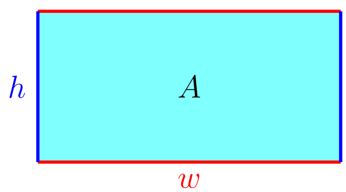
5.1 Areas and distances

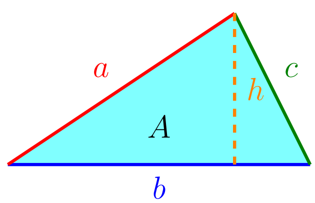
1. area problem 面積問題
2. distance problem 距離問題

0.1 Area problem



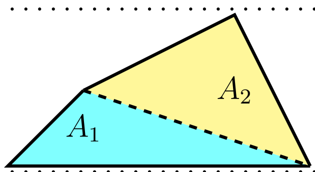
Rectangle:

$$\boxed{A} = wh$$



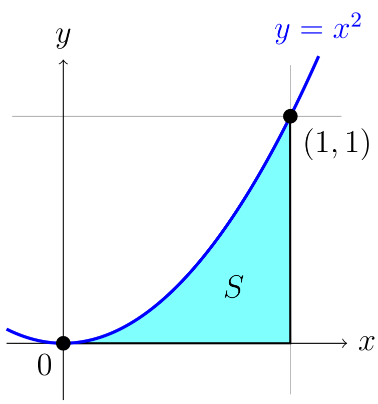
Triangle:

$$\begin{aligned} \boxed{A} &= \frac{1}{2}bh \\ &= \sqrt{s(s-a)(s-b)(s-c)}, \\ s &= \frac{a+b+c}{2} \quad \text{(Heron)} \end{aligned}$$

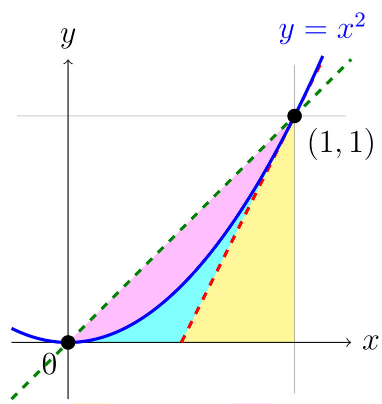


Polygon:

$$A = \boxed{A_1} + \boxed{A_2}$$

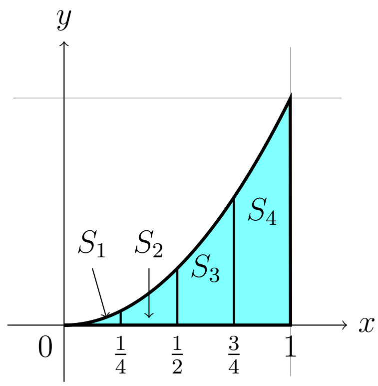


$S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.



$$\boxed{\frac{1}{4}} < \boxed{A} < \boxed{\frac{1}{2}}.$$

Question: Let $\boxed{A}$ be the area of S, $A =$?

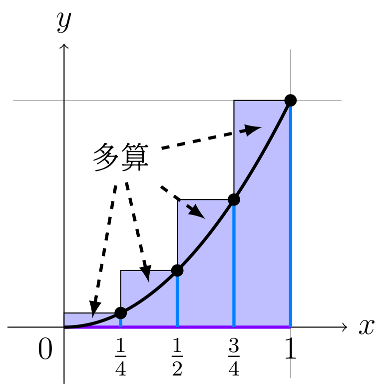


把 $[0,1]$ 均分成 4 段:

$$\Big[0,\frac{1}{4}\Big],\quad \Big[\frac{1}{4},\frac{1}{2}\Big],\quad \Big[\frac{1}{2},\frac{3}{4}\Big],\quad \Big[\frac{3}{4},1\Big];$$

