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Brahmagupta's propositions on the perpendiculars of cyclic quadrilaterals

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Abstract

We continue a recent analysis of Propositions XII.21–28 of Brahmagupta's *Brāhma-sphuṭa-siddhānta* (India, 628 A.D.), on the area and diagonals of the cyclic quadrilateral, by examining Propositions XII.29–32, that explain how to determine the perpendiculars as well as all the portions of diagonals and perpendiculars. These results include the result nowadays referred to as "Brahmagupta's theorem" (XII.30–31). Brahmagupta describes both the geometric situation and the key elements of the derivation of his results. We analyze the expression of hypotheses and derivations, using only Brahmagupta's conceptual framework, that does not include the notion of angle, and uses proportion only in a standard form (XII.25).

Résumé

On prolonge une analyse récente des Propositions XII.21–28 du*Brāhma-sphuṭa-siddhānta* de Brahmagupta (Inde, 628 A.D.), sur l'aire et les diagonales du quadrilatère cyclique, par une étude des Propositions XII.29–32, qui expliquent comment déterminer les hauteurs, ainsi que toutes les portions des diagonales et des hauteurs. Ces résultats comprennent ce que l'on appelle de nos jours le "Théorème de Brahmagupta" (XII.30–31). Brahmagupta décrit la situation géométrique et par là même suggère une dérivation des résultats. On montre que cette dérivation, dans le cadre conceptuel de l'auteur—duquel la notion d'angle est absente—repose sur l'utilisation systématique de proportions dans des triangles rectangles sous une forme standardisée (XII.25), comme le texte l'indique.

MSC: 01A32; 01A35; 51-03

Keywords: Geometry; India; Medieval; Cyclic quadrilaterals

1. Introduction

Propositions XII.21–27 of Brahmagupta's *Brāhma-sphuṭa-siddhānta* (*BSS*, India, 628 A.D.) constitute a coherent mathematical discourse leading to the derivation of the expression

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of the area of an arbitrary cyclic quadrilateral; Proposition XII.28, giving the expressions of its diagonals, follows from it.[1] The purpose of this paper is to extend this analysis to XII.29–32, that show how to determine the lengths of all the portions of diagonals and perpendiculars defined by their intersections. This passage contains the result now known as "Brahmagupta's theorem," and stated as follows: *in a cyclic quadrilateral with perpendicular diagonals, the line through the point of intersection of the diagonals, perpendicular to one side, bisects the opposite side*. In this form, the result was first stated by Chasles [1837, 435] in his exposition of Brahmagupta's work, as a restatement of Proposition XII.30–31. To the best of our knowledge, apart from commentaries on *BSS*,[2] and unlike Brahmagupta's area formula, no mathematical treatise, in India or elsewhere, seems to mention this property until Colebrooke's translation [1817] and Chasles' report on Indian Geometry [1837, 417–456]. [3] The issues considered in this paper are as follows: (i) Do Brahmagupta's XII.29–32 form a natural continuation of Propositions XII.21–28? (ii) How did he derive his results using only the tools at his disposal—in particular, without the notion of angle?

The article is organized as follows. The rest of this Introduction summarizes Propositions XII.21–28, focusing on the tools that will be used in this paper, and briefly reviews earlier translations. Next, Propositions XII.29, XII.30–31[4] and XII.32 are analyzed. The conclusions of the study are summarized in the final section.

The analysis of each proposition is divided into four parts: translation; gloss, clarifying technical terms and drawing attention to the structure of the text; comments, explicating what comes out if one carries out the suggestions in the text; and temporary conclusions drawn from the analysis of each proposition. To make the structure of the text more explicit, we have translated each verse in four lines, because the Sanskrit verse Brahmagupta uses is divided into two half-verses, each of which is split into two quarter-verses by an optional break. Each line in the translation corresponds to one quarter-verse, in the same order as in the original.

1.1. Terminology and notation

Propositions XII.21–32 give relations involving the sides, diagonals and perpendiculars of a cyclic quadrilateral[5] $ABCD$; here, perpendiculars may be dropped to a side, or to a diagonal. To describe the results of XII.21–28, "... we must bring out a few expressions from the Hindus' mathematical nomenclature, of which they make a very fortunate use

[1] [Kichenassamy, 2010]. The reader is referred to this paper for background information. We merely recall that *BSS* consists of Sanskrit verses in *āryā* meter, in 24 chapters. Each chapter, apart from chapter summaries and conclusions, contains statements comprising one or several verses. Each of these statements will be called a proposition, followed by the appropriate verse numbers. They may include results, formulae expressed in words, or brief indications on the logical dependence of propositions. Their arrangement and structure also strongly constrains possible derivations.

[2] Only one, by Pṛthūdakasvāmi (IXth century A.D.) seems extant. For the translation of the passages relevant to the present study, see [Colebrooke, 1817, 302–305].

[3] For studies using modern methods, see for instance [Altshiller-Court, 1952, 137] for XII.30–31 and [Askey, 2010] for XII.32.

[4] XII.30–31 form a single proposition.

[5] A cyclic quadrilateral is the figure formed by four points on a circle. The circumcircle of a triangle is the circle that goes through all its vertices. The radius of the circle through the vertices of a cyclic quadrilateral or a triangle is called the circumradius.

to state theorems in a concise manner and without the help of figures; which gives them a character of generality…"[6] In any given geometric situation, one line is identified as the base;[7] it is generally represented horizontally in our figures. The perpendiculars[8] are then vertical, while diagonals (*karṇa*) are neither horizontal nor vertical. The intersection of diagonals determines two portions on each diagonal, one called "upper," and the other one "lower" (*ūrdhva, adhara*). Similarly, the intersection of a diagonal and perpendicular determines lower and upper portions of diagonal and perpendicular. Like Brahmagupta, we shall restrict the word "segments" (*āvādhā*) to projections on the base, keeping "portion" (*khaṇḍa*) for parts of diagonals or perpendiculars. While there are terms for perpendiculars and for segments cut off by a perpendicular, the text never expresses directly that two lines cut each other at right angles; in fact, the concept of angle is not used at all in Brahmagupta's geometry. Thus, what we call right triangles are thought of as half-oblongs, and their hypotenuse is called *karṇa*, diagonal; their base and upright have special names (*bhuja*, *koṭi*).

