Almost-sure convergence $\Rightarrow$ Convergence in probability

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables

Let X also be a random variable

- Want to show:

$$\text{If } X_n \xrightarrow{a.s.} X, \text{ then } X_n \xrightarrow{P} X$$

Pf: Recall that if $X_n \xrightarrow{a.s.} X$, then for any $\varepsilon > 0$

$$P\left(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\} \text{ infinitely often}\right) = 0 \quad (1)$$

(this is the alternative definition discussed in Lecture 2?)

For each $n \in \mathbb{N}$, define event $A_n = \{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}$

Then (1) can be written as

$$P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) = 0 \quad (2)$$

Next, we try to connect (2) with WLLN:

$$0 = P\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n\right)$$

this is a decreasing sequence

$$= P\left(\lim_{k\to\infty} \bigcap_{m=1}^{k}\bigcup_{n=m}^{\infty} A_n\right)$$

$$= \lim_{k\to\infty} P\left(\bigcap_{m=1}^{k}\bigcup_{n=m}^{\infty} A_n\right)$$ ← by continuity of probability function (in Lecture 3)

$$= \lim_{k\to\infty} P\left(\bigcup_{n=k}^{\infty} A_n\right)$$

$$\geq \lim_{k\to\infty} P(A_k) = \lim_{k\to\infty} P\left(\{\omega : |X_k(\omega) - X(\omega)| > \varepsilon\}\right)$$

Hence, we have $\lim_{k\to\infty} P\left(\{\omega : |X_k(\omega) - X(\omega)| > \varepsilon\}\right) = 0$,

which suggests that $X_k \xrightarrow{P} X$.

□