3.2 The product and quotient rules

1. derivative of product 函數乘積的導函數 $(fg)' = f'g + fg'$.
2. derivative of quotient 函數除商的導函數 $(f/g)' = (f'g - fg')/g^2$.

0.1 Derivative of product

The Product rule: .... $\boxed{(fg)' = f'g + fg'}$

口訣: 前面的微分乘以後面加前面乘以後面的微分。

If f and g are differentiable, then $f(x)g(x)$ is differentiable and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \qquad (\text{一加一減 } f(x)g(x+h))\\
= &\lim_{h\to 0}\frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h}\\
= &\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\lim_{h\to 0}g(x+h) + \lim_{h\to 0}f(x)\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\\
= &f'(x)g(x) + f(x)g'(x). \qquad (\text{與課本順序前後不同})
\end{aligned}$$

($\because$ g 可微分 $\implies$ 連續 $\iff$ $\lim_{h\to 0} g(x+h) = g(\lim_{h\to 0}(x+h)) = g(x)$.)

♦**:** 課本上是用 $u = f(x)$, $v = g(x)$, $\Delta u = f(x+h) - f(x)$, $\Delta v = g(x+h) - g(x)$, 得到 $(fg)' = fg' + gf'$. 符號順序不太一樣, 原理是一樣的。

Example 0.1 *$f(x) = xe^x$, find f' and $f^{(n)}$.*

$f' = (xe^x)' = (x)'e^x + x(e^x)' = 1e^x + xe^x = (x+1)e^x$.
$f'' = ((x+1)e^x)' = (x+1)'e^x + (x+1)(e^x)' = 1e^x + (x+1)e^x = (x+2)e^x$.
Use mathematical induction on n we have $f^{(n)} = (x+n)e^x$. ■

Example 0.2 *$f(t) = \sqrt{t}(a+bt)$, find f'.*

[Sol 1]: $f' = (\sqrt{t})'(a+bt) + \sqrt{t}(a+bt)' = \dfrac{1}{2\sqrt{t}}(a+bt) + \sqrt{t}b = \dfrac{a+3bt}{2\sqrt{t}}$.

[Sol 2]: $f' = (at^{1/2} + bt^{3/2})' = \dfrac{1}{2}at^{-1/2} + \dfrac{3}{2}bt^{1/2}$. ■

Note: $f' = f'(t) = \dfrac{d}{dt}f(t) \neq \dfrac{d}{d\boldsymbol{x}}f(t)(=?)$.

0.2 Derivative of quotient

The Quotient rule: .... $\boxed{(f/g)' = (f'g - fg')/g^2}$

口訣: 上面的微分乘以下面減上面乘以下面的微分再除以下面兩次。

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}.$$

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+h)/g(x+h) - f(x)/g(x)}{h} \qquad (\text{通分}) \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \qquad (\text{一加一減 } f(x)g(x)) \\
&= \lim_{h\to 0}\frac{[f(x+h) - f(x)]g(x) - f(x)[g(x+h) - g(x)]}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\lim_{h\to 0}\frac{g(x)}{g(x+h)g(x)} \\
&\qquad - \lim_{h\to 0}\frac{f(x)}{g(x+h)g(x)}\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \qquad (\text{有減號, 注意順序。})
\end{aligned}$$

Example 0.3 *Find* $y = e^x/(1+x^2)$ 在 $x = 1$ 的切線。

$$y' = \frac{(e^x)'(1+x^2) - e^x(1+x^2)'}{(1+x^2)^2} = \frac{e^x(1+x^2) - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}.$$

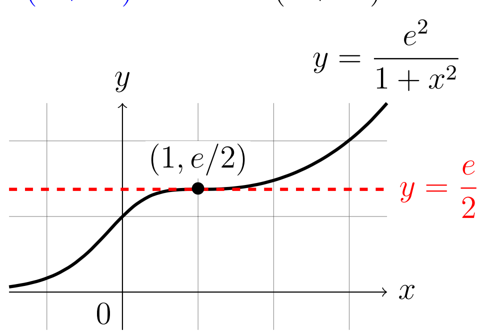
$y\Big|_{x=1} = \frac{e}{2}, \implies$ 切點 $(1, \frac{e}{2})$.

$y'\Big|_{x=1} = \frac{e^1(1-1)^2}{(1+1^2)^2} = 0.$

切線: $y = y'\Big|_{x=1}(x-1) + y\Big|_{x=1}$

$= 0(x-1) + \frac{e}{2} = \frac{e}{2}.$ ■

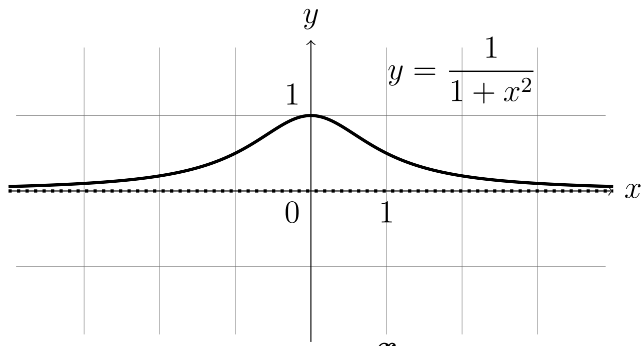
(法線?)



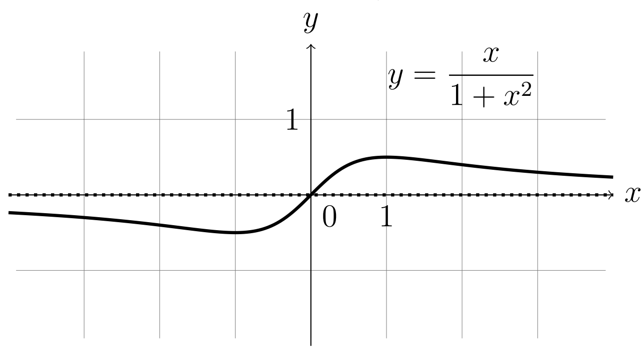
Skill: 公式怎麼背? 寫題目, 先抄公式, 再把每項換上去。

⧫ Additional: Stories in Exercises

(Exercise 3.2.35) 曲線 $\boldsymbol{y = \dfrac{1}{1+x^2}}$ 稱爲 ***witch of (Maria) Agnesi*** 阿涅西的女巫/箕舌線。 — Maria Gaetana Agnesi, 義大利女數學家。
1630 Fermat 費馬首先發現。
1718 Grandi 葛蘭迪命名“versoria”, 意爲帆繩。
1748 Agnesi 阿涅西書中譯爲“versiera”[義大利文], 與女巫同義。
1801 Colson 柯爾森誤譯爲“witch”。



(Exercise 3.2.36) 曲線 $\boldsymbol{y = \dfrac{x}{1+x^2}}$ 稱爲 ***serpentine*** 蛇狀線 (/蛇紋石)。



(Exercise 3.2.64) ***Reciprocal Rule*** 倒數律

$$\frac{d}{dx}\Big[\frac{1}{g(x)}\Big] = -\frac{g'(x)}{[g(x)]^2}.$$