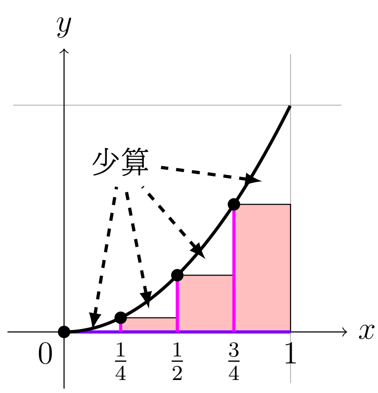
S 也被分成 4 塊寬度一樣是 $\frac{1}{4}$ 的區域:

$$S_1,\qquad S_2,\qquad S_3,\qquad S_4.$$



考慮用每塊的右端點 (***right endpoint***)
爲高度的方塊來估計: $\boxed{R_4}$.

$$\boxed{R_4} = \frac{1}{4}\cdot\Big(\frac{1}{4}\Big)^2 + \frac{1}{4}\cdot\Big(\frac{1}{2}\Big)^2 + \frac{1}{4}\cdot\Big(\frac{3}{4}\Big)^2 + \frac{1}{4}\cdot(1)^2 = \frac{15}{32} \approx 0.46875.$$

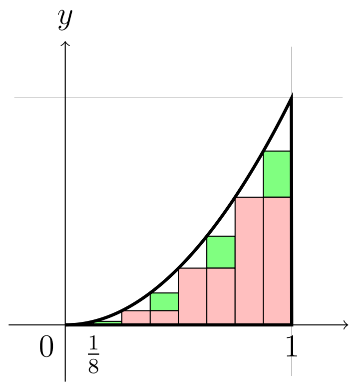


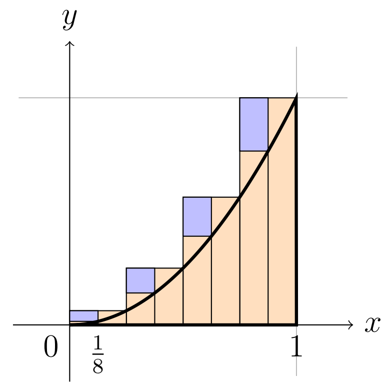
考慮用每塊的左端點 (***left endpoint***)
爲高度的方塊來估計: $\boxed{L_4}$.

$$\boxed{L_4} = \frac{1}{4}\cdot(0)^2 + \frac{1}{4}\cdot\Big(\frac{1}{4}\Big)^2 + \frac{1}{4}\cdot\Big(\frac{1}{2}\Big)^2 + \frac{1}{4}\cdot\Big(\frac{3}{4}\Big)^2 = \frac{7}{32} \approx 0.21875.$$

$$\Longrightarrow\ 0.21875 \approx \boxed{L_4} < \boxed{A} < \boxed{R_4} \approx 0.46875.$$

把 $[0,1]$ 均分成 8 段: $\implies$ $\boxed{L_4} < \boxed{L_8} < \boxed{A} < \boxed{R_8} < \boxed{R_4}$.





Observation: 分越多段 (n 越大), R_n 與 L_n 的估計越準 (誤差越小)。這個例子中, 隨著 $n \to \infty$, L_n 遞增, R_n 遞減, 而且總是有 $L_n < A < R_n$.

Example 0.1 $\lim_{n\to\infty} R_n = ?$ $\lim_{n\to\infty} L_n = ?$

Divide $[0,1]$ *into* n *intervals:*

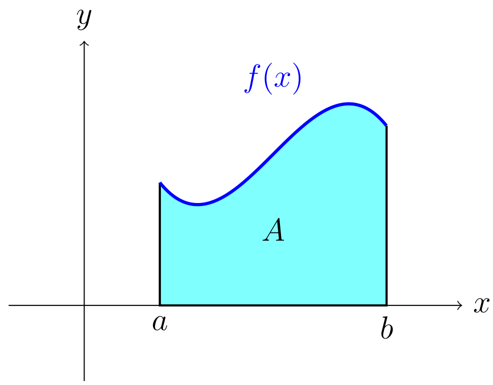
$$\left[0, \frac{1}{n}\right], \quad \left[\frac{1}{n}, \frac{2}{n}\right], \quad \cdots, \quad \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

