14.8 Lagrange multipliers

有條件的極值, 如: 到曲面的最短距離, 限制面積的最大容積。
§ 14.7 使用代入限制條件減少變數作微分, 但只能解最多三變數。
拉格朗日乘數法可以解多條件多變數的極值。

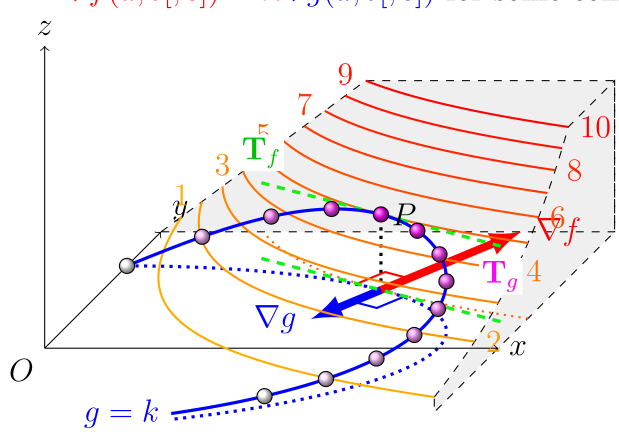
0.1 One constraint

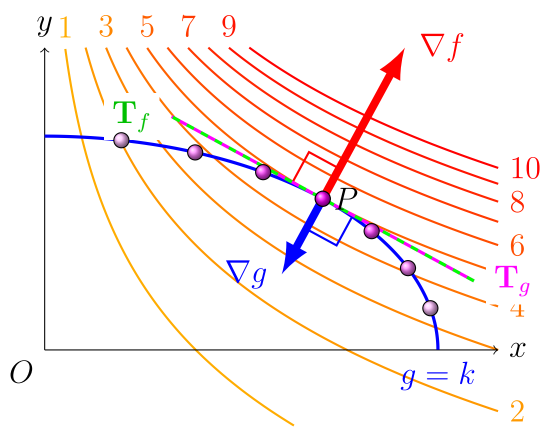
Recall: 梯度垂直等高[面]線: $\nabla f \perp f=c$, 所以也垂直切[面]線。

如果想找 $f(x,y[,z])$ 在 $g(x,y[,z])=k$ 限制下的極值, 可以先畫出 $f=\ell$ 的等高面/線 跟 $g=k$, 極值就發生在 $f=\ell$ 與 $g=k$ 相切的點 $P(a,b[,c])$; 這時候 $f=\ell$ 與 $g=k$ 在 P 的切線[平面]<u>平行</u>(相同)。

因爲 $\nabla f(a,b[,c])$ 與 $f=\ell$ 的切線[平面]<u>垂直</u>, 而且 $\nabla g(a,b[,c])$ 與 $g=k$ 的切線[平面]<u>垂直</u>, 所以 $\nabla f(a,b[,c])$ 與 $\nabla g(a,b[,c])$ <u>平行</u>, 也就是:

$\nabla f(a,b[,c])=\lambda\nabla g(a,b[,c])$ for some constant λ ("lambda"[爛打])。





$$\implies \begin{aligned} \mathbf{T}_f &\parallel \mathbf{T}_g, \\ \nabla f &\perp \mathbf{T}_f, \\ \nabla g &\perp \mathbf{T}_g, \\ \nabla f &\parallel \nabla g. \end{aligned}$$

Theorem 1 (Method of Lagrange Multipliers) *(一個限制)*
To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$ [assuming extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x,y,z)=k$]:
(a) Find all values of x, y, z, and λ such that

$$\boxed{\boldsymbol{\nabla f(x,y,z) = \lambda \nabla g(x,y,z)} \quad \& \quad \boldsymbol{g(x,y,z)=k}}$$

where λ is called a ***Lagrange multiplier*** 拉格朗日乘數.
(b) 比大小, 最大/小的一定是絕對最大/小。

Note: 1. 雙 [三]變數會有 3[4] 個式子: (方程式與變數一樣多!)

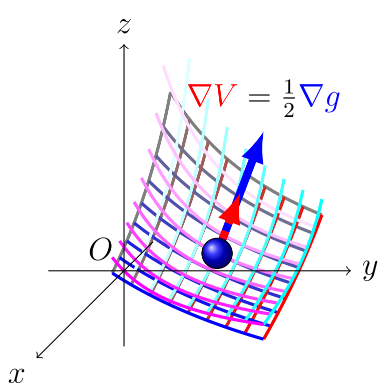
$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad [f_z = \lambda g_z] \qquad g(x,y,z)=k$$