Brahmagupta uses specific terms for cyclic quadrilaterals. The "triquadrilateral" (*tricaturbhuja*) is obtained from a triangle[9] by adding a fourth point on its circumcircle, and by connecting it to the two closest vertices of the triangle to form two new sides, as in Fig. 1. One of the sides of the triangle then becomes a diagonal of the triquadrilateral. The fourth point may be chosen arbitrarily on the circumcircle.[10] When it is obtained from a vertex of the triangle by mirror symmetry with respect to the vertical diameter of the circumcircle, one obtains the "symmetric (quadrilateral)" (*aviṣama*); see $ABEC$ on Fig. 1. The sides AB and EC are equal, and the fourth (top) side BE is called the "face" (*mukha*). In modern terms, $ABEC$ is an isosceles trapezium. All other choices for the fourth point D lead to an asymmetric quadrilateral (*viṣama*). To every symmetric quadrilateral $ABCE$, one associates a special asymmetric quadrilateral $ABCF$, in which F is the mirror image of B with respect to the second (horizontal) diameter. In modern terms, $ABCF$ has perpendicular diagonals.[11] The radius r of the circumcircle is called the heart-cord (*hṛdayarajju*).

1.2. Summary of Propositions XII.21–28

XII.21 gives Brahmagupta's formula for the area of a triquadrilateral $ABCD$:

$$\mathrm{Area}(ABCD) = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{1}$$

[6] "… il nous faut faire connaître quelques expressions de la nomenclature mathématique des Hindous, dont ils font un usage très-heureux pour énoncer les théorèmes de manière concise et sans le secours de figures; ce qui leur donne un caractère de généralité…" [Chasles, 1837, 421].

[7] Literally, the ground, the Earth (*bhūmi, bhū*).

[8] In Sanskrit: *lamba, avalamba, lambaka* or *avalambaka*. We follow the usual practice of translating all of them by "perpendicular;" this is sufficient for our purposes.

[9] More precisely, a "trilateral" or "three-armed." In Sanskrit: *tribhuja*.

[10] Thus, the triquadrilateral consists of a triangle and a quadrilateral that have two sides in common. As pointed out by P.-S. Filliozat (private communication), *tricaturbhuja* is a possessive compound (in Sanskrit grammatical terminology, it is a *bahuvrīhi*). It should be split as *tricatur-bhuja*, and represents a object having "three and four" (*tricatur*) sides (*bhuja*). One may further analyze *tri-catur* as a copulative compound (or *dvandva*).

[11] Chasles defines a *trapèze* to be a cyclic quadrilateral with perpendicular diagonals, and uses *tétragone inscrit*, *quadrilatère inscriptible* and variants, for the general cyclic quadrilateral.

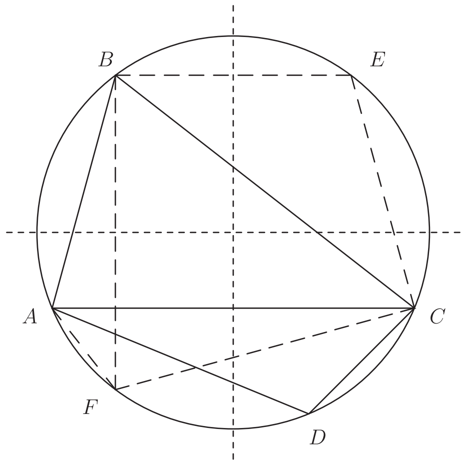

Figure 1. General triquadrilateral $ABCD$, derived from triangle ABC by adding one point D on its circumcircle, together with the distinguished quadrilaterals $ABEC$ and $ABCF$. The points E and F are obtained from B by symmetry with respect to the axes. $a = AB$, $b = BC$, $c = CD$, $d = DA$, $f = BE$, $\gamma = AC$, $\delta = BD$.

where s is half the sum of the four sides $a = AB$, $b = BC$, $c = CD$, $d = DA$. It also gives the approximate value $\frac{1}{2}(a + c) \times \frac{1}{2}(b + d)$. Propositions XII.22–27 introduce tools that may be used to derive this result.

The first set of tools concerns triangles (XII.22–24). Of them, XII.22 will be used extensively in this article. With reference to Fig. 2, consider triangle ABC, with AC as base, and drop perpendicular BX from B to AC. Proposition XII.22 expresses the segments (*āvādhā*) $\alpha = AX$, $\beta = XC$ of the base $\gamma = AC$, and the perpendicular $h = BX$, in terms of the sides of ABC:

$$\alpha = \frac{1}{2}\left[\gamma + \frac{a^2 - b^2}{\gamma}\right], \quad \beta = \frac{1}{2}\left[\gamma - \frac{a^2 - b^2}{\gamma}\right], \tag{2}$$

and

$$h = \sqrt{a^2 - \alpha^2} = \sqrt{b^2 - \beta^2}. \tag{3}$$

XII.23 applies these relations to the isosceles trapezium $ABEC$, and XII.24 to a triangle in which one side is a diameter of the circumcircle; it states that $\gamma^2 = a^2 + b^2$ in that case. The converse of this proposition follows from XII.27.

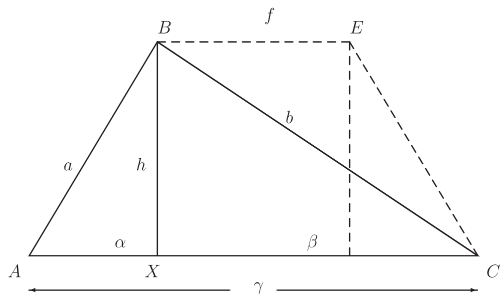

Figure 2. Triangle and associated symmetric quadrilateral.

