11.4 The comparison tests

1. comparison test $a_n \le b_n$
2. limit comparison test $\lim \frac{a_n}{b_n} = c$
3. estimate sums 大估包小估

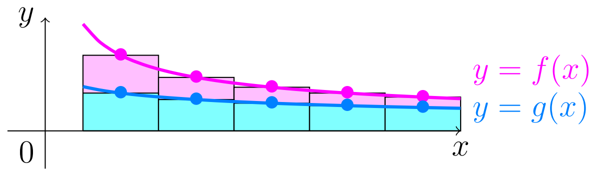
0.1 Comparison test

Recall: Comparison Theorem: Continuous $f(x) \ge g(x) \ge 0$ for $x \ge a$,

$$\int_a^\infty f(x)\,dx \text{ converges} \implies \int_a^\infty g(x)\,dx \text{ converges; 大收就小收}$$

$$\int_a^\infty g(x)\,dx \text{ diverges} \implies \int_a^\infty f(x)\,dx \text{ diverges. 小發就大發}$$

(**Attention:** 反向不保證。)



Theorem 1 (Comparison Test)
Suppose that $\sum a_n$ and $\sum b_n$ are series with ***positive*** *terms, and $a_n \le b_n$ for all n, then*

(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Proof. Let $\{s_n\}$ and $\{t_n\}$ be partial sums of $\sum a_n$ and $\sum b_n$, resp., then they are increasing ($\because$ positive).

(i) Let $\sum b_n = t$, then $s_n \le t_n \le t$, bounded above by t.

By the Monotone Convergence Theorem, $\{s_n\}$, and hence $\sum a_n$, converges.

(ii) $t_n \ge s_n \to \infty$ as $n \to \infty$, so $\sum b_n$ diverges. ■

Note: 1. 非負級數也可以。(有負的不行, 不會遞增。)
2. 不用通通 $a_n \le b_n$, 只要 ultimately ($\exists\, N \in \mathbb{N} \ni n > N \implies a_n \le b_n$)。
3. 反過來跟瑕積分一樣不保證。

Skill: 會找知道收斂/發散的級數來比, 兩個常用來比較的級數:

1. p-series $\boxed{\sum \frac{1}{n^p} \text{ converges } \iff p > 1}$.
2. geometric series $\boxed{\sum ar^{n-1} \text{ converges } \iff |r| < 1}$.

Example 0.1 *Determine whether the series* $\sum \frac{5}{2n^2+4n+3}$ *converges or diverges.*

(找級數) Let $a_n = \frac{5}{2n^2+4n+3}$ *and* $b_n = \frac{5}{2n^2}$.

(比大小) $\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$ *for* n, *and*

(驗收發) $\sum \frac{5}{2n^2} = \frac{5}{2}\sum \frac{1}{n^2}$, *a p-series with* $p = 2 > 1$, *converges.*

(做判斷) By the Comparison Test, $\sum \frac{5}{2n^2+4n+3}$ *converges.* ∎

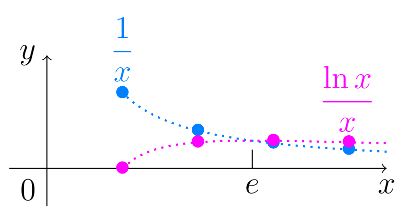
[Integral Test] $\int_1^{\infty} \frac{5}{2x^2+4x+3}\,dx = \lim_{t\to\infty}\int_1^t \frac{5}{2(x+1)^2+1}\,dx$

$= \lim_{t\to\infty}\int_1^t \frac{5}{\sqrt{2}}\frac{d[\sqrt{2}(x+1)]}{[\sqrt{2}(x+1)]^2+1} = \lim_{t\to\infty}\left[\frac{5}{\sqrt{2}}\tan^{-1}(\sqrt{2}(x+1))\right]_1^t$

$= \lim_{t\to\infty}\left[\frac{5}{\sqrt{2}}(\tan^{-1}(\sqrt{2}(t+1)) - \tan^{-1}2\sqrt{2})\right] = \frac{5}{\sqrt{2}}(\frac{\pi}{2} - \tan^{-1}2\sqrt{2}).$ ∎

Example 0.2 *Test the series* $\sum \frac{\ln n}{n}$ *for convergence or divergence.*

(Integral Tested: divergent.)



