3.4 The chain rule

1. chain rule 連鎖律 $(f(g))' = f'(g)g'$

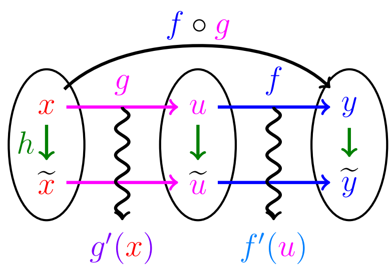
連鎖律 – 微分的超必殺技! 必學!
微積至尊, 寶刀連鎖, 微遍天下, 莫敢不微! 分部不出, 誰與爭鋒?
連鎖用的好, 微分沒煩惱; 微分學得好, 積分不苦惱。

0.1 Chain Rule

Recall: 合成函數: $f(g(x)) = (f \circ g)(x)$ & $g(f(x)) = (g \circ f)(x)$ are composite functions (composition) of functions $f(x)$ & $g(x)$.

The Chain Rule: 連鎖律, 鏈鎖律, 鏈鎖法則
If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}.$$



In Leibniz notation, $y = f(u)$ and $u = g(x)$ are both differentiable, then $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}}$.

Proof. (跟書上使用 Leibniz 的寫法不同, 但精神上是一樣的。)
$\because$ g is differentiable at x, by Theorem, $\Longrightarrow$ g is continuous at x
$\Longleftrightarrow$ $g(x+h) \to g(x)$ as $h \to 0$, $\lim\limits_{h\to 0} \overset{(\not\Leftarrow)}{\Longrightarrow} \lim\limits_{g(x+h)\to g(x)}$. (反過來不保證。)

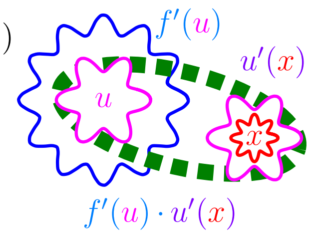
$$\begin{aligned}
[f(g(x))]' &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \qquad (\text{一乘一除 } g(x+h) - g(x)) \\
&= \lim_{h\to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{g(x+h)\to g(x)} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(g(x)) \cdot g'(x). \qquad (f'(A) = \lim_{X\to A} \tfrac{f(X)-f(A)}{X-A}.) \qquad \blacksquare
\end{aligned}$$

Note: $f'(g(x))$ 是先把 $f(x)$ 微分得到導函數 $f'(x)$, 再把 $g(x)$ 代入 $f'(x)$。

♦ **Additional:** 如果 $g(x+h) - g(x) = 0$ 怎麼辦? 這時候 $g(x) = c$ 常數。

How to use the Chain Rule:

Let $u = u(x)$, then $f = f(u)$ and (用新變數簡化函數。)



$$f'(x) = f'(u)u'(x) = f'(u(x))u'(x).$$

Step 1. 計算 $f'(u)$: 把 f 看成 u 的函數, 對 u 微分;

Step 2. 計算 $u'(x)$: 把 u 看成 x 的函數, 對 x 微分;

Step 3. 把結果相乘, 把 u 代回 x 的函數 $u(x)$, 就是 f 對 x 的微分。

Example 0.1 $f(x) = \sqrt{x^2+1}$, $f'(x) = ?$

Let $u = u(x) = x^2 + 1$, *then* $(y =)f(u) = \sqrt{u}$ *(*變成 u 的函數*) and*

$$\begin{aligned} f'(x) &= \left(\frac{dy}{du}\frac{du}{dx} = \right) f'(u)u'(x) \\ &= \frac{d}{du}(\sqrt{u}) \cdot \frac{d}{dx}(x^2+1) \;=\; \frac{1}{2\sqrt{u}} \cdot 2x \\ (\text{代回 } u = u(x)) &= \frac{1}{2\sqrt{x^2+1}} \cdot 2x \;=\; \frac{x}{\sqrt{x^2+1}}. \qquad \blacksquare \end{aligned}$$

Example 0.2 *(a)* $f(x) = \sin(x^2)$, $f'(x) = ?$ *(b)* $f(x) = \sin^2 x$, $f'(x) = ?$

(a) Let $u = u(x) = x^2$, *then* $f(u) = \sin u$ *and*

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du}(\sin u) \cdot \frac{d}{dx}(x^2) \;=\; \cos u \cdot 2x \\ (\text{代回 } u = u(x)) &= \cos(x^2) \cdot 2x \;=\; 2x\cos(x^2). \end{aligned}$$

*(*如果不會弄錯, $\frac{d}{du}f(u) \cdot \frac{d}{dx}u(x)$ 可以省略 "$\cdot$" 成 $\frac{d}{du}f(u)\frac{d}{dx}u(x)$。*)*

(b) Let $u = u(x) = \sin x$, *then* $f(u) = u^2$ *and*

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du}(u^2)\frac{d}{dx}(\sin x) \;=\; 2u \cdot \cos x \\ (\text{代回 } u = u(x)) &= 2\sin x \cdot \cos x \;=\; 2\sin x \cos x. \qquad \blacksquare \end{aligned}$$

