2.6 Limit at infinity; horizontal asymptotes

1. limit at infinity 無限處極限
2. horizontal asymptote 水平漸近線
3. infinite limit at infinity 無限處無限極限

$\lim\limits_{t\to\infty}$ 夕陽 = 黃昏 — 夕陽無限好, 只是近黃昏。

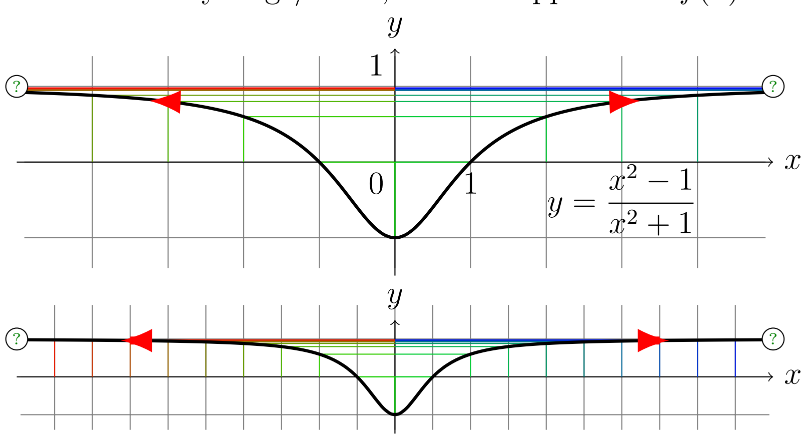
之前都是討論函數在某個點附近的趨勢傾向 ($f(x)$ 在 a 的極限), 如果函數在實數上都有定義 (ex: polynomial), 在 (正負) 無限遠處的趨勢傾向是什麼呢?

Where $f(x)$ goes when x goes to (negative) infinity and beyond?

0.1 Limit at infinity

Let $f(x) = \dfrac{x^2-1}{x^2+1}$. When x is very large/small, what's happened to $f(x)$?

x	$f(x)$
0	-1
± 1	0
± 2	0.6
± 3	0.8
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998



觀察: 如果 x 越大/小, 則 $f(x)$ 越大, 但是都不會超過 1 ($\dfrac{x^2-1}{x^2+1} < 1$)。

Question: Where does $f(x)$ go when x goes to (negative) infinity?
Answer: 1. (要怎麼簡單表示? 還是用極限。)

Question: $\lim\limits_{\boxed{?}} \dfrac{x^2-1}{x^2+1} = 1$, $\boxed{?}$ 怎麼寫?

Define: f is defined on (a, ∞).

$$\boxed{\lim_{x\to\infty} f(x) = L} \quad \text{or} \quad \boxed{f(x) \to L \text{ as } x \to \infty}$$

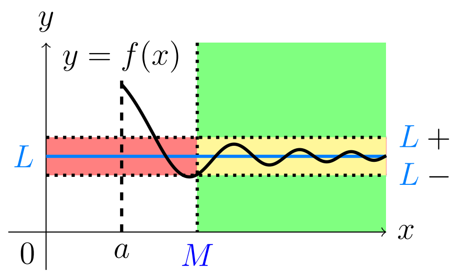
$$\text{if } \boxed{\forall\, \varepsilon > 0,\ \exists\ M > 0,\ \ni x > M \implies |f(x) - L| < \varepsilon.}$$

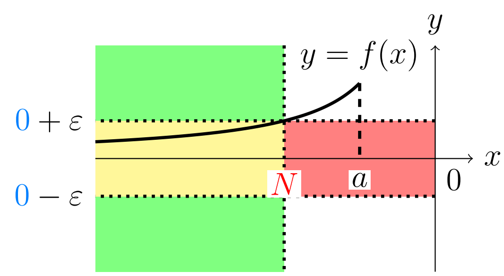
$f(x)$ approaches L as x ***sufficiently*** 充分足夠 large.
(只要 x 夠大, $f(x)$ 就會夠靠近 L。)
f is defined on $(-\infty, a)$.