(找級數) Let $b_n = \frac{\ln n}{n}$ *and* $a_n = \frac{1}{n}$.

(比大小) $\because$ $\ln n > 1$ *for* $n \geq 3$, $\frac{1}{n} < \frac{\ln n}{n}$, *and*

(驗收發) $\sum \frac{1}{n}$ *is a p-series with* $p = 1$ *(harmonic series), diverges.*

(做判斷) By the Comparison Test, $\sum \frac{\ln n}{n}$ *diverges.* ∎

Skill: 使用技巧: 要證收斂, 找比他大且收斂; 要證發散, 找比他小且發散。

Theorem 2 (Limit Comparison Test)
Suppose that $\sum a_n$ *and* $\sum b_n$ *are series with* ***positive*** *terms. If*

$$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = c}$$

where c *is a finite number and* $c > 0$*, then either both series converge or both diverge. (比值極限存在而且正, 同收同發。)*

Proof. $\forall\ 0 < m < c < M\ \exists\ N > 0 \ni n > N \implies 0 < m < \dfrac{a_n}{b_n} < M$.

If $\sum b_n$ converges, so does $\sum Mb_n$ and hence $\sum a_n$; if $\sum b_n$ diverges, so does $\sum mb_n$ and hence $\sum a_n$ by the Comparison Test. ∎

Note: 如果極限是 0 或 ∞, 只能保證一邊。(Exercise 11.4.40 & 11.4.41)
If $c = 0$ (想像 $a_n \ll b_n$), $\sum b_n$ converges $\implies$ $\sum a_n$ converges.
If $c = \infty$ (想像 $a_n \gg b_n$), $\sum b_n$ diverges $\implies$ $\sum a_n$ diverges.

Example 0.3 *Test the series* $\sum \dfrac{1}{2^n - 1}$ *for convergence or divergence.*

Let $a_n = \dfrac{1}{2^n - 1}$. *(*$\displaystyle\int_1^\infty \frac{1}{2^x - 1}\, dx = ?$ *不會算就不要用 Integral Test!)*

(找級數) Let $b_n = \dfrac{1}{2^n}$. *(*$\sum b_n$ *收斂, 但是* $\dfrac{1}{2^n - 1} > \dfrac{1}{2^n}$ *(小收),*
大小方向相反不能用 *Comparison Test!)* Dio: 無駄無駄 (沒用沒用)...

(求極限) $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = \frac{1}{1 - 0} = 1 (> 0)$.

(驗收發) $\because \sum \dfrac{1}{2^n}$ *is a geometric series with* $|r| = \left|\dfrac{1}{2}\right| < 1$*, converges.*

(做判斷) By the Limit Comparison Test, $\sum \dfrac{1}{2^n - 1}$ *converges.* ∎

Additional: 1. $\displaystyle\int \frac{dx}{2^x - 1} = \lg\left(\frac{2^x - 1}{2^x}\right) + C$, $\displaystyle\int_1^\infty \frac{dx}{2^x - 1} = 1$.

2. Choose $b_n = \dfrac{1}{2^{n-1}} \geq \dfrac{1}{2^n - 1}$ & $\sum \dfrac{1}{2^{n-1}} = 2$. (找對人就可比。)

Skill: 爲什麼找 $\dfrac{1}{2^n}$? 因爲當 n 很大, $2^n - 1 \approx 2^n$.
用這招找到的 b_n 得到的比值極限通常都是 $1 (> 0)$。

Example 0.4 *Determine whether the series* $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ *converges or diverges.*

Let $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$. *When* n *large,* $2n^2+3n \approx 2n^2$, $\sqrt{5+n^5} \approx \sqrt{n^5}$, $\frac{2n^2+3n}{\sqrt{5+n^5}} \approx \frac{2n^2}{\sqrt{n^5}} = \frac{2}{\sqrt{n}}$. *(找級數) Choose* $b_n = \frac{2}{\sqrt{n}}$.

(求極限) $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{2/\sqrt{n}} = \lim_{n\to\infty} \frac{2+3/n}{2\sqrt{5/n^5+1}} = \frac{2+3\cdot 0}{2\sqrt{5\cdot 0+1}} = 1$.

