1179: Probability
Lecture 19 — Joint Distributions and Conditional Distributions

Bayesian perspective
ML

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Review

$P(X=x) = \sum_y P(X=x, Y=y)$

Joint PMF
$P(X = x, Y = y)$

Marginal PMF
$\underbrace{P(X = x), P(Y = y)}$

$F_X(x) = F_{XY}(x, \infty)$

$P(X \le x, Y \le y)$

Joint CDF
$F_{XY}(x, y)$

Marginal CDF
$\underline{F_X(x)}, F_Y(y)$

$f_{XY} = \frac{\partial^2 F_{XY}}{\partial x \partial y}$

Joint PDF
$f_{XY}(x, y)$

Marginal PDF
$f_X(x), f_Y(y)$

$\underbrace{f_X(x)} = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$

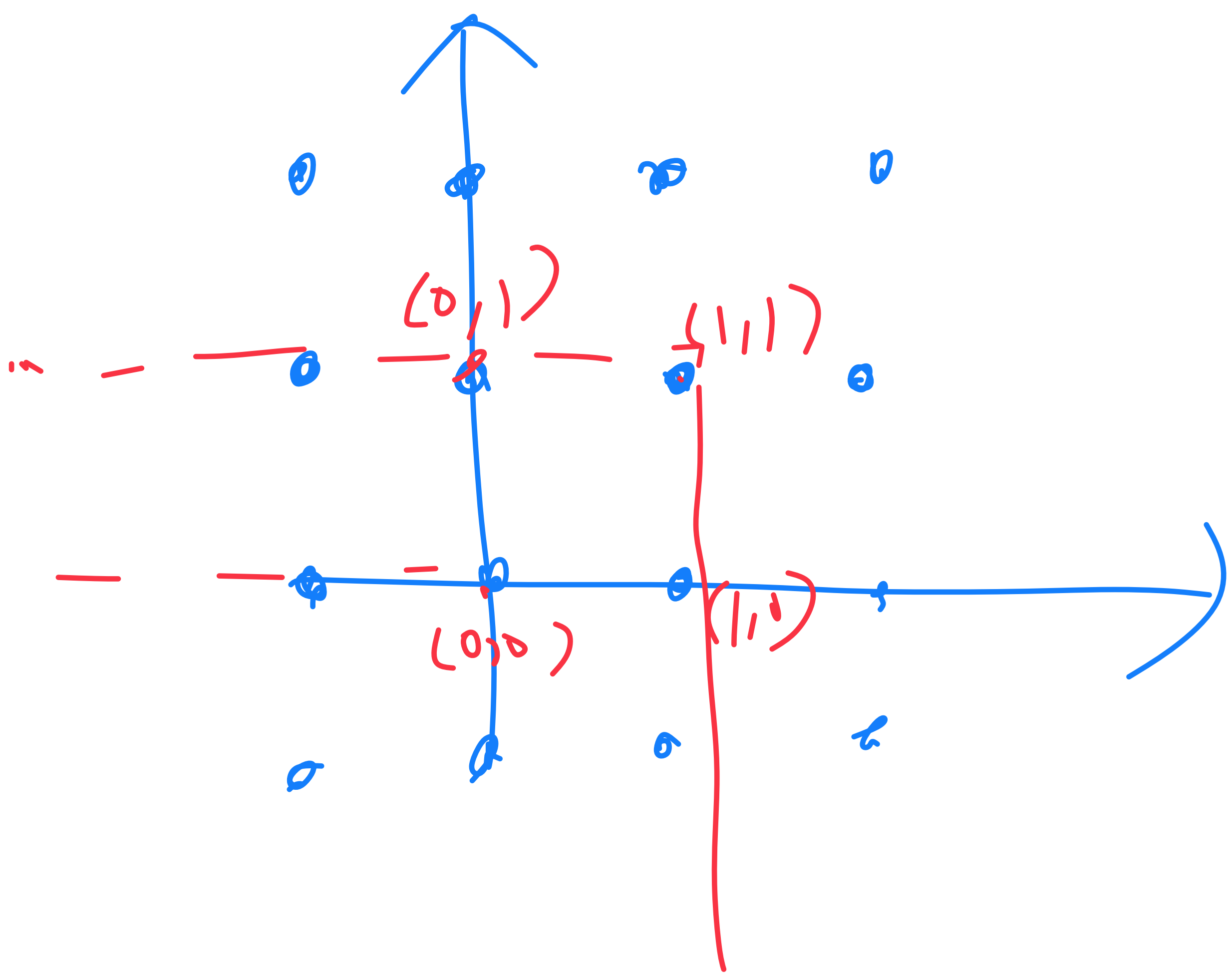
Q: How to find joint PMF based on joint CDF :

