Chapter 5
The Time Value of Money

Financial Management (MGCM10018)

Preview

① $FV = PV(1+r)^t$

② $PV = \frac{C}{r}$

③ $PV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right]$

- Investments and returns often come from different time periods.
  - How do we measure and compare them?
  - It is important to understand the relationship between the values of dollars today and dollars in the future.
    - Money has a time value. It can be expressed in multiple ways:
      - A dollar today held in savings will grow.
      - A dollar received in a year is not worth as much as a dollar received today.

Outline

- Future Value and Compound Interest
- Present Value
- Multiple Cash Flows
- Level Cash Flows: Perpetuities and Annuities
- Annuities Due
- Effective Annual Interest Rates
- Inflation and the Time Value of Money

Future Values (5.1)

- Future value (FV) is the amount to which an investment will grow after earning interest.
- Let r = annual interest rate and let t = # of years:
  - Simple interest

$$\text{FV}_{Simple} = \text{Initial investment} \times (1 + r \times t)$$

  - Compound interest

$$\text{FV}_{Compound} = \text{Initial investment} \times (1 + r)^t$$

Simple Interest: Example

Interest earned at a rate of 7% for five years on a principal balance of $100.

Example - Simple Interest

	Today	Future Years				
		1	2	3	4	5
Interest Earned		*7*	*7*	*7*	*7*	*7*
Value	100	***107***	***114***	***121***	***128***	***135***

Compound Interest: Example

Interest earned at a rate of 7% for five years on the previous year's balance.

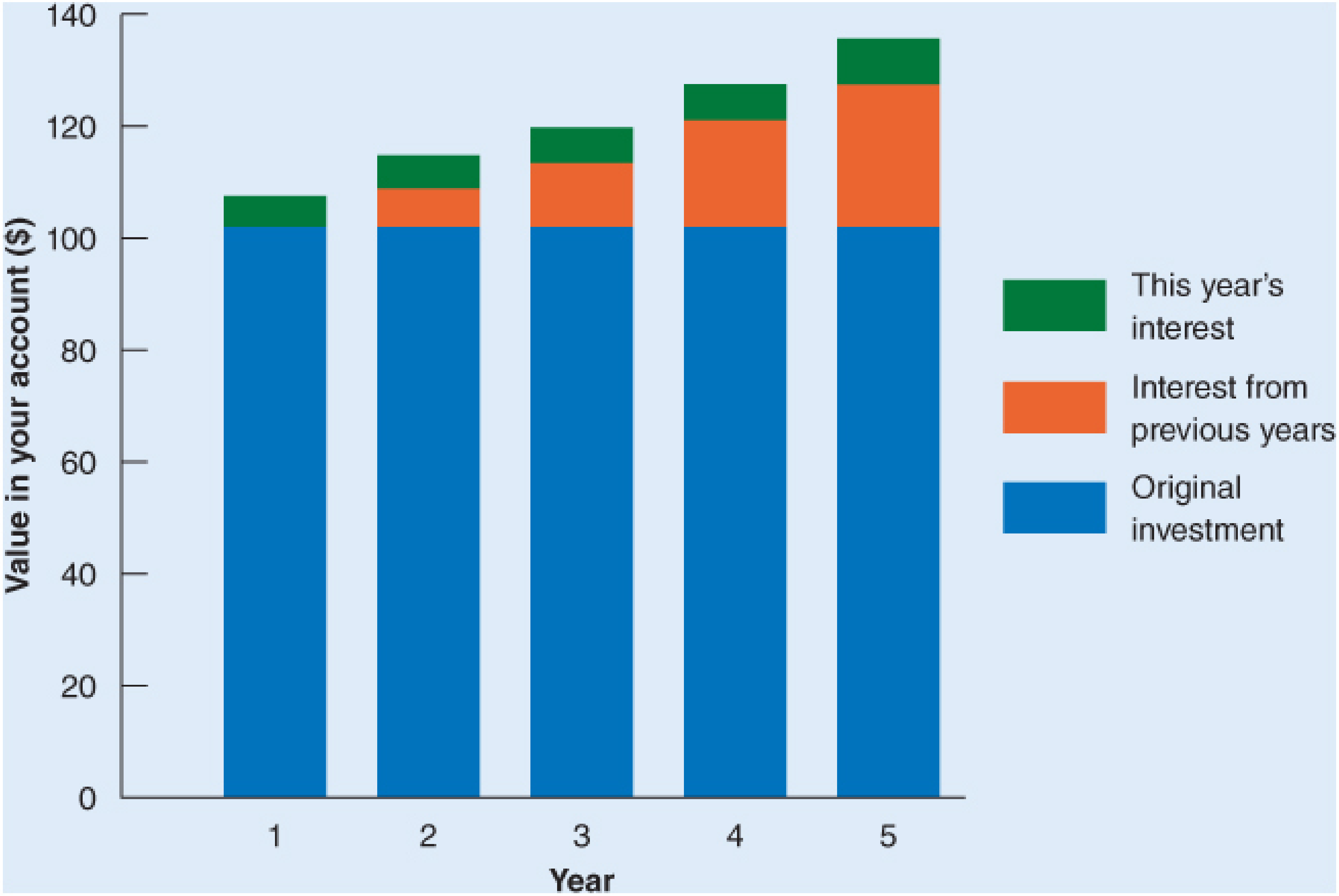
Example - Compound Interest

	Today	Future Years				
		1	2	3	4	5
Interest Earned		*7*	*7.49*	*8.01*	*8.58*	*9.18*
Value	100	*107*	*114.49*	*122.50*	*131.08*	*140.26*

