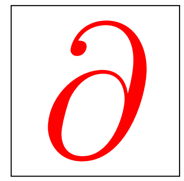
14.3 Partial derivatives

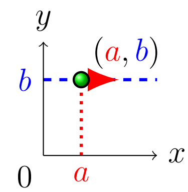
1. partial derivatives (definition, notation, rule)
2. more than two variables and higher derivatives
3. partial differential equation (PDE)



0.1 Partial derivatives

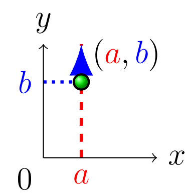
Define: If f is a function of two variables, then
the ***partial derivative of*** f ***with respect to*** x ***at*** (a,b)
(f 對 x 在 (a,b) 的偏導數)

$$\boxed{f_x(a,b)=\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}}$$



(固定 $y=b$, 看成單變數函數 $g_b(x)=f(x,b)$, $g_b'(a)=f_x(a,b)$.)
the ***partial derivative of*** f ***with respect to*** y ***at*** (a,b)
(f 對 y 在 (a,b) 的偏導數)

$$\boxed{f_y(a,b)=\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}}$$



(固定 $x=a$, 看成單變數函數 $h_a(y)=f(a,y)$, $h_a'(b)=f_y(a,b)$.)
the ***partial derivatives*** of f (f 的偏導 (函) 數 (們)) are functions

$$\boxed{\boldsymbol{f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}}}$$

$$\boxed{\boldsymbol{f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}}}$$

Notation: If $z=f(x,y)$, (∂: rounded d, 唸作 "partial".)

$$\boxed{\boldsymbol{f_x(x,y)=f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial z}{\partial x}}=f_1=D_1f=D_xf;}$$

$$\boxed{\boldsymbol{f_y(x,y)=f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}f(x,y)=\frac{\partial z}{\partial y}}=f_2=D_2f=D_yf.}$$

(偏導函數有時會省略 (x,y), 但如果是在某點 (a,b) 的偏導數不可以省略。)

Rule: 找偏導函數的規則 — $\boxed{\text{把其他 (自) 變數當常數做微分}}$。

找 f_x, 把 y 當常數: $\dfrac{\partial}{\partial x}[yg(x)] = yg'(x)$, $\dfrac{\partial y}{\partial x} = 0$;

找 f_y, 把 x 當常數: $\dfrac{\partial}{\partial y}[xh(y)] = xh'(y)$, $\dfrac{\partial x}{\partial y} = 0$.

Example 0.1 *$f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.*

$$
\begin{array}{rcccccc|rcccccc}
f(x,y) & = & x^3 & + & x^2y^3 & - & 2y^2 & f(x,y) & = & x^3 & + & x^2y^3 & - & 2y^2 \\
f_x(x,y) & = & 3x^2 & + & 2xy^3 & - & 0 & f_y(x,y) & = & 0 & + & 3x^2y^2 & - & 4y \\
 & = & 3x^2 & + & 2xy^3 & & & & = & 3x^2y^2 & - & 4y & & \\
f_x(2,1) & = & 3\cdot 2^2 & + & 2\cdot 2\cdot 1^3 & = & 16. & f_y(2,1) & = & 3\cdot 2^2\cdot 1^2 & - & 4\cdot 1 & = & 8.
\end{array}
$$

(偏導數一樣要先微分再代入, 先代入再微分都會變零。) ■

Note: 有時候求偏導數用定義比較好算。(See Exercise 14.3.103.)

