7.1 Integration by parts

1. indefinite integration version $\int fg'\ dx = fg - \int f'g\ dx$
2. definite integration version $\int_a^b fg'\ dx = fg|_a^b - \int_a^b f'g\ dx$

相愛容易相處難, 微分容易積分難。

鏈鎖律 Chain Rule	$\longleftrightarrow$	變數變換 Substitution Rule
乘積律 Product Rule	$\longleftrightarrow$	分部積分 Integration by Parts

SOP—積分 123:

1. 積分公式 (Antiderivative) 有沒有?
   如果是基本函數 x^n, e^x, $\ln x$, $\sin x$, $\sin^{-1} x$,... 的導函數,
   by TFTC: $F' = f \implies \int f\ dx = F + C$.
   如果是他們的加減常數倍, $\int (cf \pm g)\ dx = c\int f\ dx \pm \int g\ dx$.
2. 變數變換 (Substitution Rule) 換不換?
   如果是剛好可以換乾淨&變簡單, $\int f'(g)g'\ dx = f(g) + C$.
3. 分部積分 (Integration by Part) 分一分? $\int fg'\ dx = fg - \int f'g\ dx$.

0.1 Indefinite integral version

$$\boxed{\int \boldsymbol{f(x)g'(x)\ dx} = \boldsymbol{f(x)g(x)} - \int \boldsymbol{g(x)f'(x)\ dx}}$$

Proof. Recall Product Rule: $(fg)' = f'g + fg'$.

By TFTC, $fg = \int (fg)'\ dx = \int (f'g + fg')\ dx = \int f'g\ dx + \int fg'\ dx$,

$$\int fg'\ dx = fg - \int f'g\ dx.$$

(不用 $+C$, 不定積分本身就是最一般的反導數 (有 C)。) ■

Skill: 記憶法: Let $u = f(x)$ and $v = g(x)$, then differentials $du = f'(x)\ dx$ and $dv = g'(x)\ dx$. (把 du, dv 當作 differential 微分。) By Substitution Rule:

$$\boxed{\int \boldsymbol{u\ dv} = \boldsymbol{uv} - \int \boldsymbol{v\ du}}$$

Example 0.1 $\int x \sin x \, dx = ?$

[Ver 1: 正式]

Let $f(x) = x$ and $g'(x) = \sin x$, then $f'(x) = 1$ and $g(x) = -\cos x$.

$$\int \underbrace{x}_{f(x)} \underbrace{\sin x}_{g'(x)} \, dx = \underbrace{x}_{f(x)} \underbrace{(-\cos x)}_{g(x)} - \int \underbrace{-\cos x}_{g(x)} \cdot \underbrace{1}_{f'(x)} \, dx$$

$$= -x\cos x + \int \cos x \, dx = -x\cos x + \sin x + C.$$

[Ver 2: 非正式] 令 u 是其中一個函數, 剩下 (含 dx) 令爲 dv, 找出 du & v.

Let $u = x$ and $dv = \sin x \, dx$, then $du = dx$ and $v = -\cos x$.

$$\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv} \boxed{= \int \underbrace{x}_{u} \underbrace{d(-\cos x)}_{dv}} = \underbrace{x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{-\cos x}_{v} \underbrace{dx}_{du}$$

$$= -x\cos x + \sin x + C. \qquad \blacksquare$$

Attention: 1. [Ver 2] 中 "$d(-\cos x)$" 是$\boxed{\text{非正式}}$的寫法, 但是推薦使用。
2. 下括號 "$\underbrace{}_{\dots}$" 是注釋, 不用寫。
3. 非證明題可以省略寫 "Let $u = \cdots$" 節省時間。

Note: 1. 別忘了 $+C$;
2. 怎麼檢查對不對? 還是一樣用微分! (這時候會用上乘積律)
$(-x\cos x + \sin x + C)' = -\cos x - x(-\sin x) + \cos x + 0 = x \sin x$;
3. 換人積積看?

if let $u = \sin x$ and $dv = x \, dx$, then $du = \cos x \, dx$ and $v = \dfrac{x^2}{2}$.

$$\int \underbrace{\sin x}_{u} \cdot \underbrace{x \, dx}_{dv} \boxed{= \int \underbrace{\sin x}_{u} \underbrace{d(\frac{x^2}{2})}_{dv} = \underbrace{\sin x}_{u} \cdot \underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v} \underbrace{d\sin x}_{du}}$$

$= \underbrace{\sin x}_{u} \cdot \underbrace{\dfrac{x^2}{2}}_{v} - \int \underbrace{\dfrac{x^2}{2}}_{v} \underbrace{\cos x \, dx}_{du}$, 變得更複雜難積 (X)。

Attention: 不保證一定算得出來, 只是換個函數積分。

Skill 1: 通常 $u = f(x)$ 會選擇 <u>$f'(x)$(導數) 變簡單</u>的。
推薦: 多項式 x^n (x^{-n} 不算), $\ln x$.

