通常寫成 $\boldsymbol{z} = \boldsymbol{f(x,y)}$.
$\boldsymbol{x}, \boldsymbol{y}$ 叫做 ***independent variables*** 獨立 (自, 因) 變數 (量, 項)。
$\boldsymbol{z}$ 叫做 ***dependent variable*** 相依 (依, 應) 變數 (量, 項)。

Note: 定義域沒講就是取最大可能。

Example 0.1 *Evaluate $f(3,2)$ and find the domain of*
(a) $f(x,y)=\dfrac{\sqrt{x+y+1}}{x-1}$; *(b)* $f(x,y)=x\ln(y^2-x)$.

(a) $f(3,2)=\dfrac{\sqrt{3+2+1}}{3-1}=\dfrac{\sqrt{6}}{2}$,
domain $D=\{(x,y): x+y+1\geq 0,\ x\neq 1\}$. *(range $R=\mathbb{R}$)*
(b) $f(3,2)=3\ln(2^2-3)=0$,
domain $D=\{(x,y): y^2-x>0\}$. *(range $R=\mathbb{R}$)* ■





numerically (數字表格)

Example 0.2 *Wind-chill index* 風寒指數 $W=W(T,v)$, T: 溫度, v: 速度。

$T\backslash v$	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

Wind-chill index: T(°C), v(km/h).
$W(T,v)\approx 13.12+0.6215T-11.37v^{0.16}+0.3965Tv^{0.16}$ (2001 US&Canada).

algebraically (明確的公式表示)

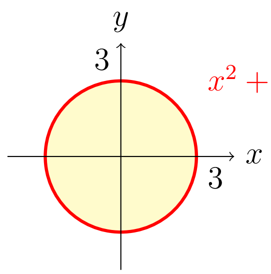
Example 0.3 *Cobb-Douglas production function*

$$P(L,K) = bL^{\alpha}K^{1-\alpha}$$

P: total production 產量*;*
L: amount of labor 工時*;*
K: amount of capital invested 資本投資*;*
domain $\{(L,K) : L \geq 0, K \geq 0\}$.

Example 0.4 *Find the domain and range of $g(x,y) = \sqrt{9-x^2-y^2}$.*

Domain $D = \{(x,y) : 9-x^2-y^2 \geq 0\} = \{(x,y) : x^2+y^2 \leq 9\}$,
Range $R = \{\sqrt{9-x^2-y^2} : x^2+y^2 \leq 9\} = [0,3]$.



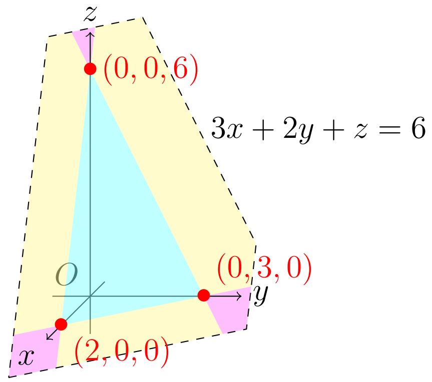
Note: 注意, 由 $g(x,y)$ 只知道值域包含在 $[0,3]$ 之中 (Range of $g \subseteq [0,3]$), 但是不保證所有值都有。(要證明)
Proof. $\forall\ t \in [0,3]$, $g(\sqrt{9-t^2},0) = t \implies$ Range of g is $[0,3]$.

visually ($\overset{\text{3D}}{\text{圖}}$ 或$\overset{\text{2D}}{\text{等高曲線}}$)

Define: The ***graph*** 圖 of a function f of two variables with domain D is $\{(x,y,z) : z = f(x,y), (x,y) \in D\}$.

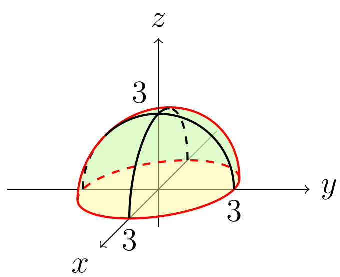
(先找 domain, 再畫 $z = f(x,y)$.)

Example 0.5 *Sketch the graph of $f(x,y) = 6 - 3x - 2y$.*



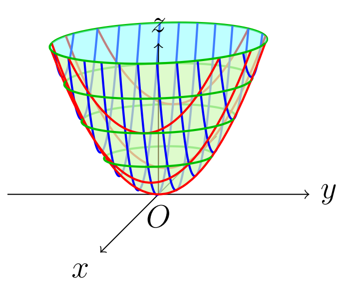
Note: $f(x,y) = ax + by + c$ is called a ***linear function*** 線性函數 whose graph $z = ax + by + c$ or $ax + by - z + c = 0$, a plane 平面。

Example 0.6 *Sketch the graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$.*



(top-half of sphere 上半球 of $x^2 + y^2 + z^2 = 9$.)