(驗收發) $\because \sum \frac{1}{\sqrt{n}}$ *is a p-series with* $p = \frac{1}{2} < 1$, *diverges.*

(做判斷) By the Limit Comparison Test, $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ *diverges.* ■

0.2 Estimate sums

當 $a_n \le b_n$ for n, 且是用 Comparison Test 以 $\sum b_n$ 收斂來證明 $\sum a_n$ 收斂。
Let $s_n = \sum_{i=1}^{n} a_n$, $s = \sum a_n$, $R_n = s - s_n$, $t_n = \sum_{i=1}^{n} b_n$, $t = \sum b_n$, $T_n = t - t_n$.
Then $R_n = a_{n+1} + a_{n+2} + \ldots \le b_{n+1} + b_{n+2} + \ldots = T_n$. (小的比大的剩得少。)

1. 如果 $\sum b_n$ 剛好是 p-series ($p > 1$), 就可以用 T_n 的 Estimate for the Integral Test 去幫忙估計 R_n: $R_n \le T_n \le \int_n^\infty \frac{1}{x^p}\,dx = \frac{1}{(p-1)n^{p-1}}$.

2. 如果剛好是 geometric series ($|r| < 1$) 怎麼估? 用積分? 錯!

直接算: $T_n = \sum_{i=n+1}^{\infty} ar^{i-1} = \frac{ar^n}{1-r}$.

Example 0.5 *Use* s_{100} *to approximate* $\sum \frac{1}{n^3+1}$. *Estimate the error involved in the approximation.*

$$a_n = \frac{1}{n^3+1} < \frac{1}{n^3} = b_n,\ R_{100} \le T_{100} \le \int_{100}^\infty \frac{1}{x^3}\,dx = \frac{1}{2(100)^2} = 0.00005.$$

$\sum \frac{1}{n^3+1} \approx s_{100} \approx 0.6864538$ *with error less than* 0.00005. ■

Additional: Series with $\ln n$

Recall: p**-series** $\sum \frac{1}{n^p}$ converges $\iff p > 1$.

T4D	: Test for Divergence	$\lim_{n\to\infty} a_n \neq 0$ diverges.
IT	: Integral Test	$\int_1^{\infty} f(x)\, dx \iff \sum_{n=1}^{\infty} f(n)$.
CT	: Comparison Test	大收就小收, 小發就大發。
LCT	: Limit Comparison Test	$\lim_{n\to\infty} \frac{a_n}{b_n} = \begin{cases} c > 0 & \sum b_n \iff \sum a_n \\ = 0 & \sum b_n \text{ 大 } \sum a_n \text{ 小} \\ = \infty & \sum b_n \text{ 小 } \sum a_n \text{ 大} \end{cases}$.

$$\boxed{1.\ \sum_{n=3}^{\infty} \frac{1}{\ln \ln n} \text{ diverges}}$$

Proof. $\ln \ln n < \ln n < n$, $\frac{1}{\ln \ln n} > \frac{1}{\ln n} > \frac{1}{n}$.

$\because \sum_{n=3}^{\infty} \frac{1}{n}$ diverges, $\therefore \sum_{n=3}^{\infty} \frac{1}{\ln \ln n}$ (also $\sum_{n=3}^{\infty} \frac{1}{\ln n}$) diverges by CT. ■

$$\boxed{2.\ \sum_{n=2}^{\infty} \frac{1}{(\ln n)^p} \text{ diverges for all } p}$$

Proof. For $p \leq 0$, $\lim_{n\to\infty} \frac{1}{(\ln n)^p} \neq 0$, diverges by T4D.

For $p > 0$, let $b_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{1/(\ln n)^p}{1/n} = \lim_{n\to\infty} \frac{n}{(\ln n)^p} \overset{l'H}{=} \lim_{n\to\infty} \frac{n}{p(\ln n)^{p-1}}$

$\overset{l'H}{=} \cdots \overset{l'H}{=} \lim_{n\to\infty} \frac{n(\ln n)^{\lceil p \rceil - p}}{p(p-1)\cdots(p-\lceil p \rceil+1)} = \infty$. ($\ell$'Hospital's Rule $\times \lceil p \rceil$)

$\because \sum_{n=2}^{\infty} \frac{1}{n}$ diverges, $\therefore \sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ diverges by LCT.