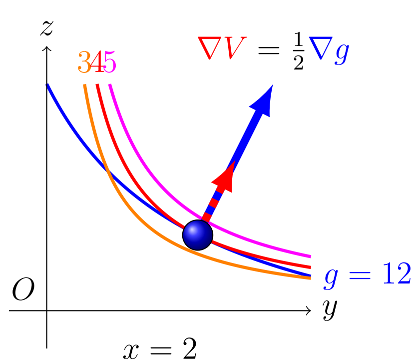
Attention: 1. $\nabla g \neq \mathbf{0}$, 當 $\nabla g = \mathbf{0}$ 時的點要另外判斷。(Ex 14.8.25)
♦ 2. 其實找到的是奇異點 (critical points), 有可能是局部極大/小或鞍點。
Ex: $f=y$, $g=y-x^3=0$. *(try it yourself.)*

Example 0.1 *(無蓋盒)* $V(x,y,z)=xyz$, $g(x,y,z)=2xz+2yz+xy=12$.

$$\nabla V = \lambda \nabla g \implies \begin{cases} V_x = \lambda g_x \\ V_y = \lambda g_y \\ V_z = \lambda g_z \\ g = 12 \end{cases} \implies \begin{cases} yz = \lambda(2z+y) \\ xz = \lambda(2z+x) \\ xy = \lambda(2x+2y) \\ 2xz+2yz+xy = 12 \end{cases}$$

If $\lambda = 0$, 代入得到 $g=0$, *contradiction* 矛盾, *so* $\lambda \neq 0$.
約掉 $\lambda \implies 2xz+xy = 2yz+xy = 2xz+2yz \implies xy = 2yz = 2xz$.
If $x=0$, $y=0$, *or* $z=0$ *then* $V(x,y,z)=0$. *(這也是個極值)*
Otherwise, $\implies x=y=2z$, 代入 $g(2z,2z,z)=12z^2=12$, $z=1$ *(負不合)*.
So $x=y=2$, $z=1$, $(\lambda = \frac{1}{2},)$ $V(2,2,1)=4$. ■



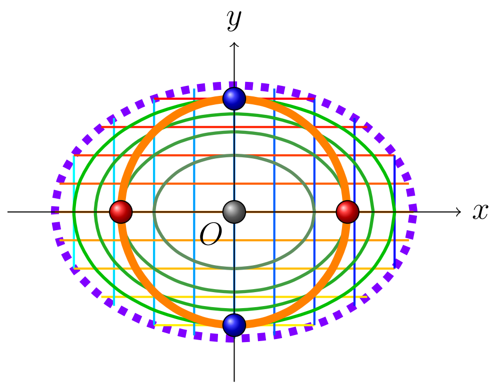


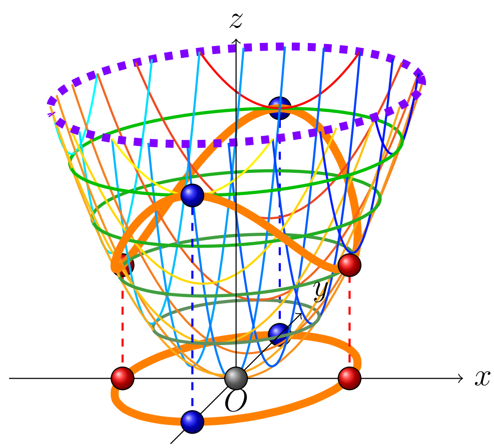
Example 0.2 *Find the extreme value of* $f(x,y) = x^2 + 2y^2$ *on the circle* $x^2 + y^2 = 1$.

Let $g(x,y) = x^2 + y^2 = 1$.

$$\nabla f = \lambda \nabla g \implies \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 1 \end{cases} \implies \begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases} \implies \begin{array}{l} x = 0 \text{ or} \\ \lambda = 1 \end{array}$$

If $x = 0$*, then* $y = \pm 1$*; if* $\lambda = 1$*, then* $y = 0$*,* $x = \pm 1$.
So the absolute maximum value of f *on the circle is* $f(0, \pm 1) = 2$*,*
and the absolute minimum value is $f(\pm 1, 0) = 1$. ■





Example 0.3 *Find the extreme value of* $f(x,y) = x^2 + 2y^2$ *on the disk* $x^2 + y^2 \leq 1$.

