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BME 548
HW # 1

1) a) $u^T v = [2\ 0\ 4\ 3]\begin{bmatrix}3\\5\\1\\2\end{bmatrix} = 6+0+4+6 = \boxed{16}$

b) $uv^T = \begin{bmatrix}2\\0\\4\\3\end{bmatrix}[3\,5\,1\,2] = \begin{bmatrix}6 & 10 & 2 & 4\\0 & 0 & 0 & 0\\12 & 20 & 4 & 8\\9 & 15 & 3 & 6\end{bmatrix}$

c) $u \circ v = \begin{bmatrix}2\\0\\4\\3\end{bmatrix} \circ \begin{bmatrix}3\\5\\1\\2\end{bmatrix} = \begin{bmatrix}6\\0\\4\\6\end{bmatrix}$

d) $a^H b = [e^{-\frac{i\pi}{2}}\ 1\ e^{-\frac{i\pi}{4}}\ e^{-\frac{i\pi}{4}}]\begin{bmatrix}2e^{i\pi}\\0\\1e^{i\pi}\\3e^{-i\pi}\end{bmatrix} = 2e^{\frac{i\pi}{2}} + 0 + e^{\frac{3i\pi}{4}} + 3e^{-\frac{5i\pi}{4}}$

e) $a^H a = [e^{-\frac{i\pi}{2}}\ 1\ e^{-\frac{i\pi}{4}}\ e^{-\frac{i\pi}{4}}]\begin{bmatrix}e^{\frac{i\pi}{2}}\\1\\e^{\frac{i\pi}{4}}\\e^{\frac{i\pi}{4}}\end{bmatrix} = 1+1+1+1 = 4$

$b^H b = [2e^{-i\pi}\ 0\ 1e^{-i\pi}\ 3e^{i\pi}]\begin{bmatrix}2e^{i\pi}\\0\\1e^{i\pi}\\3e^{-i\pi}\end{bmatrix} = 4+0+1+9 = \boxed{14}$

f) $a \circ a^* = \begin{bmatrix}e^{\frac{i\pi}{2}}\\1\\e^{\frac{i\pi}{4}}\\e^{\frac{i\pi}{4}}\end{bmatrix} \circ \begin{bmatrix}e^{-\frac{i\pi}{2}}\\1\\e^{-\frac{i\pi}{4}}\\e^{-\frac{i\pi}{4}}\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

g) $b \circ b^* = \begin{bmatrix}2e^{i\pi}\\0\\1e^{i\pi}\\3e^{-i\pi}\end{bmatrix} \circ \begin{bmatrix}2e^{-i\pi}\\0\\1e^{-i\pi}\\3e^{i\pi}\end{bmatrix} = \begin{bmatrix}4\\0\\1\\9\end{bmatrix}$

$y = a^T I b$ where $a = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$y = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \\ 4 & 2 & 8 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10/3 & \frac{10}{3} & \frac{14}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{10/3}$$

Row major I: $I = [3\ 6\ 1\ 4\ 2\ 8\ 3\ 2\ 5]^T$

$y = c^T I$ where $c = [\frac{1}{3}\ 0\ 0\ \frac{1}{3}\ 0\ 0\ \frac{1}{3}\ 0\ 0]^T$

$$y = \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 1 \\ 4 \\ 2 \\ 8 \\ 3 \\ 2 \\ 5 \end{bmatrix} = \frac{3}{3} + \frac{4}{3} + \frac{3}{3} = \boxed{\frac{10}{3}}$$

2 a) $u = [-1, 1]$

$v = [7, 6, 5, 4, 3, 2]$

	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
v[k]						-1	1						
v[k-1]	2	3	4	5	6	7							
v[k-2]		2	3	4	5	6	7						
v[k-3]			2	3	4	5	6	7					
v[k-4]				2	3	4	5	6	7				
v[k-5]					2	3	4	5	6	7			
v[k-6]						2	3	4	5	6	7		
v[k-7]							2	3	4	5	6	7	

$= [-7, -6+7, -5+6, -4+5, -3+4, -2+3, 2]^T$

$= [-7, 1, 1, 1, 1, 1, 2]^T$

b) $W = Uv \Rightarrow \begin{bmatrix} -7 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = U \begin{bmatrix} 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \end{bmatrix} \Rightarrow \overset{w}{\begin{bmatrix} -7 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}} = \overset{U}{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \overset{v}{\begin{bmatrix} 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}}$

U is performing a difference operation between two adjacent elements in vector v. Therefore, kernel U is the difference kernel or 1^{st} order derivative kernel.

c) $w = Uv$

$U^T w = (U^T U)v$

$(U^T U)^{-1} U^T w = v$ therefore $D = (U^T U)^{-1} U^T$

$$U^T U = \overset{U}{\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}} \overset{U^T}{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}} = \overset{U^T U}{\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}}$$