$$\boxed{\lim_{x\to-\infty} f(x) = L} \quad \text{or} \quad \boxed{f(x) \to L \text{ as } x \to -\infty}$$

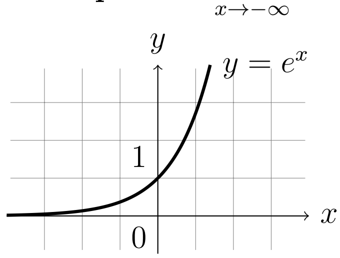
$$\text{if } \boxed{\forall\, \varepsilon > 0,\ \exists\ N < 0,\ \ni x < N \implies |f(x) - L| < \varepsilon.}$$

$f(x)$ approaches L as x sufficiently small.
(只要 x 夠小, $f(x)$ 就會夠靠近 L。)



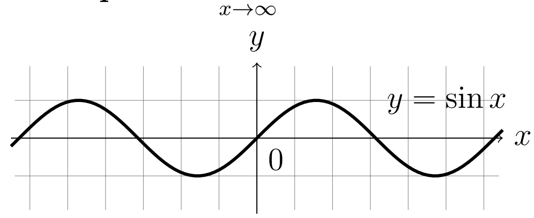


Example 0.1 $\lim_{x\to-\infty} e^x = ?$



$e^x \to 0$ as $x \to -\infty$,
$\therefore \lim_{x\to-\infty} e^x = 0.$ ■
(when $x \to \infty$?)

Example 0.2 $\lim_{x\to\infty} \sin x = ?$



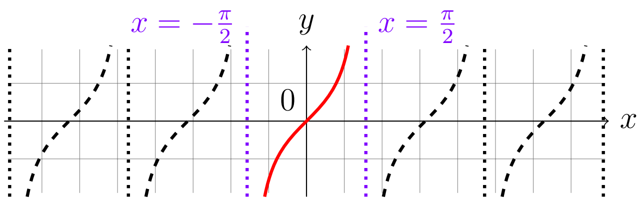
$\sin x$ 會在 $[-1, 1]$ 不斷變化,
所以不存在極限. *(也沒有水平漸近線)*
$\therefore \lim_{x\to\infty} \sin x$ *does not exist.* ■

Attention: 沒有 $e^{-\infty}$, 也沒有 $e^{-\infty} = 0$.

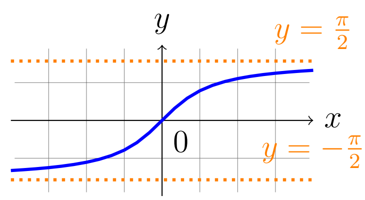
0.2 Horizontal asymptote

Define: $y = L$ is a ***horizontal asymptote*** 水平漸近線 of $y = f(x)$ if the limit (L) exists at the (negative) infinity $\infty/-\infty$. 當無限處極限的2種情形之一發生時。Ex: $y = 0$ is an H.A. of $y = e^x$, and $\sin x$ has no H.A..

Example 0.3 $\lim\limits_{x\to-\infty} \tan^{-1} x = -\dfrac{\pi}{2}$, $\lim\limits_{x\to\infty} \tan^{-1} x = \dfrac{\pi}{2}$, *horizontal asymptotes:* $y = -\dfrac{\pi}{2}$, $y = \dfrac{\pi}{2}$.



$\tan x : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$



$\tan^{-1} x : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

Attention: 沒有 $\tan^{-1} \pm\infty$, 也沒有 $\tan^{-1} \pm\infty = \pm\dfrac{\pi}{2}$.

Note: 水平漸近線最多只有兩條 (as $x \to \infty$ and $x \to -\infty$). (垂直的呢?)

Note: $y = f(x)$ 有$\begin{Bmatrix} \text{V.A.} & x = a \\ \text{H.A.} & y = L \end{Bmatrix} \iff y = f^{-1}(x)$ 有$\begin{Bmatrix} \text{H.A.} & y = a \\ \text{V.A.} & x = L \end{Bmatrix}$.