Compound Interest: Example

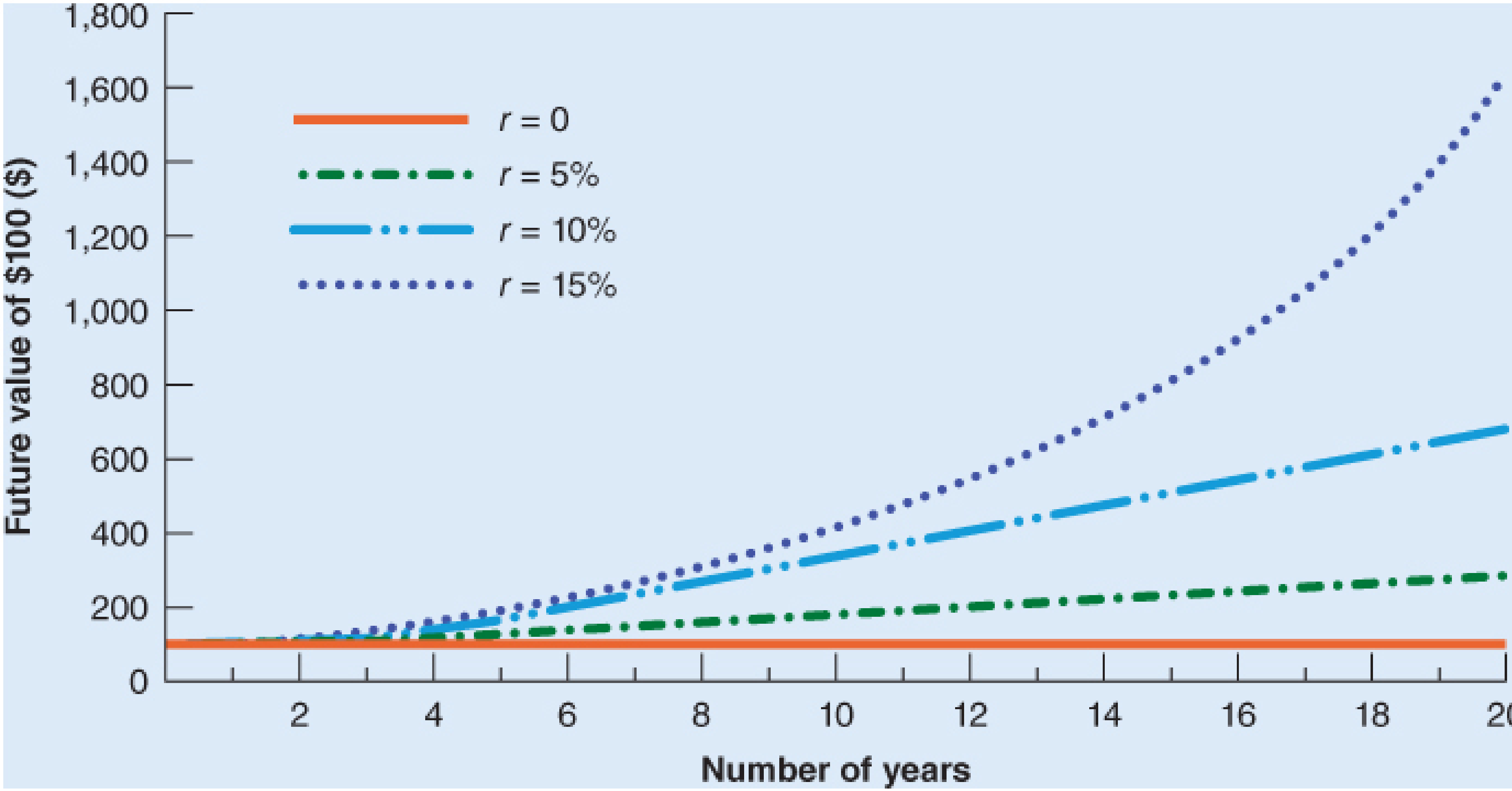
Year	Balance at Start of Year	Interest Earned during Year	Balance at End of Year
1	\$100.00	.06 × \$100.00 = \$6.00	\$106.00
2	\$106.00	.06 × \$106.00 = \$6.36	\$112.36
3	\$112.36	.06 × \$112.36 = \$6.74	\$119.10
4	\$119.10	.06 × \$119.10 = \$7.15	\$126.25
5	\$126.25	.06 × \$126.25 = \$7.57	\$133.82

- With 6% compound interest rate



140
120
100
80
60
40
20
0
Value in your account ($)
1
2
3
4
5
Year
This year's interest
Interest from previous years
Original investment

The Power of Compounding



Example of Future Values

Number of Years	Interest Rate per Year					
	5%	6%	7%	8%	9%	10%
1	1.0500	1.0600	1.0700	1.0800	1.0900	1.1000
2	1.1025	1.1236	1.1449	1.1664	1.1881	1.2100
3	1.1576	1.1910	1.2250	1.2597	1.2950	1.3310
4	1.2155	1.2625	1.3108	1.3605	1.4116	1.4641
5	1.2763	1.3382	1.4026	1.4693	1.5386	1.6105
10	1.6289	1.7908	1.9672	2.1589	2.3674	2.5937
20	2.6533	3.2071	3.8697	4.6610	5.6044	6.7275
30	4.3219	5.7435	7.6123	10.0627	13.2677	17.4494

- With $1 Initial Investment

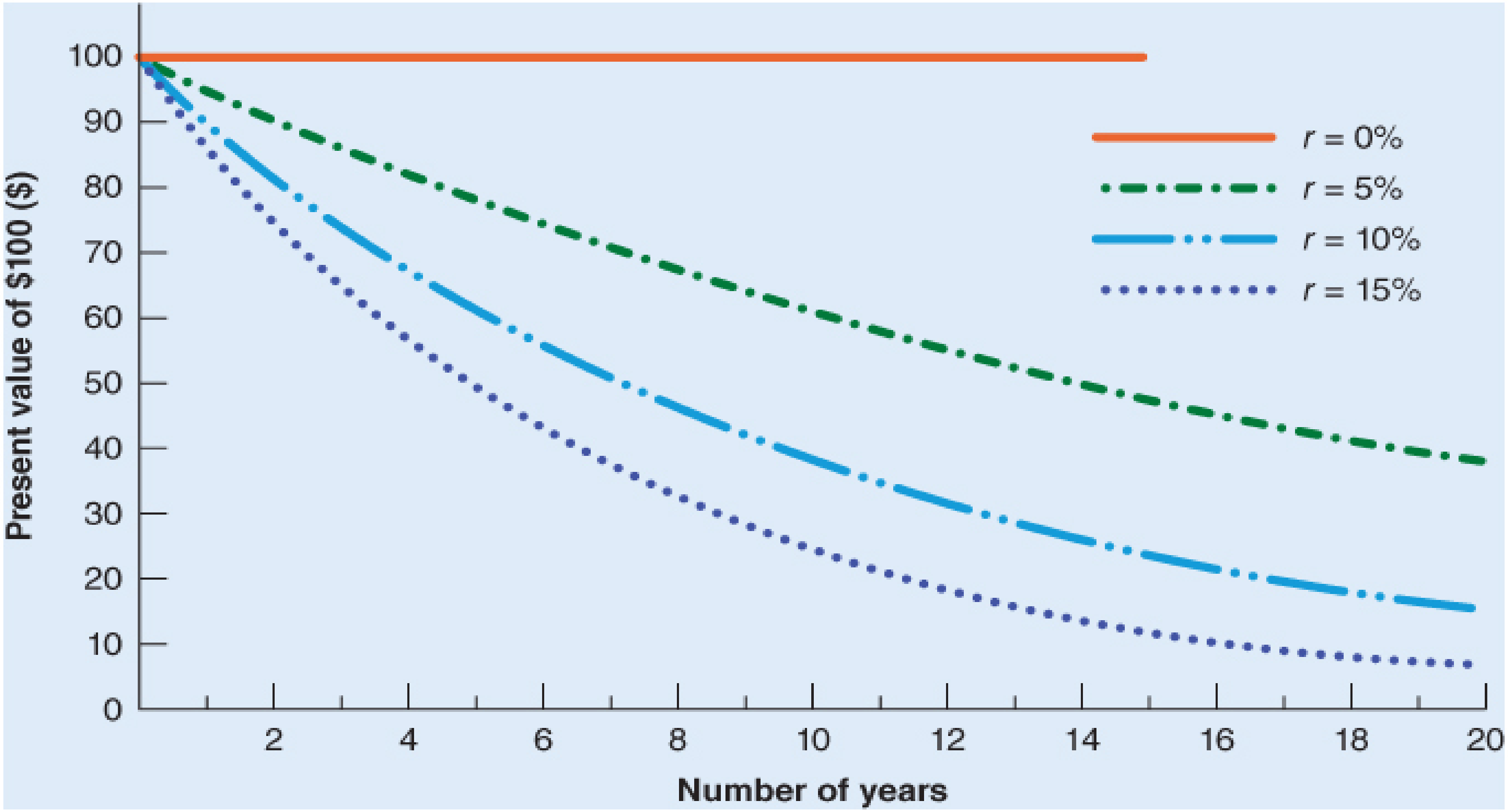
Present Values (5.2)