1. 找奇異點*:* $f_x = 2x = 0$*,* $f_y = 4y = 0$.
critical point: $(0,0)$ *and* $f(0,0) = 0$.
2. 找邊點*:* 由上題 $f(\pm 1, 0) = 1$ *and* $f(0, \pm 1) = 2$.
3. 比大小*: the maximum value of* f *on the disk is* $f(0, \pm 1) = 2$ *and the minimum value is* $f(0,0) = 0$. ■

Example 0.4 *Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and the farthest from the point $(3,1,-1)$.*

Let d be the distance, $f(x,y,z)=d^2=(x-3)^2+(y-1)^2+(z+1)^2$, $g(x,y,z)=x^2+y^2+z^2=4$.

$$\nabla f=\lambda\nabla g \implies \begin{cases} 2(x-3) = \lambda 2x \\ 2(y-1) = \lambda 2y \\ 2(z+1) = \lambda 2z \\ x^2+y^2+z^2 = 4 \end{cases} \implies \begin{cases} x = \dfrac{3}{1-\lambda} \\ y = \dfrac{1}{1-\lambda} \\ z = \dfrac{-1}{1-\lambda} \end{cases} \quad (\lambda\neq 1)$$

$$\frac{3^2+1^2+(-1)^2}{(1-\lambda)^2}=4,\ \lambda=1\pm\frac{\sqrt{11}}{2},$$

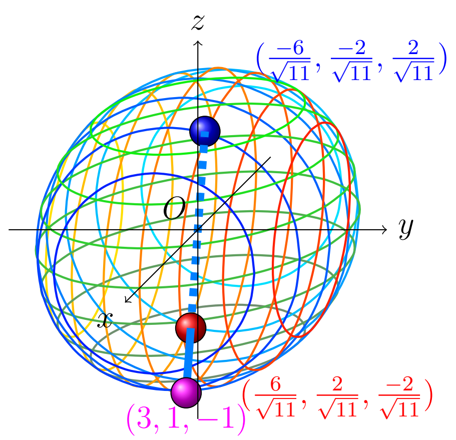
$\implies (x,y,z)=\left(\dfrac{-6}{\sqrt{11}},\dfrac{-2}{\sqrt{11}},\dfrac{2}{\sqrt{11}}\right)$ *(when $\lambda=1+\dfrac{\sqrt{11}}{2}$),*

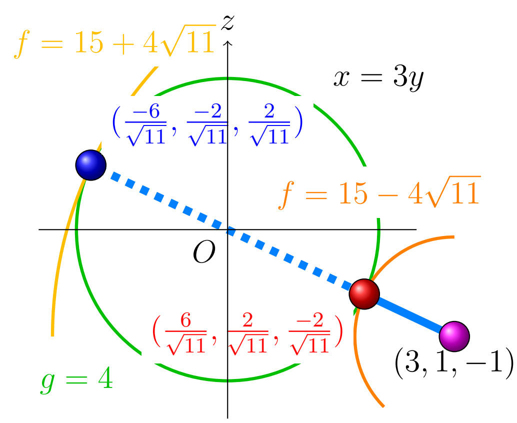
and $(x,y,z)=\left(\dfrac{6}{\sqrt{11}},\dfrac{2}{\sqrt{11}},\dfrac{-2}{\sqrt{11}}\right)$ *(when $\lambda=1-\dfrac{\sqrt{11}}{2}$).*

$$\begin{aligned} f(x,y,z) &= (x-3)^2+(y-1)^2+(z+1)^2 \ (\text{用 } \nabla f=\lambda\nabla g \text{ 換}) \\ &= x^2\lambda^2+y^2\lambda^2+z^2\lambda^2=4\lambda^2=(2\pm\sqrt{11})^2=15\pm4\sqrt{11}. \end{aligned}$$

So the closest point is $\left(\dfrac{6}{\sqrt{11}},\dfrac{2}{\sqrt{11}},\dfrac{-2}{\sqrt{11}}\right)$ *(with distance $\sqrt{11}-2$, $\sqrt{11}>2$),*

and the farthest point is $\left(\dfrac{-6}{\sqrt{11}},\dfrac{-2}{\sqrt{11}},\dfrac{2}{\sqrt{11}}\right)$ *(with distance $2+\sqrt{11}$).* ■





Note: $\pm$ 代表有正跟負兩個答案, 同時也用 $\mp=-(\pm)$ 表示對應的負與正。
ex: $x=a\pm b\mp c$ 代表 $x=a+b-c$ and $x=a-b+c$.