X, Y integers

$$P(X=1, Y=1) = F_{XY}(1,1) - F_{XY}(1,0)$$

$$- F_{XY}(0,1)$$

$$+ F_{XY}(0,0)$$



This Lecture

1. Independent Random Variables

2. Expected Value Regarding 2 Random Variables

3. Conditional Distributions

- Reading material: Chapter 8.2~8.3

Given Joint CDF: Find Joint PDF

Partial Derivative of Joint CDF is Joint PDF:

X and Y are two continuous random variables.

Let $F_{XY}(x, y)$ be the joint CDF of X and Y.

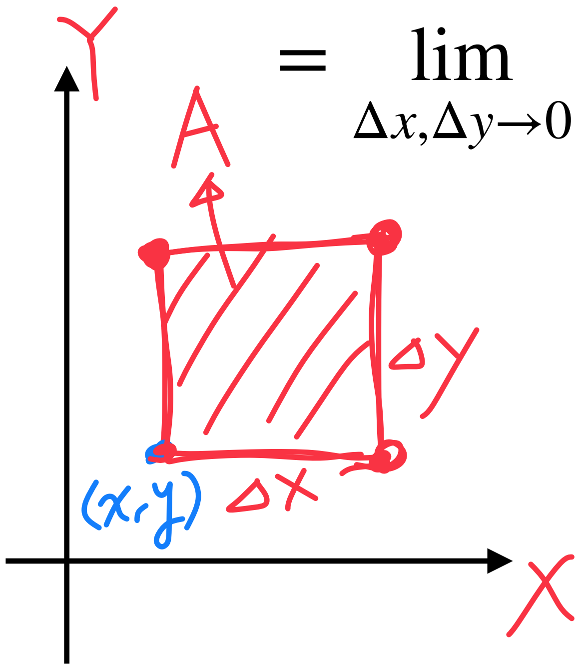
Assume the partial derivatives of $F_{XY}(x, y)$ exist. Then, one valid choice of PDF can be

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

Joint PDF: Interpret "Density" Using Limits

$$f_{XY}(x,y) \equiv \lim_{\Delta x, \Delta y \to 0} \frac{P(x < X \le x + \Delta x, y < Y \le y + \Delta y)}{\Delta x \Delta y}$$

$$= \lim_{\Delta x, \Delta y \to 0} \frac{F_{XY}(x+\Delta x, y+\Delta y) - F_{XY}(x, y+\Delta y) - F_{XY}(x+\Delta x, y) + F_{XY}(x,y)}{\Delta x \Delta y}$$



Main idea:

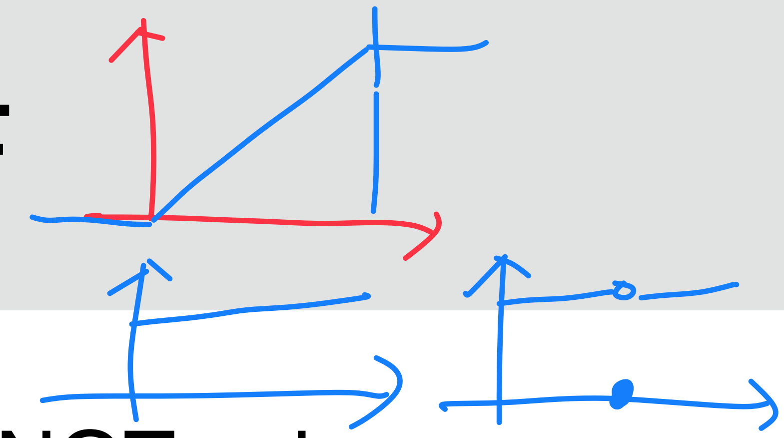
Density at (x,y) $\approx \boxed{\dfrac{\text{Probability mass on } A}{\text{Area of } A}} \stackrel{\Delta}{=} d_{\Delta x, \Delta y}(x,y)$

more accurate if A is getting smaller

$$d_{\Delta x,\Delta y}(x,y) = \frac{F_{XY}(x+\Delta x, y+\Delta y) - F_{XY}(x+\Delta x, y) - F_{XY}(x, y+\Delta y) + F_{XY}(x,y)}{\Delta x \cdot \Delta y}$$