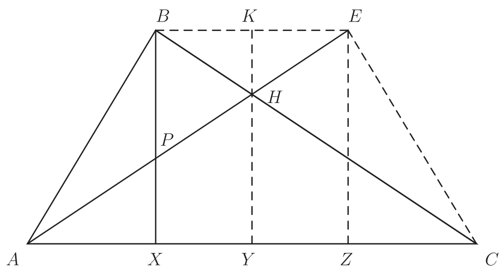

Figure 3. Symmetric quadrilateral with the intersections of diagonals and perpendiculars.

XII.25 gives, in the symmetric quadrilateral $ABCE$, all the portions of diagonals or perpendiculars determined by their points of intersection, such as P and H on Fig. 3. They are determined by a single proportion (*anupāta*) that will be used repeatedly in the analysis of XII.29–32. For any line, be it a diagonal or perpendicular, it reads:

$$\text{portion of line} = \text{entire line} \times \frac{\text{own segment}}{\text{own connection}}.$$

Here, the own segment (*svāvādhā*) is the projection on the base of the portion of diagonal under consideration; the own connection (*svayuti*) is the projection of the entire diagonal.[12] Thus, considering the lower portion AP of diagonal AE, the own segment is AX and the own connection AZ, and

$$AP = AE \times \frac{AX}{AZ}. \tag{4}$$

The lower portion PX of perpendicular BX is similarly related to the entire perpendicular:

$$PX = BX \times \frac{AX}{AZ}. \tag{5}$$

Note that the two lower portions AP and PX have the same own segment (AX) and the same own connection (AZ). Similarly, the upper portions PE and BP both have the own segment XZ, and own connection AZ. The own segment of AH and HY is AY, and so forth.

Propositions XII.26–27 give the common circumradius r of $ABEC$, $ABCF$, ABC and $ABCD$, see Fig. 2. Of relevance in the following are the relations

$$r = \frac{1}{2}\sqrt{a^2 + c'^2} = \frac{1}{2}\sqrt{b^2 + d'^2}, \tag{6}$$

where $c' = CF$ and $d' = FA$, and $r = \frac{ab}{2h}$, where h is the perpendicular of ABC (dropped to AC); hence, the diameter of the general triquadrilateral $ABCD$ is

$$2r = \frac{ab}{h}. \tag{7}$$

Proposition XII.28 expresses the two diagonals of the general triquadrilateral in terms of its sides:

$$\gamma = \sqrt{\frac{ad + bc}{ab + cd}(ac + bd)} \quad \text{and} \quad \delta = \sqrt{\frac{ab + cd}{ad + bc}(ac + bd)}. \tag{8}$$

[12] It connects the foot of the diagonal and the foot of the perpendicular dropped from the end of the diagonal.

1.3. Earlier translations

BSS was edited by Dvivedin [1902], and by Sharma [1966]. The translation by Colebrooke [1817] formed the basis of most subsequent work. His work also gives extracts from the only extant commentary, by Pṛthūdakasvāmi (IX$^{\text{th}}$ century); for the propositions under consideration, the commentary illustrates them on one numerical example, for which all the lines may be computed without square roots; it does not discuss derivations.[13] While Colebrooke does not analyze the assumptions underlying this passage, Chasles [1837], working on the basis of Colebrooke's translation, stresses that XII.30–31 specifically requires a cyclic quadrilateral with perpendicular diagonals. Another translation is due to Sarasvati Amma [1999, 81–82], who does not require the diagonals to be perpendicular in XII.29 and XII.32. Other translations may be found in Datta and Singh [1980, 145] and Plofker [2007, 424–425] (see also [Plofker, 2008, 146–147]), without detailed analysis.

2. Propositions on the perpendiculars of quadrilaterals (XII.29–32)

2.1. Outline

XII.29 refers to the general triquadrilateral $ABCD$, see Fig. 4. Its sides will be called $a = AB$, $b = BC$, $c = CD$ and $d = DA$; its diagonals $\gamma = AC$ and $\delta = BD$. The base of $ABCD$ is AD. We shall also need the perpendiculars dropped from the two upper vertices to the base AD: their feet are X and Z; similarly, Y denotes the foot of the perpendicular dropped to the base from the point H where the diagonals intersect, see Fig. 4. It is the lower portion of the full "middle perpendicular" KY. XII.29 determines the "extreme" perpendiculars BX and CZ.

XII.30–31 determines KY and KH when the diagonals are perpendicular. The discussion will be based on Fig. 5.

XII.32 determines the portions of diagonals and perpendiculars in the general triquadrilateral $ABCD$, by introducing auxiliary constructions. The discussion will be based on Fig. 6.

2.2. Proposition XII.29

2.2.1. Text and translation

> XII.29: *viṣamacaturasramadhye viṣamatribhujadvayaṃ prakalpya pṛthak karṇadvayena pūrvavadāvādhe lambakau ca pṛthak*
> In the middle of an asymmetric quadrilateral,
> Consider severally the asymmetric trilaterals of the pair [determined]
> By the pair of diagonals; [obtain] as before
> Two segments, and two perpendiculars, severally.

2.2.2. Gloss

The derivation of XII.21 was based on the introduction of a pair of triangles, such as ABC and ACD, whose common base was one diagonal of $ABCD$. By contrast,

[13] The only part that could have dealt with a derivation is, according to Colebrooke [1817, 304, Footnote] "irretrievably corrupt."