- Present value (PV) is the value today of a future cash flow.
  - Recall that a dollar today is worth more than a dollar tomorrow
  - Why is it useful?
- Let r be the discount rate. The discount factor would be $DF = \frac{1}{(1+r)^t}$
- Thus, $PV = FV \times \frac{1}{(1+r)^t}$

Present Value: Example

Always ahead of the game, Tommy, at 8 years old, believes he will need $100,000 to pay for college. If he can invest at a rate of 7% per year, how much money should he ask his rich Uncle GQ to give him?

Present Value of a Future Cash Flow



The present value of **$100** to be received in 1 to 20 years at varying discount rates:

Example of Present Values

Number of Years	Interest Rate per Year					
	5%	6%	7%	8%	9%	10%
1	0.9524	0.9434	0.9346	0.9259	0.9174	0.9091
2	0.9070	0.8900	0.8734	0.8573	0.8417	0.8264
3	0.8638	0.8396	0.8163	0.7938	0.7722	0.7513
4	0.8227	0.7921	0.7629	0.7350	0.7084	0.6830
5	0.7835	0.7473	0.7130	0.6806	0.6499	0.6209
10	0.6139	0.5584	0.5083	0.4632	0.4224	0.3855
20	0.3769	0.3118	0.2584	0.2145	0.1784	0.1486
30	0.2314	0.1741	0.1314	0.0994	0.0754	0.0573

- The value today of $1 received in the future.

Present Values (continued)

- Note that we should never compare cash flows occurring at different times without first discounting them to a common date.
  - By calculating present values, we see how much cash must be set aside today to pay future bills.

Present Values (continued)

- Finding the interest rate
- Suppose that a PV of $58.9 will generate FV of $1,000 in 46 years.
    - What is the interest rate?

$$PV = 58.9 = 1{,}000 \times \frac{1}{(1+r)^{46}}$$

$$(1+r)^{46} = \frac{1000}{58.9} = 16.978$$

$$(1+r) = 16.978^{1/46} = 1.0635$$

$$r = 0.0635$$

Present Values (continued)

- Finding the interest rate

$$r = \left(\frac{FV}{PV}\right)^{\frac{1}{t}} - 1$$

Present Values (continued)

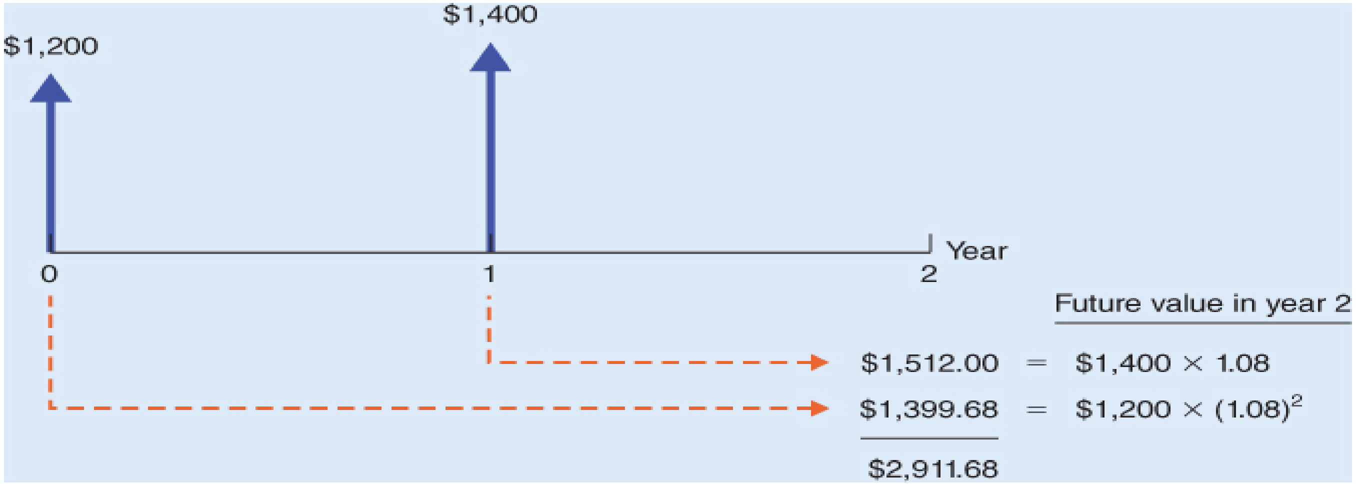
- Finding the time period:

$$t = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1+r)}$$

72法则，1%複利72年，本金会翻成2倍

Multiple Cash Flows (5.3)

- Suppose you want to buy a new computer, and you will be able to save $1,200 now and $1,400 in 1 year.
  - If you earn 8% rate of interest, how much will you have to spend on the computer?



Multiple Cash Flows (continued)

- FV formula of multiple cash flows.

$$FV = C_1(1+r)^1 + C_2(1+r)^2 + \ldots + C_t(1+r)^t$$

Multiple Cash Flows (continued)

For PV of multiple cash flows:

$Denote:$

$C_1 =$ The cash flow in year 1

$C_2 =$ The cash flow in year 2

$C_t =$ The cash flow in year t (with any number of cash flows in between)

$$PV = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_t}{(1+r)^t}$$

Multiple Cash Flows: Example

Your auto dealer gives you the choice to pay $15,500 cash now or make three payments: $8,000 now and $4,000 at the end of the following two years. If your cost of money (discount rate) is 8%, which do you prefer?

Initial Payment* 8,000.00

$$PV\ of\ C_1 = \frac{4{,}000}{(1+.08)^1} = 3{,}703.70$$

$$PV\ of\ C_2 = \frac{4{,}000}{(1+.08)^2} = 3{,}429.36$$

Total PV = $15,133.06

installment plan 分期付款

Level Cash Flows (5.5)

- Suppose we want to value a stream of equal cash flows.
  - For example, a home mortgage might require the homeowner to make equal monthly payments for the life of loan.
    - This involves 360 equal payments if the loan lasts 30 years.
    - Any such sequence of equally spaced level cash flows is called annuity.
    - If the payment stream lasts forever, it is called perpetuity. 永續年金(每年的利息正好拿去使用)

Perpetuities

Suppose you could invest $100 at an interest rate of 10%. You could earn annual interest of $10 per year forever.