Example 0.4 *Evaluate* $\lim\limits_{x\to 0^-} e^{\frac{1}{x}} =?\ 0$.

Let $t = \dfrac{1}{x}$. $t \to -\infty \iff x \to 0^-$. $\therefore \lim\limits_{x\to 0^-} e^{\frac{1}{x}} = \lim\limits_{t\to-\infty} e^t = 0$. ■

Note: 無限處極限是一種單邊極限:
因爲不可能從 ∞ 的右邊靠近, 也不可能從 $-\infty$ 左邊靠近。

Skill: 我們可以利用這個方法換成 0^+ & 0^-: 天涯若比零

Let $t = \dfrac{1}{x}$, then $x \to \infty/-\infty \iff t = \dfrac{1}{x} \to 0^+/0^-$. Then

$$\boxed{\lim_{x\to\infty/-\infty} f(x) = \lim_{t\to 0^+/0^-} f(\frac{1}{t}).}$$

Note: 極限律: "加減乘除常數倍, 冪次開根 c&x." 只要極限是存在的, 單邊極限也能用 = 無限處極限也能用。

Example 0.5 *Find infinite limits and limits at infinity of* $f(x) = \frac{1}{x}$, *and find asymptotes of* $y = \frac{1}{x}$.

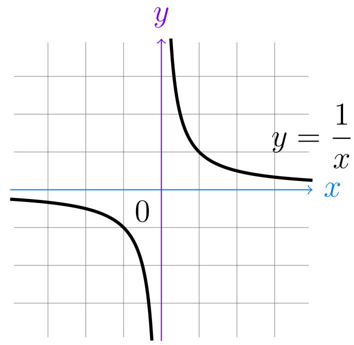
From graph $y = \frac{1}{x}$ *or compute:*

$$\left.\begin{array}{l} \lim\limits_{x\to 0^+} \frac{1}{x} = \infty \\ \lim\limits_{x\to 0^-} \frac{1}{x} = -\infty \end{array}\right\} \text{ infinite limits}$$

$\Longrightarrow\ x = 0$ *vertical asymptote*

$$\left.\begin{array}{l} \lim\limits_{x\to \infty} \frac{1}{x} = 0 \\ \lim\limits_{x\to -\infty} \frac{1}{x} = 0 \end{array}\right\} \text{ limits at infinity}$$

$\Longrightarrow\ y = 0$ *horizontal asymptote(s?)* ■



(共有幾條漸近線?)

Example 0.6 *Prove* $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ *by definition. (Also,* $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$.*)*

1. $\left|\frac{1}{x} - 0\right| < \varepsilon \iff |x| > \frac{1}{\varepsilon}$, *guess* $M = \frac{1}{\varepsilon}$. $(N = -\frac{1}{\varepsilon})$

2. *Given* $\varepsilon > 0$, *choose* $M = \frac{1}{\varepsilon} > 0$. $(N = -\frac{1}{\varepsilon} < 0)$

If $\left\{\begin{array}{c} x > M \\ x < N \end{array}\right\}$, *then* $\left|\frac{1}{x} - 0\right| < \left\{\begin{array}{c} |1/M| \\ |1/N| \end{array}\right\} = \frac{1}{1/\varepsilon} = \varepsilon.$

Therefore, by the definition, $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ *(also,* $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$*).* ■

Proposition 1 *If* $r > 0$ *is a rational number, then*

$$\boxed{\lim_{x\to\infty} \frac{1}{x^r} = 0.}$$

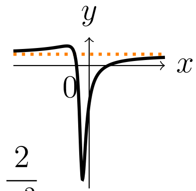
If $r > 0$ *is a rational number and* x^r *is defined* $((-\infty, a))$, *then*

$$\boxed{\lim_{x\to-\infty} \frac{1}{x^r} = 0.}$$

Proof. 利用 $\lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$ 與極限律 (冪次&開根)。

Tool: 利用$\frac{1}{x^r}$ 來計算有理函數的極限!