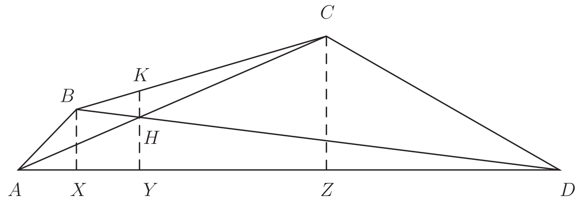

Figure 4. Cyclic quadrilateral $ABCD$, with sides $a = AB$, $b = BC$, $c = CD$ and $d = DA$, diagonals $\gamma = AC$ and $\delta = BD$, and perpendiculars BX and CZ dropped to AD. The "middle perpendicular" KHY is also indicated. The center of the circumcircle is located below the line AD.

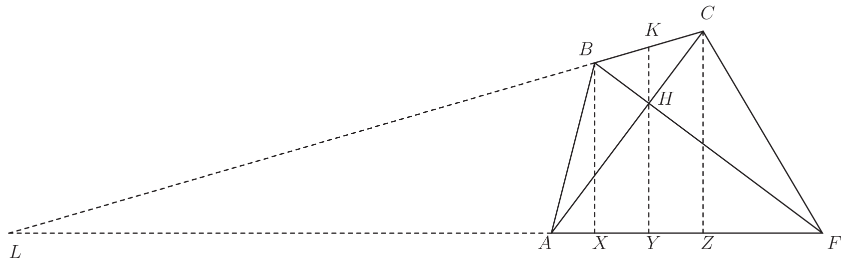

Figure 5. Cyclic quadrilateral $ABCF$ with perpendicular diagonals, and its perpendiculars dropped to the base AF. Its sides are $a = AB$, $b = BC$, $c' = CF$, $d' = FA$, and its diagonals are $\gamma = AC$, $\delta = BF$. The portions of diagonals are $\alpha = AH$, $\beta = HC$, $h = BH$, $k = HF$.

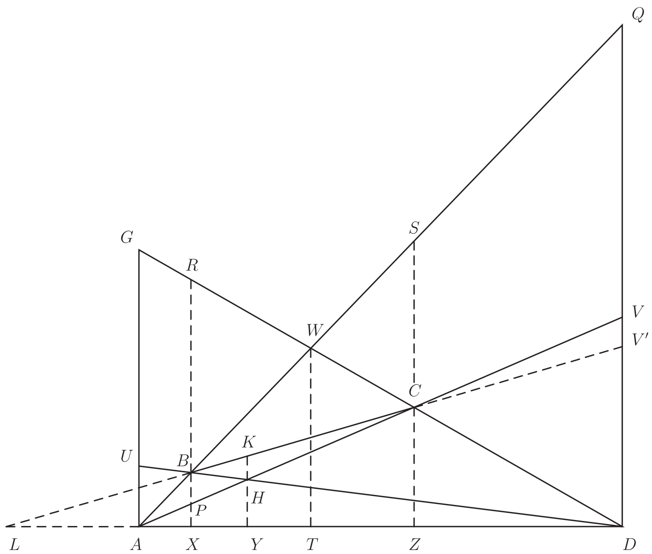

Figure 6. Cyclic quadrilateral $ABCD$, with non-perpendicular diagonals, and its perpendiculars dropped to AD. The needle is AWD, and the figure with produced sides and perpendiculars is formed by completing $ABCD$ by the triangles $AUGRD$ and $ASQVD$.

Brahmagupta considers here a pair (*dvaya*) of triangles within the quadrilateral $ABCD$, corresponding to the pair of diagonals: the parallel formulation[14] suggests that each triangle in

[14] "... of the pair... ", "... the pair of...".

the pair has a different diagonal as one of its sides. We therefore consider the pair of triangles ABD, ACD, whose common base is side AD, as in Fig. 4. If $ABCD$ were symmetric, the two triangles would be mirror images of one another: their perpendiculars or segments would be the same, and there would be no need to consider each of them separately. Hence the mention that the quadrilateral should be asymmetric (*viṣama*). The mention of the pair of diagonals also recalls that $\gamma = AC$ and $\delta = BD$ are known, by XII.28. Hence, the sides of ABD and ACD are known: they are (a,δ,d) and (γ,c,d) respectively. From them, we may determine the segments, and then the perpendicular of each of these triangles using XII.22 "as before" (*pūrvavat*). Each diagonal determines two segments and one perpendicular; but since we know the sum of the segments (it is equal to AD), only one segment needs to be found. Hence, for the pair, two segments and two perpendiculars must be found. For instance, the segment AX of ABD is

$$AX = \frac{1}{2}\left[d + \frac{a^2 - \delta^2}{d}\right] = \frac{1}{2d}[d^2 + a^2 - \delta^2], \tag{9}$$

and the perpendicular BX is $\sqrt{a^2 - AX^2}$. The segment XD is simply $AD - AX = d - AX$. Similarly, using XII.22 in ACD,

$$ZD = \frac{1}{2}\left[d + \frac{c^2 - \gamma^2}{d}\right] = \frac{1}{2d}[d^2 + c^2 - \gamma^2],$$

$CZ = \sqrt{c^2 - ZD^2}$ and $AZ = d - ZD$. The text mentions the segments before the perpendiculars, since the segments are determined first.

The word "consider" on the second line translates the indeclinable past participle *prakalpya*, cognate to *kalpanā*, imagination; the literal meaning is "having represented mentally." Since the quadrilateral and its diagonals have already been introduced in previous propositions, no new construction is required here: we are merely expected to direct our attention to ABD and ACD.