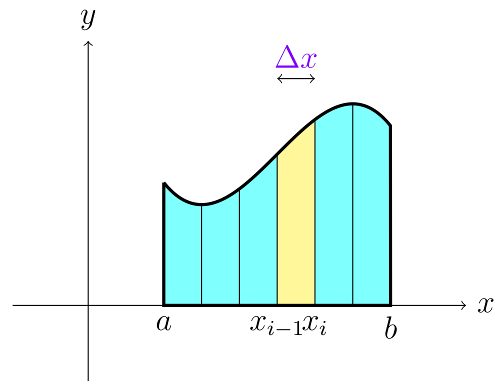
$$\begin{aligned}
R_n &= \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^2 \\
&= \frac{1}{n}\cdot\frac{1}{n^2}(1^2+2^2+\cdots+n^2) \\
&= \frac{1}{n}\cdot\frac{1}{n^2}\frac{n(n+1)(2n+1)}{6} \\
&= \frac{1}{6}\frac{n+1}{n}\frac{2n+1}{n} \\
&= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right), \\
\lim_{n\to\infty} R_n &= \lim_{n\to\infty}\left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{6}\cdot 1\cdot 2 = \frac{1}{3}.
\end{aligned}$$

Similarly, $\lim_{n\to\infty} L_n = \lim_{n\to\infty}\left[\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right] = \frac{1}{3}$. ∎

Answer: $\frac{1}{3} = \lim_{n\to\infty} L_n \le A \le \lim_{n\to\infty} R_n = \frac{1}{3} \implies A = \frac{1}{3}$.

Suppose $f(x)$ is continuous(連續) and nonnegative(非負) on $[a, b]$.



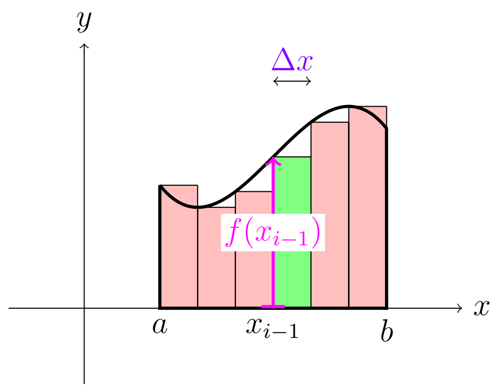


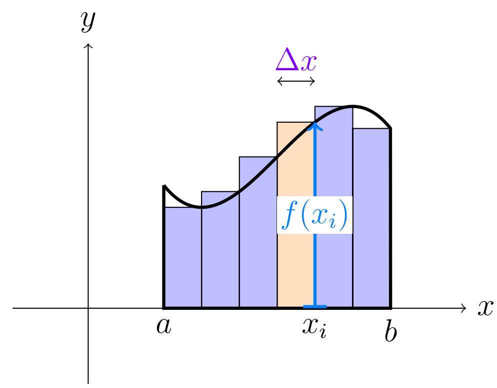
Similarly, the area A of the region under f can be estimate by:
Dividing $[a, b]$ into n intervals: $[x_{i-1}, x_i]$, where

$$a = x_0 < x_1 < \cdots < x_n = b,$$

$$\Delta x = x_i - x_{i-1} = \frac{b-a}{n}, \qquad i = 1, 2, \ldots, n.$$

$$(x_i = a + i\Delta x, \qquad i = 0, 1, 2, \ldots, n.)$$





Then A is approximated by the sum of the area of these rectangles:

$$\begin{aligned} R_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x; \\ L_n &= f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x. \end{aligned}$$

Question: How do we define area?
Answer: Limit!

