3.10 Linear approximations and differentials

1. linear (tangent line) approximation 線性 (切線) 逼近
   $f(x) \approx f(a) + f'(a)(x-a) := L(x)$
2. differentials 微分 $dy = f'(x)\,dx$, $\Delta y = f(x+\Delta x) - f(x)$

0.1 Linear approximation

Recall: The tangent line equation of a curve $y = f(x)$ at $(a, f(a))$ is

$$y = f(a) + f'(a)(x-a)$$

Define: The ***linear (tangent line) approximation*** of f at a is

$$\boxed{\boldsymbol{f(x) \approx f(a) + f'(a)(x-a)}}.$$

Define: The ***linearization*** 線性化 (其實就是切線的函數) of f at a is

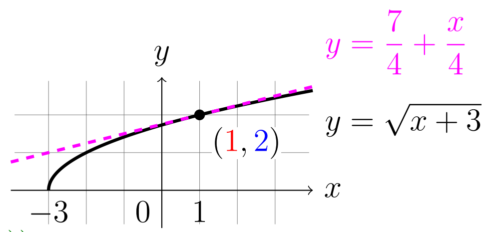
$$\boxed{\boldsymbol{L(x) = f(a) + f'(a)(x-a)}}.$$

Example 0.1 *Find the linearization of $f(x) = \sqrt{x+3}$ at 1, and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates* 高估 *or underestimates* 低估*?*

$f'(x) = \dfrac{1}{2\sqrt{x+3}}$, $f'(1) = \dfrac{1}{4}$, $f(1) = 2$.

$L(x) = f(1) + f'(1)(x-1)$

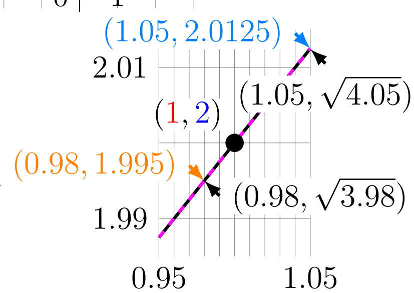
$= 2 + \dfrac{1}{4}(x-1) = \dfrac{7}{4} + \dfrac{x}{4}$,



$\implies (f(x) =)\sqrt{x+3} \approx \dfrac{7}{4} + \dfrac{x}{4} (= L(x))$.

$\sqrt{3.98} = \sqrt{0.98+3} \approx \dfrac{7}{4} + \dfrac{0.98}{4} = 1.995$.

$\sqrt{4.05} = \sqrt{1.05+3} \approx \dfrac{7}{4} + \dfrac{1.05}{4} = 2.0125$.



$\sqrt{3.98} = 1.99499373... < 1.995$,

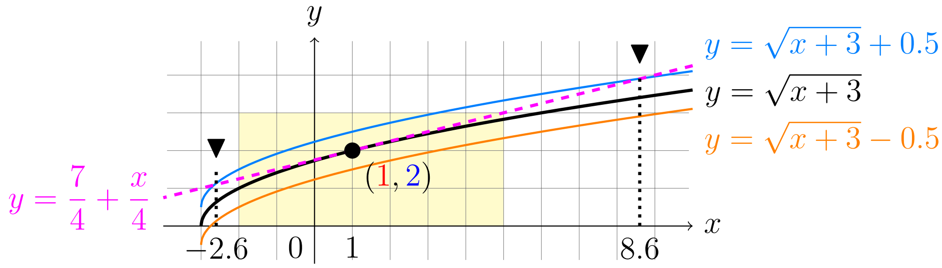
$\sqrt{4.05} = 2.01246117... < 2.0125$.*(*皆高估*)*

*(*從圖形看, 切線在上面就會高估, 在下面就低估。*)*

Ans: $\underline{\dfrac{7}{4} + \dfrac{x}{4},\ \sqrt{3.98} \approx 1.995,\ \sqrt{4.05} \approx 2.0125,}$ *both overestimated.* ■

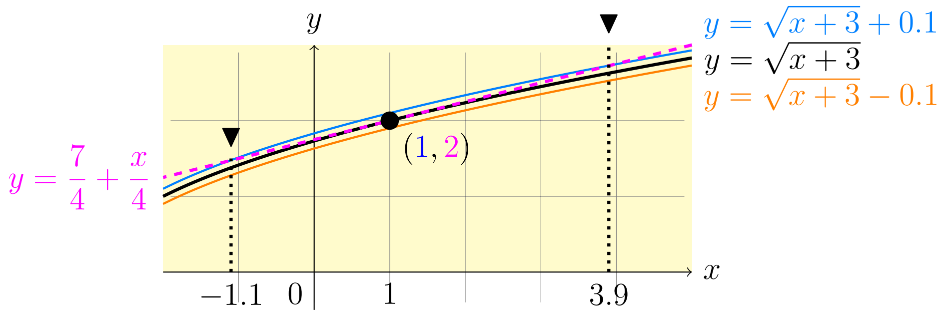
Example 0.2 *When does* $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ *accurate to within 0.5? 0.1?*