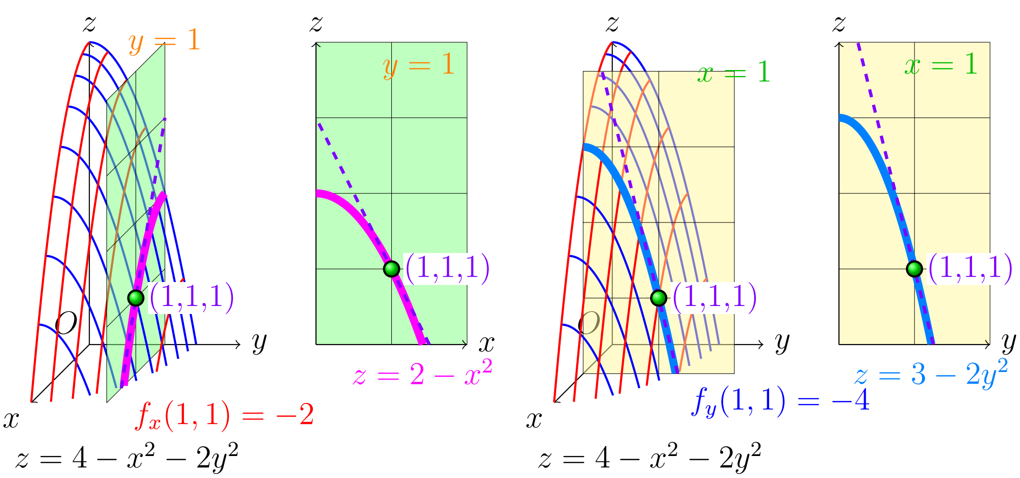
Geometric interpretation of partial derivatives 偏導數的幾何意義:

$f_x(a,b)$ 是曲面 $z = f(x,y)$ 與平面 $y = b$ 的交集曲線 $\{(x, b, f(x,b))\}$ 在點 $(a, b, f(a,b))$ 往正 x-軸方向的切線斜率。

$f_y(a,b)$ 是曲面 $z = f(x,y)$ 與平面 $x = a$ 的交集曲線 $\{(a, y, f(a,y))\}$ 在點 $(a, b, f(a,b))$ 往正 y-軸方向的切線斜率。

Example 0.2 *$f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$.*

$f_x(x,y) = -2x$, $f_x(1,1) = -2$; $f_y(x,y) = -4y$, $f_y(1,1) = -4$. ■



Example 0.3 *(Chain rule)* $f(x,y)=\sin\left(\frac{x}{1+y}\right)$, *calculate* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$.

Let $u=\frac{x}{1+y}$.

$$f_x=\frac{d}{du}\sin u\cdot\frac{\partial u}{\partial x}=\cos u\cdot\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right)=\cos\left(\frac{x}{1+y}\right)\cdot\frac{1}{1+y}$$
$$=\frac{1}{1+y}\cdot\cos\frac{x}{1+y}. \qquad \text{(“(,), ·” 可省略。)}$$
$$f_y=\frac{d}{du}\sin u\frac{\partial u}{\partial y}=\cos u\frac{\partial}{\partial y}\frac{x}{1+y}=\cos\frac{x}{1+y}\cdot\frac{-x}{(1+y)^2} \qquad \text{(“·” 不可省略。)}$$
$$=-\frac{x}{(1+y)^2}\cos\frac{x}{1+y}.$$
■

Note: 單項 (cx^n, $\frac{\cdots}{\cdots}$) 常省略“(,)”, 放 cos, ln, ..., 前面常省略 “·”. 怕混淆可以留著, 但不要用 “×”, 容易跟外積與 x 混淆。

Example 0.4 *(Implicit)* $x^3+y^3+z^3+6xyz=1$, *find* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$.

(同時對 x 偏微, 把 y 當常數, 把 z 當成 x 與 y 的函數。)

$$\begin{array}{rcl} x^3+y^3+z^3+6xyz &=& 1, \\ 3x^2+0+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x} &=& 0, \end{array}\quad\Longrightarrow\quad \frac{\partial z}{\partial x}=-\frac{x^2+2yz}{z^2+2xy}.$$

Similarly, $\frac{\partial z}{\partial y}=-\frac{y^2+2xz}{z^2+2xy}$.

(方程式中 x 與 y 角色一樣, $\frac{\partial z}{\partial y}$ 等於把 $\frac{\partial z}{\partial x}$ 裡的 x 與 y 交換。) ■

Attention: 在另外定義點的偏導數一定要用定義算 — 求極限。

Example 0.5 (extra♡) *Find* $f_x(0,0)$ *and* $f_y(0,0)$ *for*

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

$$f_x(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{h\cdot0}{h^2+0^2}-0}{h}=0,$$
$$f_y(0,0)=\lim_{h\to0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{0\cdot h}{0^2+h^2}-0}{h}=0.$$
■

