4.1 Maximum and minimum values

微分應用之一: 找極值。

1. Extreme Value Theorem 極值定理 (存在性)
2. Fermat's Theorem 費馬定理 (找極值)

0.1 Extreme Value Theorem

Define: A function $f : D \to R$ and $c \in D$. $f(c)$ is the

- ***absolute maximum*** value 絕對最大值 of f on D
  if $f(c) \geq f(x)$ for all $x \in D$. (maximum a., n., pl.: -ima)
- ***absolute minimum*** value 絕對最小值 of f on D
  if $f(c) \leq f(x)$ for all $x \in D$. (minimum a., n., pl.: -ima)

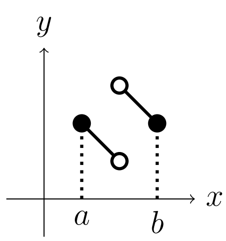
  (absolute max/min is sometimes called ***global*** max/min, and both called ***extreme values*** 極值 of f.)
- ***local maximum*** value 相對 (局部) 極大值 of f
  if $f(c) \geq f(x)$ when x is near c. (有些書用 maximal a.)
- ***local minimum*** value 相對 (局部) 極小值 of f
  if $f(c) \leq f(x)$ when x is near c. (有些書用 minimal a.)

Attention: 絕對最大/小值發生在端點時不是相對極大/小值。

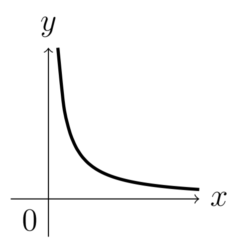
Theorem 1 (Extreme Value Theorem) 閉連續有極值
*If f is **continuous** on a **closed** interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$. (f has **extreme** values.)*

Proof. (省略: 閉區間 = 頭尾固定; 連續 = 中間沒斷; $\Longrightarrow$ 有高有低。) ∎

Note:



not continuous, no max.



not closed, no extreme values.

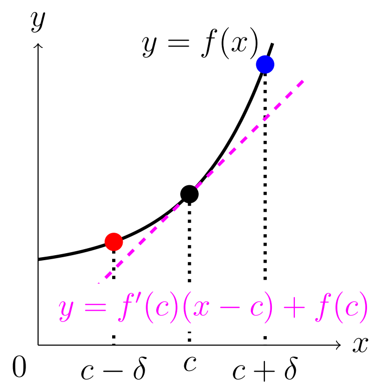
0.2 Fermat's Theorem

Theorem 2 (Fermat's Theorem) 極處可微導數爲零
If f has a ***local maximum or minimum at*** *c, and if* ***$f'(c)$ exists****, then* ***$f'(c) = 0$****.*

Proof. [反證法]

If $f'(c) > 0$, then $\lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h} > 0$,
$\Longrightarrow \; f(c+\delta) > f(c) > f(c-\delta)$
for some $\delta > 0$ small enough.

(The case of $f'(c) < 0$ is similar.)
$\Longrightarrow \; f(c)$ is not a local max/min, a contradiction.



Therefore, $f'(c) = 0$. ∎

[直證法] Suppose $f(c)$ is a local max, $f(c) \geq f(c+h)$.

When $h > 0$, $\dfrac{f(c+h) - f(c)}{h} \leq 0$. $\left(\dfrac{(-)}{(+)}\right)$

$\because f'(c)$ exists, $f'(c) = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h} \leq 0.$

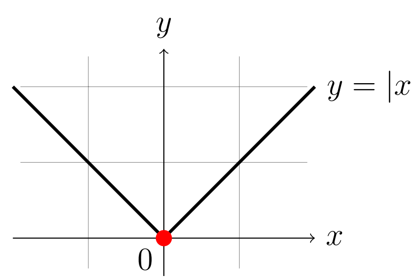
When $h < 0$, $\dfrac{f(c+h) - f(c)}{h} \geq 0$. $\left(\dfrac{(-)}{(-)}\right)$

$\because f'(c)$ exists, $f'(c) = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \dfrac{f(c+h) - f(c)}{h} \geq 0.$

Therefore, $f'(c) = 0$. ∎

Remark: f has local max/min: $\begin{cases} f'(c) \text{ exists} \implies f'(c) = 0; \\ f'(c) \text{ does not exist} \implies ? \end{cases}$

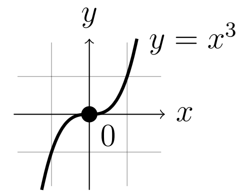
Ex: $f(x) = |x|$ has a (absolute) min(imum value) $f(0) = 0$ at 0, but $f'(x)$ does not exist) at 0 ($\nexists\, f'(0)$).