(從 $(a, f(a))$ 沿切線找第一次跑出 $y = f(x) + \varepsilon$ 與 $y = f(x) - \varepsilon$ 包圍的 x.)



Solve $\left|\sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right)\right| < 0.5$

$-2.66 \approx 3 - \sqrt{32} < x < 3 + \sqrt{32} \approx 8.66,$ *choose* $-2.6 < x < 8.6.$



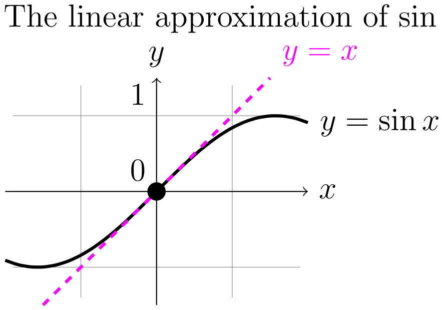
Solve $\left|\sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right)\right| < 0.1$

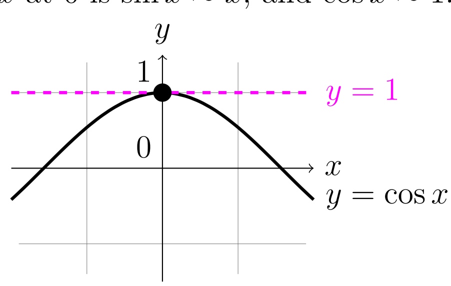
$-1.13 \approx 1.4 - \sqrt{6.4} < x < 1.4 + \sqrt{6.4} \approx 3.93,$ *choose* $-1.1 < x < 3.9.$

Ans: $\underline{-2.6 < x < 8.6,\ -1.1 < x < 3.9}$. ■

Application to physics:

The linear approximation of $\sin x$ and $\cos x$ at 0 is $\sin x \approx x$, and $\cos x \approx 1$.





Remark: $L(x)$ 與 a 有關, 切點$(a, f(a))$不同, 切線與 $L(x)$ 也不同。

0.2 Differential

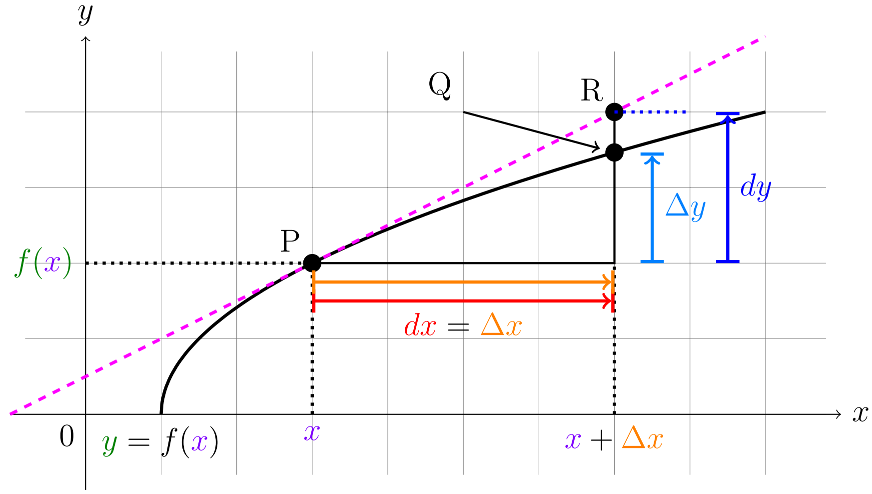
Define: If $y = f(x)$ and f is differentiable, then the ***differential*** 微分 dx is an independent variable, and the ***differential*** 微分 dy is defined by

$$\boxed{\boldsymbol{dy = f'(x)\ dx}}.$$

Let Δx be the change in (***increment*** 增量 of) x , then the change in y is

$$\boxed{\boldsymbol{\Delta y = f(x + \Delta x) - f(x)}}$$

The linear approximation: $f(a + dx) \approx f(a) + f'(a)\ dx = f(a) + dy$.