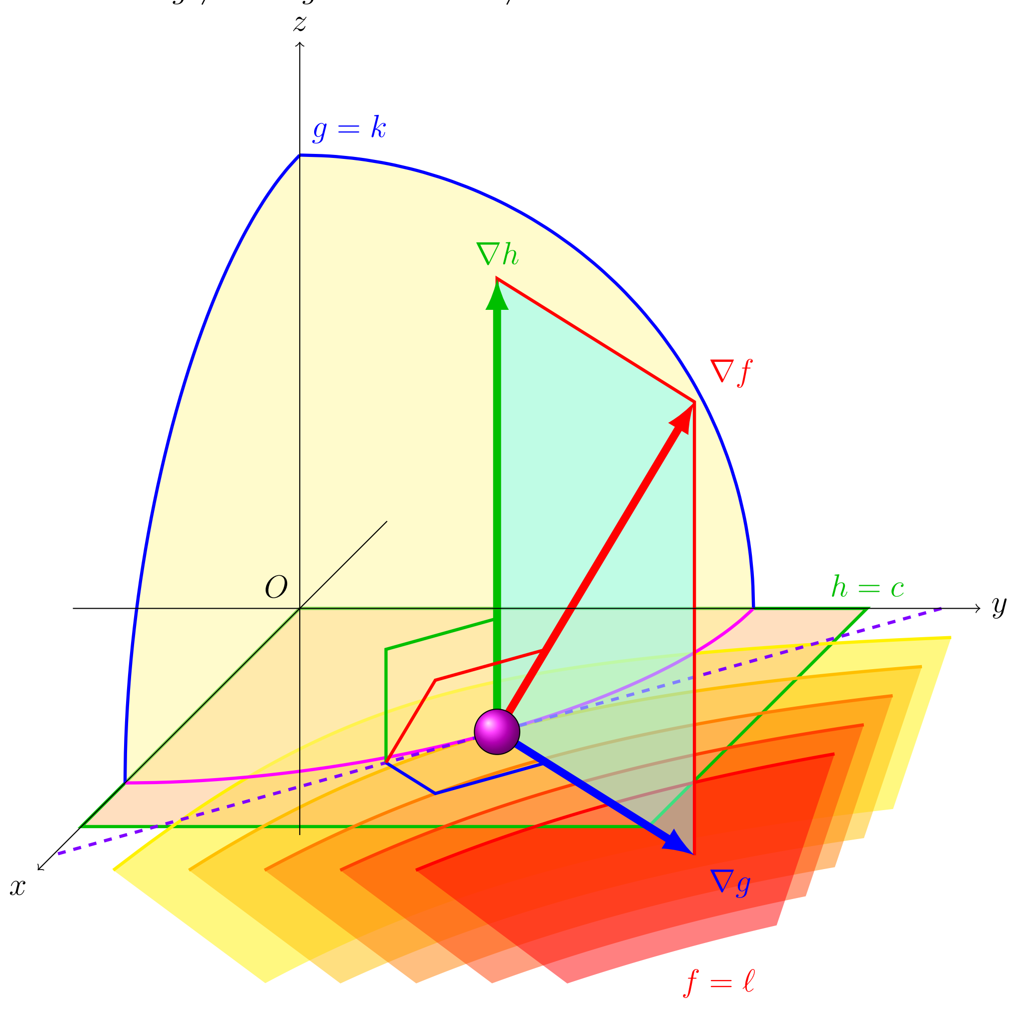
0.2 Two constraints

極值發生在 f 的 level curves(surfaces) C 與 $g=k$ 與 $h=c$ 的交線 C' 相切的點 P。這時後 C 與 C' 在 P 的切向量平行。因爲 ∇f 與 C 在 P 的切向量垂直, ∇g 與 ∇h 都與 C' 在 P 的切向量垂直, 所以 ∇f 在 ∇g 與 ∇h 展開的平面上, 也就是

$$\boxed{\boldsymbol{\nabla f = \lambda \nabla g + \mu \nabla h} \quad \& \quad \boldsymbol{g = k} \quad \& \quad \boldsymbol{h = c}}$$

where λ, μ ("mu"[喵]) are called ***Lagrange multipliers***.

Note: $\nabla g \neq \mathbf{0}$ on $g=k$ and $\nabla h \neq \mathbf{0}$ on $h=c$.



Example 0.5 *Find the maximum value of $f(x,y,z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.*

Let $g(x,y,z) = x - y + z = 1$, $h(x,y,z) = x^2 + y^2 = 1$.

$$\nabla f = \lambda \nabla g + \mu \nabla h \implies \begin{cases} 1 = \lambda + \mu 2x \\ 2 = -\lambda + \mu 2y \\ 3 = \lambda \\ x - y + z = 1 \\ x^2 + y^2 = 1 \end{cases} \implies \begin{cases} \lambda = 3 \\ x = \dfrac{-1}{\mu} \\ y = \dfrac{5}{2\mu} \end{cases}$$

(代 h:) $(\frac{1}{\mu})^2 + (\frac{5}{2\mu})^2 = 1$, $\mu = \pm\frac{\sqrt{29}}{2}$,

$\implies x = \mp\frac{2}{\sqrt{29}}$, $y = \pm\frac{5}{\sqrt{29}}$, *(代 g:)* $z = 1 - x + y = 1 \pm \frac{7}{\sqrt{29}}$,

$$f(\mp\frac{2}{\sqrt{29}}, \pm\frac{5}{\sqrt{29}}, 1\pm\frac{7}{\sqrt{29}}) = \mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}.$$

So the maximum value of f on the curve is

$f(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}) = 3 + \sqrt{29}$. ■

(And the minimum value of f on the curve is

$f(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}) = 3 - \sqrt{29}$.*)*