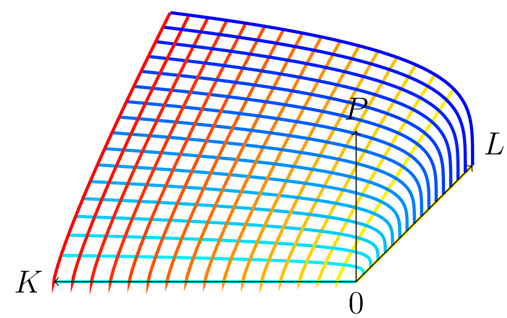
Example 0.7 *Find the domain and range and sketch the graph of $h(x,y) = 4x^2 + y^2$.*



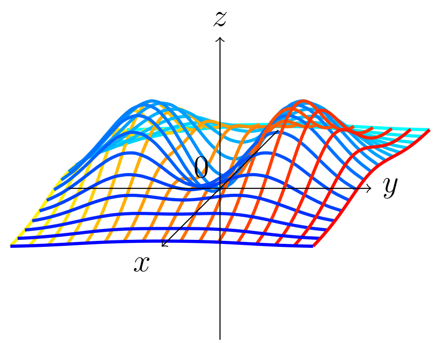
(elliptic paraboloid 橢圓拋物面 $x^2 + \frac{y^2}{4} = \frac{z}{4}$.)

Domain $\mathbb{R}^2$, range $[0,\infty)$. ($\forall\ t \geq 0,\ h(0,\sqrt{t}) = t$.) ■

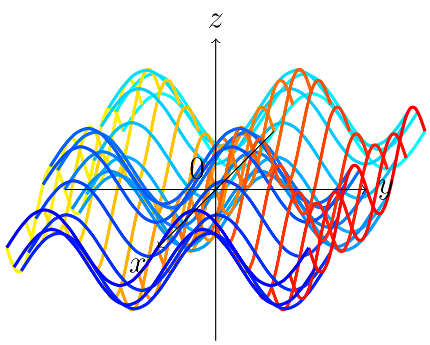
大多的電腦繪圖都是畫出 $x = k$ 與 $y = k$ 的曲線 (等間隔的 k), 再把格子填色並移除被遮住的部分.



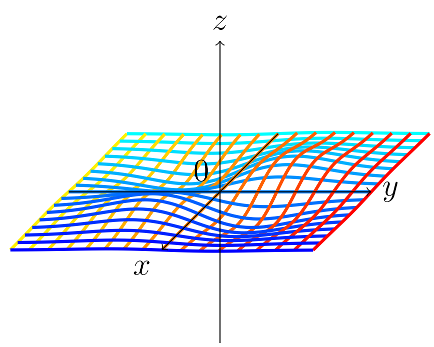
$P(L, K) = 1.01L^{0.75}K^{0.25}$



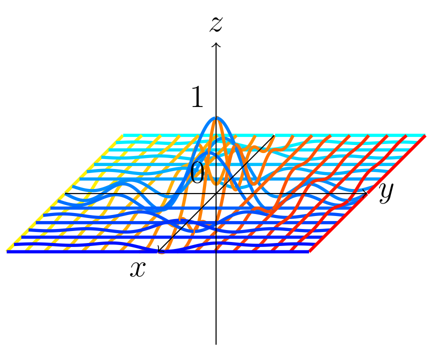
$f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



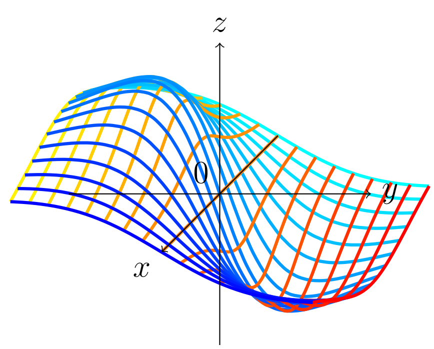
$f(x, y) = \sin x + \sin y$



$f(x, y) = -xye^{-x^2-y^2}$



$$f(x, y) = \frac{\sin x \sin y}{xy}$$

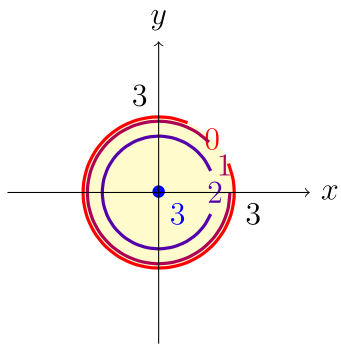


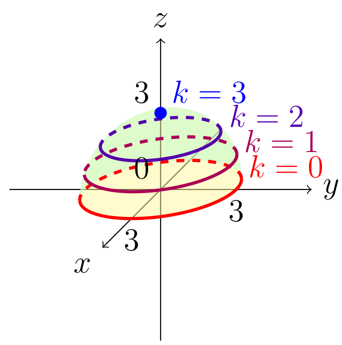
$$f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$$

Define: the ***level curves*** 等高 (曲) 線 (or ***contour lines*** 輪廓線) of a function of two variables f are the curves with equations $f(x, y) = k$, where k is a constant in the range of f.