Example 0.7 *Evaluate* $\lim\limits_{x\to\infty}\dfrac{3x^2-x-2}{5x^2+4x+1}$.



Assume $x>0$ *when* $x\to\infty$.

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}\overset{(\div x^2)}{=}\lim_{x\to\infty}\frac{\frac{3x^2-x-2}{x^2}}{\frac{5x^2+4x+1}{x^2}}=\lim_{x\to\infty}\frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}$$

$$=\frac{\lim\limits_{x\to\infty}(3-\frac{1}{x}-\frac{2}{x^2})}{\lim\limits_{x\to\infty}(5+\frac{4}{x}+\frac{1}{x^2})}=\frac{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\frac{1}{x}-2\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}5+4\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{1}{x^2}}=\frac{3-0-0}{5+0+0}=\frac{3}{5}.\quad\blacksquare$$

Skill: 計算有理函數的無限處極限時, 同除分母函數的 x 的最高次。

Example 0.8 *Find asymptotes of* $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5}$.

$$(x>0)\ \lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\overset{(\div x)}{=}\lim_{x\to\infty}\frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}}=\lim_{x\to\infty}\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}$$

$$=\frac{\lim\limits_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\lim\limits_{x\to\infty}(3-\frac{5}{x})}=\frac{\sqrt{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}\frac{1}{x^2}}}{\lim\limits_{x\to\infty}3-5\lim\limits_{x\to\infty}\frac{1}{x}}=\frac{\sqrt{2+0}}{3-0}=\frac{\sqrt{2}}{3}.$$

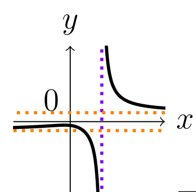
When $x<0$, $\dfrac{\sqrt{2x^2+1}}{x}=\boxed{-}\sqrt{2+\dfrac{1}{x^2}}$ *(是負的!)*, $\lim\limits_{x\to-\infty}f(x)=-\dfrac{\sqrt{2}}{3}$.

Therefore, $y=\dfrac{\sqrt{2}}{3}$ *and* $y=-\dfrac{\sqrt{2}}{3}$ *are horizonal asymptotes.*

$3x-5=0\iff x=\dfrac{5}{3}$.

Since $\sqrt{2x^2+1}>0$, $\lim\limits_{x\to\frac{5}{3}^-}f(x)=-\infty$, *and* $\lim\limits_{x\to\frac{5}{3}^+}f(x)=\infty$.

Therefore, $x=\dfrac{5}{3}$ *is a vertical asymptote.* $\blacksquare$



Note: 同除 x 的奇數次時要注意正負號, (熊出) 沒注意會差一條。

Skill: 垂直漸近線 — 多發生於當分母爲 0 處, 考慮 $\lim\limits_{x\to a^+}\dfrac{P(x)}{Q(x)}$ and $\lim\limits_{x\to a^-}\dfrac{P(x)}{Q(x)}$ for a with $Q(a)=0$. (可以只算左右極限, 省略計算 $\lim\limits_{x\to a}\dfrac{P(x)}{Q(x)}$。)

0.3 Infinite limit at infinity

Define: f is defined on $(a,\infty)/(-\infty,a)$.

$$\boxed{\lim_{\substack{x\to\infty\\ -\infty}} f(x)=\infty} \quad \text{or} \quad \boxed{f(x)\to\infty \text{ as } x\to\infty/-\infty}$$

$$\text{if } \boxed{\begin{array}{l}\forall\ M>0,\ \exists\ \begin{array}{c}N>0\\ N<0\end{array},\ \ni\ \begin{array}{c}x>N\\ x<N\end{array} \implies f(x)>M.\end{array}}$$

$$\boxed{\lim_{\substack{x\to\infty\\ -\infty}} f(x)=-\infty} \quad \text{or} \quad \boxed{f(x)\to-\infty \text{ as } x\to\infty/-\infty}$$

$$\text{if } \boxed{\begin{array}{l}\forall\ M<0,\ \exists\ \begin{array}{c}N>0\\ N<0\end{array},\ \ni\ \begin{array}{c}x>N\\ x<N\end{array} \implies f(x)<M.\end{array}}$$

$f(x)$ becomes arbitrarily {large, small} as x sufficiently {large, small}.
(當 x 足夠{大,小}, $f(x)$ 會任意{大,小}。) (你們都太任性了!)