Let C = Yearly Cash Payment
PV of Perpetuity:

$$PV = \frac{C}{r}$$

The earliest example of perpetuities: Consol Bond issued by UK in 1751

Perpetuities: Example

In order to create an endowment, which pays \$185,000 per year forever, how much money must be set aside today if the rate of interest is 8%?

$$PV = \frac{185{,}000}{.08} = \$2{,}312{,}500$$

What if the first payment won't be received until 3 years from today (a delayed perpetuity)?

$$PV = \frac{2{,}312{,}500}{1.08^2} = 1{,}982{,}596$$

Present Value of an Annuity

C = yearly cash payment
r = interest rate
t = number of years cash payment is received

$$PV = C \times \underbrace{\left[\frac{1}{r} - \frac{1}{r \times (1+r)^t} \right]}_{\text{annuity factor}}$$

Present Value of an Annuity

C = yearly cash payment
r = interest rate
t = number of years cash payment is received

$$PV = C\left[\frac{1-\frac{1}{(1+r)^t}}{r}\right]$$

Annuities: Example 1

You are purchasing a home and are scheduled to make 30 annual installments of $10,000 per year. Given an interest rate of 5%, what is the price you are paying for the house (*i.e.* what is the present value)?

$$PV = 10{,}000\left[\frac{1}{0.05} - \frac{1}{0.05(1+0.05)^{30}}\right]$$

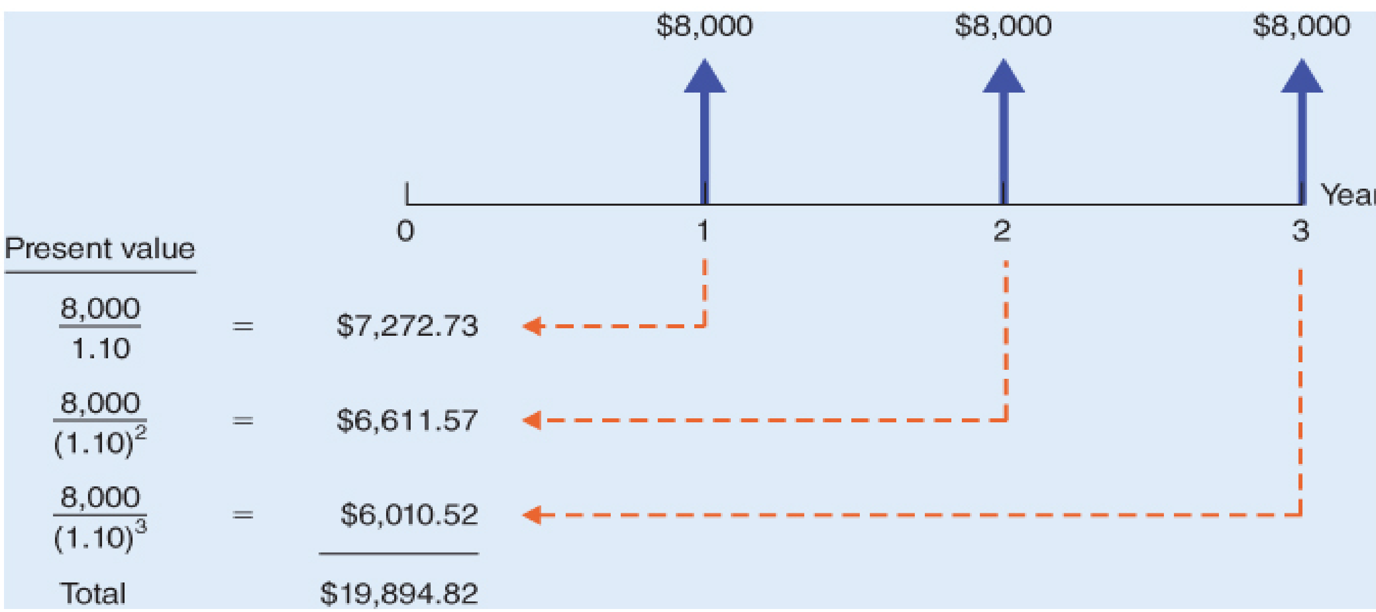
$$= 153{,}724.51$$

$$\frac{\frac{8000}{1+r}\left(1-\frac{1}{(1+r)^n}\right)}{1-\frac{1}{(1+r)}} = \frac{8000\left(1-\frac{1}{(1+r)^n}\right)}{1+r-1} = 8000\left(\frac{1-\frac{1}{(1+r)^n}}{r}\right)$$

Annuities: Example 2

$$PV = 8,000\left(\frac{1-\frac{1}{1.1^3}}{0.1}\right) = 19894.82$$

Suppose a car dealer offer you an installment plan to pay for the car with $8,000 a year at the end of each year of the next 3 years. What's the PV with r = 10%?



Note that the value of an annuity is equal to the difference between the value of two perpetuities:

	Cash Flow						
Year:	1	2	3	4	5	6 . . .	Present Value
1. Perpetuity A	\$1	\$1	\$1	\$1	\$1	\$1 . . .	$\frac{1}{r}$
2. Perpetuity B				\$1	\$1	\$1 . . .	$\frac{1}{r(1+r)^3}$
3. Three-year annuity	\$1	\$1	\$1				$\frac{1}{r} - \frac{1}{r(1+r)^3}$

Example of Annuity

Number of Years	Interest Rate per Year 5%	6%	7%	8%	9%	10%
1	0.9524	0.9434	0.9346	0.9259	0.9174	0.9091
2	1.8594	1.8334	1.8080	1.7833	1.7591	1.7355
3	2.7232	2.6730	2.6243	2.5771	2.5313	2.4869
4	3.5460	3.4651	3.3872	3.3121	3.2397	3.1699
5	4.3295	4.2124	4.1002	3.9927	3.8897	3.7908
10	7.7217	7.3601	7.0236	6.7101	6.4177	6.1446
20	12.4622	11.4699	10.5940	9.8181	9.1285	8.5136
30	15.3725	13.7648	12.4090	11.2578	10.2737	9.4269

- PV of $1 a year received for each of *t* years.

Example of Home Mortgage

- Suppose a house costs \$125,000 and the buyer makes 20% down payment. The buyer obtains a mortgage for remaining cost and re-pay the mortgage with monthly payment for next 30 years (an amortizing loan).
  - What would be the mortgage payment with interest rate of 1% per month?