2.2.3. Comment

Let us carry out the determination of AX, BX, ZD and CZ outlined in this proposition. Inserting the value of δ^2 from (8) into (9), we obtain, after multiplication by the denominator $2d(ad + bc)$,

$$\begin{aligned}2d(ad+bc)AX &= (d^2+a^2)(ad+bc) - (ab+cd)(ac+bd)\\ &= (d^2+a^2)ad + a^2bc + bcd^2 - a^2bc - bcd^2 - adb^2 - adc^2\\ &= ad(a^2 - b^2 - c^2 + d^2).\end{aligned}$$

Therefore,

$$AX = \frac{a(a^2 - b^2 - c^2 + d^2)}{2(ad + bc)}.$$

Since $BX^2 = a^2 - AX^2$,

$$\begin{aligned}4(ad+bc)^2BX^2 &= a^2\{4(ad+bc)^2-(a^2-b^2-c^2+d^2)^2\}\\ &= a^2[2(ad+bc)+(a^2+d^2-b^2-c^2)][2(ad+bc)-(a^2+d^2-b^2-c^2)]\\ &= a^2[(a+d)^2-(b-c)^2][(b+c)^2-(a-d)^2]\\ &= a^2(a+d+c-b)(a+d+b-c)(b+c-a+d)(b+c+a-d)\\ &= 16a^2\text{Area}(ABCD)^2,\end{aligned}$$

where XII.21 has been used in the last step. We therefore have a very simple expression for the perpendicular BX:

$$BX = 2a\frac{\text{Area}(ABCD)}{ad+bc}. \tag{10}$$

Similarly,

$$CZ = 2c\frac{\text{Area}(ABCD)}{ab+cd}. \tag{11}$$

2.2.4. Conclusion

While previous propositions led up to the complete description of the lines associated with the two triangles into which a cyclic quadrilateral is decomposed by a diagonal, this proposition achieves the same for the segments and perpendiculars of the two trilaterals having a side of the quadrilateral as base. The possibility of this determination is an immediate consequence of earlier results. Brahmagupta states the method, rather than the result that follows by implementing it.

2.3. Proposition XII.30–31

2.3.1. Text and translation

> XII.30: *viṣamabhujāntastribhuje prakalpya karṇau bhuvau tadāvādhe pṛthagūrdhvādhara-khaṇḍe karṇayutau karṇayoradhare*
>
> XII.31: *tribhuje bhujau tu bhūmistallambo*[15] *lambakādharaṃkhaṇḍam ūrdhvamavalamba-khaṇḍaṃ lambakayogārdhamadharonam*
>
> Within the asymmetric arms, two trilaterals
> Are considered; the two diagonals are their two bases; their two segments
> Are severally the lower and upper portions
> [Determined by] the meeting of the diagonals. Beneath both diagonals,
> In the trilateral [they bound, the portions] are, however, the two arms; the base [is the same].
> Its perpendicular is the perpendicular's lower portion;
> The upper portion of the perpendicular
> Is the perpendiculars' half-sum less the lower [portion].

[15] A variant for the quarter-verse "*tribhuje bhujau tu bhūmis*" is "*tribhujabhujaubhūrbhūmis*" [Sharma, 1966, I, p. 160]. This would mean: "they are the two arms of a triangle; the ground (line of the triangle) is the base (of the quadrilateral)."

2.3.2. Gloss

Brahmagupta considers now a quadrilateral $ABCF$ called *viṣamabhuja* (literally, "asymmetric-arm"); see Fig. 5.[16] It contains two triangles that have the diagonals as bases. Since they should stand above their bases AC and BF, these triangles are ABC and BCF. Calling H the point where the diagonals meet, AH and HC are the portions of diagonal AC; similarly, BH and HF are the portions of diagonal BF. Brahmagupta requires that these portions be segments of ABC and BCF. Now, if AH and HC are segments of ABC, necessarily, BH is the perpendicular dropped from B to AC. In modern terms, the lines BH and AC are perpendicular. Since BH is a portion of diagonal BF, the two diagonals are perpendicular. The same argument applies to BCF. Since Brahmagupta's geometry does not involve the concept of angle—not even that of right angle—his formulation is the most convenient way for him to express that the diagonals are perpendicular. Brahmagupta then points out that the lower segments AH and HF may also be considered as the arms of the trilateral AHF "under both diagonals," with base AF: the same as the base of the quadrilateral. The perpendicular HY of AHF may be viewed as the lower portion of the perpendicular KY of the quadrilateral. The extreme perpendiculars BX and CZ have already been determined in XII.29. Brahmagupta now states that

$$KH = (BX + CZ)/2 - HY. \tag{12}$$

He therefore suggests to determine first the entire middle perpendicular $KY = KH + HY$ as the half-sum of the extreme perpendiculars BX and CZ (*lambakayogārdham*), and then to remove the lower portion HY (*adharonam*).

2.3.3. Comment

Brahmagupta's focus here is on the "middle perpendicular" KHY, assuming $ABCF$ has perpendicular diagonals, or more precisely, that AHB, BHC, CHF, FHA are all half-oblongs. Unlike the case of the isosceles trapezium (XII.25), Brahmagupta does not handle the lower and upper portions of the perpendicular in a symmetric fashion, but obtains the upper portion KH as $KY - HY$. We proceed to determine HY, KY and KH. To this end, we determine the segments of the base corresponding to the three perpendiculars: AX, XY, YZ and ZF.

Let us call $\alpha = AH$, $\beta = HC$, $h = BH$ and $k = HF$ the portions of the diagonals. By assumption, (α,β) and (h,k) are segments of ABC and AFC respectively, so that h and k are their respective perpendiculars. Since the arms of ABC and AFC are (a,b) and (c',d') respectively, Eq. (3) yields

$$a^2 = \alpha^2 + h^2;\ b^2 = \beta^2 + h^2;\ c'^2 = \beta^2 + k^2;\ d'^2 = \alpha^2 + k^2.$$

Since AHF is a half-oblong, with perpendicular HY, its area may be expressed either as half the product of its sides, or as half the product of base and perpendicular: $\frac{1}{2}d' \times HY = \frac{1}{2}\alpha k$. Therefore,

$$HY = \frac{\alpha k}{d'}.$$

[16] It is generally assumed that *viṣamabhuja* is short for *viṣamacaturbhuja* (literally, "asymmetric-four-arm" or asymmetric quadrilateral), a term not used by Brahmagupta here. The term *viṣamabhuja*, by stressing the asymmetry of the arms, may suggest that the equality of base and face ($BC = AF$) is allowed.