Note: 給定 $dx = \Delta x$, Δy 是實際差值, dy 是 (線性) 估計差值。

Example 0.3 *Compare Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.*

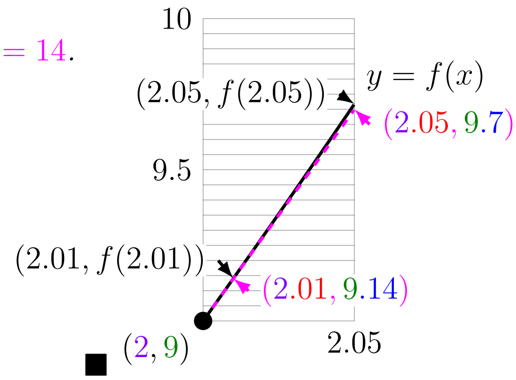
$f(2) = 9$, $f'(x) = 3x^2 + 2x - 2$, $f'(2) = 14$.

(a) $f(2.05) = 9.71765$,

$$\begin{aligned} dx &= \Delta x = 2.05 - 2 = 0.05, \\ \Delta y &= f(2.05) - f(2) = 0.71765, \\ dy &= f'(2)\ dx = 14 \cdot 0.05 = 0.7. \end{aligned}$$

(b) $f(2.01) = 9.140701$,

$$\begin{aligned} dx &= \Delta x = 2.01 - 2 = 0.01, \\ \Delta y &= f(2.01) - f(2) = 0.140701, \\ dy &= f'(2)\ dx = 14 \cdot 0.01 = 0.14. \end{aligned}$$



■

Application: 用微分 (dy) 來估計誤差 (Δy): $\Delta y \approx dy$.

Example 0.4 *The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?*

$$V = V(r) = \frac{4}{3}\pi r^3. \text{ (球體積公式)}$$
$$\Delta V \approx dV = 4\pi r^2\, dr = 4\pi(21)^2(0.05) \approx 277.$$

And: *The maximum error is about 277 cm^3.* ■

Note: 算誤差 ± 0.05 都要看。
$\Delta V = \frac{4}{3}\pi(21+0.05)^3 - \frac{4}{3}\pi(21)^3 \approx \quad 277.748730103,$
$\Delta V = \frac{4}{3}\pi(21-0.05)^3 - \frac{4}{3}\pi(21)^3 \approx -276.429261188,$
max error $= 277.748730103 (> 276.429261188)$.
Think it yourself: Why use $dr = 0.05$ not -0.05?

Errors 誤差: $y = f(x)$ at $x = a$,
the ***maximum error*** 最大誤差 is $\Delta y (\approx dy)$,
the ***relative error*** 相對誤差 is $\frac{\Delta y}{y} (\approx \frac{dy}{y})$ which can be expressed as
the ***percentage error*** 百分誤差 $\frac{\Delta y}{y} \times 100\%$.