0.2 More than two variables and higher derivatives

Partial derivatives of function of n variables. $\left[\text{多變數常用向量寫法}\right]$

$$\begin{aligned} u &= f(x_1, x_2, \ldots, x_n)\left[= f(\mathbf{x})\right], \\ \frac{\partial u}{\partial x_i} &= \lim_{h\to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h} \\ \Bigg[&= \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{e}_i) - f(\mathbf{x})}{h}, \text{ where } \mathbf{e}_i = \langle 0, \ldots, 0, \overset{i\text{-th}}{1}, 0, \ldots, 0\rangle \Bigg] \\ &= \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f. \end{aligned}$$

Example 0.6 *Find f_x, f_y and f_z of $f(x, y, z) = e^{xy}\ln z$.*

$f_x = ye^{xy}\ln z,\ f_y = xe^{xy}\ln z,\ f_z = e^{xy}/z.$ ■

Second derivatives of $z = f(x, y)$: (有四個)

$$\begin{aligned} \boldsymbol{f_{xx}} &= f_{11} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \boldsymbol{\frac{\partial^2 f}{\partial x^2}} = \boldsymbol{\frac{\partial^2}{\partial x^2} f(x, y)} = \boldsymbol{\frac{\partial^2 z}{\partial x^2}}, \\ \boldsymbol{f_{xy}} &= f_{12} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \boldsymbol{\frac{\partial^2 f}{\partial y \partial x}} = \boldsymbol{\frac{\partial^2}{\partial y \partial x} f(x, y)} = \boldsymbol{\frac{\partial^2 z}{\partial y \partial x}}, \\ \boldsymbol{f_{yx}} &= f_{21} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \boldsymbol{\frac{\partial^2 f}{\partial x \partial y}} = \boldsymbol{\frac{\partial^2}{\partial x \partial y} f(x, y)} = \boldsymbol{\frac{\partial^2 z}{\partial x \partial y}}, \\ \boldsymbol{f_{yy}} &= f_{22} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \boldsymbol{\frac{\partial^2 f}{\partial y^2}} = \boldsymbol{\frac{\partial^2}{\partial y^2} f(x, y)} = \boldsymbol{\frac{\partial^2 z}{\partial y^2}}. \end{aligned}$$

Third derivatives of $f(x, y)$: (有八個)

$$\begin{aligned} &\boldsymbol{f_{xxx}} = \boldsymbol{\frac{\partial^3 f}{\partial x^3}}, \quad \boldsymbol{f_{xxy}} = \boldsymbol{\frac{\partial^3 f}{\partial y \partial x^2}}, \quad \boldsymbol{f_{xyx}} = \boldsymbol{\frac{\partial^3 f}{\partial x \partial y \partial x}}, \quad \boldsymbol{f_{xyy}} = \boldsymbol{\frac{\partial^3 f}{\partial y^2 \partial x}}, \\ &\boldsymbol{f_{yxx}} = \boldsymbol{\frac{\partial^3 f}{\partial x^2 \partial y}}, \quad \boldsymbol{f_{yxy}} = \boldsymbol{\frac{\partial^3 f}{\partial y \partial x \partial y}}, \quad \boldsymbol{f_{yyx}} = \boldsymbol{\frac{\partial^3 f}{\partial x \partial y^2}}, \quad \boldsymbol{f_{yyy}} = \boldsymbol{\frac{\partial^3 f}{\partial y^3}}. \end{aligned}$$

Note: 先後順序有差, 記得近的先微: $f_{xy}(x, y) = \dfrac{\partial^2}{\partial y \partial x} f(x, y)$ (x 比 y 近)。

Note: 連續相同變數的偏微分 (∂x) 才能合併 (∂x^n)。

★ 差異之二: 一個 (高階) 導數 v.s 多個 (高階) 偏導數。

Example 0.7 *Find the second derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$.*

$$
\begin{array}{ccc}
 & f(x,y) = x^3 + x^2y^3 - 2y^2 & \\
\frac{\partial}{\partial x} \swarrow & & \searrow \frac{\partial}{\partial y} \\
f_x = 3x^2 + 2xy^3, & & f_y = 3x^2y^2 - 4y. \\
\frac{\partial}{\partial x} \swarrow \quad \searrow \frac{\partial}{\partial y} & & \frac{\partial}{\partial x} \swarrow \quad \searrow \frac{\partial}{\partial y}
\end{array}
$$

$$f_{xx} = 6x + 2y^3, \quad f_{xy} = 6xy^2, \quad f_{yx} = 6xy^2, \quad f_{yy} = 6x^2y - 4. \qquad \blacksquare$$

(Notice that $f_{xy} = f_{yx}$.)

