Report-Group3

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1 Simulation of a positron-induced Muon Source

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The main goal of this project is to produce a Monte Carlo simulation for the scattering $e^+ + e^- \to \mu^+ + \mu^-$, with $\sqrt{s} \sim 2m_\mu$ close to the muon production threshold.

In the following, all quantities denoted with a star (*) are measured in the center of mass (CM) frame of reference, while quantities without it refer to the laboratory frame. Also, $c = 1$ for all computations.

The leading-order **differential cross-section** for the unpolarized scattering $e^+ + e^- \to \mu^+ + \mu^-$ in the CM frame has the following expression Section **??**, as function of the total energy $\sqrt{s}$ in the CM frame, and the emission angles θ^* and φ^*:

$$\frac{\mathrm{d}\sigma^*}{\mathrm{d}\Omega^*}(\sqrt{s}, \theta^*, \varphi^*) = \frac{\alpha^2}{4s}\left(1 - \frac{4m_\mu^2}{s}\right)^{1/2}\left(1 - \frac{4m_e^2}{s}\right)^{-1/2}\left[1 + \frac{4}{s}(m_e^2 + m_\mu^2) + \left(1 - \frac{4m_e^2}{s}\right)\left(1 - \frac{4m_\mu^2}{s}\right)\cos^2\theta^*\right] \tag{1.1}$$

Integration over the sphere leads to the **total cross section**:

$$\sigma^*(\sqrt{s}) = \int_0^\pi \sin\theta^* \mathrm{d}\theta^* \int_0^{2\pi} \mathrm{d}\varphi^* \frac{\mathrm{d}\sigma^*}{\mathrm{d}\Omega^*}(\sqrt{s}, \theta^*, \varphi^*) = \frac{4\pi\alpha^2}{3s^3}\frac{\sqrt{1 - \frac{4m_\mu^2}{s}}}{\sqrt{1 - \frac{4m_e^2}{s}}}(2m_e^2 + s)(2m_\mu^2 + s) \tag{1.2}$$

For a fixed energy $\sqrt{s}$, the angle distribution $f(\theta^*, \varphi^*)$ of the scattered particles is given by normalizing the differential cross-section:

$$f(\theta^*, \varphi^*) = \frac{1}{\sigma} \frac{\mathrm{d}\sigma^*}{\mathrm{d}\Omega^*} (\sqrt{s}, \theta^*, \varphi^*) \tag{1.3}$$

As the scattering process possesses cylindrical symmetry, the differential cross section does not depend on φ^*. This means that $f(\theta^*, \varphi^*)$ can be factored into two independent **angular distributions**, $g(\theta^*)$ for the θ^* angle, and a uniform $h(\varphi^*)$ for the φ^* angle:

$$g(\theta^*) = \frac{\mathrm{d}\sigma^*}{\mathrm{d}\theta^*} = \int_0^{2\pi} \mathrm{d}\varphi^* \frac{1}{\sigma^*} \frac{\mathrm{d}\sigma^*}{\mathrm{d}\Omega^*} \sin\theta^* = \frac{2\pi \sin\theta^*}{\sigma^*} \frac{\mathrm{d}\sigma^*}{\mathrm{d}\Omega^*}; \qquad h(\varphi^*) = \int_0^{\pi} \sin(\theta^*) \mathrm{d}\theta^* \frac{\mathrm{d}\sigma^*}{\mathrm{d}\Omega^*} = \frac{1}{2\pi} \tag{1.4}$$

We consider the e^- stationary in the laboratory frame, so that $\beta_{e^-} = 0$. This means that the velocity β of the CM frame with respect to the laboratory is $\beta = \beta^*_{e^-}$. We can express it as a function of $\sqrt{s}$ as follows. First, note that the electron/positron 4-momenta in the CM frame are:

$$p^*_{e^\pm} = (\mathcal{E}^*_{e^\pm}, \pm\vec{p}^*_{e^+}); \qquad \mathcal{E}^*_{e^\pm} = \sqrt{m_e^2 + ||\vec{p}^*_e||^2}$$

where $\vec{p}^*_{e^+}$ is the positron 3-momentum in the CM frame.

So s is equal to:

$$s = (p^*_{e^-} + p^*_{e^+})^2 = 4(m_e^2 + ||\vec{p}^*_e||^2) = 4(\mathcal{E}^*_e)^2 \Rightarrow \mathcal{E}^*_e = \frac{\sqrt{s}}{2} \tag{1.5}$$

Then:

$$\beta^*_e = \beta = \frac{||\vec{p}^*_e||^2}{\mathcal{E}^*_e} = \frac{\sqrt{(\mathcal{E}^*_e)^2 - m_e^2}}{\mathcal{E}^*_e} = \sqrt{1 - \frac{m_e^2}{(\mathcal{E}^*_e)^2}} = \sqrt{1 - \frac{4m_e^2}{s}} \tag{1.6}$$