$125{,}000 \times \frac{4}{5}$

$$100{,}000 = C\left(\frac{1}{0.01} - \frac{1}{0.01(1.01)^{360}}\right)$$

$$C = \frac{r \cdot PV}{1-(1+r)^{-t}}$$

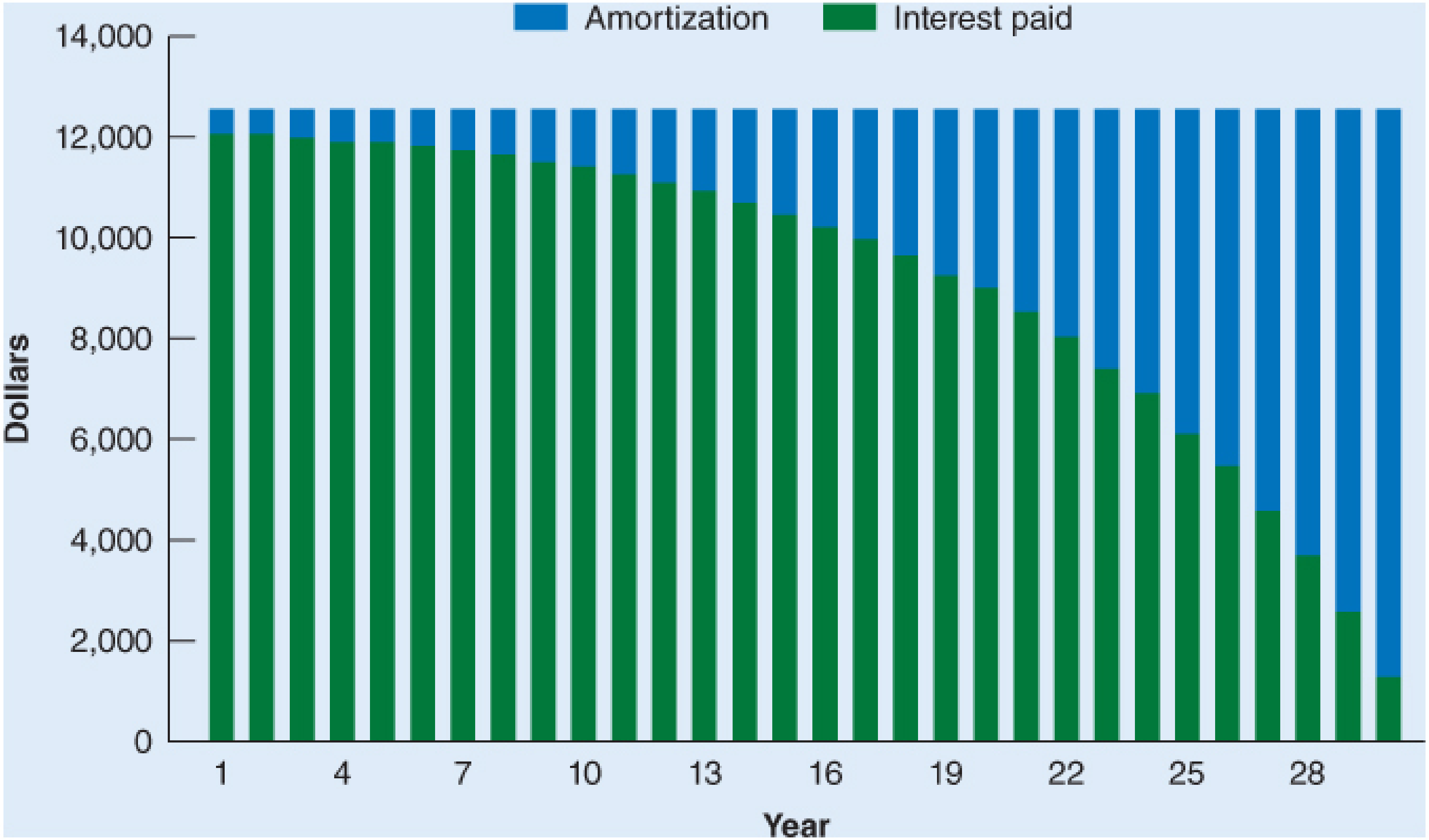
$$C = \frac{100{,}000}{97.218} = \$1{,}028.61$$

Example of Amortizing Loan

Year	Beginning-of-Year Balance	Year-End Interest Due on Balance	Year-End Payment	Amortization of Loan	End-of-Year Balance
1	$1,000.00	$100.00	$315.47	$215.47	$784.53
2	$784.53	$78.45	$315.47	$237.02	$547.51
3	$547.51	$54.75	$315.47	$260.72	$286.79
4	$286.79	$28.68	$315.47	$286.79	$0

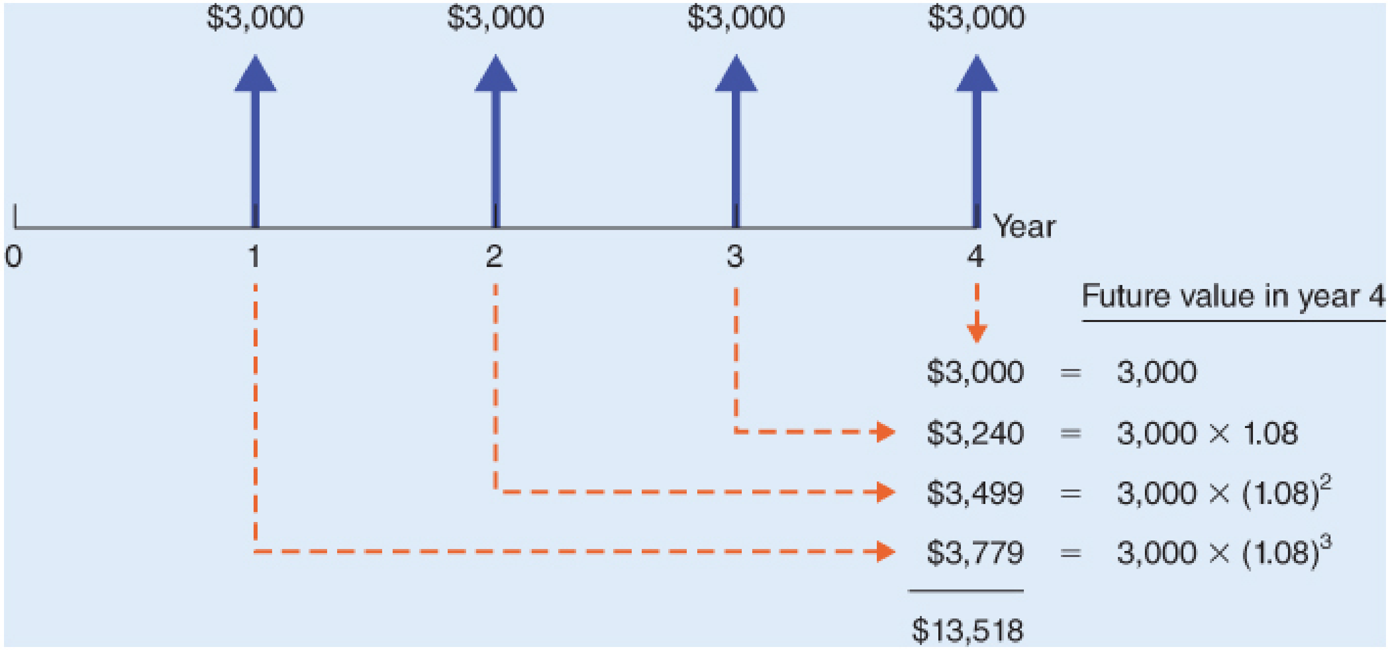
- If we borrow $1,000 at an interest rate of 10%.
- We would need to make an annual payment of $315.47 over 4 years to repay the loan.