Skill: 計算 $\left\{\begin{array}{c}\lim\limits_{x\to\infty} f(x)\\ \lim\limits_{x\to-\infty} f(x)\end{array}\right\}$時, 可以只考慮 $\left\{\begin{array}{c}x>0\\ x<0\end{array}\right\}$, 不要經過 0。

Example 0.9 $\lim\limits_{x\to\infty}(\sqrt{x^2+1}-x)=?$

$\because \sqrt{x^2+1}\to\infty$ *as* $x\to\infty$,

$\therefore \lim\limits_{x\to\infty}(\sqrt{x^2+1}-x)=\lim\limits_{x\to\infty}\sqrt{x^2+1}-\lim\limits_{x\to\infty}x=\infty-\infty$. *(***Wrong!***)*

注意! 不能用極限減法: 因為 *infinite limit* 不算極限存在.

正確的方法是 — 同乘 $\sqrt{x^2+1}+x$ $(\neq 0$ *as* $x>0)$:

$$\lim_{x\to\infty}(\sqrt{x^2+1}-x)=\lim_{x\to\infty}\left[(\sqrt{x^2+1}-x)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right]=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

(課本直接 $=0$ *since* $\sqrt{x^2+1}+x\to\infty$ *as* $x\to\infty$. — 不夠嚴謹, 再同除 x:)

$$=\lim_{x\to\infty}\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1}=\frac{\lim\limits_{x\to\infty}\frac{1}{x}}{\sqrt{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{1}{x^2}}+\lim\limits_{x\to\infty}1}=\frac{0}{\sqrt{1+0}+1}=0. \qquad \blacksquare$$

Example 0.10 $\lim_{x\to\infty}(x^2-x)=?$

注意! 不能用極限減法, $\lim_{x\to\infty}(x^2-x)=\lim_{x\to\infty}x^2-\lim_{x\to\infty}x=\infty-\infty$. *(**Wrong**)*

[Sol 1]: $\because x\to\infty \implies x-1\to\infty \implies x(x-1)\to\infty$,
Because when x becomes large, $x-1$ becomes large, and so does $x(x-1)$.
$\therefore \lim_{x\to\infty}(x^2-x)=\lim_{x\to\infty}x(x-1)=\infty$.

[Sol 2]: (Prove by definition 較嚴謹, $\boxed{\text{不}}$限數學系。)
Given $M>0$, choose $N=\max\{2,M\}>0$.
If $x>N$, then $x-1>1$ and $x>M$, $x^2-x=x(x-1)>M\cdot 1=M$.
Therefore, by definition $\lim_{x\to\infty}(x^2-x)=\infty$. ■

Example 0.11 $\lim_{x\to\infty}\dfrac{x^2+x}{3-x}=?$

$$\lim_{x\to\infty}\frac{x^2+x}{3-x}\overset{(\div x)}{=}\lim_{x\to\infty}\frac{x+1}{\frac{3}{x}-1}$$ *(同除分母最高次)*

$\because x\to\infty \implies x+1\to\infty$ *and* $\dfrac{3}{x}-1\to-1 \implies \dfrac{x+1}{\frac{3}{x}-1}\to-\infty$.