The same calculations can be done for the $\mu^\pm$, leading to:

$$\beta^*_{\mu^\pm} = \sqrt{1 - \frac{4m_\mu^2}{s}} \tag{1.7}$$

Then, the norm of the muon 3-momentum as function of $\sqrt{s}$ is:

$$||\vec{p}^*_\mu|| = \sqrt{(\mathcal{E}^*_\mu)^2 - m_\mu^2} = \sqrt{\frac{s}{4} - m_\mu^2} \tag{1.8}$$

Knowing $\sqrt{s}$, θ^* and φ^* uniquely identifies the muon 4-momentum components:

$$p^*_\mu = \left(\frac{\sqrt{s}}{2}, \vec{p}^*_\mu\right) \qquad \vec{p}^*_\mu = ||\vec{p}^*_\mu||(\cos\theta^*, \sin\theta^*\cos\varphi^*, \sin\theta^*\sin\varphi^*) \tag{1.9}$$

Finally, we can boost the angle θ^* and the 4-momentum to the laboratory frame:

$$\tan\theta_\mu = \frac{\sin\theta^*_\mu}{\gamma\left(\cos\theta^*_\mu + \frac{\beta}{\beta^*_\mu}\right)} \tag{1.10}$$

$$\begin{pmatrix} \mathcal{E} \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{E}^* \\ p^*_x \\ p^*_y \\ p^*_z \end{pmatrix} \tag{1.11}$$

```
In [1]: %matplotlib inline
        import numpy as np
        import pandas as pd
        import matplotlib
        import matplotlib.pyplot as plt
        import seaborn as sns
        from scipy.interpolate import interp1d

        #Constants
        alpha = 0.00729735257
        mmu = 105.66 #MeV
        me = 0.511 #MeV

        #Cross-section
        def dsigma(s, theta):
            """Differential cross-section (1) [MeV^-2 sr^-1]"""
            return (alpha**2 / (4*s)) * (beta_mu(s) / beta_e(s)) * (1. + 4. * (me**2 + mmu**2)

        def sigma(s):
            """Total cross section (2) [MeV^-2]"""
            return (4 * np.pi * alpha**2) / (3 * s**3) * (beta_mu(s) / beta_e(s)) * (2 * me**2

        #Distributions
        def g(s, theta):
            """Distribution for theta* [First. eq. in (4)]"""

            return (2 * np.pi) * np.sin(theta) * (3 * s * (4 * (me**2 + mmu**2) + s) + 3 * (-4
                   (16 * np.pi * (2 * me**2 + s) * (2 * mmu**2 + s))

        def g_cdf(s, theta):
            """CDF for the theta* distribution"""

            return (4 * (2 * me**2 + s) * (2 * mmu**2 + s) - 3 * s * (4 * me**2 + 4 * mmu**2 +
                   (8 * (2 * me**2 + s) * (2 * mmu**2 + s))

        #Kynematics functions
        def beta_mu(s):
            """beta_mu* (eq. 7)"""
            return np.sqrt(1 - 4*(mmu**2)/s)

        def beta_e(s):
            """beta_e* = beta (eq. 6)"""
            return np.sqrt(1 - 4*(me**2)/s)

        def gamma_e(s):
            """gamma_e^* = gamma"""
            return 1 / np.sqrt(1 - beta_e(s)**2)
```

```python
def p_mu_cm(s):
    """Norm of muon 3-momentum in CM (eq. 8)"""
    return np.sqrt(s/4 - mmu**2)

#Boosts
def theta_boost(s, theta_cm):
    """Converts theta* (measured in CM) for a muon into tan(theta) (measured in labora

    return np.sin(theta_cm) / \
           (gamma_e(s) * (np.cos(theta_cm) + beta_e(s)/beta_mu(s)))

def energy_x_boost(s, x_cm):
    """Boosts E* => E, and x^* => x (from CM to lab frame), with the following formula
        E = gamma * ( E_cm         + beta * x_cm )
        x = gamma * ( beta * E_cm + x_cm         )
        with beta = beta_e(s), and E_cm = sqrt(s)/2
    """

    E_cm  = np.sqrt(s) / 2
    gamma = gamma_e(s)
    beta  = beta_e(s)

    return (gamma * ( E_cm + beta * x_cm ), gamma * (beta * E_cm + x_cm))

#Utility
def to_cartesian(norm, theta, phi):
    """Returns cartesian components of 3-vector (eq. 9)"""
    sintheta = np.sin(theta)
    return (norm * np.cos(theta), norm * sintheta * np.cos(phi), norm * sintheta * np.s
```

```python
In [2]: #Plot settings

        from matplotlib.ticker import AutoMinorLocator, MultipleLocator, FuncFormatter
        from matplotlib import rc

        rc('font',**{'family':'sans-serif','sans-serif':['Helvetica'], })
        rc('xtick', labelsize=12)
        rc('ytick', labelsize=12)
        rc('axes', titlesize=16, labelsize=14)
        rc('legend', fontsize=12)
```

4 Distribution plots

4.1 a. Cross-section

The plot of the cross section $\sigma(\sqrt{s})$ and of (a section of) the differential cross section dœ/d■(`*,'*) are shown below.