Define: A ***critical number*** 臨界值 (奇異點) of a function f is a number c in the ***domain*** of f such that either $\boldsymbol{f'(c) = 0}$ or $\boldsymbol{f'(c)}$ ***does not exist***.

Note: f has local max/min at $c \implies c$ is a critical number of f.
but c is a critical number of $f \cancel{\implies} f$ has local max/min at c.

兇手　　　我們之　　　　　　　戀曲　　有美好回憶
極值就在臨界值中!　　不是每個臨界都是極值。



Ex: $f(x) = x^3$ has a critical number 0, but $f(0)$ is not an extreme value.

Closed interval method: 閉區間法

f is continuous on a closed interval $[a, b]$ 閉連續。(先保證有極值再講究方法)

Step 1. 找臨界值: $f(c)$ for critical number $c \in (a, b)$ of f. (沒有不在場證明)

Step 2. 找邊界點: $f(a)$, $f(b)$.

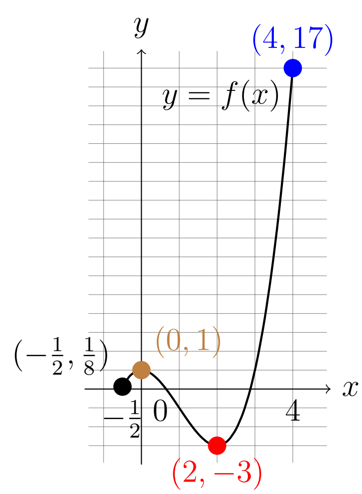
Step 3. 比大小: The largest/smallest value in Steps 1 and 2 is the absolute maximum/minimum value of f.

Example 0.1 *Find max/min of* $f(x) = x^3 - 3x^2 + 1,\ -\dfrac{1}{2} \le x \le 4$.

1. $f'(x) = 3x^2 - 6x = 3x(x-2)$,
$f'(x) = 0$ *when* $x = 0$ *or* $x = 2$, $(\in (-\dfrac{1}{2}, 4))$
臨界值*:* $f(0) = 1,\ f(2) = -3$.

2. 邊界點*:* $f(-\dfrac{1}{2}) = \dfrac{1}{8},\ f(4) = 17$.

3. Choose the largest/smallest in $\{-3, \dfrac{1}{8}, 1, 17\}$.



Ans: <u>*The absolute maximum value is* $f(4) = 17$,</u>
<u>*the absolute minimum value is* $f(2) = -3$.</u> ■

Note: 要寫清楚極值在哪值多少 $f(\cdots) = \cdots$。

♦ Additional: Proof of Extreme Value Theorem

極值定理的完整證明, 需要學習進階 (advanced) 微積分, 從 實數的完備性公設 (Completeness Axiom of Real Numbers) 開始, 一路證明: 區間套定理 (Nested Intervals Thoerem), 波爾札諾-魏爾斯特拉斯定理 (Bolzano-Weierstrass Theorem), 有界定理 (Bounding Theorem); 這裡只給最後一步的證明。

Theorem 3 (Completeness Axiom of Real Numbers (§11))
Every nonempty set of real numbers that has an upper bound has a least upper bound (lub).

$$\Downarrow$$

Theorem 4 (Nested Interval Theorem)
Intervals $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots \implies \bigcap_{n=1}^{\infty} I_n \neq \emptyset.$

$$\Downarrow$$

Theorem 5 (Bolzano-Weierstrass Theorem)
Every bounded sequence has a convergent subsequence.

$$\Downarrow$$

Theorem 6 (Bounding Theorem) *(閉區間連續函數的值域有界。)*
If f is continuous on closed interval $[a, b]$, then the range of f is bounded.

$$\Downarrow$$

Proof. (Extreme Value Theorem)
By Bounding Theorem and Completeness Axiom, the range of f has a least upper bound M, claim that "$\exists\ c \in [a, b] \ni f(c) = M$".

Suppose not, consider $g(x) = \dfrac{1}{M - f(x)}$ is continuous on $[a, b]$, by Bounding Theorem, $g(x)$ is bounded above by $L > 0$. Then $\dfrac{1}{M - f(x)} < L$, $f(x) < M - \dfrac{1}{L}(< M)$, a lower upper bound than M, a contradiction.

Similarly, the range of f has a greatest lower bound L and $\exists\ d \in [a, b] \ni f(d) = L$. ■