Question: 先微後微都會一樣嗎? 不一定!
什麼時候會一樣? 天才兒童告訴你。(13歲發論文, 18歲出書。)

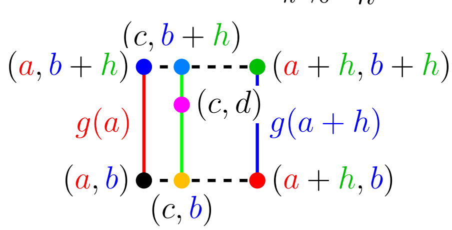
Theorem 1 (Clairaut's Theorem) *Suppose f is defined on a disk D that contains the point (a,b). If the function f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$. (偏導數要連續, 偏導順序才會沒差。)*

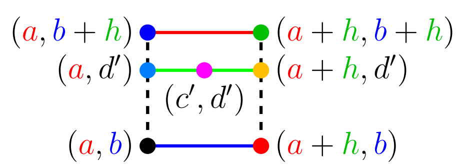
♦ **Proof.** Consider $g(x) = f(x, b+h) - f(x,b)$. By Mean Value Theorem, there exist c between a and $a+h$ and d between b and $b+h$.

$$
\begin{aligned}
\Delta(h) &= \overbrace{[f(a+h,b+h) - f(a+h,b)]}^{g(a+h)} - \overbrace{[f(a,b+h) - f(a,b)]}^{g(a)} \\
&= h\overbrace{[f_x(c,b+h) - f_x(c,b)]}^{g_x(c)} \;=\; h^2 f_{xy}(c,d)
\end{aligned}
$$

Since $(a+h,b+h) \to (a,b)$ and hence $(c,d) \to (a,b)$ as $h \to 0$, and by the continuity, $\displaystyle\lim_{h\to 0} \frac{\Delta(h)}{h^2} = \lim_{(c,d)\to(a,b)} f_{xy}(c,d) = f_{xy}(a,b)$.

Similarly, $\displaystyle f_{xy}(a,b) = \lim_{h\to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a,b)$. $\blacksquare$





By Clairaut's Theorem, $f_{xyy} = f_{yxy} = f_{yyx}$ if they are continuous. (Exercise 14.3.101.)

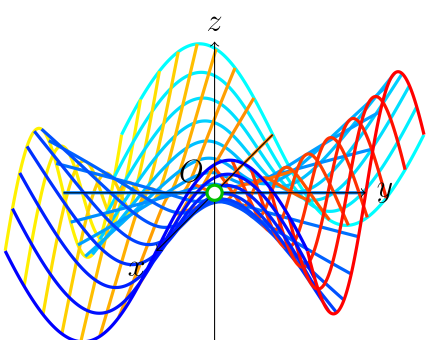
Example 0.8 **(偏微分順序不同答案也不同的例子)** *(Exercise 14.3.105.)*

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

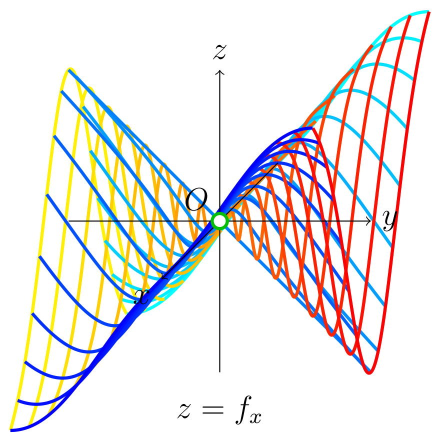
$$\begin{aligned}
f_x(0,k) &= \lim_{h\to 0}\frac{f(h,k)-f(0,k)}{h} &&= \lim_{h\to 0}\frac{hk\frac{h^2-k^2}{h^2+k^2}-0}{h} &&= -k, (\text{含 } k=0)\\
f_{xy}(0,0) &= \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} &&= \lim_{k\to 0}\frac{-k-0}{k} &&= -1;\\
f_y(h,0) &= \lim_{k\to 0}\frac{f(h,k)-f(h,0)}{k} &&= \lim_{k\to 0}\frac{hk\frac{h^2-k^2}{h^2+k^2}-0}{k} &&= h, (\text{含 } h=0)\\
f_{yx}(0,0) &= \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} &&= \lim_{h\to 0}\frac{h-0}{h} &&= 1.
\end{aligned}$$