Mortgage Amortizing



Future Value of Annuities

You plan to save $3,000 at the end of every year for 4 years. Given a 8% rate of interest, how much will you have saved?



Future Value of Annuities

You plan to save $3,000 at the end of every year for 4 years. Given a 8% rate of interest, how much will you have saved?

Future Value of Annuities

You plan to save $4,000 every year for 20 years and then retire. Given a 10% rate of interest, how much will you have saved by the time you retire?

Future Value of Annuities

General Formula:

$$FV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right] \times (1+r)^t$$

$$FV = C\left[\frac{(1+r)^t - 1}{r}\right]$$

Example of FV from Annuity

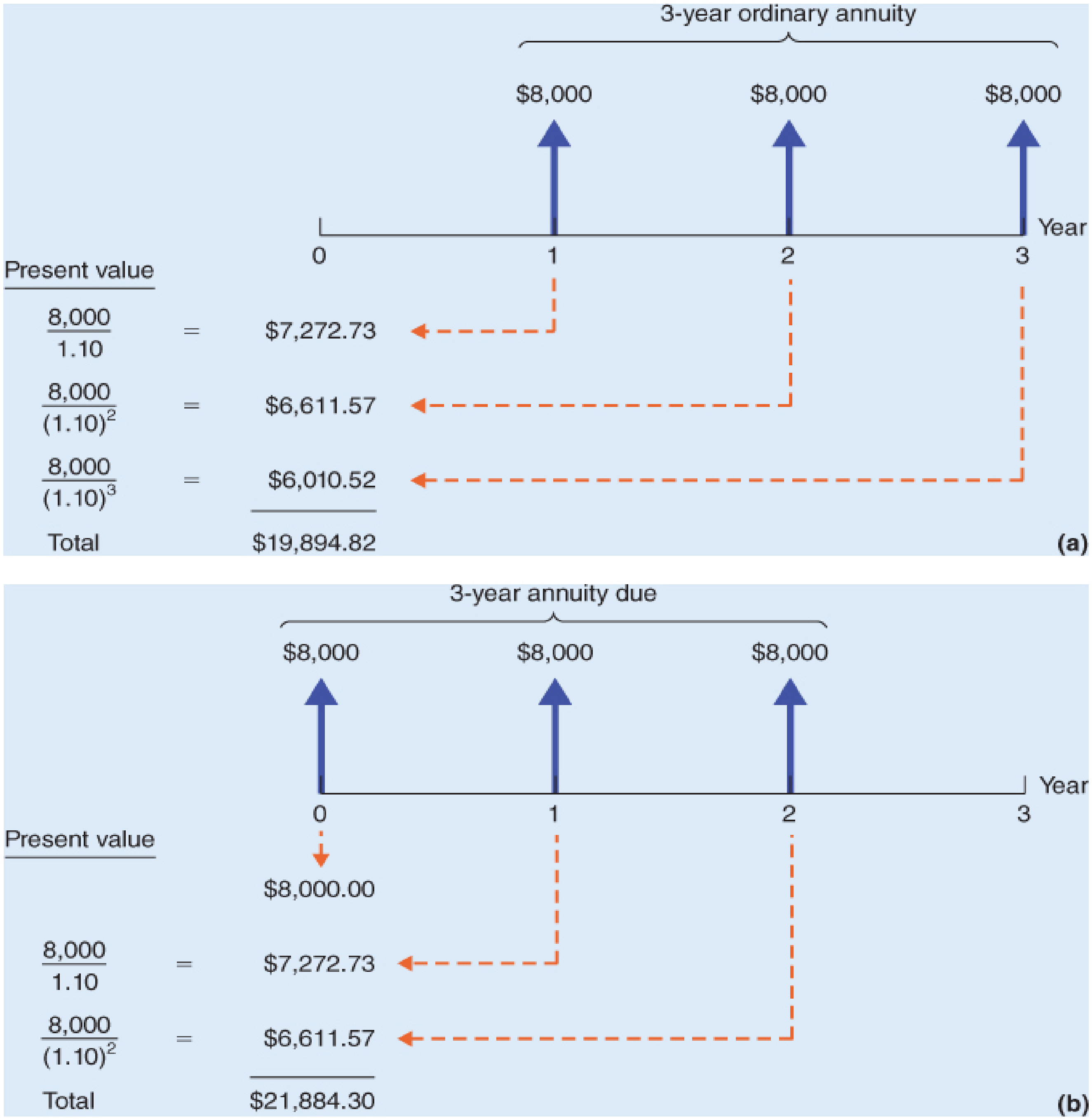
Number of Years	Interest Rate per Year					
	5%	6%	7%	8%	9%	10%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0500	2.0600	2.0700	2.0800	2.0900	2.1000
3	3.1525	3.1836	3.2149	3.2464	3.2781	3.3100
4	4.3101	4.3746	4.4399	4.5061	4.5731	4.6410
5	5.5256	5.6371	5.7507	5.8666	5.9847	6.1051
10	12.5779	13.1808	13.8164	14.4866	15.1929	15.9374
20	33.0660	36.7856	40.9955	45.7620	51.1601	57.2750
30	66.4388	79.0582	94.4608	113.2832	136.3075	164.4940

- An investment of $1 a year for each of *t* years.

Annuities Due

- Annuity due: level stream of cash flows starting immediately.
  - For example, many auto dealers sell we a car on credit, they might insist that the first payment be made at the time of the sale.
- How does it differ from an ordinary annuity?