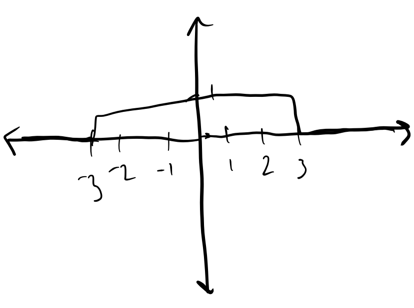
$$(U^T U)^{-1} U^T = \begin{bmatrix} 6/7 & 5/7 & 4/7 & 3/7 & 2/7 & 1/7 \\ 5/7 & 10/7 & 8/7 & 6/7 & 4/7 & 2/7 \\ 4/7 & 8/7 & 12/7 & 9/7 & 6/7 & 3/7 \\ 3/7 & 6/7 & 9/7 & 12/7 & 8/7 & 4/7 \\ 2/7 & 4/7 & 6/7 & 8/7 & 10/7 & 5/7 \\ 1/7 & 2/7 & 3/7 & 4/7 & 5/7 & 6/7 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ -5/7 & -5/7 & 2/7 & 2/7 & 2/7 & 2/7 & 2/7 \\ -4/7 & -4/7 & -4/7 & 3/7 & 3/7 & 3/7 & 3/7 \\ -3/7 & -3/7 & -3/7 & -3/7 & 4/7 & 4/7 & 4/7 \\ -2/7 & -2/7 & -2/7 & -2/7 & -2/7 & 5/7 & 5/7 \\ -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & 6/7 \end{bmatrix} = D$$

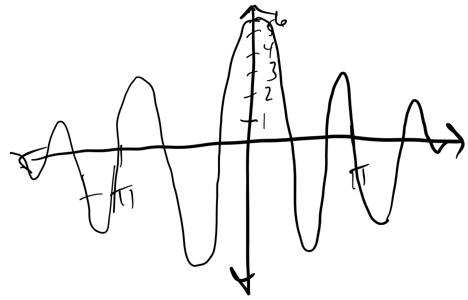
Since D is recovering v from w, it must be an integral kernel.

d) DD^T is invertible because it is a square matrix and has 6 pivot positions. $DD^T = (U^T U)^{-1} U^T$ so $(DD^T)^{-1} = U^T U$

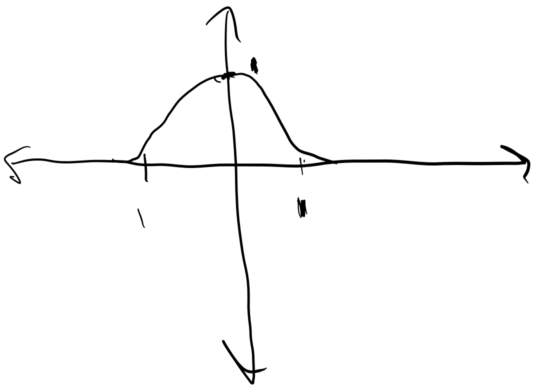
3 a) $U(x) = \text{rect}\left(\frac{x}{6}\right)$