Define: The ***area*** A of a region S that lies under the graph of the continuous function f (nonnegative on $[a, b]$) is the limit of the sum of the areas of approximating rectangles: (面積就是近似長方形面積和的極限)

$$\boxed{A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]},$$

where $a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$.

$$\boxed{A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]}.$$

We could choose any number $x_i^* \in [x_{i-1}, x_i]$ instead of x_{i-1} or x_i.

$$\boxed{A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]},$$

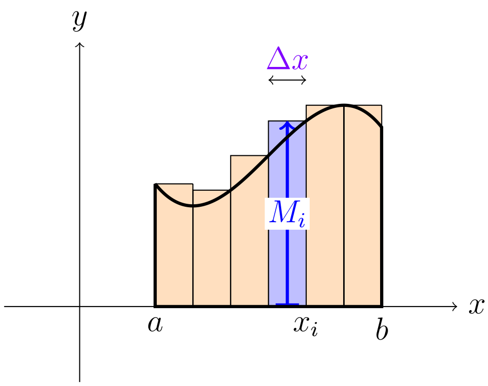
where ***sample point***(樣本點) $\boxed{\boldsymbol{x_i^*}}$ $\in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$.

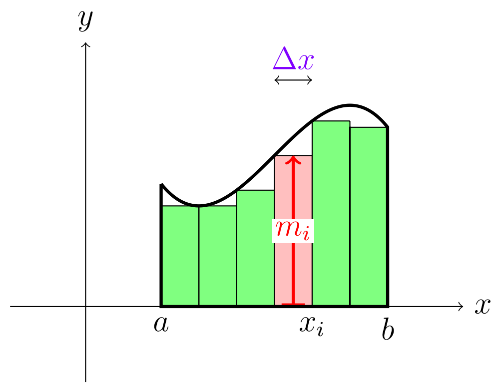
$$A = \lim_{n\to\infty} U_n = \lim_{n\to\infty} [M_1\Delta x + M_2\Delta x + \cdots + M_n\Delta x],$$

where M_i is the absolute maximum of f on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$.

$$A = \lim_{n\to\infty} D_n = \lim_{n\to\infty} [m_1\Delta x + m_2\Delta x + \cdots + m_n\Delta x],$$

where m_i is the absolute minimum of f on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$.





Note: 一般而言, 不一定會有: L_n increases, R_n decreases, $L_n < A < R_n$. 但是一定會有: D_n increases, U_n decreases, $D_n < A < U_n$; 可是不容易求極值。

Note: 因爲 f 連續, 這些極限都存在!

Note: 目前只考慮非負函數。

Notation: Summation, sum of many terms: $\boxed{\sum}$ (capital sigma)

$$\sum_{i=m}^{n} i\text{-term} = m\text{-term} + (m+1)\text{-term} + \cdots + n\text{-term}.$$

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$$

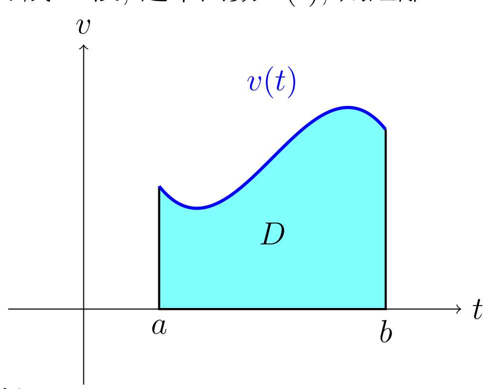
Recall: Area

$$\begin{aligned}
A &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x && \text{(右端)}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x && \text{(左端)}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x && \text{(樣本)}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} M_i\Delta x && \text{(最大)}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} m_i\Delta x && \text{(最小)}
\end{aligned}$$

0.2 Distance problem

Distance = velocity × time, 把時間均分成 n 段, 速率函數 $v(t)$, 則距離

$$\begin{aligned}
D &= \lim_{n\to\infty}\sum_{i=1}^{n} v(t_i)\Delta t\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} v(t_{i-1})\Delta t\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} v(t_i^*)\Delta t
\end{aligned}$$



Note: 這是沒有回頭 ($v(t) \geq 0$) 的情況: traveled distance = displacement.