[Another proof]

$p > 0$, $\frac{1}{p} \ln n = \ln n^{1/p} < n^{1/p}$, $\ln n < pn^{1/p}$, $\frac{1}{(\ln n)^p} > \frac{1}{p^p n}$.

$\because \sum_{n=2}^{\infty} \frac{1}{n}$ diverges, $\therefore \sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ diverges by CT. ■

Skill: When n large ($> e$), $1 < \ln n = p \ln n^{1/p} < pn^{1/p}$ for $p > 0$.

$$\boxed{3.\ \textbf{(ln-series)}\ \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}\ \text{converges}\iff p>1}\ \text{(Exercise 11.3.29)}$$

Proof. $f(x)=\dfrac{1}{x(\ln x)^p}$ 連正遞減 for $x>e^{-p}$. $\left(f'=-\dfrac{(p+\ln x)}{x^2(\ln x)^{p+1}}\right)$.

$$\int_2^\infty\frac{1}{x(\ln x)^p}\,dx=\begin{cases}\lim\limits_{n\to\infty}\ln\ln x\Big|_2^t=\infty & \text{if } p=1,\\ \lim\limits_{n\to\infty}\dfrac{(\ln x)^{1-p}}{1-p}\Big|_2^t=\infty & \text{if } p<1,\\ \lim\limits_{n\to\infty}\dfrac{-1}{(p-1)(\ln x)^{p-1}}\Big|_2^t=\dfrac{1}{(p-1)(\ln 2)^{p-1}} & \text{if } p>1.\end{cases}$$

$$\therefore\int_2^\infty\frac{1}{x(\ln x)^p}\,dx\ \left(\overset{u=\ln x}{=}\int_{\ln 2}^\infty\frac{du}{u^p}\right)\iff\sum_{n=2}^\infty\frac{1}{n(\ln n)^p}\ \text{by IT.}\qquad\blacksquare$$

$$\boxed{4.\ \sum_{n=1}^{\infty}\frac{\ln n}{n^p}\ \text{converges}\iff p>1}\ \text{(Exercise 11.3.32)}$$

Proof. $f(x)=\dfrac{\ln x}{x^p}$ 連正遞減 for $x>e^{1/p}$ $\left(f'=\dfrac{1-p\ln x}{x^{p+1}}\right)$.

$$\int_1^\infty\frac{\ln x}{x^p}\,dx=\begin{cases}\lim\limits_{n\to\infty}\dfrac{(\ln x)^2}{2}\Big|_1^n=\lim\limits_{n\to\infty}\dfrac{(\ln n)^2}{2}=\infty & \text{if } p=1,\\ \lim\limits_{n\to\infty}\dfrac{x^{1-p}[(1-p)\ln x-1]}{(1-p)^2}\Big|_1^n=\infty & \text{if } p<1,\\ \lim\limits_{n\to\infty}\dfrac{-(p-1)\ln x-1}{(p-1)^2x^{p-1}}\Big|_1^n=\dfrac{1}{(p-1)^2} & \text{if } p>1.\end{cases}$$

$$\therefore\int_1^\infty\frac{\ln x}{x^p}\,dx\iff\sum\frac{\ln n}{n^p}\ \text{by IT.}$$

[Alternating proof]

For $n\ge 1$ and $\varepsilon>0$, $\varepsilon\ln n=\ln n^\varepsilon<n^\varepsilon$, $\ln n<\dfrac{1}{\varepsilon}n^\varepsilon$, $\dfrac{\ln n}{n^p}<\dfrac{n^\varepsilon}{\varepsilon n^p}=\dfrac{1}{\varepsilon n^{p-\varepsilon}}$.

$\because\sum\dfrac{1}{\varepsilon n^{p-\varepsilon}}$ converges for $p-\varepsilon>1\iff p>1+\varepsilon\iff p>1$,

$\therefore\sum\dfrac{\ln n}{n^p}$ converges for $p>1$ by CT.

For $n\ge 3$, $\dfrac{\ln n}{n^p}>\dfrac{1}{n^p}$, $\sum\dfrac{1}{n^p}\implies\sum\limits_{(n=3)}^{(\infty)}\dfrac{\ln n}{n^p}$ diverges for $p\le 1$ by CT. $\blacksquare$