$$\lim_{\Delta y \to 0} d_{\Delta x,\Delta y}(x,y) = \frac{1}{\Delta x}\left(\underbrace{\lim_{\Delta y \to 0} \frac{F_{XY}(x+\Delta x, y+\Delta y) - F_{XY}(x+\Delta x, y)}{\Delta y}}_{= \frac{\partial F_{XY}(x+\Delta x, y)}{\partial y}} - \underbrace{\lim_{\Delta y \to 0} \frac{F_{XY}(x, y+\Delta y) - F_{XY}(x,y)}{\Delta y}}_{= \frac{\partial F_{XY}(x,y)}{\partial y}} \right)$$

Technical Issues With Joint PDF



1. Given joint CDF $F_{XY}(x, y)$, the joint PDF is <u>NOT unique</u>

2. Suppose the partial derivatives of $F_{XY}(x, y)$ exist, then $\frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$ is a valid joint PDF

$P(X \geq y)$

3. In this class, we usually assume (unless stated otherwise):
   1. The partial derivatives of $F_{XY}(x, y)$ exist
   2. $\frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$ is the joint PDF to operate with

Marginal CDF/PDF to Joint CDF/PDF?

- Question: Could we get joint CDF/PDF from marginal CDF/PDF?

bivariate normal

$$\text{Joint CDF } F_{XY}(x,y) \xleftarrow{\text{NOT}} \text{Marginal CDF } F_X(x), F_Y(y)$$

$$\text{Joint PDF } f_{XY}(x,y) \longleftarrow \text{Marginal PDF } f_X(x), f_Y(y)$$

Independent Random Variables

Recall: Independence of 2 Random Variables

$(0,1)$ $[0,1]$ $[0,1] \cup [2,3]$

Definition: Two random variables X, Y are said to be **independent** if for arbitrary sets of real numbers A, B, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent, i.e.

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

- Remark: Same definition for both discrete and continuous random variables

- Question: What if we choose the sets as $A = (-\infty, t]$ and $B = (-\infty, u]$?

Property: Independence of 2 Random Variables

Independence ≡ joint CDF is the product of the marginal CDFs:

Two random variables X, Y are **independent** if and only if

$$F_{XY}(t, u) = F_X(t) \cdot F_Y(u)$$

marginal CDF of Y

joint CDF

marginal CDF of X

for every $t, u \in \mathbb{R}$

$X \in (-\infty, t]$

- Remark: This property holds for both discrete and continuous random variables

$F_{XY}(t,u) = P(X \le t, Y \le u)$

$Y \in (-\infty, u]$

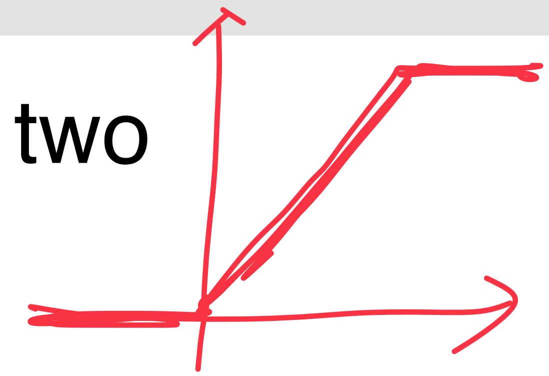
$F_X(t) = P(X \le t)$

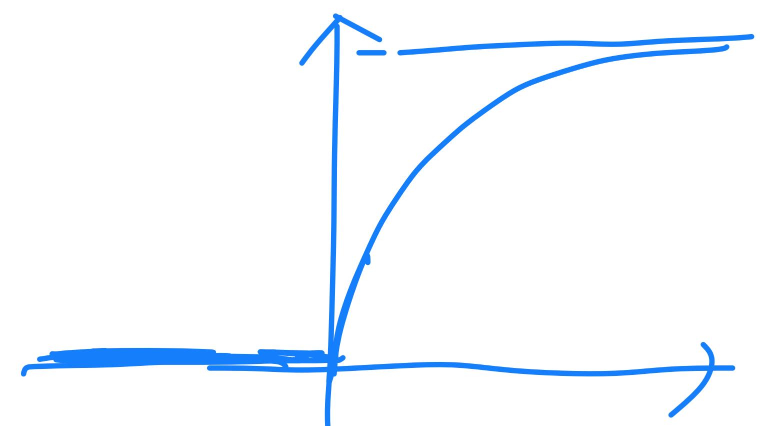
$F_Y(u) = P(Y \le u)$

Example: Continuous Uniform and Exponential

(continuous)