Example 0.5 *(Continuous) relative error in V and r.*

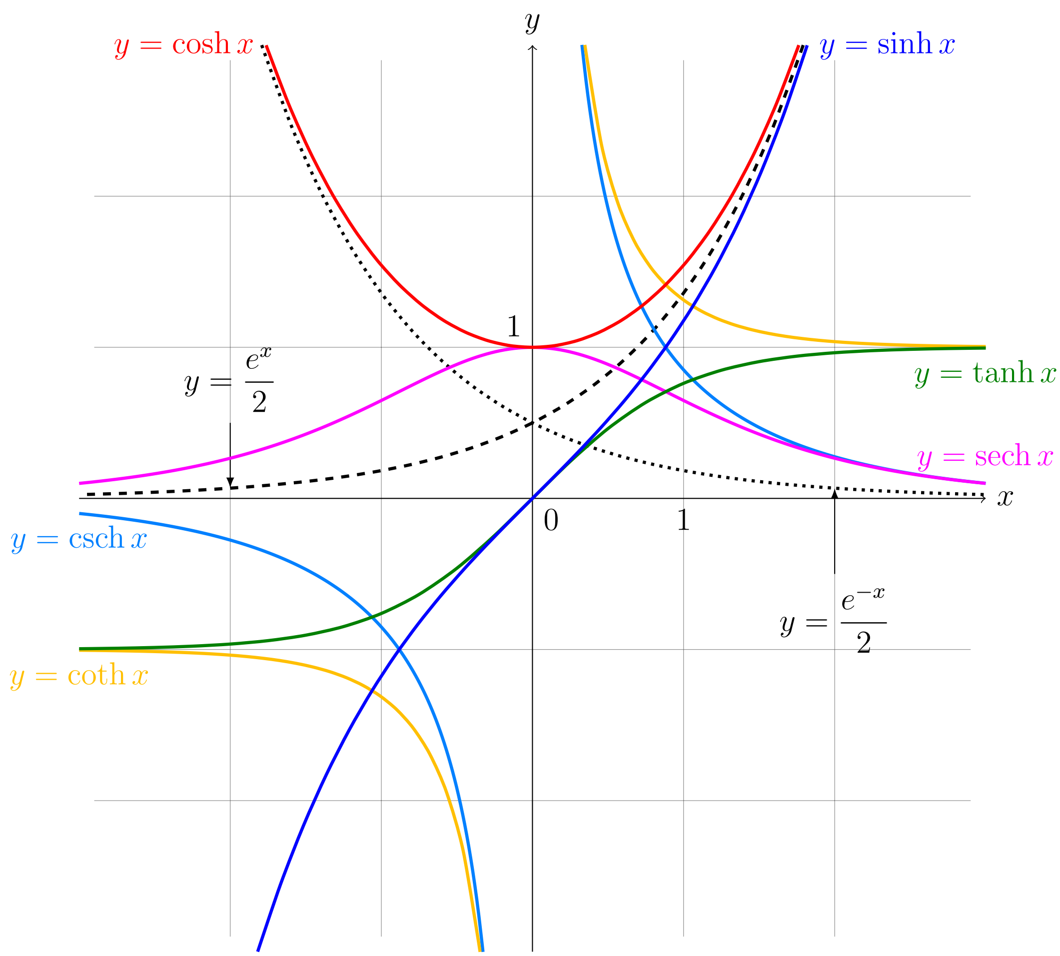
$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2\, dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}.$$
(The relative error in V is about 3 times the one in r.)

$$\frac{dr}{r} = \frac{0.05}{21} (\approx 0.00238095238) \approx 0.0024, \text{ and hence}$$
$$\frac{dV}{V} (\approx 0.00714285714) \approx 3 \times 0.0024 \approx 0.007.$$

Ans: *The percentage errors are 0.24% in radius and 0.7% in volume.* ■

♦ 3.11 Hyperbolic functions (optional)

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x},$$
$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$



♦: Euler's formula: $e^{ix} = \cos x + i \sin x$, $i = \sqrt{-1}$.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$
$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

- **Identity:**
  $\sinh(-x) = -\sinh x,\ \cosh(-x) = \cosh x.$

  $\cosh^2 x - \sinh^2 x = 1,\ 1 - \tanh^2 x = \operatorname{sech}^2 x,\ \coth^2 x - 1 = \operatorname{csch}^2 x,$

  $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$
  $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$

- **Derivative:**
  $(\sinh x)' = \cosh x;$
  $(\cosh x)' = \sinh x;$
  $(\tanh x)' = \operatorname{sech}^2 x;$
  $(\coth x)' = -\operatorname{csch}^2 x;$
  $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x;$
  $(\operatorname{csch} x)' = -\operatorname{csch} x \coth x.$

- **Antiderivative:**
  $\int \sinh x\ dx = \cosh x + C;$
  $\int \cosh x\ dx = \sinh x + C;$
  $\int \tanh x\ dx = -\ln|\operatorname{sech} x| + C;$
  $\int \coth x\ dx = -\ln|\operatorname{csch} x| + C;$
  $\int \operatorname{sech} x\ dx = \tan^{-1}(\sinh x) + C;$
  $\int \operatorname{csch} x\ dx = \ln|\coth x - \operatorname{csch} x| + C.$

- **Inverse:**
  $\sinh^{-1} x = \ln(x + \sqrt{x^2+1}),\ x \in \mathbb{R};$
  $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}),\ x \geq 1$ (limited);
  $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ -1 < x < 1;$
  $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right),\ |x| > 1;$
  $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ 0 < x \leq 1$ (limited);
  $\operatorname{csch}^{-1} x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right),\ x \neq 0.$

- **Derivative of inverse:**
  $(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}};$
  $(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}};$
  $(\tanh^{-1} x)' = \frac{1}{1-x^2};$
  $(\coth^{-1} x)' = -\frac{1}{x^2-1};$
  $(\operatorname{sech}^{-1} x)' = -\frac{1}{x\sqrt{1-x^2}};$
  $(\operatorname{csch}^{-1} x)' = -\frac{1}{|x|\sqrt{x^2+1}}.$