13.3 Arc length and curvature

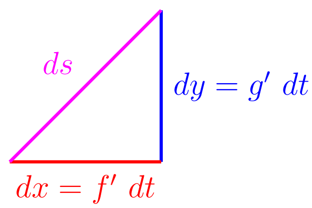
1. arc length & arc length function $L = \int |\mathbf{r}'(t)|\ dt$ & $s(t) = \int_a^t |\mathbf{r}'(u)|\ du$
2. ♦ curvature, & normal & binormal vectors

0.1 Arc length & arc length function

Recall: The arc length 弧長 of a curve with parametric equations $x = f(t)$, $y = g(t)$, where f, g are smooth (f', g' are continuous), is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

and arc length function 弧長函數 is



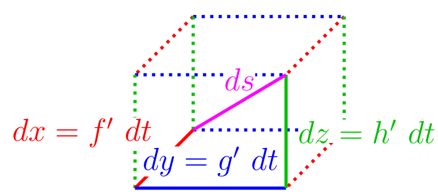
$$s(t) = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2}\ du$$

$ds = s'(t)\ dt = \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt$, $(ds)^2 = (dx)^2 + (dy)^2$, $L = \int_a^b\ ds$.

Define: The ***arc length*** of a curve with parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f, g, h are smooth (f', g', h' are continuous), is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

and ***arc length function*** is



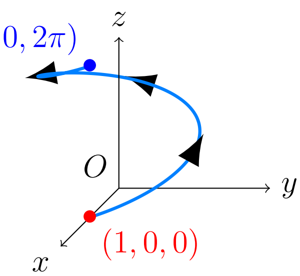
$$s(t) = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}\ du$$

$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt = |\mathbf{r}'(t)|\ dt$, (位置向量的導數的長度)
$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$, $L = \int\ ds$.

$$\boxed{L = \int_a^b |\mathbf{r}'(t)|\ dt} \qquad \boxed{s(t) = \int_a^t |\mathbf{r}'(u)|\ du} \qquad \boxed{\frac{ds}{dt} = |\mathbf{r}'(t)|}$$

Example 0.1 *Find the length of the arc of the circular helix with vector equation* $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ *from* $(1, 0, 0)$ *to* $(1, 0, 2\pi)$.

$\mathbf{r}(t) = \langle 1, 0, 0\rangle \implies t = 0,$
$\mathbf{r}(t) = \langle 1, 0, 2\pi\rangle \implies t = 2\pi,$
$0 \le t \le 2\pi,$
$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k},$
$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2},$
$L = \displaystyle\int_0^{2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$ ■



A curve can be represented by more than one vector function:

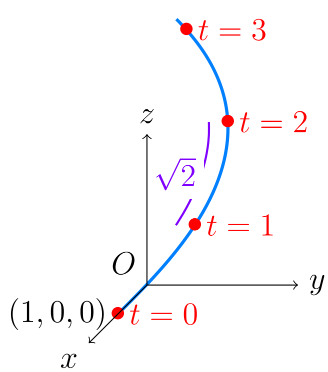
$$\mathbf{r_1}(t) = \langle t, t^2, t^3\rangle \text{ and } \mathbf{r_2}(u) = \langle e^u, e^{2u}, e^{3u}\rangle$$

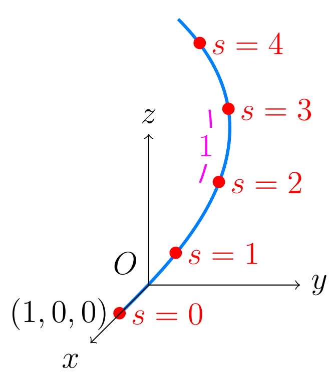
are parametrizations of the twisted cubic.

Parametrize a curve with respect to arc length 以弧長爲參數:
A curve $\mathbf{r}(t)$ with parameter t, and arc length function $s(t)$. If $t = t(s)$, then the curve can be reparametrized in terms of s as $\mathbf{r}(t(s))$.

Example 0.2 *Reparametrize the helix* $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ *w.r.t. arc length from* $(1, 0, 0)$.

$\dfrac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2},\ s = \displaystyle\int_0^t |\mathbf{r}'(u)|\, du = \sqrt{2}t,\ t = \frac{s}{\sqrt{2}},$
$\mathbf{r}(t(s)) = \cos \dfrac{s}{\sqrt{2}}\mathbf{i} + \sin \dfrac{s}{\sqrt{2}}\mathbf{j} + \dfrac{s}{\sqrt{2}}\mathbf{k}.$ ■





0.2 ♦ Curvature, & normal & binormal vectors

Define: A curve defined by $\mathbf{r}$ is ***smooth*** on an interval I if $\mathbf{r}'$ continuous on I and $\mathbf{r}'(t) \neq 0$ (except possibly at endpoint of I). A smooth curve has no sharp corners 轉角 or cusps 尖頭; tangent vector turns continuously.