- Example: $X \sim \text{Unif}(0,1)$ and $Y \sim \text{Exp}(\lambda = 1)$ be two independent continuous random variables.
  - Joint CDF of X and Y?





$$F_{XY}(t,u) = \underbrace{F_X(t)} \cdot \underbrace{F_Y(u)}$$

$$F_X(t) = \begin{cases} 0 & , \ t \leq 0 \\ t & , \ 0 < t < 1 \\ 1 & , \ t \geq 1 \end{cases}$$

$$F_Y(u) = \begin{cases} 0 & , \ u < 0 \\ 1 - e^{-\lambda t} & , \ u \geq 0 \end{cases}$$

$$F_{XY}(t,u) = \begin{cases} 0 & , \ t \leq 0, u < 0 \\ 0 & , \ t \leq 0, u \geq 0 \\ & , \ \vdots \\ & , \ \vdots \\ & , \end{cases}$$

Property: Independence of 2 Discrete Random Variables

Joint PMF is the product of the marginal PMFs under independence:

If two discrete random variables X, Y are **independent**, then the joint PMF satisfies that

$$\underbrace{p_{XY}(x, y)}_{\text{joint PMF}} = \underbrace{p_X(x)}_{\text{marginal PMF}} \cdot p_Y(y)$$

- Proof:

$$p_{XY}(x,y) = P(\underbrace{X=x, Y=y}_{})$$

$$p_X(x) = P(X=x)$$

$$p_Y(y) = P(Y=y)$$

Choose $A = \{x\}$

$B = \{y\}$

This follows from the definition of independence

Property: Independence of 2 Continuous Random Variables

Joint PDF is the product of the marginal PDFs under independence:

If two continuous random variables X, Y are **independent**, then the joint PDF satisfies that

$$f_{XY}(t, u) = \underbrace{f_X(t) \cdot f_Y(u)}_{\text{independence}}$$

- Proof:

$$f_{XY}(t,u) = \frac{\partial^2 F_{XY}(t,u)}{\partial x \partial y} = \frac{\partial^2 \left(F_X(t) F_Y(u) \right)}{\partial x \partial y}$$

$$= f_X(t) \cdot f_Y(u).$$

Summary

"Independence".

$P_{XY}(x,y) = P_X(x) \cdot P_Y(y)$

Joint PMF
$P(X = x, Y = y)$

Marginal PMF
$P(X = x), P(Y = y)$

$F_{XY}(t,u) = F_X(t) \cdot F_Y(u)$

Joint CDF
$F_{XY}(x, y)$

Marginal CDF
$F_X(x), F_Y(y)$

$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

Joint PDF
$f_{XY}(x, y)$

Marginal PDF
$f_X(x), f_Y(y)$

Expected Value Regarding Two Random Variables

Recall: LOTUS for 1 Discrete Random Variable

Expected Value of a Function of Discrete R.V.:

1. Let X be a discrete random variable with
   - the set of possible values S
   - PMF of X is $p_X(x)$
2. Let $g(\cdot)$ be a real-valued function

The expectation of $g(X)$ is

$$E[g(X)] = \sum_{x \in S} g(x) \cdot p_X(x)$$

LOTUS for 2 Discrete Random Variables

Expected Value of a Function of 2 Discrete RVs:

1. Let X, Y be 2 discrete random variables with sets of possible values S_X, S_Y and joint PMF $p(x, y)$
2. Let $g(\,\cdot\,,\,\cdot\,)$ be a function from $\mathbb{R}^2 \to \mathbb{R}$

The expected value of $g(X, Y)$ is

$$E[g(X, Y)] =$$

Example: Using Joint PMF to Find Expected Value

- Example: Bus #2 (NCTU - Mackay - Train Station)
  - $X =$ traveling time from NCTU to Mackay
  - $Y =$ traveling time from Mackay to Train Station
  - $E[X + Y]$ = ?

Joint PMF	X=10	X=15	X=20
Y=10	0.1	0.1	0.05
Y=15	0.1	0.3	0.1
Y=20	0.05	0.1	0.1

Conditional Distributions

Example: Using Joint PMF to Find Conditional PMF

- Example: Bus #2 (NCTU - Mackay - Train Station)
  - $X =$ traveling time from NCTU to Mackay
  - $Y =$ traveling time from Mackay to Train Station
  - $P(X = 10 \mid Y = 15)$ = ?

