Problem 1

(a)

$$P_X(k) = \frac{e^{-\lambda T} \cdot (\lambda T)^k}{k!} , \quad k^* = \lfloor \lambda T \rfloor$$

$$P_X(k) = e^{-\lambda T} \cdot \frac{\lambda T}{k} \cdot \frac{\lambda T}{k-1} \cdot \cdots \cdot \frac{\lambda T}{1}$$

for all $x \in N \leq k^*$, $\frac{\lambda T}{x} \geq 1$, so $P_X(k)$ is non-decreasing

for all $x \in N > k^*$, $\frac{\lambda T}{x} < 1$, so $P_X(k)$ is decreasing

$\Rightarrow$ when $x = k^*$, $P_X(k)$ has the maximum value #

(b) For all i, $X_i = (1-p)^{k_i - 1} \cdot p$

$$\Rightarrow F_{X_i}(t) = \sum_{i=1}^{t} (1-p)^{t-1} \cdot p = \frac{p[1-(1-p)^t]}{1-(1-p)} = 1-(1-p)^t$$

$$\Rightarrow P(X_i > t) = (1-p)^t$$

$$P(X > t) = P(X_1 > t) \cdot P(X_2 > t) \cdot \cdots \cdot P(X_n > t) = [(1-p)^t]^n = (1-p)^{nt}$$

$$\Rightarrow F_X(t) = 1-(1-p)^{nt}$$

$\Rightarrow P(X=t) = [(1-p)^n]^{t-1} \cdot [1-(1-p)^n]$, which is a geometric r.v. #

Problem 2

(a) By discrete uniform probability law, $q_k = \dfrac{\text{occurrences of } k \text{ ball in } 1^{st} \text{ cell}}{\text{all occurrences}}$

the number of all the occurrences is $H^n_r = C^{n+r-1}_r$

the number of occurrences of k ball in 1^{st} cell

= the number of occurrences of $(r-k)$ ball in other $(n-1)$ cells

$= H^{n-1}_{r-k} = C^{n+r-k-2}_{r-k}$

$$\Rightarrow q_k = \frac{C^{n+r-k-2}_{r-k}}{C^{n+r-1}_r} \quad \#$$

(b) $$q_k = \frac{\frac{(n+r-k-2)!}{(r-k)!\,(n-2)!}}{\frac{(n+r-1)!}{r!\,(n-1)!}} = \frac{(n+r-k-2)!\cdot r!\cdot \overset{n-1}{\cancel{(n-1)!}}}{(r-k)!\cdot \cancel{(n-2)!}\cdot (n+r-1)!}$$

$$= \frac{\overbrace{r(r-1)\cdots(r-k+1)}^{k}\cdot(n-1)/n^{k+1}}{\underbrace{(n+r-1)(n+r-2)\cdots(n+r-k-1)}_{k+1}/n^{k+1}}$$

$$= \frac{\frac{r}{n}\left(\frac{r}{n}-\frac{1}{n}\right)\cdots\left(\frac{r}{n}-\frac{k-1}{n}\right)\cdot\frac{n-1}{n}}{\left(1+\frac{r}{n}-\frac{1}{n}\right)\left(1+\frac{r}{n}-\frac{2}{n}\right)\cdots\left(1+\frac{r}{n}-\frac{k+1}{n}\right)} \xRightarrow{\lim_{n\to\infty}} \frac{\lambda^k}{(1+\lambda)^{k+1}} \quad \#$$

$q_k \longrightarrow \dfrac{\lambda^k}{(1+\lambda)^{k+1}} = \dfrac{1}{1+\lambda}\cdot\left(\dfrac{\lambda}{1+\lambda}\right)^k$, Let $\dfrac{1}{1+\lambda} = p$:

$q_k = (1-p)^k\cdot p \Rightarrow$ a geometric random variable #

Problem 3

(a) $P(X=k)=\sum_{n=0}^{\infty} P(X=k \mid V=k+n)\cdot P(V=k+n)$

$$=\sum_{n=0}^{\infty} C_k^{k+n}\cdot p^k\cdot(1-p)^n\cdot\frac{e^{-\lambda T}\cdot(\lambda T)^{k+n}}{(k+n)!}$$

$$=\sum_{n=0}^{\infty}\frac{\cancel{(k+n)!}}{k!\,n!}\cdot p^k\cdot(1-p)^n\cdot\frac{e^{-\lambda T}\cdot(\lambda T)^{k+n}}{\cancel{(k+n)!}}$$

$$=\frac{e^{-\lambda T}\cdot p^k(\lambda T)^k}{k!}\cdot\sum_{n=0}^{\infty}\frac{(1-p)^n\cdot(\lambda T)^n}{n!}$$

$$=\frac{e^{-\lambda T}\cdot(\lambda p T)^k}{k!}\cdot e^{\lambda T(1-p)}$$

$$=\frac{e^{-\lambda p T}\cdot(\lambda p T)^k}{k!}$$

$\Rightarrow$ it has a Poisson PMF with average rate λp #

(b) $\gamma = P(1\text{ received}) = P(1\text{ received}\cap 1\text{ transmitted}) + P(1\text{ received}\cap 0\text{ transmitted})$

$$=\frac{e^{-\lambda p T}\cdot(\lambda p T)^k}{k!}\cdot\alpha_1+\frac{e^{-\lambda(1-p)T}\cdot(\lambda(1-p)T)^k}{k!}\cdot(1-\alpha_0)$$