It follows that $AY^2 = \alpha^2 - HY^2 = (\alpha^2/d'^2)[d'^2 - k^2] = (\alpha^2/d'^2)\alpha^2$, hence

$$AY = \frac{\alpha^2}{d'}.$$

Consider now ACZ; the portions of diagonal AC determined by the point H are $AH = \alpha$ and $HC = \beta$. Therefore, applying XII.25, we have

$$AY = AZ \times \frac{AH}{AC} = AZ \times \frac{\alpha}{\gamma}.$$

Therefore, $\alpha^2/d' = \alpha \times AZ/\gamma$, so that $AZ = \frac{\alpha\gamma}{d'}$. Since $\gamma = \alpha + \beta$, this takes the form

$$AZ = \frac{\alpha^2 + \alpha\beta}{d'}.$$

It follows that

$$YZ = AZ - AY = \frac{\alpha\beta}{d'}.$$

This may be expressed in terms of the sides: since the circumcircle of $ABCF$ is also the circumcircle of ABF or BFC, its radius r satisfies, by XII.27 (Eq. (7)),

$$2r = \frac{ad'}{\alpha} = \frac{bc'}{\beta}.$$

Therefore,

$$YZ = \frac{ad'}{2r} \times \frac{bc'}{2r} \times \frac{1}{d'} = \frac{abc'}{4r^2}.$$

Similarly, considering XBF, we obtain

$$YF = XF \times \frac{HF}{BF} = XF \times \frac{k}{\delta}.$$

This yields $XF = YF \times \delta/k$. Now, $YF = AF - AY = d' - \alpha^2/d' = [d'^2 - \alpha^2]/d' = k^2/d'$. Therefore, $XF = k\delta/d'$. Since $\delta = h + k$,

$$XF = \frac{k^2 + hk}{d'},$$

hence

$$XY = XF - YF = \frac{hk}{d'}.$$

Since

$$2r = \frac{ab}{h} = \frac{c'd'}{k},$$

we obtain

$$XY = \frac{ab}{2r} \times \frac{c'd'}{2r} \times \frac{1}{d'} = \frac{abc'}{4r^2}.$$

Therefore,

$$XY = YZ. \tag{13}$$

The perpendicular BX may now be found by expressing the area of ABF in two ways: using the perpendicular dropped to BF (viz. $AH = \alpha$), or to AF (viz. BX):

$$\text{Area}(ABF) = \frac{1}{2}\alpha\delta = \frac{1}{2}d' \times BX.$$

This yields

$$BX = \frac{\alpha\delta}{d'}.$$

Similarly, $\text{area}(ACF) = \frac{1}{2}k\gamma = \frac{1}{2}d' \times CZ$ yields

$$CZ = \frac{k\gamma}{d'}.$$

We now turn to the determination of KY. To apply XII.25, we introduce the intersection L of lines AF and BC (produced); see Fig. 5. Since Brahmagupta requires that the quadrilateral be asymmetric, the extensions of AF and BC must intersect. Let us apply XII.25 to triangle LCZ. The own segments corresponding to BX and KY are LX and LY respectively, and the own connection is LZ. Since $XZ = XY + YZ = 2XY$ (by (13)), $LY = LX + XY$ and $LZ = LX + XZ = LX + 2XY$, we have

$$LY = \frac{1}{2}(LX + LZ).$$

Application of (5) now yields

$$BX = CZ \times \frac{LX}{LZ},$$

and

$$KY = CZ \times \frac{LY}{LZ} = \frac{1}{2}CZ\frac{LX + LZ}{LZ} = \frac{1}{2}\left[CZ \times \frac{LX}{LZ} + CZ\right] = \frac{1}{2}(BX + CZ).$$

The upper portion of the middle perpendicular of $ABCF$ is therefore

$$KH = KY - HY = (BX + CZ)/2 - HY;$$

hence Brahmagupta's formulation of his result (Eq. (12)). A further application of XII.25 yields

$$BK = XY \times \frac{LC}{LZ}, \text{ and } KC = YZ \times \frac{LC}{LZ},$$

because XY, YZ are the segments corresponding to BK and KC respectively. Since $XY = YZ$ (by (13)), we obtain $BK = KC$: K is the midpoint of BC: this is the form in which "Brahmagupta's theorem" is stated nowadays.

One may now complete Brahmagupta's results by expressing KH in terms of the sides alone. The result turns out to be particularly simple. Brahmagupta does not mention it, and may not have been keen on giving out all lines in terms of the sides, since he does not do so in XII.27 or XII.29 for instance. We have, since $\gamma = \alpha + \beta$ and $\delta = h + k$,

$$\begin{aligned} KH &= \frac{1}{2}(BK + CZ) - HY \\ &= \frac{1}{2}(\alpha\delta/d' + k\gamma/d') - \alpha k/d' \\ &= \frac{1}{2d'}[\alpha(h+k) + (\alpha+\beta)k - 2\alpha k] \\ &= \frac{\alpha h + \beta k}{2d'}. \end{aligned}$$

Since $2r = ab/h = ad'/\alpha$ (by (7)),

$$\alpha h = \frac{ab}{2r} \times \frac{ad'}{2r} = \frac{a^2bd'}{4r^2}.$$

Similarly, $2r = bc'/\beta = c'd'/k$, hence

$$\beta k = \frac{bc'}{2r} \times \frac{c'd'}{2r} = \frac{c'^2bd'}{4r^2}.$$

Therefore,

$$KH = \frac{1}{2d'} \times \frac{(a^2 + c'^2)bd'}{4r^2} = \frac{b}{2} \times \frac{(a^2 + c'^2)}{4r^2}.$$

Now, from Eq. (6), the circumradius r is given by

$$2r = \sqrt{a^2 + c'^2} = \sqrt{b^2 + d'^2}.$$

Therefore,

$$KH = b/2. \tag{14}$$

Another method to derive (14) is to observe that BHC is a half-oblong: the perpendicular dropped from C to BH is equal to the side CH; XII.27 then shows that BC is a diameter of the circumcircle of BHC; hence, the midpoint of BC, namely K, is its center. This proves $KH = BH = HC = \frac{1}{2}BC$, or $KH = b/2$. However, this argument may not be in the spirit of Brahmagupta's text, since he specifically states that one should derive KH from HY, BX and CZ.