$$PV_{Annuity\ Due} = PV_{Annuity} \times (1+r)$$

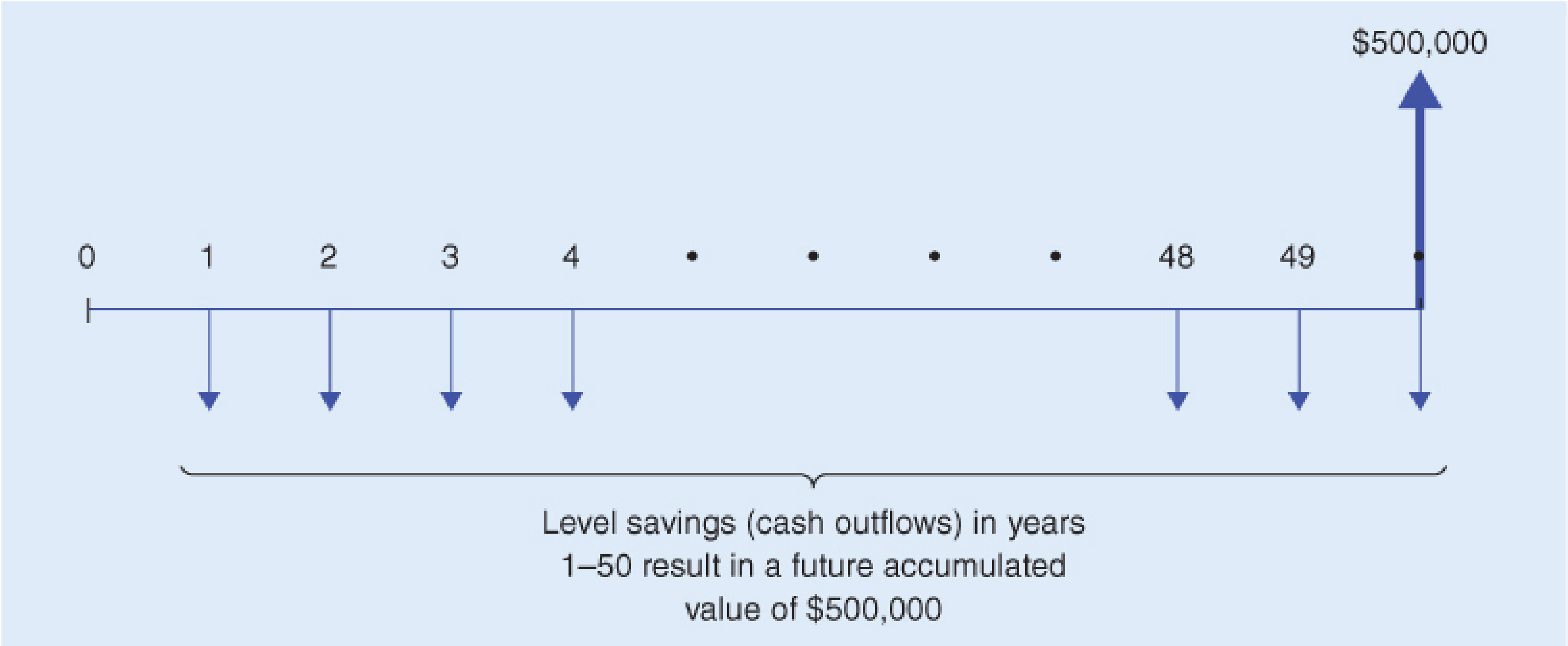

3-year ordinary annuity
$8,000
$8,000
$8,000
Year
0
1
2
3
Present value
8,000 / 1.10 = $7,272.73
8,000 / (1.10)^2 = $6,611.57
8,000 / (1.10)^3 = $6,010.52
Total $19,894.82
(a)
3-year annuity due
$8,000
$8,000
$8,000
Year
0
1
2
3
Present value
$8,000.00
8,000 / 1.10 = $7,272.73
8,000 / (1.10)^2 = $6,611.57
Total $21,884.30
(b)

Annuities Due (continued)

- For future value of annuity due, similar logic can be applied.

$$FV_{Annuity\,Due} = FV_{Annuity} \times (1 + r)$$

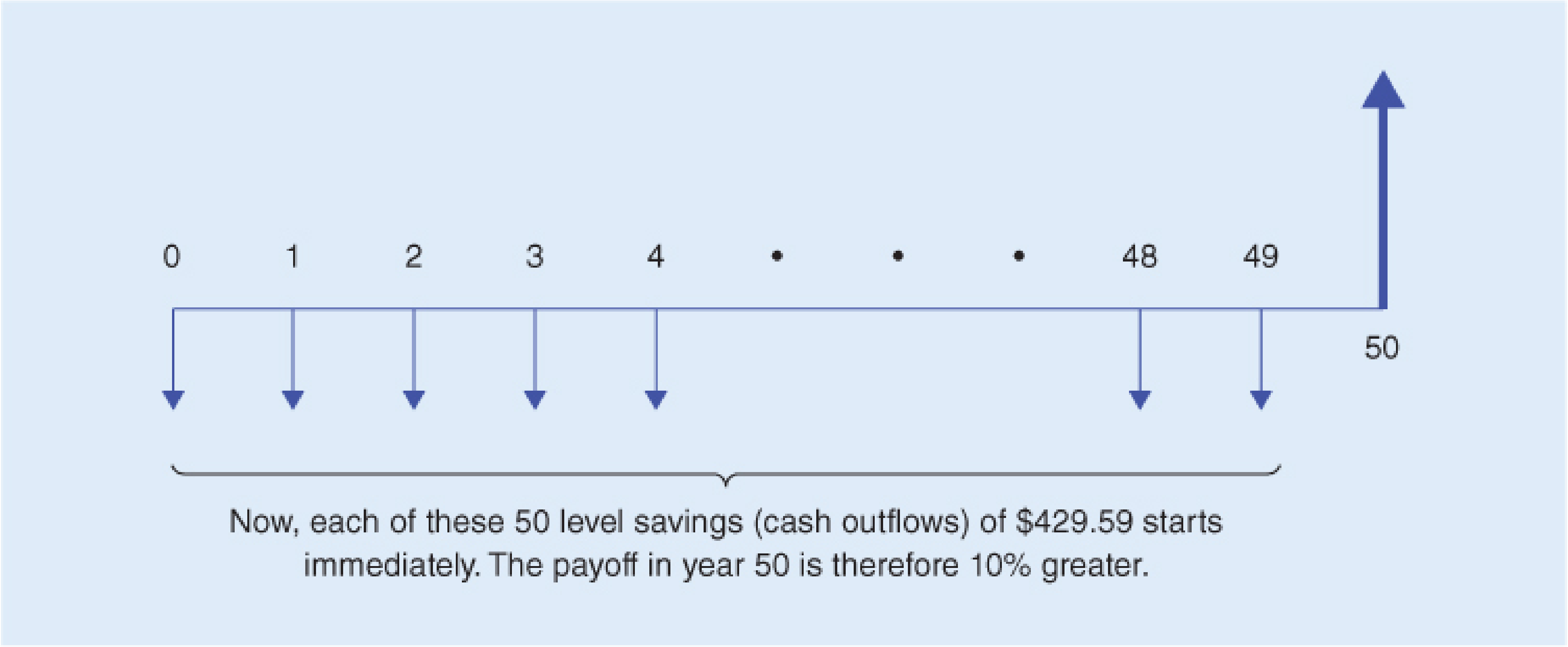
- Suppose we believe we need $500,000 by the retirement date in 50 years. How much saving each year would be necessary with 10% interest rate?
  - What if we can save the money at the beginning of each year?



- If we save \$1 in 50 years with 10% rate, we'll have

$$FV = \frac{(1+r)^t - 1}{r} = \frac{(1.1)^{50} - 1}{0.10} = \$1{,}163.91$$

- Thus, the C that ensure \$500,000 would be \$500,000 / 1,163.91 = \$429.59



- Now we save $429.59 at the beginning of each year, the $FV_{\text{Annuity Due}} = FV_{\text{Annuity}} \times (1 + r)$
- Thus, we will have $550,000 in 50 years.

Effective Annual Interest Rate (5.7)

- Effective annual interest rate: the interest rate that is annualized using compound interest.
  - For example, if we borrow \$100 at 1% rate per month for one year, we need to repay $\$100 \times (1.01)^{12} = \$112.68$.
  - Thus, the 1% monthly rate is equivalent to an effective annual interest rate (or annually compounded rate) of 12.68%.
  - Yet, the annual percentage rate (APR) is simply 1% times 12 as 12%.