Define: The ***curvature*** [ˈkɝvətʃɚ] 曲率 $\boldsymbol{\kappa}$ (“kappa”) of a curve is the magnitude of the rate of change of the unit tangent vector with respect to arc length. 單位切向量對弧長變化率的量, 也是彎曲的程度 ($\kappa = 0 \iff$ 直線)。

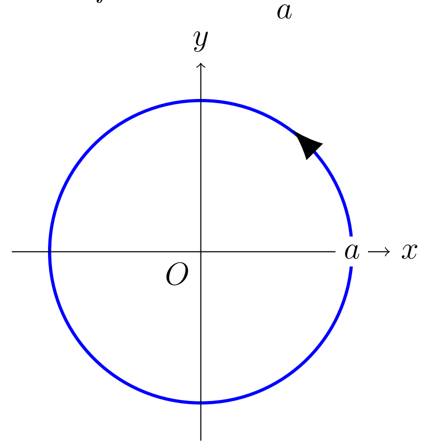
$$\boxed{\kappa = \left|\frac{d\mathbf{T}}{ds}\right|}$$

By Chain Rule, $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right|$ and $\frac{ds}{dt} = |\mathbf{r}'(t)|$, $\boxed{\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}}$.

Example 0.3 *Show that the curvature of a circle of radius a is $\frac{1}{a}$.*

$$\begin{aligned}
\mathbf{r}(t) &= a\cos t\mathbf{i} + a\sin t\mathbf{j},\\
\mathbf{r}'(t) &= -a\sin t\mathbf{i} + a\cos t\mathbf{j},\\
|\mathbf{r}'(t)| &= \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a,\\
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t\mathbf{i} + \cos t\mathbf{j},\\
\mathbf{T}'(t) &= -\cos t\mathbf{i} + -\sin t\mathbf{j},\\
|\mathbf{T}'(t)| &= \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1,\\
\kappa &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}. \qquad \blacksquare
\end{aligned}$$

(豆知識: 半徑 a 的圓曲率是 $1/a$。)



Define: When $\kappa(t) \neq 0$ (有彎曲), the ***(principal unit) normal vector*** (主單位) 法向量 is (單位切向量的單位切向量)

$$\boxed{\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}},$$

and the ***binormal vector*** 次法向量 is

$$\boxed{\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)}.$$

Define: The ***torsion*** [ˋtɔrʃən] 扭率 $\boldsymbol{\tau}$ ("tau") (or second curvature 或稱第二曲率) of a curve is the rate of change of the curve's osculating plane. (曲線扭轉的速率, 右旋爲正, 左旋爲負。)

Define: Frenet frame : **TNB** frame. (Exercise 13.3.59, 13.3.61 & 13.3.62) Frenet-Serret formula 弗萊納 (-夕瑞) 公式:

$$\begin{aligned} \frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N} \\ \frac{d\mathbf{N}}{ds} &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \frac{d\mathbf{B}}{ds} &= -\tau\mathbf{N} \end{aligned}$$

Define: At a point P on a curve C, the plane defined by **N** and **B** is called the ***normal plane*** 法面 of C at P, and the plane defined by **T** and **N** is called the ***osculating plane*** 密切面 of C at P. (osculate[ˋɑskjəlet]: from Latin "ōsculum"[ˋoskulum], means "kiss") The circle lying in the osculating plane of C at P has the same tangent as C at P, lies on the concave side of C (**N** points toward), and has radius $\rho = \dfrac{1}{\kappa}$, is called the ***osculating circle*** 密切圓 (or ***circle of curvature***) 曲率圓 of C at P.

Note: $\rho = \dfrac{1}{\kappa}$: radius of curvature 曲率半徑, also the radius of osculating circle 密切圓半徑。
$\sigma = \dfrac{1}{\tau}$: radius of torsion 扭率半徑。
1. $\kappa = 0 \iff$ 是直線。
2. $\tau = 0 \iff$ 曲線在平面上。
3. κ nonzero constant, $\tau = 0$ $\iff$ part of circle (of radius $\dfrac{1}{\kappa}$).
4. κ, τ nonzero constant $\iff$ circular helix.
5. $\kappa \neq 0$, 往法向量 **N** 彎。
6. $\mathbf{T} \perp \mathbf{N}$, $\mathbf{B} \perp \mathbf{T}$, $\mathbf{B} \perp \mathbf{N}$ and $|\mathbf{T}| = |\mathbf{N}| = |\mathbf{B}| = 1$.
右手定則: 拇指 **T**, 食指 **N**, 中指 **B**.