Because when x becomes large, $x+1$ becomes large and $\frac{3}{x}-1$ approaches $-1(\neq 0)$, $\frac{x+1}{\frac{3}{x}-1}$ becomes small. $\therefore \lim_{x\to\infty}\dfrac{x^2+x}{3-x}=\lim_{x\to\infty}\dfrac{x+1}{\frac{3}{x}-1}=-\infty$. ■

Attention: 沒有 $\infty\cdot\infty=\infty$, $\dfrac{1}{\pm\infty}=0$, $\dfrac{\infty}{-1}=-\infty$; $\because \infty$ 只是符號。

Question: 無限處極限 $\lim_{x\to\pm\infty}\dfrac{P(x)}{Q(x)}=\{0,\pm\infty\}$ 應該怎麼寫? 怎麼看?

Answer: 如果 $P(x),Q(x)\to\pm\infty$ as $x\to\pm\infty$:

1. 先同除 $Q(x)$ 的最高次 x^n, 這時分母會有極限 $=c(\neq 0)$。
2. 如果分子也有極限 $=d$ (可能是0), 就使用極限律除法得到 $\dfrac{d}{c}$;
3. 如果分子是 $\pm\infty$, 就用討論的說明:

Because when x becomes {large, small}, $\dfrac{P(x)}{x^n}$ becomes {large, small} and $\dfrac{Q(x)}{x^n}$ approaches c, $\dfrac{P(x)/x^n}{Q(x)/x^n}$ becomes {large($=-$small), small($=-$large)}.

Therefore, $\lim_{x\to\pm\infty}\dfrac{P(x)}{Q(x)}=\lim_{x\to\pm\infty}\dfrac{P(x)/x^n}{Q(x)/x^n}=\pm\infty$.

Remark: 極限已經教到極限了, 來個總複習。

Limit:

- Limit $\lim_{x\to a} f(x) = L \iff f(x) \to L$ as $x \to a$.
- One-side limit $\lim_{x\to a^{\pm}} f(x) = L \iff f(x) \to L$ as $x \to a^{\pm}$.
- ∞ limit $\lim_{x\to a, a^{\pm}} f(x) = \pm\infty \iff f(x) \to \pm\infty$ as $x \to a, a^{\pm}$.

  Vertical Asymptote[ˋ弗替摳 ˋ耶神,討特] 垂直漸近線: $x = a$.
- Limit @ ∞ $\lim_{x\to\pm\infty} f(x) = L \iff f(x) \to L$ as $x \to \pm\infty$.

  Horizontal Asymptote[,吼李ˋ讓倫 ˋ耶神,討特] 水平漸近線: $y = L$.
- ∞ limit @ ∞ $\lim_{x\to\pm\infty} f(x) = \pm\infty \iff f(x) \to \pm\infty$ as $x \to \pm\infty$.

- p.s. $x \to a, a^{\pm}$: x approaches[阿婆落去] a (from the right/left),
  $x \to \pm\infty$: x sufficiently[捨非選特李] large/small;
  $f(x) \to L$: f approaches L,
  $f(x) \to \pm\infty$: f becomes arbitrarily[阿比踹了李] large/small.

Evaluate limit:

- Limit laws 極限律: 極限存在, “加減乘除常數倍, 冪次開根 c & x”。
- Left-/right-hand limits 左右極限:
  $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \iff \lim_{x\to a} f(x) = L$.
- Squeeze Theorem 夾擠定理:
  $f \le g \le h$, $\lim f = \lim h = L \implies \lim g = L$.
- $\lim_{x\to\pm\infty} f(x) = \lim_{t\to 0^{\pm}} f(\frac{1}{t})$. $\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\pm\infty} \frac{1}{x^r} = 0$, $r \in \mathbb{Q}^+$.

Continuity:

- f is continuous 連續 at $a \iff \lim_{x\to a} f(x) = f(a)$ 極限就是函數值。
- Intermediate Value Theorem 中間值定理: 閉連續, 頭尾異, 中間值。
  Locating Root Theorem 勘根定理: 閉連續, $f(a)f(b) < 0$, 開有解。
- 基本的連續函數 (開根有理多項式, 指對三角反三角),
  及其“加減乘除常數倍, 冪次開根與組合 (連續函數的連續函數)”。