Example 0.8 *Sketch the level curve of* $g(x, y) = \sqrt{9 - x^2 - y^2}$ *for* $k = 0, 1, 2, 3$.

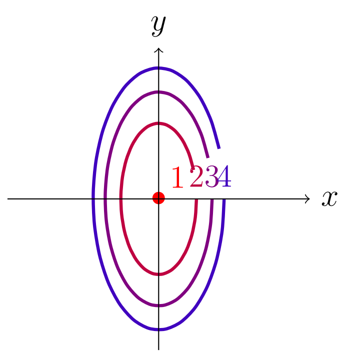
$\sqrt{9 - x^2 - y^2} = k$, $x^2 + y^2 = (\sqrt{9 - k^2})^2$, *circles of radius* $3, 2\sqrt{2}, \sqrt{5}$, *and a point* $(0, 0)$. ∎

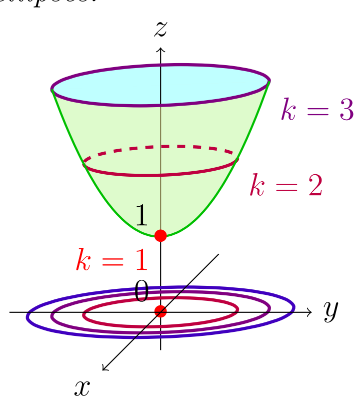




Example 0.9 *Sketch the level curve of* $h(x, y) = 4x^2 + y^2 + 1$.

$4x^2 + y^2 + 1 = k$, $\dfrac{x^2}{(\sqrt{k-1}/2)^2} + \dfrac{y^2}{\sqrt{k-1}^2} = 1$.
$k = 1$ *a point* $(0, 0)$, $k > 1$ *ellipses.* ∎





Note: 1. 常用於: 高度 (contour line 等高 (輪廓) 線), 氣壓 (isobar 等壓線), 體積 (isochore 等容線), 溫度 (isotherm 等溫線)。
2. 不同 k 值的等高線不會相交, 標出 k 值很重要 (是山是谷)。
3. 通常會取 k 成等差, 這時候線越密代表變化越快。

0.2 Function of more than two variables

Define: A ***function of three variables*** 三變數函數 f is a rule that assigns to each ordered triple (x, y, z) in a domain D a unique real number denoted by $f(x, y, z)$.

$$f : D \subseteq \mathbb{R}^3 \to R \subseteq \mathbb{R}$$

Example 0.10 *Find the domain of $f(x, y, z) = \ln(z - y) + xy \sin z$.*

*Domain $D = \{(x, y, z) : z > y\}$, a **half space** consisting of all points lying above the plane $z = y$.* ■

Note: 三個變數的叫做 ***level surfaces*** 等高 (曲) 面 $f(x, y, z) = k$。

Example 0.11 *Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.*

$x^2 + y^2 + z^2 = (\sqrt{k})^2$, *spheres of center O and radius $\sqrt{k}$, $k \geq 0$.* ■

Define: A ***function of*** n ***variables*** f is a rule that assigns to each an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers in a domain D a unique real number denoted by $f(x_1, x_2, \ldots, x_n)$.

$$f : D \subseteq \mathbb{R}^n \to R \subseteq \mathbb{R}$$

Let $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle \in V_n$ be the position vector of $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. 常常寫 $f(\mathbf{x})$ 代替 $f(x_1, x_2, \ldots, x_n)$. Linear function $f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ 也可以寫成 $f(\mathbf{x}) = \mathbf{c} \bullet \mathbf{x}$, where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$.

Note: 有三種觀點: 可以看成 $\left\{\begin{array}{ll} n \text{ 個變數} & x_1, x_2, \ldots x_n \\ n \text{ 維點} & (x_1, x_2, \ldots x_n) \\ n \text{ 維向量} & \langle x_1, x_2, \ldots x_n \rangle \end{array}\right.$ 到實數的函數。

Recall:
單變數函數 $f(x) : \mathbb{R} \to \mathbb{R}$,
向量函數 $\mathbf{r}(x) : \mathbb{R} \to V_n \cong \mathbb{R}^n$,
多變數函數 $f(\mathbf{x}) : V_n \cong \mathbb{R}^n \to \mathbb{R}$。