Joint PMF	X=10	X=15	X=20
Y=10	0.1	0.1	0.05
Y=15	0.1	0.3	0.1
Y=20	0.05	0.1	0.1

Conditional PMF (Formally)

- **Conditional PMF**: Let X, Y be two discrete random variables with joint PMF $p_{XY}(x, y)$. When $P(Y = y) > 0$, the conditional PMF of X given $Y = y$ is

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

- Question: Conditional PMF of Y given $X = x$?

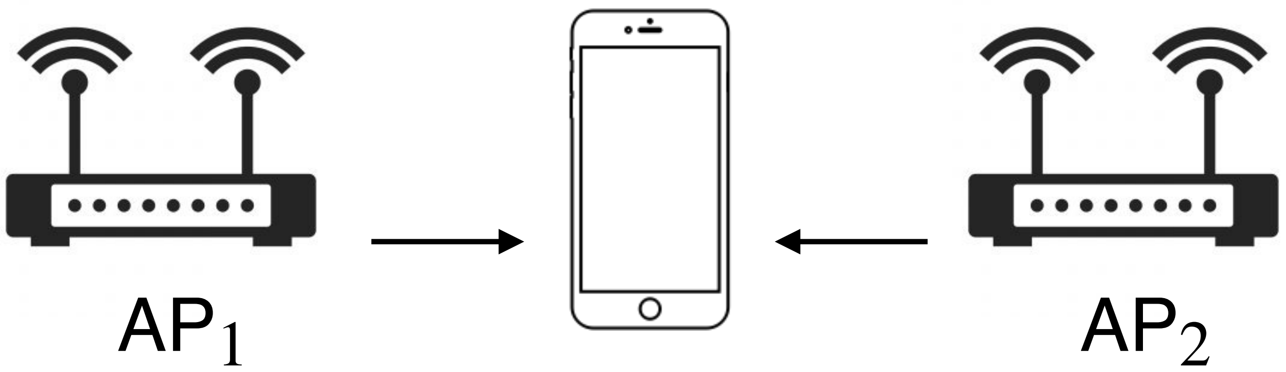
- Question: $\sum_x p_{X|Y}(x \mid y) =$

Conditional CDF of Discrete Random Variables

- **Conditional CDF**: Let X, Y be two discrete random variables with joint PMF $p_{XY}(x, y)$ and marginal PMFs $p_X(x), p_Y(y)$. When $P(Y = y) > 0$, the conditional CDF of X given $Y = y$ is

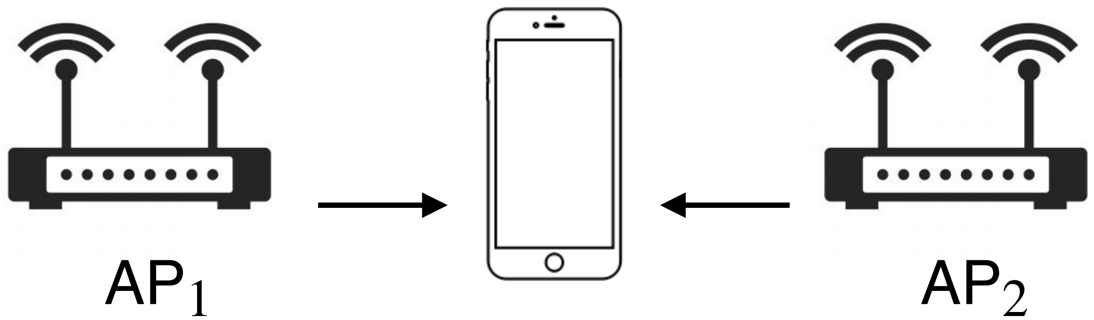
$$F_{X|Y}(x\,|\,y) := P(X \leq x|Y = y) = \sum_{t \leq x} p_{X|Y}(t|y) = \sum_{t \leq x} \frac{p_{XY}(t, y)}{p_Y(y)}$$

Example: Conditioning and Sum of Poisson



- Let N_1 and N_2 be the # of bits transmitted by AP_1 and AP_2 in a time interval T, respectively
  - N_1 and N_2 are Poisson with rates λ_1 and λ_2, respectively.
  - Moreover, N_1 and N_2 are independent
  - Define $M = N_1 + N_2$
  - Question: Conditional PMF $p_{N_1|M}(n \mid m)$ = ?

Example: Conditioning and Sum of Poisson



- Conditional PMF $p_{N_1|M}(n \mid m)$