Problem 4

(a) $E(x) = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p$

$= p \cdot \left(\sum_{k=1}^{\infty} -(1-p)^{k} \right)'$

$= (-p) \cdot \left(\frac{1-p}{1-(1-p)} \right)'$

$= (-p) \cdot (-p^{-2})$

$= \frac{1}{p}$ #

$Var(x) = E(x^2) - E(x)^2$

$= \sum_{k=1}^{\infty} k^2 \cdot (1-p)^{k-1} \cdot p - \frac{1}{p^2}$

$= p \cdot \left(\sum_{k=1}^{\infty} -k(1-p)^{k} \right)' - \frac{1}{p^2}$

$= p \cdot \left(\sum_{k=1}^{\infty} -(k+1)(1-p)^{k} + \sum_{k=1}^{\infty} (1-p)^{k} \right)' - \frac{1}{p^2}$

$= p \cdot \left(\left(\sum_{k=1}^{\infty} (1-p)^{k+1} \right)' + \sum_{k=1}^{\infty} (1-p)^{k} \right)' - \frac{1}{p^2}$

$= p \cdot \left(\left(\frac{(1-p)^2}{1-(1-p)} \right)' + \frac{1-p}{1-(1-p)} \right)' - \frac{1}{p^2}$

$= p \cdot \left(1 - \frac{1}{p^2} + \frac{1-p}{p} \right)' - \frac{1}{p^2}$

$= p \cdot \left(\frac{p-1}{p^2} \right)' - \frac{1}{p^2}$

$= p \cdot \left(\frac{2-p}{p^3} \right) - \frac{1}{p^2} = \frac{1-p}{p^2}$ #

(b) $E[X^m] = E[Y^m]$

$\Rightarrow \sum_{t\in X} t^m \cdot P(X=t) = \sum_{t\in Y} t^m \cdot P(Y=t)$

$\Rightarrow \sum_{t\in X} t^m \cdot (P(X=t) - P(Y=t)) = 0$ (as X, Y share the same set)

Suppose there are k numbers such that $P(X=t) \neq P(Y=t)$

and let $d_i = P(X=t_i) - P(Y=t_i)$:

$$\begin{cases} \sum_{i=1}^{k} t_i^m \cdot d_i = 0 \text{ for } m \in [1, n-1] \\ \sum_{i=1}^{k} d_i = 0 \text{ (because the total probability of X and Y are the same)} \end{cases}$$

$\Rightarrow d_i = 0$ for $i \in [1, k]$

contradict to the assumption that $P(X=t) \neq P(Y=t)$

$\Rightarrow P(X=t) = P(Y=t)$ for $t \in [0, n-1]$ #

(c)

i) $Var(Z) = E(x^2) - E(x)^2$

$$= \sum_{n=1}^{\infty} ((-1)^n \sqrt{n})^2 \cdot \frac{6}{(\pi n)^2} - \sum_{n=1}^{\infty} (-1)^n \cdot \sqrt{n} \cdot \frac{6}{(\pi n)^2}$$

$$= \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{6}{\pi^2} \cdot \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{\frac{3}{2}}}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ does not exist $\Rightarrow E(x^2)$ does not exist

$\lim_{n\to\infty} \left| (-1)^n \cdot \frac{1}{n^{\frac{3}{2}}} \right| = 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{\frac{3}{2}}}$ converges $\Rightarrow E(x) =$ constant k

$\Rightarrow Var(Z)$ does not exist #

ii) $$\sum_{n=1}^{\infty} [(-1)^n \sqrt{n}]^3 \cdot \frac{6}{(\pi n)^2}$$

$$= \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$= \frac{6}{\pi^2} \cdot (-1+\sqrt{2}) \zeta\left(\frac{1}{2}\right)$$

#

iii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges, while $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

$\Rightarrow$ by Riemann Rearrangement Theorem, there exists a rearrangement $\{b_n\}$ of $\left\{\frac{(-1)^n}{\sqrt{n}}\right\}$ such that $\sum_{n=1}^{\infty} b_n$ equals any real number

$\Rightarrow E[Z^3]$ does not exist #

iv) $E[Z^{10}] = \sum_{n=1}^{\infty} [(-1)^n \sqrt{n}]^{10} \cdot \frac{6}{(\pi n)^2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} n^3$ diverges

$\Rightarrow E[Z^{10}]$ does not exist #