Example 0.2 $\boxed{\int \ln x \; dx}=?$

Let $u=\ln x$ *and* $dv=dx$*, then* $du=\frac{1}{x}\;dx$ *and* $v=x$*.*

$$\int \underbrace{\ln x}_{u}\underbrace{dx}_{dv} = \underbrace{\ln x}_{u}\cdot\underbrace{x}_{v} - \int \underbrace{x}_{v}\underbrace{d\ln x}_{du} = \underbrace{\ln x}_{u}\cdot\underbrace{x}_{v} - \int \underbrace{x}_{v}\cdot\underbrace{\frac{1}{x}\;dx}_{du}$$

$$= x\ln x - \int \; dx = \boxed{x\ln x - x + C}.$$ *(加入你的不定積分表)* ■

Note: 1. $\boxed{\int \; dx = \int 1 \; dx} = x + C$, 不是漏打, 是積 1。

2. x 乘在 $\sin x$, $\ln x$ ... 等後面要加“·”區隔, 乘前面可以省略。
Ex: $\ln x \cdot x = x\ln x = \ln x^x \neq \ln(x\cdot x) = \ln x^2 = 2\ln x \neq (\ln x)^2$.

Example 0.3 $\int t^2 e^t \; dt = ?$

Let $u=t^2$ *and* $dv=e^t\;dt$*, then* $du=2t\;dt$ *and* $v=e^t$*.*

$$\int \underbrace{t^2}_{u}\underbrace{e^t\,dt}_{dv} = \int \underbrace{t^2}_{u}\underbrace{de^t}_{dv} = \underbrace{t^2}_{u}\underbrace{e^t}_{v} - \int \underbrace{e^t}_{v}\underbrace{dt^2}_{du}$$

$$= \underbrace{t^2}_{u}\underbrace{e^t}_{v} - \int \underbrace{e^t}_{v}\underbrace{2t\;dt}_{du} = t^2e^t - 2\int te^t\;dt$$ *(雖然沒解決, 但是函數變簡單。)*

再對 $\int te^t\;dt$ 用一次分部積分法: ($\because u, v$ 用過了, *let* $U=t$, $dV=e^t\;dt$.)

$$\int \underbrace{t}_{U}\underbrace{e^t\,dt}_{dV} = \int \underbrace{t}_{U}\underbrace{de^t}_{dV} = \underbrace{t}_{U}\underbrace{e^t}_{V} - \int \underbrace{e^t}_{V}\underbrace{dt}_{dU} = te^t - e^t + C.$$

$$\int t^2e^t\;dt = t^2e^t - 2\int te^t\;dt = t^2e^t - 2te^t + 2e^t + C_1,\ \textit{where}\ C_1 = -2C.$$ ■

Note: 分部積分可以用了再用。 — 一次分不夠, 你可以分第二次。
不用加那麼多種 C, 最後的答案 $+C$ 就好: $\int t^2e^t\;dt = t^2e^t - 2te^t + 2e^t + C$.

Skill 2: 通常 $dv = g'(x)\;dx$ 會選擇 $g(x)$ (反導數) 不變難的。
推薦: x^{-n}, e^x, $\sin x$, $\cos x$, $\sec^2 x$, $\sec x\tan x$.

Example 0.4 $\int e^x \sin x \; dx = ?$

(熟練後可以不用 *Let* $f(x)/u = ...,\ g'(x)/dv = ...$)

$$\int \underbrace{e^x}_{u} \underbrace{\sin x \; dx}_{dv} \boxed{= \int \underbrace{e^x}_{u} \underbrace{d(-\cos x)}_{dv} = \underbrace{e^x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \underbrace{de^x}_{du}}$$

$$= \underbrace{e^x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \underbrace{e^x \; dx}_{du} = -e^x \cos x + \int e^x \cos x \; dx,$$

(和原積分相似, 變成 $\cos x$, 再做一次。)

$$\int \underbrace{e^x}_{U} \underbrace{\cos x \; dx}_{dV} \boxed{= \int \underbrace{e^x}_{U} \underbrace{d \sin x}_{dV} = \underbrace{e^x}_{U} \underbrace{\sin x}_{V} - \int \underbrace{\sin x}_{V} \underbrace{de^x}_{dU}}$$

$$= \underbrace{e^x}_{U} \underbrace{\sin x}_{V} - \int \underbrace{\sin x}_{V} \cdot \underbrace{e^x \; dx}_{dU} = e^x \sin x - \int e^x \sin x \; dx,$$

(還是沒解決, 但是變出了負的$(-)$原式, 可以做!)

$$\begin{aligned}
\int e^x \sin x \; dx &= -e^x \cos x + \int e^x \cos x \; dx \\
&= -e^x \cos x + e^x \sin x - \int e^x \sin x \; dx, \\
2\int e^x \sin x \; dx &= e^x(\sin x - \cos x), \quad (\text{移項}) \\
\int e^x \sin x \; dx &= \frac{1}{2}e^x(\sin x - \cos x) + C. \quad (\text{除 } 2)
\end{aligned}$$