2.3.4. Conclusion

Brahmagupta considers here an asymmetric quadrilateral with perpendicular diagonals. This condition is expressed by the identity of the segments and the portions of the diagonals—a natural formulation, given that the notion of angle was not available. His main result is the determination of the upper portion of the middle perpendicular. As in XII.29, he suggests the steps that lead to the result, including the auxiliary result that the middle perpendicular is the average of the extreme perpendiculars, a statement equivalent to "Brahmagupta's theorem".

2.4. Proposition XII.32

2.4.1. Text and translation

XII.32: *karṇāvalambakayutau khaṇḍe karṇāvalambayoradhare anupātena tadūne ūrdhve sūcyāṃ sapāṭāyām*

Where diagonals and perpendiculars meet,
The two lower portions of diagonal and perpendicular
Are found by proportion; subtracting these,
The two upper portions [are found] in the needle with [the figure with] intersection.

2.4.1.1. Gloss. Brahmagupta now considers new constructions determined by triquadrilateral $ABCD$. The needle (*sūcī*) is the triangle formed by producing two arms until they intersect, giving triangle AWD in Fig. 6, if AD is the base. (If CD were the base, the needle would be LCD.) The *pāṭa* is the point obtained by producing an arm and a perpendicular until they intersect, as well as the figure formed by such an intersection [Colebrooke, 1817, 303, Footnote 3].[17] First produce one side, say AB, until it meets the perpendicular that goes through C, at point S; then, further produce AB until it meets the perpendicular QD erected from the endpoint D of the base, to form the "flank intersection."[18] Thus, with reference to Fig. 6, one first completes quadrilateral $ABCD$ into the needle AWD, and then further adds the two triangles AGD and AQD. In the process, the lines AG, XR, WT, ZS and DQ are introduced; each of them meets the two diagonals AC and BD at points such as U and V.[19] The lines AW and DW are the sides of the needle, and WT is its perpendicular. Therefore, the perpendiculars Brahmagupta has in mind are QD, GA, RX, SZ, UA, VD, WT and KY. The extension of KY could be handled similarly, but does not seem to have been considered in later Indian texts. Brahmagupta states that the portions determined by the intersection of perpendiculars with each of the two (produced) diagonals may be determined: for the lower portions, one should use proportion (*anupāta*) as in XII.25; for the upper ones, one should subtract the lower portion from the full perpendicular. Therefore, the full perpendiculars should also be determined.

2.4.2. Comment

Let us carry out the steps outlined in the text, using the standardized proportion argument in XII.25 repeatedly. Intersections of the diagonals with BX or CZ are simplest, since these perpendiculars and the associated segments AX and AZ have been determined in XII.29. For instance, the point P where AC and BX meet satisfies

$$PX = CZ \times \frac{AX}{AZ} \quad \text{and} \quad AP = AC \times \frac{AX}{AZ}.$$

[17] The word *sapāṭa* means "with *pāṭa*;" here, *sa* is short for *saha* ("with"), in accordance with rule VI.3.80 in Pāṇini's grammar, that allows this replacement when referring to an object that is not directly perceived. This is consistent with the absence of figures in the text.

[18] In Sanskrit, *pārśva-pāṭa* [Colebrooke, 1817, 305, Footnote].

[19] The term *sūcī* is found in later literature, but *pāṭa* in this sense seems to be found only in *BSS* and its commentary. Figures involving two overlapping triangles such as ASZ and XRD were considered by later authors in India, but with a different terminology; the study of these constructions is beyond the scope of this paper. See for example Mahāvīra's *Gaṇitasārasaṁgraha* $179\frac{1}{2}$ sqq. (X$^{\text{th}}$ century) [Raṅgācārya, 1912; Padmavathamma, 2000, p. 555 sqq] and Bhāskara II's *Līlāvatī* 194–200 [Altshiller-Court, 1952, 83–86] (XII$^{\text{th}}$ century). Both authors use this construction to determine the perpendicular dropped from the point of intersection of diagonals. See also [Altshiller-Court, 1952, 82–84], where related work by Nārāyaṇa Paṇḍita (XIV$^{\text{th}}$ century) is also described, with excerpts from the Sanskrit text. Sarasvati Amma does not seem to have had access to [Colebrooke, 1817]: her list of references does not include it; she only mentions the second volume of Colebrooke's selected papers, which merely includes a reprint of the introduction of [Colebrooke, 1817].

For the other intersections, it suffices to determine the full perpendiculars and their segments: the portions are obtained by the argument we just used for P. We therefore only need to determine SZ, QD, RX, GA, UA, VD and KY, and segments AT and AY.

2.4.3. Determination of SZ, QD, RX, GA, UA and VD

By XII.25, the lower portions of diagonals and perpendiculars are in proportion of the corresponding portions of the base. Now, AX and AZ are respectively the own segment and own connection for BX in the figure ASZ. Similarly, AX and AD are respectively the own segment and own connection for BX in the figure AQD. Application of XII.25 yields the proportions

$$BX = SZ \times \frac{AX}{AZ} = QD \times \frac{AX}{AD}.$$

From them, we obtain SZ and QD. To determine RX and GA, we consider XRD and AGD, and the proportions

$$CZ = RX \times \frac{ZD}{XD} = GA \times \frac{ZD}{AD}.$$

Similarly, in AVD, CZ has AZ and AD as own segment and own connection, hence

$$CZ = VD \times \frac{AZ}{AD}$$

This yields VD. Similarly, UA is found from the proportion

$$BX = UA \times \frac{XD}{AD},$$

in AUD.