f, f_x, f_y are continuous at $(0,0)$, but $f_{xy}\neq f_{yx}$ at $(0,0)$. ■

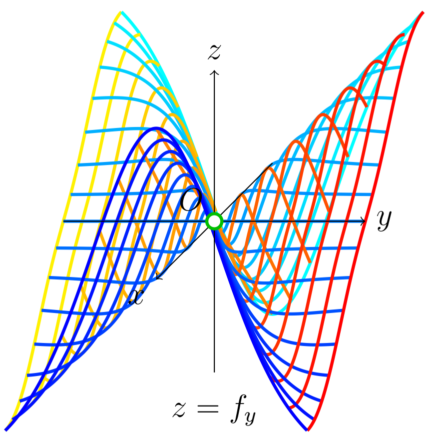
(有偏導數 = 只有 x 或 y 方向的極限 $\cancel{\Longrightarrow}$ 有極限, 所以並不代表會連續。)



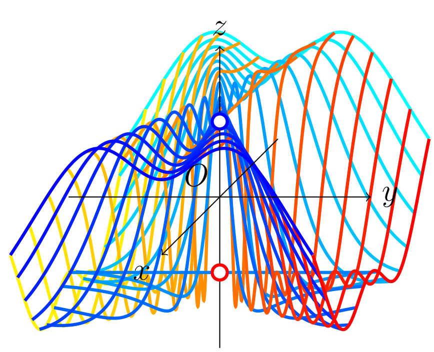
$z = xy\dfrac{x^2-y^2}{x^2+y^2}$



$z = f_x$



$z = f_y$



$z = f_{xy} = f_{yx}$

0.3 Partial Differential Equation (PDE)

1. ***Laplace's equation:***

$$(\Delta u(x,y) =)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Δ (∇^2 or $\nabla \bullet \nabla$): Laplace operator 拉普拉斯算子。
The **Laplacian** (function) Δf of a function f.
Solutions are called ***harmonic functions*** 調和函數。

- Heat Conduction 熱力學–熱傳導:
heat equation 熱方程 (diffusion equation 擴散方程式)

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$u(x,y,z,t)$: temperature 溫度 of x, y, z 座標與 t 時間,
k: thermal conductivity 熱傳導係數。

- Fluid Flow 流體動力學–流體流動:
無黏性、不可壓縮、無旋性的流場稱爲 potential flow.

$$\Delta \psi = 0$$

ψ: flow function.

- Electric Potential 靜電學–電勢 (位):

$$\Delta \phi = -\rho/\epsilon_0$$

ϕ: electric potential 電勢,
ρ: charge density 電荷密度,
$\epsilon_0 = 8.854187817\ldots \times 10^{-12}[F/m]$ (farads per meter): vacuum permittivity 眞空電容率, permittivity of free space 眞空介電係數, or electric constant 電常數。

Example 0.9 *$u(x,y) = e^x \sin y$ is a solution of Laplace's equation.*

$u_x = e^x \sin y (= u)$, $u_{xx} = u_x = u$.

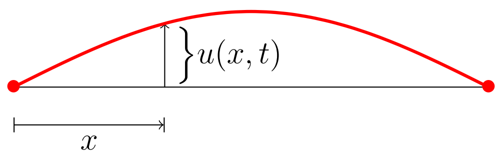
$u_y = e^x \cos y$, $u_{yy} = -e^x \sin y = -u$.

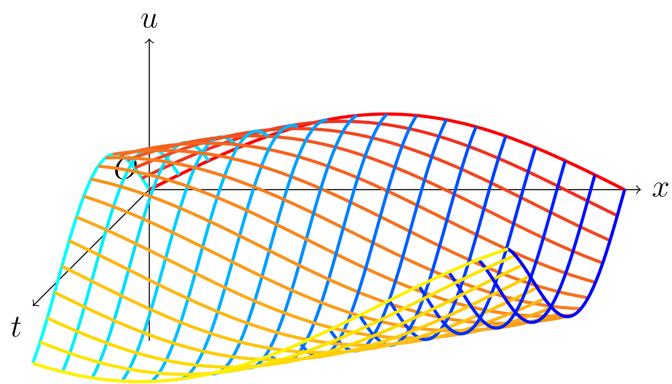
$u_{xx} + u_{yy} = 0$. ■

2. ***Wave equation* 波動方程式*:*** (telegraph equation 電報方程式)

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$: displacement of distance x and time t.