*(*如果 $f(x)$ 只有一項而且不會弄錯, $\frac{d}{dx}(\ldots)$ 可以省略括號 "$(\,)$"。*)*

Note: $\sin(x^2) = \sin x^2 \neq \sin^2 x = (\sin x)^2$, 位置不同, 函數不同。

The Power Rule Combined with the Chain Rule:
For $n \in \mathbb{R}$, if $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x).$$ (不建議背)

Example 0.3 $y = (x^3 - 1)^{100}$, $y' = ?$

Let $u = u(x) = x^3 - 1$, *then* $f(u) = u^{100}$ *and*

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du}u^{100}\frac{d}{dx}(x^3 - 1) = 100u^{99} \cdot 3x^2 \\ \text{(代回 } u = u(x)\text{)} &= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}. \quad \blacksquare \end{aligned}$$

Note: $(f/g)' = [f \cdot (g)^{-1}]' = f'(g)^{-1} + (-1)f(g)^{-2}g' = (f'g - fg')/(g)^2$.
(書上 Examples(4–6): $\dfrac{1}{\sqrt[3]{x^2 + x + 1}}$, $\left(\dfrac{t-2}{2t+1}\right)^9$, $(2x + 1)^5(x^3 - 1 + 1)^4$ 略.)

Example 0.4 $f(x) = e^{\sin x}$, $f'(x) = ?$

Let $u = u(x) = \sin x$, *then* $f(u) = e^u$ *and*

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du}e^u\frac{d}{dx}\sin x = e^u \cdot \cos x \\ \text{(代回 } u = u(x)\text{)} &= e^{\sin x} \cdot \cos x = e^{\sin x}\cos x. \quad \blacksquare \end{aligned}$$

Example 0.5 $f(x) = a^x$, $a > 0$, $f'(x) = ?$ *(用定義證過了)*

$a^x = e^{\ln a^x} = e^{x\ln a}$. *Let* $u = u(x) = x\ln a$, *then* $f(u) = e^u$ *and*

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du}e^u\frac{d}{dx}(x\ln a) = e^u \cdot \ln a \\ \text{(代回 } u = u(x)\text{)} &= e^{x\ln a} \cdot \ln a = a^x\ln a. \quad \blacksquare \end{aligned}$$

Question: 怎麼選擇 $u = u(x)$? **Answer:** 寫習題累積經驗。

Note: 多重連鎖律: $f = f(u)$, $u = u(v)$, $v = v(x)$,

$$\boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dv}\frac{dv}{dx}}.$$

(四重、五重可不可以? 可以! 九九重陽都沒問題。)

Example 0.6 $f(x) = \sin(\cos(\tan x))$, $f'(x) = ?$

Let $v = v(x) = \tan x$ *and* $u = u(v) = \cos v$, *then* $f(u) = \sin u$ *and*

$$\begin{aligned}
f'(x) &= \left(\frac{df}{du}\frac{du}{dv}\frac{dv}{dx} = \right) f'(u)u'(v)v'(x) \\
&= \frac{d}{du}\sin u \frac{d}{dv}\cos v \frac{d}{dx}\tan x \\
&= \cos u \cdot (-\sin v) \cdot \sec^2 x \\
\text{(全部代回)} &= \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \cdot \sec^2 x \\
&= -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x. \qquad \blacksquare
\end{aligned}$$

Example 0.7 $f(\theta) = e^{\sec 3\theta}$, $f'(\theta) = ?$

Let $v = v(\theta) = 3\theta$ *and* $u = u(v) = \sec v$, *then* $f(u) = e^u$ *and*

$$\begin{aligned}
f'(\theta) &= f'(u)u'(v)v'(\theta) \\
&= \frac{d}{du}e^u \frac{d}{dv}\sec v \frac{d}{d\theta}3\theta \\
&= e^u \cdot \sec v \tan v \cdot 3 \\
\text{(全部代回)} &= e^{\sec 3\theta} \cdot \sec 3\theta \tan 3\theta \cdot 3 \\
&= 3e^{\sec 3\theta}\sec 3\theta \tan 3\theta. \qquad \blacksquare
\end{aligned}$$

Skill: 目標: 簡化函數。$e^{\sec \boxed{3\theta}} = e^{\boxed{\sec v}} = e^u \to$ Let $v = 3\theta$, $u = \sec v$.

Question: 爲什麼要令 $u = u(x)$ (變數名 = 函數名)?
Answer: 代回時比較不會找錯。
Question: 一定要 Let $u = ...$, $f'(x) = f'(u)u'(x)$? 可不可以直接算出來?
Answer: 當然可以直接算! 如果是證明題, 爲了部分分數還是要寫。
但是因爲常常$\cdots$常會忘記 $f'(u)$ 還要乘 $u'(x)$, 寫出來可以避免漏掉。