2.4.4. Determination of WT and AT

We now write the proportions in AQD and AGD involving WT:

$$WT = QD \times \frac{AT}{AD} = GA \times \frac{TD}{AD}. \tag{15}$$

Therefore

$$\frac{WT}{QD} + \frac{WT}{GA} = \frac{AT}{AD} + \frac{TD}{AD} = \frac{AT + TD}{AD} = 1,$$

hence

$$WT \times \frac{QD + GA}{QD \times GA} = 1,$$

and finally,

$$WT = \frac{QD \times GA}{QD + GA}.$$

Once WT is known, the location of T on the base is derived from either of the two relations in (15); thus,

$$AT = WT \times \frac{AD}{QD}.$$

The sides AW, DW of the needle are also given by proportion:

$$AW = AB \times \frac{WT}{BX} \quad \text{and} \quad DW = DC \times \frac{WT}{CZ}.$$

2.4.5. Determination of HY, KY and KH

The method applied to the figure made of AQD and AGD may now be applied in the figure in the *pāṭa* made of the two triangles AUD and AVD: it gives the lower portion HY of the middle perpendicular KY. The result is

$$HY = \frac{UA \times VD}{UA + VD}.$$

From it, we determine the location of the point Y using the proportion

$$AY = AZ \times \frac{HY}{CZ}.$$

To obtain the full perpendicular KY, consider the intersection L of the extensions of AD and BC. We have the proportions

$$LX = LZ \times \frac{BX}{CZ}, \text{ and } LY = LZ \times \frac{KY}{CZ}.$$

Since $LX = LZ - XZ$, and $XZ = AZ - AX$ is known, we may determine LZ:

$$LZ - XZ = LZ \times \frac{BX}{CZ},$$

hence

$$LZ \times \left[1 - \frac{BX}{CZ}\right] = XZ,$$

or

$$LZ = \frac{CZ \times XZ}{CZ - BX}.$$

We then have $LY = LZ - YZ = LZ - (AZ - AY)$, and finally,

$$KY = CZ \times \frac{LY}{LZ},$$

hence the value of $KH = KY - HY$.

2.4.6. Conclusion

The consideration of the extended figures introduced in XII.32 suggests a method for finding the perpendicular in a triangle in which the intersections of the produced sides with the perpendiculars erected on the endpoints of the base are known. This method, applied to the triangle AWD (the *sūcī*) yields its perpendicular WT. Applied to the triangle AHD, viewed as part of the *pāṭa* figure, it yields the perpendicular dropped from the point H of intersection of the diagonals. From it, the location of the foot Y of the perpendicular on the base is obtained. By a further use of proportion, the entire middle perpendicular KY is determined. Thus, Brahmagupta concludes his study of the general cyclic quadrilateral by indicating the additional constructions needed to obtain the portions of diagonals and perpendiculars.

3. Conclusion

Propositions XII.29–32 complete the investigation of the general cyclic quadrilateral by showing how to determine the lines obtained by the intersection of each of its diagonals with the other diagonal, or with a perpendicular. They give the following results.

(XII.29): The perpendiculars of a cyclic quadrilateral dropped to the base may be determined. Brahmagupta states that this is achieved "as before" (*pūrvavat*). This may indeed be accomplished using XII.22 and the expression of the diagonals (XII.28). The expression of the perpendiculars takes a simple form [Eqs. (10) and (11)] if one uses his area formula (XII.21).

(XII.30–31): In case the portions of the diagonals are also segments, the middle perpendicular, through the point where diagonals meet, is the half-sum of the perpendiculars in the preceding proposition [Eq. (12)]. The usual form of "Brahmagupta's theorem" follows from this. Brahmagupta expresses here the condition that diagonals are perpendicular by the identity of portions and segments, because the notion of right angle is not part of his conceptual framework.

(XII.32): For the general cyclic quadrilateral, Brahmagupta states that all the portions of diagonals and perpendiculars may be determined by proportion (*anupātena*) in two new configurations defined by the intersections of produced sides and perpendiculars. This may indeed be accomplished by repeated use of the proportions given in XII.25 for similar half-oblongs in a special configuration.

This paper and [Kichenassamy, 2010] establish two facts about Brahmagupta's discourse on the cyclic quadrilateral: it contains indications about its correct meaning, and possible derivations, some of which were ignored by later authors, even in India; it is also sufficiently detailed for us to determine which mathematical tools were available to him, and which were not [Kichenassamy, 2010, Section 3]. The method that led to these results may be of wider applicability. It consists in treating Brahmagupta's text as tightly woven discourse that possesses both small-scale structure and large-scale consistency. The requirement of consistency is a strong constraint, that helps keep in check possible tendencies to read into the text one's own idea of what it could mean. The similarity to literary analysis has been developed elsewhere [Kichenassamy, in press]; this method has also been applied to an earlier Indian text [Kichenassamy, 2006].

One often finds, in general Histories of Mathematics, expression of a feeling that "the Indians were in the habit of putting into verse all mathematical results they obtained, and of clothing them in obscure and mystic language [...] Although the great Hindu mathematicians doubtless reasoned out most or all of their discoveries, yet they were not in the habit of preserving the proofs, so that the naked theorems and processes of operation are all that have come down to our time. Very different in this respect are the Greeks" [Cajori, 1991, 83]. In the light of the present analysis, we see that two factors may have contributed to this feeling: First, a partial breach of continuity in Indian Mathematics would explain why Brahmagupta and earlier authors were misunderstood in India. Second, the wide dissemination, in the last two centuries, of Modern Geometry at elementary levels may have promoted the idea that, to be valid, a mathematical argument had to be couched in hypothetico-deductive style. As a result, ancient texts came to be evaluated, rather than understood, in terms of an often anachronistic standard. However, an appropriate analysis shows that Brahmagupta organized his discourse to convey how one result leads to another, rather than to establish that all results ultimately depend on a small number of common notions.

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