Example 0.10 $u(x,t) = \sin(x - \alpha t)$ *satisfies the wave equation.*

$u_x = \cos(x - \alpha t)(= u)$, $u_{xx} = -\sin(x - \alpha t)(= -u)$,
$u_t = -\alpha \cos(x - \alpha t)$, $u_{tt} = -\alpha^2 \sin(x - \alpha t)$,
$u_{tt} = \alpha^2 u_{xx}$. ■

♦: 二階偏微分方程類型 (Types of Second-Order Equations):
hyperbolic: wave equation $u_{tt} - u_{xx} = 0$,
elliptic: Laplace equation $u_{xx} + y_{yy} = 0$,
parabolic: heat equation $u_t - u_{xx} = 0$.

3. ***Cobb-Douglas production function* 產量函數*:***

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

Proof. 1. $L = 0$ or $K = 0$ then $P = 0$;

2. the ***marginal productivity of labor*** is proportional to the amount of production per unit of labor:
(勞動的邊際產量與每單位勞動的產量成正比。)
$\dfrac{\partial P}{\partial L} = \alpha \dfrac{P}{L} \implies P(L, K_0) = C_1(K_0)L^{\alpha}.$

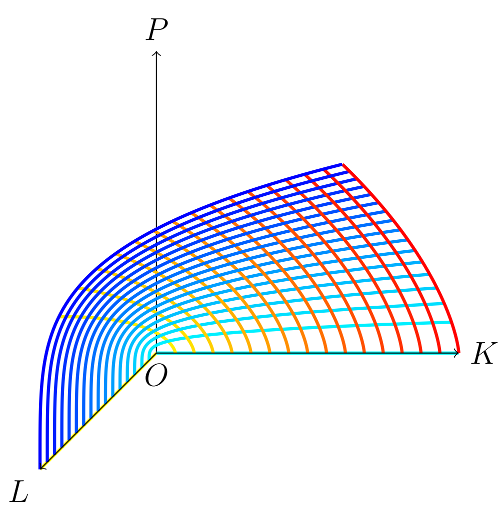
3. the ***marginal productivity of capital*** is proportional to the amount of production per unit of capital:
(資本的邊際產量與每單位資本的產量成正比。)
$\dfrac{\partial P}{\partial K} = \beta \dfrac{P}{K} \implies P(L_0, K) = C_2(L_0)K^{\beta}.$

From (2) & (3), $P(L, K) = bL^{\alpha}K^{\beta}$, from (1), $\alpha > 0$ & $\beta > 0$.

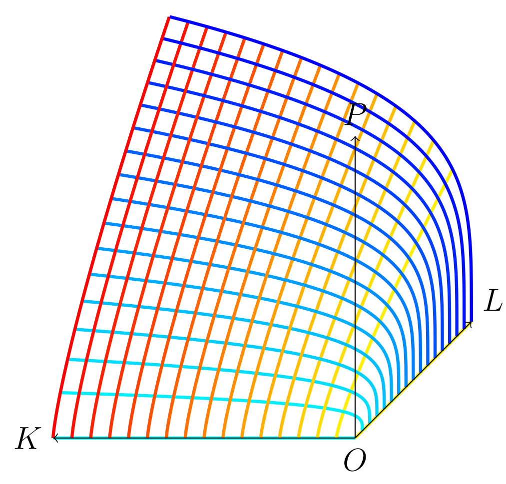
(當勞資都變 m 倍, 產量也變 m 倍。)

$P(mL, mK) = m^{\alpha+\beta}bL^{\alpha}K^{\beta} = m^{\alpha+\beta}P(L, K)$

$\implies \alpha + \beta = 1, \implies P(L, K) = bL^{\alpha}K^{1-\alpha}.$ ■



$P(L, K) = 1.01L^{0.75}K^{0.25}$



$P(L, K) = 1.01L^{0.75}K^{0.25}$