■

Attention: 選擇要一致, 如果第二次用 $\int \cos x \; de^x$ 就會變回原題目。

Skill 3: 分部積分完又<u>出現原式</u>, 可以<u>移項合併</u>, <u>最後</u>再一起 <u>$+C$</u>。

Question: 這一題可以挑 $u = \sin x$, $v = e^x$ (換人積) 嗎?
Answer: 可以, 請務必試試: $\int \sin x \cdot e^x \; dx = \int \sin x \; de^x = ...$
Question: Who is u and who is v?
Answer: 積不下去就換人積積看。$\int$經驗 d作業

Example 0.5 *Prove the reduction formula*

$$\int \sin^n x\ dx \;=\; -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\ dx$$

where $n \geq 2$ *is an integer.*

Proof. *Let* $u = \sin^{n-1}x$ *and* $dv = \sin x\ dx$,
then $du = (n-1)\sin^{n-2}x\cos x\ dx$ *and* $v = (-\cos x)$.

$$\begin{aligned}
\int \sin^n x\ dx &= \int \sin^{n-1}x\sin x\ dx = \boxed{\int \sin^{n-1}x\ d(-\cos x)} \\
&= \boxed{\sin^{n-1}x\,(-\cos x) - \int (-\cos x)\ d\sin^{n-1}x} \\
&= \sin^{n-1}x\,(-\cos x) - \int (-\cos x)\,(n-1)\sin^{n-2}x\cos x\ dx \\
&= -\cos x\sin^{n-1}x + (n-1)\int \cos^2 x\sin^{n-2}x\ dx \\
&= -\cos x\sin^{n-1}x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x\ dx \\
&= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\ dx - (n-1)\int \sin^n x\ dx, \\
n\int \sin^n x\ dx &= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\ dx, \\
\int \sin^n x\ dx &= -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\ dx.
\end{aligned}$$

(不用 $+C$, *因為還有不定積分。)* ∎

Note: 減化公式說明 $\int \sin^n x\ dx$ 最後可以變成 $\int \sin x\ dx$ (if n is odd) 或是 $\int\ dx$ (if n is even) 與 $\sin x, \cos x$ 的組合。

補充: (Exercise 7.1.48.) integer $n \geq 2$,

$$\int \cos^n x\ dx \;=\; \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\ dx$$

(More reduction formula see Exercise 7.1.51–54.)

0.2 Definite integral version

$$\boxed{\int_a^b f(x)g'(x)\ dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\ dx}$$

Note: 差別在代入上下界, 沒有 $+C$。

Example 0.6 $\displaystyle\int_0^1 \tan^{-1} x\ dx = ?$

$$\int_0^1 \tan^{-1} x\ dx \boxed{= \tan^{-1} x \cdot x\Big|_0^1 - \int_0^1 x\ d\tan^{-1} x}$$

$$= \tan^{-1} x \cdot x\Big|_0^1 - \int_0^1 x \frac{1}{1+x^2}\ dx \qquad \left((\tan^{-1} x)' = \frac{1}{1+x^2}.\right)$$

$$= x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\ dx$$

$$= (1\cdot\tan^{-1} 1 - 0\cdot\tan^{-1} 0) - \int_0^1 \frac{x}{1+x^2}\ dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\ dx.$$

Use Substitution Rule: let $t = 1+x^2$ ($\because u, v$ *are used*),
then $dt = 2x\ dx$, $x\ dx = \frac{1}{2}\ dt$, *when* $x=0$, $t=1$, *and when* $x=1$, $t=2$.

$$\int_0^1 \frac{x}{1+x^2}\ dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\Big[\ln|t|\Big]_1^2 \qquad \left(\int \frac{dt}{t} = \ln|t| + C, \text{因爲 } t>0, \text{這裡可以用 } \ln t\text{。}\right)$$

$$= \frac{1}{2}(\ln 2 - \ln 1) \qquad (\ln 1 = 0, \text{不要沒事寫一堆。}) \text{ 成步堂 } \ln 1\text{: 異議阿里!}$$

$$= \frac{\ln 2}{2}.$$

$$\therefore \int_0^1 \tan^{-1} x\ dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\ dx = \frac{\pi}{4} - \frac{\ln 2}{2} \left(= \frac{\pi}{4} - \ln\sqrt{2}\right). \qquad \blacksquare$$

♦: [Another sol] $\int \tan^{-1} x\ dx = \boxed{\int \theta\ d\tan\theta} = \theta\tan\theta - \int \tan\theta\ d\theta$
$= \theta\tan\theta - \ln|\sec\theta| + C = \tan^{-1} x \cdot x - \ln\sqrt{1+x^2} + C.$