$\hat{U}(f_x) = 6\,\text{sinc}\left(\frac{6w}{2}\right) = 6\,\text{sinc}(3w)$

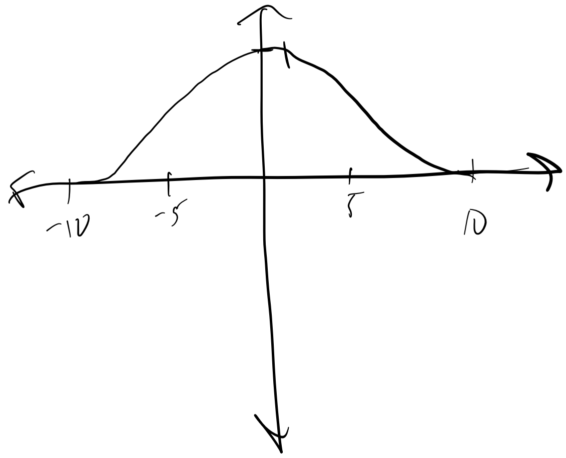




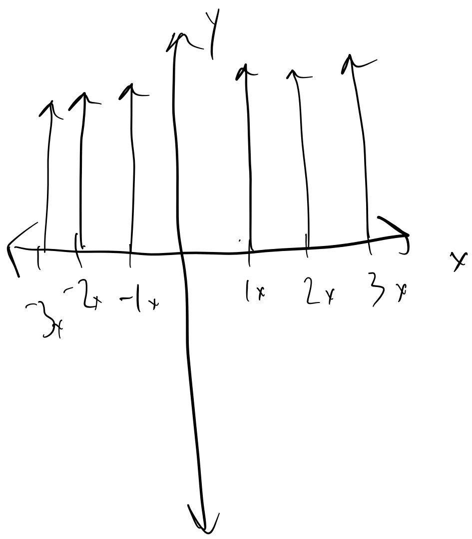
b) $V(x) = e^{-3x^2}$



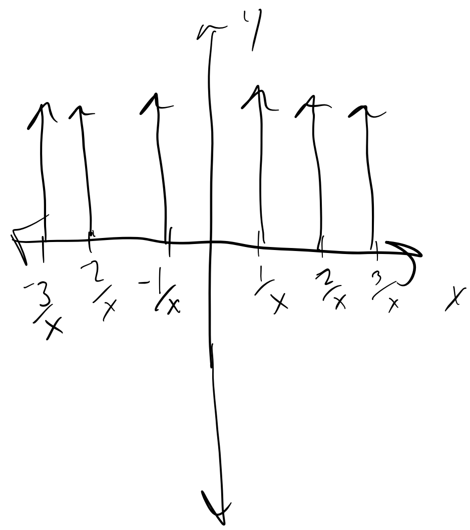
$\hat{V}(f_x) = \sqrt{\frac{\pi}{3}}\, e^{-\omega^2/12}$



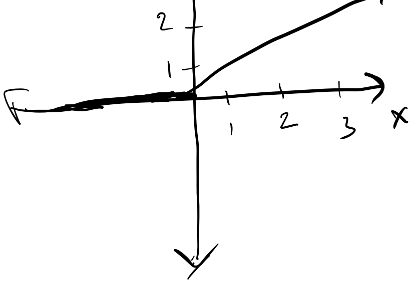
c) $V(x) = \text{III}(x)$



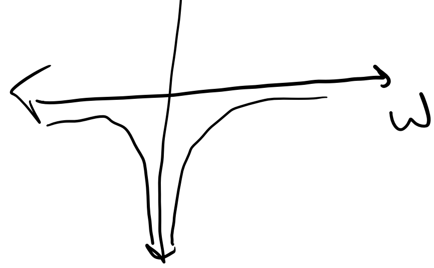
$V(f_x) = \text{III}\left(\frac{1}{x}\right)$



d) $V(x) = \text{ReLU}(x)$



$\hat{V}(f_x) = -\frac{1}{\sqrt{2\pi}\,\omega^2} \cdot -i\pi\,\delta'(\omega)$



e) $F[U(x)V(x)] = F[U(x)] * F[V(x)]$

$$F[U(x)V(x)] = \int_{-\infty}^{\infty} U(x)V(t-x)\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x)V(t-x)\,dx\, e^{2\pi i s t}\,dt$$

Substitute
$w = t - x$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x)V(w)\,e^{2\pi i s(x+w)}\,dx\,dw$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x)e^{2\pi i s(x)}\,V(w)e^{2\pi i s w}\,dx\,dw$$

$$= \int_{-\infty}^{\infty} U(x)e^{2\pi i s x}\,dx \int_{-\infty}^{\infty} V(w)e^{2\pi i s w}\,dw$$

We arbitrarily assigned variables, and the variables can be anything we choose, so we can sub w for x and we recover the convolution theorem

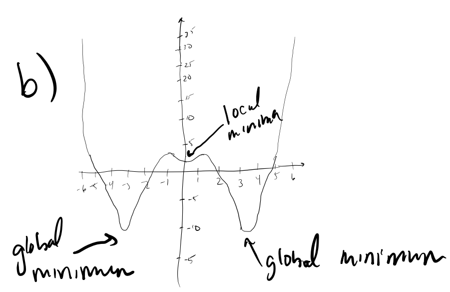
$$= \underbrace{\int_{-\infty}^{\infty} U(x)e^{2\pi i s x}\,dx}_{F[U(x)]} \underbrace{\int_{-\infty}^{\infty} V(x)e^{2\pi i s x}\,dx}_{F[V(x)]}$$

$$= F[U(x)] * F[V(x)]$$ ✓

4 a) Frequency $= 0.2/\mu m$
Period $= \frac{1\mu m}{0.2} = 5\,\mu m$

We need to sample twice per period, therefore we need to sample at least every $2.5\,\mu m$ to faithfully reconstruct $U(x)$, so a pixel pitch of $10\,\mu m$ in both x and y won't work. If the pixel pitch is $2\,\mu m$, then we would sample sufficiently and be able to faithfully reconstruct $U(x)$.

b)



A good initial value of x to begin gradient descent would be at $x = 2.5$.

A bad initial value would be at $x = 0$.

We are not always guaranteed to find the global minima because our gradient descent could get "stuck" in the local minima and incorrectly assign it as the global minimum.

c) $w = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$ $f(x) = \text{sign}(w^T x)$

$$w^T x = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = 3 + 2x_1 + x_2$$

$f(x) = 3 + 2x_1 + x_2$

$0 = 3 + 2x_1 + x_2$

$-3 = 2x_1 + x_2$ $f(x) = 0$ when $2x_1 + x_2 = -3$

$f(x) = 1$ when $2x_1 + x_2 > -3$

$f(x) = -1$ when $2x_1 + x_2 < -3$

$x_2 = -3 - 2x_1$