EAR and APR

- Let *MR* be the monthly interest rate.
- Effective annual interest rate (EAR):

$$EAR = (1 + MR)^{12} - 1$$

- Annual percentage rate (APR):

$$APR = MR \times 12$$

EAR and APR: Example

Given a monthly rate of 1%, what is the Effective Annual Rate(EAR)? What is the Annual Percentage Rate (APR)?

$$EAR = (1.01)^{12} - 1 = 12.68\%$$

$$APR = (0.01) \times (12) = 12.00\%$$

General formula with *m* compounding periods:

$$EAR = \left(1 + \frac{APR}{m}\right)^m - 1$$

EAR vs. APR

Compounding Period	Periods per Year (m)	Per-Period Interest Rate	Growth Factor of Invested Funds	Effective Annual Rate
1 year	1	6%	1.06	6.0000%
Semiannually	2	3	$1.03^2 = 1.0609$	6.0900
Quarterly	4	1.5	$1.015^4 = 1.061364$	6.1364
Monthly	12	.5	$1.005^{12} = 1.061678$	6.1678
Weekly	52	.11538	$1.0011538^{52} = 1.061800$	6.1800
Daily	365	.01644	$1.0001644^{365} = 1.061831$	6.1831
Continuous			$e^{.06} = 1.061837$	6.1837

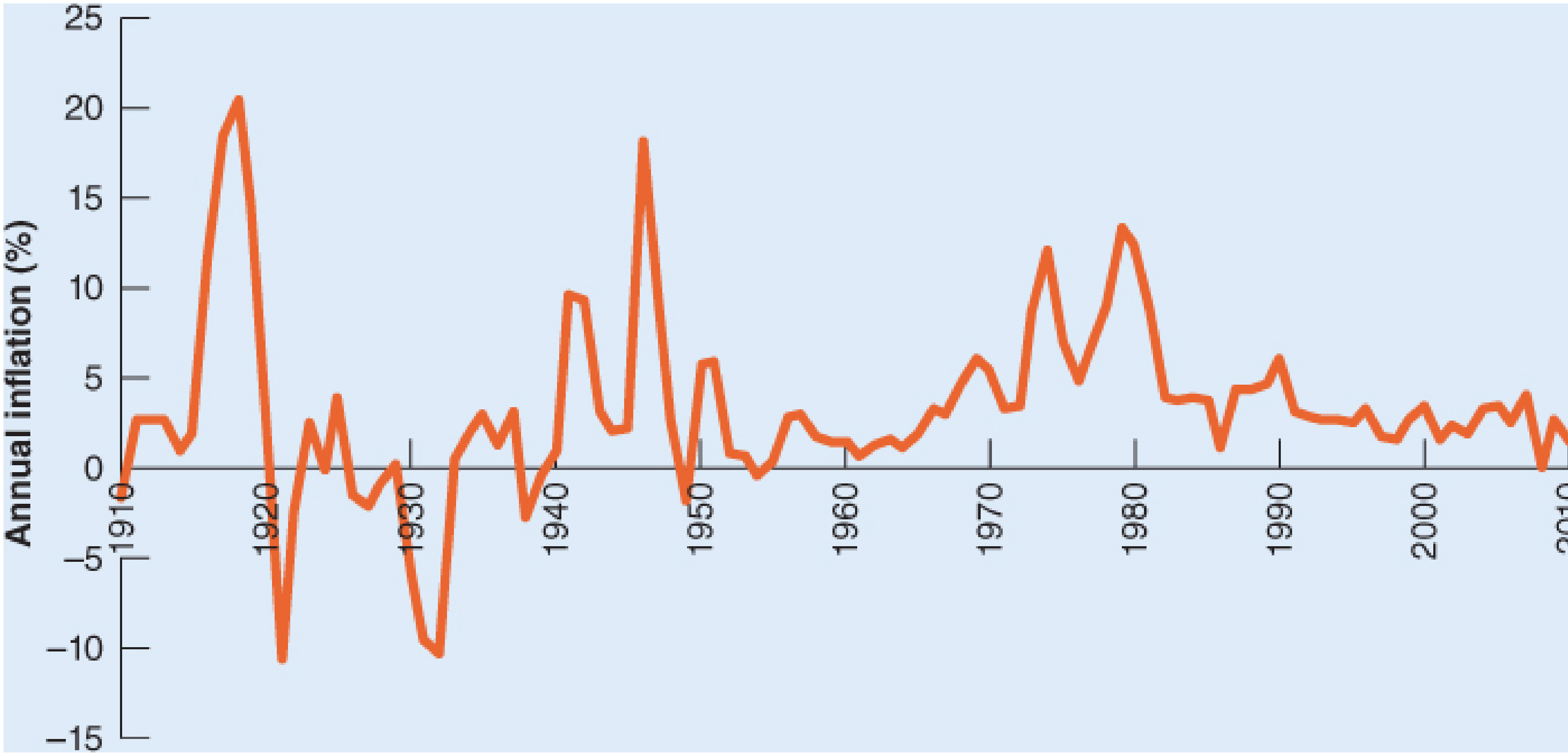
- These investments all have an APR of 6%, but the more frequently interest is compounded, the higher is the EAR.

Inflation (5.8)

- Inflation: rate at which prices as a whole are increasing.
  - The increase in general level of prices means that the purchasing power of money has eroded.

	CPI	Percent Change since 1950
1950	25.0	
1960	29.8	+ 19.2%
1970	39.8	+ 59.2
1980	86.3	+ 245.2
1990	133.8	+ 435.2
2000	174.0	+ 596.0
2010	219.2	+ 776.8

Annual rates of inflation



Source: Bureau of Labor Statistics

Inflation (continued)

- From the figure we can see that there are some periods where the prices were falling.
  - This is called deflation.
- Nominal interest rate: rates at which money invested grows.
- Real interest rate: rates at which the purchasing power of an investment increases.

$$1 + \text{real interest rate} = \frac{1+\text{nominal interest rate}}{1+\text{inflation rate}}$$

Inflation (continued)

- The relation between nominal interest rate and real interest rate can be approximately shown:
  - Real interest rate ≈ Nominal interest rate – Inflation rate.
  - For example, if the banks offer 5% interest rate and the inflation is also 5%, both formula will show that the real increase in the savings would be zero percent.

Inflation: Example

If the nominal interest rate on your interest-bearing savings account is 2.0% and the inflation rate is 3.0%, what is the real interest rate? −0.97 %

Valuing Real Cash Payment

- Given 10% nominal interest rate, how much do we need to invest now to produce \$100 in a year?
    - PV = \$100 / 1.10 = \$90.91
    - Suppose inflation is 7%, the real value of that \$100 is only \$100 / 1.07 = \$93.46.
    - We can see that the real interest rate would be (1.10 / 1.07) – 1 = 2.8%.
        - Thus, discount the \$93.46 by 2.8%, we have
        - PV = \$93.46 / 1.028 = \$90.91